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Symplectic Linear Algebra and the Affine Non-Squeezing Theorem

Bachelor Thesis

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Contents

Abstract

In this thesis we give the basic definitions of a symplectic vector space and symplectic form, which are the foundations of symplectic linear algebra. We then focus mainly on the types of subspaces and structures that can be defined on symplectic vector spaces.

Introduction

Symplectic linear algebra is a branch of mathematics necessary to introduce and study symplectic manifolds and, more generally, symplectic geometry. The main objects of study are the so-called symplectic vector spaces, i.e., real vector spaces equipped with additional structures called symplectic forms.

This thesis is primarily concerned with the study of symplectic vector spaces, the interactions between their subspaces and the structures that can be defined on them.

In the first chapter we introduce the basic definitions useful for all chapters. We define the symplectic group, i.e., morphisms that preserve the symplectic form, and study the properties of its elements, such as their eigenvalues. We also study the subspaces of a symplectic vector space and how they behave under the action of the symplectic group. We then discuss a particular type of subspaces: Lagrangian subspaces.

In the second and third chapters, we focus on some specific functions and structures. First the Maslov index, which can be defined for loops of symplectic matrices and Lagrangian subspaces. Then we define what is a complex structure on a vector space and, more specifically, a compatible complex structure defined on a symplectic vector space. We study the characteristics of the space composed of all these compatible complex structures and, in particular, the important fact that this space is contractible.

In the last chapter, we focus on the affine non-squeezing theorem, which sheds light on one of the first counter-intuitive aspects of symplectic geometry. In fact, this theorem states that a ball can only be embedded in a symplectic cylinder via a symplectic map, if its radius is less or equal to the radius of the cylinder. In other words, the ball cannot be squeezed more than its initial width, which would instead be possible with a non-symplectic volume-preserving map. We then introduce the concept of linear symplectic width and study in detail the case of ellipsoids.

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Chapter 1

Basics of Symplectic Linear Algebra

1.1 Symplectic Vector Spaces

The first important notions that we introduce are the *symplectic form* and the *symplectic vector space*. We also define the concept of canonical form of a symplectic form and the symplectic basis of a symplectic vector space. We let V be a finite dimensional real vector space.

Definition 1.1.1. Let

$$\omega : V \times V \to \mathbb{R}$$

be a non-degenerate, skew-symmetric bilinear form, then ω is called *symplectic form.*

Remark. It follows from the definition that $\omega(v, v) = 0 \ \forall v \in V$, and if $\omega(v, w) = 0 \ \forall v \in V$, then $w = 0$

Definition 1.1.2. Let V be vector space equipped with a symplectic form ω, then (V, ω) is called *symplectic vector space.*

Example 1.1.3. Let V= $\mathbb{R}^{2n}$ with basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ and define ω such that $\omega(e_i, e_j) = 0$, $\omega(f_i, f_j) = 0$, $\omega(e_i, f_j) = \delta_{i,j}$. Then (V, ω) is a symplectic vector space.

Remark. We can define the same concepts for a finite dimensional vector space over a field K with characteristic 0.

Example 1.1.4. Let V be a vector space of dimension n and V^* its dual. If we define $U := V \oplus V^*$ and $\omega : U \to U$ such that $\omega((v, \alpha), (v', \alpha')) = \alpha'(v) - \alpha(v')$ then (U, ω) is a symplectic vector space.

Definition 1.1.5. Let (V, ω) be a symplectic vector space, then for any subspace $U \subseteq V$ we define the ω-orthogonal space $U^\omega := \{v \in V : \omega(v, w) = 0, \forall w \in U\}$.

Proposition 1.1.6. Let V be a finite dimensional real vector space of dimension m and ω a bilinear form, then

1. if ω is symmetric with rank r there exits a basis $\underline{e}$ of V such that the representation of ω relative to $\underline{e}$ is

$$\begin{pmatrix} \epsilon_1 & & & & & \\ & \ddots & & & & \\ & & \epsilon_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \text{ where } \epsilon_i = \pm 1$$

2. if ω is skew-symmetric with rank r there exists a basis $\underline{e}$ of V such that the representation of ω relative to $\underline{e}$ is
$\begin{pmatrix} 0 & Id_n & 0 \\ -Id_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ where Id_n is the identity matrix of dimension $n = r/2$

Proof. 1. A proof can be found in [3]

2. Let $e_1, e_{n+1} \in V$ such that $\omega(e_1, e_{1+n}) \neq 0$ (such vectors must exist if $\omega \neq 0$). By rescaling e_1 we can assume $\omega(e_1, e_{1+n}) = 1$. Since ω is skew-symmetric we have $\omega(e_1, e_1) = 0$ and $\omega(e_{1+n}, e_{1+n}) = 0$. So the restriction of ω on the space U_1 spanned by e_1 and e_{1+n} is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let $U_2 = U_1^\omega$, then the intersection $U_1 \cap U_2$ is trivial and for any $v \in V$ we have

$$v - \omega(v, e_{1+n})e_1 + \omega(v, e_1)e_{1+n} \in U_2$$

So $V = U_1 \oplus U_2$. We can then repeat the procedure on U_2 and find e_2 and e_{2+n} such that $\omega(e_2, e_{2+n}) = 1$. Inductively we find the basis $\underline{e}$.

□

Remark. Since we focus on non-degenerate skew-symmetric bilinear form, i.e. with rank $r = 2n = m$, we can consider only the case with matrix representation $\begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$ and V must have an even dimension.

Corollary 1.1.7. *Every finite dimensional symplectic vector space (V, ω) has even dimension.*

We can identify the space of skew-symmetric bilinear form as the space $\wedge^2 V^*$. So if $\underline{e} = \{e_1, ...e_{2n}\}$ is a basis of V and $\underline{e}^*$ its dual, then for any $\omega \in \wedge^2 V^*$ represented by the matrix $A_e = (a_{ij})$ relative to $\underline{e}$ we can also write ω as

$$\omega = \textstyle\sum_{i<j} a_{ij} e_i^* \wedge e_j^*$$

Remark. Since elements of $\wedge^2 V^*$ are represented by anti-symmetric matrices and with all the entries of the main diagonal equal to 0, for a vector space V of dimension $2n$ we have dim $\wedge^2 V^* = \frac{(2n)(2n-1)}{2}$.

Corollary 1.1.8. *For every skew-symmetric bilinear form $\omega \in \wedge^2 V^*$ there exits a basis $\underline{e}$ of V such that the representation of ω relative to $\underline{e}$ is*

$$\omega = \textstyle\sum_{i<j} e_i^* \wedge e_j^*$$

This representation is called a canonical form of ω and $\underline{e}$ a symplectic basis of V. This representation also allows us to identify every symplectic vector spaces (V, ω) of dimension $2n$ with the one presented in Example 1.1.3 (later we will re-prove this statement with the definition of symplectomorphism).

We now introduce a map that will be useful in the next sections. For a symplectic form ω we define $\omega^\flat$ as follows

$$\omega^\flat : V \to V^*$$
$$v \mapsto \omega^\flat(v)$$

where $\omega^\flat(v)(w) = \omega(v, w)\ \forall w \in V$

Since we work with finite dimensional V it follows that $\omega^\flat$ is an isomorphism if and only if ω is non-degenerate. It also follows that for any subspace $U \subseteq V$ U^ω is equal to the pre-image of $ann(U) \subseteq V^*$ under $\omega^\flat$, so $\mathrm{dimU}^\omega = \mathrm{dimV} - \mathrm{dimU}$ and, since per definition follows that $\mathrm{U} \subseteq (\mathrm{U}^\omega)^\omega$, we have $(\mathrm{U}^\omega)^\omega = \mathrm{U}$.

1.2 Symplectomorphisms and Symplectic Groups

Definition 1.2.1. Let (V_1, ω_1) and (V_2, ω_2) be symplectic vector spaces. Then a linear map $\phi : V_1 \to V_2$ is called *symplectic* if $\phi^*\omega_2 = \omega_1$, i.e. if for any v and w in V_1 it holds $\omega_1(v, w) = \omega_2(\phi(v), \phi(w))$. If ϕ is also bijective it is called *symplectomorphism.* In this case (V_1, ω_1) and (V_2, ω_2) are called *symplectomorphic.*

Remark. If V_1 and V_2 have same dimension and the linear map ϕ satisfies $\phi^*\omega_2 = \omega_1$ then it is also injective, since ω_1 and ω_2 are non-degenerate and so $\phi(v) = 0$ implies $v = 0$. From a dimension argument follows that ϕ is also surjective, so it is a symplectormorphism.

It is easy to see that the set of symplectomorphisms of a symplectic vector space (V, ω) forms a group under the usual composition.

Definition 1.2.2. The group of symplectomorhisms of a symplectic vector space (V, ω) is called *symplectic group* and denoted $Sp(V)$.

As said before with Proposition 1.1.6 we can conclude the following very important theorem.

Theorem 1.2.3. *Every symplectic vector space (V, ω) of dimension 2n is symplectomorphic to $\mathbb{R}^{2n}$ with the symplectic form of Example 1.1.3.*

In Section 1.4 we will also report another proof for this theorem.

With this result we can consider $Sp(V) \cong Sp(\mathbb{R}^{2n})$ a subset of the general linear group $GL_{2n}(\mathbb{R})$ (i.e. the group of invertible $2n \times 2n$ matrices). Then using the second part of Proposition 1.1.6 we get

$$\omega(v, w) = v^T \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix} w = v^T Jw, \quad \text{where } J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$$

So we can conclude that an element M of $GL_{2n}(\mathbb{R})$ is in $Sp(\mathbb{R}^{2n})$ if and only if it satisfies the condition

$$v^T Jw = \omega(v, w) = \omega(Mv, Mw) = v^T M^T JMw, \, \forall v, w \in V$$

And that is if and only if

$$M^T JM = J.$$

This has as immediate consequence that for every $M \in Sp(\mathbb{R}^{2n})$ we have $(\det M)^2 = 1$, we will later show that is indeed $\det M = 1$ since $Sp(\mathbb{R}^{2n})$ is a connected space. Another consequence is that, if M is symplectic, so is M^T. This can be seen using the fact that $Sp(\mathbb{R}^{2n})$ is a group, so also M^{-1} is symplectic and then taking the inverse of $(M^{-1})^T JM^{-1} = J$ we get $MJM^T = J$, since $J^{-1} = -J$

We conclude this section focusing on the dimension of $Sp(V)$. Saying that all symplectic vector spaces of same dimension are symplectomorphic to $\mathbb{R}^{2n}$, and so also to each other, means that $GL_{2n}(\mathbb{R})$ acts transitively on $\wedge^2 V^*$. The stabiliser at ω is $Sp(V)$. Therefore if dim $V = 2n$ we get

$$\dim\, Sp(V) = \dim\, GL_{2n}(\mathbb{R}) - \dim\, \wedge^2 V^* = (2n)^2 - \tfrac{(2n)(2n-1)}{2} = 2n^2 + n$$

1.3 Eigenvalues of Symplectic Matrices

We can say something about the eigenvalues of elements in $Sp(\mathbb{R}^{2n})$.

