ETH
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Lights Out: An Application of Linear Algebra over a Finite Field

Bachelor's Thesis

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February 28, 2023

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Abstract

The main goals of this thesis are to explain the mathematical background of the game *Lights Out* and provide an analytical technique such that the player is able to:

1. verify if a game is solvable
2. find a solution to a solvable game

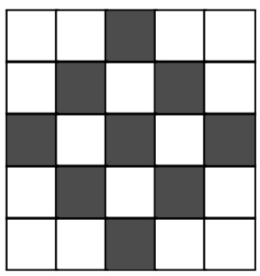
Introduction

Lights Out is an electronic game released by Tiger Electronics that found its popularity in the mid to late 90s. The game consists of a 5×5 board of luminous buttons. When the game starts, a random pattern of these lights is switched on and the goal is to eliminate all lights.

This thesis goes over the mathematical framework of the game *Lights Out*.

In Chapter 1, some fundamental understanding on the underlying linear algebra of the game is established. Although all findings hold true for general fields, we will solely focus on the finite field $\mathbb{F}_2$. All definitions of this chapter stem from G.Fischer[1]. In the second chapter, the application of the previously acquired mathematical tools will take place. We will elaborate in detail on how the game works and how we can apply linear algebra over a finite field in order to find a solution to a given game. Some of the results in this chapter are based on the works of M.A. Madsen[2] and M. Anderson[3]. In the final chapter, we will explore some special properties of *Lights Out* that are not only interesting from a mathematical point of view, but helpful for judging the solvability simply by eye.

Basic knowledge in linear algebra is the only prerequisite for reading this thesis.



Example of a game of *Lights Out*

Contents

Chapter 1

Linear algebra background

1.1 Field axioms

Definition 1.1 *A* field *is a tuple* $(F, +, \cdot, 0, 1)$ *consisting of a set* F *with two mappings*

$$+ : F \times F \longrightarrow F, \quad (x, y) \mapsto x + y$$

$$\cdot : F \times F \longrightarrow F, \quad (x, y) \mapsto x \cdot y$$

and neutral elements $0, 1 \in F$*, such that the following field axioms hold:*

(F1) $\forall x, y, z \in F : x + (y + z) = (x + y) + z$ *(Associativity of addition)*

(F2) $\forall x, y \in F : x + y = y + x$ *(Commutativity of addition)*

(F3) $\forall x \in F : x + 0 = x$ *(Neutral element of addition)*

(F4) $\forall x \in F \; \exists x' \in F : x + x' = 0$ *(Inverse element of addition)*

(F5) $\forall x, y, z \in F : x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *(Associativity of multiplication)*

(F6) $\forall x, y \in F : x \cdot y = y \cdot x$ *(Commutativity of multiplication)*

(F7) $\forall x \in F : x \cdot 1 = x$ *(Neutral element of multiplication)*

(F8) $\forall x \in F \setminus \{0\} \; \exists x' \in F : x \cdot x' = 1$ *(Inverse element of multiplication)*

(F9) $\forall x, y, z \in F : \begin{cases} x \cdot (y + z) = x \cdot y + x \cdot z \\ (y + z) \cdot x = y \cdot x + z \cdot x \end{cases}$ *(Distributivity)*

(F10) $1 \neq 0$ *(Non-triviality)*

Remark 1.2 *It is a direct consequence of the non-triviality axiom that every field must have at least two elements.*

Example 1.3 *The rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$, where both sets are endowed with the usual arithmetic operations of multiplication and addition are fields.*

1.2 The field $\mathbb{F}_2$

With the introduction of the field axioms and a few examples one will rightfully assume that there exist many more different fields,
but when trying to understand a given problem it is generally a good idea to not just use any arbitrary field but to use something that takes as little as possible and as much as necessary.
The field of choice for this thesis is the field $\mathbb{F}_2$. Not only is it finite, but it is also the smallest field (up to isomorphism). What this means exactly will be discussed in the following.

Definition 1.4 *Let X be a set and $n \in \mathbb{N}$. We say the set X has cardinality n and write $|X| = n$, if X admits a bijection to $\{1, \dots, n\}$. In this case, we call X a finite set and write $|X| < \infty$. If X is not finite, we call X an infinite set.*

Definition 1.5 *A field $(F, +, \cdot, 0, 1)$ is called finite, if the set F is finite.*

Proposition 1.6 *There exists exactly one field (up to isomorphism) with two elements $\{a, b\}$. We denote this field by $\mathbb{F}_2 := \{0, 1\}$.*

Proof Without loss of generality, we choose $a = 0$ and $b = 1$, where 0 is the neutral element of addition and 1 is the neutral element of multiplication (the same proof can be performed by exchanging the values of a and b).
Due to the neutral element property of a and b and by applying the axioms of definition 1.1 we can fill in the operation tables for $\{a, b\}$ as follows:

