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Borsuk–Ulam Theorem and Applications

Bachelor Thesis

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Abstract

The Borsuk{Ulam theorem from algebraic topology states that for every continuous function from the n-dimensional unit sphere to the $(n + 1)$-dimensional Euclidean space there are two antipodal points on the sphere that get mapped to the same point. Using the notion of transversality from di erential topology, we prove a version of the statement based on the mod 2 winding number of a function. Then we apply it to solve combinatorial problems such as the necklace splitting problem and the ham sandwich theorem.

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Chapter 1

Introduction

The Borsuk{Ulam theorem is named after the mathematicians Karol Borsuk and Stanislaw Ulam. In 1933, Karol Borsuk found a proof for the theorem conjectured by Stanislaw Ulam. However, as Jir Matousek mentioned in [Mat03, Chapter 2, Section 1, p. 25], an equivalent theorem in the setting of set coverings, namely the Lusternik | Schnirelmann theorem, appeared already three years before in an article written by Lazar Aronovich Lyusternik and Lev Genrikhovich Schnirelmann ([LS47]).

The chapters 2 to 4 of this Bachelor thesis are mainly based on chapters 1 and 2 of the book [GP10] by Victor Guillemin and Alan Pollack. In these chapters of the book, fundamental concepts from analysis, topology and di erential geometry are recalled and the notion of transversality is introduced.
The nal chapter of this thesis is based on chapter 3 of the book [Mat03] by Jir Matousek. This book approaches the Borsuk{Ulam theorem from a combinatorial viewpoint by proving Tucker's lemma and showing that it is equivalent to the Borsuk{Ulam theorem. Furthermore, it contains a geometric proof of the Borsuk{Ulam theorem.

In Chapter 2, we introduce preliminary material such as manifolds, tangent spaces, derivatives and homotopy and prove some theorems that we will need. In Chapter 3, we de ne the concept of transversality and prove some of its properties. Then we extend the main results from the second chapter to manifolds with boundary to obtain theorems needed to de ne the mod 2 winding number at the end of the chapter. In Chapter 4, we state and prove the Borsuk{Ulam theorem and derive another version of the theorem, which will be more handy in Chapter 5 on applications of the theorem. There we discuss the ham sandwich theorem, multicolor partitions and the necklace splitting problem.

I wish to express my sincere thanks to my adviser, Professor Ana Cannas da Silva, for supervising this thesis. She spent a lot of time answering my questions concerning the material as well as the writing process and guided me when I lost sight of the goal. I also wish to thank Manuel Wiedmer for discussing some of the exercises in the book with me. Moreover, I wish to thank him and Nils Jensen for proofreading this thesis and giving me valuable comments.

Chapter 2

Preliminaries

In this chapter we state the basic things we need in this thesis. For a more detailed introduction we refer the reader to [GP10, Chapter 1, Sections 1–4]. The statements we prove are mostly exercises from this book.

2.1 Manifolds

We start with defining the spaces we are going to work on, namely manifolds. To do so we need the following definitions, which are taken from [GP10, Chapter 1, Section 1]. In this book as well as in this thesis, the words "map" and "function" are used equivalently.

Definition 2.1. A neighborhood of a point $x \in \mathbb{R}^n$ is an open set $V \subset \mathbb{R}^n$ such that $x \in V$.

Definition 2.2. A map $f\colon U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is an open set, is called smooth if it has continuous partial derivatives of all orders.

Definition 2.3. Let $f\colon X \to \mathbb{R}^m$ be defined on an arbitrary subset $X \subset \mathbb{R}^n$. Then f is called smooth if for each $x \in X$ there exists an open set $U \subset \mathbb{R}^n$ containing x and a smooth map $F\colon U \to \mathbb{R}^m$ such that F and f agree on $U \cap X$. In this case we say that f admits a local extension to a smooth map on open sets.

Definition 2.4. Let X and Y be subsets of two Euclidean spaces. A smooth map $f\colon X \to Y$ is called a diffeomorphism if it is bijective and the inverse map $f^{-1}\colon Y \to X$ is also smooth.

Definition 2.5. A subset $X \subset \mathbb{R}^N$ is a k-dimensional manifold if it is locally diffeomorphic to $\mathbb{R}^k$, that is, if for each $x \in X$ there exists a neighborhood $V \subset X$ that is diffeomorphic to an open set $U \subset \mathbb{R}^k$. Such a diffeomorphism $\phi\colon U \to V$ is called a parametrization of the neighborhood V and its inverse

$\varphi^{-1}\colon V \to U$ is called a coordinate system on V. If we write the map φ^{-1} in coordinates $\varphi^{-1} = (x_1, \dots, x_k)$, then the k smooth functions $x_1, \dots, x_k$ on V are called coordinate functions.

A parametrization of the neighborhood $V \subset X$ of x is called a local parametrization of X about x.
Unless specified otherwise, we use the letters X, Y and Z to denote manifolds. Moreover, we implicitly assume that $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ are of dimension k and l respectively.

Definition 2.6. If X and Z are manifolds in $\mathbb{R}^N$ and $Z \subset X$, then Z is called a submanifold of X.

Notice that a manifold X in $\mathbb{R}^n$ is a submanifold of $\mathbb{R}^N$. Moreover, any open subset of X is a submanifold of X.

The following theorem tells us that we can construct new manifolds by taking the Cartesian product of manifolds.

Theorem 2.7. *If X and Y are manifolds, then so is $X \times Y$. Moreover, we have* $\dim(X \times Y) = \dim(X) + \dim(Y)$.

An outline of the proof can be found in [GP10, Chapter 1, Section 1, p. 4–5], but there are some steps — in particular the proof of the next Lemma — left for the reader. As the map (2.1) will be important for us later, we give a proof of this step.

Lemma 2.8. *If $\varphi\colon U \to X$ is a local parametrization around $x \in X$ and $\psi\colon V \to Y$ is a local parametrization around $y \in Y$, then the map*

$$\varphi \times \psi\colon U \times V \to X \times Y; (u, v) \mapsto (\varphi(u), \psi(v)) \tag{2.1}$$

is a local parametrization of $X \times Y$ around (x, y).

Proof. Notice that $U \times V \subset \mathbb{R}^k \times \mathbb{R}^l \cong \mathbb{R}^{k+l}$ is open. Let $\varphi\colon U \to X$ be a local parametrization of X around x, that is, there exists a neighborhood O_x of x in X such that $\varphi\colon U \to O_x$ is a diffeomorphism. Analogously, assume that $\psi\colon V \to O_y$ is a diffeomorphism, where O_y is a neighborhood of y in Y. We have that $O_x \times O_y \subset X \times Y$ is open.
First, the function

$$\varphi \times \psi\colon U \times V \to O_x \times O_y; (u, v) \to (\varphi(u), \psi(v))$$

is smooth by Definition 2.2 because both of its components are assumed to be smooth. Moreover, it is bijective because both its components are.
It is left to show that

$$(\varphi \times \psi)^{-1}\colon O_x \times O_y \to U \times V$$

is smooth.
By assumption, $\varphi^{-1}\colon O_x \to U$ and $\psi^{-1}\colon O_y \to V$ are smooth. The sets O_x

and O_y are open in X and Y respectively, but X and Y need not be open in $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively. Hence, O_x and O_y need not be open in $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively. The maps φ^{-1} and ψ^{-1} are smooth in the sense of Definition 2.3. That is, around each $x \in O_x$ and each $y \in O_y$ there exist neighborhoods $W_x \subset \mathbb{R}^N$ and $W_y \subset \mathbb{R}^M$ and smooth maps $\Phi : W_x \to U$ and $\Psi : W_y \to V$ such that $\Phi = \varphi^{-1}$ on $W_x \cap O_x$ and $\Psi = \psi^{-1}$ on $W_y \cap O_y$.
Then $W_x \times W_y$ is open in $\mathbb{R}^N \times \mathbb{R}^M$ and $\Phi \times \Psi$ equals the smooth map $\varphi^{-1} \times \psi^{-1}$ on $(O_x \cap W_x) \times (O_y \cap W_y)$. Since this holds for each $(x, y) \in O_x \times O_y$, the map $\varphi^{-1} \times \psi^{-1}$ is smooth on the not necessarily open subset $O_x \times O_y$ of $\mathbb{R}^N \times \mathbb{R}^M$. We are done with the proof since $(\varphi \times \psi)^{-1} = \varphi^{-1} \times \psi^{-1}$. □

2.2 Derivative and Tangent Space

This section is based on [GP10, Chapter 1, Section 2–3]. We recall the definition of the derivative of a smooth map defined on an open set in Euclidean space from analysis. Then we use it to define the tangent space of a manifold at a certain point. As we will see, tangent spaces will be the domains for the derivative of a smooth map between manifolds.

Definition 2.9. For an open set $U \subset \mathbb{R}^n$, let $f : U \to \mathbb{R}^m$ be a smooth map and let $x \in U$. The derivative of f in the direction of the vector $h \in \mathbb{R}^n$ at the point $x \in U$ is defined by

$$df_x(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

Recall that the derivative $df_x : \mathbb{R}^n \to \mathbb{R}^m$ is linear and satisfies the chain rule:

Theorem 2.10 (The Chain Rule on open subsets of Euclidean spaces)**.** *Assume that $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets and let $f : U \to V$ and $g : V \to \mathbb{R}^l$ be smooth maps. Then for each $x \in U$,*

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

Lemma 2.11. *If the smooth map $f : U \to \mathbb{R}^m$, defined on an open set $U \subset \mathbb{R}^n$, is linear, then $f \equiv df_x$ for any $x \in U$.*

Proof. Let $x \in U$ be arbitrary. We use linearity of f in Definition 2.9 to get for any $h \in \mathbb{R}^n$:

$$\begin{aligned} df_x(h) &= \lim_{t \to 0} \frac{f(x + th) - f(x)}{t} \\ &= \lim_{t \to 0} \frac{f(x) + t \cdot f(h) - f(x)}{t} \\ &= f(h), \end{aligned}$$

which completes the proof. □

Let $X \subseteq \mathbb{R}^N$ be a k-dimensional manifold. Suppose that $\phi\colon U \to X$, where $U \subseteq \mathbb{R}^k$ is open, is a local parametrization around x such that $\phi(0) = x$. The best linear approximation to $\phi\colon U \to X$ at 0 is the map

$$h\colon \mathbb{R}^k \to \mathbb{R}^N; u \mapsto \phi(0) + d\phi_0(u) = x + d\phi_0(u).$$

Definition 2.12. Let X and ϕ be as above. The tangent space of X at x is the vector subspace of $\mathbb{R}^N$ defined by the image of $d\phi_0$, namely

$$T_x(X) := \operatorname{im}(d\phi_0).$$

That the definition of the tangent space is independent of the choice of local parametrization is shown in [GP10, Chapter 1, Section 2, p. 9–10]. This justifies the article "the" in the definition above.
If $X \subseteq \mathbb{R}^k$ is an open set containing 0, then we can take the identity map $id\colon X \to X$ as our local parametrization ϕ around 0. Using the fact that the derivative of the identity map from the Euclidean space $X \subseteq \mathbb{R}^k$ to itself is the identity map from $\mathbb{R}^k$ to itself, we get

$$T_0(X) = \operatorname{im}(d\phi_0) = \mathbb{R}^k;$$

showing that for any open subset $U \subseteq \mathbb{R}^k$ containing 0, we have $T_0(U) = \mathbb{R}^k$. We need this result for Definition 2.16 of the derivative of a smooth map between manifolds.

A proof of the following Lemma can be found in [GP10, Chapter 1, Section 2, p. 9–10].

Lemma 2.13. *Let $X \subseteq \mathbb{R}^N$ be a k-dimensional manifold. If $T_x(X)$ is its tangent space at some $x \in X$, then*

$$\dim(T_x(X)) = \dim(X) = k.$$

Now we move on to show that the tangent space of a product manifold is the product of the tangent spaces.

Lemma 2.14. *Let X and Y be manifolds. For every $x \in X$ and $y \in Y$ we have*

$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y).$$

Proof. Suppose that $\phi\colon U \to X$ and $\psi\colon V \to Y$ are local parametrizations around x and y respectively such that $\phi(0) = x$ and $\psi(0) = y$. By Lemma 2.8, the map $\phi \times \psi$ defined by (2.1) is a local parametrization of $X \times Y$ around (x, y). The best linear approximation to $\phi \times \psi$ is given by

$$(u, v) \mapsto (\phi \times \psi)(0, 0) + d(\phi \times \psi)_{(0,0)}(u, v).$$

Assuming the identity $d(\phi\times\psi)_{(0,0)}(u,v) = (d\phi_0 \times d\psi_0)(u,v)$ and applying Definition 2.12, we get

$$\begin{aligned} T_{(x,y)}(X\times Y) &:= \operatorname{im}(d(\phi\times\psi)_{(0,0)}) \\ &= \operatorname{im}(d\phi_0 \times d\psi_0) \\ &= \operatorname{im}(d\phi_0) \oplus \operatorname{im}(d\psi_0) \\ &=: T_x(X) \oplus T_y(Y). \end{aligned}$$

This concludes the proof. □

Next, we show that the tangent space of $\mathbb{R}^N$ is the vector space itself.

Lemma 2.15. *Let V be a vector subspace of $\mathbb{R}^N$. Then $T_x(V) = V$ for any $x \in V$.*

Proof. Assume that V is a vector subspace of $\mathbb{R}^N$ of dimension k. Consider the basis $\{v_1, v_2, \dots, v_k\}$ of V. Now, we have that the map

$$\phi : \mathbb{R}^k \to V, (a_1, \dots, a_k) \mapsto \sum_{i=1}^{k} a_i v_i$$

is a linear diffeomorphism. Hence, Lemma 2.11 implies $\phi = d\phi_a$ for any $a \in \mathbb{R}^k$, in particular for $a = 0$. Therefore, Definition 2.12 of the tangent space tells us that

$$T_x(V) := \operatorname{im}(d\phi_0) = \operatorname{im}(\phi) =: V$$

which is what we wanted to show. □

Now we introduce the derivative of a smooth map between manifolds.

Definition 2.16. Let X and Y be manifolds of dimension k and l respectively. Suppose $f : X \to Y$ is a smooth map. Let $x \in X$ and define $y := f(x)$. Assume that $\phi : U \to X$ and $\psi : V \to Y$ are local parametrizations of X about x and Y about y respectively such that $\phi(0) = x$ and $\psi(0) = y$. Define the function $h := \psi^{-1} \circ f \circ \phi$ such that the following diagram commutes for U small enough:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ U & \xrightarrow[h]{} & V \end{array}$$

The derivative of f at x is the linear map $df_x : T_x(X) \to T_y(Y)$ between tangent spaces defined by

$$df_x := d\psi_0 \circ dh_0 \circ d\phi_0^{-1},$$

where $d\phi_0$ is the directional derivative according to Definition 2.9 of the map $\phi : U \to \mathbb{R}^N$ from an open subset in some Euclidean space to $\mathbb{R}^N$, where $N \in \mathbb{N}$ is large enough such that $X \subset \mathbb{R}^N$. Analogously, $d\psi_0$ is the derivative of the map $\psi : V \to \mathbb{R}^M$, where $M \in \mathbb{N}$ is chosen such that $Y \subset \mathbb{R}^M$. This leads to the commuting diagram:

$$\begin{array}{ccc} T_x(X) & \xrightarrow{df_x} & T_y(Y) \\ {\scriptstyle d\phi_0}\uparrow & & \uparrow{\scriptstyle d\psi_0} \\ \mathbb{R}^k & \xrightarrow[dh_0]{} & \mathbb{R}^l \end{array}$$

One may check that df_x is independent of the choice of parametrizations ϕ and ψ.

Local parametrizations enable us to construct smooth maps on open subsets of Euclidean spaces from smooth maps between arbitrary manifolds. Hence, they allow us to extend the chain rule on Euclidean spaces (Theorem 2.10) to manifolds.

Theorem 2.17 (The Chain Rule). *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of smooth maps of manifolds, then*

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

For a proof we refer the reader to [GP10, Chapter 1, Section 2, p. 10–11].

In the remainder of this section, we state and prove some further properties of tangent spaces. We apply them to show that the derivatives of the inclusion and the projection maps are identical to the inclusion and the projection maps respectively, but on the corresponding tangent spaces.

Lemma 2.18. *Assume that Z is an l-dimensional submanifold of X and let $z \in Z$. Then there exists a local coordinate system $\{x_1, \dots, x_k\}$ defined in a neighborhood U of z in X such that $U \cap X$ is defined by equations*

$$x_{l+1} = 0, \dots, x_k = 0.$$

The proof of Lemma 2.18 uses the local immersion theorem from [GP10, Chapter 1, Section 3, p. 15].