Theorem 1.3.1 (Symplectic Eigenvalue Theorem). *Let M be an element of $Sp(\mathbb{R}^{2n})$, then all its eigenvalues other than 1 and -1, which have even multiplicity, come either in pairs $\lambda, \overline{\lambda}$ with same multiplicity and $|\lambda| = 1$ or in quadruples $\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}$, with same multiplicity and $|\lambda| \neq 1$.*

Proof. For any $M \in GL_{2n}(\mathbb{R})$ the eigenvalues come in complex conjugates pairs of same multiplicity and since $Sp(\mathbb{R}^{2n}) \subseteq GL_{2n}(\mathbb{R})$ this also holds for the symplectomorphisms. Then from $M^TJM = J$ follows that M^{-1} and M^T are similar matrices, and therefore also M and M^{-1}. This means that for every λ eigenvalue of M, also λ^{-1} is an eigenvalue with same multiplicity. Finally, since the product of all eigenvalues is equal to $\det M = 1$ (this will be proved in Section 2.1) we have that the multiplicity of 1 and -1 must be even. ☐

We have also some results about the eigenspaces for a special case of symplectic matrices.

Lemma 1.3.2. *Let $M \in Sp(\mathbb{R}^{2n})$ be diagonalizable and with all eigenvalues real. We denote E_λ the eigenspace for the eigenvalue λ. Then we have*

$$E_\lambda^\omega = \bigoplus\nolimits_{\lambda\mu\neq 1} E_\mu$$

Proof. Let $v \in E_\lambda$ and $w \in E_\mu$, then we have

$$\omega(v, w) = \omega(Mv, Mw) = \lambda\mu\omega(v, w)$$

so for $\lambda\mu \neq 1$ and $v \in E_\lambda$ it holds $\omega(v, w) = 0$ $\forall w \in E_\mu$, hence $E_\mu \subseteq E_\lambda^\omega$. Then applying the formula $dimV = dimU + dimU^\omega$ for the dimension of a subspace and the fact that M is diagonalizable we finish the proof. ☐

Lemma 1.3.3. *Let $M \in Sp(\mathbb{R}^{2n})$ be diagonalizable and with all eigenvalues real. Then $\forall\alpha \geq 0$ holds $M^\alpha \in Sp(\mathbb{R}^{2n})$.*

Proof. Since M is diagonalizable with real eigenvalues there exists a orthonormal basis of eigenvectors. Hence, if we denote E_{λ_i} the eigenspace for the eigenvalue λ_i we can write

$$\mathbb{R}^{2n} = \bigoplus_{i=1}^{k} E_{\lambda_i}$$

Then for any two vectors $u, v \in \mathbb{R}^{2n}$ we can write them as

$$u = \sum\nolimits_{i=1}^k u_i \text{ and } v = \sum\nolimits_{i=k}^k v_i$$

where $u_i, v_i \in E_{\lambda_i}$. Then since M is symplectic we have

$$\omega(u_i, v_j) = \omega(Mu_i, Mv_j) = \lambda_i\lambda_j\omega(u_i, v_j)$$

hence either $\lambda_i\lambda_j = 1$ or $\omega(u_i, v_j) = 0$. Therefore for any $\alpha \geq 0$

$$\omega(M^\alpha u, M^\alpha v) = \sum\nolimits_{i,j=1}^k \omega(M^\alpha u_i, M^\alpha v_j) = \sum\nolimits_{i,j=1}^k (\lambda_i\lambda_j)^\alpha \omega(u_i, v_j) = \sum\nolimits_{i,j=1}^k \omega(u_i, v_j) = \omega(u, v)$$

☐

Remark. Lemmas analogous to the above hold when M is symmetric and positive definite.

In order to state the last result about the eigenvalues of a symplectic matrix we need to introduce a new definition.

Definition 1.3.4. Let V be a vector space and ϕ a morphism of V, then ϕ is called *stable* if $\forall\epsilon > 0$ there exists a $\theta > 0$ such that $|\phi^N v| < \epsilon$ $\forall N \in \mathbb{N}$, if $|v| < \theta$.

Lemma 1.3.5. *If $M \in Sp(\mathbb{R}^{2n})$ has an eigenvalue λ such that $|\lambda| \neq 1$, then M is not stable.*

Proof. Let $M \in Sp(\mathbb{R}^{2n})$ with eigenvalue λ such that $|\lambda| \neq 1$. We can assume without loss of generality that $|\lambda| > 1$ (otherwise we can use λ^{-1} thanks to Theorem 1.3.1). Then given $v \in E_\lambda$ we have $M^N v = \lambda^N v$, in particular $|M^N v| = |\lambda|^N |v|$, which goes to infinity as N goes to infinity. Therefore, M cannot be stable. ☐

1.4 Subspaces

In this section we will classify and analyse the different kind of subspaces of a symplectic vector space.

Definition 1.4.1. Let (V, ω) be a symplectic vector space and $U \subseteq V$ a subspace with ω-orthogonal U^ω. We then say that U is

1. *isotropic* if $U \subseteq U^\omega$, i.e. if $\omega\mid_U = 0$
2. *coisotropic* if $U^\omega \subseteq U$, i.e. if U^ω is isotropic
3. *Lagrangian* if $U = U^\omega$, i.e. U is both isotropic and coisotropic
4. *symplectic* if $U \cap U^\omega = \{0\}$, i.e. if $\omega\mid_U$ is non-degenerate

Remark. At the end of section 1.1 we have showed the formula $\dim V = \dim U + \dim U^\omega$, from which we see that if $\dim V = 2n$ then all the isotropic subspaces have dimension smaller or equal n, all the coisotropic have dimension bigger or equal n and all the Lagrangian subspace have dimension n.

Remark. If $U \subseteq V$ is a symplectic subspace it follows from the definition that $U \cap U^\omega = \{0\}$ and therefore, from the previous formula for the dimension follows that $V = U \oplus U^\omega$.

Example 1.4.2. If we use the same set up as in Example 1.1.3 and we define $U_1 = span(e_1, e_2)$, $U_2 = span(e_1, ..., e_n, f_3, ..., f_n) = U_1^\omega$, $U_3 = span(e_1, ..., e_n)$ and $U_4 = span(e_1, f_1)$. Then U_1 is isotropic, U_2 coisotropic, U_3 Lagrangian and U_4 symplectic.

If we are in a case where the conditions of Lemma 1.3.2 hold, we have as straightforward consequences of the same lemma that E_λ for $|\lambda| \neq 1$ are isotropic, because $\lambda\lambda \neq 1$ so $E_\lambda \subseteq E_\lambda^\omega$. For $\lambda = \pm 1$ E_λ is symplectic, because eigenspaces of different eigenvalues are disjoint and for all λ we have $E_\lambda \oplus E_{\lambda^{-1}}$ is symplectic, because $E_\lambda \cap E_{\lambda^{-1}}^\omega = \{0\}$ and $E_{\lambda^{-1}} \cap E_\lambda^\omega = \{0\}$.

Now we state two important lemmas.

Lemma 1.4.3. *Every symplectic vector space (V, ω) has a Lagrangian subspace.*

Proof. Since for every $v \in V$ we have $\omega(v, v) = 0$, V has an isotropic subspace. Let $L \subseteq V$ be a maximal isotropic subspace, namely that it is not contained in any isotropic subspace of larger dimension. Then L must be Lagrangian since if there exists $v \in L^\omega \backslash L$ then $L \oplus span(v)$ is a larger isotropic subspace that contains L. □

From this proof we also conclude that a maximal isotropic subspace is a Lagrangian subspace. Therefore we have the following corollary.

Corollary 1.4.4. *Every isotropic subspace is contained in a Lagrangian subspace.*

Lemma 1.4.5. *Let $L_1, ..., L_n$ be a collection of Lagrangian subspaces of V. Then exists a Lagrangian subspace L such that $L \cap L_i$={0} $\forall i$.*

Proof. Let L be an isotropic subspace such that $L \cap L_i = \{0\}$, for example span(v) for a $v \in V$ such that $v \notin L_i$ $\forall i$ (such v must exists since all the L_i have dimension equal to half of $\dim V$), and such that L is not contained in any larger isotropic subspace transversal to all L_i. We show that L is in fact Lagrangian. Assume it is not, it means that L^ω is a coisotropic subspace that strictly contains L. Define π: $L^\omega \to L^\omega/L$ the quotient map. Then all the space $\pi(L^\omega \cap L_i)$ are isotropic subspace of L^ω/L, because all L_i are Lagrangian and $(L^\omega)^\omega = L$. So we can choose a u $\in L^\omega/L$ such that u $\notin \pi(L^\omega \cap L_i)$ $\forall i$. Let L$' = \pi^{-1}(span(u)) \subseteq L^\omega$. Then L' strictly contains L and is an isotropic subspace of $L^\omega \subseteq V$ that is also transversal to all L_i, which is a contradiction to our choice of L. □

As a consequence of Lemma 1.4.3 and 1.4.5 can provide the following alternative proof of Theorem 1.2.3.

Proof. (Theorem 1.2.3) Let L_1, L_2 be Lagrangian subspaces of (V, ω) such that they are transversal, then we have that

$$L_1 \times L_2 \to \mathbb{R}$$
$$(v, w) \mapsto \omega(v, w)$$

is non-degenerate, which means that

$$L_2 \overset{\iota}{\hookrightarrow} V \xrightarrow{\omega^\flat} V^* \xrightarrow{\iota^*} L_1^*$$

is an isomorphism, where ι is the inclusion map. So we have $L_2 \cong L_1^*$. Let $\{e_1, ..., e_n\}$ be a basis of L_1 and $\{f_1, ..., f_n\}$ its dual basis for $L_1^* \cong L_2$. Define $\underline{e}$=$\{e_1, ..., e_n, f_1, ..., f_n\}$, since L_1 and L_2 are Lagrangian and transversal we get that ω with respect to $\underline{e}$ has the form of Example 1.1.3, which is called *symplectic standard form* and so $\underline{e}$ is the symplectic basis. □

This proof also shows the following corollary.

Corollary 1.4.6. *Assume we have two symplectic vector spaces (V, ω) and (V', ω') of same dimension with Lagrangian subspaces $L_i \subseteq V$ and $L_i' \subseteq V'$, $i = 1, 2$, such that $L_1 \cap L_2 = \{0\}$ and $L_1' \cap L_2' = \{0\}$. Then there is a symplectomorphism ϕ: $V \to V'$ such that $\phi(L_1) = L_1'$ and $\phi(L_2) = L_2'$.*

1.5 Radicals and Reduction

In this section we will focus on a symplectic vector space (V, ω) of dimension $2n$ and a subspace $U \subseteq V$ of dimension k. We will exhibit two invariants of U. First we claim that, for any $\phi \in Sp(V)$, we have k =dim$\phi(U)$ and, secondly, that also rank $\omega\mid_{\phi(U)}$ is invariant under any $\phi \in Sp(V)$. We will show that these are the only two symplectic invariants, namely, if two subspaces have the same values for these parameters, then there exists an element of $Sp(V)$ which takes one to the other, in this case we also say they are *symplectomorphic*. For this purpose, we need some more definitions and constructions. Here, (V, ω) always denotes a symplectic vector space of dimension $2n$ and U a subspace of V of dimension k, we also denote by 2ℓ the rank of the restriction of ω to U ($2\ell = rank\, \omega\mid_U$).