+	a	b
a	a	b
b	b	**?**

$\cdot$	a	b
a	**?**	a
b	a	b

We are now left with $a \cdot a$ and $b + b$. According to our table we do not have an additive inverse to b hence the only choice is to define $b + b = a$.
Now for $a \cdot a$ we have two options. Either $a \cdot a = b$ or $a \cdot a = a$ Assume $a \cdot a = b$. Then:

$$1 = b = a + b = (a \cdot b) + (a \cdot a) = a \cdot (b + a) = a \cdot b = a = 0$$

Which is a contradiction to the non-triviality axiom. Therefore, we conclude:

$+$	a	b
a	a	b
b	b	a

$\cdot$	a	b
a	a	a
b	a	b

This construction proves the existence and uniqueness of a field with two elements. From now on, we call this field $\mathbb{F}_2$. $\square$

Corollary 1.7 (Idempotence) $\forall x \in \mathbb{F}_2 : x \cdot x = x$

Corollary 1.8 $\forall x \in \mathbb{F}_2 : x + x = 0$

1.3 Vector spaces over $\mathbb{F}_2$

Now that we have constructed $\mathbb{F}_2$ it is time to take it one step further. Often when talking about solving a problem, it is not sufficient to solve it using only one variable at a time. *Lights Out* with its 5×5 board is no exception to this. It is therefore necessary to somehow "extend" our ability to execute calculations. This brings us to the following definition:

Definition 1.9 (Vector space) *Let F be a field. A set V with operations*

$$+ : V \times V \longrightarrow V, \quad (\vec{x}, \vec{y}) \mapsto \vec{x} + \vec{y}$$

called addition and

$$\cdot : F \times V \longrightarrow V, \quad (\lambda, \vec{x}) \mapsto \lambda \cdot \vec{x}$$

called scalar multiplication is an F-vector space (or vector space over F), if it fulfills the following properties:

(V1) V equipped with addition is an abelian group[1]. The neutral element is called zero vector and is denoted by $\vec{0}$ and the inverse of $\vec{x} \in V$ is denoted by $-\vec{x}$.

(V2) The multiplication with scalars must be compatible with the addition as follows:

$$(\lambda + \mu) \cdot \vec{x} = \lambda \cdot \vec{x} + \mu \cdot \vec{x}, \qquad \vec{x} \cdot (\lambda + \mu) = \vec{x} \cdot \lambda + \vec{x} \cdot \mu$$

$$\lambda \cdot (\mu \cdot \vec{x}) = (\lambda \cdot \mu) \cdot \vec{x}, \qquad 1 \cdot \vec{x} = \vec{x}$$

$\forall \vec{x}, \vec{y} \in V$ *and* $\lambda, \mu \in F$.

[1] An *abelian group* is an algebraic structure that satisfies the axioms F1 to F4.

Definition 1.10 (Linear subspace) *Let V be an F-vector space and $W \subset V$ a subset. W is called a* linear subspace *of V if:*

$W \neq \{\}$
$\vec{v}, \vec{w} \in W \implies \vec{v} + \vec{w} \in W$
$\vec{v} \in W, \lambda \in F \implies \lambda \cdot \vec{v} \in W$

Example 1.11 $\{\vec{0}\} \subset V$ *is a linear subspace of V.*

Remark 1.12 *$span(\vec{v}_1, \dots, \vec{v}_n)$ denotes the space that is obtained by "collecting" every possible linear combination of $\vec{v}_1, \dots, \vec{v}_n$.*

Definition 1.13 (Basis) *A family $\mathcal{B} = (\vec{v}_i)_{i \in I}$ in a vector space V is called a generating system of V if*

$$V = span(\vec{v}_i)_{i \in I}.$$

A generating system $\mathcal{B} = (\vec{v}_i)_{i \in I}$ of V is called basis of V, if it is linearly independent[2]. V is called finitely generated, if it has a finite generating system i.e. $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$. If $\mathcal{B}$ is a finite basis, we call n the length of the basis.

Definition 1.14 (Dimension) *Let V be a F-vector space. We define*

$$dim_F(V) := \begin{cases} \infty, & \text{if } V \text{ does not have a finite basis,} \\ n, & \text{if } V \text{ has a basis of length } n. \end{cases}$$

$dim_F(V)$ is called the dimension of V over F.

Proposition 1.15 *Every vector space V has a basis.*

Proposition 1.16 *Every basis $\mathcal{B}$ of a vector space V has the same length. Hence,* $\dim_F(V)$ *is well-defined.*[3]

Remark 1.17 *Let V be a F-vector space, $\mathcal{B}$ a basis of V and $dim_F(V) = n < \infty$. Then we can represent any vector $\vec{v} \in V$ in terms of its corresponding coordinates v_i.*

1. *We call $\vec{v}, \vec{w} \in V$ orthogonal with respect to $\mathcal{B}$ when*

$$\langle \vec{v}, \vec{w} \rangle := \sum_{i=1}^{n} v_i w_i = 0.$$

   We denote orthogonal vectors by $\vec{v} \perp \vec{w}$.