Lemma 2.19. *If Y is a manifold and $X \subset Y$ is a submanifold, then for any $x \in X$ we have $T_x(X) \subset T_x(Y)$.*

Proof. Let $x \in X$ be an arbitrary element of the submanifold X of $Y \subset \mathbb{R}^M$. Suppose that the map $\psi : V \to Y$ is a local parametrization of Y around x defined on an open subset $V \subset \mathbb{R}^l$ such that $\psi(0) = x$, that is, there exists a neighborhood $W \subset Y$ of x such that $\psi : V \to W$ is a diffeomorphism.
The set $W \cap X$ is a neighborhood of x in X. Define the set

$$U := \psi^{-1}(W \cap X) = \{v \in V : \psi(v) \in W \cap X\}.$$

By Lemma 2.18, ψ can be chosen in such a way that only the first k components of points in U are nonzero and the other components vanish. Hence, we may

view U as a subset of $\mathbb{R}^k$. Since φ is continuous and $W \cap X$ is open, U is also open.
The map

$$\psi := \varphi|_U : U \to W \cap X$$

is a parametrization of X around x satisfying $\psi(0) = x$.
By Definition 2.12 of the tangent space, we get

$$T_x(X) := \operatorname{im}(d\psi_0) = \operatorname{im}(d(\varphi|_U)_0) \subseteq \operatorname{im}(d\varphi_0) =: T_x(Y)$$

showing the desired inclusion. □

With Lemma 2.19 above we we are able to show that the derivative of the inclusion map is the inclusion map and, in particular, that the derivative of the identity is again the identity.

Lemma 2.20. *Let $Y \subset \mathbb{R}^N$ be a manifold and assume that $X \subset Y$ is a submanifold. Denote the inclusion map from X to Y by $i\colon X \to Y; x \mapsto x$. Then for any $x \in X$, the derivative di_x is the inclusion map from $T_x(X)$ to $T_x(Y)$. In particular, for any $x \in X$, the derivative of the identity map $id\colon X \to X$ from a manifold to itself is the identity map $d(id)_x\colon T_x(X) \to T_x(X)$ from the corresponding tangent space to itself.*

Proof. Choose an arbitrary $x \in X$ and let $\varphi\colon V \to Y$ and W be as in the proof above such that $U := \varphi^{-1}(W \cap X)$ can be viewed as a subset of $\mathbb{R}^k$. The map $\psi := \varphi|_U\colon U \to X$ parametrizes X around x. Then

$$h := \varphi^{-1} \circ i \circ \varphi|_U$$

is the inclusion map from U to V. Using the fact that the statement is true for maps between Euclidean spaces, that is, dh_0 is the inclusion from $\mathbb{R}^k$ to $\mathbb{R}^l$, we get that the derivative

$$di_x := d\varphi_0 \circ dh_0 \circ d(\varphi|_U)_0^{-1}$$

is the inclusion map from $T_x(X)$ to $T_y(Y)$. □

A consequence of Lemma 2.20 is that the derivative of the restriction of a smooth map to a submanifold is the restriction of the derivative of this map to the tangent space of the submanifold.

Lemma 2.21. *Let $f\colon X \to Y$ be a smooth map and suppose that Z is a submanifold of X. Then for any $x \in Z$ we have $d(f|_Z)_x = df_x|_{T_x(Z)}$.*

Proof. We may write the map $f|_Z$ as $f|_Z = f \circ i$, where $i\colon Z \to X$ is the inclusion map. By Lemma 2.20, its derivative $di_x\colon T_x(Z) \to T_x(X)$ at any $x \in Z$ is the inclusion map. Thus, we compute for any $x \in Z$

$$d(f|_Z)_x = d(f \circ i)_x = df_{i(x)} \circ di_x = df_x|_{T_x(Z)},$$

which is the desired identity. □

The next statement shows that the tangent space of an arbitrary open subset of a manifold at any point is the same as the tangent space of the manifold.

Lemma 2.22. *If O is an open subset of the manifold X, then $T_x(O) = T_x(X)$ for any $x \in O$.*

Proof. We claim that if X is a k-dimensional manifold in $\mathbb{R}^N$ and $O \quad X$ is an open subset, then O is a k-dimensional submanifold of X.
Since any open subset O of a manifold X is a submanifold of X, Lemma 2.20 implies that $T_x(O) \quad T_x(X)$, so proving that the dimensions of O and X coincide will conclude the proof.
Let $x \in O$. Since $O \quad X$, x possesses a neighborhood V in X that is diffeomorphic to an open set $U \quad \mathbb{R}^k$, where we denote the diffeomorphism by $\;: U \to V$. As O and V are neighborhoods of x, their intersection $O \cap V$ is a neighborhood of x in O. Thus $\;j_P : P \to O \cap V$ parametrizes $O \cap V$ around x, where $P := \;^{-1}(O)$ is open as it is the preimage of an open set under the continuous function .
Thus for $x \in O$ we found a local parametrization $\;: P \to O$ with $P \quad \mathbb{R}^k$. □

The next result will be of importance in the proof of the Borsuk–Ulam theorem.

Proposition 2.23. *Let $f : X \quad Y \to X$ be the projection map $(x, y) \mapsto x$. Then its derivative $df_{(x,y)} : T_x(X) \quad T_y(Y) \to T_x(X)$ is the analogous projection $(u, v) \to u$.*

Proof. Let $\;: U \to X$ and $\;: V \to Y$ with $U \quad \mathbb{R}^k$ and $V \quad \mathbb{R}^l$ be local parametrizations of X around $x \in X$ and Y around $y \in Y$ respectively. Define $\;:= \quad$ as in (2.1), then

$$\begin{array}{ccc} X \quad Y & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ U \quad V & \xrightarrow[h]{} & U \end{array}$$

commutes with $h := \;^{-1} \; f \;$.
If we adapt the notation of Definition 2.16 to this setting, then we define

$$df_{(x,y)} := d_{\;0} \; dh_0 \; d_{\;0}^{\;-1} : T_{(x,y)}(X \quad Y) \to T_x(X)$$

such that the diagram

$$\begin{array}{ccc} T_{(x,y)}(X \quad Y) & \xrightarrow{df_{(x,y)}} & T_x(X) \\ {\scriptstyle d_{\;0}}\uparrow & & \uparrow{\scriptstyle d_{\;0}} \\ \mathbb{R}^k \quad \mathbb{R}^l & \xrightarrow[dh_0]{} & \mathbb{R}^k \end{array}$$

commutes.
Let $(u; v)$ 2 U V be arbitrary. Then the de nitions of and f lead to

$$\begin{aligned} h(u; v) &:= \ {}^{1}(f(\ (u; v))) \\ &= \ {}^{1}(f(\ (u);\ (v))) \\ &= \ {}^{1}(\ (u)) \\ &= u. \end{aligned}$$

Hence $h\colon U \quad V$! U is the projection map in Euclidean space. Moreover, since h is linear we have $dh_0 \quad h$ by Lemma 2.11.
If we use Lemma 2.14, then the commutative diagram on the level of derivatives becomes:

$$\begin{array}{ccc} T_x(X) \quad T_y(Y) & \xrightarrow{df_{(x;y)}} & T_x(X) \\ {\scriptstyle d\ \ 0}\uparrow & & \uparrow{\scriptstyle d\ \ 0} \\ \mathbb{R}^k \quad \mathbb{R}^l & \xrightarrow[h]{} & \mathbb{R}^k \end{array}$$

Let $(x_0; y_0)$ 2 $T_x(X) \quad T_y(Y)$ be such that for some u_0 2 $\mathbb{R}^k$ and v_0 2 $\mathbb{R}^l$ we have

$$d\ \ _0(u_0; v_0) = (x_0; y_0).$$

Then

$$\begin{aligned} df_{(x;y)}(x_0; y_0) &= (d\ \ _0 \quad h \quad d\ \ _0{}^{1})(x_0; y_0) \\ &= (d\ \ _0(h(u_0; v_0))) \\ &= d\ \ _0(u_0) \\ &= x_0 \end{aligned}$$

which concludes the proof. □

The following proposition will lead us down the path to the inverse function theorem, which is applied in the proofs of several results in this thesis.

Proposition 2.24. *If $f\colon X$! Y is a di eomorphism between manifolds, then df_x is an isomorphism of tangent spaces at each x 2 X.*

Proof. If f is a di eomorphism between manifolds X and Y, then the dimensions of X and Y coincide, that is, $k = l$. Let x 2 X and de ne $y := f(x)$ 2 Y. Using De nition 2.16, we get the commutative diagram:

$$\begin{array}{ccc} T_x(X) & \xrightarrow{df_x} & T_y(Y) \\ {\scriptstyle d\ \ 0}\uparrow & & \uparrow{\scriptstyle d\ \ 0} \\ \mathbb{R}^k & \xrightarrow[dh_0]{} & \mathbb{R}^k \end{array}$$

Since linearity of $df_x := d\phi_0 \circ dh_0 \circ d\psi_0^{-1}$ follows from Definition 2.16, it is left to show that df_x is bijective.
We proceed by constructing the inverse df_x: Since f is a diffeomorphism, its inverse map $f^{-1}\colon Y \to X$ exists and is smooth. When we denote the identity map by id and take an arbitrary $x \in X$, then we get for $y = f(x)$

$$\begin{aligned} df_x \circ d(f^{-1})_y &= df_{f^{-1}(y)} \circ d(f^{-1})_y \\ &= d(f \circ f^{-1})_y && \text{by Theorem 2.17} \\ &= d(id)_y \\ &= id && \text{by Lemma 2.20} \end{aligned}$$

and

$$\begin{aligned} d(f^{-1})_y \circ df_x &= d(f^{-1})_{f(x)} \circ df_x \\ &= d(f^{-1} \circ f)_x && \text{by Theorem 2.17} \\ &= d(id)_x \\ &= id && \text{by Lemma 2.20} \end{aligned}$$

showing that $d(f^{-1})_y$ is the right and left inverse of df_x. □

The implication in the proposition above works only in a local sense in the opposite direction. But this already suffices to obtain a valuable result. We state it after introducing the local setting.

Definition 2.25. Assume that X and Y are manifolds of the same dimension and let $x \in X$. A smooth map $f\colon X \to Y$ is called a local diffeomorphism at x if there exists a neighborhood of x in X that is mapped diffeomorphically by f onto a neighborhood of $y := f(x)$ in Y.

The following version of the theorem is taken from [GP10, Chapter 1, Section 3, p. 13]

Theorem 2.26 (The Inverse Function Theorem). *Let $f\colon X \to Y$ be a smooth map between manifolds and let $x \in X$. If the derivative df_x at x is an isomorphism, then f is a local diffeomorphism around x.*

A proof of the theorem for Euclidean space can be found in [Spi65, Chapter 2, p. 35]. Using local parametrizations, one can translate the proof to smooth maps between manifolds.

2.3 Homotopy

In this tiny section, we recall a fundamental concept from topology needed in the next chapter.
In the following, we write I to denote the unit interval $[0,1] \subset \mathbb{R}$.

Definition 2.27. Two smooth maps $f_0 \colon X \to Y$ and $f_1 \colon X \to Y$ are called homotopic, abbreviated by $f_0 \sim f_1$, if there exists a smooth map $F \colon X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. The map F is called a homotopy between f_0 and f_1.

One can verify that homotopy is an equivalence relation. The equivalence class of a mapping is called its homotopy class.

2.4 The Preimage Theorem

As we will see in the statement of the namesake theorem of this section, we would like to consider values y in the target manifold Y of a smooth map $f \colon X \to Y$ for which the derivative df_x at every x in the preimage of y under f is surjective. In this section, we give these values a special name. But first we give a name to points in the domain manifold for which the derivative is surjective. The following definitions are taken from [GP10, Chapter 1, Section 4 and 7].

Definition 2.28. For a smooth map $f \colon X \to Y$, a point $x \in X$ is called regular point of f if $df_x \colon T_x(X) \to T_y(Y)$ is surjective.

Instead of saying that x is a regular point of f one can say that f is regular at x.

Definition 2.29. If $x \in X$ is not a regular point of $f \colon X \to Y$, then x is a critical point of f.

Sometimes, we also want to put the function in the foreground:

Definition 2.30. A smooth map $f \colon X \to Y$ is a submersion at $x \in X$ if $df_x \colon T_x(X) \to T_{f(x)}(Y)$ is surjective. If f is a submersion at every $x \in X$, then f is called submersion.

Note that f is a submersion at x if and only if x is a regular point of f.

The next definitions transfer this notion of "regularity" to the target manifold.

Definition 2.31. Let $f \colon X \to Y$ be a smooth map. We call $y \in Y$ a regular value for f if $df_x \colon T_x(X) \to T_y(Y)$ is surjective for every $x \in f^{-1}(y)$.

An arbitrary value $y \in Y$ is a regular value of f if every $x \in f^{-1}(y)$ is a regular point of f. We point out that if $y \in Y$ is not contained in the image of f, that is, $f^{-1}(y) = \emptyset$, then y belongs to the set of regular values of f.

Definition 2.32. Any $y \in Y$ that is not a regular value of f is called a critical value.

If at least one $x \in f^{-1}(y)$ is a critical point of f, then y is a critical value of f.

While the second statement of the next proposition is a consequence of the inverse function theorem (Theorem 2.26), the first and third implication follow directly from the definitions above. But since the third statement will play a role in the proof of the Borsuk–Ulam theorem, we point it out.

Proposition 2.33. *Assume that $f: X \to Y$ is a smooth map between manifolds and let $y \in Y$ be a regular value.*

1. *If $\dim(X) > \dim(Y)$, then f is a submersion at each $x \in f^{-1}(y)$.*
2. *If $\dim(X) = \dim(Y)$, then f is a local diffeomorphism at each $x \in f^{-1}(y)$.*
3. *If $\dim(X) < \dim(Y)$, then every point in $f(X)$ is a critical value and the regular values are those contained in $Y \setminus f(X)$.*

We will elaborate the second statement in the last section of this chapter, where we state and prove the stack of records theorem.

The inverse function theorem is also needed to prove the next theorem from [GP10, Chapter 1, Section 4, p. 20].

Theorem 2.34 (Local Submersion Theorem). *Suppose that $f: X \to Y$ is a submersion at x and define $y := f(x)$. Then there exist local coordinates around x and y such that*

$$f(x_1, \ldots, x_k) = (x_1, \ldots, x_l).$$

Finally, we state the theorem which designates this section. This fundamental prerequisite for the next chapter and already for the final section of this chapter is taken from [GP10, Chapter 1, Section 4, p. 21]. On this page also a proof using the local submersion theorem can be found.

Theorem 2.35 (The Preimage Theorem). *Suppose that y is a regular value of $f: X \to Y$. Then its preimage $f^{-1}(y)$ is a submanifold of X of dimension*

$$\dim(f^{-1}(y)) = \dim(X) - \dim(Y).$$

We conclude this section by stating the following proposition from [GP10, Chapter 1, Section 4, p. 24], which will be important for us in the beginning of the next chapter.

Proposition 2.36. *Let y be a regular value of the smooth map $f: X \to Y$ and define $Z := f^{-1}(y)$, which is a submanifold by the preimage theorem. Then the kernel of the derivative $df_x: T_x(X) \to T_y(Y)$ at any $x \in Z$ is precisely the tangent space to Z, namely*

$$\ker(df_x) = T_x(Z).$$

2.5 Sard's Theorem

In the previous section, we simply stated the preimage theorem, but we did not discuss how useful it is when we consider an arbitrary element y in the codomain of a smooth function $f: X \to Y$. If there are only a few regular values, then

we could not gain much from the theorem. However, we will see in this section that this is not the case.
We first need some definitions, which are taken from [GP10, Chapter 1 section 7, p. 39].

Definition 2.37. A rectangular solid in $\mathbb{R}^l$ is a cartesian product of l intervals in $\mathbb{R}^l$ and its volume is the product of the lengths of the intervals.

Definition 2.38. A set $A \subset \mathbb{R}^l$ has measure zero if it can be covered by a countable number of rectangular solids with arbitrary small total volume, that is, if for every $\varepsilon > 0$ there exists a countable collection $\{S_1, S_2, \ldots\}$ of rectangular solids in $\mathbb{R}^l$ such that

$$A \subset \bigcup_{i \in \mathbb{N}} S_i \text{ and } \sum_{i \in \mathbb{N}} \operatorname{vol}(S_i) < \varepsilon.$$

Using local parametrizations, we are able to extend the definitions above to manifolds.

Definition 2.39. Let Y be a manifold. A subset $C \subset Y$ is said to have measure zero if for every local parametrization ϕ of Y, the preimage $\phi^{-1}(C)$ in Euclidean space has measure zero.

