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Classification of Surfaces

Bachelor Thesis

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Aknowledgments

I thank my supervisor Ana Cannas da Silva for her guidance and support throughout the whole process. She chose a topic matching my interests and stood by me the whole time giving me enough space to follow my way, while kindly guiding me when needed. I appreciate her effort to take time to satiate my curiosity, even when my questions diverged from the main topic. The way she sees mathematics inspires me and I hope that the style of this thesis reflects a bit of the joy she transmits.

Contents

Introduction

Humans have developed an intrinsic ability to classify almost every aspect of life. This ability turned out to be an indispensable tool in the modern culture, offering the capacity to understand and handle properly the most diverse situations. Mathematicians tend to go beyond personal, cultural and social needs. In particular, the wish to study and understand mathematical objects leads inevitably to the need of their classification, creating what turned out to be a centuries-old quest.

With the beginning of the study of manifolds in the 19th century, the attempt to classify them arose naturally, marking its presence in history throughout the two last centuries until this day. Spotlighting compact surfaces, a rather simple classification is given by their orientability. Travelling along a closed loop on some surface and finding ourselves mirrored, when coming back to the start, is a sufficient indication that we were travelling on a non-orientable surface. We rapidly see, that this is not the end of the story, for instance by considering the sphere and the torus. They are orientable surfaces, but are obviously not homeomorphic. Indeed, the theorem for the classification of surfaces takes the following form.

> *Every compact surface is homeomorphic to either a sphere, or to a connected sum of tori, or to a connected sum of projective planes.*

The notion of an abstract surface not embedded in the Euclidean 3-dimensional space, as the real projective plane, was not elaborated when the first proofs of the classification of surfaces appeared. A. Möbius showed in 1861 the classification theorem for orientable surfaces [14], followed by C. Jordan in 1866 [6] and W. van Dick in 1888 [18], who included non-orientable surfaces. The first rigorous proofs were done by M. Dehn and P. Heegaard in 1907 [15] and by Brahana in 1922 [1]. Both proofs rely on the crucial assumption that a surface is triangulable, a technical difficulty that was not overcome until 1925 by T. Radó [16].

In this thesis, we present the proof of the classification of surfaces by William S. Massey [12], published in 1991 in his book 'A Basic Course in Algebraic Topology'. Moreover, we complement it with background material, an extension to manifolds with boundary and other viewpoints on the proof of the classification theorem.

Chapter 1, 2 and 3 contain the necessary preliminaries to understand surfaces, their triangulations and connected sums, following the approach taken by Massey.

Chapter 4 is the highlight of this thesis. Applying the knowledge from the previous chapters, the main proof reduces itself to two main steps. As for most proofs on the classification of surfaces, it relies first on a combinatorial step, showing that every surface admits a polygonal presentation in the real plane. The application of an elaborated algorithm allows, after a finite number of manipulations, to classify the surface into the three classes of the sphere, the connected sum of tori or the connected sum of projective planes. The second step relies more on topological invariants such as the Euler characteristic and orientability of a surface. Combining these invariants leads to the conclusion that the three classes are indeed disjoint.

In Chapter 5, we will discuss the classification of compact bordered surfaces, which will turn out to be surprisingly simple. For this, we rely on an early version of the book mentioned above 'Algebraic Topology: An Introduction' [11] of Massey.

Chapter 6 is dedicated to alternative viewpoints. With the birth of algebraic topology in the beginning of the last century, very useful tools emerged and mathematicians did not miss the chance to use them for the classification of manifolds. Wanting to present a slightly different point of view, we give in Section 6.1 an outline of the proof presented in John Lee's 'Introduction to topological manifolds' written in 2011 [8]. This proof relies partially on Massey's approach, while making use of cell complexes for the triangulation of surfaces. The final part of this proof deals with the fundamental group, a topological feature that distinguishes compact surfaces, which can also be found in Massey's book. Written a whole century after the appearance of the first proof, this proof is presented in a more complete and exhaustive manner, leaving few room for uncertainties for the attentive reader. We hope this to be a welcomed addition for the reader to deepen the understanding acquired in the preceding chapters.

Balancing out the formal strictness of the last section, we end this thesis with a more visually appealing proof by John Conway dating 1999 in Section 6.2. Though Conway did not write down the proof himself, we are guided through his ideas with beautiful drawings by G. Francis and J. Weeks [4]. Substituting tori by handles, and projective planes by cross caps, we end up with a different formulation of the classification theorem.

> *Every compact surface is homeomorphic to either a sphere with handles or a sphere with crosscaps.*

Chapter 1

Compact surfaces

To understand this thesis it is required to have some understanding of topological spaces, homeomorphisms and manifolds. Our objects of study will be compact surfaces which can be orientable or non-orientable. In this chapter, we will remind of some basic topological definitions, which can be looked up in [7], [13] or [9]. For the definition of orientability we will follow Massey's book [12].

1.1 Topological manifolds

Definition 1.1 A **Hausdorff space** is a topological space where for every two distinct points there exists two disjoint neighborhoods of each point.

While the Hausdorff property ensures that the topological space contains enough open sets to work with, we want to restrict the amount of all possible sets by defining them in terms of a countable basis, which motivates the following definition.

Definition 1.2 A topological space is **second countable** if the topology admits a countable basis of open sets.

Definition 1.3 A topological space is called **locally Euclidean** when every every point of the space has a neighborhood that is homeomorphic to the open ball $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 < 1\}$.

This will ensure in some way, that the topological spaces we are working with are free of some kind of singularities or intersections.

Definition 1.4 A topological space X is **connected** if it there are no non-empty and disjoint open sets $A \subset X$ and $B \subset X$ such that $X = A \cup B$.

Definition 1.5 A n**-dimensional topological manifold** is a locally Euclidean Hausdorff space of dimension n, where every connected component is second countable.

In this thesis, we will refer to it in the short form as *n-manifold*.

Massey does not require second countability in his definition of a topological manifold. Nevertheless, he writes that he will restrict his attention to manifolds with a countable basis. For this reason, we require second countability as an axiom in the above definition.

Notice further that this is the definition for manifolds without a boundary. On a manifold with boundary each point must have a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ or to a relatively open subset in the Euclidean half space $\{(x_1, \ldots, x_n \in \mathbb{R}^2 : x_n \geq 0\}$.

In Section 2, we will talk about the triangulation of surfaces and that every connected 2-manifold is triangulable. Though we will not prove this statement, the need of second countability plays an important role. As Tibor Radó explains in [16], a triangulation of 2-manifolds is possible as soon as the second countability axiom is required.

Example 1.6 The Hausdorff property is not something that follows from other properties like locally Euclidean or connected. For example, consider the space $\mathbb{R} \times \{a\} \cup \mathbb{R} \times \{b\}$ with the equivalence relation $(x, a) \sim (x, b)$ for all $x \neq 0$ and $a \neq b$. This space is clearly locally Euclidean. But the intersection of every open neighborhood of the point $(0, a)$ with an open neighborhood of the point $(0, b)$ will always be non-empty, violating the Hausdorff property.

Now, we have all the pieces to finally define the objects we will study throughout this thesis, namely compact surfaces.

Definition 1.7 A **surface** is a connected 2-dimensional topological manifold.

Notice that also here, some are authors understand a surface to be just a synonym for a 2-manifold. We include connectedness in the definition because of practical reasons, since we will only work with connected 2-manifolds. The best known example of a surface would be the two-dimensional sphere. Other surfaces we will work with in this thesis are the torus and the real projective plane.

Definition 1.8 A topological space is **compact** if for every open cover of this space, there exists a finite subcover.

For the classification theorem we study compact surfaces without boundary, which in some literature are called closed manifolds. Notice that the definition of surface is not the same for all authors of topology related books. Some

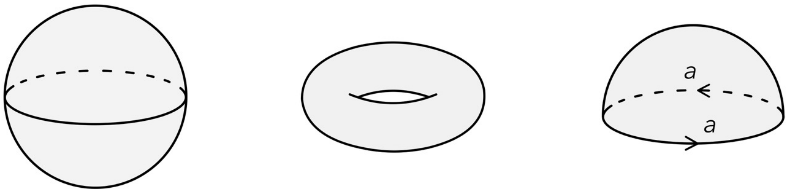

Figure 1.1: The sphere and the torus are orientable surfaces. The real projective plane, represented here as the upper hemisphere with its boundary identified, is a non-orientable surface; see Section 1.2.

authors require a surface to be further orientable and a closed manifold. We will use the above definition and, from here on, the use of the word *surface* in this thesis will implicitly contain the assumption, that the manifold has no boundary. Otherwise, we will explicitly refer to it as a bordered manifold or a manifold with boundary.

An important thing to have in mind is that compactness is a topological invariance. Given two homeomorphic spaces if we know that one of them is compact, then we can immediately conclude, that the second space is also compact.

1.2 Orientability

Orientability will play an important role throughout this thesis. We will divide our objects of study into orientable and non-orientable surfaces. A well-known non-orientable topological space is the Möbius strip, which will serve us to define orientability.

A Möbius strip is constructed by gluing the ends of a rectangular strip with a twist in it (see Figure 1.2).

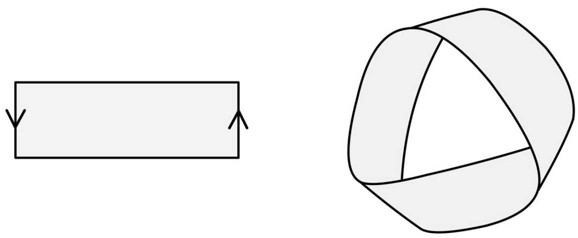

Figure 1.2: A Möbius strip is a non-orientable manifold with boundary.

The center line of the strip will thus form a closed loop. By placing an object on this line, like some arrow perpendicular to it, we notice that after one loop the arrow will point in the contrary direction as at the beginning. Going twice through the loop, the arrow appears with the same orientation as in the beginning. Thus, the Möbius strip is non-orientable, since there exists such an orientation-reversing path. If the orientation of such an arrow is preserved after going through a closed path, then we call it an orientation-preserving path. This motivates the following definition.

Definition 1.9 A surface is **orientable** if every closed path is orientation-preserving. If there exists an orientation-reversing closed path, then the surface is called **non-orientable**.

An equivalent definition would be:

Definition 1.10 A surface is **orientable** if there exists no embedding of a Möbius strip into it. Otherwise the surface is called **non-orientable**.

Notice that, by Definition 1.7, the Möbius strip is not a surface since it has a boundary. We call it a bordered surface.

We will see in Section 3.4 that the real projective plane (defined in Section 2.3.3) contains an embedding of a Möbius strip.

Chapter 2

Triangulation and polygonal presentation

Our goal throughout this thesis is to establish certain rules to manipulate a given surface in order to find another simpler surface, that is homeomorphic to it and which we can easily classify. To achieve this, we want to construct a model of the surface on the $\mathbb{R}^2$-plane, built up by triangles. Gluing the triangles together by their edges in the right way, we get a surface homeomorphic to the surface we started with. This method will be intensely described in Section 4.1.1. There, the surface is going to be identified as the quotient space of a polygonal model in $\mathbb{R}^2$ with a relation on its edges. In this chapter, we want to describe the meaning of *triangulation* and learn how to interpret representations of surfaces by polygons, in particular, those of the sphere, the torus and the real projective plane.

2.1 Triangulation of compact surfaces

Definition 2.1 Let S be a compact surface. A **triangulation** of S is a family of finitely many closed subsets $\{T_1, ...T_n\}$ that cover S and a family of homeomorphisms $\varphi_i : T_i' \longrightarrow T_i$, where T_i' is a triangle in the $\mathbb{R}^2$-plane. The images of vertices resp. edges of any T_i' under φ_i are called *vertices* resp. *edges* of T_i. For this reason the closed sets $T_i \in S$ are also called *triangles*. Moreover, any two distinct triangles on S are either disjoint or they share exactly one vertex or exactly one edge.

An example for the violation of the last condition is presented in Figure 2.1, since the lower triangle on the right intersects the other triangles, but neither exactly in one edge nor at exactly one vertex.

Since every point of a given surface has a locally Euclidean neighborhood, each edge of the triangulation is shared exactly by two distinct triangles.

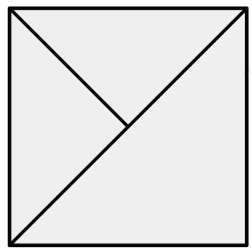

Figure 2.1: This does not represent a triangulation of any surface.

This will be an important fact for the proof of Theorem 4.1. Notice further that triangles sharing the same vertex v can be enumerated in cyclic order $T_0, T_1, ..., T_n = T_0$, s.t. T_i and T_{i-1} share the same edge for every $1 \leq i \leq n$. A set of n triangles around the vertex v is unique since, if there was another set of triangles sharing v, disjoint from the first set, then this would be homeomorphic to two discs glued at their center. Consequently, no neighborhood of this vertex v would be homeomorphic to an open disc in $\mathbb{R}^2$, which violates the definition of a surface.

Figure 2.2 shows an example of a triangulation on a tetrahedron, which we easily recognize as being homeomorphic to a sphere. In this case, a triangulation is naturally given by the shape of the tetrahedron. Passing every triangle to the $\mathbb{R}^2$-plane and gluing them along corresponding edges leads finally to a polygon in the $\mathbb{R}^2$-plane. If the pairs of boundary edges of this polygon are glued back together, then it will give a surface homeomorphic to a sphere.

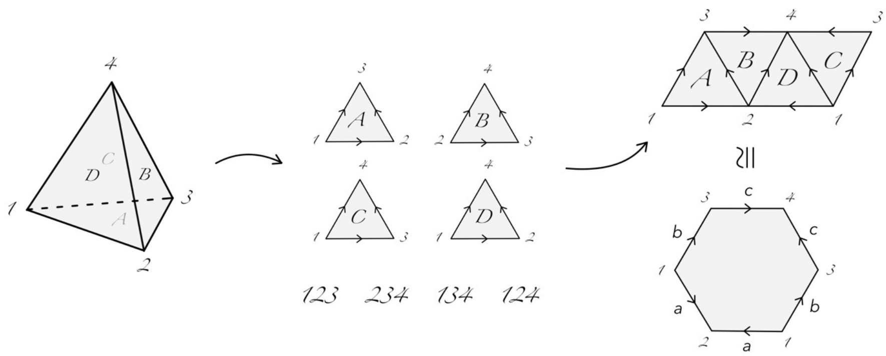

Figure 2.2: Construction of a polygon representing a tetrahedron with boundary edges identified. The cycle $aa^{-1}bcc^{-1}b^{-1}$ represents the boundary.

The exact process for every step is described in Section 4.1.1, as already hinted above. For better understanding, the reader can always take this figure as a reference.

One important question is if every surface or even every n-manifold allows a triangulation. In the beginning of the 20th century mathematicians first thought that every manifold would be triangulable. This was the so called triangulation conjecture. Surprisingly, the classification theorem was first proved by Max Dehn and Poul Heegaard in 1907 assuming that surfaces are triangulable, before the conjecture was proved by Tibor Radó in 1925 for 2-manifolds [8].

Theorem 2.2 *Every two dimensional manifold allows for a triangulation.*

Going through such a technical proof would go beyond the scope of this thesis, so we will skip it, giving [3] or [17] as a reference, the first being a short proof by P. H. Doyle and D. A. Moran relying on Radós proof. The triangulation for 0- and 1-manifolds is, on the other hand, trivially given.