First we observe from Proposition 1.1.6 that for any symplectic vector space of dimension two we can find a basis $\{e_1, e_2\}$, so that the space is represented as $span(e_1, e_2)$, where $\omega(e_1, e_1) = \omega(e_2, e_2) = 0$ and $\omega(e_1, e_2) = 1$. If we have two such spaces that are ω-orthogonal to each other we get that their sum is still a symplectic vector space of dimension four. Applying this procedure iteratively on any symplectic vector space of dimension $2r$ we arrive to write it as an ω-orthogonal sum of r spaces, each of whom of dimension two.

Definition 1.5.1. The *radical* of U is defined by

$$rad\, U = U \cap U^\omega$$

It is therefore equal to the kernel of the restriction of ω to U, i.e.

$$rad\, U = \{u \in U \mid \omega(u, v) = 0\ \forall v \in U\}$$

It follows from the definition that $rad\, U$ is an isotropic subspace and

$$\dim(rad\, U) = \dim\, U - rank\, \omega\mid_U = k - 2\ell$$

With this concept we can state another definition.

Definition 1.5.2. The *symplectic space associated to* U, also called *reduced space*, is given by

$$U^{red} = U/rad\, U$$

It is clear that this space is symplectic, i.e. it inherits a symplectic form ω_U which is non-degenerate. Its dimension is a direct consequence of the definition and it holds

$$\dim\, U^{red} = \dim\, U - \dim\,(rad\, U) = k - (k - 2\ell) = 2\ell$$

Moreover, from the definitions of $rad\, U$ and U^{red}, we have that there exists a subspace $(W, \omega\mid_W)$ of (U, ω) which is isomorphic to U^{red}. Therefore W is symplectic and ω-orthogonal to $rad\, U$, and it satisfies

$$U = rad\, U \oplus W$$

We can finally state the theorem, which will give us the desired results.

Theorem 1.5.3. *Let (V, ω) be a symplectic vector space, U a subspace and W another subspace ω-orthogonal to $rad\, U$ such that*

$$U = rad\, U \oplus W$$

Let $e_1, ..., e_r$ be a basis of $rad\, U$, then we can find $f_1, ..., f_r \in V$ such that $\omega(e_i, f_i) = 1$ and $U_i = span(e_i, f_i)$ are symplectic subspace ω-orthogonal to W for any i and ω-orthogonal to each other for different indices. So if we define the subspace $\overline{U} := U_1 \oplus ... \oplus U_r \oplus W$ it contains U per construction and is symplectic. Moreover, for any symplectic vector space (V', ω') any injective and symplectic linear map $\phi : U \to V'$ can be extended to $\overline{\phi} : \overline{U} \to V'$ which is still symplectic.

Before the proof we state and prove an important corollary that will allow us to conclude what we wanted to show.

Corollary 1.5.4. *Let V and V' be symplectomorphic symplectic vector spaces, $U \subseteq V$ and $\phi : U \to V'$ an injective and symplectic linear map. Then ϕ can be extended to a symplectomorphism $\overline{\phi} : V \to V'$.*

Proof. From Theorem 1.5.3 we can always extend ϕ to $\overline{\phi} : \overline{U} \to V'$ so without loss of generality we can assume that U is symplectic. Therefore, we can write $V = U \oplus U^\omega$. Let $U' := \phi(U)$ and define $U'' \subseteq V'$ to be ω-orthogonal to U' and such that $V' = U' \oplus U''$. Since U and U' are symplectic it follows that U^ω and U'' are both symplectic and have same dimension $(= \dim\, V - \dim\, U)$ so from Theorem 1.2.3 they are symplectomorphic. We can therefore extend $\overline{\phi}$ to a symplectomorphism between V and V'. $\square$

From this corollary we see that for a symplectic vector space $(V,\ \omega)$ with subspaces U_1 and U_2 with same dimension and such that $rank\ \phi\mid_{U_1} = rank\ \phi\mid_{U_2}$ follows that these subspaces are symplectomorphic. In fact ,we can write $U_i = rad\ U_i \oplus W_i$ as in Theorem 1.5.3, where $\dim\ rad\ U_1 = \dim\ rad\ U_2$ and $\dim\ W_1 = \dim\ W_2$, then map the basis of $rad\, U_1$ to the basis of $rad\, U_2$ and W_1 to W_2. Finally we can, as shown in the corollary, extend this map to a symplectomorphism of V.

For completeness we provide now a proof of Theorem 1.5.3.

Proof. First we prove that we can construct $\overline{U}$ which satisfies the desired properties. We note from the definition that W must be isomorphic to U^{red}, hence it is symplectic. We proceed by induction over the dimension of $rad U$. If this is 0 we have nothing to show because U is already symplectic. Assume the statement holds for $\dim\ rad\, U < r$. Now let $\dim\ rad\, U = r$ and define

$$U_0 := span(e_1, ..., e_{r-1}) \oplus W \qquad .$$

From this construction follows that U_0 is $\omega-$orthogonal to e_r, since $e_r \in rad\, U = span(e_1, ..., e_{r-1}, e_r)$ and $U_0 \subseteq U$. So we have $e_r \in U_0^\omega$. It also holds

$$rad\, U_0^\omega = rad\, U_0 = span(e_1, ..., e_{r-1})$$

so $e_r \in U_0^\omega \backslash rad\, U_0^\omega$. That means we can find a $v \in U_0^\omega$ such that $\omega(e_r, v) \neq 0$. By rescaling v we can call it f_r and get $\omega(e_r, f_r) = 1$. Let $U_r := span(e_r, f_r)$ then, since $U_r \subseteq U_0^\omega$ we have $U_0 \subseteq U_r^\omega$.
Now we can apply the induction hypothesis to U_0 as a subspace of U_r^ω, since dim $rad\, U_0 = r - 1$ and $U_0 = rad\, U_0 \oplus W$. That means we have $f_1, ..., f_{r-1} \in U_r^\omega \subseteq V$ such that $U_i := span(e_i, f_i)$ are symplectic subspaces $\omega-$orthogonal to each other, to W and of course to U_r. So we can define the space

$$\overline{U} := U_1 \oplus ... \oplus U_r \oplus W$$

which has the desired properties.
We now prove the second part of the theorem. Let $\phi : U \to V'$ be an injective and symplectic linear map. Define $e_i' := \phi(e_i)$ for $i \in \{1, ..., r\}$ and $W' := \phi(W)$. Since ϕ is symplectic follows that W' is still symplectic and $\omega'-$orthogonal to $span(e_1', ..., e_r')$. We can apply the first part of the theorem to $\phi(U) = span(e_1', ..., e_r') \oplus W'$ and find $f_1', ..., f_r' \in V'$ such that $\omega'(e_i', f_i') = 1$ and $U_i' := span(e_i', f_i')$ are $\omega'-$orthogonal to W' and to each other. Finally, we can set $\overline{\phi}(f_i) := f_i'$ to extend ϕ to $\overline{U}$ while staying symplectic.

□

1.6 Lagrangian Grassmannian

We now focus more on the set of Lagrangian subspaces.

Definition 1.6.1. The set of all the Lagrangian subspaces of (V, ω) is called *Lagrangian Grassmannian* and is denoted by *Lag(V, ω)*, or simply *Lag(V)*.

In particular we have the following proposition.

Proposition 1.6.2. Let V be a real vector space of dimension $2n$ endowed with two symplectic forms ω_1 and ω_2, such that $Lag(V, \omega_1) = Lag(V, \omega_2)$. Then there exists a real number $\lambda \in \mathbb{R}^*$ such that $\omega_1 = \lambda\omega_2$.

Proof. Let ω_1 and ω_2 be two symplectic forms on V that satisfy $Lag(V, \omega_1) = Lag(V, \omega_2)$. This condition, together with Corollary 1.4.4, implies that the pairs of vector which are symplectically orthogonal to each other with respect to ω_1, respectively ω_2, are the same. Now let $\underline{e} = \{e_1, ..., e_n, f_1, ..., f_n\}$ be a symplectic basis of V with respect to ω_1, that means

$$\omega_1(e_i, f_j) = \delta_{ij} \text{ and } \omega_1(e_i, e_j) = 0 = \omega_1(f_i, f_j) \ \ \forall i, j \in \{1, ..., n\}.$$

Hence, we have

$$\omega_2(e_i, f_j) = \lambda_i \cdot \delta_{ij} \text{ and } \omega_2(e_i, e_j) = 0 = \omega_2(f_i, f_j) \ \ \forall i, j \in \{1, ..., n\}$$

where $\lambda_i \in \mathbb{R}^*\ \forall i \in \{1, ..., n\}$. Now for any $i \neq j$ we can compute

$$\omega_1(e_i + e_j, f_i - f_j) = \omega_1(e_i, f_i) + 0 - 0 - \omega_1(e_j, f_j) = 1 - 1 = 0.$$

Therefore, we also have

$$0 = \omega_2(e_i + e_j, f_i - f_j) = \omega_2(e_i, f_i) + 0 - 0 - \omega_2(e_j, f_j) = \lambda_i - \lambda_j.$$

It follows that for any i and j we have $\lambda_i = \lambda_j =: \lambda$ and $\omega_2(e_i, f_j) = \lambda\delta_{ij} = \lambda\omega_1(e_i, f_j)$. Since $\underline{e}$ is a basis of V we can conclude that for any $u, v \in V$ we have $\omega_2(u, v) = \lambda\omega_1(u, v)$. $\square$

We have seen that for any $L \in Lag(V)$ we have $dim\, L = \frac{1}{2}\, dimV = n$ and $rank\, \omega \mid_L = 0$, hence, from Corollary 1.5.4, we have that for any pair of subspaces L_1, L_2 elements of $Lag(V)$ they are symplectomorphic. That means that the group $Sp(V)$ acts transitively on $Lag(V)$. Thus if we denote by G_L the isotropy group (or stabiliser) of $L \in Lag(V)$, i.e. the subgroup of $Sp(V)$ for which we have $\phi(L) = L$, $\forall \phi \in G_L$, we get to write $Lag(V)$ in the form

$$Lag(V) \cong Sp(V)/G_L$$

To find the matrix representation of an element in G_L we can consider a symplectic basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ of V such that $L = span(e_1, ..., e_n)$. Then if M is an element of $G_L \subseteq Sp(V)$ its representation relative to this basis must be

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $C = 0$ because it must fix L, $A^T D = Id_n$ and $B^T D - D^T B = 0$ because it must be symplectic.

Another set of interesting subspaces of V for a given element $L \in Lag(V)$ is the following

$$\tau(L) := \{L' \in Lag(V) : L \oplus L' = V\}$$

We can see that given two elements of $\tau(L)$, denoted L' and L'', they are symplectomorphic since they both are in $Lag(V)$. If we call ϕ the map that takes L' to L'' we can actually restrict it to L', indeed we get $\phi' : L' \to V$, with $\phi' L' = L''$. Now from Corollary 1.5.4 we can extend ϕ' to a map $\phi'' : V \to V$ which leaves unchanged L, since $L \oplus L' = V = L \oplus L''$. Moreover, ϕ'' is still a symplectomorphism, therefore is an element of G_L. That means G_L acts transitively on $\tau(L)$, indeed, the following statement holds.