2. *Let $U \subset V$ be a linear subspace, then its orthogonal complement is defined as*

$$U^{\perp} := \{\vec{v} \in V : \vec{v} \perp \vec{u} \quad \forall \vec{u} \in U\}.$$

[2] Linear independence of $\mathcal{B}$ means that for any finite subset $\{\vec{v}_1, \dots, \vec{v}_r\}$ of $\mathcal{B}$, if $\sum_{j=1}^{r} \lambda_j \cdot \vec{v}_j = \vec{0}$ it follows that $\lambda_1 = \dots = \lambda_r = 0$.

[3] More about Propositions 1.15 and 1.16 can be found in G.Fischer[1] chapter 2.4.

1.4 Systems of linear equations over $\mathbb{F}_2$

Having stated the most important concepts in connection to vector spaces, we now consider mappings between them. Of particular interest are those mappings, where the structures (addition and multiplication with scalars) are respected. With the help of the previously developed techniques, we can now introduce linear maps and define systems of linear equations.

1.4.1 Linear maps and equation systems

Definition 1.18 *Let V, W be $\mathbb{F}_2$-vector spaces. A map $A : V \longrightarrow W$ is said to be linear if:*

(L1) $A(\vec{x} + \vec{y}) = A(\vec{x}) + A(\vec{y})$

(L2) $A(\lambda\vec{x}) = \lambda A(\vec{x})$

$\forall \vec{x}, \vec{y} \in V$ and $\lambda, \mu \in \mathbb{F}_2$.

Definition 1.19 *Let A be a linear map. Then the set*

$$Im(A) := \{\vec{b} \mid \exists \vec{x} \text{ with } A \cdot \vec{x} = \vec{b}\}$$

is called the image of A and

$$Null(A) := \{\vec{x} \mid A \cdot \vec{x} = \vec{0}\}$$

is called the null space of A.

Proposition 1.20 *Let V, W be vector spaces over $\mathbb{F}_2$. For a linear map $A : V \longrightarrow W$ the following formula holds true:*

$$dim(V) = dim(Null(A)) + dim(Im(A))^4$$

Once bases for the vector spaces V and W have been chosen, a linear map $A : V \longrightarrow W$ may be (uniquely) represented by a matrix, which we will denote by $A = (a_{ij})$ where i denotes the row entry and j denotes the column entry.

Henceforward, we use the terms *linear map* and *matrix* synonymously.

Example 1.21 *Let $V = \mathbb{F}_2^2$ and $id_V : V \longrightarrow V$, $\vec{v} \mapsto \vec{v}$. Then*

$$id_V = (\delta_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We call id_V the identity matrix.

[4] Chapter 3.2 of G.Fischer[1].

Definition 1.22 (Matrix addition) *Let $A, B \in \mathbb{F}_2^{m\times n}$, then*

$$A + B := (a_{ij} + b_{ij}) \in \mathbb{F}_2^{m\times n}.$$

Definition 1.23 (Matrix multiplication) *Let $A \in \mathbb{F}_2^{m\times n}, B \in \mathbb{F}_2^{n\times l}$, then*

$$A \cdot B := (\sum_{j=1}^{n} a_{ij}b_{jk}) \in \mathbb{F}_2^{m\times l}$$

with $i \in \{1 \dots m\}$, $k \in \{1 \dots l\}$.

Definition 1.24 (Systems of linear equations) *A system of linear equations is a set of linear equations with one or more unknowns that are all supposed to be satisfied simultaneously i.e.:*

For $A \in \mathbb{F}_2^{m\times n}$, $\vec{b} \in \mathbb{F}_2^m$ and $\vec{x} \in \mathbb{F}_2^n$ we get the following equation system:

$$A \cdot \vec{x} = \vec{b}, \quad i.e. \quad (\sum_{j=1}^{n} a_{ij}x_j) = b_i \quad for\ i = 1, \dots, m.$$

If $\vec{b} \neq \vec{0}$ we call the system inhomogeneous, otherwise the system is called homogeneous. The set

$$Sol(A, \vec{b}) := \{\vec{x} \in \mathbb{F}_2^n : A \cdot \vec{x} = \vec{b}\}$$

is called solution set *of the equation system $A \cdot \vec{x} = \vec{b}$.*

Remark 1.25 *The dimension of the set of all $\vec{b}$ such that $A \cdot \vec{x} = \vec{b}$ is solvable is called the rank*[5] *of A. Therefore the dimension of the image of A is equal to the rank of A. Thus the formula from remark 1.20 yields:*

$$dim(Null(A)) = n - r.$$

Where n is the dimension of V.