With the development of the knowledge of topological spaces in higher dimensions, the presumption arised, that the triangulation conjecture could be false. Indeed, Michael Freedman constructed in 1982 a 4-manifold for which a few years later Andrew Casson proved not to be triangulable. Thus the triangulation conjecture was proved to be wrong for at least dimension 4 [5]. Three decades after the counterexamples were found, Ciprian Manolescu was able to prove in 2013, that indeed for every dimension $n \geq 4$ there exists a n-manifold that does not allow a triangulation. An interesting fact that can be found in his 'Lectures on the triangulation conjecture" [10] is that all non-triangulable 5-manifolds are non-orientable. This is just some peculiarity of the fifth dimension, in dimension 6 there are also some examples of orientable non-triangulable manifolds. With this, the conjecture was finally disproved and we know today for sure that only up to dimension three every manifold allows for a triangulation (when no further restriction is added). Leaving one last interesting remark with no explanation, we want to highlight, that smooth manifolds are triangulable in every dimension.

We will address the triangulation of a sphere, a torus and a real projective plane in Section 2.3, when we have properly defined and represented these spaces as quotient spaces of some polygon. In the same section, we will give an example in Figure 2.7 of a triangular subdivision on a torus which happens not to be a triangulation. This will show that a naive approach of subdividing a surface into triangles can lead to mistakes.

The triangulation of surfaces is a crucial step for the proof of the classification theorem. Every proof done until now relies on it. First, there is the need for manipulating a polygonal presentation of the surface, which is obtained from the triangulation. Secondly, the triangulation is used to calculate the Euler characteristic, being itself a topological invariant, which we shall see in Section 4.2

2.2 Labelling scheme of polygons

In the previous section we defined a triangulation on a surface. It gives us homeomorphisms from the triangles on the surface to triangles on the plane $\mathbb{R}^2$. Before starting the construction of polygons based on these triangles, we want to be able to represent polygons properly.

When triangulating our surface, we can label every edge of each triangle and give the edges an orientation. The orientation is represented by an arrow on the edge that points from one vertex of the edge (initial vertex) to the second one (terminal vertex). This helps to keep track of the orientation of every edge, when passing to the model of the surface in $\mathbb{R}^2$. Two edges with the same label are identified in a way such that both arrows point in the same direction, matching the initial resp. terminal vertices of both edges.

The boundary of the polygonal model is composed by the edges of the outer triangles, each edge appearing exactly twice as a pair. Thus the boundary is a $2n$-gon and it can be represented symbolically by recording the letters that appear on it. Starting at some vertex, we can note down every label of each successive edge going around the boundary once. An edge a, whose orientation is against the labeling direction, is denoted by its 'inverse' a^{-1}. Notice that going around the polygon clockwise or counterclockwise is indifferent for its labelling scheme and so is the choice of the vertex where the labelling starts. We obtain a symbol that represents the $2n$-sided polygon, called a *labelling scheme* of the form

$$a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \dots a_{i_{2n}}^{\varepsilon_{2n}},$$

where a_{i_k} is the label of the k-th edge with $i_k \in \{1, \dots, n\}$ and $\varepsilon_k \in \{\pm 1\}$ its orientation. This contains every information of the polygon, as the number of edges, their labels and orientations. When the number of edges is small enough, instead of enumerating the edges by $a_1, \dots, a_n$ we denote them by $a, b, c, \dots$ for practical reasons. We also omit the positive exponents. When referring to the polygonal model that represents the surface on the $\mathbb{R}^2$-plane, constructed from the triangulation of the surface, we will call the labelling scheme *polygonal presentation*.

The polygon constructed in Figure 2.2 offers an example for such labelling scheme. Starting at the bottom and going clockwise, we identify this polygon by $aa^{-1}bcc^{-1}b^{-1}$. As we already mentioned, every scheme with different initial vertex or orientation reprsents the same polygon.

In the next chapter, we will see concrete examples of polygonal presentations of surfaces.

2.3 Representation of the standard surfaces

As indicated in the theorem for the classification of surfaces, our main surfaces of study are: the sphere, the torus and the real projective plane. Throughout the next chapters, we will often designate them as *standard surfaces*. In this chapter, we first define them, and then study their polygonal representation.

2.3.1 The sphere

The sphere is represented by the set $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. For its polygonal representation we can first imagine the sphere with a zipper on the surface. When we open the zipper, the sphere looks like an opened purse. This idea can be represented as indicated in Figure 2.3, where the boundary represents the opened zipper.

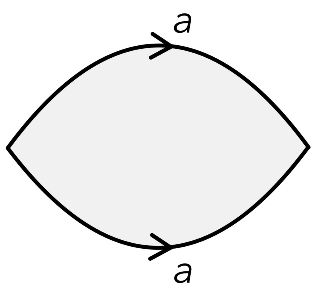

Figure 2.3: Representation of the sphere as a 2-gon by aa^{-1}.

Thus, the sphere is homeomorphic to the quotient space of a polygon with its boundary edges identified in pairs. The labelling scheme is aa^{-1}.

2.3.2 The torus

Let X be the square in the $\mathbb{R}^2$ plane defined by

$$\{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 1\},$$

Then the torus $\mathbb{T}^2$ is defined as the quotient space of X under the following equivalence relation, which is visually shown in Figure 2.4 on the left:

$$(x, 0) \sim (x, 1) \quad \forall x \in [0, 1]$$

$$(0, y) \sim (1, y) \quad \forall y \in [0, 1].$$

This identification can be visualised by the reader by taking a square sheet of paper and folding the edges as indicated by the Figure 2.4 on the left. The polygonal presentation of the torus is given by $aba^{-1}b^{-1}$.

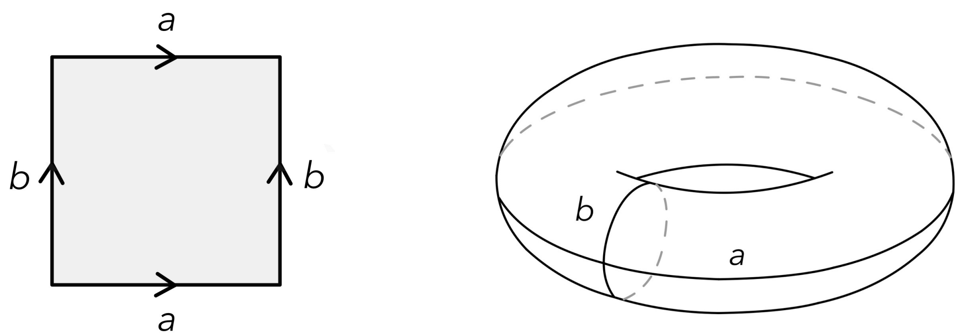

Figure 2.4: Representation of the torus with the labelling scheme $aba^{-1}b^{-1}$.

The torus is homeomorphic to the space $\mathbb{S}^1 \times \mathbb{S}^1$. Furthermore it can be shown to be also homeomorphic to the following set in $\mathbb{R}^3$:

$$\{(x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2}-2\right)^2 + z^2 = 1\}.$$

2.3.3 The projective plane

The real projective plane $\mathbb{R}P^2$ is the image of the quotient map of the sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ under the equivalence:

$$(x,y,z) \sim -(x,y,z) \quad \forall (x,y,z) \in S^2,$$

where two diametrically opposite points on the sphere, also called antipodal points, are being identified.

We observe that the upper hemisphere of $\mathbb{S}^2$, namely $H = \{(x,y,z) \in S^2 : z \geq 0\}$, contains every equivalence class of the relation above. On the interior of H every point represents a pair of two antipodal points on the sphere. On the boundary of H, i.e. the equator of the sphere, we have that $(x,y,0) \sim (-x,-y,0)$. With this observation, the projective plane is homeomorphic to the quotient of H under the relation on the boundary, as shown in the Figure 2.5 on the right. Its labelling scheme is *aa*.

Since H is homeomorphic to a square in $\mathbb{R}^2$, the relation on the boundary of H devolves to the relation on the boundary of the square as indicated in Figure 2.5 on the left, where its labelling scheme is *abab*. In Section 4.1.4, 'making pairs of the second kind adjacent', we will show how the polygonal presentation *abab* is homeomorphic to *aa* by cutting and gluing the polygon in a proper way.

We notice that by identifying every point on a line in $\mathbb{R}^3$ going through the origin, i.e.

$$\forall \lambda \in \mathbb{R} \backslash \{0\} : \quad x \sim \lambda x \quad \forall x \in \mathbb{R}^3 \backslash \{0\},$$

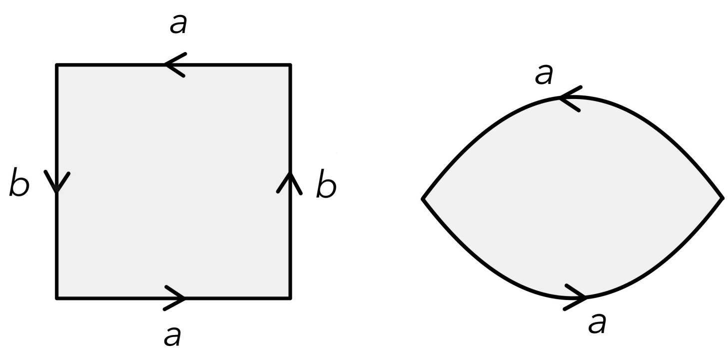

Figure 2.5: Two homeomorphic polygonal representations of the real projective plane by *abab* (left) and *aa* (right).

we get a space homeomorphic to $\mathbb{R}P^2$. Every space homeomorphic to a projective plane will also be denoted by the same name.

2.3.4 Triangulation of the standard surfaces

Now that we know how to represent the standard surfaces as quotient spaces of discs and circles, we can show, how one of many triangulations of these surfaces looks like in Figure 2.6.

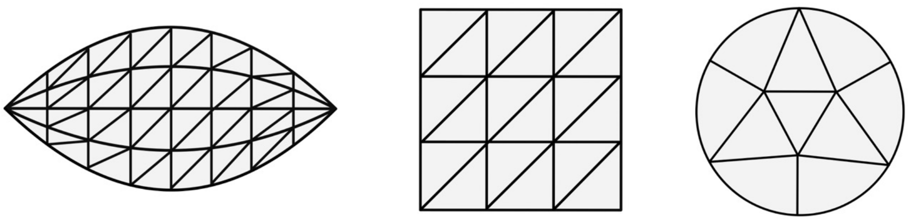

Figure 2.6: Triangulation of the sphere, the torus and the real projective plane.

As promised in the chapter on triangulation, Figure 2.7 shows subdivisions of a torus which are not triangulations. For instance, on the right one, the vertex in the middle of the left boundary edge does not appear in the right boundary edge. By definition of a triangulation this cannot happen, as the bigger right triangle shares one edge and three vertices with the second triangle from on the bottom on the left half. Similarly for the torus on the left, the top left and the bottom right triangle share the same three vertices.

2.4 The Euler characteristic

The Euler characteristic of a surface can be simply calculated from its triangulation. It will give us a very strong criterion to help us to distinguish between

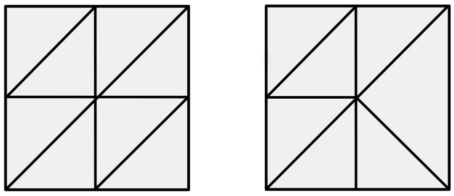

Figure 2.7: These figures do not represent a triangulation of a torus in spite of being a subdivision into triangles.

some, but sadly not all compact surfaces, as we will see in Section 4.2.

Definition 2.3 The **Euler characteristic** of a surface S with triangulation $\{T_1, \dots T_n\}$ is given by the formula

$$\chi(M) = v - e + t,$$

where

v = total number of vertices,

e = total number of edges,

t = total number of triangles of M (in this case, $t = n$).

But will the most complicated triangulation of any surface result in the same Euler characteristic as any other triangulation? The following lemma gives us the answer.

Lemma 2.4 *The Euler characteristic of a surface is independent of the chosen triangulation.*

Proof: Given two different triangulations $\{T_1, \dots, T_m\}$ and $\{T'_1, \dots, T'_n\}$ of a surface S, we want to be able to go from the first triangulation to the second, with a finite number of manipulations. First, we have to set up the rules for allowed manipulations.

This time, we want to allow for subdivisions of the surface into arbitrary polygons, not just triangles, and also subdivisions of the kind represented in 2.8, where edges must not subdivide a region. Moreover, we require the following three things:

1. the interior of a polygon must be homeomorphic to an open disc;
2. the closure of an edge is homeomorphic to a closed interval in $\mathbb{R}$;
3. the number of faces, edges and vertices stays always finite.

Now we allow the following operations:

- Subdivision of an edge into two by placing a vertex in its interior or

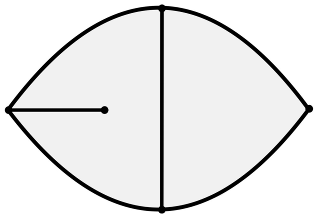

Figure 2.8: Edges ending in the interior of a polygon are allowed.

- removal of a vertex that belongs exactly to two edges, by merging the these edges together (inverse operation).
- Subdivision of a polygon into two sections by connecting two of its boundary vertices with an edge or
- amalgamation of two polygons that share an edge, by removing it (inverse operation).
- Placing a new edge in a region starting at a boundary vertex and ending freely in the interior or
- removal of an edge whose vertex ends in the interior of a polygon (inverse operation).

We observe that the quantity $v - e + f$, where v, e, f are respectively the number of vertices, edges and faces, remains the same after each operation. For example subdividing a region with an edge augments the number of edges and faces respectively by one. Since both are counted with different signs, $v - e + f$ remains unchanged.

Assume now we lay both triangulations $\{T_1, \ldots, T_m\}$ and $\{T'_1, \ldots, T'_n\}$ of the surface S on top of each other. If there are finitely many intersections, then reasonably, we will only need finitely many moves to deform the first triangulation into the second, using the above operations. If there are infinitely many intersections, then we can get around this problem by moving one of edges slightly. Proving this second case is not easy and, like Massey, we will skip it for now. There is a nicer way to prove it using homology groups in Chapter 13 of [8] or Chapter 9 in Masssey's book [12]. But if we believe this to be possible, then the lemma is proved. $\square$

This lemma shows the invariance of the Euler characteristic. Assume we have two homeomorphic spaces X and Y, which allow for a triangulation T_X and respectively T_Y. Then T_X is mapped to a triangulation T'_Y of the second space by the given homeomorphism between the two spaces. Using the preceding lemma, we can transform this triangulation T'_Y into T_Y using the allowed manipulations presented in the proof of the lemma. This proves that the two spaces must have the same Euler characteristic.

As mentioned at the end of the above proof, for those readers involved in the theory of homology, a much simpler proof could be achieved using the following equation for a *CW-complex* X (see Section 6.1.1 for its definition):

$$\chi(X) = \sum_n (-1)^n rank H_n(X).$$

Now, we know that whatever triangulation we happened to choose, it will always give the same Euler characteristic. Leaving things simple, if we want to determine the Euler characteristic of the sphere, the torus or the projective plane, we can use the simplest triangulation possible. We leave the reader to determine the Euler characteristics with help of Figure 2.6 in Section 2.3.4 and we present just the results with the remark, that identified points and edges are counted exactly once.

- Sphere: $\chi(\mathbb{S}^2) = 2$;
- Torus: $\chi(\mathbb{T}^2) = 0$;
- Projective plane: $\chi(\mathbb{R}P^2) = 1$.

Notice that two non-homeomorphic spaces can have the same Euler characteristic. Compare, for example, the real projective plane and one single point in $\mathbb{R}^3$, both having characteristic of 1. The Euler characteristic only reveals its importance when two spaces have different characteristics, making them not homeomorphic, since the Euler characteristic is a topological invariant. In the case of our three main surfaces, they have distinct Euler characteristics, making them non-homeomorphic to each other. But what about their connected sums? This we will prove in Section 3.2.