Proposition 1.6.3. Let $L \in Lag(V)$, $\tau(L)$ and G_L defined as above, then

$$\tau(L) \cong G_L/GL_n(\mathbb{R})$$

Proof. Let $L' \in \tau(L)$. Find a symplectic basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ of V such that $L = span(e_1, ..., e_n)$ and $L' = span(f_1, ..., f_n)$. Then let $M \in G_L$ which also fixes L', so its matrix representation must have the form

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

for some $A, B \in GL_n(\mathbb{R})$. Since it is in $Sp(V)$, for all $v, w \in V$ we have that

$$v^T M^T \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix} Mw = v^T \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix} w$$

Therefore we deduce $B = (A^T)^{-1}$. That means, if we denote by $G_{L,L'}$ the stabiliser of L' in G_L, that $G_{L,L'} \cong GL_n(\mathbb{R})$ and the proposition follows. $\square$

Corollary 1.6.4. *$\tau(L)$ is an affine space isomorphic to a $\frac{n(n+1)}{2}$−dimensional subspace of $\mathbb{R}^{n^2}$.*

Proof. Form the above statement we can identify $\tau(L)$ with the elements of G_L modulo $GL_n(\mathbb{R})$. We have seen that the matrix representation for $M \in G_L$ is

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

which modulo $G_{L,L'} \cong GL_n(\mathbb{R})$ becomes

$$\begin{pmatrix} Id_n & B' \\ 0 & Id_n \end{pmatrix}$$

where B' is symmetric. So $\tau(L)$ is isomorphic to the space of real symmetric $n \times n$ matrices, which has dimension $\frac{n(n+1)}{2}$. $\square$

Chapter 2

Maslov Indices

2.1 Relations among Classical Lie Groups

For this section we need the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the classical way, i.e. a vector $z = (x, y)$ with $x, y \in \mathbb{R}^n$ corresponds to the vector $x + iy \in \mathbb{C}^n$. The multiplication by $J_0 := -J$, where J is the matrix introduced in Section 1.2, corresponds with the multiplication by i in $\mathbb{C}^n$. With this identification we can consider the group $GL_n(\mathbb{C})$ as a subgroup of $GL_{2n}(\mathbb{R})$. Moreover, we can see the unitary group $U(n) = \{U \in \mathbb{C}^{n\times n} : UU^\dagger = Id_n\}$ as a subgroup of $Sp(\mathbb{R}^{2n})$ in the following way: write $U \in U(n)$ as $U = X + iY$ where $X, Y \in \mathbb{R}^{n\times n}$ and satisfy $XY^T = YX^T$ and $XX^T + YY^T = Id_n$. Then we can define

$$M = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

and it follows from the discussion in Section 1.2 and the properties of X and Y that $M \in Sp(\mathbb{R}^{2n})$. We denote by $O(2n)$ the orthogonal group, which is also subgroup of $GL_{2n}(\mathbb{R})$.

Lemma 2.1.1. *It holds*

$$Sp(\mathbb{R}^{2n}) \cap O(2n) = Sp(\mathbb{R}^{2n}) \cap GL_n(\mathbb{C}) = O(2n) \cap GL_n(\mathbb{C}) = U(n)$$

Proof. Let $M \in GL_{2n}(\mathbb{R})$, then

$$M \in GL_n(\mathbb{C}) \iff MJ_0 = J_0M$$

$$M \in Sp(\mathbb{R}^{2n}) \iff M^TJ_0M = J_0$$

$$M \in O(2n) \iff M^TM = Id_n$$

and any two of these conditions imply the third. For the last equality, we can write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then if $M \in Sp(\mathbb{R}^{2n}) \cap O(2n)$ we have from above that M is also in $GL_n(\mathbb{C})$, it means that $MJ_0 = J_0M$, i.e. $A = D$ and $B = -C$. We also have $M^TJ_0M = J_0$ and $M^TM = Id_n$, so by a straightforward calculation we have $A^TC = C^TA$ and $A^TA + C^TC = Id_n$, which are exactly the condition given above for $U := A + iC$ to be unitary. □

Proposition 2.1.2. The inclusion of $U(n)$ into $Sp(\mathbb{R}^{2n})$ is a homotopy equivalence, in particular $Sp(\mathbb{R}^{2n})$ is a connected space.

It follows a corollary that we have already anticipated in Section 1.2.

Corollary 2.1.3. *For any* $M \in Sp(\mathbb{R}^{2n})$ *we have* $\det M = 1$.

Proof. We need to define a homotopy inverse of the inclusion $\iota : U(n) \hookrightarrow Sp(\mathbb{R}^{2n})$. We define

$$F : Sp(\mathbb{R}^{2n}) \times [0,1] \to Sp(\mathbb{R}^{2n})$$

$$F(M,t) = f_t(M) = M(M^T M)^{-t/2}$$

then $M^T M$, and therefore also its inverse, is symplectic, symmetric and positive definite, so from Lemma 1.3.3 follows that $(M^T M)^{-t/2} \in Sp(\mathbb{R}^{2n})$ for any $t \geq 0$. Hence f_t is well defined and is continuous. We have $f_0 = id$ and $f_t \circ \iota = id$ since for $M \in U(n)$ follows $M^T M = Id$. We also have $f_1(Sp(\mathbb{R}^{2n}) \subseteq U(n)$ because $f_1(M)$ is symplectic and orthogonal. So if we define $g := f_1 : Sp(\mathbb{R}^{2n}) \to U(n)$ we get

$$g \circ \iota = id_{U(n)} \text{ and } \iota \circ g \simeq f_0 = id_{Sp(\mathbb{R}^{2n})}$$

hence g is the homotopy inverse of ι and $Sp(\mathbb{R}^{2n}) \simeq U(n)$. □

Corollary 2.1.4. *For every* $M \in Sp(\mathbb{R}^{2n})$, *there exists a unique symplectic polar decomposition, namely we can write*

$$M = UP$$

where

$$U := M(M^T M)^{-1/2} \in U(n) \text{ and } P := (M^T M)^{1/2}$$

is symmetric and positive definite.

Proof. Since $M \in Sp(\mathbb{R}^{2n}) \subseteq GL_{2n}(\mathbb{R})$ we know that M has a unique polar decomposition, which is the one written above. From the proof of previous proposition we see that both U and P are indeed in $Sp(\mathbb{R}^{2n})$. □

Proposition 2.1.5. The fundamental group of $U(n)$ is isomorphic to the integers.

The idea of the proof is to show that the map $det : U(n) \to S^1$ induces an isomorphism $\pi_1(U(n)) \to \pi_1(S^1) \cong \mathbb{Z}$. An explicit proof can be found in [5].

Corollary 2.1.6. *The fundamental group of* $Sp(\mathbb{R}^{2n})$ *is isomorphic to the integers.*

This follows from the homotopy invariance of the fundamental group. An explicit isomorphism between $\pi_1(Sp(\mathbb{R}^{2n}))$ and $\mathbb{Z}$ is given by the so called Maslov index.

2.2 Maslov Index for Loops of Matrices

Theorem 2.2.1 (Maslov Index for loops of symplectic matrices)**.** *There exists a unique function* μ *called Maslov index, which assigns to every loop*

$$\alpha : [0,1] \to Sp(\mathbb{R}^{2n})$$

an integer $\mu(\alpha) \in \mathbb{Z}$ *and satisfies the following axioms:*

1. *Two loops are homotopic relative to endpoints if and only if they have the same Maslov index.*

2. *For any two loops* α *and* β *we have* $\mu(\alpha\beta) = \mu(\alpha) + \mu(\beta)$

3. *For* $n' + n'' = n$ *we can identify* $Sp(\mathbb{R}^{2n'}) \oplus Sp(\mathbb{R}^{2n''})$ *as a subgroup of* $Sp(\mathbb{R}^{2n})$ *then* $\mu(\alpha' \oplus \alpha'') = \mu(\alpha') + \mu(\alpha'')$

4. The loop $\alpha : [0,1] \to Sp(\mathbb{R}^2)$ where

$$\alpha(t) = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}$$

has Maslov index 1.

Hence, μ induces an isomorphism between $\pi_1(Sp(\mathbb{R}^{2n}))$ and $\mathbb{Z}$.

Proof. We first define a map $\rho : Sp(\mathbb{R}^{2n}) \to S^1$ in the following way: let M be in $Sp(\mathbb{R}^{2n})$ and U be the orthogonal part in its polar decomposition. We can define

$$\rho(M) := \det(U) \qquad .$$

Then for a loop $\alpha : [0,1] \to Sp(\mathbb{R}^{2n})$ we define

$$\mu(\alpha) := deg(\rho \circ \alpha) \qquad .$$

An equivalent definition of this degree is the following: let g be the function

$$\begin{aligned} g : \mathbb{R} &\to S^1 \\ t &\mapsto e^{2\pi i t} \end{aligned}$$

and let $h : [0,1] \to \mathbb{R}$ be the lift of $\rho \circ \alpha$ such that $g \circ h = e^{2\pi i h(t)} = \rho \circ \alpha(t) = \det(U(t))$, where $U(t)$ is the orthogonal part in the polar decomposition of $\alpha(t) \in Sp(\mathbb{R}^{2n})$. Then we have $\mu(\alpha) = h(1) - h(0)$. Since α is a loop $\mu(\alpha)$ is in $\mathbb{Z}$. It follows from Propositions 2.1.2 and 2.1.5 that ρ induces an isomorphism of fundamental groups $\pi_1(Sp(\mathbb{R}^{2n}) \to \mathbb{Z}$, this proves the first axiom. The second axiom holds for loops α and β in $\pi_1(U(n))$, since then also $\alpha\beta$ is in $\pi_1(U(n))$, and the orthogonal part of its polar decomposition is the product of the ones of α and β. Since $\pi_1(U(n)) \cong \pi_1(Sp(\mathbb{R}^{2n}))$ the axiom holds for every symplectic loops. The third axiom follows from the construction of ρ and μ, and the fourth axiom is obvious because $\alpha(t) \in Sp(\mathbb{R}^{2n}) \cap O(2n) = U(n)$ so the orthogonal part in its polar decomposition is $U(t) = \cos(2\pi t) + i\, \sin(2\pi t) = e^{2\pi i t}$ so $\mu(\alpha) = h(1) - h(0) = 1 - 0$. So we have constructed a function μ which satisfies the required properties. We still have to prove that this function is indeed unique. Let us assume that it exists another function μ' which satisfies the four axioms and we'll prove by induction on n that $\mu = \mu'$. For the basic case $n = 1$ we have that $Sp(\mathbb{R}^2) = O(2)$ and therefore from the fourth axiom follows that for every loop α which takes values in $Sp(\mathbb{R}^2)$ must holds $\mu'(\alpha) = 1 = \mu(\alpha)$. Now we can assume that $\mu'(\alpha) = \mu(\alpha)$ for every loop that takes values in $Sp(\mathbb{R}^{2k})$ for some $1 \leq k < n$. Let $\alpha : [0,1] \to Sp(\mathbb{R}^{2n})$ be a loop, and let $s = \mu(\alpha)$. Since we have proved that $Sp(\mathbb{R}^{2n})$ retracts to $U(n)$ we can homotopically deform α such that it takes values in $U(n)$. Next we define the loop

$$u(t) = \begin{pmatrix} e^{2\pi i s t} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in U(n)$$

which takes values in $U(n)$. It follows from the definition of μ that $\mu(u) = s$ so from axiom one we have that α and u are homotopic. We can now decompose $u(t)$ in a sum of two elements $u = u_1 \oplus u_2$ where u_1 is the element

$$u_1 = \begin{pmatrix} \cos(2\pi s t) & -\sin(2\pi s t) \\ \sin(2\pi s t) & \cos(2\pi s t) \end{pmatrix} \in Sp(\mathbb{R}^2)$$

and u_2 is a constant loop in $Sp(\mathbb{R}^{2(n-1)})$. We can now apply axiom three and the induction hypothesis to get

$$\mu'(\alpha) = \mu'(u) = \mu'(u_1) + \mu'(u_2) = s + 0 = \mu(\alpha).$$

□

2.3 Maslov Index for Loops of Lagrangian Subspaces

Next, we look at the Maslov index for loops of Lagrangian subspaces. First, we need a new way to identify $Lag(V)$. Since we have proved that any symplectic space is symplectomorphic to $\mathbb{R}^{2n}$ we look at the elements of $Lag(V)$ as subspaces of $\mathbb{R}^{2n}$. In section 1.4 we have seen that given two elements L_1, L_2 of $Lag(V)$ they are symplectomorphic to each other, we now prove that the element of $Sp(\mathbb{R}^{2n})$ that take L_1 to L_2 is in $U(n)$.