It is important to notice that we only need to check that $\phi^{-1}(C)$ has measure zero for some parametrizations and not all of them: If there exists a countable collection of local parametrizations ϕ_α such that $\bigcup_\alpha \phi_\alpha(\phi_\alpha^{-1}(C))$ covers $C \subset Y$ and $\phi_\alpha^{-1}(C)$ has measure zero for all α, then C has measure zero. Indeed, since $\bigcup_\alpha \phi_\alpha(\phi_\alpha^{-1}(C))$ is a cover of C, we may write $C = \bigcup_\alpha \phi_\alpha(\phi_\alpha^{-1}(C))$. Let ψ be an arbitrary parametrization (not necessarily one of the ϕ_α). Then

$$\psi^{-1}(C) = \psi^{-1}\left(\bigcup_\alpha \phi_\alpha(\phi_\alpha^{-1}(C))\right) = \bigcup_\alpha (\psi^{-1} \circ \phi_\alpha)(\phi_\alpha^{-1}(C)).$$

Since $\phi_\alpha^{-1}(C)$ has mesure zero for each α, and $\psi^{-1} \circ \phi_\alpha$ is a smooth map from $\mathbb{R}^k$ to itself for some $k \in \mathbb{N}$, C has measure zero. This follows from the fact that if $A \subset \mathbb{R}^l$ has measure zero and $g \colon \mathbb{R}^l \to \mathbb{R}^l$ is smooth, then $g(A)$ has measure zero. A proof of this statement can be found in [GP10, Appendix 1, p. 204–205].

The next theorem, whose proof can be found in [GP10, Appendix 1, p. 205–207], tells us that the Preimage Theorem (Theorem 2.35) cannot only be applied in some edge cases but "nearly all the time".

Theorem 2.40 (Sard's Theorem for manifolds). *The set of critical values of a smooth map of manifolds has measure zero.*

2.6 The Stack of Records Theorem

We conclude this chapter by proving the stack of records theorem from [GP10, Chapter 1, Section 4, p. 26].

Theorem 2.41 (Stack of Records Theorem). *Suppose that y is a regular value of $f : X \to Y$, where X is compact and has the same dimension as Y. Then $f^{-1}(y)$ is a finite set $\{x_1, \dots, x_L\}$ for some $L \in \mathbb{N}$ and there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \dots \cup V_L$, where V_i is a neighborhood of x_i in X and f maps each V_i diffeomorphically onto U for any $i \in \{1, \dots, L\}$.*

Proof. First we show that $f^{-1}(y)$ is a finite set. Using results from topology, we infer this from the following two assertions.

Claim 1. The set $f^{-1}(y)$ is compact.

Claim 2. The set $f^{-1}(y)$ is a 0-dimensional manifold.

Proof of Claim 1. We view $\mathbb{R}^M$ as a metric space with the standard topology, that is, a set is open in $\mathbb{R}^M$ if it can be written as a union of open balls. Since the manifold Y is a subspace of $\mathbb{R}^M$, it is metrizable, that is, there exists a metric on Y which induces the standard topology on it. Thus the point set $\{y\}$ is a closed subset of Y. Since f is continuous, the preimage $f^{-1}(\{y\})$ is closed in X. Now we use the fact that a closed subset of a compact space is compact to conclude that $f^{-1}(\{y\})$ is compact. □

Proof of Claim 2. Since y is a regular value of f, the preimage theorem (Theorem 2.35) implies that $f^{-1}(y)$ is a submanifold of X. Moreover, $f^{-1}(y)$ is 0-dimensional as $\dim(X) = \dim(Y)$. □

From topology we know that the single possible connected 0-dimensional manifold is the point and disconnected 0-dimensional manifolds are discrete sets. Hence, $f^{-1}(y)$ is discrete and compact. Therefore, it is a finite set, say

$$f^{-1}(y) = \{x_1, \dots, x_L\}.$$

Now we prove the second part of the theorem. Since y is a regular value of f, the derivative df_{x_i} is surjective for each $i \in \{1, \dots, L\}$. In particular, as the dimensions of X and Y coincide, df_{x_i} is an isomorphism.
By the inverse function theorem (Theorem 2.26), f is a local diffeomorphism at x_i for each $i \in \{1, \dots, L\}$, that is, for each $i \in \{1, \dots, L\}$ there exists a neighborhood S_i of x_i in X and a neighborhood T_i of $f(x_i) = y$ such that $f|_{S_i} : S_i \to T_i$ is a diffeomorphism. Since $f^{-1}(y)$ is finite, we may shrink the S_i

if needed such that we have $S_i \cap S_j = \emptyset$ for $i \neq j \in \{1, \dots, L\}$.
The set

$$\tilde{U} := \bigcap_{i=1}^{L} T_i$$

is an open set containing y since it is a finite intersection of neighborhoods of y. Now we restrict the S_i such that their image under f is $\tilde{U}$. For each $i \in \{1, \dots, L\}$ we define

$$\tilde{V}_i := S_i \cap f^{-1}(\tilde{U}).$$

Since the sets S_i and U are open and f is continuous, these sets $\tilde{V}_i$ are open. Then each $\tilde{V}_i$ is diffeomorphic to $\tilde{U}$, where the diffeomorphism is given by

$$f|_{\tilde{V}_i} : \tilde{V}_i \to \tilde{U}.$$

The union of the sets $\tilde{V}_i$ is an open set and the set

$$X \setminus \bigcup_{i=1}^{L} \tilde{V}_i$$

is a closed subset of the compact manifold X, hence it is compact itself. Using the fact that the image of a compact set under a continuous map is compact, the set

$$f(X \setminus \bigcup_{i=1}^{L} \tilde{V}_i)$$

is compact, in particular closed. Moreover, it does not contain y. Indeed, if y was contained in $f(X \setminus \bigcup_{i=1}^{L} \tilde{V}_i)$, then there must be a preimage point of y that lies in $X \setminus \bigcup_{i=1}^{L} \tilde{V}_i$. That is, there exists a preimage point of y for which we did not remove an ambient neighborhood, a contradiction.
One may think that the $\tilde{V}_i$ and U are the desired sets. But this is not the case due to the following issue: We know that the preimage points of y are contained in U, but if we take some $\tilde{y} \in Y \setminus \{y\}$, then it might occur that some of its preimages are contained in $X \setminus \bigcup_{i=1}^{L} \tilde{V}_i$.
We solve this problem as follows: Since $f(X \setminus \bigcup_{i=1}^{L} \tilde{V}_i)$ is closed and does not contain y, its complement $Y \setminus f(X \setminus \bigcup_{i=1}^{L} \tilde{V}_i)$ is open and contains y. Now we take an arbitrary neighborhood $R \subset Y \setminus f(X \setminus \bigcup_{i=1}^{L} \tilde{V}_i)$ of y. Thus

$$R \cap f(X \setminus \bigcup_{i=1}^{L} \tilde{V}_i) = \emptyset, \tag{2.2}$$

which guarantees that no point in $X \setminus \bigcup_{i=1}^{L} \tilde{V}_i$ gets mapped to R. We define

$$U := \tilde{U} \cap R \text{ and } V_i := \tilde{V}_i \cap f^{-1}(R),$$

which are open sets since R is open and f is continuous. Then the map $f|_{V_i} : V_i \to U$ is a diffeomorphism for each $i \in \{1, \dots, L\}$. Moreover, we have

$$f^{-1}(U) = \bigcup_{i=1}^{L} (V_i \cap f^{-1}(R)).$$

On the one hand, we have $f^{-1}(U) = f^{-1}(\overline{U}) \cap f^{-1}(R)$ and each V_i is a subset of $f^{-1}(\overline{U})$.
On the other hand, if $x \in f^{-1}(U)$, then (2.2) implies that

$$f^{-1}(R) \cap (X \setminus \bigcup_{i=1}^{L} \overline{V}_i) = \emptyset.$$

Hence, $x \in \bigcup_{i=1}^{L} V_i$. This concludes the proof. □

Chapter 3

Transversality

In this chapter, we introduce the notion of transversality and state some of its properties. Transversality is the key to extend the results about manifolds in the first two sections to manifolds with boundary in the third section of this chapter. Moreover, it will be a valuable tool in the proof of the Borsuk–Ulam theorem. This chapter is based on [GP10, Chapter 1, Sections 5–7 and Chapter 2, Sections 1–5].

3.1 Definition and Properties

Definition 3.1. Let $f: X \to Y$ be a smooth map and let Z be a submanifold of Y. The map f is transversal to Z, written $f \pitchfork Z$, if and only if for every $x \in f^{-1}(Z)$

$$\operatorname{im}(df_x) + T_y(Z) = T_y(Y). \tag{3.1}$$

If f is transversal to $Z = \{0\}$, then (3.1) becomes $\operatorname{im}(df_x) = T_y(Y)$, which means that f is a submersion at every $x \in f^{-1}(Z)$.
If f is a submersion, then $df_x: T_x(X) \to T_y(Y)$ is surjective for every $x \in X$. In this case, (3.1) is satisfied for any submanifold Z of Y. This shows that for an arbitrary function, being a submersion is more powerful than being transversal to some submanifolds of the codomain of the function.

Lemma 3.2. *If $f: X \to Y$ is a submersion, then f is transversal to any submanifold Z of Y.*

We now define what it means for two submanifolds to be transversal.

Definition 3.3. Let X and Z be submanifolds of Y. Then X and Z are said to be transversal, written $X \pitchfork Z$, if for every $x \in X \cap Z$ we have

$$T_x(X) + T_x(Z) = T_x(Y). \tag{3.2}$$

Note that if X and Z are disjoint submanifolds of Y, then (3.2) is automatically satisfied and $X \pitchfork Z$. On the other hand, if X and Z are transversal and not disjoint, then $\dim(X) + \dim(Z) \geq \dim(Y)$, as otherwise (3.2) would not hold. This condition on the dimensions is the reason why two curves in $\mathbb{R}^3$ can never intersect transversally, except when they do not intersect at all.

Before we formulate the theorem extending the preimage theorem (Theorem 2.35), we introduce the term "codimension" to write it in a more compact way.

Definition 3.4. The codimension of an arbitrary submanifold $Z \subset X$ is defined by

$$\operatorname{codim}_X(Z) := \dim(X) - \dim(Z).$$

Theorem 3.5. *If the smooth map $f : X \to Y$ is transversal to a submanifold $Z \subset Y$, then the preimage $f^{-1}(Z)$ is a submanifold of X. Moreover, the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y, that is,*

$$\operatorname{codim}_X(f^{-1}(Z)) = \operatorname{codim}_Y(Z).$$

If Z is just a single point, then the theorem becomes the preimage theorem (Theorem 2.35). The proof we give below follows the one in [GP10, Chapter 1, Section 5, p. 27–28]. It uses independent functions, which are defined as follows.

Definition 3.6. Let X be a k-dimensional manifold and let $l \leq k$. The smooth functions $g_1, \dots, g_l : X \to \mathbb{R}$ are said to be independent at some $x \in X$ if their derivatives $d(g_1)_x, \dots, d(g_l)_x : T_x(X) \to \mathbb{R}$ are linearly independent on $T_x(X)$.

If we define $g := (g_1, \dots, g_l) : X \to \mathbb{R}^l$, then $dg_x : \mathbb{R}^k \to \mathbb{R}^l$ is an $l \times k$ matrix at each $x \in X$. Since $l \leq k$, dg_x is surjective if and only if $\operatorname{rank}(dg_x) = l$, that is, if $g_1, \dots, g_l$ are independent at x.
Before we prove Theorem 3.5, we state the next lemma, which is a consequence of Lemma 2.18.

Lemma 3.7. *Let Z be a submanifold of X with codimension m and let $z \in Z$ be an arbitrary point. Then there exist m independent functions $g_1, \dots, g_m : U \to \mathbb{R}$, where U is neighborhood of z in X, such that $g := (g_1, \dots, g_m)$ vanishes exactly on $Z \cap U$.*

For further properties of independent functions we refer the reader to [GP10, Chapter 1, Section 4, p. 23–24].

Proof of Theorem 3.5. We use the observation that $f^{-1}(Z)$ is an r-dimensional manifold if and only if each point $x \in f^{-1}(Z)$ has a neighborhood U in X such that $f^{-1}(Z) \cap U$ is an r-dimensional manifold.

Lemma 3.7 implies that we may write the submanifold Z of Y with codimension m in a neighborhood U of each $y \in Z$ in Y as the zero set of m independent functions

$$g_1, \dots, g_m : U \to \mathbb{R},$$

where $m = \operatorname{codim}_Y(Z)$.
Define $g := (g_1, \dots, g_m) : U \to \mathbb{R}^m$. Then $Z \cap U = g^{-1}(0)$. Take $x \in X$ such that $f(x) = y$. Then $x \in f^{-1}(Z)$. Moreover, we have $f^{-1}(Z \cap U) = f^{-1}(Z) \cap f^{-1}(U)$. Since $f^{-1}(U)$ is a neighborhood of x in X, our goal is to show that $f^{-1}(Z) \cap f^{-1}(U)$ is a manifold. Then we can apply the observation from above to conclude the proof.
Notice that

$$f^{-1}(Z) \cap f^{-1}(U) = f^{-1}(Z \cap U) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0).$$

Thus the preimage theorem (Theorem 2.35) implies that $f^{-1}(Z) \cap f^{-1}(U)$ is a manifold if 0 is a regular value of $g \circ f : X \to \mathbb{R}^m$, that is, if

$$d(g \circ f)_x = dg_y \circ df_x : T_x(X) \to T_{g(y)}(\mathbb{R}^m) = \mathbb{R}^m$$

is surjective for every $x \in (g \circ f)^{-1}(0)$.
Since $g_1, \dots, g_m$ are independent at y, $dg_y : T_y(Y) \to \mathbb{R}^m$ is surjective. Moreover, since $g|_{Z \cap U} \equiv 0$ by assumption, $dg_y : T_y(U) \to \mathbb{R}^m$ satisfies

$$\begin{aligned} dg_y|_{T_y(Z)} &= dg_y|_{T_y(Z \cap U)} && \text{by Lemma 2.22} \\ &= d(g|_{Z \cap U})_y && \text{by Lemma 2.21} \\ &= 0. \end{aligned}$$

Hence, $T_y(Z) \subseteq \ker(dg_y)$. We even have

$$T_y(Z) = \ker(dg_y) \tag{3.3}$$

since the dimensions coincide:

$$\begin{aligned} \dim(\ker(dg_y)) &= \dim(Y) - \dim(\operatorname{im}(dg_y)) \\ &= \dim(Y) - \dim(\mathbb{R}^m) \\ &= \dim(Y) - m \\ &= \dim(Z) \\ &= \dim(T_y(Z)). \end{aligned}$$

Thus the composition $dg_y \circ df_x$ is surjective at x precisely if the image of df_x and $T_y(Z)$ span $T_y(Y)$, namely

$$\operatorname{im}(df_x) + T_y(Z) = T_y(Y).$$

But this equation holds at each $x \in f^{-1}(Z)$ by our assumption that f is transversal to Z.

To complete the proof we need to show the identity for the codimensions.
The preimage theorem (Theorem 2.35) for the regular value 0 of $g \circ f$ implies that

$$\dim((g \circ f)^{-1}(0)) = \dim(X) - \dim(\mathbb{R}^m).$$

Using the fact that the dimensions of $f^{-1}(Z)$ and $f^{-1}(Z \cap U)$ coincide, we get

$$\begin{aligned}
\operatorname{codim}_X(f^{-1}(Z)) &= \dim(X) - \dim(f^{-1}(Z)) \\
&= \dim(X) - \dim(f^{-1}(Z) \cap f^{-1}(U)) \\
&= \dim(X) - \dim((g \circ f)^{-1}(0)) \\
&= \dim(X) - (\dim(X) - \dim(\mathbb{R}^m)) \\
&= \operatorname{codim}_Y(Z),
\end{aligned}$$

as $m = \operatorname{codim}_Y(Z)$. This concludes the proof. □

From Theorem 3.5, one can derive the following theorem (taken from [GP10, Chapter 1, Section 5, p. 30]).

Theorem 3.8. *The intersection of two transversal submanifolds X and Z of Y is a submanifold. Moreover,*

$$\operatorname{codim}_Y(X \cap Z) = \operatorname{codim}_Y(X) + \operatorname{codim}_Y(Z). \tag{3.4}$$

If we write (3.4) in terms of the dimension, then it still depends on $\dim(Y)$. Therefore, whether X and Z are transversal depends on the manifold Y they are contained in. In particular, if X and Z are transversal for some ambient manifold Y, then we can find a manifold W of larger dimension containing both X and Z such that X and Z are not transversal as submanifolds of W.

The next propositions are exercises in [GP10, Chapter 1, Section 5, p. 32–33]. The first one is needed to prove the latter one, which will be applied twice in the proof of the Borsuk–Ulam theorem.

