ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Lagrangian cutting

Bachelor Thesis

Giovanni Ambrosioni

May 29, 2020

Supervisor: Prof. Dr. Ana Cannas da Silva
Department of Mathematics, ETH Zürich

Abstract

In this thesis we first recall well known concept and results regarding group actions of symplectic manifolds, especially the symplectic cutting procedure introduced by Lerman in [Ler95]. We study how lagrangian submanifolds behave with respect to the cutting procedure.

Introduction

Simplectic geometry is a branch of mathematics that naturally arises in hamiltonian classical mechanics, where certain systems are associated to phase spaces that take the form of what is called a symplectic manifold.

This thesis is mainly concerned with the kind of symplectic geometry which involves a certain type of smooth action of a toric group on a symplectic manifold. The first chapter aims to introduce all the basic notions and notations needed.

We first study how we can reduce symmetries of a symplectic manifold. In mathematical terms we can describe symmetries of a manifold by studying invariance properties of smooth Lie group actions on it. It is natural to ask wheater and how we can exploit symmetries in order to reduce the number of variables when handling a physical problem. In the symplectic setting, this passes under the name *symplectic reduction* and is the content of the second chapter of this thesis. There we also study how a class of submanifolds, the so called *lagrangian submanifolds*, of a symplectic manifold behaves under this process and we provide an important example.

In the third chapter we introduce *symplectic cutting*, a procedure invented in the nineties that uses symplectic reduction to decompose a symplectic manifolds into two symplectic manifolds. This is the central notion of this thesis.

In the fourth chapter we show that the cutting procedure preserves a special class of lagrangian submanifolds of symplectic toric manifolds.

Acknowlegments

My gratitude to Prof. Ana Cannas Da Silva is twofold: I thank her for introducing me to symplectic geometry and toric manifolds trought the seminar she organizes during the spring semester 2019, and I also thank her for supervising me for the (long) duration of this thesis. We had many valuable conversations, both on mathematics and non-mathematics, and I'm very thankful for the time she spent for me.

Thanks also to Yael Karshon for her interest and for sharing with me a draft of a preprint.

Contents

Contents **v**

1 Preliminaries **1**
- 1.1 Basics of symplectic geometry . . . 1
- 1.2 Hamiltonian actions . . . 4
- 1.3 Symplectic toric manifolds . . . 5

2 Symplectic reduction **9**
- 2.1 Marsden-Weinstein-Mayer Theorem . . . 9
- 2.2 Reduction in stages . . . 11
- 2.3 Lagrangian reduction . . . 13
- 2.4 Reduction of the real part of a symplectic manifold . . . 17

3 Symplectic cutting **21**
- 3.1 Intuition . . . 21
- 3.2 Lerman's original symplectic cutting . . . 23
- 3.3 Multi-dimensional cutting . . . 26
- 3.4 Karshon's cutting . . . 31

4 Lagrangian cutting **35**
- 4.1 Toric lagrangian cutting . . . 35
- 4.2 Circle cutting . . . 39

5 Final remarks **43**

Bibliography **47**

Chapter 1

Preliminaries

We start this paper by introducing basic definitions and facts about symplectic geometry, in order to fix notations and provide all the necessary knowledge to understand what follows. Some basic knowledge of differential manifolds and Lie theory is assumed.

1.1 Basics of symplectic geometry

Definition 1.1 *Let M be a C^∞-differentiable manifold (or smooth). A non-degenerate closed two form $\omega \in \Omega^2(M)$ is called symplectic form, and the couple (M, ω) is said to be a symplectic smooth manifold.*

Examples 1.2 *On $\mathbb{R}^{2n}$ with coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ we have the symplectic form*

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Under the canonical identification with $\mathbb{C}^n$, the above form becomes

$$\omega_0 = \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i$$

Symplectic forms are obviously not unique on a specified manifold, e.g. on $\mathbb{C}^*$ we have the symplectic form $d\log(|z|) \wedge d\theta$.

In the symplectic category, we define isomorphism exactly as one would expect.

Definition 1.3 *Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds and let $f : M_1 \to M_2$. We say that f is a symplectomorphism if it is a diffeomorphism and $f^*\omega_2 = \omega_1$.*

We define some special submanifolds.

Definition 1.4 *Let (M, ω) be a symplectic manifold, $N \subset M$ a submanifold and $i : N \hookrightarrow M$ the inclusion. Given $x \in N$ we define the symplectic orthogonal of T_xN as*

$$T_xN^\omega := \{v \in T_xM : \forall w \in T_xN \ \omega_x(v, w) = 0\}$$

We call N:

1. *an isotropic submanifold if each tangent space of N is contained in its symplectic orthogonal;*
2. *a coisotropic submanifold if each tangent space of N contains its symplectic orthogonal;*
3. *a lagrangian submanifold if each tangent space of N equals its symplectic orthogonal;*
4. *a symplectic submanifold if $(N, i^*\omega)$ is a symplectic manifold.*

A manifold N included in N as set is called an immersed lagrangian if each tangent space of N equals its symplectic orthogonal, and the inclusion of N in M is an immersion.

We state the famous Darboux Theorem, which tells us that for symplectic manifolds, dimension is the only thing that matters locally. In some sense, symplectic geometry is very topological.

Theorem 1.5 (Darboux) *Let (M, ω) be a symplectic manifold and $x \in M$. Then there are local coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ on an open neighborhood U of x, such that*

$$\omega|_U = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Proof. A proof can be found in [Can08]. □

One of the first thing that comes to mind looking at the definition is to compare symplectic forms with Riemannian metrics[1]. First, Darboux Theorem distinghuishes symplectic and Riemannian geometry a lot: in the

[1] A Riemannian metric is a section of $T^*M \otimes T^*M$ which is an inner product at any point of M.

latter case we have got local invariants, e.g. the curvature; in fact thanks to Darboux we could say that symplectic manifolds behave similarly to flat manifolds: in this sense we may interpret closedness as a symplectic analogue of flatness in the riemannian setting. Second, not every manifold admits a symplectic structure: necessary conditions for a manifold to be symplectic are even dimensionality, as linear algebra tells us, and orientability, since the top power of the symplectic form is a volume form. There are also similarities; here we're going to build one, which will be central in our definition of moment map. Given a smooth function $f \in C^\infty(M)$ from a Riemannian manifold (M, m) to $\mathbb{R}$, we have the gradient $\mathrm{grad}_m f \in \Gamma(TM)$ defined through $\imath_{\mathrm{grad}_m f} m = df$. Similarly, given a smooth function, or, as symplectic geometers like to call functions, hamiltonian, $H \in C^\infty(M)$, we define the Hamiltonian vector field $X_H \in \Gamma(TM)$ associated to H through

$$\imath_{X_H} \omega = dH$$

Both the gradient and the hamiltonian vector field constructions are well-defined because both the tensor structures are non-degenerate by defintion, so that they induce an isomorphism between vector fields and one forms on M.

Let's consider an example to visualize the situation.

Examples 1.6 *Consider the height function $H : (\theta, h) \in S^2 \to h \in \mathbb{R}$ on the sphere $(S^2, d\theta \wedge dh)$. We have $dh = \imath_{X_H}(d\theta \wedge dh) = d\theta(X_H) \wedge dh$, i.e. $d\theta(X_H) = 1$, so that $X_H = \partial_\theta$. Similarly, if we endow S^2 with the induced metric from the euclidean one on $\mathbb{R}^2$, we get $\imath_{grad_m H} m = dh$, so $grad_m H = \partial_h$.*

Here we notice something: gradient and hamiltonian differ by a right angle, in particular the hamiltonian vector fields goes along level sets of the height function (horizontal circles of the sphere), suggesting a preservation. Next lemma shows this and moreover tells us that hamiltonian vector fields maintain the form of the manifold.

Definition 1.7 *Let (M, ω) be a symplectic manifold. A vector field on M is called symplectic if its flow preserves the symplectic form. A vector field on M is called hamiltonian if it is the hamiltonian vector field of a smooth function on M.*

Lemma 1.8 *Let (M, ω) be a symplectic manifold. Any hamiltonian* *vector field on M is symplectic and in addition its integral curves are contained in a level set of its hamiltonian function.*

Proof. Take $X_H \in \Gamma(TM)$ be the hamiltonian vector field of the map $H \in C^\infty(M)$ and denote by φ the flow of X_H. Then, as $\varphi_0 = \mathrm{id}$, $\varphi_0^*\omega = \omega$, and

$$\frac{d}{dt}\varphi_t^*\omega = \varphi_t^*\mathcal{L}_{X_H}(\omega) = \varphi_t^*(d\imath_{X_H}\omega + \imath_{X_H}d\omega) = 0$$

In alternative, we can prove the first claim by noticing that for a vector field being symplectic amounts to say that its contraction with the symplectic form is closed, while it is exact in the hamiltonian case. It is also $X_H(H) = \mathcal{L}_{X_H}(H) = \imath_{X_H}dH = \imath_{X_H}\imath_{X_H}\omega = 0$ so that straight from the definitions, for any time t: $H(x) = \varphi_t^*H(x) = H(\varphi_t(x))$. □

We conclude that hamiltonian vector fields are nicer objects than gradients. In this thesis we will almost always assume a further property of hamiltonian vector fields: time-periodicity of flows. The functions satisfying this condition are a particular instance of what we will call "moment maps".

1.2 Hamiltonian actions

Consider a smooth action of a Lie group G on a symplectic manifold (M, ω). We will denote the stabilizer of a point $x \in M$ by G_x and its orbit by $G \cdot x$. Given an element $X \in \mathfrak{g}$ of the Lie algebra of G, we define its fundamental vector field through

$$_MX^\#(x) := \frac{d}{dt}|_{t=0}\exp(-tX)\cdot x$$

for $x \in M$.

Definition 1.9 *If the action preserves ω, i.e. if at each point of G the action is a symplectomorphism, we say that it is symplectic. Notice that this is equivalent to say that $i_{X^\#}\omega$ is closed for any $X \in \mathfrak{g}$.*

We now recall the construction of the coadjoint representation, which will be our fixed action of G on $\mathfrak{g}^*$, the dual of the Lie algebra of G.

Definition 1.10 *Let ψ be the conjugation action of G on itself. We define the adjoint representation $Ad : G \to GL(\mathfrak{g})$ through $Ad_g := D\psi_g(e)$. We will call coadjoint representation $Ad^* : G \to GL(\mathfrak{g}^*)$ the dual of the adjoint representation. In a formula:*

$$\langle Ad_g^*\xi, X\rangle = \langle \xi, Ad_{g^{-1}}X\rangle$$

The proof that these maps are indeed representations can be found looking for Proposition 10.20 in [Mer19].

Back to our original action. Let's suppose that the contraction is exact, i.e. for any Lie vector X there is $\mu^X \in C^\infty(M)$ s.t. $X^\#$ is the hamiltonian vector field for μ^X.

Definition 1.11 *The action of G on (M, ω) is hamiltonian if for any $X \in \mathfrak{g}$ there is $\mu^X \in C^\infty(M)$ s.t. $i_{X^\#}\omega = d\mu^X$ and if $\mu \in C^\infty(M, \mathfrak{g}^*)$ defined through $\langle \mu, X \rangle := \mu^X$ is equivariant with respect to the coadjoint representation on $\mathfrak{g}^*$. We then call (M, ω, G, μ) hamiltonian G-space.*

Remark 1.12 *Not every symplectic group action is hamiltonian: for example, the action of the two torus $\mathbb{T}^2$ with standard symplectic form on itself by translation is symplectic but not hamiltonian, as orbits are symplectomorphic to the original torus, and this contradicts Theorem 1.15.*
An example of obstruction for a symplectic action on a compact connected manifold to be hamiltonian is the first de Rahm cohomology (see [Can08]).

1.3 Symplectic toric manifolds

Consider an hamiltonian $\mathbb{T}^n$-space $(M, \omega, \mathbb{T}^n, \mu)$. A cornerstone theorem in the study of these spaces is the following theorem due to Atiyah[Ati82] and Guillemin-Sternberg[GS82].

Theorem 1.13 (Atiyah, Guillemin-Sternberg) *If M is connected and compact, then the levels of μ are connected and the image of μ is the convex hull of the fixed points of the action.*

Proof. A proof can be found in [Aud91]. □

In what follows we will constantly assume that our action moves every point.

Definition 1.14 *An action of a group G on a manifold M is said to be effective if the intersection of all stabilizers $G_x := \{g \in G : gx = x\}$, for $x \in M$, is trivial.*

We motivate the main definition thanks to next two facts.

Theorem 1.15 *Let the action of $\mathbb{T}^n$ on M be effective, then:*

1. *there is an n-dimensional orbit;*
2. *orbits are isotropic embedded submanifolds of (M, ω).*

Proof. A proof can be found in [Can08]. □

We will concentrate on the limit case.

Definition 1.16 *A symplectic toric manifold is a connected hamiltonian $\mathbb{T}^n$-space $(M, \omega, \mathbb{T}^n, \mu)$, where the torus action is effective and $\dim(M) = 2n$.*

Compact symplectic toric manifold have been classified by Delzant in [Del88] in the late 80's by simple, rational and smooth polytopes.

Definition 1.17 *A polytope in $\mathbb{R}^n$ is a bounded polyhedral set in $\mathbb{R}^n$. A polytope is said to be:*

1. *simple, if exactly n edges meet at each vertex;*
2. *rational, if each edge is of the form $x + tv$ for a vertex x and $v \in \mathbb{Z}^n$;*
3. *smooth, if it is rational and if for each vertex x, with corresponding edges $\{x + tv_i\}_{i=1}^n$, the set $(v_1, ..., v_n) \subset \mathbb{Z}^n$ forms a basis of $\mathbb{Z}^n$.*

Theorem 1.18 (Delzant) *There is a bijection between symplectic toric manifold modulo equivariant symplectomorphisms and simple, rational and smooth polytopes modulo translation given by the image of the moment map.*

Proof. This is proved in [Del88]. □

Karshon and Lerman then classified all symplectic toric manifolds in [KL15]. It's worth mentioning that in general the moment map gives us little information about the hamiltonian space. Karshon and Lerman showed an equivalence of categories between symplectic toric manifolds and symplectic toric bundles to prove the following theorem. This equivalence of categories is closely related to the notion of symplectic cut introduced by Lerman in [Ler95] and to its generalization later made by Karshon in [Kar20].