Lemma 2.3.1. *Let L_1 and L_2 be in $Lag(\mathbb{R}^{2n})$, then there exists a symplectic matrix $M \in Sp(\mathbb{R}^{2n}) \cap O(2n) = U(n)$ such that $L_2 = ML_1$.*

Proof. We consider our vector space as $\mathbb{R}^{2n}$ with the symplectic basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ and we show that for every $L \in Sp(\mathbb{R}^{2n})$ there exists an element M of $U(n)$ that take the Lagrangian subspace $L_0 := span(e_1, ..., e_n)$ to L. We construct the matrix M in the following way, let X, Y be real matrices in $\mathbb{R}^{n \times n}$ and define the matrix

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^{2n \times n}$$

so that $A := Im(Z)$ is a subspace of $\mathbb{R}^{2n}$. The elements of A are of the form $z = (Xu, Yu)$, where u is a vector in $\mathbb{R}^{2n}$. The conditions necessary and sufficient for Z such that A is a Lagrangian subspace are: $rankZ = n$ and $X^TY = Y^TX$, so that for any $z = (Xu, Yu)$ and $z' = (Xu', Yu')$ we have $\omega(z, z') = u^T(X^TY - Y^TX)u' = 0$. With a rescaling of X and Y we can get the columns of Z to be an orthonormal basis of A, in this case holds also

$$U := X + iY \in U(n)$$

Now we can define Z as above such that its columns are an orthonormal basis of L and let

$$M := \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

then $M \in Sp(\mathbb{R}^{2n}) \cap O(2n)$ and $ML_0 = L$. $\square$

Corollary 2.3.2. *$U(n)$ acts transitively on $Lag(V)$ and we can write $Lag(V) \cong U(n)/O(n)$.*

This follows from the proof of previous lemma and the fact that $U = X + iY \in U(n)$ is determined by L up to right multiplication by an element of $O(n)$.

From Corollary 2.3.2, Proposition 2.1.5 and the homotopy long exact sequence

$$... \to \pi_1(O(n)) \to \pi_1(U(n)) \to \pi_1(Lag(\mathbb{R}^{2n})) \to 0$$

where can be computed that the map $\pi_1(O(n)) \to \pi_1(U(n))$ is the zero map, it follows that also $\pi_1(Lag(V))$ is isomorphic to $\mathbb{Z}$. Also in this case the explicit isomorphism is a function called Maslov Index.

Theorem 2.3.3 (Maslov Index for loops of Lagrangian subspaces)**.** *There exists a unique function μ called Maslov Index which assigns to any loop*

$$\alpha : [0,1] \to Lag(\mathbb{R}^{2n})$$

an integer $\mu(\alpha) \in \mathbb{Z}$ and satisfies the following axioms:

1. *Two loops are homotopic relative to endpoints if and only if they have the same Maslov index.*
2. *For any two loops $\alpha : [0,1] \to Lag(\mathbb{R}^{2n})$ and $\beta : [0,1] \to Sp(\mathbb{R}^{2n})$ we have*

$$\mu(\beta\alpha) = \mu(\alpha) + 2\mu(\beta)$$

3. *For $n' + n'' = n$ we can identify $Lag(\mathbb{R}^{2n'}) \oplus Lag(\mathbb{R}^{2n''})$ as a subspace of $Lag(\mathbb{R}^{2n})$, then*

$$\mu(\alpha' \oplus \alpha'') = \mu(\alpha') + \mu(\alpha'')$$

4. *A constant loop α_0 has Maslov index zero.*

5. *The loop*

$$\alpha : [0,1] \to Lag(\mathbb{R}^2)$$
$$\alpha(t) := e^{\pi i t}\mathbb{R}$$

has Maslov index one.

Proof. The construction is very similar to the one of Theorem 2.2.1. Again we first define a function $\rho : Lag(\mathbb{R}^{2n}) \to S^1$ in the following way: let $L \in Lag(\mathbb{R}^{2n})$ and $Z = \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^{2n\times n}$ as in the proof of Lemma 2.3.1 such that $L = Im(Z)$ and $U := X + iY \in U(n)$. Then we define

$$\rho(L) := \det(U^2)$$

Again for a loop $\alpha : [0,1] \to Lag(\mathbb{R}^{2n})$ we let the Maslov index to be

$$\mu(\alpha) := deg(\rho \circ \alpha)$$

or equivalently for a lift $h : [0,1] \to \mathbb{R}$ such that

$$e^{2\pi i h(t)} = \rho \circ \alpha(t) = \det(U(t)^2)$$

we have

$$\mu(\alpha) = h(1) - h(0)$$

It follows that μ takes values in $\mathbb{Z}$ and it depends only on the homotopy class of the loop α. Conversely if we have two loops α_1 and α_2 such that $\mu(\alpha_1) = \mu(\alpha_2)$ we show that they are indeed homotopic. Without loss of generalities we can assume that $\alpha_1(0) = \alpha_2(0) = \mathbb{R}^n \times \{0\}$. Let $U_j(t) = X_j(t) + i\,Y_j(t)$ a path in U(n) such that $\alpha_j(t) = Im\begin{pmatrix} X_j(t) \\ Y_j(t) \end{pmatrix}$ for $j = 1, 2$ and $U_j(0) = Id$. With right multiplication by an orthogonal matrix we can get

$$U_j(1) = \begin{pmatrix} \pm 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

so we can define $U(t) := U_2(t)U_1(t)^{-1}$ which is a loop in $\pi_1(U(n))$. We also have that this loop is contractible since $deg(det(U_1(t)^2)) = \mu(\alpha_1) = \mu(\alpha_2) = deg(\det(U_2(t)^2))$ which implies that the loop $\det(U(t)) \in \pi_1(S^1)$ is contractible and so, with Proposition 2.1.5 we have that $U(t)$ is contractible. That means that $U_1(t)$ and $U_2(t)$ are homotopic and therefore so are α_1 and α_2. This proves axiom 1. The second axiom follows from the construction of ρ and μ and the fact that a loop $\beta \in \pi_1(Sp(\mathbb{R}^{2n}))$ is homotopic to a loop in $\pi_1(U(n))$, this follows from Proposition 2.1.2. The third, fourth and fifth axioms follow from straightforward calculation. Again the axioms define uniquely the Maslov index.

□

Remark. An alternative definition, actually the first one that was introduced, of the Maslov index for loops of Lagrangian subspaces is the (signed) number of intersection of a loop α with the so called Maslov cycle $\Sigma(n)$. This one is defined as the set of all the Lagrangian subspaces that intersect non transversally the subspace $\{0\} \times \mathbb{R}^n$, i.e.,

$$\Sigma(n) := \{L \in Lag(\mathbb{R}^{2n}) : L \cap (\{0\} \times \mathbb{R}^n) \neq \{0\}\}$$

It can then be proved that this definition also satisfies the five axioms of Theorem 2.3.3.

Chapter 3

Compatible Triples

3.1 Additional Structures on a Symplectic Vector Space

In this chapter we focus on the complex structures and how they act on a symplectic vector space. First we give the definition of complex structure.

Definition 3.1.1. A *linear complex structure* on a vector space V is an automorphism

$$J : V \to V$$

such that

$$J^2 = -Id \qquad .$$

A real vector space V of even dimension (like a symplectic vector space) with such a structure J becomes a complex vector space, where the multiplication by i corresponds to the multiplication by J. So the scalar multiplication over $\mathbb{C}$ is given by

$$\mathbb{C} \times V \to V$$

$$(x + iy, v) \mapsto (x + yJ)v$$

Example 3.1.2. The automorphism of $\mathbb{R}^{2n}$ given by the matrix

$$J_0 = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$$

is a linear complex structure and is called the standard complex structure.

Now we show that every linear complex structure is isomorphic to the standard complex structure of Example 3.1.2.

Proposition 3.1.3. Let V be a real vector space of dimension $2n$ and let J be a complex structure on V, then there exists an isomorphism

$$\Phi : \mathbb{R}^{2n} \to V$$

such that

$$J\Phi = \Phi J_0$$

Proof. From linear algebra we know that every complex vector space has a complex basis, i.e. there exists a basis $\{v_1, ..., v_n\}$ over $\mathbb{C}$. Then $\{v_1, Jv_1, ..., v_n, Jv_n\}$ is a basis over $\mathbb{R}$ and we can define Φ

in the following way

$$\Phi u = \sum_{j=1}^{n}(x_j v_j + y_j J v_j)$$

where $u = (x_1, ..., x_n, y_1, ..., y_n)$. It is then easy to check that this map satisfies the required property. □

Next we look at the case where a complex structure is defined on a symplectic vector space V.

Definition 3.1.4. A complex structure J on a symplectic vector space (V, ω) is called ω-compatible (or just compatible) if

$$g(v, w) := \omega(v, Jw)$$

defines a positive definite inner product.

Remark. It follows from the definition that a compatible complex structure on V is also an element of $Sp(V)$. Indeed using the symmetry property of an inner product we have

$$\omega(Jv, Jw) = g(Jv, w) = g(w, Jv) = \omega(w, J^2 v) = -\omega(w, v) = \omega(v, w).$$

As before we have a symplectic vector space (V, ω). If it has a compatible complex structure J we can consider it as a complex vector space as done above. Moreover, it can become a Hermitian vector space (i.e. a complex vector space with a Hermitian inner product) where the Hermitian inner product is given by

$$h(v, w) := g(v, w) + i\omega(v, w)$$

where g is the inner product induced by ω and J. h is complex linear with respect to the second entry and complex anti-linear with respect to the first entry:

$$h(v, Jw) = g(v, Jw) + i\omega(v, Jw) = -\omega(v, w) + ig(v, w) = ih(v, w)$$

$$h(Jv, w) = g(Jv, w) + i\omega(Jv, w) = \omega(v, w) - ig(v, w) = -ih(v, w)$$

moreover $h(v, v) > 0$ for $v \neq 0$. It follows from the construction of h that an element $A \in Sp(V)$ that preserves h must also preserve the positive definite inner product g, hence, by Lemma 2.1.1, that is an element of the unitary group.