1.4.2 Gauss-Jordan elimination

We will now introduce the *Gauss-Jordan elimination* over $\mathbb{F}_2$. It will become apparent, that it is the key procedure to finding a solution for the game *Lights Out*. Let us therefore consider the following system of linear equations:

$$A \cdot \vec{x} = \vec{b}.$$

[5] We use $r := rank(A)$.

We call $A \in \mathbb{F}_2^{m\times n}$ the coefficient matrix of the system. By appending $\vec{b} \in \mathbb{F}_2^m$ to A we get the following matrix

$$(A, \vec{b}) = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix}$$

This matrix is called the *extended coefficient matrix* and contains all information about the system of equations.

Definition 1.26 *A $m \times n$ - matrix A is said to have row-echelon form, if it has the following form:*

$$A = \begin{pmatrix} 1 & & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & & \\ 0 & \dots & & & & \dots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \dots & & & & \dots & 0 \end{pmatrix}$$

Hence, A is in row-echelon form if the following applies:

(1) *There exists a number $r \in \{0, \dots, n\}$ such that for every entry $a_{i,j}$ with $i > r$ it holds that $a_{i,j} = 0$.*

(2) *For every i with $1 \leq i \leq r$ we take the smallest index j_i of the column in which there exists an entry unequal to zero i.e.*

$$j_i := min\{j : a_{i,j} \neq 0\}$$

obviously we have that $1 \leq j_i \leq n$ and an additional echelon condition is that

$$j_1 < j_2 < \ldots < j_r.$$

As for the $r = 0$, we define the case in which all entries of A are equal to zero ($A = 0_M$). We call the entries

$$a_{1,j_1},\ a_{2,j_1},\ \dots\ , a_{r,j_r}$$

the pivots of A.

Now we will provide a procedure for solving a linear system of equations where the coefficient matrix A is in row-echelon form. This exact procedure can later be applied to find a solution for the game *Lights Out*. For simplicity reasons, we assume that the pivots are in the first r columns. Then the expanded coefficient matrix has the form

$$(A, \vec{b}) = \begin{pmatrix} a_{1,1} & & & & & b_1 \\ & a_{2,2} & & & & \vdots \\ & & \ddots & & & \\ & & & a_{r,r} & & b_r \\ 0 & \dots & & \dots & 0 & b_{r+1} \\ \vdots & & & & & \vdots \\ 0 & \dots & & \dots & 0 & b_m \end{pmatrix}$$

such that $a_{1,1} \neq 0, \dots, a_{r,r} \neq 0$.

Theorem 1.27 *$A \cdot \vec{x} = \vec{b}$ with $(A, \vec{b})$ in row-echelon form is solvable if and only if $b_i = 0$ for $r + 1 \leq i \leq m$*

Proof Assume that there exists a $b_i \neq 0$ such that $r + 1 \leq i \leq m$. Then the $i-$th equation is

$$0 \cdot x_1 + \dots + 0 \cdot x_n = b_i \neq 0.$$

There does not exist such an $\vec{x} \in \mathbb{F}_2^n$.

Now for the reverse direction we assume

$$b_{r+1} = \dots = b_m = 0$$

We will provide a method such that one can construct a solution for $A \cdot \vec{x} = \vec{b}$. For this purpose, we distinguish between two types of variables:

$$x_{r+1}, \dots, x_n$$

which are called free variables, they can obtain arbitrary values and:

$$x_1, \dots, x_r$$

are called bound variables. They are uniquely determined by the choice of the free variables i.e. set $k := n - r$ as the number of the free variables and choose $\lambda_1, \dots, \lambda_k \in \mathbb{F}_2$ as parameters and set

$$x_{r+1} = \lambda_1, \quad x_{r+2} = \lambda_2, \quad \dots \quad , \quad x_{r+k} = \lambda_k.$$

In order to calculate $x_1, \dots, x_r$ we start by taking the r-th equation

$$a_{r,r}x_r + a_{r,r+1}\lambda_1 + \dots + a_{r,r+k}\lambda_k = b_r.$$

Due to $a_{r,r} \neq 0$ we get the following representation

$$x_r = \frac{1}{a_{r,r}}(b_r - a_{r,r+1}\lambda_1 - \dots - a_{r,r+k}\lambda_k)$$

$$= \frac{1}{a_{r,r}}b_r + \frac{-a_{r,r+1}}{a_{r,r}}\lambda_1 + \dots + \frac{-a_{r,r+k}}{a_{r,r}}\lambda_k$$