In the following theorem, the definition of unimodular local embedding requires some further notions on manifold with corners, so we will not define it in this thesis. However, the content of the theorem is understandable if we understand unimodular local embeddings as local diffeomorphisms that send corners of a manifold with corners to smooth cones in $\mathbb{R}^n$, that is, they straighten corners. Details may be found in [KL15].

Theorem 1.19 *Consider a unimodular local embedding $\Psi : W \to \mathbb{R}^n$ of a manifold with corners W, then:*

1. *there is a symplectic toric manifold $(M, \omega, \mathbb{T}^n, \mu)$ s.t. $M/\mathbb{T}^n = W$ and μ descends to Ψ;*
2. *the equivalence classes of symplectic toric manifolds with $M/\mathbb{T}^n = W$ and moment map descending to Ψ are in bijective correspondence with $H^2(\mathbb{Z}^n \times \mathbb{R})$.*

Proof. This is proved in [KL15]. □

In this work we'll deal most of the time with symplectic toric manifolds whose moment map describes it well, i.e. when the the moment map is proper as a map to a convex set: in this case the image (see Theorem 4.2 in [Ler+96]) is convex and the induced map on the orbit space is an embedding, so that the above cohomology vanishes; and so:

Theorem 1.20 *Consider a connected symplectic toric manifold such that the moment map is proper as a map into a convex open set. Its image determines the manifold up to equivariant symplectomorphism.*

Proof. This is proved in [KL15]. □

Definition 1.21 *Fix n primitive lattice vectors $\{v_1, ..., v_n\}$ spanning the integral lattice Λ of the n-torus, i.e. the kernel of the Lie exponential map $\mathfrak{t}^n \cong \mathbb{R}^n \to \mathbb{T}^n$. A rational polyhedral set in $\mathfrak{t}^*$ is a set of the form $\Delta = \bigcap_{i=1}^k \{x \in \mathfrak{t}^* : \langle x, v_i \rangle \leq \lambda_i\}$ with non-empty interior, where $\lambda_i \in \mathbb{R}$. Such a polyhedral set is called Delzant if it also satisfies a smoothness condition, i.e. whenever for an index set $I \subset \{1, ..., k\}$ one has[2] $\Delta_I := \{x \in \mathfrak{t}^* : \forall i \in I : \langle x, v_i \rangle = \lambda_i\} \neq \emptyset$, then*

$$span_{\mathbb{Z}}\{v_i : i \in I\} = \Lambda \cap span_{\mathbb{R}}\{v_i : i \in I\}$$

Notice that fixing our choice of the primitive lattice vectors ensures simplicity of the polyhedral set. Smoothness in the "non polytope" case is very similar to the polytope case, however it can happen that a polyhedral set doesn't have a vertex, or that it is smooth at a vertex, but not everywhere: for instance, consider the cone hull of $(1, 2, 1)$, $(0, 1, 1)$ and $(1, 0, 1)$.

[2] Define $\Delta_{\emptyset} := \mathrm{int}(\Delta)$.

Theorem 1.22 *Consider a connected symplectic toric manifold such that the moment map is proper as a map into a convex open set. Its image is the intersection of that convex set with a Delzant polyhedral set.*

Proof. This is a clear consequence of last two theorems. □

Here there are two basic examples.

Example 1.23 *Consider the standard symplectic structure on $\mathbb{C}^n$ and consider the action of the n-torus $\mathbb{T}^n$ by coordinatewise rotation. This action gives $\mathbb{C}^n$ the structure of a symplectic toric manifold, with standard moment map given by*

$$\mu : (z_1, ..., z_n) \in \mathbb{C}^n \longmapsto \frac{1}{2}(|z_1|^2, ..., |z_n|^2)$$

Indeed, if $v = (v_1, ..., v_n) \in \mathbb{R}^n$, then if we put $z_j = x_j + iy_j$ for $x_j, y_j \in \mathbb{R}$ it is easy to check that (up to the cononical isomorphism for tangent spaces of vector spaces)

$$d\mu^v(z) = \sum_{i=1}^{n} v_i(x_i dx(x_i) + y_i dx(y_i)) = \frac{d}{dt}\bigg|_0 (e^{itv_1}z_1, ..., e^{itv_n}z_n)$$

showing that the action is hamiltonian.

Example 1.24 *Similary to what above, $\mathbb{C}P^n$ with Fubini-Study symplectic structure is a symplectic toric manifold when endowed with the action of $\mathbb{T}^n$ on the last n homogeneous coordinates. The moment map is*

$$\mu : [z_0, ..., z_n] \in \mathbb{C}P^n \mapsto \frac{1}{2}\left(\frac{|z_1|^2}{\sum_{i=0}^n |z_i|^2}, ..., \frac{|z_n|^2}{\sum_{i=0}^n |z_i|^2}\right)$$

Chapter 2

Symplectic reduction

In this chapter we introduce the famous Marsden-Weinstein-Mayer symplectic reduction, a procedure to destroy symmetries of a hamiltonian space. We then study how reduction behaves under subgroup actions. In the end we try to establish under which condition the reduction property keeps the lagrangian property of a submanifold, and one important example is then presented in detail.

2.1 Marsden-Weinstein-Mayer Theorem

We recall a fundamental construction of hamiltonian geometry, which is called *symplectic reduction* and is due to the work of Marsden–Weinstein in [MW74] and of Meyer in [Mey73]. The idea is to kill the simmetries of an hamiltonian space in order to lower the dimension of our original manifold.

We could naively hope that the restriction of a symplectic form to a level set of a regular value of the moment map mantains non-degeneracy, however its's not hard to see that the null foliation of the form is the collection of the coadjoint stabilizer orbits. Since the level set is coisotropic (this is part of Lemma 2.5), symplectic linear algebra then tells us that we get linear symplectic forms on the quotient of the tangent spaces, and the reduction theorem will tell us that under some assumptions we get a symplectic forms on the orbit spaces of regular levels.

We will cover symplectic reduction for a general Lie group G. However, our furthere applications only involve the compact connected abelian Lie group $\mathbb{T}^n$.

Definition 2.1 *Consider a smooth action of a Lie group G on a manifold M. The action is free if given $g_1, g_2 \in G$ there is $x \in M$ such that $g_1 \cdot x = g_2 \cdot x$, then $g_1 = g_2$. The action is proper if it is proper as a map $(g, x) \in G \times M \longmapsto (g \cdot x, x) \in M \times M$.*

It is not hard to see that actions of compact Lie groups are proper actions. Let's first recall the quotient manifold theorem from differential geometry.

Theorem 2.2 *If a Lie group G acts smoothly, freely and properly on a manifold M, then the orbit space M/G is a manifold and the projection $\pi : M \to M/G$ is a principal G-bundle.*

Proof. A proof can be found in [Can08]. □

Let G be a Lie group and let $(M, \omega, G, \mu : M \to \mathfrak{g}^*)$ be a hamiltonian G-space. Let $v \in \mathfrak{g}^*$ such that the action of the coadjoint stabilizer G_v on $\mu^{-1}(v)$ is free. Define the orbit space $M_v^G := \mu^{-1}(v)/G_v$. By quotien $i_v : \mu^{-1}(v) \hookrightarrow M$ be the inclusion and $\pi_v : \mu^{-1}(v) \to M_v^G$ be the projection. Notice that the G_v is in fact the largest subgroup of G acting on $\mu^{-1}(v)$, by definition of moment map. We want π_v to have the structure of a principal G_v-bundle in order to pushforward $i_v^*\omega$ to M_v^G: the pushforwarded form will be symplectic since it is symplectic on the tangent space as we noted above.

Theorem 2.3 (Marsden-Weinstein, Mayer) *Let $v \in \mathfrak{g}^*$ be a regular value of the moment map μ. If the action of G_v on $\mu^{-1}(v)$ is proper, then:*

1. *M_v^G is a manifold;*
2. *π_v is a principal G_v-bundle;*
3. *there is a unique symplectic form ω_v on M_v^G satisfying $\pi_v^*\omega_v = i_v^*\omega$.*

Proof. A proof can be found in [Can08]. □

An accurate analysis of theorem above suggests us that a "reduction" construction can be done in a more general setting. One can show that for any arbitrary coisotropic submanifold $C \subset (M, \omega)$, $(TC)^\omega \subset TC$ is an integrable $(2n - m)$-distribution on C, so that by Frobenius theorem it is induced by a foliation on C, whose partition is made of isotropic manifolds. In general, the leaf space of the foliation is not a manifold, however, the construction can be done locally: linear algebra tells us that tangent spaces of the leaf space inherit a symplectic structure which are independent from the chosen points on the leaves. The interested reader may have a look at Lecture 3 in [Wei77] for more details.

2.2 Reduction in stages

We will now investigate how this construction behaves with respect to product groups. What follows is an extension of chapter 24.3 in [Can08], as suggested by the last line of it, with some aid from [Mar+06]. For simplicity we will work with 0-levels, where the coadjoint stabilizer is the full group. This is not a great restriction as in this thesis we will mostly deal with the abelian case.

Consider a compact connected Lie group G and a normal Lie subgroup H. Define the inclusion $\imath : H \hookrightarrow G$ and the quotient $p : G \to G/H$. Let (M, ω) be a connected symplectic manifold, and suppose that (M, ω, G, μ_G) is a hamiltonian G-space. We know that (M, ω, H, μ_H), with $\mu_H := D\imath(e)^* \circ \mu_G$, is a hamiltonian H-space. In the remaining of this section we will specify the group for the inclusions and projections coming from reduction.

Corollary 2.4 *Assume that the action of H on $\mu_H^{-1}(0)$ is free. Then there is an action of G/H on M_0^H which is hamiltonian and whose moment map $\mu_{G/H}$ satisfies $Dp(e)^* \circ \mu_{G/H} \circ \pi_0^H = \mu_G \circ i_0^H$.*

Proof. It follows by equivariance of μ_G and normality of H in G, that G acts on $\mu_H^{-1}(0) = \mu_G^{-1}(\ker(D\imath(e)^*))$, so we have an action of G on $\mu_H^{-1}(0)/H = M_0^H$ given by $g \cdot (H \cdot x) := H \cdot (g \cdot x)$. Notice that it is π_0^H-equivariant. This action of course passes to an action of G/H on M_0^H.
We construct the moment map $\mu_{G/H} : M_0^H \to (\mathfrak{g}/\mathfrak{h})^*$. Notice that by equivariance $\mu_G \circ i_0^H$ is constant on H, since H acts trivially on $\ker(D\imath(e)^*) = im(Dp(e)^*)$, and it factors trought the dual of the Lie algebra of the quotient group since $\ker(D\imath(e)^*) = \mathrm{im}(Dp(e)^*)$.
We show that this is indeed a moment map. Let $X = Dp(e)[X'] \in \mathfrak{g}/\mathfrak{h}$, first we show that $-i_X^{\#}\omega^{\mathrm{red},H} = d\mu_{G/H}^X$. Let $x \in \mu_H^{-1}(0)$ and $D\pi_0^H(x)[v] \in$

$T_{H\cdot x}\big(M_0^H\big)$, then:

$$\begin{aligned}\omega^{\mathrm{red},H}_{H\cdot(x)}(X^{\#}(H\cdot x), D\pi_0^H(x)[v]) &= \omega^{\mathrm{red},H}_{\pi_0^H(x)}(D\pi_0^H(x)[X'^{\#}(x)], D\pi_0^H(x)[v])\\ &= ((\pi_0^H)^*\omega^{\mathrm{red},H})_x(X'^{\#}(x), v)\\ &= ((i_0^H)^*\omega)_x(X'^{\#}(x), v)\\ &= d\mu_G^{X'}|_x(v) = Dp(e)^*\circ d\mu_{G/H}^{X'}|_{\pi_0^H(x)}(d\pi_0^H|x(v))\\ &= d\mu_{G/H}^{X}|_{H\cdot x}(D\pi_0^H(x)(v))\end{aligned}$$

working with the natural identifications of tangent spaces of vector spaces. Equivariance is a similar mess of notation and follows by equivariance of p and μ_G. $\square$

A consequence is that if we reduce first with respect to H then with respect to G/H we get a symplectic manifold symplectomorphic to the G-reduction.

Corollary 2.5 *There is a symplectomorphism between $(M_0^H)_0^{G/H}$ and M_0^G.*

Proof. This is just a chasing game. $\square$

Corollary 2.6 *Let G_1, G_2 be two Lie groups, and suppose that a symplectic manifold carries both the structures of a G_1-hamiltonian space and of a G_2-hamiltonian space. If the two action commute and the moment maps are invariant with respect to the other action, then we have that the reduced space with respect to the G_1 action is naturally a hamiltonian G_2-space (and viceversa).*

Proof. We just have to prove that $(M, \omega, G_1\times G_2, (\mu_1, \mu_2))$ is a hamiltonian space under these assumption, then the claim follows by Corollary 2.4. This is easy to see, as the only thing to check is equivariance. $\square$

The results of this section may seem basic to the experts. I initially struggled to understand why Lerman's manifold (cfr. Theorem 3.4) actually carried its structure, and in fact didn't know why as I presented it last year during a seminar held by Prof. Ana Cannas da Silva on symplectic toric manifolds. I added these results and their proofs more to let a trace of my initial difficulty than out of mathematical interest.

2.3 Lagrangian reduction

What follows is mainly a revisitation of [Wei77].

Our aim is to reduce a lagrangian submanifold of a hamiltonian S^1-space (M, ω, S^1, μ) to get a lagrangian submanifold of the reduced manifold at the zero level $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, S^1, \mu_{\mathrm{red}})$. We assume that 0 is a regular value of μ. Fix a lagrangian submanifold $L \subset M$. Denote $\pi : \mu^{-1}(0) \to M$ the orbit projection and $i : \mu^{-1}(0) \hookrightarrow M$, $j : \mu^{-1}(0) \cap L \hookrightarrow \mu^{-1}(0)$ and $\overline{j} : L_{\mathrm{red}} := \pi(L \cap \mu^{-1}(0)) \hookrightarrow M_{\mathrm{red}}$ the inclusions (the last two are for the moment map of sets).