Chapter 3

Connected sums

We want to understand how two compact surfaces can be glued together forming another compact surface. For this we define the concept of connecting surfaces. Then we study connected sums of the standard surfaces.

Definition 3.1 The **connected sum** $T_1 \# T_2$ of two surfaces T_1 and T_2 is defined by removing a small disc from both surfaces and identifying their boundaries.

More precisely, we define a homeomorphism on the boundaries of the removed discs:

$$h : \partial D_1 \to \partial D_2,$$

which gives an equivalence relation

$$x \sim h(x) \quad \forall x \in \partial D_1.$$

The connected sum of T_1 and T_2 is then described by the quotient of their union under this relation, i.e.

$$T_1 \# T_2 = (T_1 \cup T_2)/\sim .$$

We remark here that the connected sum is an associative and commutative operation. This means that the order which surfaces are connected to each other does not matter.

Further, we acknowledge that the connected sum of two orientable surfaces is again orientable. If one of the surfaces is non-orientable, then the connected sum is also a non-orientable surface. This comes from the definition of orientability, since at least one of the surfaces in the sum contains an embedded Möbius strip.

3.1 Euler characteristic of connected sums

As the Euler characteristic will play an important role in determining whether or not two surfaces are homeomorphic, we study in this chapter, how the Euler characteristic changes, when building connected sums of the standard surfaces.

Proposition 3.2 *Let S_1 and S_2 be compact surfaces. Then*

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

Proof: Assuming that both S_1 and S_2 are triangulated, we determine their Euler characteristics $\chi(S_1)$ and $\chi(S_2)$. The connected sum is built by removing the interior of a disc, in this case of a triangle given by the triangulation, from both surfaces and by identifying their boundaries. Thus from the total amount of triangles of $S_1 \cup S_2$, we remove two faces and identify three vertices, respectively three edges, reducing the number of vertices by three and augmenting the contribution of edges by three. Thus we get:

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 3 + 3 - 2 = \chi(S_1) + \chi(S_2) - 2,$$

which proves the claim. □

3.2 Connected sum of spheres

A sphere, where the interior of a closed disc has been removed, is homeomorphic to the unit disc on the $\mathbb{R}^2$-plane. By identifying the boundaries of two unit discs we get a surface homeomorphic to a sphere. Thus, the labelling scheme is just aa^{-1} independent of the number of connected spheres.

3.2.1 A sphere connected to an arbitrary surface

For the same reason, connecting a sphere to any other surface will lead to the surface itself, since the sphere will fill in the hole created on the surface by the removed disc. Hence, we see how the sphere plays the role of a neutral element in the monoid of compact surfaces up to homeomorphism.

The Euler characteristic of a connected sum of a sphere with any compact surface S is given by:

$$\chi(\mathbb{S}^2 \# S) = \chi(\mathbb{S}^2) + \chi(S) - 2 = \chi(S),$$

using the formula above.

3.3 Connected sum of tori

In order to form the connected sum of two tori, we take from each one the interior of a small disc and designate their boundaries by c resp. c′. Figure 3.1 shows how, after a hole is formed by removing a disc, each torus can be continuously deformed to a pentagon. Finally by identifying c and c′ together, we get an octagonal shape representing the connected sum.

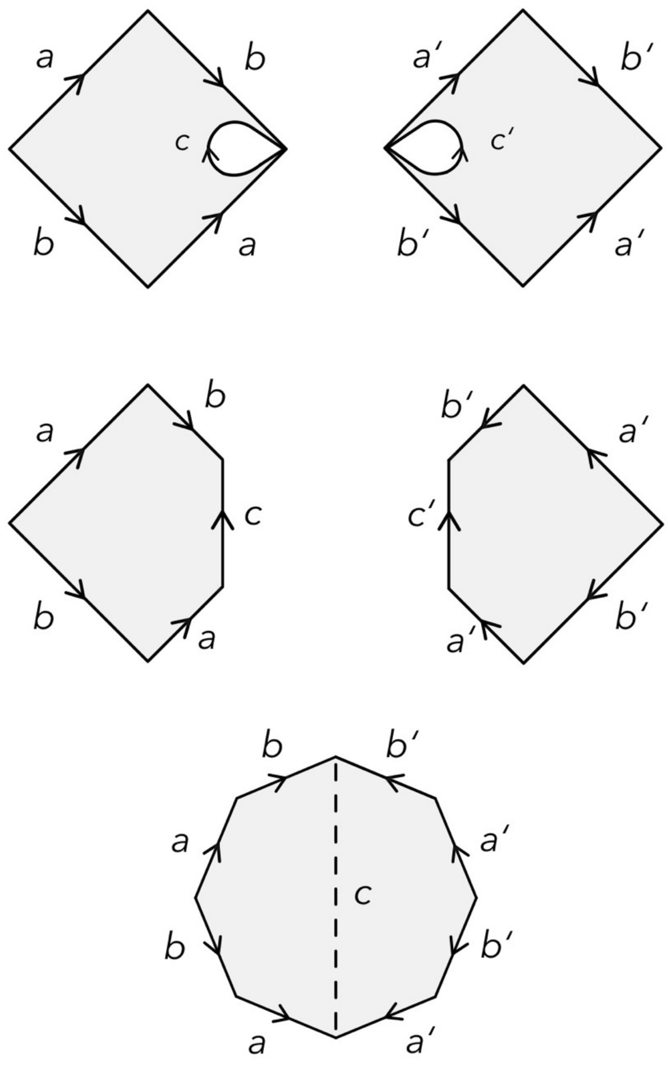

Figure 3.1: Construction of two connected tori.

By removing a disc from each torus, we actually add a new edge to the polygon. But since their boundaries are identified and lay in the interior of the new connected surface, we neither gain nor loose any edges when connecting tori.

Applying this conclusion inductively, we observe that after connecting n tori $T_1\#\cdots\#T_n$, we end up with a polygon with $4n$ edges, i.e. a $4n-$gon. A connected sum of n tori is also called an *orientable surface of genus n*.

The labelling scheme of a polygon representing the connected sum of n tori is given by $a_1b_1a_1^{-1}b_1^{-1}\dots a_nb_na_n^{-1}b_n^{-1}$. The Euler characteristic for $n \geq 2$ is, using the formula in Proposition 3.2, given by:

$$\chi(\mathbb{T}_1^2 \,\#\, \dots \,\#\, \mathbb{T}_n^2) = 2 - 2n \in \{-2, -4, -6, \dots\}$$

3.4 Connected sum of projective planes

As a matter of simplicity we take the labelling scheme aa for the real projective plane as seen in Figure 2.5 on the right. The connected sum of n such spaces can be done in the same manner of that of the torus explained above in 3.3. Then, we immediately see that we end up with a $2n$-gon with the labelling scheme given by $a_1a_1\dots a_na_n$. This connected sum can be called a *non-orientable surface of genus n*.

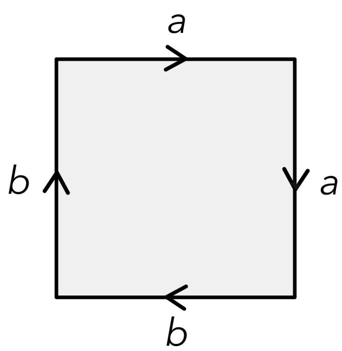

Figure 3.2: Polygon representing the connected sum of two real projective planes with labelling $aabb$.

The Euler characteristic of the connected sum of n projective planes for $n \geq 2$ can be any integer smaller or equal to 0:

$$\chi(\mathbb{R}P_1^2 \,\#\, \dots \,\#\, \mathbb{R}P_n^2) = 2 - n \in \{0, -1, -2, -3 - 4, \dots\}$$

For even numbers of the Euler characteristic, it is impossible to distinguish between a connected sum of tori or a connected sum of projective planes. Thus, we already foresee, that the Euler characteristic will be an important criterion to classify surfaces, reveling its worth, when combined with another criterion, which solves this mentioned issue.

3.5 The Möbius strip in the projective plane

When connecting a real projective plane to a surface, we first remove the interior a disc out. Surprisingly, a projective plane with a removed disc is homeomorphic to a well-known topological object: the Möbius strip. We leave the reader to discover this interesting fact with help of Figure 3.3.

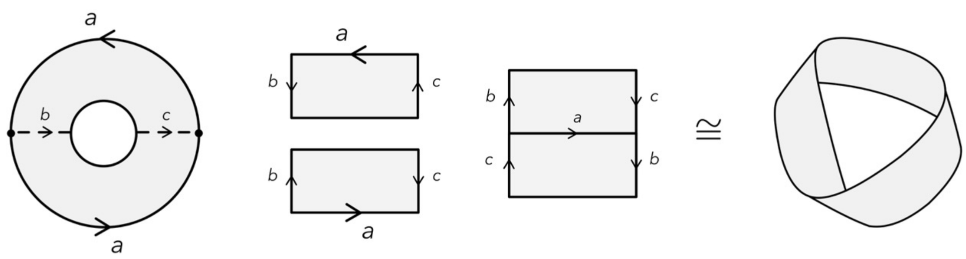

Figure 3.3: The real projective plane with a removed disc is homeomorphic to the Möbius strip.

This also shows that a Möbius strip is embedded in the real projective plane, making it a non-orientable surface by Definition 1.10.

3.6 The Klein bottle or two connected projective planes

From the previous paragraph we observe that the connected sum of two projective planes is the sum of two Möbius strips. How do two Möbius strips glued together look like? As for the real projective plane, there is no embedding in $\mathbb{R}^3$, thus we expect the surface to have self-intersections or singularities.

Amazingly, there is a nice visualisation of this surface that proudly carries the name of its discoverer, namely the Klein Bottle. The Klein Bottle is constructed similarly to the torus, but with one edge orientation reversed, as it can be observed in Figure 3.4. Its polygon is represented by $aba^{-1}b$ instead of $aba^{-1}b^{-1}$ as for the torus. First, two opposite edges of a square are identified, thus forming a cylinder. Then, one end will have a reversed orientation relative to the other. In order to visualise this surface in a three dimensional world, we allow one of the ends to go through the surface, creating an intersection, such that its orientation matches with the other end (see first part of Figure 3.5).

Now, what does a Klein bottle have to do with projective planes and Möbius strips? Well, cutting a Klein bottle in half, as represented in Figure 3.5, shows that both halves are indeed homeomorphic to Möbius strips, where the boundaries generated by the cut are the boundaries of Möbius strips. Thus, a Klein bottle is a sum of two connected real projective planes, as it

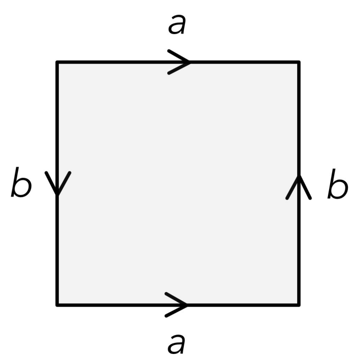

Figure 3.4: The Klein bottle is represented by $aba^{-1}b$.

is a sum of two Möbius strips. Notice that Figure 3.5 of the Klein bottle is just an immersion with nice intersections, as no embedding in $\mathbb{R}^3$ would be possible.

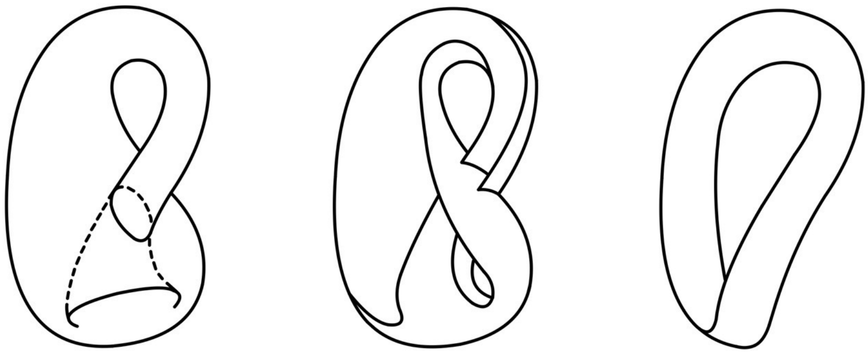

Figure 3.5: The Klein bottle (left) cut in half (middle) is homeomorphic to a Möbius strip (right).

If this visual proof of two projective planes being homeomorphic to a Klein bottle is not so appealing for the reader, then we warmly recommend to take a look to the approach taken in Chapter 4 of [2]. Starting from the sum of two projective planes with the polygonal presentation *ababcdcd*, by cutting and gluing the polygon we get the presentation $abab^{-1}$ of the Klein bottle.

3.7 A torus connected to a projective plane

As we mentioned previously, a surface is homeomorphic to exactly one of the three classes of the classification theorem. It is quite intriguing to which of such classes a torus connected to a projective plane belongs to.

Lemma 3.3 *The connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.*

Proof: In this proof, we want to show the following isomorphisms:

$$T^2 \# \mathbb{R}P^2 \overset{(1)}{\cong} K^2 \# \mathbb{R}P^2 \overset{(2)}{\cong} \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$$

Isomorphism (2) arises due to the fact that a Klein bottle $\mathbb{K}^2$ is homeomorphic to two connected projective planes, as seen in Section 3.4

We know that, after a disc is removed from the real projective plane (in particular when building a connecting sum), this space becomes homeomorphic to a Möbius strip $\mathbb{M}$. Thus isomorphism (1) will follow easily from a much simpler isomorphism:

$$\mathbb{T}^2 \# \mathbb{M} \overset{(3)}{\cong} \mathbb{K}^2 \# \mathbb{M}$$

For this isomorphism (3), we want to first construct $\mathbb{T}^2 \# \mathbb{M}$ and $\mathbb{K}^2 \# \mathbb{M}$ separately. To build the connected sum of a torus with a Möbius strip we remove the interior of a disc from the torus and cut it into two sections along the edge c. The section with the removed disc is marked with (II) as indicated in Figure 3.6 on the left.

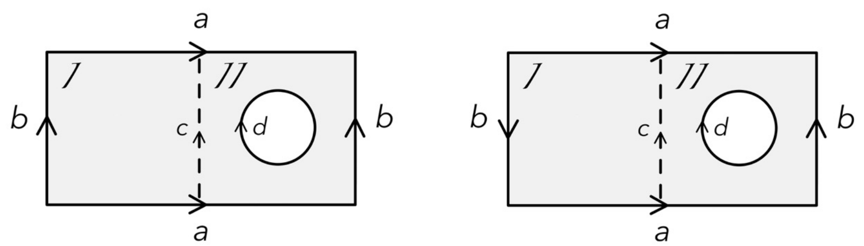

Figure 3.6: Torus (left) and Klein bottle (right), where the interior of a disc with boundary b is removed.

We first connect region II (separately from region I) to a Möbius strip. Notice that region II is homeomorphic to a cylinder with a removed disc. The boundary of the disc is identified with that of a removed disc of the Möbius strip. A cylinder is homeomorphic to a sphere with two holes. We already know that a sphere connected to a surface is just homeomorphic to the surface itself. Thus a cylinder connected to a surface is homeomorphic to the surface with two holes (see Figure 3.7, where the dark shaded discs represent removed discs from the surfaces). We conclude that the connection of region II to the Möbius strip is just the Möbius strip perforated twice.

Region (I) is also homeomorphic to a cylinder. We identify region I with this perforated Möbius strip by the boundaries of the cylinder and get a surface as in Figure 3.8 on the left.