Which gives us

$$x_r = c_{r,r}b_r + d_{r,1}\lambda_1 + \dots + d_{r,k}\lambda_k.$$

Plugged in to the $(r-1)$-th equation gives analogously

$$x_{r-1} = c_{r-1,r-1}b_{r-1} + c_{r-1,r}b_r + d_{r-1,1}\lambda_1 + \dots + d_{r-1,k}\lambda_k$$

and finally

$$x_1 = c_{1,1}b_1 + c_{1,2}b_2 + \dots + c_{1,r}b_r + d_{1,1}\lambda_1 + \dots + d_{1,k}\lambda_k.$$

The numbers $c_{i,j}$ and $d_{i,j}$ are only dependent on the entries $a_{i,j}$ but not on $b_1, \dots, b_r$ and $\lambda_1, \dots, \lambda_k$. With the help of matrices, this can be represented as follows:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{pmatrix} = \left(\begin{array}{ccc} c_{11} & \dots & c_{1r} \\ & \ddots & \vdots \\ \mathbf{0} & & c_{rr} \\ \hline 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{array}\right) \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} + \left(\begin{array}{ccc} d_{11} & \dots & d_{1k} \\ \vdots & & \vdots \\ d_{r1} & \dots & d_{rk} \\ \hline 1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1 \end{array}\right) \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix}$$

or $$\vec{x}(\lambda) = C \cdot \vec{b} + D \cdot \vec{\lambda}$$

where C is a $(n \times r)$ - matrix and D is a $(n \times k)$ - matrix. □

Having seen how a system of linear equations in row-echelon form can be solved, we now try to put any system into this form. For this purpose, we use two types of *elementary row operations* on the extended coefficient matrix:

1. Swapping two rows
2. Adding λ times the $i-th$ row to the $k-th$ row where $i \neq k$ and $\lambda \neq 0$

Theorem 1.28 *Every matrix A can be transformed by elementary row operations into a matrix $\tilde{A}$ in row-echelon form. We call this transformation the Gauss-Jordan elimination method.*

Proof We give a concrete procedure, which is carried out step by step and is structured in such a way that a computer program can be made from it without great difficulty.
Let A be a $m \times n$- matrix. If $A = 0_M$ then by definition it already has row-echelon form with $r = 0$.
If $A \neq 0_M$ there exists at least one entry $a_{i,j} \neq 0$. Therefore, there exists a column which is nonzero. We choose the one with the smallest index j_1:

$$j_1 = min\{j : \exists i \text{ such that } a_{i,j} \neq 0\}.$$

If $a_{1,j_1} \neq 0$ we can choose it as pivot, else we search for $a_{i_1,j_1} \neq 0$ and exchange the first row with row i_1. Using this procedure, we obtain the first row of $\tilde{A}$. Therefore, it holds that for the first pivot:

$$\tilde{a}_{1,j_1} = a_{i_1,j_1}.$$

Through elementary row operations of *type* 2 we can erase all entries below $\tilde{a}_{1,j_1}$. If a is one of the entries below then

$$a + \lambda \tilde{a}_{1,j_1} = 0$$

so that we can set

$$\lambda = -\frac{a}{\tilde{a}_{1,j_1}}.$$

The result of these operations is of the following form:

$$\tilde{A}_1 = \left(\begin{array}{cccc|c} 0 & \cdots & 0 & \tilde{a}_{1j_1} & * \cdots\cdots * \\ \hline \vdots & & \vdots & 0 & \\ \vdots & & \vdots & \vdots & A_2 \\ 0 & \cdots & 0 & 0 & \end{array}\right)$$

where the entries marked as $*$ stand for arbitrary entries. The matrix A_2 has $m-1$ rows and $n-j_1$ columns. In the second step, one has to do the same

procedure with A_2 as was done with $A = A_1$:
If $A_2 = 0_M$ then $\tilde{A}_1$ is in row-echelon form. Else, we search for $j_2 > j_1$ and the pivot $\tilde{a}_{2,j_2}$. The required row operations of A_2 can be extended from row 2 to m of $\tilde{A}_1$ without changing the columns from 1 to j_1 because they are all equal to zero. Having transformed A_2, we obtain A_3 and repeat the procedure. The reason why this procedure is terminal is that either the rank of the A_k's gets smaller or the case $A_k = 0_M$ occurs. The final result is:

$$\tilde{A} = \begin{pmatrix} & \tilde{a}_{1j_1} & * & \cdots & \cdots & \cdots & * \\ & & \tilde{a}_{2j_2} & & & & \vdots \\ & & & \ddots & & & \vdots \\ & 0 & & & \tilde{a}_{rj_r} & \cdots & * \\ & & & & & & \end{pmatrix}$$

$\square$

Chapter 2

Solving Lights Out

2.1 The game

The game *Lights Out* consists of a 5×5 matrix with a total of 25 buttons. Each button has the function of a light, which means that it can either be on or off. The game starts by a given board of an initial button configuration. The goal is to turn off all the lights. But there is a catch. Pushing an arbitrary button will not only change its own state, but also the state of all its vertical and horizontal neighbours.
A complete strategy for solving the game can be obtained by using linear algebra over $\mathbb{F}_2$.