We start from the simplest case. The assumption that L is contained in the zero level set is necessary in this case, because lagrangianity and invariance imply that the moment map is constant on L.
We need a preliminary lemma.

Lemma 2.7 *For any $x \in \mu^{-1}(0)$ it is*

$$\ker\big(D\pi(x)\big) = T_x\mu^{-1}(0)^{\omega_x}$$

In particular the foliation induced by the symplectic orthogonal is the partition of $\mu^{-1}(0)$ into the isotropic submanifolds $G \cdot x$, $x \in M$, of M.

Proof. Denote by ψ the action of G on M. Counting dimension we get that for any $x \in \mu^{-1}(0)$ it is $\ker\big(D\pi(x)\big) = D\psi_x(e)[\mathfrak{g}] = T_x(G \cdot x)$, but for any $X \in \mathfrak{g}$

$$D\psi_x(e)[X] = \frac{d}{dt}|_{t=0}\psi_x(\exp(-tX)) = X^{\#}(x)$$

so that $\ker\big(D\pi(x)\big) = \{X^{\#}(x) : \ X \in \mathfrak{g}\}$. On the other side again by dimension count we have $T_x\mu^{-1}(0) = \ker\big(D\mu(x)\big)$, since 0 is a regular value of 0. Let $v \in T_x\mu^{-1}(0)$, then for any $X \in \mathfrak{g}$:

$$0 = D\mu(x)[v](X) = d\mu^X|_x(v) = \omega(X^{\#}(x), v)$$

that means $v \in T_x(G \cdot x)^{\omega_x}$. Looking a last time at dimensions, we conclude the proof. $\square$

Lemma 2.8 *If the lagrangian L is contained in $\mu^{-1}(0)$ and invariant under the circle action, then L_{red} is a lagrangian submanifold of M_{red}.*

To prove this lemma we recall that an injective immersion is an embedding if and only if it is open onto its image.

Proof. Constant rank theorem tells us that L_{red} is an immersed lagrangian, indeed
$$\ker(D(\pi \circ j)(x)) = T_xL \cap T_x\mu^{-1}(0)^{\omega_x} = T_x\mu^{-1}(0)^{\omega_x}$$
has constant dimension $\dim(M) - \dim(\mu^{-1}(0))$. We show that the inclusion $\bar{j}$ is open onto its image. Let $O \subset L_{\text{red}}$ be open, i.e. $\pi|_L^{-1}(O) = U \cap L$ for some $U \subset M$ open, then $\pi(U) \subset M_{\text{red}}$ is open. Then notice that $\pi^{-1}(\pi(U) \cap L_{\text{red}}) = S^1 \cdot U \cap L$ and hence $\pi(U \cap L) = \pi(S^1 \cdot U \cap L) = \pi(U) \cap L_{\text{red}}$, by invariance of the lagrangian. This implies $\bar{j}(U) = \pi(U) \cap L_{\text{red}}$, which is open in L_{red}. □

We would like to emulate this proof in the general case, but there are problems: is the intersection $L \cap \mu^{-1}(0)$ a manifold? Does the projection have constant rank? To handle correctly this questions we recall two definitions from differential geometry.

Definition 2.9 *Let N_1 and N_2 be submanifolds of M. We say that they have clean intersection if $N_1 \cap N_2$ is a manifold and if for any $x \in N_1 \cap N_2$ it holds that*
$$T_xN_1 \cap T_xN_2 = T_x(N_1 \cap N_2)$$

Definition 2.10 *Let $f : M \to M'$ be a smooth map and let N be a submanifold of M'. We say that f is transverse to N if for any $x \in f^{-1}(N)$ it holds that*
$$T_{f(x)}M' = T_{f(x)}N + Df(x)[T_xM]$$
We say that two submanifolds N_1 and N_2 of M intersect transversally if the inclusion $i : N_1 \hookrightarrow M$ is transverse to N_2.

One can count dimensions to notice that transverse intersection is a particular case of clean intersection.

Back to our problem. The core of the first part of the previous proof was that the projection $\pi|_L$ had constant rank. In the general case,
$$\ker D(\pi \circ j)(x) = T_xL \cap T_x\mu^{-1}(0)^{\omega_x}$$
for $x \in L \cap \mu^{-1}(0)$: we hence want $TL \cap T\mu^{-1}(0)^{\omega}$ to be a bundle.

Lemma 2.11 *If the intersection of the lagrangian L and the zero level $\mu^{-1}(0)$ is clean, then L_{red} is an immersed lagrangian in M_{red}.*

Proof. Thanks to the previous lemma, all that remains is to prove that $TL \cap T\mu^{-1}(0)^{\omega}$ is a fiber bundle. It is clear that $TL \cap T\mu^{-1}(0)$ is a bundle.

$TM \cap T\mu^{-1}(0)^\omega$ is a subbundle of the latter: if $: U \to \mathbb{R}^n$ is a bundle chart for $TL \cap T\mu^{-1}(0)$, then if we restrict it to $TM \cap T\mu^{-1}(0)^\omega$, we can set as image a diffeomorphic copy of some $\mathbb{R}^m$ into $\mathbb{R}^n$, as $\mu^{-1}(0)$ is coisotropic in M, i.e. we found a bundle chart. □

We proceed as Lemma 2.8 suggests and we look at the alleged failure of $\overline{j}$ to be an open map in the situation of last lemma. Let $U \subset L_{\text{red}}$ be open, then by definition

$$(\pi|_{L\cap\mu^{-1}(0)})^{-1}(U) = O \cap L \subset L \cap \mu^{-1}(0)$$

for some $O \subset \mu^{-1}(0)$ open. In particular $V := \pi(O) \subset M_{\text{red}}$ is open and $\pi(O \cap L) = \overline{j}(U)$. The problem here is that we don't know a priori that $O \cap L$ is open in $\mu^{-1}(0)$, altough it is open in $L \cap \mu^{-1}(0)$. We will ensure that the image is open by requiring π to be injective on $L \cap \mu^{-1}(0)$. We proved the following:

Proposition 2.12 *If the intersection of the lagrangian L and the zero level $\mu^{-1}(0)$ is clean, and if L touches each leaf of π contained in $\mu^{-1}(0)$ at most once, then L_{red} is a lagrangian submanifold of M_{red}.*

We have the following immediate corollary.

Corollary 2.13 *If the intersection of the lagrangian L and the zero level $\mu^{-1}(0)$ is transverse, and if L touches each leaf of π contained in $\mu^{-1}(0)$ at most once, then $L \cap \mu^{-1}(0)$ can be embedded as a lagrangian submanifold of M_{red}.*

Proof. Transversality of the intersection means that at each allowed point $x \in M$, $T_xL + T_x\mu^{-1}(0) = T_xM$, so linear algebra tells us that

$$T_xL \cap T_x\mu^{-1}(0)^{\omega_x} = \left(T_xL + T_x\mu^{-1}(0)\right)^{\omega_x} = T_xM^{\omega_x} = \{0\}$$

i.e. $D\pi(x)$ is injective. We proved that $\pi|_{L\cap\mu^{-1}(0)}$ is an injective immersion. This is enough as it is well known that submersions are open maps. □

It is important to point out that the condition that the lagrangian should be preserved only by the trivial subgroup is quite restrictive; one usually has to go for other directions in order to show that the lagrangian is embedded (see e.g. Section 2.4 and Chapter 4).

We have thus shown that under some conditions symplectic reduction keeps the lagrangian property. The reverse operation works too, and its proof is much easier, as next proposition points out.

Proposition 2.14 *Let $L \subset M_{red}$ be a lagrangian submanifold. Then $i(\pi^{-1}(L)) \subset M$ is a lagrangian submanifold.*

Proof. Recall that the preimage of a submanifold under a submersion still is a submanifold. Since π is a principal circle bundle, counting dimensions we get $\dim(\pi^{-1}(L)) = \frac{1}{2}\dim(M)$ and by Marsden-Weinstein-Mayer theorem, the restriction of the symplectic form on M to $\pi^{-1}(L)$ is the restriction to L of the pullback of the reduced form, which vanishes. $\square$

Clearly, last proposition tells us that each lagrangian in M_{red} comes from a reduction of a lagrangian of M.

We also look at the converse operation: in the nicest possible situation, if we first reduce a lagrangian and then take its preimage under π, we will get a "smaller" lagrangian contained in the zero level set. An interesting question is: how does the new lagrangian look like? It is not hard to see that this lagrangian is the union of the orbits intersecting the original one.

Here there is a basic, but already quite interesting, example.

Example 2.15 *We build $\mathbb{C}P^n$ out of $\mathbb{C}^{n+1}$. Let S^1 act on $\mathbb{C}^{n+1}$ via*

$$e^{i\theta} \cdot (z_0, ..., z_n) := (e^{i\theta} z_0, ..., e^{i\theta} z_n)$$

for $e^{i\theta} \in S^1$ and $(z_0, ..., z_n \in \mathbb{C}^{n+1}$. This action has a moment map given by

$$\mu(z_1, ..., z_n) = \frac{1}{2}\sum_{i=0}^{n} |z_i|^2 - \frac{1}{2}$$

for $(z_0, ..., z_n) \in \mathbb{C}^{n+1}$, as shown in Example 1.23, and hence $\mu^{-1}(0) \cong S^{2n+1} \subset \mathbb{C}^{n+1}$. Clearly, as manifold, the reduced space is diffeomorphic to $\mathbb{C}P^n$. We will call the reduced symplectic form the Fubini-Study form on $\mathbb{C}P^n$.

We reduce the lagrangian $\mathbb{R}^{n+1}$ of $\mathbb{C}^{n+1}$. The intersection of S^{2n+1} and $\mathbb{R}^{n+1}$ is transverse, as, for an x in the intersection,

$$T_x S^{2n+1} \cong (\mathbb{R} \cdot x)^{\perp} \text{ and } T_x \mathbb{R}^{n+1} \cong (\mathbb{R} \cdot x)^{n+1}$$

but they share n directions, as the tangent space to the sphere contains the perpendicular complex line trough x (2n-dimensions) and the real *line trough ix (which is the line real line trough the origin which intersect the real line trough x perpendicularly). We can reduce without further doubts. The intersection of the two submanifolds $\mathbb{R}^{n+1}$ and S^{2n+1} is "the middle" n-sphere, on which S^1 acts by mirroring the points, hence the reduced space is the real projective n-space $\mathbb{R}P^n$ in $\mathbb{C}P^n$, which of course is a lagrangian submanifold.*

In the next section we will generalize this example.

2.4 Reduction of the real part of a symplectic manifold

We introduce a special class of lagrangian submanifolds, which will turn out to be useful also in the fourth chapter. I followed [Sja10] for definitions and intuitions.

Definition 2.16 *A real structure on a symplectic manifold (M, ω) is a smooth map $\sigma : M \to M$ that is an involution, i.e. $\sigma^2 = id_M$, and anti-symplectic, i.e. $\sigma^*\omega = -\omega$. Its fixed point set M^σ is called the σ-real part of M.*

Lemma 2.17 *M^σ is a lagrangian submanifold of M.*

Proof. A proof can be found in [Sja10]. □

We want σ to descend to a real structure σ_{red} on the reduced manifold, so from now on we will assume that

$$\sigma(e^{i\theta} \cdot x) = e^{-i\theta} \cdot \sigma(x)$$

for any $e^{i\theta} \in S^1$ and $x \in M$. This immediately implies that the maximal subgroup acting on the real part of M is $\{-1, 1\}$.

Proposition 2.18 *The real part of M satisfies $\mu(M) = \mu(M^\sigma)$.*

Proof. This is proved in [Dui83]. □

Lemma 2.19 *There is a unique well defined real structure σ_{red} on M_{red} induced from σ.*

Proof. By assumption, σ takes circle orbits to circle orbits. By last proposition in every orbit there is a fixes point of σ, so that if $x \in \mu^{-1}(0)$ there is $y \in S^1 \cdot x \subset \mu^{-1}(0)$, say $y = e^{i\theta} \cdot x$, such that $\sigma(y) = y$ and hence

$$\mu(\sigma(x)) = \mu(e^{i\theta} \cdot \sigma(y)) = \mu(y) = \mu(x)$$

which means that σ preserves the zero level set. So we have an induced map $\sigma_{\text{red}} : M_{\text{red}} \to M_{\text{red}}$. It is clear that it is an involution, since σ is. Let's prove it is anti-symplectic. To pass to σ, we apply π^*, which is injective.

$$\begin{aligned}\pi^*\sigma^*_{\text{red}}(\omega_{\text{red}}) &= (\sigma_{\text{red}} \circ \pi)^*\omega_{\text{red}} = (\pi \circ \sigma)^*\omega_{\text{red}} = \\ &= \sigma^* i^* \omega = -i^*\omega = -\pi^*\omega_{\text{red}}\end{aligned}$$

so $\sigma^*_{\text{red}}\omega_{\text{red}} = -\omega_{\text{red}}$. □

We now prove that we can reduce the real part.

Lemma 2.20 *The real part M^σ intersects the zero level set transversally.*

Proof. We use the strategy applied in [Can18] for the proof of Lemma 4.1, i.e. we show that the restriction of $d\mu$ to the points in M^σ never vanishes on the interior of the moment polytope. We compute the tangent space to the real part at one of its points x:

$$\begin{aligned}T_xM^\sigma &= \{\gamma'(0) : \ \gamma \in C^\infty([0,1], M^\sigma)\} = \\ &= \{\gamma'(0) : \ \gamma \in C^\infty([0,1], M) \text{ and } \sigma \circ \gamma = \gamma\} = \\ &= \{v \in T_xM : \ d\sigma_x(v) = v\}\end{aligned}$$

Let $x \in M^\sigma$ and $w \in T_xM$, then, since σ is an involution, $w + d\sigma_x(w) \in T_xM^\sigma$ and hence

$$d\mu_x(w + d\sigma_x(w)) = 2d\mu_x(w)$$

but we know that critical points of the moment map are vertices. □

This ensures that the image of $M^\sigma \cap \mu^{-1}(0)$ under the projection $\pi : \mu^{-1}(0) \to M_{\text{red}}$ is a manifold (in fact an immersed lagrangian), but we can't tell for sure that is a lagrangian submanifold, as M^σ can intersect orbits twice.