The same procedure can be done for the Klein bottle with a removed disc (Figure 3.6, right). Again, we obtain a twice perforated Möbius strip (see

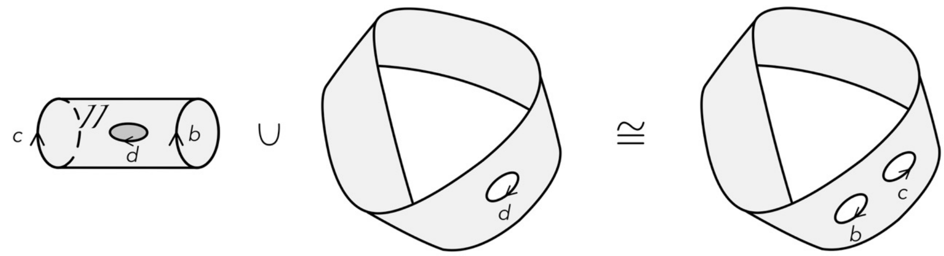

Figure 3.7: Region II attached to the Möbius strip along d, leading to a Möbius strip with two holes.

Figure 3.7). Because two opposite boundaries of region I have reversed orientation, attaching this region to the Möbius strip has to be done more carefully, as represented in the Figure 3.8 on the right.

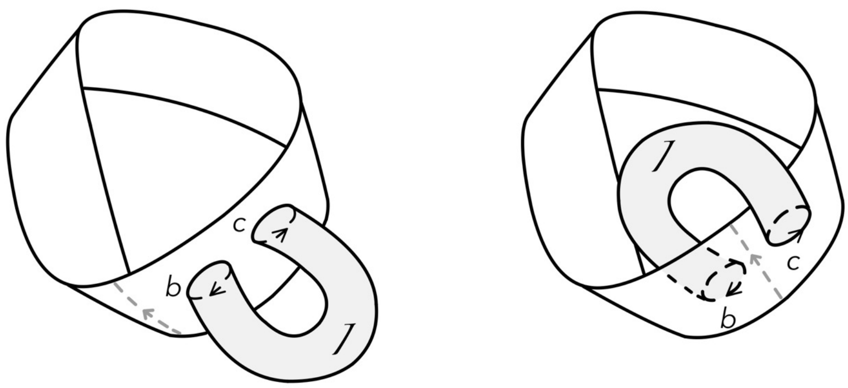

Figure 3.8: Region I of the torus (left) resp. Klein bottle (right) attached to a Möbius strip along the corresponding boundaries.

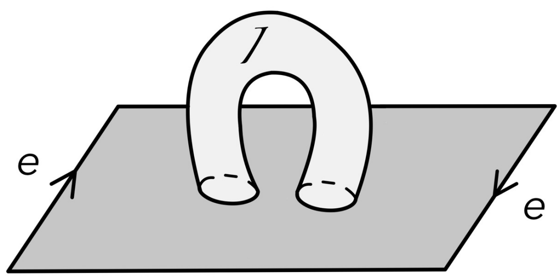

Figure 3.9: Illustration of the cut made in Figure 3.8.

Now, it only remains to show that both spaces $\mathbb{T}^2 \# \mathbb{M}$ and $\mathbb{K}^2 \# \mathbb{M}$ are indeed homeomorphic. This follows immediately by doing a smart cut on each Möbius strip as indicated by the gray dashed line in Figure 3.8. With

Figure 3.9, we see that both are homeomorphic to a torus connected to a rectangle, which has its boundary identified with a twist. Thus, we get isomorphism (3).

As stated previously, isomorphism (1) is proved by attaching a disc along the boundary of the Möbius strips of both $\mathbb{T}^2 \# \mathbb{M}$ and $\mathbb{K}^2 \# \mathbb{M}$. This proves the lemma. $\square$

We conclude from this lemma that every connected sum of n tori with m projective planes is homeomorphic to a connected sum of $2n + m$ projective planes (for every torus we get two additional projective planes).

$$\underbrace{\mathbb{T}^2 \# \ldots \# \mathbb{T}^2}_{\text{n times}} \# \mathbb{R}P^2 \cong \underset{2\cdot n+1}{\#} \mathbb{R}P^2.$$

$$\underbrace{\mathbb{T}^2 \# \ldots \# \mathbb{T}^2}_{\text{n times}} \# \mathbb{K}^2 \cong \underset{2\cdot n+2}{\#} \mathbb{R}P^2.$$

$$\underbrace{\mathbb{T}^2 \# \ldots \# \mathbb{T}^2}_{\text{n times}} \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{\text{m times}} \cong \underset{2\cdot n+m}{\#} \mathbb{R}P^2.$$

We use this result to calculate the Euler characteristics of a connected sum of a projective plane and n tori for $n \geq 1$:

$$\chi(\mathbb{R}P^2 \# \mathbb{T}^2_1 \# \ldots \# \mathbb{T}^2_n) = 1 - 2n \in \{-1, -3, -5, -7, \ldots\},$$

and for a connected sum of a Klein bottle (or 2 connected projective planes) and n tori for $n \geq 1$:

$$\chi(\mathbb{K}^2 \# \mathbb{T}^2_1 \# \ldots \# \mathbb{T}^2_n) = -2n \in \{-2, -4, -6, -8, \ldots\}$$

Every negative integer represents the Euler characteristic of some connected sum of tori and projective planes. Depending if the number of projective planes is odd or even, we get an odd resp. an even number for the Euler characteristic.

3.8 The monoid of compact surfaces up to homeomorphism

The compact surfaces up to homeomorphism form a commutative monoid with the operation given by the connected sum. The sphere is, as we know from Section 3.2.1, the neutral element. From the classification theorem, we conclude that the torus and the real projective plane generate this monoid with the relation $T^2 \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

Visually, we can represent the monoid as in Figure 3.10. The left column together with the sphere represents all orientable compact surfaces and the right column represents all non-orientable compact surfaces.

As a final remark, note that, with the knowledge from the two preceding chapters, we can give an equivalent formulation of the classification theorem:

> *Let S be a compact surface. If S is orientable, then it is homeomorphic to a sphere connected to n tori for $n \geq 0$. If S is non-orientable, it is homeomorphic to either a real projective plane or to a Klein Bottle connected to n tori for $n \geq 0$.*

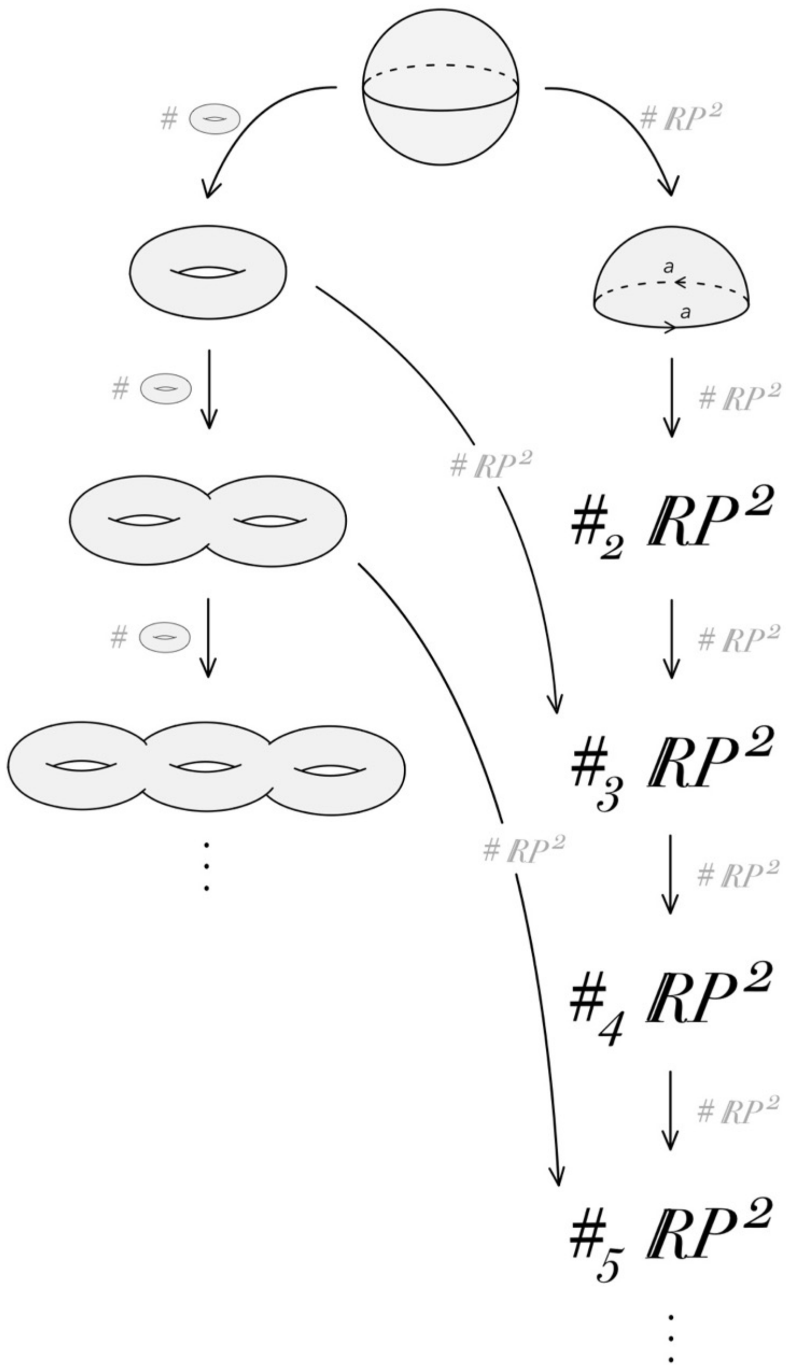

Figure 3.10: Visualisation of different possible connected sums of the standard surfaces given by the classification theorem.

Chapter 4

Classification of surfaces

We finally have all the necessary tools to prove the theorem of the classification of surfaces.

Theorem 4.1 *Every compact surface is homeomorphic to either a sphere, or to a connected sum of tori, or to a connected sum of projective planes.*

Sometimes we will refer to these three cases as *classes*: the class of the sphere, of the connected tori and of the connected projective planes. Further, when we say that a surface *belongs* to some class, we mean that it is homeomorphic to some surface in one of the classes. In the previous chapters we denoted the generators of the monoid of compact surfaces up to homeomorphism as standard surfaces. We extend this designation also for connected sums of standard surfaces, when we refer to surfaces presented in the form given by the classification theorem.

The theorem can be divided into two statements, which we are going to prove separately in part I (Section 4.1) and part II (Section 4.2). First, the theorem states that every compact surface belongs to one of the three classes. For this we will develop an algorithm to deform the polygonal presentation of a surface, until we get a presentation in the form of the standard surfaces. Notice that connected sums between surfaces of different classes will belong again to one of the classes. As seen in Section 3.7, tori connected to projective planes land in the class of connected projective planes. Secondly, the theorem makes a stronger requirement, namely that every surface is homeomorphic to one surface, contained in *exactly* one of the classes. For this, we need to find some topological invariants, meaning some identities that remain the same for every homeomorphic surface. As we will see, the Euler characteristic and orientability will be sufficient to prove this second part.

4.1 Part I: Classification of a surface

Let S be a given compact surface. Our goal is to show that S is the quotient space of some polygon in the euclidean plane with identifications on its boundary edges. Then, we want to be able to manipulate the polygon in certain allowed ways, such that the class to which the surface is homeomorphic to, reveals itself, when gluing the boundary edges according to their identifications. For this we divide this proof in 5 steps:

1. Construction of a polygonal presentation of the surface;
2. Elimination of adjacent edges of the first kind;
3. Identification to a single vertex;
4. Making pairs of the second kind adjacent;
5. Transforming pairs of the first kind into adjacent groups.

Before continuing the proof, we want to name the two possible kind of pairs that are contained in the boundary of a polygonal model of a surface:

1. Pairs of the first kind are of the form aa^{-1} (or $a^{-1}a$);
2. Pairs of the second kind are of the form aa (or $a^{-1}a^{-1}$).

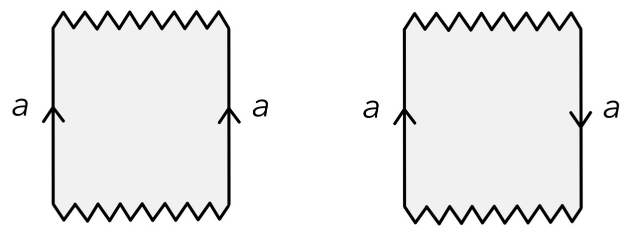

Figure 4.1: Pairs of the first kind (left) and pairs of the second kind (right).

After every step, we want analyse if the polygon fits the classification theorem, before continuing with the next step. The polygonal presentation should finally take one of the standard forms below in order to be categorized:

- A sphere: aa;
- Connected sum of n tori: $a_1b_1a_1^{-1}b_1^{-1}\dots a_nb_na_n^{-1}b_n^{-1}$;
- Connected sum of n real projective planes: $a_1a_1\dots a_na_n$;

4.1.1 Step 1: Polygonal presentation of the surface

First, we assume that S is triangulated by n triangles, which is possible by Theorem 2.2. Then, we enumerate every triangle T_i for $i = 1, \dots, n$ following an ordering given by the rule below:

> Every triangle $T_k \subseteq S$ for $k = 2, \ldots, n$ shares at least one edge, denoted by e_k, with some preceding triangle, i.e. with some T_i for $i = 1, ..., k-1$.

This 'enumeration rule' is always fulfilled in the process of enumerating the triangles. Otherwise, our surface would split into two disjoint non-empty closed sets, which contradicts the assumption that a surface is connected.

Now given the triangles $T_1, \ldots T_n$ and the shared edges $e_2, \ldots e_n$, we start to construct the model of the surface on the euclidean space. To every triangle $T_i \subseteq S$ we assign some triangle $T_i' \subseteq \mathbb{R}^2$ and a homeomorphism $h_i : T_i' \to T_i$, which is given by the triangulation. We do this for every triangle, while being careful, such that the assigned triangles T_i' do not intersect each other. If so, then we can translate them on the $\mathbb{R}^2$-plane.

Each triangle T_i' is a closed and bounded set in $\mathbb{R}^2$. Hence, the union of finitely many such sets is the compact set:

$$T' = \bigcup_{i=1}^{n} T_i'.$$

We define the map

$$\varphi : T' \to S$$

such that

$$\varphi|_{T_i'} = h_i$$

and study its properties.

Claim: *φ is a continuous and surjective map.*

Proof: The surface S is locally Euclidean. Thus for every point $x \in S$ it exists an open ball $B \subseteq S$ with $x \in B$. Every intersection of this ball with any triangle $T_i \in S$ will be an open set for $i = 1, \ldots, n$. We see that the preimage of B under φ is open as the union of open sets (notice that h_i is an isomorphism for every $i = 1, \ldots, n$).

$$\varphi^{-1}(B) = \bigcup_{i=1}^{n} h_i^{-1}(B \cap T_i).$$

This proves that φ is continuous. Surjectivity follows again automatically from the definition of φ. □

Claim: *φ is a closed map.*

Proof: Let $C \subseteq T'$ be a closed subset of the compact set T'. It follows that C is itself compact and its image under a continuous map is again compact (see [7], page 26). Hence $\varphi(C)$ is a compact subset of the Hausdorff space S. Thus we conclude that $\varphi(C)$ is a closed set (see [7], page 28). □

Claim: *S has the quotient topology determined by φ.*

Proof: We need to show that φ is a surjective quotient map. Surjectivity of φ follows from the first claim above. According to the definition of a quotient map, we want to show that a set $V \subseteq S$ is closed if and only if the preimage $\varphi^{-1}(V)$ is closed. Assuming that $V \subseteq S$ is closed, then by continuity of φ, the set $\varphi^{-1}(V)$ is closed. On the other hand, if $\varphi^{-1}(V)$ is closed, then, since φ is a closed map, it will map this set to a closed set. By surjectivity, we get $\varphi(\varphi^{-1}(V)) = V$. Thus V is closed and we conclude the claim. $\square$

An equivalence relation arises naturally from the identification of the preimages of the shared edges $e_2, \ldots e_n$, chosen through the enumeration rule. Notice that, from the triangulation of S and the enumeration rule, it follows that every edge e_i for some $i \in \{2, \ldots, n\}$ is shared exactly by two distinct triangles T_i and T_j on S for some $j \in \{1, \ldots, i-1\}$. Hence $\varphi^{-1}(e_i)$ is an edge of both the triangles T'_i and T'_j in T'. By identifying these two edges and doing this for every edge $e_2, \ldots, e_n$, we can build the quotient space D, which corresponds intuitively to gluing the triangles in $\mathbb{R}^2$ together. Each edge on the boundary appears twice as a pair. With one final claim, we can show that this space D is in fact the polygonal model we were looking for.