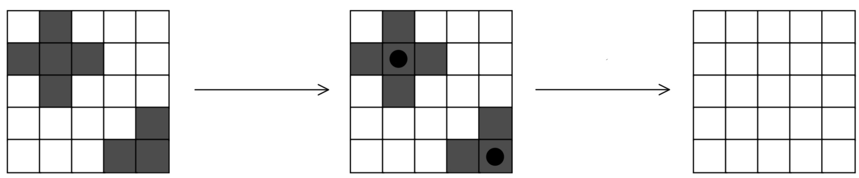


Figure 2.1: Example of a winning game of *Lights Out*.

Before elaborating on the mathematical approach to solve the game let us take a closer look at the game and state some observations.

2.2 Observations

Observation 1: Each button needs to be pressed no more than once. This observation comes from the fact that pressing a button twice is like not pressing it at all. Since pressing a button changes the state of that button and of its immediate vertical and horizontal neighbours, pressing that same button again will reverse the states and switch the buttons back to their original states.

Observation 2: The state of each button only depends on how many times it and its immediate vertical or horizontal neighbours are pressed. This observation indicates that the order in which you press the buttons is irrelevant.

Observation 3: If we start with the board where all lights are off and press a sequence of buttons to get a configuration, then starting with that configuration and pressing the same sequence of buttons will result in all the lights turned off[2].

2.3 Linear algebra setup

In the following, we will use linear algebra to provide a solution to the game. This can be done with basic matrix operations, *Gauss-Jordan* elimination and an understanding of the column and null space of a matrix. Since there are only two possible states to the buttons, we can do all of our calculations over the previously introduced field $\mathbb{F}_2$. Therefore, let *1* represent *on* and *0* represent *off*. Without loss of generality, we can think of the 5×5 matrix $A \in \mathbb{F}_2^{5\times5}$ as a vector $\vec{b} \in \mathbb{F}_2^{25}$

$$\vec{b} = (b_1,\ b_2, \ldots,\ b_{25})^T$$

where $b_1,\ b_2, \ldots,\ b_{25}$ denote whether or not the status of the $i - th$ light must be changed. We refer to this vector $\vec{b}$ as the *configuration* of the matrix A.

Example 2.1 *The configuration vector for Figure 2.1 is*

$$\vec{b} = (0,1,0,0,0,1,1,1,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,1,1)^T$$

The configuration $\vec{b}$ is obtained by pressing a sequence of buttons which we will denote as

$$\vec{x} = (x_1,\ x_2, \ldots,\ x_{25})^T$$

where $x_1,\ x_2, \ldots,\ x_{25}$ represent a strategy needed to obtain configuration $\vec{b}$.

Example 2.2 *The strategy vector for Figure 2.1 is*

$$\vec{x} = (0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1)^T$$

Since pushing one button does not only change its own status, but also its horizontal and vertical neighbours. We get the following system

$$
\begin{aligned}
b_1 &= x_1 + x_2 + x_6 \\
b_2 &= x_1 + x_2 + x_3 + x_7 \\
&\vdots \\
b_7 &= x_2 + x_6 + x_7 + x_8 + x_{12} \\
&\vdots \\
b_{25} &= x_{20} + x_{24} + x_{25}
\end{aligned}
$$

It is now straightforward to rewrite this system of linear equations in matrix form as $A \cdot \vec{x} = \vec{b}$. The coefficient matrix $A \in \mathbb{F}_2^{25\times 25}$ is the following block matrix

$$A = \begin{pmatrix} K & I_5 & O & O & O \\ I_5 & K & I_5 & O & O \\ O & I_5 & K & I_5 & O \\ O & O & I_5 & K & I_5 \\ O & O & O & I_5 & K \end{pmatrix}$$

where I_5 is the 5×5 identity matrix, O is the 5×5 zero matrix and K is defined as

$$K := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

2.4 Solvability of the game

Since solving the game means that we have to find a strategy $\vec{x}$ such that $A \cdot \vec{x} = \vec{b}$, we apply our findings from Section 1.4. Thus, a configuration $\vec{b}$ is solvable if and only if it belongs to the column space of the matrix A. To analyse Col(A), we perform *Gauss-Jordan* elimination on A. This will yield $RA = E$, where E is in row-echelon form and R is the product of the elementary matrices which perform the reducing row operations.[1]
The matrices R and E will not be displayed here, but we invite the reader to calculate them using a computer algebra system. Having done this calculation, we see that the matrix E is of rank 23 and has the following form:

[1] More about elementary matrices in G.Fischer[1] chapter 3.7.

$$E = \begin{pmatrix} 1 & 0 & \cdots & & \cdots & 0 & | & | \\ & 1 & \ddots & & & \vdots & | & | \\ & & \ddots & & & & | & | \\ & & & \ddots & \ddots & \vdots & * & * \\ & & & & 1 & 0 & | & | \\ & & & & & 1 & 1 & 1 \\ 0 & \cdots & & & & & \cdots & 0 \\ 0 & \cdots & & & & & \cdots & 0 \end{pmatrix}$$

where the last two columns of E are:

$$(e_{i,24}) = (0,1,1,1,0,1,0,1,0,1,1,1,0,1,1,1,0,1,0,1,0,1,1,0,0)^T$$

$$(e_{i,25}) = (1,0,1,0,1,1,0,1,0,1,0,0,0,0,0,1,0,1,0,1,1,0,1,0,0)^T.$$

Furthermore, we have that A is a symmetric matrix such that the column space denotes the same space as the row space. Also $Row(A)$ is the orthogonal complement of the null space of A, which in turn equals $Null(E)$. Hence, to describe $Col(A)$, we only need to determine a basis for $Null(E)$. Since E is in row-echelon form, it is easy to find an orthogonal basis for $Null(E) := span\{\vec{n}_1, \vec{n}_2\}$. By examining the last two columns of E we find the following two vectors:

$$\vec{n}_1 = (0,1,1,1,0,1,0,1,0,1,1,1,0,1,1,1,0,1,0,1,0,1,1,1,0)^T$$

$$\vec{n}_2 = (1,0,1,0,1,1,0,1,0,1,0,0,0,0,0,1,0,1,0,1,1,0,1,0,1)^T$$

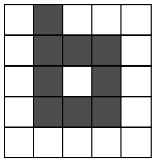
where we constructed $\vec{n}_1$ out of $(e_{i,24})$ by changing the last two entries from $(\ldots,0,0)$ to $(\ldots,1,0)$ and $\vec{n}_2$ out of $(e_{i,25})$ by changing $(\ldots,0,0)$ to $(\ldots,0,1)$.

This brings us to the following result:

Theorem 2.3 *A configuration $\vec{b}$ is solvable if and only if it is orthogonal to $Null(E) := span\{\vec{n}_1, \vec{n}_2\}$.*

Proof This follows from the fact that if $\vec{b}$ is orthogonal to $Null(E) = Null(A)$, then $\vec{b} \in Row(A) = Col(A) = Im(A)$. That in turn is equivalent to saying that $Sol(A, \vec{b}) = Sol(E, \vec{b}) \neq \{\}$. $\square$

Example 2.4 *Let us have a look at the following game of Lights Out:*



Translating this board to the configuration $\vec{b} \in \mathbb{F}_2^{25}$ *yields:*

$$\vec{b} = (0,1,0,0,0,0,1,1,1,0,0,1,0,1,0,0,1,1,1,0,0,0,0,0,0)^T$$

We take the dot product of the configuration $\vec{b}$ *with* $\vec{n}_1$ *and get*

$\langle \vec{b}, \vec{n}_1 \rangle = 1 \neq 0$ *which means that the configuration* $\vec{b}$ *is not solvable.*

2.5 Finding a solution

We have now found a simple way to check whether or not a configuration is solvable. So the last question that we have to answer is, how one finds a solution.
We propose to use the following three steps:

1. Check if a configuration is solvable by applying Theorem 2.3.
2. If it is solvable, continue by performing *Gauss-Jordan* elimination on the system $(A, \vec{b})$ by applying the same procedure that was used in proof of Theorem 1.28.
3. Construct a solution by applying the technique that was provided in the proof of Theorem 1.27 on the system that was obtained in the previous step.

Since finding a solution $\vec{x} \in \mathbb{F}_2^{25}$ means that one has to compute a number of operations on a 25×25 matrix, we suggest to use for this matter a computer system.

Chapter 3

Some special properties of Lights Out

3.1 The eigenvectors of Lights Out

For the following part, we assume that the reader already has some knowledge of the eigenvalue problem of a matrix. Otherwise, we advise to study chapter 4 of G.Fischer [1] before continuing.

The question we will answer in this section is the following:
What is the meaning of a game configuration $\vec{b}$, where the associated strategy vector $\vec{x}$ is in fact an *eigenvector* of A with *eigenvalue* λ? Since we are doing all calculations over $\mathbb{F}_2$, the only possible eigenvalues are 0 or 1.

First case when $\lambda = 0$.