Proposition 2.21 *Reduction and real part commute, that means*

$$M_{red}^{\sigma_{red}} = (M^\sigma)_{red}$$

$\|$

Proof. We show there is a bijective immersion (hence a diffeomorphism) $(M^\sigma)_{\text{red}} \to M_{\text{red}}^{\sigma_{\text{red}}}$. Consider $j : \mu^{-1}(0) \cap M^\sigma \hookrightarrow \mu^{-1}(0)$, which passes to $\overline{j} : (M^\sigma)_{\text{red}} \hookrightarrow M_{\text{red}}$. Consider an orbit $\{-1, 1\} \cdot x \in M_{\text{red}}^\sigma$, for $x \in M^\sigma$, then $\overline{j}(\{-1, 1\} \cdot x)$ is in $M_{\text{red}}^{\sigma_{\text{red}}}$ in the sense that $\{-1, 1\} \cdot x$ can be extended to an orbit $S^1 \cdot x$. Since $M_{\text{red}}^{\sigma_{\text{red}}}$ is (in particular) an immersed submanifold of M_{red}, we can regard $\overline{j}$ as a smooth injective immersion from $(M^\sigma)_{\text{red}}$ to $M_{\text{red}}^{\sigma_{\text{red}}}$. To prove surjectivity, consider the orbit $S^1 \cdot x \in M_{\text{red}}^{\sigma_{\text{red}}}$, that means $\sigma_{\text{red}}(S^1 \cdot x) = S^1 \cdot x$: we show it is the extension of a smaller orbit in the sense above, i.e. we want an element $y \in M^\sigma$ such that

$$\overline{j}(\{-1, 1\} \cdot y) = S^1 \cdot x$$

Note that there is an element $e^{2i\theta} \in S^1$ satisfying $\sigma(x) = e^{2i\theta} \cdot x$ by assumption, so that $e^{-i\theta} \cdot \sigma(x) = e^{i\theta} \cdot x$ which, since σ behaves well with respect to the action, means

$$\sigma(e^{i\theta} \cdot x) = e^{i\theta} \cdot x$$

Let $y = e^{i\theta} \cdot x$ and consider $\{-1, 1\} \cdot y$, then of course extending the latter to a circle orbit will give us $S^1 \cdot x$ by the above computations. $\square$

Chapter 3

Symplectic cutting

As Lerman points out in [Ler95], Guillemin and Sternberg in [GS89] drawed a connection between symplectic reductions and symplectic blowups: they described the blow-up of $\mathbb{C}^n$ at the origin as a reduction of $\mathbb{C}^{n+1}$ with respect to a circle action. In [Ler95], Lerman generalized their idea to general hamiltonian circle spaces introducing an elegant symplectic surgery called symplectic cutting.

3.1 Intuition

In the last chapter we focused on a small portion of a given hamiltonian space, namely regular level sets. In this chapter we will instead consider the whole manifold during the reduction process; in other words, we will cut a hamiltonian space at the height of a regular value in two pieces, obtaining two manifold with boundary, and then try to get rid of the boundary to get new manifolds which behave well in symplectic terms.

In synthesis, we will prove the following. Given a hamiltonian circle space (M, ω, S^1, μ) with free action on the zero level set, define an equivalence relation on M as follows: for $x, y \in \mu^{-1}(0)$, $x \sim y$ if there is $e^{i\theta} \in S^1$ such that $e^{i\theta} \cdot x = y$; for $x, y \in M - \mu^{-1}(0)$, $x \sim y$ if $x = y$. Then by the following three quotient have a manifold structure:

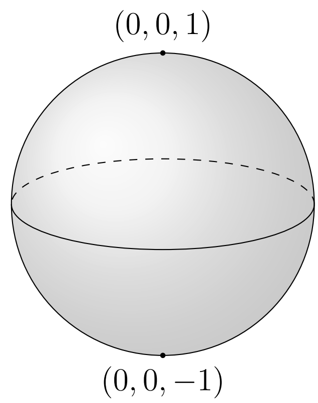
$$\overline{M_{\geq 0}} := \mu^{-1}[0, \infty)/\sim, \quad \overline{M_{\mu=0}} := \mu^{-1}/\sim \text{ and } \overline{M_{\leq 0}} := \mu^{-1}(-\infty, 0]/\sim$$

Then there are symplectic structures on these three manifold such that everything is as expected, i.e.: $\overline{M_{\mu=0}}$ is the symplectic reduction of (M, ω, S^1, μ) at zero, $\overline{M_{\mu=0}}$ is a 2-codimensional symplectic submanifold of both $\overline{M_{\geq 0}}$ and

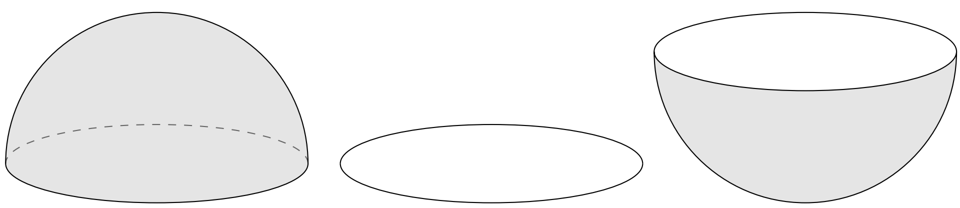
$\overline{M_{\leq 0}}$, $\overline{M_{\geq 0}} - \overline{M_{\mu=0}}$ is symplectomorphic to $\mu^{-1}(0,\infty)$ and $\overline{M_{\leq 0}} - \overline{M_{\mu=0}}$ is symplectomorphic to $\mu^{-1}(-\infty,0)$. The intuition presented in these lines inspired Yael Karshon's generalization of symplectic cutting that will appear in [Kar20] and is presented at the end of this chapter.

The main problem to handle this problem symplectically is to find the correct way to apply reduction. The construction is due to Eugene Lerman and first appeared in [Ler95].

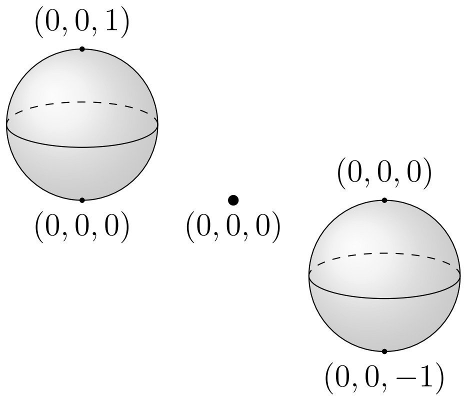
Let's see a very easy example of the cutting procedure. Take the sphere $S^2 := \{z \in \mathbb{C} : \ |z| = 1\}$, on which the circle acts by horizontal rotation, with moment map $(\varphi, h) \in S^2 \mapsto x \in [-1,1]$:



Cut it in three parts according to the recipe above:



And quotient by the equivalence relation to get two smaller spheres and a point:



3.2 Lerman's original symplectic cutting

Lerman proved the following theorem using the Marsden-Weinstein-Mayer reduction theorem (Theorem 2.3); in the remaining of this thesis we will be mainly using the construction contained in the proof instead of the intuitive one given in the introduction of this chapter. We will not in general work with both cut spaces, but as Lerman points out in [Ler95] both spaces may be interesting depending on the context.

Theorem 3.1 *Consider a hamiltonian circle space* (M, ω, S^1, μ) *and suppose that the circle acts freely on the zero level set of the moment map. Then there exist other two hamiltonian circle spaces, denoted by* $(\overline{M_{\geq 0}}, \overline{\omega_{\geq 0}}, S^1, \overline{\mu_{\geq 0}})$ *and* $(\overline{M_{\leq 0}}, \overline{\omega_{\leq 0}}, S^1, \overline{\mu_{\leq 0}})$*, such that:*

1. *there are equivariant symplectic embeddings*
$$i_+ : \mu^{-1}(0, \infty) \hookrightarrow \overline{M_{\geq 0}}$$
and
$$i_- : \mu^{-1}(-\infty, 0) \hookrightarrow \overline{M_{\leq 0}}$$
2. *there are equivariant symplectomorphisms*
$$M_{red} \cong \overline{M_{\geq 0}} - i_+\big(\mu^{-1}(0, \infty)\big)$$
and
$$M_{red} \cong \overline{M_{\leq 0}} - i_-\big(\mu^{-1}(-\infty, 0)\big)$$

We call these manifolds symplectic cuts of (M, ω, S^1, μ) *at* 0.

Before proving Theorem 3.1 we do some preliminary work. We have already seen that the tuple $(\mathbb{C}, dz \wedge d\overline{z}, S^1, z \mapsto \frac{1}{2}|z|^2)$ with diagonal action is a symplectic toric manifold. One can endow the product manifold $M \times \mathbb{C}$ with the product form $\hat{\omega} := \pi_1^*\omega + \pi_2^*(dz \wedge d\overline{z})$, where $\pi_1 : M \times \mathbb{C} \to M$ and $\pi_2 : M \times \mathbb{C} \to \mathbb{C}$ are the natural projections, to get a third hamiltonian circle space, whose moment map will be denoted $\psi_{\leq 0}(x, z) := \mu(x) + \frac{1}{2}|z|^2$. One can consider the skew-diagonal action of the circle on S^1, which also gives us a toric structure on the standard complex plane, whose moment map differs by the one above by a minus sign. Similarly to the diagonal case, we end up with a hamiltonian space $M \times \mathbb{C}$ with moment map $\psi_{\geq 0}(x, z) = \mu(x) - \frac{1}{2}|z|^2$

Lemma 3.2 *The circle acts freely on $\psi_{\leq 0}^{-1}(0)$ and $\psi_{\geq 0}^{-1}(0)$.*

Proof. Let $(x, z) \in M \times \mathbb{C}$. If $z \neq 0$, S^1 clearly moves (x, z). If $z = 0$, then $x \in \mu^{-1}(0)$ and S^1 moves x by assumption. $\square$

Hence we can reduct $M \times \mathbb{C}$ at zero in two ways. We now prove Theorem 3.1.

Proof. (of Theorem 3.1) First, define $\overline{M_{\geq 0}}$ as the symplectic reduction of $M \times \mathbb{C}$ at zero considering the skew-diagonal action and $\overline{M_{\leq 0}}$ as the reduction of $M \times \mathbb{C}$ considering the diagonal action. By Theorem 2.3, $\hat{\omega}$ passes to symplectic forms $\overline{\omega_{\geq 0}}$ on $\overline{M_{\geq 0}}$ and $\overline{\omega_{\leq 0}}$ on $\overline{M_{\leq 0}}$.

To get the hamiltonian structure on the cut spaces we use Corollary 2.6. The action of S^1 on M can be extended to $\mathbb{C}$ by requiring it to be trivial there, and preserves $\psi_{\leq 0}^{-1}(0)$ and $\psi_{\geq 0}^{-1}(0)$, since it preserves any level set of μ. The resulting circle action on the cut spaces has μ restricted to the zero level sets and pushed down as moment map.

We will prove the rest just for $\overline{M_{\geq 0}}$, as the other one is pretty identical. Notice that we have a decomposition:

$$\psi_{\geq 0}^{-1}(0) = \{(x, z) \in M \times \mathbb{C}^* : \mu(x) = \frac{1}{2}|z|^2\} \cup \{(x, 0) \in M \times \mathbb{C} : \mu(x) = 0\}$$

as manifold with circle action. So we define

$$\imath : x \in \mu^{-1}(0, \infty) \to \left(x, \sqrt{2\mu(x)}\right) \in \psi_{\geq 0}^{-1}(0)$$

which is clearly an embedding. We define i_+ as $\imath$ composed with the orbit map $\psi_{\geq 0}^{-1}(0) \to \overline{M_{\geq 0}}$; it follows that to show that i_+ is an embedding we have to prove that the image of $\imath$ intersects any circle orbit in $\psi_{\geq 0}^{-1}(0)$ at most once (cfr. Section 2.3): this is clear. Pulling back $\overline{\omega_{\geq 0}}$ via i_+ we first pull back via the orbit map, hence getting the restriction of $\hat{\omega}$ to the level set $\psi_{\geq 0}^{-1}(0)$, and then we pull back via $\imath$, hence getting the restriction of ω to $\mu^{-1}(0,\infty)$, since the second component of $\imath$ maps to the real line. This proves that i_* is a symplectic embedding.

To prove the last statement one notices that the decomposition above induces a commutative diagram:

$$\begin{array}{ccc} \mu^{-1}(0) & \xrightarrow{x \mapsto (x,0)} & \psi_{\geq 0}^{-1}(0) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} \\ M_{\text{red}} & \longrightarrow & \overline{M_{\geq 0}} \end{array}$$

Which shows that the pullback of $\overline{\omega_{\geq 0}}$ is ω_{red} since the forms on the reduced spaces are the forms on the levels sets pushed down. Notice that

$$\begin{aligned} M_{\text{red}} &\cong \overline{\pi}\big\{(x,0) \in \mu^{-1}(0) \times \mathbb{C}\}\big) = \\ &= \overline{\pi}\big(\psi_{\geq 0}^{-1}(0) - \{(x,z) \in M \times \mathbb{C}^* : \mu(x) = |z|^2\}\big) = \\ &= \overline{M_{\geq 0}} - i_+\big(\mu^{-1}(0,\infty)\big) \end{aligned}$$

This ends the proof. $\square$

The proof of Theorem 3.1 also sheds light on the relationship between the two constructions of cut spaces (the one in Section 3.1 and the one using reduction): extending i_+ to the zero level set $\mu^{-1}(0)$ one gets a surjection $\mu^{-1}[0,\infty) \to \overline{M_{\geq 0}}$ which as said above is injective on $\mu^{-1}(0,\infty)$ and is completely non-injective on orbits in the zero level set.