Claim: *The space D is homeomorphic to a closed disc in $\mathbb{R}^2$.*

Proof: The triangles $T'_1 \subseteq T'$ and $T'_2 \subseteq T'$ from above are disjoint and both homeomorphic to a closed disc $\mathbb{D}^2$. By the enumeration rule there exist edges $d' \in T'_1$ and $e' \in T'_2$, such that $\varphi(d') = \varphi(e') = e_2$. Notice that d' and e' are both homeomorphic to the interval $[0, 1]$. We can define a homeomorphism $h : T'_1 \to T'_2$ such that $h(d') = e'$. Then, we make the identification $x \sim h(x)$ for every $x \in d'$. It follows that $(T'_1 \cup T'_2)/\sim$ is homeomorphic to a closed disc $\mathbb{D}^2$.

Then we proceed by attaching T'_3 to this space. Again by the same argument

$$(((T'_1 \cup T'_2)/\sim) \cup T'_3)/\sim \;\cong \mathbb{D}^2,$$

since T'_3 has an edge e'_3, whose image is the same as the image of an edge (neither d' nor e') of one of the previous triangles T'_1 or T'_2.

The enumeration rule enables to successively repeat this process by identifying edges of every new triangle to one contained in the union of the preceding triangles. This proves the claim. $\square$

With the equivalence relations given on the edges of the triangles, the map $\varphi : T' \to S$ induces a map $\psi : D \to S$. Then S has the quotient topology induced by ψ. Thus, we can view S has the quotient space of a polygon D with its edges identified.

Now that we have the polygonal presentation of the surface, our goal is to manipulate the polygon in allowed ways, such that we can finally categorize it into one of the three classes.

4.1.2 Step 2: Elimination of adjacent edges of the first kind

Adjacent pairs of the first kind can be immediately eliminated by gluing them together as indicated in Figure 4.2. This is certainly possible, since the boundary must contain more than two edges, otherwise we would have been able to finish the classification process in step 1. If we have an adjacent pair of the first kind, where an edge and its inverse appear in succession as represented by the scheme $a_1 \ldots a_{k-1} a_k a_k^{-1} a_{k+1} \ldots a_n$, then we can eliminate the pair $a_k a_k^{-1}$ and remain with $a_1 \ldots a_{k-1} a_{k+1} \ldots a_n$.

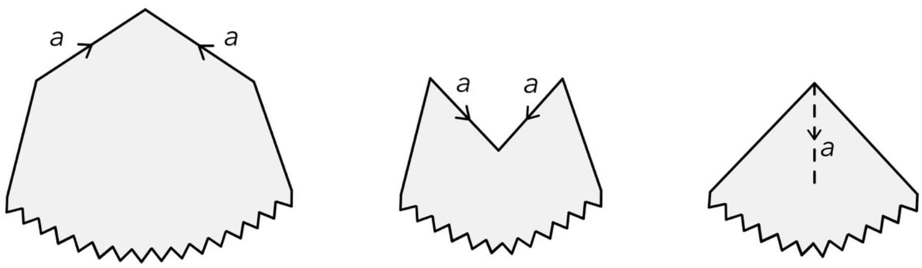

Figure 4.2: Elimination of pairs of the first kind.

We eliminate every adjacent pair of the first kind, until the polygonal presentation consists either of exactly one single pair (aa^{-1} or aa) or until all remaining pairs consist of non-adjacent pairs of the first kind and pairs of the second kind. If the remaining boundary consists of only two edges, where its presentation is of the form aa^{-1} (i.e. first kind) or aa (i.e. second kind), then the surface is homeomorphic to a sphere resp. a connected plane. Otherwise we proceed to apply step 3.

In the example of the tetrahedron in Figure 2.2, we ended up with the polygonal presentation $aa^{-1}bcc^{-1}b^{-1}$. The pairs aa^{-1} and cc^{-1} are adjacent pairs of the first kind. After eliminating them, we end up with bb^{-1}. Thus the tetrahedron is isomorphic to a sphere, which matches also our intuition.

4.1.3 Step 3: Identification to a single vertex

This step is the most ingenious one. The goal here is to identify all vertices of the polygon into a single one. Even if the purpose of this step is a priori not clear, it will turn out to be very important for step 4.

Each edge has a start and end point, which may differ. Assume the edge

a starts at P and ends at Q. If P is not to be identified with Q, then they belong to different equivalence classes of vertices $[P]$ and $[Q]$. The number of vertices contained in each equivalence class depends on the identifications made on the model of the surface and on the number of existent triangles. Our plan is to eliminate all but one equivalence class of vertices. To simplify the understanding of this process, we recommend the reader to follow the steps explained below based on Figure 4.3.

Given a polygonal presentation that has passed through the manipulation process in step 2, we assume that there are at least two different equivalence classes of vertices and denote a to be an edge belonging to two different classes of vertices $[P]$ and $[Q]$. Let b be an adjacent edge that shares the vertex $[P]$ with the edge a. We can assume that the edge b is unequal a, since the polygon went already through step 2, without revealing its form in terms of standard surfaces.

We want to eliminate one vertex of the equivalence class $[P]$. This is achieved by making a cut going from Q to the vertex of the adjacent edge b, which is not shared by the edge a. We designate the new emerging edge by c. We know that on the boundary of the polygon the edge b appears again somewhere, and we glue the piece we cut out to this second representative edge of b, immerging it in the interior of the polygon.

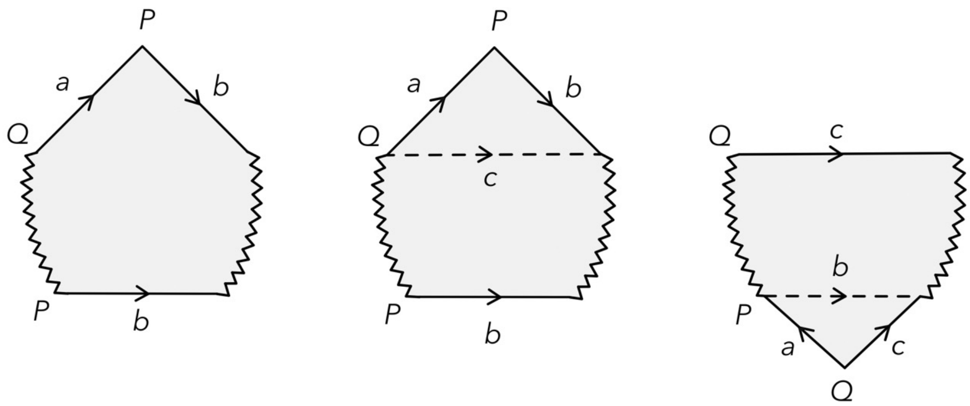

Figure 4.3: Visualisation of step 3.

Now, we have reduced the number of vertices in the class $[P]$ by one and increased $[Q]$ by one and we get new edge identifications on the boundary. Thus we apply step 2 again, in order to eliminate any adjacent pairs of the first kind, that may have been created.

We repeat the same procedure until the equivalence class of $[P]$ is eliminated, and continue afterwards with eliminating all equivalence classes of vertices, until all vertices belong to exactly one class. Finally, we apply again step 2 to ensure that no adjacent edges of the first kind exist.

Notice that a surface represented by aa^{-1} has exactly two different vertices, which cannot be reduced to one. But if we would have such a surface, then the classification process would have terminated before reaching step 3.

If we still cannot classify the surface by now, we continue with step 4.

4.1.4 Step 4: Making pairs of the second kind adjacent

Now our polygon looks quite different from the beginning. We reduced all equivalence classes of vertices to exactly one, and there are no adjacent pairs of the first kind. Still the labelling scheme may be a wild sequence of non-adjacent pairs of edges.

In this step, we want to make any pairs of the second kind adjacent. The reason for this is that we know that the sum of projective planes have the polygonal presentation $a_1a_1 \dots a_na_n$, where all pairs are adjacent pairs of the second kind. Hence this step could be seen as a seek for a sum of projective planes.

Following Figure 4.4, we first identify some pair aa of edges of the second kind. If they are adjacent, then we are finished. Otherwise we make a cut connecting both end points of the edges. Then we glue the two pieces together through the edge a, transforming the non-adjacent pair aa into an adjacent one bb.

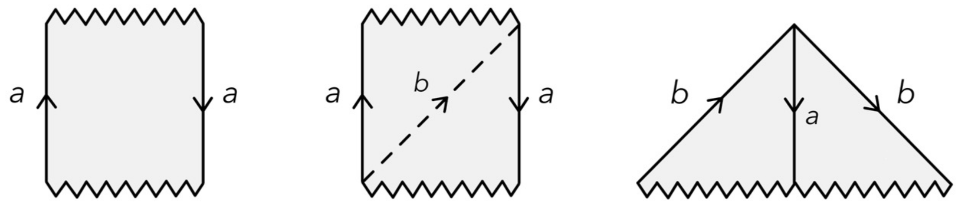

Figure 4.4: Visualisation of step 4.

This step must be repeated until all edges of the second kind are adjacent. At the end, we analyse the remaining polygon.

If we have no pairs of the first kind, then the polygonal presentation is of the form: $a_1a_1a_2a_2 \dots a_na_n$. We conclude that the surface is homeomorphic to a connected sum of projective planes.

If there is one pair of the first kind, then it implies directly the existence of at least a second pair of the first kind, that appears alternately to the first pair, such that the polygonal presentation is of the form $\dots a \dots b \dots a^{-1} \dots b^{-1} \dots$. The reason for the existence of such pairs has to do with step 3. Assume by contradiction that no such pair appears alternately to the first pair as represented in Figure 4.5. Then the edge a separates the boundary of the polygon into two disjoint sets of edges A and B, where every pair of edges appears solely as a pair in A or solely in B. In step 3, we identified every vertex into one class. Thus the set A and the set B meet at one vertex, but apart from that, they are disjoint. At such vertex, the surface would locally be homeomorphic to two discs connected to each other only at their centers. But this contradicts the definition of a surface, which must be locally homeomorphic to a disc. Thus, the labelling scheme must be of the above form. If the surface is not classifiable by now, we continue with step 5.

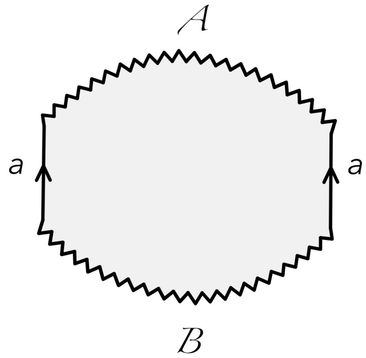

Figure 4.5: A pair of edges of the first kind cannot separate the boundary into two sections that do not share another pair of the first kind.

4.1.5 Step 5: Transforming pairs of the first kind into adjacent groups

All that remains after step 4 is a polygonal presentation of the kind

$$\ldots a \ldots b \ldots a^{-1} \ldots b^{-1} \ldots$$

containing at least two pairs of edges of the first kind. This presentation suggests the existence of a torus and it is exactly the aim of this step: to identify connected sums of tori.

Starting with $\ldots a \ldots b \ldots a^{-1} \ldots b^{-1}$ as represented in Figure 4.6, we make a cut c going through the end points of the edges a and a^{-1} and glue it along the edge b. This allows a connection of the first pair through an edge. Now, we want to connect the second pair. For this we do a second cut d from the end vertex of the edge c to the corner formed by $\ldots ac \ldots$. Gluing this piece along edge a, we finally get the polygonal presentation $\ldots cdc^{-1}d^{-1} \ldots$, which represents a connected sum with one torus. We repeat this step as long as necessary, until all tori are identified.

If there are no pairs of the second kind in the presentation, then the surface is homeomorphic to a connected sum of tori. Otherwise, at some position

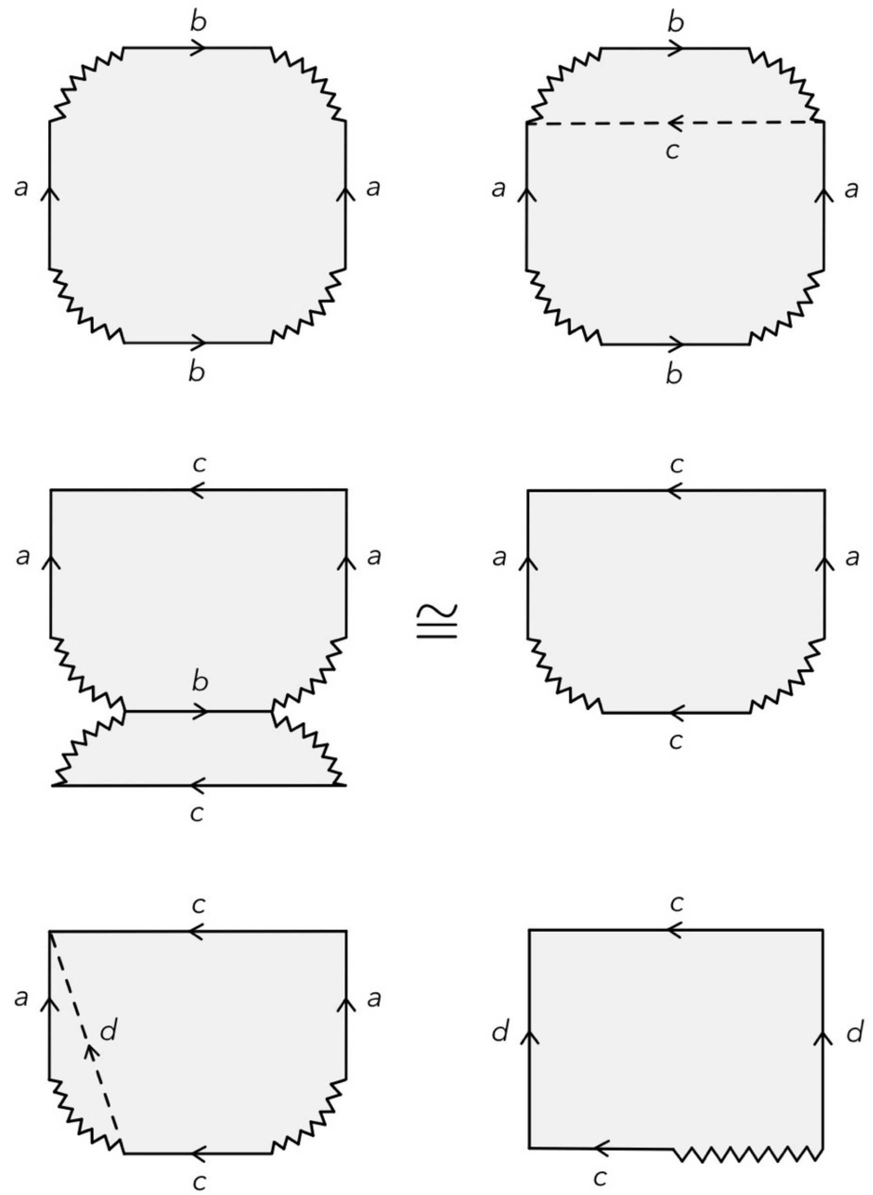

Figure 4.6: Visualisation of step 5.

of the polygonal presentation, a pair of the first kind will meet a pair of the second kind. With Section 3.7, we know that this is homeomorphic to three pairs of the second kind, allowing us to change the polygonal presentation. Every appearance of a connected sum with a torus near a projective plane can be transformed into two projective planes, and we see that the surface is a connected sum of projective planes (see Figure 4.7). This concludes part I of the proof of Theorem 4.1.