Proposition 3.9. *Let $f: X \to Y$ be a map transversal to a submanifold Z in Y. Then for any $x \in X$, $T_x(f^{-1}(Z))$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x: T_x(X) \to T_{f(x)}(Y)$, that is,*

$$T_x(f^{-1}(Z)) = (df_x)^{-1}(T_{f(x)}(Z)).$$

Proof. For any point $y \in Z$, where Z is a submanifold of Y with codimension m, we can find U as well as $g_1, \ldots, g_m: U \to \mathbb{R}$ and $g: U \to \mathbb{R}^m$ as in the proof of the extension of the preimage theorem (Theorem 3.5). As we have seen in the proof, the transversality assumption $f \pitchfork Z$ is equivalent to 0 being a regular value of $F := g \circ f: X \to \mathbb{R}^m$.
If we apply Proposition 2.36 to the regular value 0 and the smooth map F, then for any $x \in F^{-1}(0)$ we get

$$\ker(dF_x) = T_x(F^{-1}(0)).$$

Notice that $F^{-1}(0) = f^{-1}(g^{-1}(0)) = f^{-1}(Z \cap U)$, where $Z \cap U$ is a neighborhood of y. For $x \in F^{-1}(0)$ with $f(x) = y$ we have $dF_x = dg_y \circ df_x$, which leads us to the expression

$$\begin{aligned}\ker(dF_x) &= \{v \in T_x(X) : df_x(v) \in \ker(dg_y)\} \\ &= \{v \in T_x(X) : df_x(v) \in T_y(Z)\} && \text{by (3.3)} \\ &= (df_x)^{-1}(T_y(Z))\end{aligned}$$

Thus we get

$$\begin{aligned}(df_x)^{-1}(T_y(Z)) &= \ker(dF_x) \\ &= T_x(F^{-1}(0)) \\ &= T_x(f^{-1}(Z \cap U)) \\ &= T_x(f^{-1}(Z) \cap f^{-1}(U)) \\ &= T_x(f^{-1}(Z)),\end{aligned}$$

where the last step is Lemma 2.22 applied to the open subset $f^{-1}(Z) \cap f^{-1}(U)$ of $f^{-1}(U)$. This uses that f is continuous and U is open in Y. □

Proposition 3.10. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps of manifolds. Suppose that g is transversal to a submanifold W of Z. Then $f \pitchfork g^{-1}(W)$ if and only if $g \circ f \pitchfork W$.*

Proof. Since g is transversal to W, every $y \in g^{-1}(W)$ satisfies

$$\operatorname{im}(dg_y) + T_{g(y)}(W) = T_{g(y)}(Z). \tag{3.5}$$

Suppose that f is transversal to $g^{-1}(W)$. Then every $x \in f^{-1}(g^{-1}(W))$ satisfies

$$\operatorname{im}(df_x) + T_{f(x)}(g^{-1}(W)) = T_{f(x)}(Y). \tag{3.6}$$

We want to show that for any such x we have

$$\operatorname{im}(d(g \circ f)_x) + T_{g(f(x))}(W) = T_{g(f(x))}(Z). \tag{3.7}$$

Let $x \in f^{-1}(g^{-1}(W))$ be arbitrary. If we define $y := f(x) \in g^{-1}(W)$ and $z := g(f(x)) \in W$, then our goal (3.7) becomes

$$\operatorname{im}(dg_y \circ df_x) + T_z(W) = T_z(Z). \tag{3.8}$$

By (3.5) we know that for every $v_1 \in T_z(Z)$ we can find some $v_2 \in T_y(Y)$ and $v_3 \in T_z(W)$ such that

$$v_1 = dg_y(v_2) + v_3.$$

Now (3.6) implies that we may write v_2 as $v_2 = df_x(v_4) + v_5$ for $v_4 \in T_x(X)$ and $v_5 \in T_y(g^{-1}(W))$. Using linearity of the derivative dg_y, we get

$$v_1 = dg_y(df_x(v_4) + v_5) + v_3 = dg_y(df_x(v_4)) + dg_y(v_5) + v_3. \tag{3.9}$$

By Theorem 3.5, $g^{-1}(W)$ is a submanifold of Y. Since we may view the derivative of the restriction of g to $g^{-1}(W)$ at y as a map $dg_y\colon T_y(g^{-1}(W)) \to T_z(W)$, we have that $dg_y(v_5)$ is an element of $T_z(W)$.
If we define

$$v_6 := dg_y(v_5) + v_3 \in T_z(W)$$

and insert it into (3.9), then v_1 can be written as

$$v_1 = dg_y(df_x(v_4)) + v_6,$$

that is, v_1 can be decomposed into a sum of an element in $T_z(W)$ and an element in the image of $T_x(X)$ under $d(g \circ f)_x$. Since $v_1 \in T_z(Z)$ was arbitrary, this proves (3.8) and finishes the first direction of the proof.

Now assume that $g \circ f \pitchfork W$, that is, (3.8) holds for any $x \in f^{-1}(g^{-1}(W))$. Let $x \in f^{-1}(g^{-1}(W))$ be such an element. Our goal is to prove (3.6), where we define $y := f(x) \in g^{-1}(W)$ and $z := g(f(x)) \in W$ as before. Take an arbitrary element $w_1 \in T_y(Y)$. Then $dg_y(w_1) \in T_z(Z)$ and by (3.8), this can be written as

$$dg_y(w_1) = dg_y(df_x(w_2)) + w_3$$

for some $w_2 \in T_x(X)$ and $w_3 \in T_z(W)$. Linearity of dg_y leads us to

$$dg_y(w_1 - df_x(w_2)) = dg_y(w_1) - dg_y(df_x(w_2)) = w_3 \in T_z(W),$$

showing that $w_1 - df_x(w_2)$ is contained in $(dg_y)^{-1}(T_z(W))$, which is equal to $T_y(g^{-1}(W))$ by Proposition 3.9.
If we define $w_4 := w_1 - df_x(w_2)$, then we may write w_1 as

$$w_1 = df_x(w_2) + w_4,$$

which is (3.6). This concludes the proof. □

3.2 Extension to Manifolds with Boundary

The following section is based on [GP10, Chapter 2 Section 1]. Our goal is to make our results applicable to more general spaces by allowing manifolds to have a boundary. To do this we introduce the notion of a manifold with boundary and show that our previous results for manifolds can be extended to manifolds with boundary. For this extension, we use the notion of transversality.

Definition 3.11. We define the upper half-space in $\mathbb{R}^k$ by

$$H^k := \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_k \geq 0\}.$$

One can show that H^k is among other spaces like the closed unit ball not a manifold. In order to treat these spaces, we need the following definition.

Definition 3.12. A set $X \subset \mathbb{R}^N$ is a k-dimensional manifold with boundary if every point of X possesses a neighborhood that is diffeomorphic to an open set in the space H^k. As in the definition of manifolds, such a diffeomorphism is called a local parametrization of X.
The boundary of X is the set of points that are contained in the image of the boundary of H^k under some local parametrization. The boundary is denoted by ∂X.
The interior of X is its complement, namely

$$\text{int}(X) := X \setminus \partial X.$$

We want to point out that the boundary and the interior of a manifold X are not the same as the topological boundary and interior of X viewed as a subset of $\mathbb{R}^N$.
Note that the manifolds from Definition 2.5 are manifolds with boundary, where the boundary is the empty set. Manifolds without boundary are called boundaryless.

As before, we want to obtain new manifolds with boundary by putting two of them together via the Cartesian product. However, there are cases in which the Cartesian product of two manifolds with boundary is not a manifold with boundary. For example, the unit square $[0,1] \times [0,1]$ fails to be a manifold with boundary because none of its corner points has a neighborhood that is diffeomorphic to an open set in H^k. But the next proposition guarantees us a possibility to generate new manifolds with boundary.

Proposition 3.13. *The product of a boundaryless manifold X and a manifold with boundary Y is a manifold with boundary. Moreover,*

$$\partial(X \times Y) = X \times \partial Y$$

and

$$\dim(X \times Y) = \dim(X) + \dim(Y).$$

A proof of this statement can be found in [GP10, Chapter 2, Section 1, p. 58]. For a verification that we may extend the notions of tangent spaces and derivatives to manifolds with boundary we refer the reader to [GP10, Chapter 2, Section 1, p. 59].

Another application of the inverse function theorem (Theorem 2.26) is the proof of the following result:

Proposition 3.14. *Assume that X is a k-dimensional manifold with boundary. Then* $\text{int}(X)$ *is a k-dimensional boundaryless manifold and ∂X is a* $(k-1)$*-dimensional boundaryless manifold.*

In particular, the proposition tells us that if $x \in \partial X$, then $T_x(\partial X)$ is a linear subspace of $T_x(X)$ of codimension 1. We introduce the following function and highlight a result concerning its derivative, which follows immediately from Lemma 2.21.

Definition 3.15. Let $f: X \to Y$ be a smooth map of manifolds. Then we define $\partial f := f|_{\partial X} : \partial X \to Y$.

Lemma 3.16. *Let $f: X \to Y$ be a smooth map of manifolds. Then at any point $x \in \partial X$, the derivative of ∂f is the restriction of df_x to the subspace $T_x(\partial X) \subset T_x(X)$, namely*

$$d(\partial f)_x = df_x|_{T_x(\partial X)}.$$

As argued in [GP10, Chapter 2, Section 1, p. 60], a transversality assumption along the boundary is required to extend Theorem 3.5 to manifolds with boundary. This is realized by demanding the transversality condition on the function f as well as additionally on the function ∂f.

Theorem 3.17. *Suppose that f is a smooth map from a manifold with boundary X to a boundaryless manifold Y. If both $f: X \to Y$ and $\partial f: \partial X \to Y$ are transversal to a boundaryless submanifold Z in Y, then $f^{-1}(Z)$ is a manifold with boundary, where the boundary is given by*

$$\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X.$$

Moreover, as in Theorem 3.5, we have

$$\operatorname{codim}_X(f^{-1}(Z)) = \operatorname{codim}_Y(Z).$$

A proof of this theorem can be found in [GP10, Chapter 2, Section 1, p. 61–62].

We need the following generalized version of Sard's Theorem, which is taken from [GP10, Chapter 2, Section 1, p. 62]. Since it will play a crucial role in the proof of the Borsuk–Ulam theorem, we prove it afterwards.

Theorem 3.18 (Sard's Theorem). *Let $f: X \to Y$, where X is a manifold with boundary and Y is a boundaryless manifold. Then almost every point of Y is a regular value of both $f: X \to Y$ and $\partial f: \partial X \to Y$.*

Proof. We show that the set $\{\text{crit. values of } f \text{ or } \partial f\}$, where we abbreviate "critical" by "crit.", has measure zero.
Lemma 3.16 implies that if $d(\partial f)_x$ is surjective at some $x \in \partial X$ (that is, $d(\partial f)_x$ is regular at x), then

$$df_x|_{T_x(\partial X)} : T_x(\partial X) \to T_y(Y)$$

is surjective. Then also df_x is regular at x. Hence, if $y \in Y$ is a critical value of f, then there exists some $x \in f^{-1}(y)$ such that df_x is not surjective. Either $x \in \partial X$, then y is a critical value of ∂f, or $x \in \operatorname{int}(X)$, in which case y is a critical value of $f_{\operatorname{int}(X)}$. Thus

$$\{\text{crit. values of } f\} \subset \{\text{crit. values of } \partial f\} \cup \{\text{crit. values of } f_{\operatorname{int}(X)}\}. \tag{3.10}$$

Notice that we can replace $\{$crit. values of $f\}$ by $\{$crit. values of f or $\partial f\}$ and the inclusion still holds.
Sard's theorem for manifolds (Theorem 2.40) applied to both of the boundaryless manifolds $\text{int}(X)$ and $\partial(X)$ tells us that both sets on the right hand side of (3.10) have measure zero. Since the union of two sets of measure zero has measure zero, $\{$crit. values of f or $\partial f\}$ is contained in a set of measure zero. Therefore, it has itself measure zero, which finishes the proof. □

We conclude this section by stating the classification theorem of manifolds from [GP10, Chapter 2, Section 2, p. 65] and a valuable corollary on the page after.

Theorem 3.19 (The Classification of one-dimensional Manifolds). *Every compact, connected, 1-dimensional manifold with boundary is diffeomorphic to* $[0, 1]$ *or* S^1. *In particular, compact, connected, 1-dimensional manifolds without boundary are diffeomorphic to* S^1.

For a proof of the theorem we refer the reader to [GP10, Appendix 2, p. 208–211].

Corollary 3.20. *The boundary of any compact 1-dimensional manifold with boundary consists of an even number of points.*

3.3 The Transversality Homotopy Theorem

This section contains a series of theorems that are prerequisites to prove the transversality homotopy theorem. In the long run, this theorem leads us to the notion of the mod 2 winding number, which is needed to even formulate our version of Borsuk–Ulam theorem.

Let X, Y and S be manifolds, where Y and S are boundaryless and X is supposed to have boundary. By Proposition 3.13, $X \times S$ is a manifold with boundary, where the boundary is given by $\partial(X \times S) = \partial X \times S$. Suppose that $f_s \colon X \to Y$ with $s \in S$ is a family of smooth maps. We consider the map

$$F \colon X \times S \to Y, F(x, s) := f_s(x)$$

from the manifold with boundary $X \times S$ to the boundaryless manifold Y.

With this setting, we are able to state the transversality theorem. The version below is taken from [GP10, Chapter 2, Section 3, p. 68].

Theorem 3.21 (The Transversality Theorem). *In the setting above, assume* $F \colon X \times S \to Y$ *to be smooth. Let* Z *be a boundaryless submanifold of* Y. *If both* F *and* ∂F *are transversal to* Z, *then for almost every* $s \in S$, *both* f_s *and* ∂f_s *are transversal to* Z.

The proof given in [GP10, Chapter 2 Section 3, p. 68–69] uses main results from the previous section, namely Theorem 3.17 and Sard's theorem (Theorem 3.18).

The next proposition is a corollary of the ε-neighborhood theorem in [GP10, Chapter 2 Section 3, p. 69–70].

Proposition 3.22. *Let $f\colon X \to Y$ be a smooth map, where Y is a boundaryless manifold. Then there exists an open ball S with $0 \in S$ in some Euclidean space and a smooth map $F\colon X \times S \to Y$ such that $F(x, 0) = f(x)$, and for any fixed $x \in X$ the map $S \to Y, s \mapsto F(x, s)$ is a submersion. In particular, both F and ∂F are submersions.*

Consider an arbitrary smooth map $f\colon X \to Y$, where Y is a boundaryless manifold, and let Z be a boundaryless submanifold of Y. Proposition 3.22 tells us that there exists a smooth map $F\colon X \times S \to Y$ such that $F(x, 0) = f(x)$ and both F and ∂F are submersions. By Lemma 3.2, F and ∂F are transversal to Z. Now we go the other way around and define the family of smooth maps

$$f_s\colon X \to Y, f_s(x) := F(x, s)$$

with $s \in S$. We point out that

$$f_0(x) := F(x, 0) = f(x).$$

Theorem 3.21 implies that for almost every $s \in S$, both f_s and ∂f_s are transversal to Z. Each f_s is homotopic to f, where the homotopy is given by

$$H\colon X \times I \to Y, (x, t) \mapsto F(x, ts).$$

Therefore, we obtain the transversality homotopy theorem, which allows us to introduce the mod 2 intersection number in the following section.

Theorem 3.23 (Transversality Homotopy Theorem). *For any smooth map $f\colon X \to Y$ and any boundaryless submanifold Z of the boundaryless manifold Y, there exists a smooth map $g\colon X \to Y$ homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$.*

3.4 The Mod 2 Intersection Number

In the whole section we consider the following setting: We denote by X and Y manifolds, where X is compact and not necessarily contained in Y. We assume Z to be a closed submanifold of Y such that $\dim(X) + \dim(Z) = \dim(Y)$. Suppose that $f\colon X \to Y$ is a smooth map. If f is transversal to Z, then $f^{-1}(Z)$ is a closed 0-dimensional submanifold of X by Theorem 3.5. Indeed,

$$\begin{aligned} \dim(X) - \dim(f^{-1}(Z)) &= \operatorname{codim}_X(f^{-1}(Z)) \\ &= \operatorname{codim}_Y(Z) && \text{by Theorem 3.5} \\ &= \dim(Y) - \dim(Z) \\ &= \dim(X) \end{aligned}$$

shows that $\dim(f^{-1}(Z)) = 0$. Moreover, since Z is closed and f is continuous, $f^{-1}(Z)$ is compact because it is a closed subset of a compact set. Since 0-dimensional manifolds are discrete sets, $f^{-1}(Z)$ is a finite set since it is compact.

Definition 3.24. Two submanifolds X and Z of Y have complementary dimension if $\dim(X) + \dim(Z) = \dim(Y)$.

This allows us to give the following defintion.

Definition 3.25. For X, Y, Z and f as in the setting above, we define the mod 2 intersection number of f with Z by

$$I_2(f, Z) := |f^{-1}(Z)| \mod 2.$$

Definition 3.26. We define the mod 2 intersection number of an arbitrary smooth map $g\colon X \to Y$, with Z as before, by

$$I_2(g, Z) := I_2(f, Z),$$

where f is homotopic to g and transversal to Z.

We are always able to find such a map f as above by the transversality homotopy theorem (Theorem 3.23).

The next theorem tells us that if we find two smooth functions $f_0, f_1\colon X \to Y$ that are transversal to Z and homotopic to a given smooth map g, then we do not get different values for $I_2(g, Z)$ depending on whether we choose f_0 or f_1.