Example 3.3 *Consider the symplectic toric manifold $\mathbb{C}^2$. We are going to cut the moment image $\mathbb{R}^2_{\geq 0}$ along the line $y = \frac{1}{2} - x$, i.e. with respect to the circle generated by the element $(1,1) \in \mathbb{T}^2$ and with a translation of $-\frac{1}{2}$ appearing in the moment map, in two different ways: we hence have moment maps $\psi_{\leq 0}(z_1,z_2,w) := \frac{1}{2}(|z_1|^2 + |z_2|^2 + |w|^2 - 1)$ and $\psi_{\geq 0}(z_1,z_2,w) := \frac{1}{2}(|z_1|^2 + |z_2|^2 - |w|^2 - 1)$. One expects to get (something equivariantly symplectomorphic to) the standard $\mathbb{C}P^2$ cutting below the line. We have level sets*

$$\psi_{\leq 0}^{-1}(0) = \{(z_1,z_2,w) \in \mathbb{C}^2 \times \mathbb{C} : |z_1|^2 + |z_2|^2 + |w|^2 = 1\}$$

and

$$\psi_{\geq 0}^{-1}(0) = \{(z_1, z_2, w) \in \mathbb{C}^2 \times \mathbb{C} : |z_1|^2 + |z_2|^2 - |w|^2 = 1\}$$

The first reduction gives us the standard $\overline{\omega_{\mu\leq 0}} \cong S^5/S^1 \cong \mathbb{C}P^2$*: one can calculate that the restriction of the standard form on the complex plane restricts to*

$$-\frac{i}{2|z|^4} z_1 \overline{z}_2 dz_1 \wedge d\overline{z}_2 - \frac{i}{2|z|^4} z_2 \overline{z}_1 dz_2 \wedge d\overline{z}_1 = \frac{i}{2} \partial\overline{\partial} \log(|z|^2|)$$

which is exactly $\frac{1}{2}$ *times the pullback of the Fubini Study form with respect to the Hopf fibration, so our cut space and* $\mathbb{C}P^2$ *are symplectomorphic. Indeed, they are equivariantly symplectomorphic since* $\mathbb{T}^2$ *acts on* S^5 *on the first two coordinates. The second one is the so called blow-up of* $\mathbb{C}^2$ *at the origin, i.e. the manifold*

$$L := \{([p], z) \in \mathbb{C}P^1 \times \mathbb{C}^2 : \ \exists \lambda \in \mathbb{C} : \ z = \lambda p\}$$

with symplectic form $\frac{1}{2}\pi_1^* \omega_{FS} + \pi_2^*(dz \wedge d\overline{z})$*. To prove this we use the decomposition of the level set and the decomposition of the blow up in the exceptional divisor and* $\mathbb{C}^2 - \{0\}$*. We have*

$$(z_1, z_2) \in \mathbb{C}^2 - \{0\} \longmapsto (z_1, z_2, \sqrt{\mu(z_1, z_2)}) \in \mu^{-1}(0, \infty) \times S^1$$

and, viewing $\mathbb{C}P^1$ *as* S^3/S^1*:*

$$[z_1, z_2] \longmapsto (z_1, z_2, 0) \in \mu^{-1}(0) \times \{0\}$$

This basic example shed also light on the announced relationship between cutting and blow-ups: the resulting cut space is the $\frac{1}{2}$-blow-up of $\mathbb{C}^2$ at the origin. Of course, one may blow up at any regular level $0 \neq c \in \mathbb{R}$ of the moment map to get a blow-up on whose exceptional subset the symplectic form will be the standard Fubini-Study form multiplied by a c factor.

3.3 Multi-dimensional cutting

Lerman's simplectic toric manifold We construct a symplectic toric manifold with a given Delzant polyhedral set

$$\Delta := \{x \in (\mathfrak{t}^n)^* : \ \forall k = 1, ..., d, \ \langle x, v_k \rangle \geq \lambda_k\}$$

as moment image, where $v_1, ..., v_d$ are primitive lattice vectors and $\lambda = (\lambda_1, ..., \lambda_k) \in \mathbb{R}^k$. What we are going to do is to take the euclidean part of the tangent bundle of the torus and *cut* it into the interior $\mathrm{int}(\Delta)$ of our polyhedral set. We will close the resulting manifold in such a way that the induced form from the symplectic form keeps non-degeneracy.

onsider the cotangent bundle $T^*\mathbb{T}^n$ of the n-torus, trivialized via left invariant vector fields as $\mathbb{T}^n \times (\mathfrak{t}^n)^*$ with action-angle coordinates (θ, ξ), and equip it with the symplectic form

$$\omega_{T^*\mathbb{T}^n} := \sum_{k=1}^{n} d\xi_k \wedge d\theta_k$$

Cotangent lifting the multiplication on the torus we get an action of $\mathbb{T}^n$ on $T^*\mathbb{T}^n$ given by $\theta' \cdot (\theta, \xi) = (\theta\theta', \xi)$, for $\theta' \in \mathbb{T}^n$ and $(\theta, \xi) \in T^*\mathbb{T}^n$, which is hamiltonian with respect to the projection onto the second factor.

We define an homomorphism

$$\rho_\Delta : \theta \in \mathbb{T}^d \longmapsto \exp\left(\sum_{k=1}^{d} \theta'_k v_k\right) \in \mathbb{T}^n$$

and an action of $\mathbb{T}^d$ on $T^*\mathbb{T}^n$ as

$$\theta' \cdot (\theta, \xi) := \big(\rho_\Delta(\theta')\theta, \xi\big)$$

This action is hamiltonian with moment map $\mu := D\rho_\Delta(e)^* \circ \mathrm{pr}_2 - \lambda$. Notice that $D\rho_\Delta(e)[X] = \sum_{k=1}^{d} X_k v_k \in \mathfrak{t}^n$, for $X \in \mathfrak{t}^d$, by well-known properties of the exponential, so that $D\rho_\Delta(e)^*[\xi] = \sum_{k=1}^{d} \langle \xi, v_k \rangle e_k \in (\mathfrak{t}^d)^*$, for $\xi \in (\mathfrak{t}^n)^*$.

Furthermore, we let $\mathbb{T}^d$ act on $(\mathbb{C}^d, \omega_{\mathbb{C}^d} := \frac{i}{2} \sum dz_k \wedge d\bar{z}_k)$ skew-diagonally:

$$(\theta_1, ..., \theta_d) \cdot (z_1, ..., z_d) := (e^{-i\theta_1} z_1, ..., e^{-i\theta_d} z_d)$$

for $(\theta_1, ..., \theta_d) \in \mathbb{T}^d$ and $(z_1, ..., z_d) \in \mathbb{C}^d$. This action is hamiltonian with moment map $(z_1, ..., z_d) \longmapsto -\frac{1}{2}(|z_1|^2, ..., |z_d|^2)$.

Putting all together we get an action of the d-torus on $T^*\mathbb{T}^n \times \mathbb{C}^d$ with moment map

$$\alpha : ((\theta, \xi), z) \in T^*\mathbb{T}^n \times \mathbb{C}^d \longmapsto \sum_{k=1}^{d} \langle \xi, v_k \rangle - \lambda - \frac{1}{2}(|z_1|^2, ..., |z_d|^2)$$

i.e. in the end we've got the hamiltonian space $(T^*\mathbb{T}^n \times \mathbb{C}^d, \omega_{T^*\mathbb{T}^n} \oplus \omega_{\mathbb{C}^d}, \mathbb{T}^d, \alpha)$.

Theorem 3.4 *The product $\mathbb{T}^d$ action is free on the zero level set of α, so the reduced space of $(T^*\mathbb{T}^n \times \mathbb{C}^d, \omega_{T^*\mathbb{T}^n} \oplus \omega_{\mathbb{C}^d}, \mathbb{T}^d, \alpha)$ at zero is a symplectic manifold. Moreover this space is naturally a $2n$-symplectic toric manifold with moment map image Δ.*

Definition 3.5 *We denote $(L_\Delta, \omega_\Delta)$ the reduced space of $(T^*\mathbb{T}^n \times \mathbb{C}^d, \omega_{T^*\mathbb{T}^n} \oplus \omega_{\mathbb{C}^d}, \mathbb{T}^d, \alpha)$ at zero and we call it Lerman's symplectic toric manifold with respect to the Delzant polyhedral set Δ.*

To prove the theorem we state this simple lemma:

Lemma 3.6 *For an index set $I \subset \{1, ..., d\}$ let*

$$\mathbb{T}_I := \{(\theta_1, ..., \theta_d) \in \mathbb{T}^d : \forall j \notin I : t_j = 0\}$$

Then the smoothness condition on Δ is equivalent to $\rho_\Delta|_{\mathbb{T}_I}$ being injective for any $I \subset \{1, ..., d\}$ such that $\Delta_I \neq 0$.

Proof. Let $I \subset \{1, ..., d\}$ and $(\theta_1, ..., \theta_d) \in \ker(\rho_\Delta|_{\mathbb{T}_I})$, i.e. $\exp\left(\sum \theta_k v_k\right) = (0, ..., 0)$, then $\sum_{k \in I} \theta_k v_k \in \mathbb{Z}^n$, and hence $\theta_k \in \mathbb{Z}$ for any $k \in I$. This is exactly injectivity[1]. The converse is proven in the same way. $\square$

Proof. (of Theorem 3.4) 1. Let $((\theta, \xi), z) \in \alpha^{-1}(0)$. If $z_k \neq 0$, then the k-th circle of $\mathbb{T}^d$ acts freely on $\mathbb{C}^n$, hence also on $T^*\mathbb{T}^n \times \mathbb{C}^d$. Let $I := \{k : \ z_k = 0\} = \{k : \ \langle \xi, v_k \rangle = \lambda_k\}$[2]; then by last lemma, $\rho_\Delta|_{\mathbb{T}_I} : \mathbb{T}_I \to \mathbb{T}^n$ is injective. This proves that $\mathbb{T}_I$ acts freely on $T^*\mathbb{T}^n$, since $\mathbb{T}^n$ does by multiplication. This proves that the reduced space is a symplectic manifold.
2. The standard $\mathbb{T}^n$-action on $T^*\mathbb{T}^n$ descends to a $\mathbb{T}^n$-action on L_Δ with moment map $\widetilde{pr}_2 : [(\theta, \xi), z] \mapsto \xi$ by Corollary 2.6. Let $\xi \in \mathfrak{t}^*$, then

$$\xi \in \widetilde{pr}_2(L_\Delta) \iff \xi \in \alpha^{-1}(0) \iff \xi \in \Delta$$

by definition of α. $\square$

We analyze freeness of the resulting $\mathbb{T}^n$-action on L_Δ.

[1] We're always pretending that $1 = 2\pi$, rule that should have been set much time ago...

[2] This step is legal because of simplicity: there are at most n elements in I.

Corollary 3.7 *The torus action is free on $\mu_\Delta^{-1}(int(\Delta))$. More generally the (preimages of) points on a open face all have the same stabilizer, which is the perpendicular subtorus to that face. In precise terms if an open face F is characterized by $\{x \in \Delta : \forall j \in I_F : \langle x, v_j \rangle = \lambda_j\}$, then all the points in $\mu_\Delta^{-1}(F)$ are stabilized by*

$$\mathbb{T}_F := \exp\Big(span_{\mathbb{R}}\{v_j : j \in I_F\}\Big)$$

Proof. Let $[\theta, \xi, z] \in L_\Delta$ and $\theta' \in \mathbb{T}^n$ such that $[\theta, \xi, z] = [\theta\theta', \xi, z]$. Then there is $t \in \mathbb{T}^d$ such that $\theta' = \exp(\sum_{j=1}^d t_j v_j)$ and $z_j = z_j e^{-it_j}$. If we let as before $I := \{k : \ z_k = 0\}$ we get that $\theta' = \exp(\sum_{j \in I} t_j v_j)$ □

General cutting construction We extend the construction in last subsection to general a hamiltonian toric space $(M, \omega, \mathbb{T}^n, \mu)$.

Definition 3.8 *We define the cut space with respect to the Delzant polyhedral set $\Delta \subset (\mathfrak{t}^n)^*$ of the hamiltonian $\mathbb{T}^n$-space $(M, \omega, \mathbb{T}^n, \mu)$ as the symplectic reduction of $M \times L_\Delta$ at the zero level with respect to the skew-diagonal action of $\mathbb{T}^n$. Assuming that the $\mathbb{T}^n$-action is free on the zero level of the product moment map, we denote the resulting symplectic manifold as $(M^\Delta, \omega^\Delta)$.*

Of course M^Δ inherits the structure of hamiltonian toric space from M: we denote the resulting space by $(M^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$.

Lemma 3.9 *Let $(x, y) \in M \times L_\Delta$ lie in the zero level set of the product action, and assume that y lies on an open face F. The stabilizer of (x, y) is $\mathbb{T}_x \cap \mathbb{T}_F$.*

Remark 3.10 *The original cutting construction described in Section 3.2 is the cut space of a hamiltonian circle space with respect to the Delzant polyhedral set $[0, +\infty)$.*

Remark 3.11 *The construction described at the beginning of this section is also a special case of cut space, with $M = T^*\mathbb{T}^n$.*

Remark 3.12 *One can also see M^Δ as the symplectic reduction at 0 of $M \times T^*\mathbb{T}^n \times \mathbb{C}^d$ by $\mathbb{T}^n \times \mathbb{T}^d$ and hence as the symplectic reduction at 0 of $M \times \mathbb{C}^d$ by $\mathbb{T}^d$, where in the second case $\mathbb{T}^d$ acts on M via the exponential map as in the first paragraph of Section 3.3.*

What we did is hence circle-cutting many times a fixed hamiltonian toric space. In the following we work out the details of these sub-cuttings. This is useful both for understanding what is going on and to fix a notation for the following chapter. We will work with ≥ 0 circle cutting, but what is described works analogously for ≤ 0 cutting.