We get $A \cdot \vec{x} = \lambda\vec{x} = \vec{0}$. This result gives us two types of information about the game. Firstly, that the strategy vector $\vec{x}$ is in the null space of the matrix A and secondly, that the game configuration $\vec{b} = \vec{0}$ is trivial i.e. a configuration where no button had to be pushed.
What is interesting for the case $\lambda = 0$ is that we have a nonzero strategy vector $\vec{x}$. Hence, we obtain a strategy where we will be able to press a number of buttons, which results in having changed nothing at all i.e. the board before executing the strategy $\vec{x}$ looks the same as after the execution of $\vec{x}$. The two vectors $\vec{n}_1, \vec{n}_2$ from section 2.4 form a basis of the $\lambda = 0$ eigenspace.

Second case when $\lambda = 1$.

For $\lambda = 1$ we get $A \cdot \vec{b} = \vec{b}$ or $(A - I_{25}) \cdot \vec{b} = \vec{0}$. Hence, we have to find the null space of the matrix $(A - I_{25})$. We get the following matrix for $(A - I_{25})$:

$$(A - I_{25}) = \begin{pmatrix} \tilde{K} & I_5 & O & O & O \\ I_5 & \tilde{K} & I_5 & O & O \\ O & I_5 & \tilde{K} & I_5 & O \\ O & O & I_5 & \tilde{K} & I_5 \\ O & O & O & I_5 & \tilde{K} \end{pmatrix}$$

where I_5 is the 5×5 identity matrix, O is the 5×5 zero matrix and $\tilde{K}$ is defined as

$$\tilde{K} := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Remark 3.1 *It can be verified with a computer algebra system that* $(A - I_{25})$ *has rank* 20.

Calculating $Null(A - I_{25})$ we find 5 linearly independent eigenvectors that satisfy $A \cdot \vec{b} = \vec{b}$.

These vectors have the interesting property that one can solve the game by just pressing all the lights that are turned on.

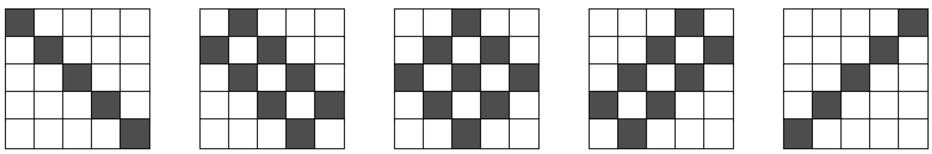


Figure 3.1: Illustration of the boards where the configuration is a eigenvector with eigenvalue 1.

3.2 Configurations with palindromic symmetries

For this section, we want to point out another special property of the game *Lights Out*, related to solutions of a configuration with *palindromic symmetry*.

Definition 3.2 *We say that a vector* $\vec{b} \in \mathbb{F}_2^n$ *is a palindrome or has palindromic symmetry if*

$$\vec{b}_i = \vec{b}_{n-i+1}.$$

Intuitively speaking, this means that it does not matter if you read the vector from bottom to top or top to bottom. One may have heard of a palindromic word or phrase where reading it backwards says the same as reading it in regular order.

Example 3.3 *The vector* $(1, 0, 0, 1)$ *and the name* ANNA *are palindromes.*

Theorem 3.4 *Every configuration* $\vec{b}$ *with palindromic symmetry is solvable.*

Remark 3.5 *Hence, if one is able to detect this kind of symmetry in a board configuration then its solvability is a given.*

Proof of the Theorem First note that the two vectors $\vec{n}_1, \vec{n}_2 \in Null(A)$ are palindromes. Furthermore, we notice that both vectors have a 0 at their $13th$ entry. Applying the property of palindromic vectors on the dot product gives us:

$$\langle \vec{n}, \vec{b} \rangle = \sum_{i=1}^{12} n_i b_i + \sum_{i=14}^{25} n_{25-i+1} b_{25-i+1} + \underbrace{n_{13} b_{13}}_{=0} = \overbrace{\sum_{i=1}^{12} n_i b_i + \sum_{i=1}^{12} n_i\, b_i}^{\textit{Corollary 1.8}} = 0$$

Hence, $\vec{n} \perp \vec{b}$. It follows from Theorem 2.3 that $\vec{b}$ is solvable. $\square$

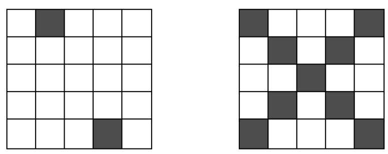


Figure 3.2: Two configurations with palindromic symmetry.

Acknowledgement

I would like to thank *Roman Waldmeyer, Tiffany Bucheli, Sabina Bremer, Anna Coulibaly* and *Michał Mikuta* for their constructive criticism of the manuscript. Moreover, I thank *Prof. Dr. Ana Cannas da Silva* for her suggestion on writing this thesis and support throughout this project.

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