Theorem 3.27. *If $f_0, f_1\colon X \to Y$ are homotopic smooth maps that are both transversal to Z, where X, Y and Z are manifolds with the properties specified in the beginning of this section, then $I_2(f_0, Z) = I_2(f_1, Z)$.*

We give an outline of the proof in [GP10, Chapter 2, Section 4, p. 78–79]:
We denote the homotopy of f_0 and f_1 by $F\colon X \times I \to Y$. By Proposition 3.13, $X \times I$ is a manifold with boundary and the boundary is given by

$$\partial(X \times I) = X \times \{0\} \cup X \times \{1\}. \qquad (3.11)$$

Then $\partial F \pitchfork Z$ follows from (3.11), as $\partial F \equiv f_0$ on $X \times \{0\}$ and $\partial F \equiv f_1$ on $X \times \{1\}$ and both are transversal to Z.
An extended version of the transversality homotopy theorem (Theorem 3.23) which is called "Extension Theorem" in [GP10, Chapter 2, Section 3, p. 72] lets us assume that $F \pitchfork Z$.
By Theorem 3.17, $F^{-1}(Z)$ is a submanifold of $X \times I$ with boundary

$$\partial(F^{-1}(Z)) = F^{-1}(Z) \cap \partial(X \times I) = f_0^{-1}(Z) \times \{0\} \cup f_1^{-1}(Z) \times \{1\} \qquad (3.12)$$

of dimension 1 because

$$\begin{aligned}
\dim(X) + 1 &= \dim(X) + \dim(I) \\
&= \dim(X \times I) && \text{by Proposition 3.13} \\
&= \operatorname{codim}_{X \times I}(F^{-1}(Z)) + \dim(F^{-1}(Z)) \\
&= \operatorname{codim}_Y(Z) + \dim(F^{-1}(Z)) && \text{by Theorem 3.17} \\
&= \dim(Y) - \dim(Z) + \dim(F^{-1}(Z)) \\
&= \dim(X) + \dim(F^{-1}(Z)). && (3.13)
\end{aligned}$$

Then the corollary to the classification of 1-manifolds (Corollary 3.20) implies that $\partial(F^{-1}(Z))$ consists of an even number of points. Then (3.12) gives the desired result.

Since homotopy is an equivalence relation, we get:

Corollary 3.28. *If $g_0, g_1 : X \to Y$ are arbitrary homotopic maps, then we have $I_2(g_0, Z) = I_2(g_1, Z)$.*

We need the next theorem from [GP10, Chapter 1, Section 4, p. 80] for the proof of final theorem of this chapter.

Theorem 3.29 (Boundary Theorem). *Suppose that X is the boundary of some compact submanifold W of Y and $g : X \to Y$ is a smooth map. If g may be extended to all of W, then $I_2(g, Z) = 0$ for any closed submanifold Z of Y of complementary dimension with respect to X, that is, Z satiefies the identity $\dim(X) + \dim(Z) = \dim(Y)$.*

The proof that can be found on the same page as the theorem proceeds by applying the transversality homotopy theorem (Theorem 3.23) to the extension $G : W \to Y$. For an arbitrary closed submanifold Z of Y this theorem provides a smooth map $F : W \to Y$ homotopic to G such that both F and $f := \partial F$ are transversal to Z. Since $g = \partial G$, f is homotopic to g. By Definition 3.26 of the mod 2 intersetion number for an arbitrary smooth map, we have

$$I_2(g, Z) = I_2(f, Z) = |f^{-1}(Z)| \mod 2.$$

By Proposition 3.14, $\dim(W) = \dim(X) + 1$ and Theorem 3.17 implies that $F^{-1}(Z)$ is a manifold with boundary of dimension 1 since

$$\begin{aligned}
\dim(X) &= \dim(Y) - \dim(Z) \\
&= \operatorname{codim}_Y(Z) \\
&= \operatorname{codim}_W(F^{-1}(Z)) \\
&= \dim(W) - \dim(F^{-1}(Z)) \\
&= \dim(X) + 1 - \dim(F^{-1}(Z)).
\end{aligned}$$

Moreover, $F^{-1}(Z)$ is compact because Z is closed. But then $|f^{-1}(Z)| = |(\partial F)^{-1}(Z)|$ is an even number by Corollary 3.20. Thus the boundary theorem follows.

The following theorem shows that the next object we will introduce is well-defined. Hence, we include a proof of it, where both the theorem and its proof are taken from [GP10, Chapter 2, Section 4, p. 80–81].

Theorem 3.30. *If $f : X \to Y$ is a smooth map of a compact manifold X into a connected manifold Y and $\dim(X) = \dim(Y)$, then $I_2(f, \{y\})$ is the same for all $y \in Y$.*

Proof. Let $f : X \to Y$ be a smooth map between manifolds with the properties from the statement. Take an arbitrary $y \in Y$ and consider it as a submanifold $\{y\}$ of Y. If f is not transversal to $\{y\}$, then, by the transversality homotopy theorem (Theorem 3.23), we can replace it by a map that is homotopic to f and transversal to $\{y\}$.
The stack of records theorem (Theorem 2.41) tells us that there exists a neighborhood U of y such that its preimage $f^{-1}(U)$ can be decomposed into a disjoint union

$$f^{-1}(U) = V_1 \cup \cdots \cup V_n,$$

where V_i is an open subset of X and $f|_{V_i} : V_i \to U$ is a diffeomorphism for each $i \in \{1, \ldots, n\}$. Thus each $z \in U$ has n preimages and satisfies

$$I_2(f, \{z\}) = n \mod 2.$$

Therefore, the function

$$h : Y \to \mathbb{N}, y \mapsto I_2(f, \{y\})$$

is locally constant. But since Y is connected, h must be globally constant. □

The next tool is needed to define the mod 2 winding number in the following section.

Definition 3.31. Let X and Y be manifolds such that X is compact, Y is connected and their dimensions coincide, that is, $\dim(X) = \dim(Y)$. We define the mod 2 degree of a smooth map $f : X \to Y$ by

$$\deg_2(f) := I_2(f, \{y\}),$$

where it does not matter which $y \in Y$ we take due to the theorem above.

We determine the mod 2 degree of a smooth map f as above by taking an arbitrary regular value y of f and counting its preimage points, namely,

$$\deg_2(f) = |f^{-1}(y)| \mod 2. \tag{3.14}$$

Since the mod 2 degree is a mod 2 intersection number, we may apply Corollary 3.28 to get the following result:

Theorem 3.32. *Homotopic maps have the same mod 2 degree.*

3.5 The Mod 2 Winding Number

Assume that X is a compact, connected $(n-1)$-dimensional manifold and let $f: X \to \mathbb{R}^n$ be a smooth map. Let $z \in \mathbb{R}^n$ be not contained in the image of f. To study how f wraps X around in $\mathbb{R}^n$ we count how often the unit vector that points from z in the direction of $f(x)$, namely the vector

$$u(x) := \frac{f(x) - z}{\|f(x) - z\|},$$

points in a given direction. This leads us to the final definition we need for the Borsuk–Ulam theorem.

Definition 3.33. Let the map f be as above. The mod 2 winding number of f around z is

$$W_2(f; z) := \deg_2(u).$$

The next theorem is an exercise from [GP10, Chapter 2, Section 5, p. 87]. We need it for the proof of the Borsuk–Ulam theorem.

Theorem 3.34. *Let X and f be as above. Suppose that there exists a compact manifold D with boundary X, and let $F: D \to \mathbb{R}^n$ be a smooth map extending f, that is, $\partial F = f$. Suppose that z is a regular value of F that does not belong to the image of f. Then $F^{-1}(z)$ is a finite set and $W_2(f; z) = |F^{-1}(z)| \mod 2$. That is, f winds X around z as often as F hits z, modulo 2.*

An instruction for the proof can be found in the book.

Chapter 4

The Borsuk–Ulam Theorem

In this chapter, we introduce and prove the following version of the theorem from [GP10, Chapter 2, Section 6, p. 91]. It will be a tool to deduce a more functional version at the end of this chapter.

Theorem 4.1. *Let $f : S^k \to \mathbb{R}^{k+1}$ be a smooth map whose image does not contain the origin. If f is an odd function in the sense that for all $x \in S^k$ we have*

$$f(-x) = -f(x); \tag{4.1}$$

then $W_2(f; 0) = 1$.

The proof given in this chapter follows the inductive proof in [GP10, Chapter 2, Section 6, p. 91–92]. The authors leave parts of the proof as an exercise for the reader. We fill in more details, for example as in the next section.

4.1 Prerequisites

The following lemmas are prerequisites for the base case and the inductive step respectively of the proof.

Lemma 4.2. *Let $f : S^1 \to S^1$ be a smooth map. Then there exists a smooth map $g : \mathbb{R} \to \mathbb{R}$ satisfying $f(\cos t; \sin t) = (\cos g(t); \sin g(t))$ and $g(2\pi) = g(0) + 2\pi q$ for some integer q. Moreover, $\deg_2(f) = q \mod 2$.*

Proof. Let $f : S^1 \to S^1$ be a smooth map. The map $p : \mathbb{R} \to S^1$ given by $p(t) := (\cos t; \sin t)$ is a local diffeomorphism winding the real line around the circle. In particular, p is surjective.
In the first step, we find a smooth map $g : [0; 2\pi] \to \mathbb{R}$ such that $p \circ g = f \circ p$ and $g(2\pi) = g(0) + 2\pi q$. Then, in the second step, we will show that g can be extended to the entire real line by requiring $g(t + 2\pi) = g(t) + 2\pi q$ for all $t \in \mathbb{R}$. In the last step, we will compute the mod 2 degree of f.

Step 1: As p is locally bijective, we would like to set

$$g := p^{-1} \circ f \circ p. \tag{4.2}$$

We now show the following two claims.

Claim 1. For every $t \in [0, 2\pi]$ there exists a connected neighborhood U_t of t such that on U_t we can define g by (4.2).

Claim 2. The map g can be defined by (4.2) on the interval $[0, 2\pi]$.

Proof of Claim 1. Let $t \in [0, 2\pi]$ be arbitrary. The map $f \circ p \colon \mathbb{R} \to S^1$ is smooth since it is the composition of the smooth maps f and p. Let V_t be a neighborhood of $f(p(t))$ that is not the full circle. Then p is invertible on V_t and continuity of $f \circ p$ implies that we can choose $U_t \subset (f \circ p)^{-1}(V_t)$ as a connected neighborhood of t in $\mathbb{R}$. Then we can define $g|_{U_t} := p^{-1} \circ f \circ p|_{U_t}$. □

Proof of Claim 2. We define the set

$$X := \{t \in [0, 2\pi] : g \text{ can be defined by (4.2) on } [0, t]\}.$$

- Firstly, X is nonempty: Since p is surjective, we can find a preimage of the point $f(p(0))$ under p. Then we may define $g(0) := p^{-1}(f(p(0)))$ and thus $0 \in X$.
- Secondly, X is open: Assume that we can define g by (4.2) on $[0, t]$. Then we apply Claim 1 to define g on a neighborhood of t. One can verify that the two definitions of g, namely the one for $[0, t]$ and the one for U_t, coincide on $[0, t] \cap U_t$ after possibly shifting g by a multiple of 2π. Hence, g can be defined by (4.2) on $[0, t] \cup U_t$.
- Thirdly, X is closed because $[0, 2\pi] \setminus X$ is open: If $t \notin X$, then, again by Claim 1, no point of U_t can be in X because otherwise we could extend X similar as above such that it would contain t.

Since X is nonempty, open, closed and connected, X must be the whole interval $[0, 2\pi]$. □

The map $g \colon [0, 2\pi] \to \mathbb{R}$ defined by (4.2) is smooth since it is a composition of smooth maps.
To complete this step we apply the definitions of p and f as well as periodicity of sine and cosine to get

$$\begin{aligned} (\cos(g(0+2\pi)), \sin(g(0+2\pi))) &= p(g(0+2\pi)) \\ &= f(p(0+2\pi)) \\ &= f(\cos(0+2\pi), \sin(0+2\pi)) \\ &= f(\cos(0), \sin(0)) \\ &= (\cos(g(0)), \sin(g(0))) \end{aligned} \tag{4.3}$$

Using properties of sine and cosine, one can verify that (4.3) implies that

$$g(2\pi) = g(0) + 2\pi q \tag{4.4}$$

for some $q \in \mathbb{Z}$.

Step 2: We extend the smooth map $g\colon [0, 2\pi] \to \mathbb{R}$ from the first step to the interval $[0, 4\pi]$ by defining

$$\tilde{g}\colon [0, 4\pi] \to \mathbb{R}, \quad \tilde{g}(t) = \begin{cases} g(t) & \text{if } t \in [0, 2\pi] \\ g(t - 2\pi) + 2\pi q & \text{if } t \in [2\pi, 4\pi]. \end{cases}$$

Since $g(2\pi - 2\pi) + 2\pi q = g(0) + 2\pi q = g(2\pi)$ by (4.4), $\tilde{g}$ is well-defined and continuous. Moreover, for $t \in [0, 2\pi]$, $\tilde{g}$ satisfies

$$\tilde{g}(t + 2\pi) = g(t + 2\pi - 2\pi) + 2\pi q = g(t) + 2\pi q = \tilde{g}(t) + 2\pi q.$$

We need to show that $\tilde{g}$ is even smooth. If we replace the interval $[0, 2\pi]$ by $[\pi, 3\pi]$ in the first step, then we obtain a smooth function $h\colon [\pi, 3\pi] \to \mathbb{R}$ such that $p \circ h = f \circ p$ and $h(3\pi) = h(\pi) + 2\pi r$ for some $r \in \mathbb{Z}$. For each $t \in [\pi, 3\pi]$ we have

$$p(h(t)) = f(p(t)) = p(\tilde{g}(t)$$

or, equivalently,

$$(\cos(h(t)), \sin(h(t))) = (\cos(\tilde{g}(t)), \sin(\tilde{g}(t))).$$

This implies that $\tilde{g}(t) - h(t) = 2\pi k(t)$, where $k\colon [\pi, 3\pi] \to \mathbb{Z}$ is a continuous function since $\tilde{g}$ and h are continuous. Thus k is constant and we may write $\tilde{g}(t) = h(t) - 2\pi k$, which shows that $\tilde{g}$ is smooth in a neighborhood of 2π. Therefore, $\tilde{g}$ is smooth and an extension of g. This procedure allows us to extend g to the whole real line, as desired.

Step 3: Sard's theorem (Theorem 3.18) implies the existence of some $v \in S^1$ such that $f(v)$ is a regular value for f. By surjectivity of p, there exists some $t \in \mathbb{R}$ such that $p(t) = v$. Then $f(v) = f(p(t)) = p(g(t))$.
Since S^1 is connected, the mod 2 degree of f is well-defined. We apply (3.14) to the regular value $f(v)$ to compute it:

$$\begin{aligned} \deg_2(f) &= |f^{-1}(f(v))| \mod 2 \\ &= |f^{-1}(\{p(g(t))\})| \mod 2 \\ &= |\{x \in S^1 : f(x) = p(g(t))\}| \mod 2 \end{aligned} \tag{4.5}$$

Since $p\colon [t, 2\pi + t] \to S^1$ is bijective, we are able to further transform the above such that the following equation holds modulo 2:

$$\begin{aligned} |\{x \in S^1 | f(x) = p(g(t))\}| &\equiv |\{y \in [t, 2\pi + t) : f(p(y)) = p(g(t))\}| \\ &\equiv |\{y \in [t, 2\pi + t) : p(g(y)) = p(g(t))\}| \\ &\equiv |\{y \in [t, 2\pi + t) : \exists k \in \mathbb{Z}_{\geq 0} : g(y) = g(t) + 2\pi k\}|; \end{aligned}$$

where the last equality follows from the definition of p.

If we define

$$N_k := |\{y \in [t, 2\pi + t) : g(y) = g(t) + 2\pi k\}|,$$

then we get

$$\begin{aligned} \deg_2(f) &\equiv |\{y \in [t, 2\pi + t) : \exists k \in \mathbb{Z}_{\geq 0} : g(y) = g(t) + 2\pi k\}| \\ &= \sum_{k=0}^{\infty} |\{y \in [t, 2\pi + t) : g(y) = g(t) + 2\pi k\}| \\ &= \sum_{k=0}^{\infty} N_k. \end{aligned} \tag{4.6}$$

Since g is continuous and

$$\lim_{y \to t + 2\pi} g(y) = g(t) + 2\pi q$$

the intermediate value theorem implies that any value in $[g(t), g(t) + 2\pi q)$ must be achieved by g. Thus for every $k \in \{0, 1, \ldots, q-1\}$, we have $N_k \geq 1$. Moreover, the number N_k must be odd for every $k \in \{0, 1, \ldots, q-1\}$ because there cannot be any contact points between g and the horizontal lines through the points of

$$\{g(t) + 2\pi k : k \in \mathbb{N} \cup \{0\}\}. \tag{4.7}$$

Indeed, since $f(v) = p(g(t))$ is a regular value, for any $x \in f^{-1}(p(g(t)))$ the derivative

$$df_x : T_x(S^1) \to T_{p(g(t))}(S^1)$$

must be surjective. In particular, df_x is an isomorphism because

$$\dim(T_x(S^1)) = 1 = \dim(T_{f(x)}(S^1)).$$

Thus df_x never vanishes, but this is a necessary condition for obtaining a contact point.