Let again $(M, \omega, \mathbb{T}^n, \mu)$ denote an hamiltonian toric space and consider the Delzant polyhedral set Δ defined above. We assume again that the toric action is free on the zero level set of the moment map. We work here with an arbitrary $v = v_k$ for some $k = 1, ..., d$. Consider the circle $S^1_v := \exp(\mathbb{R}v) < \mathbb{T}^n$, i.e. the image of the homomorphism ρ_Δ in a one dimensional case. Similarly to explained above, we have a circle action on $M \times \mathbb{C}$ given by $e^{i\theta} \cdot (x, z) := (\exp(v\theta) \cdot x, e^{-i\theta} z)$ for $x \in M$ and $z \in \mathbb{C}$. Then as in the first section of this chapter we can define $(M_{v,\geq 0}, \omega_{v,\geq 0}, \mathbb{T}^n, \mu_{v,\geq 0})$ to be the symplectic reduction of $M \times \mathbb{C}$ at 0. We call $(M_{v,\geq 0}, \omega_{v,\geq 0}, \mathbb{T}^n, \mu_{v,\geq 0})$ the symplectic cutting at level 0 of M along the primitive lattice vector v.

Corollary 3.13 *Let $\Delta := \{x \in (\mathfrak{t}^n)^*) : \ \forall k = 1, ..., d, \ \langle x, v_k \rangle \geq \lambda_k\}$. If we repeatedly cut the hamiltonian toric space $(M, \omega, \mathbb{T}^n, \mu)$ at level λ_i along the primitive lattice vector v_i, for $i = 1, ..., d$, we get a space equivariantly symplectomorphic to $(M^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$.*

Proof. This follows directly by repeated use of Corollary 2.6. □

Hence, we proved that toric cutting is just a chain of circle cuttings. Let's see what happens to the moment image when we cut a symplectic toric manifold.

Proposition 3.14 *Consider a symplectic toric manifold $(M, \omega, \mathbb{T}^n, \mu)$ such that its moment image is a convex rational polyhedral set. If $v \in \mathfrak{t}$ generates a circle subgroup S^1_v of $\mathbb{T}^n$, then the symplectic cutting at level 0 of M along the primitive lattice vector v has*

$$\mu(M) \cap \{X \in \mathfrak{t}^* : \langle v, X \rangle \geq 0\}$$

map.image.

Proof. The moment map of the circle action is $\mu^X = \langle \mu, X \rangle$. The two actions clearly commute with invariant moment maps. □

And we have the following corollary.

Corollary 3.15 *Consider a sympletic toric manifold $(M, \omega, \mathbb{T}^n, \mu)$ and a Delzant n-polyhedral set Δ such that the cut construction is legal. Then $M_\Delta = L^{\Delta \cap \mu(M)}$. In particular if $\Delta \subset int(\mu(M))$, then $M_\Delta = L^\Delta$. If conversely $\mu(M) \subset int(\Delta)$, then $M_\Delta = M$.*

Proof. Let $\Delta := \{x \in (\mathfrak{t}^n)^*) : \ \forall k = 1, ..., d, \ \langle x, v_k \rangle \geq \lambda_k\}$ for primitive lattice vectors This follows by the combination of Delzant theorem with Corollary 3.13 and Proposition 3.14. Indeed, as M^Δ is obtained by repeated circle cuts, by last proposition its moment image is exactly $\mu(M) \cap \Delta$. Delzant theorem concludes the proof. $\square$

3.4 Karshon's cutting

In [Kar20] Karshon develops a "cutting"[3] construction as a functor from the category of manifolds with boundary equipped with free circle actions near the boundary and whose morphisms are equivariant transverse maps, to the category of smooth manifolds by collapsing orbits on the boundary.

We resume the ideas contained in [Kar20]. In the following M will be a manifold with boundary equipped with a free cricle action on a neighbourhood U of the boundary. We define on M the following equivalence relation: two points of M are equivalent if and only if they both lie on the boundary of M and are in the same circle orbit. We denote the quotient space by M_c.

We need some technical definitions:

Definition 3.16 *An invariant boundary defining function is a smooth map $f : M \to \mathbb{R}_{\geq 0}$ such that:*

1. $f^{-1}(0) = \partial M$*;*
2. $df|_{\partial M}$ *never vanishes;*
3. *There is an invariant neighbourhood $V \subset U$ of the boundary such that $f|_V$ is S^1-invariant.*

Definition 3.17 *Consider two manifolds with boundary M and N equipped with free circle actions on neighbourhoods U_M resp. U_N of the boundary. A smooth map $\psi : M \to N$ is said to be equivariant transverse if:*

1. *there is an invariant boundary defining function f_N on N such that $f_N \circ \psi$ is an invariant boundary defining function on M;*

[3]The name might not be the same when the paper will appear in print.

2. *for any point in ∂M there is a neighbourhoof $V \subset U_M \cap \psi^{-1}(U_N)$ of it in M such that ψ_V is S^1-equivariant.*

We copy here two main theorems proved in [Kar20]. The following is Theorem 1.1. there.

Theorem 3.18 *Let f be an invariant boundary defining function on M. There is a unique manifold structure on the cut space M_c such that a function $h : M_c \to \mathbb{R}$ is smooth if and only if:*

1. *$h \circ c|_{int(M)}$ is smooth;*
2. *for each point in ∂M there is an invariant neighbourhood $V \subset U$ of it and a smooth map $H : V \times \mathbb{C} \to \mathbb{R}$ such that*
   a) *for any $^{i\theta} \in S^1$ and $(x, z) \in V \times \mathbb{C}$ we have that $H(e^{i\theta} \cdot x, z) = H(x, e^{i\theta} z)$;*
   b) *for any $x \in V$ we have that $h(c(x)) = H(x, \sqrt{f(x)}$.*

The structure is independend of the choiche of f.

Proof. This is proved in [Kar20]. □

The following is Proposition 4.1 in [Kar20].

Theorem 3.19 *Consider two manifolds with boundary M and N that are equipped with free circle actions on neighbourhoods U_M resp. U_N of the boundary and an equivariant transverse map $\psi : M \to N$. Then ψ pushes down to a unique smooth map $\psi_c : M_c \to N_c$ such that $c_N \circ \psi = \psi_c \circ c_M$. Moreover, this construction respects compositions of equivariant transverse maps.*

Proof. This is proved in [Kar20]. □

In the end we have the following result.

Corollary 3.20 *The cutting construction in this setting yields a functor from the category whose objects are manifolds-with-boundary equipped with free circle actions near the boundary and whose morphisms are equivariant transverse maps, to the standard category of smooth manifolds.*

Proof. This follows directly from the work above. □

In the remainings of this section we prove that Karshon's cutting is indeed a generalization of Lerman's symplectic cutting [Ler95]. An aim of this thesis is to use a slight modification of Theorem 3.19 to study the behaviour of certain $\mathbb{Z}_2$-invariant submanifolds of a symplectic toric manifold under the cutting construction. What follows is an expansion of some notes in a preliminary draft of [Kar20]. [Sni13], [CB97]

From here on consider an hamiltonian circle space (M, ω, S^1, μ).

> **Lemma 3.21** *Let $c \in \mathbb{R}$ be a regular value of μ. Then there is $\varepsilon > 0$ such that for any $t \in (c-\varepsilon, c+\varepsilon)$, $\mu^{-1}(t)$ is equivariantly symplectomorphic to $\mu^{-1}(c)$.*

Proof. One can find a proof in [Aud91]. □

Let $c \in \mathbb{R}$ such that the circle acts freely on $\mu^{-1}(c)$. We define[4] $N := \mu^{-1}[c, \infty)$, then as c is a regular value of μ, N is a smooth manifold with boundary, and $\partial N = \mu^{-1}(c)$. Moreover, by Lemma 3.21, N is a smooth manifold with boundary with a free action of the circle near the boundary.

We claim that $\mu_c := \mu|_{\mu^{-1}[c,\infty)} - c$ is an invariant boundary defining function $N \to \mathbb{R}_{\geq 0}$. Of course $\mu_c^{-1}(0) = \partial N$. Nonetheless, we have the following lemma.

> **Lemma 3.22** *At any point $x \in M$, the image of $d\mu(x)$ is the annihilator of Lie algebra of the stabilizer of the circle action at x.*

Proof. This follows from the defining relation of the moment map. □

This means that, as the circle action is free near the boundary, μ is a submersion near the boundary, in particular, $\mu_c := \mu|_{\mu^{-1}[c,\infty)} - c$ is an invariant boundary defining function.

Of course, the symplectic form ω on M pulls back to a symplectic form on N. We will still denote this form by ω.

> **Lemma 3.23** *The pullback of ω to ∂N is a basic two form.*

Proof. The restriction of the symplectic form ω to $\mu^{-1}(0)$ is by symplectic reduction the pullback of a symplectic form on the reduced space. □

[4]As with ordinary Lerman's cutting, we could consider $\mu^{-1}(-\infty, c]$ and end up with analogous results.

By an argument contained in Section 6 of [Kar20], the symplectic form ω on N descends to a symplectic form ω_c on N_c. By arguments contained in Section 2.2 of this thesis, the circle action on N descends to a circle action on the smooth manifold N_c, which is hence an hamiltonian circle space.

We finally prove that Karshon's cutting generalizes Lerman's symplectic cutting.

Proposition 3.24 *There is an equivariant symplectomorphism between the cut space N_c and the symplectic cutting of M at the level c.*

Proof. This is the exact same reasoning as in the comment after the proof of Theorem 3.1. We have a surjection $\mu^{-1}[c, \infty] \to \overline{M_{\geq c}}$ that is injective away from the cutting level and completely non-injective on orbits contained in the cutting level. This means that after passing to N_c, one gets a diffeomorphism. The fact that this diffeomorphism is an equivariant symplectomorphism follows directly by the construction of the hamiltonian space N_c and the defining relation of it on the boundary. $\square$

Chapter 4

Lagrangian cutting

We finally apply the cutting procedure to particular lagrangian submanifolds of symplectic toric manifolds.

4.1 Toric lagrangian cutting

This is the result of a joint work with Ana Cannas da Silva[1]. In this subsection we will exclusively work with symplectic toric n-manifolds.

Consider a symplectic toric manifold $(M, \omega, \mathbb{T}^n, \mu)$ which is either compact or the standard $\mathbb{C}^n$. We will work with a special class of lagrangian submanifold which are being classified by Cannas da Silva and Karshon in [CK20].

Definition 4.1 *Let $n \geq 0$ and $0 \leq k \leq n$. We call k-dimensional basic toric subgroup of $\mathbb{T}^n$ the subgroup*

$$T^k := (S^1)^k \times \mathbb{Z}_2^{n-k} < \mathbb{T}^n$$

This is a Lie subgroup of the standard torus and has Lie algebra $\mathfrak{t}_k^n := \mathbb{R}^k \times \pi\mathbb{Z}^{n-k} \subset \mathfrak{t}^{\mathfrak{n}}$. Recall the general linear group of the integers $GL(n, \mathbb{Z}) = \{A \in M_n(\mathbb{Z}) : \det(A) = \pm 1\}$.

Definition 4.2 *Let $n \geq 0$ and $0 \leq k \leq n$. For $A \in GL(n, \mathbb{Z})$, we call k-dimensional elementary toric subgroup given by A the subgroup $T^{k,A} :=$ $\exp(A\mathfrak{t}_k^n)$. We call connected component of the identity in $T^{k,A}$ the subtorus*

$$T_0^{k,A} := \exp(A\mathfrak{t}_{k,0}^n)$$

[1]That is, most of the ideas of the setting are due to her.

where $\mathfrak{t}^n_{k,0} = \mathbb{R}^n \times \{0\}^{n-k}$.

We introduce toric lagrangians.

Definition 4.3 *A toric lagrangian in* $(M, \omega, \mathbb{T}^n, \mu)$ *is a pair* $(L, T^{k,A})$ *where* L *is a proper connected lagrangian submanifold of* (M, ω), $0 \leq k \leq n$ *such that the intersection of* L *with each* $\mathbb{T}^n$*-orbit is clean and either exactly one* $T^{k,A}$ *orbit or empty. We call* $T^{k,A}$ *symmetry group of the toric lagrangian.*

This means in particular that L is preserved by the action of $T^{k,A}$ and no other bigger Lie subgroup of $\mathbb{T}^n$.

We relate lagrangians to affine subspaces.

Definition 4.4 *Let* $n \geq 0$ *and* $0 \leq k \leq n$. *For* $A \in GL(n, Z)$, *we denote by* $\mathcal{S}^{k,A}_b$ *the* $(n-k)$*-dimensional rational affine subspace of* $\mathbb{R}^n$ *given by the solutions* $x \in \mathbb{R}^n$ *to the equation* $(AI_{n\times k}))^T u = b$, *for* $b \in \mathbb{R}^k$.

Lemma 4.5 *Let* L *be a connected lagrangian in* $(M, \omega, \mathbb{T}^n, \mu)$ *and* $A \in GL(n, \mathbb{Z})$. *Then there exists* $b \in \mathbb{R}^k$ *such that* $\mu(L) \subset \mathcal{S}^{k,A}_b$ *if and only if* $T^{k,A}_0$ *preserves* L

Proof. A proof can be found in [CK20]. □

It follows that the moment image of a toric lagrangian is contained in some rational affine subspace.

Before stating the main result, we need the following technical lemma.

Lemma 4.6 *Consider a free and smooth action of a compact Lie group* G *on a manifold* M. *The orbit map* $\pi : M \to M/G$ *is a proper map.*

Proof. Let $C \subset M/G$ be compact. Let $(U_i)_i$ be a cover of $\pi^{-1}(C)$ such that $\overline{U_i}$ is compact[2]. Then $(\pi(U_i))_i$ covers C. We can find a finite subcover $\pi_(O_1), ..., \pi(O_n)$. Then $\pi^{-1}(C) \subset \bigcup_{i=1}^n \pi^{-1}(\pi(O_i)) \subset \bigcup_{i=1}^n G\cdot\overline{U_i}$, i.e. $\pi^{-1}(C)$ is a closed subset of a compact set. □

We apply symplectic cutting to this class of lagrangians. We work with ≥ 0-cutting; however, everything that follows holds also for ≤ 0-cutting. We will call $c \in \mathbb{R}$ a cutting level for a if S^1_a acts freely on $\mu^{-1}(a)$.