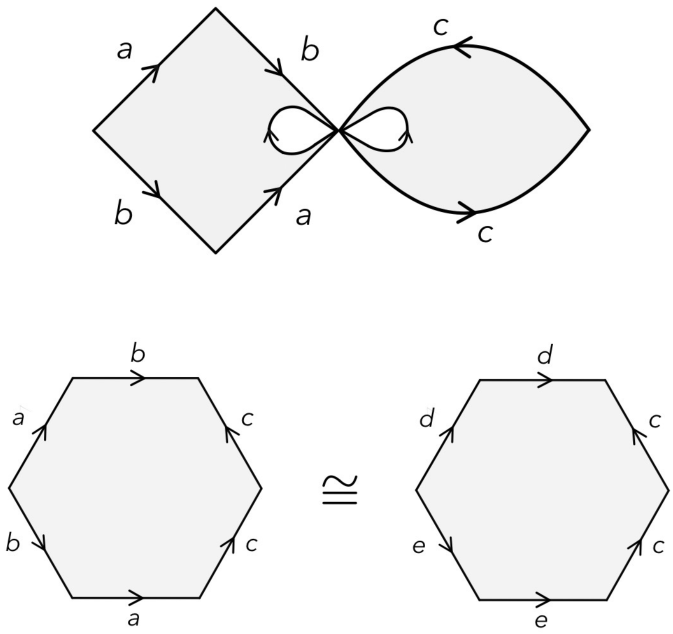

Figure 4.7: A torus connected to a projective plane (top) has a polygonal presentation as in bottom left, which is homeomorphic to the presentation as in bottom right.

4.2 Part II: Topological invariants

In part I, we proved that every compact surface can be put in one of the three classes, but it is still not clear that the classification is unique, that for example a sum of projective planes is not homeomorphic to some sum of tori. We also do not know if the algorithm in part I will always deliver the same result, when starting with another polygonal presentation for the same surface. For this, we want to find criteria that remain invariant under all the applied transformations.

The Euler characteristic is the first important topological invariant. In Section 3, we calculated the Euler characteristic for connected sums of spheres, tori and projective planes, which can be summarized in the following proposition.

Proposition 4.2 *The Euler characteristic of a compact surface S is given by*

- 2, *if* $S \cong \mathbb{S}^2$,
- $2-2n$, *if* $S \cong \#_n \mathbb{T}^2$,
- $2-n$, *if* $S \cong \#_n \mathbb{R}P^2$,

where $n > 0$ is the number of connected sums.

In Section 3.3 and 3.4 we made the following computations for $n \geq 1$.

$$\chi(\mathbb{T}_1^2 \# \cdots \# \mathbb{T}_n^2) = 2 - 2n \in \{0, -2, -4, -6, \dots\}$$

$$\chi(\mathbb{R}P_1^2 \,\#\, \cdots \,\#\, \mathbb{R}P_n^2) = 2 - n \in \{1, 0, -1, -2, -3 - 4, \dots\}$$

$$\begin{aligned}\chi(\mathbb{R}P^2 \,\#\, \mathbb{T}_1^2 \,\#\, \cdots \,\#\, \mathbb{T}_n^2) &= \chi(\mathbb{R}P_1^2 \,\#\, \cdots \,\#\, \mathbb{R}P_{2n+1}^2) \\ &= 1 - 2n \in \{-1, -3, -5, -7, \dots\}\end{aligned}$$

$$\begin{aligned}\chi(\mathbb{K}^2 \,\#\, \mathbb{T}_1^2 \,\#\, \cdots \,\#\, \mathbb{T}_n^2) &= \chi(\mathbb{R}P_1^2 \,\#\, \cdots \,\#\, \mathbb{R}P_{2n+2}^2) \\ &= -2n \in \{-2, -4, -6, -8, \dots\}\end{aligned}$$

The first observation is that connected sums of projective planes can be partitioned. Either the sum is homeomorphic to:

- the connected sum of tori with one projective plane if the Euler characteristic is odd, or
- the connected sum of tori with two projective planes (or a Klein bottle) if the Euler characteristic is even.

The second observation is that, given any even number for the Euler characteristic smaller than 0, it is impossible to distinguish if we are dealing with connected sums of projective planes or of tori.

The Euler characteristic can help us to distinguish between compact surfaces, but in most cases, it will be insufficient. That is why we need a second criterion, which arises naturally from observing the main difference between a torus and a projective plane, namely their orientability.

We can divide compact surfaces into orientable and non-orientable ones. The sphere and sum of tori we know to be orientable, and connected sums of surfaces with a real projective plane are always non-orientable.

Orientability being a topological invariant gives us a second criteria to distinguish surfaces. Together with the Euler characteristic, these two criteria are sufficient to classify compact surfaces, which we can represent in the following scheme.

$$\chi(S) \text{ is } \begin{cases} \text{even,} & \begin{cases} \text{S is orientable,} & \begin{cases} \chi(S) = 2: \\ \quad S \cong \mathbb{S}^2 \\ \chi(S) \neq 2: \\ \quad S \cong \#_t \, \mathbb{T}^2 \end{cases} \\ \text{S is non-orientable:} & \\ \quad S \cong \#_m \, \mathbb{R}P^2 & \\ \qquad \cong \mathbb{K}^2 \,\#\, (\#_n \mathbb{T}^2) & \end{cases} \\ \text{odd:} & \\ \quad S \cong \#_r \, \mathbb{R}P^2 & \\ \qquad \cong \mathbb{R}P^2 \,\#\, (\#_s \mathbb{T}^2) & \end{cases}$$

where $t = 1 - \frac{\chi(S)}{2}$, $m = 2 - \chi(S)$, $n = -\frac{\chi(S)}{2}$, $r = 2 - \chi(S)$ and $s = \frac{1-\chi(S)}{2}$.

This result together with the classification theorem imply the following theorem.

Theorem 4.3 *Let S_1 and S_2 be compact surfaces. Then S_1 is homeomorphic to S_2 if and only if their orientability agrees and $\chi(S_1) = \chi(S_2)$.*

This finally completes the proof of Theorem 4.1 for the classification of surfaces.

Chapter 5

Classification of compact surfaces with boundary

Until now, we developed tools to classify surfaces, restricting ourselves to the study of compact 2-manifolds without boundary. What happens, if a surface has a boundary? Is there still some way to classify such surfaces? The goal of this chapter is the study of compact and connected 2-manifolds with boundary, also called *bordered surfaces*, following Massey's book 'Algebraic Topology: An Introduction' [11].

5.1 Bordered surfaces

What does it mean for a surface to have a boundary? Take some surface M^* (without boundary) and remove the interior of finitely many disjoint closed discs. The space obtained is a bordered surface M. The number of bordered components of M is equal to the number of discs we initially removed. Notice that these boundary components are compact, connected 1-manifolds.

We can also do the inverse process. Take some surface M with k boundary components. Then, by attaching k closed discs, one for each boundary component, we obtain a compact surface.

From here on, we will use the symbol * to denote the surface M^* without boundary, obtained from a bordered surface M by gluing the necessary amount of closed discs. In this process, the location of the bordered components does not matter. In fact, we can state the following theorem.

Theorem 5.1 *Let M_1 and M_2 be two bordered surfaces with the same number of boundary components. Then M_1 and M_2 are homeomorphic if and only if M_1^* and M_2^* are homeomorphic.*

For the 'if' part of the theorem, we give an outline of the proof, skipping very few details regarding mostly the triangulation part, that can be read in [11].

Bordered compact surfaces can be triangulated as we defined in Section 2.1, with the difference that some edges will be the boundary of only one triangle, and not of two, as required in Section 4.1.1. This is a plausible consequence of having bordered components in our surface. At such boundary points, the surface is locally homeomorphic to the Euclidean half space.

We assume that the triangulation fulfils the condition that *no edge has both vertices contained in the boundary unless the entire edge is contained in it*. Otherwise, we can barycentrically subdivide the surface where needed. We notice that, around every boundary component, we can locally triangulate the surface along the border, such that the above condition is fulfilled. Then, these triangles form a polygon that is homeomorphic to a polygonal region in the plane with one hole in it (see Figure 5.1).

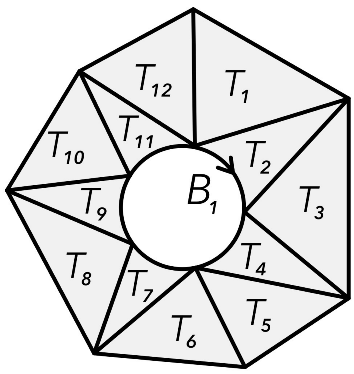

Figure 5.1: Local triangulation around a boundary component.

Assume that M_1 and M_2 are triangulated, fulfilling the above description around each boundary component. Then, we apply the algorithm in Section 4.1, while being careful when doing the necessary cuts to avoid the holes created by the boundary components.

The algorithm of the classification theorem gives finally a standard polygonal presentation of both surfaces with k holes, also called *normal form with k holes*. This normal form is given by first noting down the labelling scheme of the polygon without the holes inside. Then, starting at one certain vertex, we do a cut c_1 to the first boundary component, labelled B_1, and come back to the initial vertex along the same edge c_1. For each hole, we do such cuts $c_1, \ldots c_k$, starting every time with the same initial vertex (see Figure 5.2). These cuts must be pairwise disjoint, except for one end point.

Depending on the surface, we get the following normal forms:

1. Sphere with k holes:

$$aa^{-1}c_1B_1c_1^{-1}c_2B_2c_2^{-1}\ldots c_kB_kc_k^{-1}.$$

2. Connected sum of n tori with k holes:

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1} c_1 B_1 c_1^{-1} c_2 B_2 c_2^{-1} \dots c_k B_k c_k^{-1}.$$

3. Connected sum of n real projective planes with k holes:

$$a_1 a_1 \dots a_n a_n c_1 B_1 c_1^{-1} c_2 B_2 c_2^{-1} \dots c_k B_k c_k^{-1}.$$

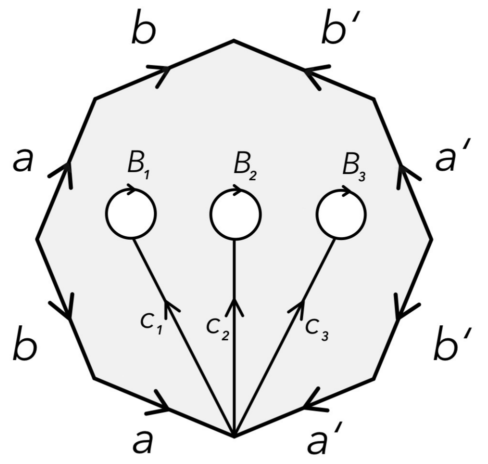

Figure 5.2: Double torus with three boundary components.

If M_1^* and M_2^* are homeomorphic, then the first part of the normal forms of M_1 and M_2, concerning the surface without holes, coincides. By assumption, M_1 and M_2 have the same number of boundary components, making the second part of the normal forms, which represents the cuts and holes, to be equal. This concludes the 'if' part of the theorem.

5.2 The classification of bordered surfaces

Orientability and the Euler characteristic are sufficient criteria to classify a compact surface, as we saw in Section 4.2. Starting with a triangulated compact surface without boundary, when deleting a face of one triangle, we obtain a compact surface with one boundary component. The Euler characteristic of this new surface is obviously reduced by one. Analogously, for every compact surface with k holes, we get that:

$$\chi(M) = \chi(M^*) - k.$$

Thus, together with the preceding theorem, we can give a general classification theorem for bordered compact surfaces.

Theorem 5.2 *Two compact bordered surfaces are homeomorphic if and only if they have the same number of boundary components, they are both orientable or non-orientable, and they have the same Euler characteristic.*

Chapter 6

Other proofs

We remind ourselves that the proof of the classification of surfaces in the previous chapter was divided into two parts. Part I, in Section 4.1, presented an algorithm, showing that any surface is homeomorphic to one of the standard ones of the theorem. Part II, in Section 4.2, shows that these standard compact surfaces are not homeomorphic to each other. For part I, most authors follow the same approach taken by Massey, using almost the same algorithm to manipulate the polygonal presentation of a given surface. Part II, on the other hand, is proved often with more modern concepts, such as the first homotopy group $\pi_1(X)$ or the first homology group $H_1(X)$. Nevertheless, the key idea remains the same for most proofs.

In the next pages, we will first study the proof presented in John Lee's book 'Introduction to topological manifolds' [8]. We want to give the reader a good understanding of this more elaborated proof, without overloading the next pages with too many technical details. We give special attention to the main differences and present all the necessary tools of algebraic topology, hoping to find an equilibrium for readers inexperienced in this matter. At the end, we will give a quick insight in the marvelous world of John Conway, whose proof differs significantly from the majority.

6.1 The classification theorem by John Lee

A big difference already in the preparation for the proof of the classification theorem is the tools which are used for the triangulation and the construction of the polygonal presentation of the surfaces. The concept used, is that of complexes, which is very useful, as we will see after a quick introduction in the basic concepts based on Chapter 5 in Lee's book.

6.1.1 CW-complexes

Some topological spaces can be obtained, starting from a discrete space, by attaching cells with successively increased dimension. This concept simplifies the handling with surfaces significantly.

An **open n-cell** is a topological space that is homeomorphic to the open unit ball in $\mathbb{R}^n$. Analogously, a **closed cell** is homeomorphic to the closed unit ball.

A topological space X can have a **cell decomposition**, which is a partition of X into open cells of various dimensions, fulfilling the following condition:

> For each cell e in the partition of dimension $n \geq 1$, there exists a continuous map from some closed n-cell D into X. This map is a homeomorphism from the interior of D onto e and it maps the boundary of D into the union of all cells of the partition of dimension strictly less then n.

A Hausdorff space X together with a specific cell decomposition is then called a **cell complex**.

A special case of a cell complex is the **CW-complex**. A cell decomposition of a topological space X is a CW-complex if, additionally to the preceding condition, the following holds:

1. The closure of each cell is contained in a union of finitely many cells;
2. A subset $U \subseteq X$ is open in X if and only if its intersection with every set $C \in F$ is open in C, where F is the family formed by the closure of every cell.

The second condition gives a *weak topology* and is necessary for spaces, where the topological space is constructed by gluing infinitely many cells.

Let's give a quick counterexample to understand the first condition [19]. Let X be a closed two dimensional disc. A possible cell decomposition would be a 2-cell for the interior of the disc and a 0-cell for every point on its boundary. Since the closure of the 2-cell is a closed disc, it intersects X at infinitely many cells, violating the first condition. Hence, such a cell decomposition is not a CW complex. A CW-complex of a disc would be a 0-cell with a 1-cell attached to it by its end points, and a 2-cell attached by its boundary along the 1-cell.

The following proposition establishes a connection between manifolds and CW-complexes.

Proposition 6.1 *A locally Euclidean space, whose CW complex has countably many cells, is a manifold.*

A special case of CW-complexes is given by simplicial complexes. A ***k*-simplex** $[v_0, \dots v_k]$ is a set spanned by points (vertices) $\{v_0, \dots, v_n\}$ in $\mathbb{R}^n$, given by

$$[v_0, \dots v_k] = \left\{ \sum_{i=0}^{k} t_i v_i : t_i \geq 0 \text{ and } \sum_{i=0}^{k} t_i = 1 \right\}.$$

For example a 2-simplex spanned by three vertices is a triangle in $\mathbb{R}^2$. A 3-simplex can be seen as a tetrahedron with one vertex at the origin of $\mathbb{R}^3$ and all the other three at each unit vector (this is the so called standard 3-simplex).