Definition 3.1.5. When we have a compatible complex structure J on a symplectic vector space (V, ω), and g is the inner product induced by ω and J, we call the triple (V, ω, J) a Kähler vector space while (ω, J, g) is called a compatible triple.

Remark. We can equivalently define a compatible complex structure in the following way: J on (V, ω) is ω-compatible if $\forall v, w \in V$ we have

$$\omega(Jv, Jw) = \omega(v, w)$$

and

$$\omega(v, Jv) > 0 \quad \forall v \neq 0.$$

We then have as a consequence that

$$g_J := \omega(v, Jw)$$

defines a positive definite inner product. The positive definiteness comes from the second condition, moreover g_J is bilinear because so is ω. Lastly the symmetry follows from

$$g_J(w, v) = \omega(w, Jv) = -\omega(Jv, w) = \omega(Jv, -w) = \omega(Jv, J^2 w) = \omega(v, Jw) = g_J(v, w)$$

3.2 The Space $\mathcal{J}(V, \omega)$

Proposition 3.2.1. Let (V, ω) be a symplectic vector space and J a linear complex structure on V. The following are equivalent

(i) J is ω-compatible

(ii) (V, ω) has a symplectic basis of the form
$$v_1, ..., v_n, Jv_1, ..., Jv_n$$

(iii) There exists an isomorphism $\Phi : \mathbb{R}^{2n} \to V$ such that

$$\Phi^* \omega = \omega_0 \text{ and } \Phi^* J = J_0$$

where ω_0 is the symplectic form of Example 1.1.3 and J_0 the standard complex structure of Example 3.1.2

(iv) J satisfies $\omega(v, Jv) > 0 \ \ \forall v \neq 0$ and $\forall L \in Lag(V)$ we have $JL \in Lag(V)$

Proof. We first prove that (i), (ii) and (iii) are equivalent, and then that so are also (i) and (iv). (i) implies (ii) because, from Lemma 1.4.3, V has a Lagrangian subspace L. Let $\{v_1, ..., v_n\}$ be a basis of L orthonormal with respect to the inner product g induced by J. Then we have

$$\omega(v_i, Jv_j) = g(v_i, v_j) = \delta_{i,j}$$

and

$$\omega(Jv_i, Jv_j) = \omega(v_i, v_j) = 0$$

Hence $\{v_1, ..., v_n, Jv_1, ..., Jv_n\}$ is a symplectic basis of (V, ω).

We see that (ii) implies (iii) by defining

$$\Phi : \mathbb{R}^{2n} \to V$$

$$z \mapsto \sum_{i=1}^{n} (x_i v_i + y_i J v_i)$$

for $z = (x_1, ..., x_n, y_1, ..., y_n)$ and (iii) follows from a simple computation. (iii) implies (i) because $J_0^* \omega_0 = \omega_0$ and Φ^* is an intertwining.

We now prove that (i) implies (iv). If J is compatible it follows from definition that $\omega(v, Jv) > 0$ for any nonzero $v \in V$. Moreover, for any $u, v \in L$ we have $\omega(Ju, Jv) = \omega(u, v) = 0$ so that $\omega\mid_{JL} = 0$. This together with the fact that J is an isomorphism proves that $JL \in Lag(V)$. Conversely if we assume (iv) we prove that $g(v, w) := \omega(v, Jw)$ defines an inner product. It is positive definite by assumption and bilinear by construction. Now assume g is not symmetric, i.e. there exist $u, v \in V$ such that

$$\omega(v, Ju) \neq \omega(u, Jv)$$

obviously v is nonzero, so $\omega(v, Jv) > 0$. Now define

$$w := u - \tfrac{\omega(v,Ju)}{\omega(v,Jv)} v$$

so we get

$$\omega(w, Jv) = \omega(u, Jv) - \omega(v, Ju) \neq 0$$

and we have that w, Jv and v, Jw are linearly independent. Moreover, since

$$\omega(v,Jw)=\omega(v,Ju)-\tfrac{\omega(v,Jv)}{\omega(v,Jv)}\omega(v,Ju)=0$$

we have a $L\in Lag(V)$ such that $v,Jw\in L$. So it follows $Jv,w\in JL$, but, since $\omega(w,Jv)\neq 0$, $JL\notin Lag(V)$, which contradicts the assumption of (iv), so g must be symmetric. Hence, we have proved that J is compatible. □

We denote by $\mathcal{J}(V,\omega)$ the space of compatible complex structures and, following the notation of [6], we denote by $Riem(V)$ the set of all positive definite inner products on V, which is a convex open subset of the space of symmetric bilinear forms S^2V^*.

Theorem 3.2.2. *Let (V,ω) be a symplectic vector space, then there exists a canonical surjective map*

$$F: Riem(V)\to\mathcal{J}(V,\omega)$$

Moreover, if we denote by G the map

$$G:\mathcal{J}(V,\omega)\to Riem(V)$$

which assigns to each $J\in\mathcal{J}(V,\omega)$ its induced inner product g we get $F\circ G(J)=J$.

Proof. To construct the map F we precede in the following way. First given a positive definite inner product $k\in Riem(V)$ we define an invertible matrix A such that $\forall v,w\in V$ we have

$$k(v,w)=\omega(v,Aw)$$

Since k is symmetric while ω is skew-symmetric follows that $A^T=-A$. So if we compute the polar decomposition of A we get

$$A=UP$$

where $U=A(A^TA)^{-1/2}=A(-A^2)^{1/2}$ and $P=(A^TA)^{1/2}=(-A)^{1/2}$. That means that U and P commute, therefore we get

$$A^2=U^2P^2=U^2(-A^2)$$

so $U^2=-Id$, which means U is a complex structure. Moreover we have

$$\omega(v,Uw)=\omega(v,AP^{-1}w)=k(v,P^{-1}w)=k(P^{-1/2}v,P^{-1/2}w)$$

so $\omega(v,Uw)$ is a positive definite inner product. We can then define $F(k)=U$, this function satisfies by construction $F\circ G=Id$ and is surjective. □

We now state and prove a proposition which actually will be a consequence of the next theorem, but that can already be proved with the previous theorem.

Proposition 3.2.3. Let (V,ω) be a symplectic vector space. Then given any two compatible complex structure $J_0,J_1\in\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ they are homotopic, i.e. there exists a function which assigns to every $t\in[0,1]$ a compatible complex structure, and such that for $t=0$ it gives J_0 and for $t=1$ it gives J_1. This function defines a path in $\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ from J_0 to J_1, so it means that $\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ is path connected.

Proof. In the previous theorem we have proved that given any $k\in Riem(V)$ it arises a compatible complex structure, which can be called J_k, and that any $J\in\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ can be generated in this way. This means given $J_0,J_1\in\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ there exist $k_0,k_1\in Riem(V)$ such that $J_{k_i}=J_i$ for $i=1,2$. Next, since $Riem(V)$ is convex, we can define

$$k_t := tk_0 + (1-t)k_1, \quad t \in [0,1]$$

and we have that the arisen compatible complex structure J_{k_t} is the function we were looking for. $\square$

3.3 Contractibility of $\mathcal{J}(V,\omega)$

Theorem 3.3.1. *Let (V,ω) be a symplectic vector space of dimension $2n$, then $\mathcal{J}(V,\omega)$ is contractible.*

For this theorem there exist many proofs. We first provide two brief proofs and then sketch another one a bit more interesting.

Proof. (1) Let $X := Riem(V)$ and $Y := \mathcal{J}(V,\omega)$. We know that X is contractible because it is a convex subset of a vector space. Let $\phi : X \times I \to X$ be a contraction, where $\phi_0 = Id_X$ and ϕ_1 is a constant map to a point of X. Then using the functions F and G from Theorem 3.2.2 and defining $\psi := F \circ \phi \circ (Id \times G)$, we get that ψ is a contraction of Y. $\square$

Proof. (2) In this proof we assume $V = \mathbb{R}^{2n}$ and $\omega = \omega_0$ (this is possible due to Theorem 1.2.3) and we prove that $\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ is diffeomorphic to the space of symmetric, positive definite, symplectic matrices in $\mathbb{R}^{2n\times 2n}$, this space is indeed contractible from Lemma 1.3.3. First we have that a matrix $J \in \mathbb{R}^{2n\times 2n}$ is in $\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ if it satisfies

$$J^2 = -Id \qquad J^T J_0 J = J_0 \qquad v^T(-J_0)Jv > 0 \quad \forall v \in V$$

where the first condition ensures that J is a complex structure, and the other two that it is ω_0-compatible. From these condition follows that

$$(J_0 J)^T = -J^T J_0 = J^T J_0 J^2 = J_0 J$$

so we can define $P := -J_0 J$ which is symplectic, symmetric and positive definite by construction. On the contrary if we have a matrix P symplectic, symmetric and positive definite and we define $J := -J_0^{-1}P = J_0 P$ follows that $J \in \mathcal{J}(\mathbb{R}^{2n},\omega_0)$. In fact we have

$$v^T(-J_0)Jv = V^T(-J_0)J_0 Pv = v^T Pv > 0$$

since P is positive definite,

$$J^T J_0 J = P^T J_0^T J_0 J_0 P = P^T J_0 P = J_0$$

because P is symplectic, and

$$J^2 = J_0 P J_0 P = J_0 P^T J_0 P = J_0 J_0 = -Id$$

because P is symmetric. Therefore we have established a diffeomorphism between these two space, which proves the theorem.

$\square$

Before we present the last proof of Theorem 3.3.1, we need to introduce the concept of Siegel upper half plane.

Definition 3.3.2. The *Siegel upper half plane* is defined as the open contractible subspace of $\mathbb{C}^{n\times n}$ composed by symmetric matrices of the form $Z = X + iY$, where the imaginary part Y is positive definite. It is denoted by $\mathcal{S}_n$.

It can then be proved (see Appendix A) that the symplectic group $Sp(\mathbb{R}^{2n})$ acts transitively on $\mathcal{S}_n$ and that the stabiliser of $(i \cdot Id) \in \mathcal{S}_n$ is the subgroup $U(n) \subseteq Sp(\mathbb{R}^{2n})$ with the usual identification. So we can write

$$\mathcal{S}_n \cong Sp(\mathbb{R}^{2n})/U(n).$$

Proof. (3) The idea of this proof (which we give in a non rigorous way) is to show that $\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ is diffeomorphic to $\mathcal{S}_n$. It follows from Proposition 3.2.1 that $Sp(\mathbb{R}^{2n})$ acts transitively on $\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ and from Lemma 2.1.1 we know that the stabiliser of J_0 is $Sp(\mathbb{R}^{2n}) \cap GL_n(\mathbb{C}) = U(n)$. Therefore there exists a bijection between $\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ and $\mathcal{S}_n$. This map can be construct in the following way:

$$J : \mathcal{S}_n \to \mathcal{J}(\mathbb{R}^{2n},\omega_0)$$

with

$$J(Z) = \begin{pmatrix} XY^{-1} & -Y - XY^{-1}X \\ Y^{-1} & -Y^{-1}X \end{pmatrix} \text{ where } Z = X + iY \in \mathcal{S}_n$$

it can then be proved that this map is actually a diffeomorphism. □

Remark. It follows from the third proof of Theorem 3.3.1 that

$$\mathcal{J}(\mathbb{R}^{2n},\omega_0) \cong Sp(\mathbb{R}^{2n})/U(n)$$

which leads to the next corollary.