[2]This assumption can be made without loss of generality as manifolds are locally compact spaces, so we can refine any basis this way.

> **Theorem 4.7** *Let $(L, T^{k,A})$ be a toric lagrangian in $(M, \omega, \mathbb{T}^n, \mu)$, $a \in (\pi\mathbb{Z}^n)$ a primitive vector and $0 \in \mathbb{R}$ a cutting level for a. If the intersection $L \cap \mu_a^{-1}(0)$ is clean and $S_a^1 \cap T^{k,A}$ is the discrete two element subgroup, then we can reduce the lagrangian submanifold $L \times \mathbb{R}$ of $M \times \mathbb{C}$ to obtain a proper lagrangian submanifold $L_{a,\geq 0}$ of $M_{a,\geq 0}$. We call $L_{a,\geq 0}$ a cut lagrangian of $M_{a,\geq 0}$.*

Proof. Proof. We prove the theorem in six steps.

Step 1: we show $\mu_a(L) \neq \{0\}$.
Assume $\mu_a(L) = \{0\}$, then by Lemma 4.5 and by definition of $\mu_a = \langle \mu, a \rangle$, $\mu(L)$ is contained in the $(n-1)$-dimensional rational affine subspace $a^T u = 0$. So that if we define $A \in GL(n, \mathbb{Z})$ trought $Ae_1 = a$, then $T_0^{1,A} = \{(e^{a_1 i\theta}, ..., e^{a_n i\theta}) : \theta \in \mathbb{R}\} = S_a^1$, which contradicts second assumption on the lagrangian. on L. This proves that $\mu_a(L) \neq \{0\}$.

Step 2: we show $L \pitchfork \mu_a^{-1}(0)$.
Notice that there is no non-empty open subset of L which is contained in $\mu_a^{-1}(c)$. Indeed, if so, $\{0\}$ would be both open and closed in $\mu(M)$, contradicting connectedness.
By assumption $L \cap \mu_a^{-1}(0)$ is clean, so in particular the intersection is a manifold. By what above it follows directly that $\dim(L \cap \mu_a^{-1}(0)) < \dim(L)$. Hence, for any $x \in L \cap \mu_a^{-1}(0)$, $\dim(T_x L + T_x \mu_a^{-1}(0)) = 2n = \dim(M)$. We conclude that the intersection is transverse.

Step 3: we show $L \times \mathbb{R} \pitchfork \psi_{\geq 0}^{-1}(0)$.
Let $(x, z) \in L \times \mathbb{R} \cap \psi_{\geq 0}^{-1}(0)$. We have $T_{(x,z)}\psi_{\geq 0}^{-1}(0) = \ker(d\psi_{\geq 0}(x, z))$. Notice that[3] the kernel of $d\psi_{\geq 0} = \pi_M^* d\mu_a + \pi_{\mathbb{C}}^* d\mu_{\mathbb{C}}$ contains $\ker(\pi_M^* d\mu_a) \cap \ker(\pi_{\mathbb{C}}^* d\mu_{\mathbb{C}})$ as a 1-codimensional susbpace, which is pointwise the direct sum of the tangent spaces to the μ_a 0-level set with the tangent space of the unit circle in $\mathbb{C}$. Hence

$$\begin{aligned} T_{(x,z)}(L \times \mathbb{R}) + T_{(x,z)}\psi_{-}^{-1}(0) &\supseteq (T_x L \oplus T_z \mathbb{R}) + (T_x \mu_a^{-1}(0) \oplus T_z S^1) = \\ &= (T_x L + T_x \mu_a^{-1}(0)) \oplus (T_z \mathbb{R} + T_z S^1) = \\ &= (T_x L + T_x \mu_a^{-1}(0)) \oplus T_z \mathbb{C} \subseteq T_{(x,z)}(M \times \mathbb{C}) \end{aligned}$$

as $\mathbb{R} \pitchfork S^1$ in $\mathbb{C}$. Since $L \pitchfork \mu_a^{-1}(c)$ by Step 2, the two inclusions are equalities (the first by dimensional reasons) and we conclude $L \times \mathbb{R} \pitchfork \psi_{\geq 0}^{-1}(c)$.

Step 4: we show that the cut lagrangian is immersed in the cut space.
This follows directly by our work on lagrangian reduction in Section 2.3.

[3]We label the moment map of $\mathbb{C}$ by $\mu_{\mathbb{C}}$ and the natural projections of $M \times \mathbb{C}$ by π_M and $\pi_{\mathbb{C}}$ respectively.

Step 5: we show that the cut lagrangian is embedded in the cut space. To do this, we prove that the inclusion of the reduced lagrangian $(L \times \mathbb{R} \cap \psi_{\geq 0}^{-1}(0))/\mathbb{Z}_2$ into the cut space is a proper map. By Lemma 4.6, the orbit map $\pi : \psi_{\geq 0}^{-1}(c) \to M_{a,\geq c}$ is a proper map. Notice that $\mathbb{R}$ is a proper lagrangian in $\mathbb{C}$, so that $L \times \mathbb{R}$ is a proper lagrangian in $M \times \mathbb{C}$. It is known that the restriction of proper maps to preimages keeps properness, so that the inclusion $L \times \mathbb{R} \cap \psi_{\geq 0}^{-1}(c) \hookrightarrow \psi_{\geq 0}^{-1}(c)$ is a proper map.
So by commutativity of

$$\begin{tikzcd}
L \times \mathbb{R} \cap \psi_{\geq 0}^{-1}(c) \arrow[r, hook] \arrow[d] & \psi_{\geq 0}^{-1}(c) \arrow[d] \\
(L \times \mathbb{R} \cap \psi_{\geq 0}^{-1}(0))/\mathbb{Z}_2 \arrow[r, hook] & M_{a,\geq c}
\end{tikzcd}$$

the inclusion downstairs is a proper map. As embeddings with closed image are exactly injective proper immersions, we conclude that the image of the inclusion, that is the cut lagrangian, is properly embedded in the cut space. This finishes the step and the proof. □

Remark 4.8 *One may prove this theorem without passing through transversality. This proof uses diffeomorphisms between level sets away from critical points (see Morse theory), but is far less elegant.*

Theorem 4.9 *In the setting of Theorem 4.7, any connected component of the cut lagrangian $(L_{a,\geq 0}, T^{k,A})$ is a toric lagrangian in the cut space.*

Proof. By step 5 of the proof of Theorem 4.7, we know that the cut lagrangian is properly embedded in the cut space. We prove the intersection properties. Following the decomposition of the cut space, it is

$$L_{a,\geq 0} \cong \frac{L \cap \mu_a^{-1}(0)}{\mathbb{Z}_2} \sqcup L \cap \mu_a^{-1}(0,\infty)$$

so that away from the cutting level, the relevant intersection condition for the cut lagrangian to be a toric lagrangian is fulfilled.

The intersection of the cut lagrangian with each $\mathbb{T}^n$-orbit in the cut space is either exactly one $T^{k,A}$-orbit or empty by construction and assumptions on L. We show that this intersection is clean. By the proof of Lemma 2.7 we get that in out toric setting, $T_x(\mathbb{T}^n \cdot x) = \{X_M^\#(x) : \ X \in \mathfrak{t}^n\}$ for any $x \in M$.

Hence the tangent space to the torus orbit in the cut space through $[x, z]$, for $(x, z) \in \psi_{\geq 0}^{-1}(0)$ is

$$T_{[x,z]}(\mathbb{T}^n \cdot [x, z]) = \{X^{\#}_{M_{a,\geq 0}}([x, z]) : \; X \in \mathfrak{t}^n\}$$

So, for any $[x, 0] \in L_{a,\geq 0}$, $\frac{d}{dt}\big|_0 [\exp(tX) \cdot x, 0]$ lies in $T_{[x,0]}L_{a,\geq 0}$ if and only if the curve $t \mapsto \exp(tX) \cdot [x, 0]$ lies in $L_{a,\geq 0}$ for any $[x, 0] \in L_{a,\geq 0}$, which happens exactly when $t \mapsto \exp(tX) \cdot x$ lies entirely in L, i.e. exactly when X lies in the Lie algebra $A\mathfrak{t}_k^n$ of $T^{k,A}$ by assumption. We conclude

$$T_{[x,0]}L_{a,\geq 0} \cap T_{[x,0]}(\mathbb{T}^n \cdot [x, 0]) = \{X^{\#}_{M_{a,\geq 0}}[x, 0] : \; X \in A\mathfrak{t}_k^n\} = T_{[x,0]}(T^{k,A} \cdot [x, 0])$$

as desired. □

Remark 4.10 *We suspect that the cut lagrangian itself is a toric lagrangian, i.e. it is connected. The proof of connectedness seems to be non-trivial and due to time constraints we will not eleborate on that further in this thesis.*

Example 4.11 *Consider a real structure σ on the symplectic toric M^{2n} and let $L = M^\sigma$. It's not hard to see that $(L, \mathbb{Z}_2^n)$ is a toric lagrangian. Then we have $\sigma^\times$ on $M \times \mathbb{C}$ defined by $\sigma^\times(x, z) := (\sigma(x), \overline{z})$; it's easy to see that $\sigma^\times$ defines a real structure on $M \times \mathbb{C}$. By Lemma 2.19 we have real structures $\overline{\sigma_{\geq 0}}$ and $\overline{\sigma_{\leq 0}}$ and by Lemma 2.20 we can reduce $(M \times \mathbb{C})^{\sigma^\times} \cong M^\sigma \times \mathbb{R}$ in both directions to get lagrangians $\overline{L_{\geq 0}} = (\overline{M_{\geq 0}})^{\overline{\sigma_{\geq 0}}}$ and $\overline{L_{\leq 0}} = (\overline{M_{\leq 0}})^{\overline{\sigma_{\leq 0}}}$.*

Corollary 4.12 *The cut lagrangian has moment image $\mu(L_{a,\geq 0}) = \mu(L) \cap \{x \in \mathfrak{t}^n : \; \langle x, a\rangle \geq 0\}$*

Proof. This is a direct consequence of Theorem 4.7 and Proposition 3.14.□

4.2 Circle cutting

We also can try to cut lagrangians in certain symplectic manifolds with respect to the lagrangian $S^1 \subset \mathbb{C}$. The substantial difference is that the moment image of the circle is a point, while the moment image of the real line is the whole non-negative real axis.
This section is in part a natural question arising from the combination of symplectic cutting and lagrangian submanifolds and in part motivated by Alessandro Fasse's Master thesis [Fas18], where in the concluding remarks

the author intuited a relationship between symplectic cutting and rational affine subspaces of $\mathbb{C}^n$.

Consider the symplectic toric manifold $(\mathbb{C}^2, dz \wedge d\bar{z}, \mathbb{T}^2, \mu_{\mathbb{C}})$, with diagonal action and moment map $\mu(z_1, z_2) = (\frac{1}{2}|z_1|^2, \frac{1}{2}|z_2|^2)$ for $(z_1, z_2) \in \mathbb{C}^2$, so that the moment map image is the first quadrant. One can ask for (immersed or embedded) toric lagrangians in $\mathbb{C}^2$ with a given affine subspace as moment image. In [Fas18], Fasse outlined the limitations of standard symplectic reduction to find such lagrangians. Circle cutting is a tool that provides another way to find these lagrangians. Unfortunately, cutting alone doesn't cover the whole proof.

The following theorem is a generalization to arbitrary finite dimension of the so called first and second ray Theoremm, proved in [Fas18]. It is worth mentioning that our notion of toric lagrangian slightly differs from the one adopted in [Fas18].

Theorem 4.13 *Let $k \in \mathbb{R}_{>0}$, $a \in \mathbb{Z}^n$ and*

$$N_k := \left\{ x \in \mathbb{R}^n_{\geq 0} : \ \langle a, x \rangle = \frac{k^2}{2} \right\}$$

a rational affine subspace of $\mathbb{R}^n_{\geq 0}$ in the sense that its linear completion is a rational afffine subspace of $\mathbb{R}^n$. If at least one coordinate of a is non negative, then there exist an immersed lagrangian L_k of $\mathbb{C}^n$ whose moment image is N_k.

Proof. Consider the half space $\mathcal{H} := \{x \in \mathbb{R}^n : \langle a, x \rangle \geq 0\}$, where without loss of generality we can assume that a is primitive. By Delzant theorem, if we cut the standard $\mathbb{C}^n$ with respect to $\mathcal{H}$ in the sense of Section 3.3, we get a space which is symplectomorphic to the original $\mathbb{C}^n$ and will hence still get denoted by $\mathbb{C}^n$.

We cut the lagrangian $\mathbb{R}^n$ at the cutting level $\mu_a^{-1}(\frac{k^2}{2})$ with respect to the lagrangian circle of radius k in the complex plane, i.e. we reduce $\mathbb{R}^n \times S^1(k)$. By the quotient manifold theorem and lagrangianity upstairs, we end up with a smooth manifold L_k inside the standard $\mathbb{C}^n$, such that $\omega|_{L_k}$ vanishes.

Under the identification of the cut space with the standard $\mathbb{C}^n$, is identified with

$$\left\{ (r_1 e^{i a_1 \theta}, ..., r_n e^{i a_n \theta}) \in \mathbb{C}^n : \ \theta \in \mathbb{R}, \ \sum_{i=1}^{n} a_i r_i^2 = \frac{k^2}{2} \right\}$$

via the map

$$[r_1, ..., r_n, ke^{i\theta}]_{\mathbb{Z}_2} \to (r_1 e^{ia_1\theta}, ..., r_n e^{ia_n\theta})$$

which is well defined as $(-1)\cdot(z_1, ..., z_n, w) = (e^{ia_1\pi}z_1, ..., e^{ia_n\pi}, -w)$ for $(z_1, ..., z_n, w) \in \mathbb{C}^n \times \mathbb{C}$.