The space spanned by a non-empty subset of the vertices of a simplex σ forms also a simplex, called a **face** of σ. A 2-simplex has three types of faces: the three vertices are 0-faces, the three edges are 1-faces and the simplex itself is a 2-face.

A **simplicial complex** is a collection of simplices in $\mathbb{R}^n$ such that every face of each simplex is also contained in the collection, and such that the intersection of two simplices of the collection is either empty or a face of each. Further it is required for the collection to be locally finite.

Taking the union of all simplices in a simplicial complex gives a **polyhedron**, which is a topological space with the topology inherited from $\mathbb{R}^n$. This definition of a polyhedron is the one we will use for the triangulation surfaces in Theorem 6.3.

It can be shown that every k-simplex is a closed k-cell. Further, an Euclidean simplicial complex gives a CW decomposition consisting of the relative interiors of the simplices.

6.1.2 Polygonal presentation and elementary transformations

Definition 6.2 A **polygonal presentation**, written

$$P = \langle S \mid W_1, \dots, W_k \rangle$$

is a finite set S together with finitely many words $W_1, \dots, W_k$ in S of length 3 or more, such that every symbol in S appears in at least one word. A **word** in S is an ordered tuple of symbols, each of the form a or a^{-1} for some $a \in S$.

Our meanwhile very well-known compact surfaces have, in this sense, the following standard presentations:

1. Sphere: $\langle a \mid aa^{-1} \rangle$ or $\langle a, b \mid abb^{-1}a^{-1} \rangle$,
2. Connected sum of tori: $\langle a_1, b_1, \dots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle$,
3. Connected sum of projective planes: $\langle a_1, \dots a_n \mid a_1 a_1 \dots a_n a_n \rangle$ or

   $\langle a_1, b_1, \dots a_n, b_n \mid a_1 b_1 a_1 b_1 \cdots a_n b_n a_n b_n \rangle$

Then, we proceed by defining the so called **elementary transformations** of a polygonal presentation. We want to give a representative of each transformation without further explanation, hoping it to be self-explanatory together with its designation. For interested readers we recommend going through Chapter 6 in Lee's book [8].

1. Relabelling:

$$\langle S \mid a_1 a_2 \cdots a_i a_1 a_{i+1} \cdots a_n \rangle \mapsto \langle S \mid b a_2 \cdots a_i b a_{i+1} \cdots a_n \rangle$$

2. Subdividing:

$$\langle S \mid a_1 \cdots a_i a_1 a_{i+1} \cdots a_n \rangle \mapsto \langle S \mid a_1 e \cdots a_i a_1 e a_{i+1} \cdots a_n \rangle$$

$$\langle S \mid a_1 \cdots a_i a_1^{-1} a_{i+1} \cdots a_n \rangle \mapsto \langle S \mid a_1 e \cdots a_i e^{-1} a_1^{-1} a_{i+1} \cdots a_n \rangle$$

3. Consolidating (replacing adjacent edges):

$$\langle S \mid a_1 a_2 a_3 \cdots a_i a_2^{-1} a_1^{-1} a_{i+1} \cdots a_n, W_2, \ldots, W_k \rangle$$

$$\mapsto \langle S \mid a_1 a_3 \cdots a_i a_1^{-1} a_{i+1} \cdots a_n, W_2, \ldots, W_k \rangle$$

4. Reflecting:

$$\langle S \mid a_1 \cdots a_n, W_2, \ldots, W_k \rangle \mapsto \langle S \mid a_n^{-1} \cdots a_1^{-1}, W_2, \ldots, W_k \rangle$$

5. Rotating:

$$\langle S \mid a_1 a_2 \cdots a_n, W_2, \ldots, W_k \rangle \mapsto \langle S \mid a_2 \cdots a_n a_1, W_2, \ldots, W_k \rangle$$

6. Cutting:

$$\langle S \mid W_1 W_2, W_3, \ldots W_k \rangle \mapsto \langle S \mid W_1 e, e^{-1} W_2, W_3, \ldots W_k \rangle$$

7. Pasting:

$$\langle S \mid W_1 e, e^{-1} W_2, W_3, \ldots W_k \rangle \mapsto \langle S \mid W_1 W_2, W_3, \ldots W_k \rangle$$

8. Folding

$$\langle S \mid W_1 e e^{-1}, W_2, W_3, \ldots W_k \rangle \mapsto \langle S \mid W_1, W_2, W_3, \ldots W_k \rangle$$

9. Unfolding

$$\langle S \mid W_1, W_2, W_3, \ldots W_k \rangle \mapsto \langle S \mid W_1 e e^{-1}, W_2, W_3, \ldots W_k \rangle$$

It is easy to prove that each elementary transformation of a polygonal presentation produces a topologically equivalent presentation.

The advantage of this notation, is that manipulations on the surface using these elementary operations can be done in written form, without relying on images. For example, showing that the Klein bottle is homeomorphic to two real projective planes as in Section 3.6, would now look as follows:

$$
\begin{aligned}
&\langle a, b \mid abab^{-1} \rangle && \text{(presentation of the Klein bottle)}\\
&\approx \langle a, b, c \mid abc, c^{-1}ab^{-1} \rangle && \text{(cut along c)}\\
&\approx \langle a, b, c \mid bca, ab^{-1}c^{-1} \rangle && \text{(rotate both triangles)}\\
&\approx \langle a, b, c \mid bca, cba^{-1} \rangle && \text{(reflect second triangle)}\\
&\approx \langle a, b, c \mid bca, a^{-1}cb \rangle && \text{(rotate second triangle)}\\
&\approx \langle a, b, c \mid bccb \rangle && \text{(paste along the edge a)}\\
&\approx \langle a, b, c \mid bbcc \rangle && \text{(rotate)}
\end{aligned}
$$

Nevertheless, having a figure in mind is really helpful to not get lost in symbolic notation.

6.1.3 Classification of a surface

In order to prove the classification theorem, first, we want to assume that every surface is triangulable. This is stated below, using the terminology given by the simplicial complexes.

Theorem 6.3 *Every 2-manifold is homeomorphic to the polyhedron of a 2-dimensional simplicial complex, in which every 1-simplex is a face of exactly two 2-simplices.*

Using this theorem, we construct a polygonal presentation from the polyhedron. Since the polyhedron consists of an union of 2-dimensional simplices, we can construct a polygonal presentation forming one word of length three for each 2-simplex (meaning we label the edges of the triangles), while taking care of the correspondences given by edges shared by the same simplices. Showing that this representation gives indeed the same polyhedron back, when the edges are identified (glued back together), we obtain the following theorem.

Proposition 6.4 *Every compact surface admits a polygonal presentation.*

With Massey's approach, we used homeomorphisms from closed sets on the surface to triangles in the real plane to construct the polygonal presentation. This established a direct link from the surface to the $\mathbb{R}^2$-plane. Lee's approach differs slightly, making a detour using polyhedrons, which already are composed of geometrically planar triangles.

Applying the algorithm of Section 4.1, we can bring the polygonal presentation into a standard one, using the elementary transformations of Section 6.1.2. We skip a detailed description of this part, since every step of the algorithm in Lee's book is very similar to those of Massey. As a side note, we want to mention that edges of the first kind are called complementary and edges of the second kind are twisted edges.

6.1.4 The Euler characteristic

If a topological space X can be represented by a finite CW complex of dimension n, then the Euler characteristic is defined by

$$\chi(X) = \sum_{k=0}^{n} (-1)^k n_k,$$

where n_k is the number of k-cells of X.

Since every polygonal presentation determines a finite CW complex, then the compact surfaces have a well-defined Euler characteristic.

Compared to what we did in the proof of Theorem 2.4, using the theory of simplices, it is very simple to show the invariance of the Euler characteristic under the elementary transformations presented in Section 6.1.2. Every polygonal presentation of a surface consists of some 0-cells, 1-cells and 2-cells. The elementary transformations *relabeling*, *reflecting* and *rotating* do not change the Euler characteristic of a presentation, as there are no changes in the number of cells for every dimension. For the other transformations, it can be easily seen that they also do not change the characteristic, since the changes cancel each other out. For example the operation *subdividing* augments the number of edges by one (a pair is counted as one) and of vertices also by one. Notice that *consolidating*, *pasting* and *folding* are inverse transformations of respectively *subdividing*, *cutting* and *unfolding*.

6.1.5 Orientability

Surprisingly, Lee's definition of orientability of a surface differs from Massey's definition, which is based on an embedded Möbius strip. In Lee's book, a compact surface is **orientable** if it admits an oriented presentation, which is a presentation with no twisted pairs (no pairs of edges of the second kind). Figure 6.1 shows a twisted pair in a polygonal presentation of some compact surface. Connecting the edges the way indicated in the figure, we observe that it contains an embedded Möbius band. On the other hand, if no twisted pair appears in the polygonal presentation, then by the discussion below and Proposition 6.5, such surface does not contain an embedded Möbius band. Hence, both definitions are equivalent.

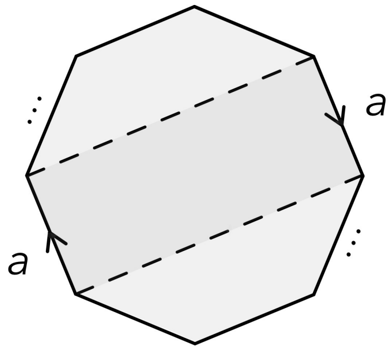

Figure 6.1: The polygonal presentation with a twisted pair shows the existence of an embedding Möbius band.

The only elementary transformation that can introduce a twist is *reflecting*. Going through the algorithm of the classification theorem, there are some steps, where a surface must be cut along an edge into two pieces (which is then represented by two words in the polygonal presentation). When both pieces are glued back together along some other edge, sometimes one piece has to be flipped, i.e. reflected, such that the corresponding edges match orientation. But this reflection only happens, when this pair is a twisted pair. For orientated pairs on the other hand, it suffices to rotate one piece, until the two corresponding edges meet.

We conclude that, if a surface is orientable, i.e. if there is no twist in its polygonal presentation, then no twists are going to be introduced during the manipulations done by the algorithm. By the classification theorem, such surface is reduced to one with no twisted pairs, namely a sphere or a sum of tori. Since the polygonal presentation of the sphere or connected sum of tori is oriented, we proved the following proposition.

Proposition 6.5 *A compact surface is orientable if and only if it is homeomorphic to the sphere or a connected sum of one or more tori.*

From the classification theorem, it follows from this proposition, that the sum of projective planes is not homeomorphic to an orientable surface.

Proposition 6.6 *A connected sum of projective planes is not orientable.*

Notice that, in Massey's approach, the Euler characteristic and orientability are sufficient to finish the proof of the classification theorem. In Lee's proof, on the other hand, they are insufficient criteria, since we cannot deduce anything about the orientability of connected sums of projective planes based on his definition of orientability. For Massey, every connected sum of projective planes contains an embedded Möbius band (see Section 3.5), making it by definition non-orientable. From Lee's perspective, it needs more sophisticated tools to show Proposition 6.6, as we will see in the next section.

6.1.6 The fundamental group

A polygonal presentation $\langle S \mid W \rangle$ gives naturally a CW-complex, where all vertices are identified to one. Thus the 1-simplices form a wedge sum of circles, one for each symbol in S. Then a 2-cell is attached along these circles, respecting the order given by the word W of the presentation.

We see that the 1-simplices form a generating set with a relation given by the attached two cell. In general, a group can be represented by a set of generators S and a set of relators R. We denote this **group presentation** by $\langle S \mid R \rangle$. Each of the generators in S determines an element in $\langle S \mid R \rangle$, while each of the relators represents a specific product of powers of the generators. The group presentation is the quotient,

$$\langle S \mid R \rangle = F(S)/\overline{R},$$

where $F(S)$, the **free group on S**, is the free product of all the infinite cyclic groups generated by elements of S. $\overline{R}$ is the normal closure of R in $F(S)$. In the quotient every relator in R is equal to 1.

Notice that the closure of R is the intersection of all normal subgroups of $F(S)$ containing R. Thus $\overline{R}$ is a normal subgroup and $\langle S \mid R \rangle$, as the quotient of a group by a normal subgroup, is again a group.

Identifying this presentation to be the **fundamental group** (the set of path classes of loops), we conclude our observations with the following theorem.

Theorem 6.7 *Let M be a topological space with the polygonal presentation given by $\langle \alpha_1, \dots, \alpha_n \mid W \rangle$ with one face, in which all vertices are identified to a single point. Then $\pi_1(M)$ has the presentation $\langle \alpha_1, \dots, \alpha_n \mid W \rangle$.*

The proof of this theorem relies superficially on the description above. We will skip the exact study of how to attach a 2-cell and the effect on the presentation of the fundamental group (see [8], page 262).

With the preceding theorem, we can easily compute the fundamental group of the standard compact surfaces of the classification theorem. Since the standard presentation identifies every vertex to a single point, their fundamental groups are given by their presentations in the following way.

$$\pi_1(\mathbb{S}^2) \cong \langle \emptyset \mid \emptyset \rangle$$

$$\pi_1(\underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{\text{n times}}) \cong \langle \alpha_1, \beta_1, \dots, \alpha_n, \beta_n \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1} = 1 \rangle$$

$$\pi_1(\underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{\text{n times}}) \cong \langle \alpha_1, \dots, \alpha_n \mid \alpha_1^2 \cdots \alpha_n^2 = 1 \rangle$$

Now, we wish to compare these groups. While this kind of presentation makes such an attempt difficult, the process gets easier with their abelianization. The abelianization $Ab(G)$ of a group G is given by the quotient

$G/[G,G]$, where $[G,G]$ is the commutator group. The commutator group contains every element of the form $aba^{-1}b^{-1}$ for $a,b \in G$.

The abelianization of the fundamental groups of the standard compact surfaces can be calculated using the *characteristic property of the abelianization* (see [8], page 266). We will skip a detailed study and present only the results:

$$Ab(\pi_1(\mathbb{S}^2)) = \{1\},$$

$$Ab(\pi_1(\underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{\text{n times}})) \cong \mathbb{Z}^{2n},$$

$$Ab(\pi_1(\underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{\text{n times}})) \cong \mathbb{Z}^{n-1} \times \mathbb{Z}/2.$$

As a side note, we want to mention that, for a path-connected space X, the abelianization of the fundamental group is isomorphic to the first homology group $H_1(X)$. This is the so called **Hurewicz Theorem**.

We see that the sphere has a trivial fundamental group. Thus it is impossible for it to be homeomorphic to a connected sum of tori or projective planes. Next, we observe that the abelianization of the fundamental group of a connected sum of projective planes contains a non-trivial torsion element, while the fundamental group of connected tori is torsion-free. We conclude that a connected sum of tori and a connected sum of projective planes cannot be homeomorphic. This proves part II of the classification theorem and Proposition 6.6, which states that a connected sum of projective planes is a non-orientable surface.

From the rank of these groups we can determine the genus of a compact surface. The rank of a *free* abelian group with a finite basis is the number of elements in the basis. A free abelian group contains no torsion elements, since a torsion element cannot be uniquely defined in terms of basis elements. As noticed previously, the abelianization of the fundamental group of the sum of projective planes contains a torsion element. For this reason, we extend the definition of the **rank of a finitely generated abelian group G** to be the rank of G/G_{tor}, where G_{tor} is the torsion subgroup containing all torsion elements.