Corollary 3.3.3. *$\mathcal{J}(\mathbb{R}^{2n},\omega_0)$ has dimension $(2n^2+n) - n^2 = n^2 + n$.*

Chapter 4

The Affine Non-Squeezing Theorem

4.1 General Statement of Theorem

The theorem we are now going to treat is very important and is one of the first results based on the theory outlined in previous chapters that shows a counter-intuitive aspect of symplectic geometry. In the whole chapter we use as symplectic vector space $\mathbb{R}^{2n}$ and denote by ω_0 the symplectic standard form as in Example 1.1.3. We also need some new definitions.

Definition 4.1.1. A map $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called *affine symplectomorphism* if it has the form

$$\phi(z) = Mz + z_0$$

where $M \in Sp(\mathbb{R}^{2n})$ and $z_0 \in \mathbb{R}^{2n}$. The group of affine symplectomorphisms is denoted by $ASp(\mathbb{R}^{2n})$.

Definition 4.1.2. Let $\{e_1, ..., e_n, f_1, ..., f_n\}$ be the standard symplectic basis of $\mathbb{R}^{2n}$ with the symplectic standard form ω_0. Then the *symplectic cylinder of radius R* is defined as

$$Z^{2n}(R) := B^2(R) \times \mathbb{R}^{2n-2} = \{z \in \mathbb{R}^{2n} : \langle e_1, z\rangle^2 + \langle f_1, z\rangle^2 \leq R^2\}$$

where $B^2(R)$ is the two dimensional ball of radius R and $\langle \cdot\,, \cdot\rangle$ the Euclidean inner product in $\mathbb{R}^{2n}$.

Remark. The Euclidean inner product is the inner product generated by ω_0 and the compatible complex structure J_0 used in Example 3.1.2, indeed, we have $\langle u\,, v\rangle = \omega_0(u, J_0 v)$ for any u and v.

We can now state the main theorem of this chapter, which claims that a ball in $\mathbb{R}^{2n}$ can only be embedded by an affine symplectomorphism into a symplectic cylinder if the radius of the ball is smaller or equal to the radius of the cylinder. In other words, it is impossible to "squeeze" a ball more than its "symplectic width" allows.

Theorem 4.1.3. *Given $\phi \in ASp(\mathbb{R}^{2n})$ and assume that*

$$\phi(B^{2n}(r)) \subset Z^{2n}(R)$$

then it follows $r \leq R$.

Proof. Without loss of generality we can assume $r = 1$ and show $R \geq 1$. We can also write ϕ in the form $\phi(z) = Az + z_0$ for a $A \in Sp(\mathbb{R}^{2n})$. Now let $\{e_1, ..., e_n, f_1, ..., f_n\}$ be a symplectic basis and define

$$u := A^T e_1 \quad v := A^T f_1 \quad a := \langle e_1, z_0\rangle \quad b := \langle f_1, z_0\rangle$$

Then the assumption $\phi(B^{2n}(r)) \subset Z^{2n}(R)$ can be written as

$$\sup_{|z|\le 1} \left(\langle e_1, \phi(z)\rangle^2 + \langle e_2, \phi(z)\rangle^2\right) \le R^2$$

$$\Rightarrow \quad \sup_{|z|\le 1} \left(\langle e_1, Az + z_0\rangle^2 + \langle e_2, Az + z_0\rangle^2\right) \le R^2$$

$$\Rightarrow \quad \sup_{|z|\le 1} \left((\langle A^T e_1, z\rangle + \langle e_1, z_0\rangle)^2 + (\langle A^T e_2, z\rangle + \langle e_2, z_0\rangle)^2 \le R^2\right.$$

$$\Rightarrow \quad \sup_{|z|\le 1} \left((\langle u, z\rangle + a)^2 + (\langle v, z\rangle + b)^2\right) \le R^2$$

Now, since $A \in Sp(\mathbb{R}^{2n})$,we also have $A^T \in Sp(\mathbb{R}^{2n})$, and it follows $1 = \omega_0(e_1, f_1) = \omega_0(u, v) = \langle u, J_0^{-1} v\rangle \le |u|\,|-J_0 v| = |u|\,|v|$, where the inequality follows from the Cauchy-Schwarz inequality. So we can assume without loss of generality $|u| \ge 1$ and choose $z_0 := \pm\frac{u}{|u|}$, where the sign of z_0 is chosen to be equal the sign of a. So we get

$$1 \le |u|^2 \le (|u| + |a|)^2 \le (\langle u, z_0\rangle + a)^2 + (\langle v, z_0\rangle + b)^2 \le R^2$$

and therefore we can conclude $R^2 \ge 1$. □

4.2 Linear Non-squeezing Property

In this section, we define a concept, the non-squeezing property, that generalises the discussion introduced with Theorem 4.1.3 and we will also see some consequences of the theorem. First we need some more definitions.

Definition 4.2.1. A matrix $M \in \mathbb{R}^{2n\times 2n}$ is said to be anti-symplectic if we have $M^*\omega_0 = -\omega_0$.

Definition 4.2.2. A set $B \subseteq \mathbb{R}^{2n}$ is called a linear (resp. affine) symplectic ball of radius r, if there exists an element $M \in Sp(\mathbb{R}^{2n})$ (resp. $M \in ASp(\mathbb{R}^{2n})$) such that $MB^{2n}(r) = B$, where $B^{2n}(r)$ is the $2n$-dimensional ball of radius r.

Definition 4.2.3. A set $Z \subseteq \mathbb{R}^{2n}$ is called a linear (resp. affine) symplectic cylinder of radius R, if there exists an element $M \in Sp(\mathbb{R}^{2n})$ (resp. $M \in ASp(\mathbb{R}^{2n})$) such that $MZ^{2n}(R) = Z$, where $Z^{2n}(R)$ is defined as in the previous section.

Definition 4.2.4. A matrix $M \in \mathbb{R}^{2n\times 2n}$ is said to have the *linear non-squeezing property* if for all linear symplectic ball B of radius r and for all linear symplectic cylinder Z of radius R we have

$$MB \subseteq Z \quad \Rightarrow \quad r \le R$$

We can now state a theorem, which says that the non-squeezing property characterises the symplectic and anti-symplectic matrices.

Theorem 4.2.5. *Let $M \in \mathbb{R}^{2n\times 2n}$ be invertible such that both M and M^{-1} have the non-squeezing property. Then M is either symplectic or anti-symplectic, i.e. $M^*\omega_0 = \pm\omega_0$.*

Proof. Let assume by contradiction that M is neither symplectic nor anti-symplectic. That means we can find some vectors $u, v \in \mathbb{R}^{2n}$ such that

$$\omega_0(Mu, Mv) \neq \pm\omega_0(u, v)$$

Since M is invertible, and if necessary rescaling u, we can assume without loss of generality $0 < |\omega_0(Mu, Mv)| < |\omega_0(u, v)| = 1$. We can write

$$0 < \lambda^2 := |\omega_0(Mu, Mv)| < \omega_0(u, v) = 1$$

Therefore we can construct two symplectic bases $\{u_1, ..., u_n, v_1, ..., v_n\}$ and $\{u'_1, ..., u'_n, v'_1, ..., v'_n\}$ of $\mathbb{R}^{2n}$ where

$$u_1 = u, \quad v_1 = v, \quad u'_1 = \lambda^{-1}Mu, \quad v'_1 = \pm\lambda^{-1}Mv$$

If we denote by $\phi \in Sp(\mathbb{R}^{2n})$ the matrix which takes the standard symplectic basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ to $\{u_1, ..., u_n, v_1, ..., v_n\}$ and by $\phi' \in Sp(\mathbb{R}^{2n})$ the matrix that takes $\{e_1, ..., e_n, f_1, ..., f_n\}$ to $\{u'_1, ..., u'_n, v'_1, ..., v'_n\}$ we can then define $A := \phi'^{-1}M\phi$. The matrix A satisfies

$$Ae_1 = \lambda e_1 \qquad Af_1 = \pm\lambda f_1$$

which means

$$AB^{2n}(1) \subseteq Z^{2n}(\lambda)$$

$$\Rightarrow \phi'^{-1}M\phi B^{2n}(1) \subseteq Z^{2n}(\lambda)$$

Hence if we denote by B the linear symplectic ball of radius 1 given by $\phi B^{2n}(1)$ and by Z the linear symplectic cylinder of radius λ given by $\phi' Z^{2n}(\lambda)$ we get

$$MB \subseteq Z$$

Since $\lambda < 1$ this is a contradiction with the assumption that M has the non-squeezing property.

$\square$

In the previous section we mentioned the "symplectic width" of a set just in an intuitive way, we can actually give a more rigorous definition.

Definition 4.2.6. Given a subset $A \in \mathbb{R}^{2n}$ we define its *linear symplectic width* as

$$w_L(A) = \sup\{\pi r^2 | \phi(B^{2n}(r)) \subseteq A, \ \phi \in ASp(\mathbb{R}^{2n})\}$$

Directly from this definition follow two properties, namely:

1. **(monotonicity)** If $\phi(A) \subseteq B$ for some $\phi \in ASp(\mathbb{R}^{2n}) \Rightarrow w_L(A) \leq w_L(B)$
2. **(conformality)** $w_L(\lambda A) = \lambda^2 w_L(A)$

Meanwhile a third property follows from Theorem 4.1.3.

3. **(non-triviality)** $w_L(B^{2n}(r)) = w_L(Z^{2n}(r)) = \pi r^2$

From monotonicity and Theorem 4.1.3 follows that elements of $ASp(\mathbb{R}^{2n})$ preserve the linear symplectic width of sets, and in a very similar way we conclude the same thing for anti-symplectic maps. Also in this case we can prove that preserving the linear symplectic width is actually a property that characterises the (anti-)symplectic maps. First we need to recall some definitions.

Definition 4.2.7. A *quadratic form in n variables* is a polynomial Q in n variables with all terms of degree two. We call Q a *positive definite quadratic form*, if for all $x \in \mathbb{R}^n \backslash \{0\}$ we have $Q(x) > 0$.

We also recall that every quadratic form Q can be represented by a unique matrix $A_Q = (a_{ij})_{ij}$ such that for any $x = (x_1, .., x_n)$ we have $Q(x) = x^T A_Q^T x = \sum_{i,j=1}^n a_{ij}x_i x_j$.

Definition 4.2.8. Let Q be a positive definite quadratic form in $2n$ variables. Then the set defined as $E_Q = \{z \in \mathbb{R}^{2n} | Q(z) \leq 1\}$ is called *ellipsoid centred at zero.*

Remark. The $2n$-dimensional ball B of radius r is an ellipsoid centred at zero defined by the positive definite quadratic form $Q(z) = \sum_{i=1}^{2n} \frac{z_i^2}{r^2}$.

Theorem 4.2.9. *Let $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear map. Then the following are equivalent.*

(i) ϕ preserves the linear symplectic width of ellipsoids centred at zero.

(ii) ϕ is either symplectic or anti-symplectic.