For $k \geq q$ we have that N_k is an even number. Indeed, there are again no contact points with the horizontal lines through the points (4.7) and that intersection points of g and an arbitrary line with $k \geq q$ come in pairs follows from applying the intermediate value theorem twice: once when increasing from a value below such a line to a value above the chosen line, and again when decreasing from the above value to a value below the line. Hence, by splitting the infinite sum from (4.6) we obtain modulo 2

$$\deg_2(f) \equiv \sum_{k=0}^{q-1} N_k + \sum_{k=q}^{\infty} N_k \equiv \sum_{k=0}^{q-1} N_k \equiv \sum_{k=0}^{q-1} 1 = q$$

which is the desired result. □

Lemma 4.3. *If a is regular value for a smooth map h satisfying the symmetry condition* (4.1)*, then $-a$ is also a regular value for h.*

Proof. If a is a regular value for h, then for every $x \in h^{-1}(a)$, we have that dh_x is surjective. Let $x \in h^{-1}(-a)$. Then $h(x) = -a$ and (4.1) implies

$$h(-x) = -h(x) = a;$$

that is, $-x \in h^{-1}(a)$. Since

$$\begin{aligned} dh_x(v) &= \lim_{t \to 0} \frac{h(x + tv) - h(x)}{t} \\ &= \lim_{t \to 0} \frac{-h(-x - tv) + h(-x)}{t} \\ &= -\lim_{t \to 0} \frac{h(-x - tv) - h(-x)}{t} \\ &= dh_{-x}(-v) \end{aligned}$$

and $-\mathbb{R}^k := \{-x : x \in \mathbb{R}^k\} = \{x \in \mathbb{R}^k\} = \mathbb{R}^k$, we get that dh_{-x} is surjective. Since $x \in h^{-1}(-a)$ was arbitrary, the statement follows. □

Lemma 4.4. *The function*

$$h\colon \mathbb{R}^{k+1} \setminus \{0\} \to S^k; x \mapsto \frac{x}{\|x\|}$$

is a submersion.

Proof. Lemma 2.21 implies that the derivative of the restriction of h to any submanifold M of $\mathbb{R}^{k+1} \setminus \{0\}$ is

$$d(h|_M)_x = dh_x|_{T_x(M)} \colon T_x(M) \to T_{h(x)}(S^k).$$

This implies that if $d(h|_M)_x$ is surjective for some $x \in M$, then dh_x is surjective as well. In other words, if $h|_M$ is a submersion at x, then h is a submersion at x.

If we consider $M = S^k$, then $h|_M$ is the identity map. This is a submersion at every $x \in M$ since its derivative is the identity map by Lemma 2.20, which is surjective.

Now we take $M_\varepsilon := \{x \in \mathbb{R}^{k+1} : \|x\| = \varepsilon\} \setminus \{0\}$ to be the sphere of radius $\varepsilon > 0$ around the origin. Then $h|_{M_\varepsilon}$ is equal to $\frac{id}{\varepsilon}$, which is again a submersion at every $x \in M_\varepsilon$.

Since $\varepsilon > 0$ was arbitrary and $\bigcup_\varepsilon M_\varepsilon = \mathbb{R}^{k+1} \setminus \{0\}$, we get that h is a submersion at every $x \in \mathbb{R}^{k+1} \setminus \{0\}$. Therefore, h is a submersion. □

4.2 Proof of the Theorem

Now we have all the tools we need to prove the Borsuk–Ulam theorem.

Theorem 4.1. *Let $f : S^k \to \mathbb{R}^{k+1}$ be a smooth map whose image does not contain the origin. If f is an odd function in the sense that for all $x \in S^k$ we have*

$$f(-x) = -f(x), \tag{4.1}$$

then $W_2(f, 0) = 1$.

Proof of Theorem 4.1. We prove the theorem by induction on the dimension k starting with the base case $k = 1$. Take a smooth map $f : S^1 \to S^1$ satisfying (4.1). Since we may write $x \in S^1$ as $x = (\cos(t), \sin(t))$ for some $t \in \mathbb{R}$, (4.1) translates to

$$f(\cos(t), \sin(t)) = f(-\cos(t), -\sin(t)) \tag{4.8}$$

Applying Lemma 4.2 twice as well as the identities $-\cos(t) = \cos(t + \pi)$ and $-\sin(t) = \sin(t + \pi)$, we get

$$f(\cos(t), \sin(t)) = (-\cos(g(t)), -\sin(g(t))) \tag{4.9}$$

and

$$\begin{aligned} f(-\cos(t), -\sin(t)) &= f(\cos(t + \pi), \sin(t + \pi)) \\ &= (\cos(g(t + \pi)), \sin(g(t + \pi))). \end{aligned} \tag{4.10}$$

Combining the equations above and using the identities again leads to

$$\begin{aligned} (\cos(g(t + \pi)), \sin(g(t + \pi))) &= (-\cos(g(t)), -\sin(g(t))) \\ &= (\cos(g(t) + \pi), \sin(g(t) + \pi)). \end{aligned} \tag{4.11}$$

Once again (4.11) implies that

$$g(t) = g(t + \pi) + 2\pi k - \pi = g(t + \pi) + \pi(2k - 1).$$

Define the odd number $n := -(2k - 1)$. Then $g(t) = g(t + \pi) - \pi n$ or

$$g(t + \pi) = g(t) + \pi n. \tag{4.12}$$

According to Lemma 4.2, g satisfies $g(2\pi) = g(0) + 2\pi q$ for some $q \in \mathbb{Z}$. If we apply (4.12) twice with $t = 0$ and $t = \pi$, then

$$\begin{aligned} g(0) + 2\pi q &= g(2\pi) \\ &= g(\pi + \pi) \\ &= g(\pi) + \pi n \\ &= g(0) + 2\pi n \end{aligned}$$

and hence $q = n$ is an odd number. Now Lemma 4.2 implies that $\deg_2(f) = 1$. This shows the base case.

For the inductive step, suppose that the theorem is true for $k-1$ and let $f\colon S^k \to \mathbb{R}^{k+1}\setminus\{0\}$ satisfy the properties from the statement. We prove the theorem holds true for the dimension k in four steps after outlining the idea of the proof.

The idea of the proof: We compute $W_2(f;0)$ by counting how often f intersects a line ℓ in $\mathbb{R}^{k+1}$. Thereby we choose a line ℓ that does not intersect the image of the equator, where we view S^{k-1} to be the equator of S^k under the embedding $(x_1,\dots,x_k)\mapsto(x_1,\dots,x_k,0)$, that is,

$$S^{k-1} := \{(x_1,x_2,\dots,x_{k+1}) \in S^k : x_{k+1} = 0\}.$$

Choosing ℓ disjoint from the image of S^{k-1} under f allows us to employ the inductive hypothesis to show that the equator winds around ℓ an odd number of times.

Define the smooth map

$$g := f|_{S^{k-1}}\colon S^{k-1} \to \mathbb{R}^{k+1}\setminus\{0\},\ x \mapsto f(x)$$

to be the restriction of f to the equator. Notice that g satisfies (4.1). In the third step, we modify g to apply the inductive hypothesis to it.

Step 1: First, we use f and g to define maps that are symmetric around the origin. With this maps and Sard's theorem we are able to find a line ℓ that is indeed never touched by g and that satisfies $f \pitchfork \ell$.

Define the maps

$$\frac{g}{\|g\|}\colon S^{k-1} \to S^k \text{ and } \frac{f}{\|f\|}\colon S^k \to S^k.$$

Both maps inherit the symmetry property (4.1) from f and g respectively. Indeed, if we have $\frac{f(x)}{\|f(x)\|} = a$ for some $x \in S^k$, then

$$-a = -\frac{f(x)}{\|f(x)\|} = \frac{f(-x)}{\|-f(-x)\|} = \frac{f(-x)}{\|f(-x)\|}$$

and analogously for $\frac{g}{\|g\|}$.

By Sard's theorem (Theorem 3.18), there exists a unit vector $a \in S^k$ that is a regular value for both maps $\frac{g}{\|g\|}$ and $\frac{f}{\|f\|}$. Lemma 4.3 implies that $-a$ is a regular value for $\frac{g}{\|g\|}$ and $\frac{f}{\|f\|}$ too.

Define the line

$$\ell := \mathbb{R}\cdot a.$$

The fact that a and $-a$ are regular values for $\frac{g}{\|g\|}$ implies that the line ℓ never hits the image of g. This follows from the third implication of Proposition 2.33. Indeed, since

$$\dim(S^{k-1}) = k-1 < k = \dim(S^k),$$

every point in $\frac{g}{\|g\|}(S^k)$ is a critical value and the regular values are the values not hit by $\frac{g}{\|g\|}$. Thus a and $-a$ are not in the image of $\frac{g}{\|g\|}$, which shows that g does not intersect the line ℓ. We need this fact in the next step.
The following claim concludes the first step of the proof.

Claim 1. The values a and $-a$ are regular values of the function $\frac{f}{\|f\|}$ if and only if $f \pitchfork \ell$.

Proof of Claim 1. Consider the sequence

$$S^k \xrightarrow{f} \mathbb{R}^{k+1} \setminus \{0\} \xrightarrow{h} S^k,$$

where

$$h : \mathbb{R}^{k+1} \setminus \{0\} \to S^k, x \mapsto \frac{x}{\|x\|}$$

denotes the division by the Euclidean norm. Let $W := \{-a, a\}$, then W is a submanifold of S^k and

$$\ell \setminus \{0\} = h^{-1}(\{-a, a\}) = h^{-1}(W).$$

Lemma 4.4 states that h is a submersion. Then Lemma 3.2 implies that h is transversal to any submanifold of S^k. In particular, h is transversal to W.
Assume now that a and $-a$ are regular values of $\frac{f}{\|f\|} = h \circ f$, that is, for any $x \in f^{-1}(h^{-1}(\{a, -a\}))$ we have

$$\operatorname{im}(d(h \circ f)_x) = T_{h(f(x))}(S^k). \tag{4.13}$$

This immediately implies $h \circ f \pitchfork W$. Since $h \pitchfork W$ and $h \circ f \pitchfork W$, Proposition 3.10 implies that $f \pitchfork h^{-1}(W)$, where $h^{-1}(W) = \ell \setminus \{0\}$. Since 0 is not in the image of f, we get $f \pitchfork \ell$.
For the other direction, suppose that f is transversal to ℓ. Since $\ell \setminus \{0\}$ is a submanifold of ℓ, we also have $f \pitchfork \ell \setminus \{0\}$. By Proposition 3.10 we have $h \circ f \pitchfork W$, that is, for any $x \in f^{-1}(h^{-1}(W))$

$$\operatorname{im}(d(h \circ f)_x) + T_{h(f(x))}(W) = T_{h(f(x))}(S^k).$$

But since W is only a finite set of points, it has dimension 0. As its tangent space $T_{h(f(x))}(W)$ is a vector space of the same dimension, it vanishes. Thus the equation above becomes (4.13), that is, a and $-a$ are regular values of $h \circ f$.
This concludes the proof of the claim. □

The fact that f is transversal to ℓ is an essential ingredient in the final step.

Step 2: We express the winding number in a different way by using the symmetry condition (4.1) for $\frac{f}{\|f\|}$ and restricting f to the upper hemisphere.

If we put $z = 0$ into Definition 3.33, we obtain a connection between the mod 2 winding number of f and the map $\frac{f}{\|f\|}$, namely

$$W_2(f;0) = \deg_2\left(\frac{f}{\|f\|}\right) = \#\left(\frac{f}{\|f\|}\right)^{-1}(a) \mod 2. \tag{4.14}$$

The symmetry condition (4.1) for $\frac{f}{\|f\|}$ implies that antipodal points have the same number of preimages under $\frac{f}{\|f\|}$, that is,

$$\#\left(\frac{f}{\|f\|}\right)^{-1}(a) = \#\left(\frac{f}{\|f\|}\right)^{-1}(-a) \tag{4.15}$$

The assumption that the origin is not contained in the image of f implies that

$$f^{-1}(\ell) = f^{-1}(\ell \setminus \{0\}). \tag{4.16}$$

The next claim allows us to further improve our expression for the winding number.

Claim 2. We have $f(x) \in \mathbb{R}_{<0}\, a$ if and only if $\frac{f}{\|f\|}(x) = -a$. Similarly, $f(x) \in \mathbb{R}_{>0}\, a$ if and only if $\frac{f}{\|f\|}(x) = a$.

Proof of Claim 2. Assume that $f(x) = -\lambda a$ for some $\lambda \in \mathbb{R}_{>0}$. Then

$$\|f(x)\| = \|-\lambda a\| = \lambda\|a\| = \lambda$$

and

$$-a = \frac{-\lambda a}{\lambda} = \frac{-\lambda a}{\|f(x)\|} = \frac{f(x)}{\|f(x)\|} = \frac{f}{\|f\|}(x).$$

For the other direction, suppose that $\frac{f}{\|f\|}(x) = -a$. Then $f(x) = -\|f(x)\|a$, thus $f(x) \in \mathbb{R}_{<0}\, a$.
The proof of the second statement can be obtained by deleting the minus signs above. □

In particular, Claim 2 implies that

$$\#\left(\frac{f}{\|f\|}\right)^{-1}(-a) = |f^{-1}(\mathbb{R}_{<0}\, a)| \text{ and } \#\left(\frac{f}{\|f\|}\right)^{-1}(a) = |f^{-1}(\mathbb{R}_{>0}\, a)|.$$

Applying this, (4.15) and (4.16), we get

$$\begin{aligned}
|f^{-1}(\ell)| &= |f^{-1}(\ell \setminus \{0\})| \\
&= |f^{-1}(\mathbb{R}_{<0} \cdot a) \cup f^{-1}(\mathbb{R}_{>0} \cdot a)| \\
&= |f^{-1}(\mathbb{R}_{<0} \cdot a)| + |f^{-1}(\mathbb{R}_{>0} \cdot a)| \\
&= \left(\frac{f}{\|f\|}\right)^{-1}(-a) + \left(\frac{f}{\|f\|}\right)^{-1}(a) \\
&= 2\left(\frac{f}{\|f\|}\right)^{-1}(a)\ ; \qquad (4.17)
\end{aligned}$$

where from the second to the third line we used the fact that preimages of disjoint sets, namely $\mathbb{R}_{<0} \cdot a$ and $\mathbb{R}_{<0} \cdot -a$, are disjoint.

The symmetry condition (4.1) tells us how f behaves on the lower hemisphere $S^k_- = \{x \in S^k : x_{k+1} \leq 0\}$ given that we know its behaviour on the upper hemisphere $S^k_+ = \{x \in S^k : x_{k+1} \geq 0\}$. Thus it suffices to investigate f on the upper hemisphere. Both hemispheres S^k_+ and S^k_- are manifolds with boundary and the boundary is in both cases given by the equator S^{k-1}. Notice that their intersection $S^k_+ \cap S^k_- = S^{k-1}$ is exactly the equator.

Let

$$f_+ := f|_{S^k_+} : S^k_+ \to \mathbb{R}^{k+1} \setminus \{0\} \text{ and } f_- := f|_{S^k_-} : S^k_- \to \mathbb{R}^{k+1} \setminus \{0\}$$

be the restrictions of f to the upper and to the lower hemisphere respectively. No point of the equator S^{k-1} gets mapped to ℓ by f since g does not intersect ℓ. Hence no point of $f^{-1}(\ell)$ can simultaneously be contained in S^k_+ and S^k_-, that is,

$$f_+^{-1}(\ell) \cap f_-^{-1}(\ell) = \emptyset. \qquad (4.18)$$

Using the symmetry condition (4.1) once again leads us to

$$|f_+^{-1}(\ell)| = |f_-^{-1}(\ell)|. \qquad (4.19)$$

Indeed, if there exists $x \in \ell$ such that $f_+(v) = x$ for some $v \in S^k_+$, then $-v \in S^k_-$ and (4.1) implies

$$-x = -f_+(v) := -f(v) = f(-v) =: f_-(-v).$$

Thus if for a point $x \in \ell$ we can find a preimage in the upper hemisphere, then its antipodal point $-x$, which is also in ℓ, is hit by the antipode of the preimage of a point in the lower hemisphere. As this argumentation works in the other direction too, the cardinality of preimage points under f_+ and f_- must be equal. Recall that $0 \in \ell$ has no preimages under f. We get

$$\begin{aligned}
|f^{-1}(\ell)| &= |f_-^{-1}(\ell) \cup f_+^{-1}(\ell)| & \\
&= |f_-^{-1}(\ell)| + |f_+^{-1}(\ell)| & \text{by (4.18)} \\
&= 2|f_+^{-1}(\ell)| & \text{by (4.19)} \qquad (4.20)
\end{aligned}$$

If we combine (4.17) and (4.20) with (4.14), then we get

$$W_2(f;0) = |f_+^{-1}(\ell)| \mod 2. \tag{4.21}$$

Step 3: We modify g by composing it with a projection map π such that we may apply the inductive hypothesis for the composition $\pi \circ g$.