See [Fas18] for a proof of the fact that L_k is an immersed submanifold of $\mathbb{C}^n$. □

Remark 4.14 *My original aim was to only use cutting. However, it turns out that in this situation the intersection $T_{(x,y)}(\mathbb{R}^n \times S^1(k) \cap \psi^{-1}_{\geq 0}(0))$ is not clean, as*

$$\mathbb{R}^n \times S^1(k) \cap \psi^{-1}_{\geq 0}(0) = \left(\mathbb{R}^n \cap \mu^{-1}\left(\frac{k^2}{2}\right)\right) \times S^1(k)$$

but

$$T_{(x,y)}(\mathbb{R}^n \times S^1(k)) \cap T_{(x,y)}\psi^{-1}_{\geq 0}(0) = (T_x\mathbb{R}^n \cap T_xM) \oplus (T_yS^1(k) \cap T_yS^1(k)$$

So the only thing we know a priori on the "cut lagrangian" is that it is a manifold contained in $\mathbb{C}^n$ and that the sympletic form of $\mathbb{C}^n$ vanishes on it.

Lemma 4.15 *There is a diffeomorphism $S^1 \times \mathbb{R}P^{n-1} \to L_k$.*

Proof. Consider coordinates $(r_1, ..., r_n)$ on $S^{n-1} \subset \mathbb{R}^n$ such that $\sum_{i=1}^n a_i r_i^2 = k$. Then we have a natural map $f : S^1 \times S^{n-1} \to L_k$ given by

$$f(e^{i\theta}, r_1, ..., r_n) := (r_1 e^{ia_1\theta}, ..., r_n e^{ia_n\theta})$$

This is a smooth surjective map. Notice that

$$f(e^{i\theta}, r_1, ..., r_n) = f(e^{i(\theta+\pi)}, e^{ia_1\pi}r_1, ..., e^{ia_n\pi}r_n)$$

that is, f is $\mathbb{Z}_2$ invariant. Hence f descends to a map from the quotient $(S^1 \times S^{n-1})/\mathbb{Z}_2 \cong S^1 \times \mathbb{R}P^{n-1}$. This induced map is a diffeomorphism. □

Remark 4.16 *The symmetry group of L_k is $\exp(\mathbb{R}a)$, that is $T^{1,A}$ for $A \in GL(n, \mathbb{Z})$ with a as first column. This means that this cutting procedure adds symmetries to the lagrangian. By the lemma above it is also clear that L_k is connected.*

Chapter 5

Final remarks

The arguments presented in this thesis may naturally be extended and completed. Here there are a few possible ways:

1. Clearly, lagrangian cutting may be naturally extended to arbitrary hamiltonian toric spaces or/and to the general cutting setting of Definition 3.8.

2. It would be interesting to investigate more on cirlce cutting. While in Remark 4.13 we pointed out that it is not as powerful as expected, I presume that there may be stronger assumption on the lagrangian so that one has not to pass trought hard machinery to prove submanifold behaviours. I add here a graphical visualization of whats going on whit circle cutting in another special case:

   Consider the cutting of $\mathbb{C}^2$ along the half space $\{(x, y) \in \mathbb{R}^2 : \ x + y \leq \frac{1}{2}\}$ to get a copy of the standard $\mathbb{C}P^2$. Fix $r \in \mathbb{R}_{>0}$. We cut the real part with the circle $S^1(r)$:

   $L \times S^1(r) \cap \psi_{\leq 0}(0) = \{(x, y, re^{i\theta}) \in L \times S^1(r) : \ x^2 + y^2 = -r^2 + 1\} \cong \mathbb{T}^2$

   which gets reduced to a torus. This torus has moment image $\left\{(x, \frac{1-r^2}{2} - x) \in \mathbb{R}^2 : \ x \in [0, \frac{r^2-1}{2}]\right\}$. Two examples are drawed in Figure 5.1.

3. It would be interesting to apply Karshon's cutting to lagrangian cutting. We have already outlined in Section 3.4 that the main problem with her work is to adeguate the setting to $\mathbb{Z}_2$ actions. Here there is a non-rigorous (an maybe completely wrong) proposal.

   Assume L intersects $L \cap \mu^{-1}(0)$ cleanly. $L \cap \mu^{-1}(0, \infty)$ is an n-dimensional submanifold of M. So, the glued manifold $L \cap \mu^{-1}[0, \infty) \cong L \cap \mu^{-1}(0) \sqcup L \cap \mu^{-1}(0, \infty)$ is indeed a manifold.

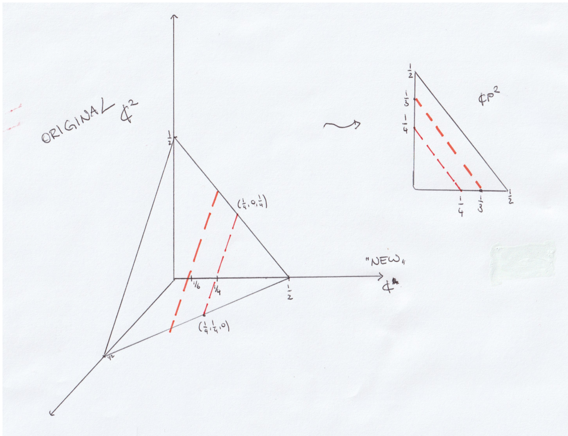


Figure 5.1: The graphical idea of the circle cutting of $\mathbb{R}^2 \subset \mathbb{C}^2$ with respect to circles with radii $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{3}}$ respectively. The symplex is the moment image of $\psi_{\leq 0}^{-1}(0)$, and one in the cutting procedure projects it to the original quadrant to get the new manifold $\mathbb{C}P^2$. Here we choose rays (moment images of $L \times S^1(r) \cap \psi_{\leq 0}^{-1}(0)$) and project them to $\mathbb{C}P^2$.

Proposition 4.1 is available just when we have *circle* action, which isn't the case in our lagrangian setting[1]. We have to develop further machinery for manifold with boundary equipped with $\mathbb{Z}_2$ actions near the boundary.

Submanifold behaviour in the Karshon setting Let N be a manifold with boundary equipped with a free circle action near the boundary. Let M be an embedded submanifold with boundary of N and denote by $i : M \to N$ the corresponding embedding. It is rather clear that by Proposition 4.1 if such a submanifold has a free circle action near the boundary it gets pushed down to an embedded submanifold of the cut space N_c. We try to study what happens if the submanifold is invariant by $\mathbb{Z}_2$ and no greater subgroup of S^1.

i restricts to a $\mathbb{Z}_2$ equivariant map $i|_{\partial M} : \partial M \to \partial N$, so it descends to a map which restricts to

$$\frac{\partial M}{\mathbb{Z}_2} \to \frac{\partial N}{\mathbb{Z}_2}$$

Recall that we have a principal circle bundle $S^1 \to \partial N \to \frac{\partial N}{S^1}$. This

[1]Recall: lagrangian preserved by the circle lie in some level set.

gives rise to a principal $\frac{S^1}{\mathbb{Z}_2} \cong S^1$-bundle

$$\frac{S^1}{\mathbb{Z}_2} \to \frac{\partial N}{\mathbb{Z}_2} \to \frac{\partial N}{S^1}$$

Composing the two we get a map $\frac{\partial M}{\mathbb{Z}_2} \to \frac{\partial N}{S^1}$[2]. Hence the embedding descends to a unique map $i_c : c(M) \to N_c$ such that the diagram commutes.

Differential structures of subsets. We need the notion of differential space, initially introduced by Sikorsky and then developed in [Sni13].

Definition 5.1 *A differential space is a pair $(P, \mathcal{F})$ where P is a topological space and $\mathcal{F}$ is a set of continuous real valued functions such that:*

- *For any open interval $I \subset \mathbb{R}$ and function $f \in \mathcal{F}$, the sets $f^{-1}(I)$ form a subbasis for the topology on P;*
- *For any $n \in \mathbb{Z}_{>0}$, g smooth real valued function on $\mathbb{R}^n$ and $f_1, ..., f_n \in \mathcal{F}$: $g \circ (f_1, ..., f_n) \in \mathcal{F}$;*
- *If $f : P \to \mathbb{R}$ is a function such that for any $p \in P$ there is an open neighbourhood U_p of p and a function $f_p \in \mathcal{F}$ such that $f_p|_{U_p} = f|_{U_p}$, then $f \in \mathcal{F}$.*

Definition 5.2 *Let F be a family of real valued functions on a set R. Endow R with the initial topology determined by F. Define $\mathcal{F}$ to be the set of real valued functions h on R such that for any $x \in R$ there is an open subset $U \subset R$, an integer $n > 0$, $h_1, ..., h_n \in F$ and $g \in C^\infty(\mathbb{R}^n)$ such that*

$$h|_U = g(h_1, ..., h_n)|_U$$

We say that F generates $\mathcal{F}$.

Lemma 5.3 *Let $S \subset R$ be a subset of a differential space $(R, \mathcal{F})$, let $i : S \hookrightarrow R$ be the inclusion and let*

$$\mathcal{R}(S) := \{f \circ i : \ f \in \mathcal{F}\}$$

[2]This is a principal bundle morphism between ∂M and ∂N with respect to the inclusion ([Mer19], D24.18)

Then $\mathcal{R}(S)$ generates a differential structure $\mathcal{F}^S$ on S such that the differential space topology of S coincides with the subspace topology. In this setting, the inclusion i is smooth.

Proof. This is Proposition 2.1.8 in [Sni13]. □

This differential strucure on $c(M)$ makes the map $c|_M : M \to c(M)$ smooth.

So, in our setting, on $c(M)$ quotient topology and subspace topology (which is the differential space topology by definition) coincide, as M is embedded in N and the diagram commutes. All is left to do is first to show that $c(M)$ is a manifold, then, show that smooth function on $c(M)$ coincide with the differential structure induced by N.

Bibliography

[Ati82] Michael Francis Atiyah. "Convexity and commuting hamiltonians". In: *Bull. London Math. Soc.* 14 (1982), pp. 1–15.

[Aud91] Michèle Audin. *The Topology of Torus Actions on Symplectic Manifolds*. Birkhäuser Basel, 1991. DOI: 10.1007/978-3-0348-7221-8.

[Can08] Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Vol. 1764. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008. ISBN: 978-3-540-42195-5. DOI: 10.1007/978-3-540-45330-7. URL: http://link.springer.com/10.1007/978-3-540-45330-7.

[Can18] Ana Cannas da Silva. "A Chiang-type lagrangian in CP2". In: *Lett. Math. Phys.* 108 (2018), pp. 699–710.

[CB97] Richard Cushman and Larry Bates. *Global Aspects of Classical Integrable Systems*. Birkhäuser Basel, 1997. DOI: 10.1007/978-3-0348-8891-2.

[CK20] Ana Cannas da Silva and Yael Karshon. *Toric lagrangians*. 2020.

[Del88] Thomas Delzant. "Hamiltoniens périodiques et images convexes de l'application moment". In: *Bulletin de la Societè Mathematique de France* 116.3 (1988), pp. 315–339. DOI: 10.24033/bsmf.2100.

[Dui83] J. J. Duistermaat. "Convexity and Tightness for Restrictions of Hamiltonian Functions to Fixed Point Sets of an Antisymplectic Involution". In: *Transactions of the American Mathematical Society* 275.1 (Jan. 1983), p. 417. ISSN: 00029947. DOI:

10.2307/1999030. URL: https://www.jstor.org/stable/1999030.

[Fas18] Alessandro Fasse. *Lagrangian Submanifolds of Symplectic Toric Manifolds*. 2018.

[GS82] Victor Guillemin and Shlomo Sternberg. “Convexity properties of the moment mapping”. In: *Invent. Math.* 67 (1982), pp. 491–513. DOI: 10.1007/BF01398933. URL: https://www.researchgate.net/publication/227035539.

[GS89] V Guillemin and S Sternberg. “Birational equivalence in the symplectic category”. In: *Invent. math* 97 (1989), pp. 485–522.

[Kar20] Yael Karshon. “Functoriality for symplectic and contact cutting”. In: Work in progress (2020).

[KL15] Yael Karshon and Eugene Lerman. “Non-compact symplectic toric manifolds”. In: *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)* 11 (July 2015). DOI: 10.3842/SIGMA.2015.055.

[Ler+96] Eugene Lerman et al. “Non-abelian convexity by symplectic cuts”. In: (Mar. 1996). URL: http://arxiv.org/abs/dg-ga/9603015.

[Ler95] Eugene Lerman. “Symplectic cuts”. In: *Mathematical Research Letters* 2.3 (Aug. 1995), pp. 247–258. DOI: 10.4310/mrl.1995.v2.n3.a2.

[Mar+06] Jerrold Marsden et al. *Hamiltonian Reduction by Stages*. Tech. rep. 2006. URL: http://springerlink.com/content/110312/.

[Mer19] Will J Merry. *Differential Geometry*. 2019. URL: https://www.merry.io/courses/differential-geometry/.

[Mey73] Kenneth R. Meyer. “Symmetries and Integrals in Mechanics”. In: *Dynamical Systems* (1973), pp. 259–272. DOI: 10.1016/b978-0-12-550350-1.50025-4.

[MW74] Jerrold Marsden and Alan Weinstein. “Reduction of symplectic manifolds with symmetry”. In: *Reports on Mathematical Physics* 5.1 (1974), pp. 121–130. ISSN: 00344877. DOI: 10.1016/0034-4877(74)90021-4.

[Sja10] Reyer Sjamaar. “Real symplectic geometry”. In: *African Diaspora Journal of Mathematics* 2 (2010), pp. 34–52.

[Sni13] J. Sniatycki. *Differential Geometry of Singular Spaces and Reduction of Symmetry*. Cambridge: Cambridge University Press, 2013. ISBN: 9781139136990. DOI: 10.1017/CBO9781139136990. URL: http://ebooks.cambridge.org/ref/id/CBO9781139136990.

[Wei77] Alan Weinstein. *Lectures on symplectic manifolds*. Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, 1977, p. 48. ISBN: 9780821816790.