For a connected sum of n tori, the rank of the abelianized fundamental group is $2n$. For the connected sum of n projective planes the rank is $n-1$. Thus, by determining the rank of the abelianization of the fundamental group, we obtain the genus of the surface. The orientability is determined by the presence or absence of a torsion subgroup.

6.2 The classification theorem by John Conway

John Conway calls his proof for the classification of surfaces the *Zero Irrelevance Proof*, short 'ZIP' proof. His goal was not to substitute the original, but rather give a different perspective and to go beyond a combinatorial proof of manipulating polygons. His proof was noted down by George K. Francis and Jeffrey R. Weeks [4] and, for some rather crucial parts, the technical details are left out, making it a not so rigorous proof of the classification theorem. To some extent, it is difficult to have a deepen understanding and it leaves room for interpretations on how to fill the existent gaps. Our goal here is to present the main idea of Conway's ZIP proof as a matter of expanding the horizon. For this, we will rely mostly on the wonderful illustrations and give a glance into this new world.

6.2.1 Ordinary surfaces and their classification

In Conway's vision, compact surfaces are just spheres with some particular objects attached to them. The proof relies, as we already have seen in previous proofs, on making cuts and gluing edges together. Conway imagines these cuts as zip pairs that are being unzipped. Zipped them up one by one, we can identify at each step, which objects are attached to the sphere. Thus the name 'ZIP' proof. The classification is going to be viewed in the following way, which we will explain in the pages to come.

Theorem 6.8 *Every compact surface is homeomorphic to either a sphere with handles or a sphere with crosscaps.*

A torus, for example, can be smoothly deformed into a sphere with a handle attached to it. Equivalently, a connected sum of n tori is homeomorphic to a sphere with n handles. In Figure 6.2 on the left, we observe that a handle attached to a sphere can be represented as a pair of zips with reversed zipping orientations relative to each other. Following the deformation steps in alphabetical order, we get a handle.

What happens if the zips have the same relative orientation? Then, in order to bring the zips together in the right manner, the 'tubes' have to intersect each other. Doing this according to Figure 6.2 on the right, we get a so called *crosshandle*. With help of Lemma 6.10, we will recognise this strange space of a crosshandle attached to a sphere as being homeomorphic to the Klein bottle.

Summarizing, for a pair of zips that occupy two boundary circles, we get either a handle or a crosshandle. For a pair occupying a single boundary circle of a perforation, we distinguish again between two cases. If the zips 'point' in the same direction, we can zip them up in one pull, 'closing' the perforation, as in Figure 6.3 on the left. A perforation with such a zip-pair,

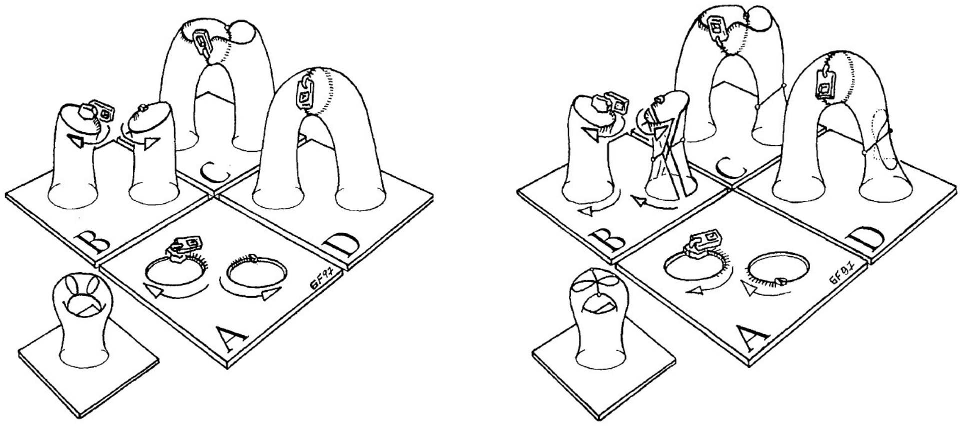

Figure 6.2: A handle (left) and a crosshandle (right) (source [4]).

we call a *cap*. If the zips have opposite orientations, then again we need to have some intersection with the surface in order to zip them up. This leaves a *crosscap* behind. In this case, it is easy to see from such a zip constellation, that a crosscap on a sphere is homeomorphic to the real projective plane.

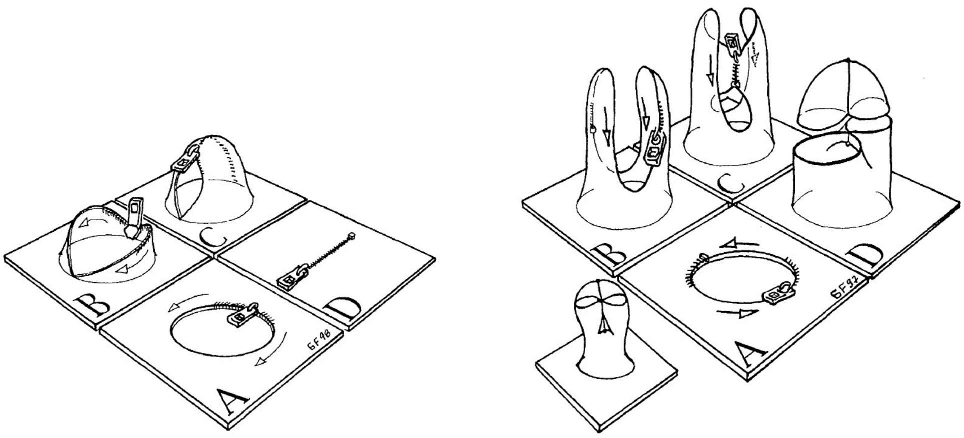

Figure 6.3: A cap (left) and a crosscap (right) (source [4]).

In this manner, we view every compact surface as a sphere with perforations, some of them with zips on their bordaries, arranged in some particular way. By zipping up the zips, we obtain a so called **ordinary** surface, which is a surface homeomorphic to a sphere with a finite number of handles, crosshandles, crosscaps, and perforations. The caps do not appear in the list, since a cap attached to a sphere is just homeomorphic to the sphere itself. The reason for perforations being on the list is that Conway also considers bordered surfaces in his proof.

As we know, after all our study on the classification of surfaces, these building blocks are more than enough to classify every surface. Indeed:

Theorem 6.9 *Every surface is ordinary.*

The proof starts as every other, namely by assuming that the surface is triangulated. What is to come, is what substantially differs Conway's proof from all the others. We imagine zips along the boundaries of these triangles, holding two adjacent triangles together, forming what seems to be a patchwork quilt (see Figure 6.4).

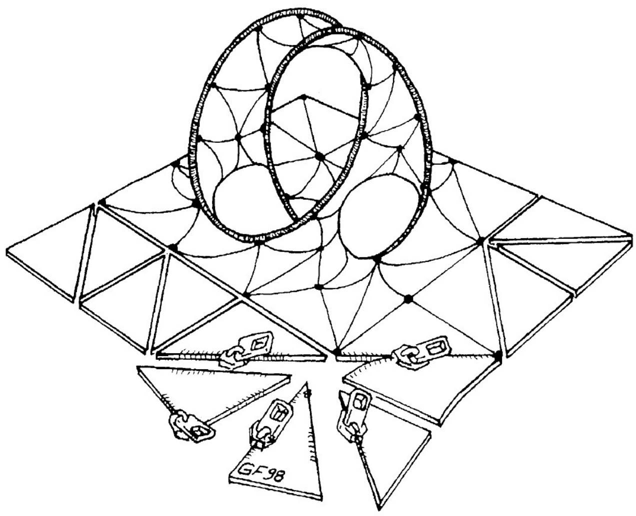

Figure 6.4: Triangulation of a surface with a zip-pair installed along each edge (source [4]).

Unzipping all the zip-pairs, we observe that we get a family of ordinary surfaces, each triangle being homeomorphic to a sphere with a single perforation. The boundary of this perforation represents the boundary of the triangle and can contain any kind of zips on it. Then, we can start zipping up the triangles back together one by one. After each addition of a new triangle, we observe that the surface remains ordinary. Take some zip-pair and assume first, that one of the zips lies in set of already zipped up triangles, while the other correspondent zip is in an unzipped triangle. Then zipping them together yields a connected space, again homeomorphic to a perforated sphere, which is an ordinary surface.

On the other hand, if for the zip pair both zips are on the boundary of the zipped up surface, then depending on their arrangement, we get a sphere with either a handle, a crosshandle, a cap or a crosscap, with or without a perforation. By definition, this is again an ordinary surface. To see this, we analyse the three different constellations, in which the zip-pair can appear. First, assume that the pair occupies a single boundary circle, as in Figure 6.3.

By zipping them together we clearly get a cap or a crosscap. Now, we assume that each zip of the pair occupies respectively a boundary circle completely. By deforming the surface, the circles can be made adjacent to each other as Figure 6.2. Zipping them together yields a handle or a crosshandle depending on their relative orientation. Finally, we assume that the two zips occupy each a boundary circle partially, leaving some gaps (see Figure 6.5, *A*). We start by pulling up the circles, forming tubes, and zip the zips together. This gives a handle with a perforation on top (*B*). Such perforation, which emerges from the gaps on the boundary circles, can be smoothly deformed away from the handle (*C*). We obtain a space homeomorphic to a sphere with a handle and a perforation. Depending on the orientation of the zips, we can also get a crosshandle attached to a sphere with a perforation.

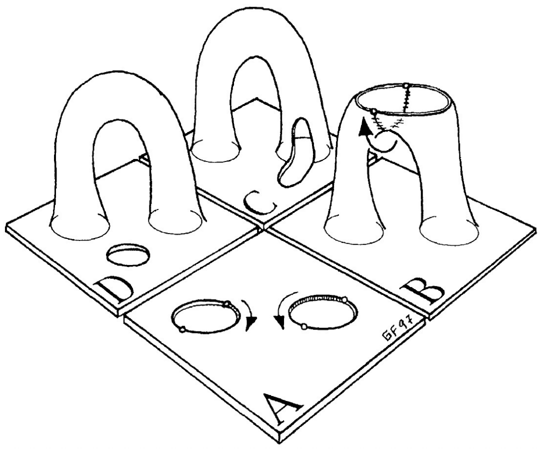

Figure 6.5: A zip-pair in this constellation yields a punctured handle (source [4]).

We conclude that, by zipping the triangles back together one by one, we can identify all handles, crosshandles, crosscaps and perforations, which classifies the surface.

6.2.2 A crosshandle or two crosscaps

From the knowledge we obtained so far from the other proofs, we know that we have too many building blocks. For instance, the crosshandle can be deleted from the list, since:

Lemma 6.10 *A crosshandle is homeomorphic to two crosscaps.*

In a more familiar language, this is equivalent to saying, that a Klein bottle is homeomorphic to the connected sum of two projective planes.

We do a proof by picture (see Figure 6.6), using the opportunity to encourage the reader to dive into a new proof of this wonderful result. We remark that the picture aims to show that there are two ways of zipping everything back together. Starting in A with the representation of the Klein bottle, we can either zip along the black arrows, leading to a crosshandle (step B and C) or along the white arrows, leading to two cross caps (step D to I in alphabetical order).

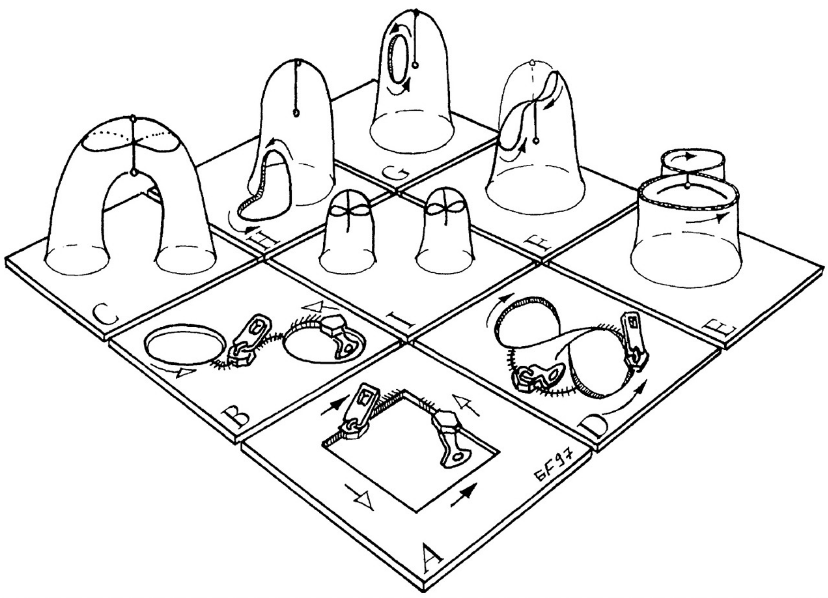

Figure 6.6: A crosshandle (C) is homeomorphic to two crosscaps (I) (source [4]).

What happens, when we have simultaneously a handle and a crosscap attached to a sphere? As we know this corresponds to the connected sum of a torus and a projective plane, which is homeomorphic to the sum of a Klein bottle and a projective plane. Translating into Conway's language, we get:

Lemma 6.11 *Handles and crosshandles are equivalent in the presence of a crosscap.*

One last time, we leave a beautiful picture for the reader to study (see Figure 6.7). Notice that, on the left-most corner, a crosscap and a handle are attached to the first square and on the third one (bottom-most), we have a crosscap and a crosshandle attached to it. Both spaces are homeomorphic to that of the second square.

From Lemma 6.10, we conclude that, whenever a crosshandle appears, it can be substituted with two crosscaps. And when a handle and a crosscap appear simultaneously, then by Lemma 6.11, they can be substituted with a crosshandle and a crosscap, which is homeomorphic to three crosscaps, again by Lemma 6.10. With this, we finally conclude the previously stated classification theorem for compact surfaces without boundary:

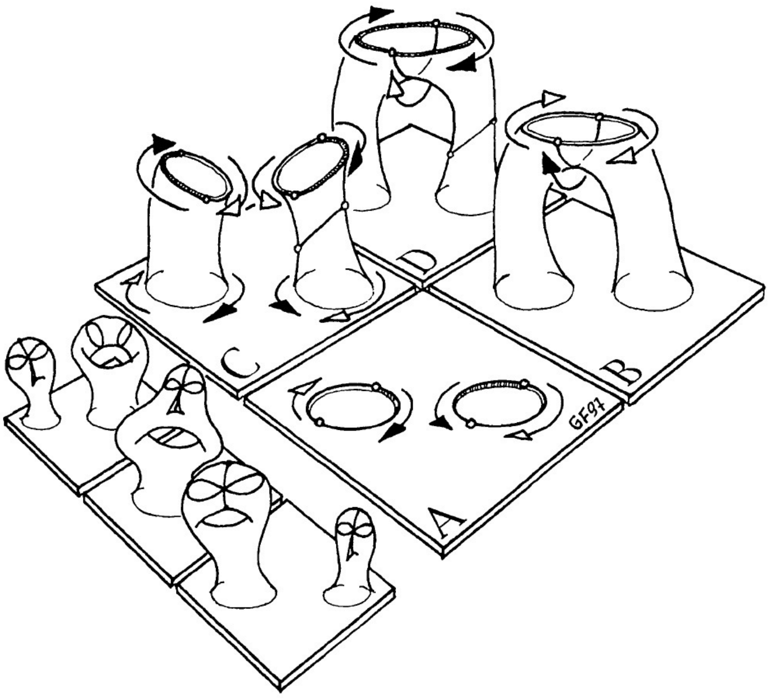

Figure 6.7: The presence of a crosscap transforms a handle into a crosshandle (source [4]).

Every compact surface is homeomorphic to either a sphere with handles or a sphere with crosscaps.

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