We are not allowed to apply the induction hypothesis for $g\colon S^{k-1} \to \mathbb{R}^{k+1} \setminus \{0\}$ because the dimension of the image domain is $k+1$ instead of k. To reduce the dimension of the image domain by 1, we define V to be the orthogonal complement of the line ℓ. As ℓ has dimension 1, V has dimension k. Let $\pi\colon \mathbb{R}^{k+1} \to V$ be the orthogonal projection and identify V with the Euclidean space $\mathbb{R}^k$.

To apply the induction hypothesis to $\pi \circ g\colon S^{k-1} \to V$, we need to show that $\pi \circ g$ satisfies the symmetry condition (4.1) and that the origin is not in the image of $\pi \circ g$. Using linearity of π and the symmetry condition (4.1) for g, we get for any $x \in S^{k-1}$:

$$-(\pi \circ g)(x) = -\pi(g(x)) = \pi(-g(x)) = \pi(g(-x)) = (\pi \circ g)(-x).$$

Moreover, if there exists $x \in S^{k-1}$ such that $\pi(g(x)) = 0$, then $g(x)$ is contained in $\pi^{-1}(0) = \ell$. But this cannot happen because ℓ and the image of g are disjoint. Hence,

$$W_2(\pi \circ g; 0) = 1. \tag{4.22}$$

Step 4: We apply the steps above and use some previous results to conclude the proof.

In the first step we proved that $f \pitchfork \ell$. Then $f_+ \pitchfork \ell$ because transversality of a function is passed on if we restrict it to any subset of the domain. Now we want to apply Proposition 3.10 to the sequence

$$S^k_+ \xrightarrow{f_+} \mathbb{R}^{k+1} \xrightarrow{\pi} V$$

to show that $\pi \circ f_+ \pitchfork \{0\}$. To do this we need to show that π is transversal to $\{0\}$, that is, for any $x \in \pi^{-1}(\{0\}) = \ell$ we have

$$\operatorname{im}(d\pi_x) + T_{\pi(x)}(\{0\}) = T_{\pi(x)}(V).$$

Since $T_{\pi(x)}(\{0\}) = 0$, this is equivalent to showing that $d\pi_x$ is surjective for any $x \in \ell$, that is, π is a submersion at each $x \in \ell$.

If we think of the projection π as the map

$$\pi\colon V \times \ell \to V; (v; r) \mapsto v;$$

then, by Proposition 2.23, its derivative

$$d\pi_{(v;r)}\colon T_v(V) \times T_r(\ell) \to T_v(V)$$

is the analogous projection. Since V and ℓ are vector spaces, we have $T_v(V) = V$ and $T_r(\ell) = \ell$ by Lemma 2.15. Hence d_x , where we defined $x := (v; r)$. Thus d_x is surjective. Thus is a submersion at every

$$y \in T_v(V) \times T_r(\ell) = T_{(v;r)}(V \times \ell) = T_x(\mathbb{R}^{k+1}).$$

As discussed above, Proposition 3.10 now implies that $f_+ \pitchfork \{0\}$, that is, for any $x \in (f_+)^{-1}(0) = f_+^{-1}(^{-1}(0))$ we have

$$\mathrm{im}(d(f_+)_x) + T_{(f_+(x))}(\{0\}) = T_{(f_+(x))}(V).$$

Once again we have $T_{(f_+(x))}(\{0\}) = 0$. Hence $d(f_+)$ is surjective at every $x \in f_+^{-1}(^{-1}(0))$. Thus 0 is a regular value of f_+.

Recall that the upper hemisphere S^k_+ is a manifold with boundary and its boundary is S^{k-1}. The function f_+ extends g, that is, $\partial(f_+) = g$. If we apply Theorem 3.34 to the regular value 0 of f_+ that is not in the image of g, then

$$W_2(g; 0) = |(f_+)^{-1}(0)| \mod 2. \tag{4.23}$$

Using the results from previous steps, we finally get

$$\begin{aligned} W_2(f; 0) &= |f_+^{-1}(l)| && \text{by (4.21)} \\ &= |f_+^{-1}(^{-1}(0)| && \\ &= W_2(g; 0) && \text{by (4.23)} \\ &= 1 \mod 2. && \text{by (4.22)} \end{aligned}$$

This concludes the proof. □

4.3 Another Version of the Theorem

For the applications in the next chapter we need the following version of the theorem.

Theorem 4.5 (The Borsuk–Ulam Theorem). *For every continuous function $f : S^k \to \mathbb{R}^k$ there exists a point $x \in S^k$ such that $f(x) = f(-x)$.*

In the remainder of this chapter we show that this version follows from Theorem 4.1.

Corollary 4.6. *Let f be as in Theorem 4.1. Then f intersects every line through the origin at least once.*

Proof. Assume by contradiction that there exists some $a \in S^k$ such that the line $\ell = \mathbb{R} \cdot a$ through the origin is not hit by f. If we go back to step 2 in the proof of Theorem 4.1 and insert (4.17) into (4.14), we obtain

$$W_2(f; 0) = \frac{1}{2}|f^{-1}(\ell)| = 0.$$

But this contradicts Theorem 4.1. Note we do not need a to be a regular value of f for the formula to be valid. □

In particular, since we may write any line ℓ through the origin as $\ell = \mathbb{R} \cdot a$ for some unit vector $a \in S^k$, any function f as in Theorem 4.1 intersects ℓ at least twice. Indeed, if $f(x) = r \cdot a \in \ell$ is an intersection point of f and ℓ, then the opposite point $(-r) \cdot a = -f(x) = f(-x)$ is another intersection point of f and ℓ.

Theorem 4.7. *Every smooth function from S^k to $\mathbb{R}^k$ satisfying the symmetry condition* (4.1) *must have a zero.*

Proof. We prove the theorem by contradiction. Assume that the smooth map $f : S^k \to \mathbb{R}^k$ satisfies (4.1) and does not have a zero, that is, $f : S^k \to \mathbb{R}^k \setminus \{0\}$. We decompose f as

$$f(x) = (f_1(x), \dots, f_k(x))$$

and define the smooth map $g : S^k \to \mathbb{R}^{k+1}$ by

$$g(x) := (f_1(x), \dots, f_k(x), 0).$$

Since f has no zeroes, g has no zeroes. By Corollary 4.6, g intersects every line through 0 at least once. But g does not intersect the x_{k+1}-axis, which leads to the desired contradiction. □

Proof of Theorem 4.5. Define the smooth map $g : S^k \to \mathbb{R}^k$ by

$$g(x) := f(x) - f(-x).$$

Then g satisfies (4.1) since

$$g(-x) := f(-x) - f(x) = -(f(x) - f(-x)) =: -g(x).$$

Theorem 4.7 implies that g has a zero x_0 and we have $f(x_0) = f(-x_0)$ by definition of g. □

We conclude this chapter with an example for $k = 2$ to illustrate Theorem 4.5: Since the temperature and the air pressure on the earth are smooth functions, there always exist two places on opposite ends of the earth having exactly the same temperature and air pressure.

Chapter 5

Applications of the Borsuk–Ulam Theorem

In this chapter, which is based on [Mat03, Chapter 3, Section 1–2], we state and prove some applications of the Borsuk–Ulam theorem.

5.1 The Ham Sandwich Theorem

We need the following definition.

Definition 5.1. A hyperplane in $\mathbb{R}^d$ is a $(d-1)$-dimensional affine subspace, that is, a set of the form

$$h = \{x \in \mathbb{R}^d : a^T x = b\}$$

for some vector $a \in \mathbb{R}^d \setminus \{0\}$ and some scalar $b \in \mathbb{R}$. Each hyperplane defines two (closed) half-spaces, namely $H_- = \{x \in \mathbb{R}^d : a^T x \leq b\}$ and $H_+ = \{x \in \mathbb{R}^d : a^T x \geq b\}$.

Now, we recall a few definitions from measure theory to state the version of the namesake theorem of this section for measures. The definitions are taken from [Sch17, Chapter 1, Section 1–2].

Definition 5.2. Let Ω be a nonempty set. A set system $\mathcal{A} \subseteq 2^\Omega$ is called a σ-field on Ω if $\Omega \in \mathcal{A}$ and $\mathcal{A}$ is closed under complements and countable unions, that is,

- if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ and
- if $A_i \in \mathcal{A}$ for all $i \in \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$.

Definition 5.3. Let Ω be a nonempty set and assume that $\mathcal{A}$ is a σ-field on Ω. A map $\mu : \mathcal{A} \to [0, \infty]$ with $\mu(\emptyset) = 0$ is called a measure if μ is σ-additive on $\mathcal{A}$, that is, for all pairwise disjoint sets $A_i \in \mathcal{A}$ with $i \in \mathbb{N}$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Definition 5.4. A probability measure on a σ-field $\mathcal{A}$ on Ω is a measure μ such that $\mu(\Omega) = 1$.

Definition 5.5. A measure $\mu : \mathcal{A} \to [0, \infty]$ on a σ-field $\mathcal{A}$ on Ω is called finite (on $\mathcal{A}$) if $|\mu(A)| < \infty$ for all $A \in \mathcal{A}$.

Definition 5.6. If $\mathcal{A} \subseteq 2^{\Omega}$ is a σ-field on a nonempty set Ω, then we call $(\Omega, \mathcal{A})$ a measurable space and $A \in \mathcal{A}$ a measurable set.

The next definitions are taken from [Mat03, Chapter 3, Section 1, p. 47].

Definition 5.7. A measure μ on $\mathbb{R}^d$ is called a finite Borel measure if all open subsets of $\mathbb{R}^d$ are measurable and $0 < \mu(\mathbb{R}^d) < \infty$.

Definition 5.8. A finite Borel measure on $\mathbb{R}^d$ assigning to every hyperplane measure 0 is called a mass distribution in $\mathbb{R}^d$.

Now, we are able to formulate the ham sandwich theorem from [Mat03, Chapter 3, Section 1, p. 47].

Theorem 5.9 (Ham Sandwich Theorem for measures)**.** *If $\mu_1, \mu_2, \ldots, \mu_d$ are mass distributions in $\mathbb{R}^d$, then there exists a hyperplane h such that for all $i \in \{1, 2, \ldots, d\}$, we have*

$$\mu_i(h^+) = \frac{1}{2}\mu_i(\mathbb{R}^d), \tag{5.1}$$

where h^+ denotes one of the half-spaces defined by h.

The ham sandwich theorem tells us that any d mass distributions in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane.
For our daily life this means that for every sandwich made of ham, cheese and bread, we are able to apply a knife in such a way that all three layers get simultaneously cut into halves by one single planar cut.
Notice that it does not matter which half-space we take since we assign measure zero to hyperplanes.

Proof. Consider an arbitrary point $u = (u_0, u_1, \ldots, u_d)$ of the sphere $S^d \subseteq \mathbb{R}^{d+1}$. If at least one of $u_1, u_2, \ldots, u_d$ is nonzero, then we assign to u the half-space

$$h^-(u) := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : u_1x_1 + \cdots + u_dx_d \le u_0\}.$$

We denote by $h^+(u)$ the other half-space defined by the hyperplane

$$\{x \in \mathbb{R}^d : u_1x_1 + \cdots + u_dx_d = u_0\}.$$

Since

$$\begin{aligned} h^-(-u) &:= \{(x_1, \ldots, x_d) \in \mathbb{R}^d : -(u_1x_1 + \cdots + u_dx_d) \le -u_0\} \\ &= \{(x_1, \ldots, x_d) \in \mathbb{R}^d : u_1x_1 + \cdots + u_dx_d \ge u_0\} \\ &= h^+(u), \end{aligned} \tag{5.2}$$

antipodal points of S^d correspond to opposite half-spaces.
If we plug the two possibilities of u excluded above, namely $u = (1, 0, \ldots, 0)$ and $u = (-1, 0, \ldots, 0)$, into the definition of $h^-(u)$, then we get

$$h^-((1, 0, \ldots, 0)) = h^+((-1, 0, \ldots, 0)) = \mathbb{R}^d$$

and

$$h^-((-1, 0, \ldots, 0)) = h^+((1, 0, \ldots, 0)) = \emptyset,$$

which are not half-spaces.
The function $f = (f_1, \ldots, f_d) : S^d \to \mathbb{R}^d$ defined by

$$f_i(u) := \mu_i(h^-(u))$$

is continuous, that is, if $(u_n)_{n \in \mathbb{N}}$ is a sequence of points converging to u, then $\mu_i(h^-(u_n))$ converges to $\mu_i(h^-(u))$. This step uses the Dominated Convergence Theorem ([Sch17, Chapter 2, Section 3, p. 62]) and is shown in [Mat03, Chapter 3, Section 1, Proof of Theorem 3.1.1].
By the Borsuk–Ulam theorem (Theorem 4.5) there exists a point $v \in S^d$ such that $f(v) = f(-v)$, that is, for every $i \in \{1, \ldots, d\}$ we have

$$\mu_i(h^-(v)) = \mu_i(h^-(-v)).$$

We point out that v cannot be $(1, 0, \ldots, 0)$ or $(-1, 0, \ldots, 0)$, as for any $i \in \{1, \ldots, d\}$

$$f_i((1, 0, \ldots, 0)) = \mu_i(\mathbb{R}^d) \in (0, \infty)$$

but

$$f_i((-1, 0, \ldots, 0)) = \mu_i(\emptyset) = 0.$$

Thus the sets $h^-(v)$ and $h^-(-v)$ are half-spaces. Using (5.2), this leads us to

$$\mu_i(h^-(v)) = \mu_i(h^+(v)) \text{ for every } i \in \{1, \ldots, d\}.$$

In words, these equations tell us that there is a pair of opposite half-spaces such that for each $i \in \{1, \ldots, d\}$, the values measure μ_i assigns to them coincide. Therefore, the boundary of the half-space $h^-(v)$ satisfies the desired property (5.1). □

Now we focus on the case where the masses are concentrated at finitely many points. Our goal is to obtain a discrete version of Theorem 5.9.

Definition 5.10. Let $A \subseteq \mathbb{R}^d$ be a finite point set. A hyperplane h bisects A if both of the open half-spaces defined by h contain at most $\lfloor \frac{1}{2}|A| \rfloor$ points of A.

In particular, if a hyperplane h bisects a finite set A containing an odd number $2k+1$ of points, then each of the open half-spaces may contain at most k points. Thus at least one point must lie on h.

Definition 5.11. A finite point set $A \subseteq \mathbb{R}^d$ is said to be in general position if no $d+1$ points lie in a common hyperplane.

The next theorem and its proof come from [Mat03, Chapter 3, Section 2, p. 48–49].

Theorem 5.12 (Ham Sandwich Theorem for point sets)**.** *For any d finite point sets in $\mathbb{R}^d$, there exists a hyperplane h simultaneously bisecting all of them.*

Proof. Let $A_1, A_2, \ldots, A_d \subseteq \mathbb{R}^d$ be finite point sets. We prove the theorem in three cases, where we weaken the assumptions on the point sets in each case.

Case 1: Assume that

1. each A_i contains an odd number of points,
2. the point sets are pairwise disjoint, that is,

$$A_i \cap A_j = \emptyset \text{ for all } i \neq j \in \{1, \ldots, d\}, \text{ and}$$

3. the disjoint union $\bigcup_{i=1}^{d} A_i$ is in general position.

The idea is to replace the points of A_i by sufficiently small balls and apply Theorem 4.5 to them.

Let A_i^{ε} be the set obtained from A_i by replacing each point by a ball of radius ε, where we choose $\varepsilon > 0$ small enough such that no $d+1$ balls of $\bigcup_{i=1}^{d} A_i^{\varepsilon}$ can be intersected by a common hyperplane. By Theorem 5.9 applied to the mass distributions $\mu_1, \ldots, \mu_d$ that distribute the mass uniformly over the $A_1^{\varepsilon}, \ldots, A_d^{\varepsilon}$, there exists a hyperplane h simultaneously bisecting the sets A_i^{ε}.
Since each A_i contains an odd number of points, each A_i^{ε} is a union of an odd number of balls. As discussed after Definition 5.10, h must then intersect at least one of the balls of each A_i^{ε}. The third assumption implies that h intersects at most d balls at all. Hence, h intersects exactly one ball of each A_i^{ε}.
Consider a ball of some A_i which is intersected by h. Since the mass of the ball is distributed uniformly over the ball, h divides the ball exactly in half. Thus h passes through the center of the ball. Therefore, h bisects A_i. Since A_i was arbitrary, we get the result.

Case 2: Assume that each A_i has odd cardinality. We reduce this case to the first one by using a perturbation argument: We shift the points a bit such that they satisfy the second and third assumption from the first case.

For every $\delta > 0$, move each point of each A_i by at most δ in such a way that the obtained sets $A_{i,\delta}$ satisfy the second and third assumption. Since we are in the previous case, there exists a hyperplane h_ε simultaneously bisecting the $A_{i,\varepsilon}$. We write

$$h_\varepsilon = \{x \in \mathbb{R}^d : a_\varepsilon^T x = b_\varepsilon\}$$

and we may assume that a_ε is a unit vector. Indeed, if this is not the case, then we simply replace b_ε by $\frac{b_\varepsilon}{\|a_\varepsilon\|}$ and a_ε by $\frac{a_\varepsilon}{\|a_\varepsilon\|}$, that is, we divide the equation by $\|a_\varepsilon\|$.
Note that the b_ε lie in a bounded interval. Indeed, since $\bigcup_{i=1}^d A_i$ is finite and the points in $\bigcup_{i=1}^d A_{i,\varepsilon}$ have at most distance ε from $\bigcup_{i=1}^d A_i$, there exists $r > 0$ such that the ball $B_r(0)$ of radius r around 0 contains $\bigcup_{i=1}^d A_{i,\varepsilon}$ for all $\varepsilon < 1$. If $|b_\varepsilon| > |r|$, then $B_r(0)$ is contained in one of the two half-spaces defined by h_ε, which is a contradiction to the definition of h_ε. This uses the fact the h_ε is orthogonal to a_ε and the distance between h_ε and 0 is $|b_\varepsilon|$. Thus $|b_\varepsilon| \leq |r|$, so the b_ε lie in a bounded interval B. Hence, the set $\{(a_\varepsilon, b_\varepsilon) : a_\varepsilon \in S^{d-1}, b_\varepsilon \in B\}$ is contained in the compact set $S^{d-1} \times B$.
Compactness implies the existence of a cluster point (a, b) of the pairs $(a_\varepsilon, b_\varepsilon)$ as $\varepsilon \to 0$. Define the hyperplane

$$h := \{x \in \mathbb{R}^d : a^T x = b\}.$$

Now, we consider a strictly decreasing sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ converging to 0 such that $(a_{\varepsilon_j}, b_{\varepsilon_j}) \to (a, b)$. If a point x has distance $\delta > 0$ from h, then it has distance at least $\frac{1}{2}\delta$ from h_{ε_j} for all sufficiently large j. Hence, if k points of A_i for $i \in \{1, \dots, d\}$ are contained in one of the two open half-spaces defined by h, then for all j large enough, the corresponding open half-space determined by h_{ε_j} contains at least k points of A_{i,ε_j}. In particular, if one half-space defined by h contains $\geq \lfloor \frac{1}{2}|A_i| \rfloor + 1$ points of A_i, then the corresponding half-space defined by h_{ε_j} contains at least $\lfloor \frac{1}{2}|A_{i,\varepsilon_j}| \rfloor + 1$ points of A_{i,ε_j}, which means that h_{ε_j} does not bisect A_{i,ε_j}, a contradiction. Therefore, each half-space defined by h cannot contain more than $\lfloor \frac{1}{2}|A_i| \rfloor$ points of A_i, which shows that h bisects each each A_i.

Case 3: We conclude the proof by allowing the sets A_i to contain an even number of points.

Suppose that we are given finite point sets $A_1, \dots, A_d$, where some of them have even cardinality. If we delete an arbitrary point from each even-sized set, then the d odd-sized sets we obtain can be bisected by a hyperplane h according to the second case. We conclude the proof by showing that this h bisects the original sets $A_1, \dots, A_d$ too:
A set A_i containing $2k$ points is bisected if both of the open half-spaces defined by h contain at most k points. If we delete exactly one arbitrary point from A_i, then we obtain a set with $2k - 1$ points, which is bisected if both half-spaces contain at most $k - 1$ points. Adding the deleted point back to its corresponding half-space leads to at most k points in each half-space. Thus A_i gets bisected. Since A_i was an arbitrary set with even cardinality, h bisects $A_1, A_2, \dots, A_d$. □

For later applications we need the following corollary of Theorem 5.12, which is taken from [Mat03, Chapter 3, Section 1, p. 49]. Its proof comes from this book as well.

Corollary 5.13 (Ham Sandwich Theorem for point sets in general position). *Let $A_1, \dots, A_d \subset \mathbb{R}^d$ be disjoint finite point sets such that $\bigcup_{i=1}^d A_i$ is in general position. Then there exists a hyperplane h that bisects each A_i such that there are exactly $\lfloor \frac{1}{2}|A_i| \rfloor$ points from A_i in each of the open half-spaces defined by h, and at most one point of A_i lies on the hyperplane h.*

Proof. The ham sandwich theorem for point sets (Theorem 5.12) provides us with a hyperplane h simultaneously bisecting $A_1, \dots, A_d$. But h may contain more than one point of some A_i. Indeed, in the worst case h may contain d points of a single A_i, but h cannot contain more than d points since $\bigcup_{i=1}^d A_i$ is in general position.
Now, we choose the coordinate system in such a way that h is the horizontal hyperplane

$$h = \{x \in \mathbb{R}^d : x_d = 0\}.$$

The set

$$B := h \cap (\bigcup_{i=1}^{d} A_i)$$

consists of at most d points. One can show that $\bigcup_{i=1}^d A_i$ being in general position implies that the points in B are affinely independent. We claim that we are able to shift h in such a way that only one point of each odd-sized A_i and no point of each even-sized A_i remains on h.
To define a new hyperplane we need d affinely independent points. Hence, we add $d - |B|$ points of h to B such that the resulting set $C \subset h$ consists of d affinely independent points. The affinely independence allows us to decide whether we want a point of h to either stay on h or to move below or above h.
For every $a \in C$ we choose a point $\tilde{a}$ as follows: If $a \notin B$ or a is a point in B that should stay on h, then we take $\tilde{a} := a$. For the other points in B we define $\tilde{a} := a + \varepsilon e_d$ or $\tilde{a} := a - \varepsilon e_d$, depending on whether we want a to lie below or above h.
Denote by $\tilde{h} = \tilde{h}(\varepsilon)$ the hyperplane determined by the d affinely independent points $\tilde{a}$. The hyperplane $\tilde{h}(\varepsilon)$ is well-defined for all sufficiently small $\varepsilon > 0$ because for such ε the $\tilde{a}$ remain affinely independent. Moreover, the motion of $\tilde{h}(\varepsilon)$ is continuous in ε. For ε small enough, the relative position to the hyperplanes of all other points, that is, points contained in the open half-spaces defined by h, does not change. This proves the existence of a desired hyperplane. □

5.2 Multicolor Partitions

The first theorem in this section is a statement about multicolored partitions. To state it we need the next definitions.

Definition 5.14. A set $C \subseteq \mathbb{R}^d$ is called convex if for each $x, y \in C$ the line segment from x to y, namely the image of the function

$$f : [0,1] \to \mathbb{R}^d, t \mapsto (1-t)x + ty$$

is entirely contained in C.

Definition 5.15. The convex hull of a set $X \subseteq \mathbb{R}^d$ is the intersection of all convex sets containing X.

The theorem is taken from [Mat03, Chapter 3, Section 2, p. 53–54]. As we will see in the proof, which follows the one from the book, the theorem is a consequence of the ham sandwich theorem for point sets in general position (Corollary 5.13).

Theorem 5.16. *Assume that $A_1, \dots, A_d \subset \mathbb{R}^d$ are point sets in $\mathbb{R}^d$, each containing n points. Let the points in A_i have color i. Then the sets A_i are disjoint since no point can be colored with two different colors. Suppose that the disjoint union $\bigcup_{i=1}^d A_i$ is in general position. Then the points of $\bigcup_{i=1}^d A_i$ can be partitioned into "rainbow" d-tuples, that is, each d-tuple contains one point of each color, with disjoint convex hulls.*

Proof. We prove the theorem by induction on n.
For $n = 1$ there is only one "rainbow" d-tuple, so there is nothing to check.
For the inductive step we assume that the statement is true for each $j < n$. We must distinguish two cases:
If n is even, then there exists a hyperplane bisecting each A_i and containing no point of $\bigcup_{i=1}^d A_i$ by Corollary 5.13. Then each of the two half-spaces defined by the hyperplane contains the half of points of each A_i, that is, $\frac{n}{2}$ points of each A_i. The inductive hypothesis tells that in each half-space we can find $\frac{n}{2}$ "rainbow" d-tuples with disjoint convex hulls. Since none of these crosses the hyperplane, we get n such d-tuples with disjoint convex hull, which is what we wanted to show.
If n is odd, then, again by Corollary 5.13, there exists a hyperplane bisecting each A_i and containing exactly one point of each color. Each of the open half-spaces defined by the hyperplane contains $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ points of each A_i. Hence, we may apply the induction hypothesis. By the same argument as above, the two open half-spaces together give us $n-1$ "rainbow" d-tuples with disjoint convex hulls. The nth "rainbow" d-tuple is formed by the points on the hyperplane and the convex hull of this d-tuple is contained in the hyperplane. The convex hulls remain disjoint since no convex hull in the open half-spaces crosses the hyperplane. □

5.3 The Necklace Theorem

We begin this section by stating the necklace splitting problem following [Mat03, Chapter 3, Section 2, p. 54].

The necklace splitting problem: Assume that two thieves steal a precious necklace consisting of different kinds of gemstones such as diamonds or rubies, which are set in pure platinum. The thieves would like to divide the stones of each kind evenly because they do not know the values of the stones. But since the chain itself is precious itself, they want to do as few cuts as possible to waste as little platinum as possible. In this section, we answer the question "How many cuts do the thieves have to make?".

We assume the necklace to be open (with two ends). Moreover, we assume that there are d kinds of gemstones and the number of each kind is even. Before we state the necklace theorem from [Mat03, Chapter 3, Section 2, p. 54], we define the moment curve, which is needed for the first of the two proofs we give, and show one of its properties.

Definition 5.17. The moment curve in $\mathbb{R}^d$ is defined by

$$\gamma : \mathbb{R} \to \mathbb{R}^d; t \mapsto (t, t^2, \dots, t^d).$$

The following lemma and its proof are taken from [Mat03, Chapter 1, Section 6, p. 17].

Lemma 5.18. *No hyperplane intersects the moment curve in $\mathbb{R}^d$ in more than d points.*

Proof. Each hyperplane h has an equation $a_1x_1 + a_2x_2 + \cdots + a_dx_d = b$ with $0 \neq a := (a_1, \dots, a_d)^T$. If a point $\gamma(t) = (t, t^2, \dots, t^d)$ of the moment curve lies on h, then it must satisfy

$$a_1t + a_2t^2 + \cdots + a_dt^d = b.$$

Hence, the values of t for which $\gamma(t)$ lies on h are the real roots of the polynomial

$$p(t) := \left(\sum_{i=1}^{d} a_it^i\right) - b$$

of degree at most d. The polynomial p has at most d roots. Thus there are at most d intersection points of h and the moment curve. Since h was arbitrary, the statement is proven. □

Theorem 5.19 (The Necklace Theorem)**.** *Every open necklace with d kinds of stones can be divided evenly between two thieves using no more than d cuts.*

We will give two proofs of the theorem, where both of them are taken from [Mat03, Chapter 3, Section 2, p. 55–56].

Proof of Theorem 5.19 using Ham Sandwich Theorem. Take a necklace with n gemstones. We place the necklace into $\mathbb{R}^d$ along the moment curve by defining the point sets

$$A_i := \{\gamma(k) : \text{the } k\text{th gemstone is of the } i\text{th kind}, k \in \{1, \dots, n\}\}.$$

We call the points in A_i the stones of the ith kind. Notice that the sets A_i are disjoint by definition since a gemstone cannot be a diamond and a ruby at the same time. By Lemma 5.18, the union $\bigcup_{i=1}^{d} A_i$ is in general position. The ham sandwich theorem for point sets in general position (Corollary 5.13) implies that there exists a hyperplane h simultaneously bisecting each A_i. By Lemma 5.18, h cuts the moment curve (and hence also the necklace) in at most d places. Since each set A_i has even cardinality, h does not cut any gemstones. Therefore, these at most d cuts divide the necklace evenly between the thieves. □

The following theorem from [Mat03, Chapter 3, Section 2, p. 55] is a preliminary result for the second proof of the necklace theorem.

Theorem 5.20 (Hobby–Rice Theorem). *Let $\mu_1, \mu_2, \dots, \mu_d$ be continuous probability measures on [0,1]. Then there exists a partition of [0,1] into $d+1$ intervals $I_0, I_1, \dots, I_d$ (using d cut points) and signs $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d \in \{-1, 1\}$ with*

$$\sum_{j=0}^{d} \varepsilon_j \mu_i(I_j) = 0 \text{ for } i \in \{1, 2, \dots, d\}.$$

We postpone the proof for a moment and focus on the second proof of the necklace theorem, which is taken from [Mat03, Chapter 3, Section 2, p. 56], instead.

Second Proof of Theorem 5.19. We denote by t_i the amount of gemstones of the ith kind. Then $n = \sum_{i=1}^{d} t_i$. We think of the necklace as the interval $[0, 1]$, where the kth stone corresponds to the segment $\left[\frac{k-1}{n}, \frac{k}{n}\right)$. We define the d characteristic functions $f_i : [0, 1] \to \{0, 1\}$ for $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$ by

$$f_i(x) = \begin{cases} 1 & \text{if the } k\text{th stone of the necklace is of the } i\text{th kind} \\ 0 & \text{otherwise.} \end{cases}$$

These functions allow us to define d measures $\mu_1, \dots, \mu_d$ on $[0, 1]$ by

$$\mu_i(A) := \frac{n}{t_i} \int_A f_i(x)dx.$$

Each $\mu_i(A)$ denotes the fraction of gemstones of the ith kind that is on part A of the necklace.

The Hobby–Rice theorem (Theorem 5.20) implies that there exists a partition of $[0,1]$ into $d+1$ intervals $I_0, I_1, \dots, I_d$ such that for each $i \in \{1, 2, \dots, d\}$

$$\sum_{j=0}^{d} \varepsilon_j \mu_i(I_j) = 0,$$

where $\varepsilon_j \in \{-1, 1\}$ for each $j \in \{0, \dots, d\}$. The intervals I_j with $\varepsilon_j = +1$ are given to the first thief, while the second thief gets the intervals I_j with $\varepsilon_j = -1$. This division is fair, but it can happen that gemstones need to get cut to achieve this division. In this case, the division is called “nonintegral”. To get a division where no gemstones get cut we use the following rounding procedure, which is based on induction on the number of “nonintegral” cuts. If a gemstone of the ith kind is cut, then either the cut is unnecessary, or there is another cut through a gemstone of this kind since the number of gemstones of each kind is even. In the latter case, we can move both cuts away from the gemstones to reduce the number of “nonintegral” cuts. Notice that moving the cuts away from the gemstones does not change the balance. Hence, we are done by induction. □

We conclude this thesis with proving the Hobby–Rice theorem. The proof follows the one in [Mat03, Chapter 3, Section 2, p. 56].

Proof of Theorem 5.20. Let $x = (x_1, x_2, \dots, x_d, x_{d+1}) \in S^d$ be an arbitrary point. We define for each $i \in \{0, 1, \dots, d+1\}$ the cut point

$$z_i := \sum_{k=1}^{i} x_k^2.$$

Notice that $0 = z_0 \le z_1 \le \dots \le z_d \le z_{d+1} = 1$. Hence, we can subdivide the interval $[0,1]$ into $d+1$ smaller intervals by associating the cuts at the points $\{z_i : i \in \{0, 1, \dots, d+1\}\}$ with x. In particular, the intervals are given by $I_j = [z_{j-1}, z_j]$ for $j \in \{1, 2, \dots, d+1\}$, and I_j has length x_j^2.
The sign for I_j is given by $\varepsilon_j := \operatorname{sign}(x_j)$. Now, we can define the continuous function

$$g = (g_1, \dots, g_d) \colon S^d \to \mathbb{R}^d \text{ with } g_i(x) := \sum_{j=1}^{d+1} \operatorname{sign}(x_j) \mu_i([z_{j-1}, z_j]).$$

The function $g_i(x)$ indicates the amount of gemstone of the ith kind given to the first thief minus the amount of this gemstone given to the second thief.

Moreover, each g_i is antipodal since

$$\begin{aligned} g_i(-x) &= \sum_{j=1}^{d+1} \operatorname{sign}(-x_j)\,\mu_i([z_{j-1}, z_j]) \\ &= -\sum_{j=1}^{d+1} \operatorname{sign}(x_j)\,\mu_i([z_{j-1}, z_j]) \\ &= -g_i(x). \end{aligned}$$

By Theorem 4.7, there exists a point $x \in S^d$ such that $g(x) = 0$. For this x a fair division is obtained. □

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