Proof. We have already said that (ii) implies (i). Now we prove the contrary. We assume that ϕ satisfies (i) and we prove that it has the non-squeezing property, which means, due to Theorem 4.2.5, that it is actually either symplectic or anti-symplectic. First, we notice that ϕ is invertible, otherwise $\phi B^{2n}(1)$ would have linear linear symplectic width zero. Moreover, also ϕ^{-1} satisfies (i), indeed let E be an ellipsoid then $w_L(E) = w_L(\phi\phi^{-1}E) = w_L(\phi^{-1}E)$. Now, we prove that ϕ (and similarly ϕ^{-1}) has the non-squeezing property. Let B be a linear symplectic ball of radius r and Z a linear symplectic cylinder of radius R such that $\phi B \subseteq Z$. Then from (i) and the properties of w_L we have

$$\pi r^2 = w_L(B) = w_L(\phi B) \leq w_L(Z) = \pi R^2$$

which means $r \leq R$. Therefore ϕ and ϕ^{-1} have the non-squeezing property, hence ϕ is either symplectic or anti-symplectic. □

4.3 Linear Symplectic Width of an Ellipsoid

The main result we want to show in this section is following theorem.

Theorem 4.3.1. *Let $E \subseteq \mathbb{R}^{2n}$ be an ellipsoid centred at zero. Then we have*

$$w_L(E) = \sup_{B \subset E} w_L(B) = \inf_{Z \supset E} w_L(Z)$$

where the supremum runs over all affine symplectic balls contained in E and the infimum runs over all affine symplectic cylinders containing E.

In order to prove this theorem we first need two lemmas.

Lemma 4.3.2. *Let (V, ω) be a symplectic vector space of dimension $2n$ and $g : V \times V \to \mathbb{R}$ be an inner product. Then there exists a basis $\{u_1, ..., u_n, v_1, ...v_n\}$ of V which is orthogonal with respect to g and a standard symplectic basis with respect to ω. Moreover it can be chosen such that*

$$g(u_i, u_i) = g(v_i, v_i), \quad \forall i \in \{1, ..., n\}$$

Proof. A proof of this lemma can be found in [5] □

For the next lemma we need to introduce some notation. Let r be an n-tuple, $r = (r_1, ..., r_n)$, such that $0 < r_1 \leq .... \leq r_n$. Then we define the ellipsoid $E(r)$ as the set

$$E(r) = \{z \in \mathbb{C}^n | \sum_{i=1}^n |\tfrac{z_i}{r_i}|^2 \leq 1\}$$

Lemma 4.3.3. *Given any compact ellipsoid*

$$E = \{w \in \mathbb{R}^{2n} | \sum_{i,j=1}^{2n} a_{ij} w_i w_j \leq 1\}$$

then there exists a matrix $\phi \in Sp(\mathbb{R}^{2n})$ such that $\phi E = E(r)$ for an n-tuple $r = (r_1, ..., r_n)$ with $0 < r_1 \leq ... \leq r_n$, which is uniquely determined by E.

The n-tuple r is then called the *symplectic spectrum* of the ellipsoid E and it is invariant under linear symplectic maps. In fact, two ellipsoids in $\mathbb{R}^{2n}$ centred at zero are linearly symplectomorphic if and only if they have the same spectrum.

Proof. We define the positive definite inner product

$$g(v,w) = \sum_{i,j=1}^{2n} a_{ij} v_i w_j$$

on $\mathbb{R}^{2n}$. Then we can write E as

$$E = \{w \in \mathbb{R}^{2n} | g(w,w) \leq 1\}$$

By Lemma 4.3.2, we have a basis $\underline{e} = \{u_1, ..., u_n, v_1, ..., v_n\}$ of $\mathbb{R}^{2n}$ which is orthogonal with respect to g and standard symplectic with respect to ω_0. We also have

$$g(u_i,u_i) = g(v_i,v_i) =: \frac{1}{r_i^2}$$

where we can assume $r_1 \leq ... \leq r_n$. Now we denote by ϕ the symplectomorphism that takes the standard basis of $\mathbb{R}^{2n}$ to $\underline{e}$, i.e.

$$\phi z = \sum_{i=1}^{n} (x_i u_i + y_i v_i)$$

where $z = (x_1, ..., x_n, y_1, ..., y_n)$. So it follows that

$$g(\phi z, \phi z) = \sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{r_i^2}$$

and we get

$$E(r) = \{z \in \mathbb{R}^{2n} | \sum_{i=1}^{n} \frac{x_i^2+y_i^2}{r_i^2} \leq 1\} = \{z \in \mathbb{R}^{2n} | g(\phi z, \phi z) \leq 1\} = \phi^{-1} E$$

We still have to prove the uniqueness of the n-tuple $0 < r_1 \leq ... \leq r_n$. Denote by $\Delta(r)$ the diagonal matrix

$$\begin{pmatrix} 1/r_1^2 & & & & & \\ & \ddots & & & & \\ & & 1/r_n^2 & & & \\ & & & 1/r_1^2 & & \\ & & & & \ddots & \\ & & & & & 1/r_n^2 \end{pmatrix}$$

We have that $z \in E$ if and only if $z^T \Delta(r) z \leq 1$. Assume there exists a $\phi \in Sp(\mathbb{R}^{2n})$ such that $E = \phi^{-1} E(r')$, we have to prove that $r = r'$. That means $z^T \phi^T \Delta(r') \phi z = z^T \Delta(r) z$, $\forall z \in \mathbb{R}^{2n}$, i.e. $\phi^T \Delta(r') \phi = \Delta(r)$. Since ϕ is symplectic we have $J_0 \phi^T = \phi^{-1} J_0$, which leads to

$$\phi^{-1} J_0 \Delta(r') \phi = J_0 \Delta(r)$$

Hence $J_0 \Delta(r)$ and $J_0 \Delta(r')$ must have the same eigenvalues. But since the eigenvalues of $J_0 \Delta(r)$ are $\pm i/r_1^2, ..., \pm i/r_n^2$ we have $r = r'$. □

Now we can prove Theorem 4.3.1.

Proof. Let $r = (r_1, ..., r_n)$ be the symplectic spectrum of E, with $0 < r_1 \leq ... \leq r_n$, from Lemma 4.3.3 we know that there exists a matrix $\phi \in Sp(\mathbb{R}^{2n})$ such that $\phi E = E(r)$. Then we have

$$B^{2n}(r_1) \subseteq E(r) \subseteq Z^{2n}(r_1)$$

$$\Rightarrow \phi^{-1} B^{2n}(r_1) \subseteq E \subseteq \phi^{-1} Z^{2n}(r_1)$$

and since

$$w_L(\phi^{-1}B^{2n}(r_1)) = \pi r_1^2 = w_L(\phi^{-1}Z^{2n}(r_1))$$

we get

$$\inf_{Z\supset E} w_L(Z) \leq \pi r_1^2 \leq \sup_{B\subset E} w_L(B)$$

Now suppose we have an affine symplectic ball B of radius r contained in E and an affine symplectic cylinder Z of radius R containing E. It follows

$$\phi B \subseteq \phi E \subseteq Z^{2n}(r_1)$$

therefore from Theorem 4.1.3 we have $r \leq r_1$. On the other hand we have

$$B^{2n}(r_1) \subseteq \phi E \subseteq \phi Z$$

still due to Theorem 4.1.3 we get $r_1 \leq R$. So we have

$$\inf_{Z\supset E} w_L(Z) \geq \pi r_1^2 \geq \sup_{B\subset E} w_L(B)$$

and we can conclude

$$\inf_{Z\supset E} w_L(Z) = \pi r_1^2 = \sup_{B\subset E} w_L(B) \quad .$$

From definition we have $w_L(E) = \sup_{B\subset E} w_L(B)$ that concludes the proof. □

Remark. If $r = (r_1, ..., r_n)$ is the symplectic spectrum of an ellipsoid E, with $0 < r_1 \leq ... \leq r_n$, then we have $w_L(E) = \pi r_1^2$

Appendix A

Siegel Upper Half Plane

In this appendix we prove the following theorem, first presented by Carl Ludwig Siegel in [7], which was cited in Chapter 3 and says that the symplectic group acts transitively on the Siegel upper half plane $\mathcal{S}_n$. We first remind the definition of $\mathcal{S}_n$. It is the subspace of $\mathbb{C}^{n\times n}$ composed by symmetric matrices $Z = X + iY$, where Y is positive definite. We use here the same notation as in Chapter 3. The following formulation is presented in [1].

Theorem. *The symplectic group $Sp(\mathbb{R}^{2n})$ acts on $\mathcal{S}_n$ by the transformation*

$$Sp(\mathbb{R}^{2n}) \times \mathcal{S}_n \to \mathcal{S}_n$$

$$(\phi, Z) \mapsto \phi(Z) := (AZ + B) \cdot (CZ + D)^{-1}$$

where $\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and A, B, C, D are $n \times n$ blocks.
This action is transitive and the stabiliser of $(i \cdot Id) \in \mathcal{S}_n$ is the unitary group $U(n)$. Therefore, we have

$$\mathcal{S}_n \cong Sp(\mathbb{R}^{2n})/U(n).$$

Proof. This proof is taken from [2] and [4]. First we need to check that this is actually a group action. In order to do this, we have to prove that the matrix $(CZ + D)$ is always invertible and that for any $Z \in \mathcal{S}_n$ and $\phi \in Sp(\mathbb{R}^{2n})$ we have $\phi(Z) \in \mathcal{S}_n$. Moreover, we have to prove that for $\phi, \psi \in Sp(\mathbb{R}^{2n})$ we have $\phi(\psi(Z))) = (\phi\psi)(Z)$. Let $Z = X + iY$ be in $\mathcal{S}_n$, that means, $Z^T - Z = 0$ and $Y > 0$. These conditions on Z can be rewritten in the following form

$$\begin{pmatrix} Z^T & Id \end{pmatrix} J \begin{pmatrix} Z \\ Id \end{pmatrix} = 0 \quad \text{and} \quad -\tfrac{1}{2i} \begin{pmatrix} \overline{Z} & Id \end{pmatrix} J \begin{pmatrix} Z \\ Id \end{pmatrix} > 0$$

where $\overline{Z} = X - iY$ denotes the complex conjugate of Z and $J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$. Then given $\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(\mathbb{R}^{2n})$ we define $E := AZ + B$ and $F := CZ + D$, or equivalently

$$\phi \begin{pmatrix} Z \\ Id \end{pmatrix} = \begin{pmatrix} E \\ F \end{pmatrix}$$

Then, since ϕ is symplectic, we have

$$\begin{pmatrix} E^T & F^T \end{pmatrix} J \begin{pmatrix} E \\ F \end{pmatrix} = \begin{pmatrix} Z^T & Id \end{pmatrix} \phi^T J \phi \begin{pmatrix} Z \\ Id \end{pmatrix} = \begin{pmatrix} Z^T & Id \end{pmatrix} J \begin{pmatrix} Z \\ Id \end{pmatrix} = 0$$

It follows that $E^T F = F^T E$. We also have

$$-\tfrac{1}{2i}\begin{pmatrix}\overline{E} & \overline{F}\end{pmatrix} J \begin{pmatrix}E\\F\end{pmatrix} = -\tfrac{1}{2i}\begin{pmatrix}\overline{Z} & Id\end{pmatrix} \phi^T J \phi \begin{pmatrix}Z\\Id\end{pmatrix} = -\tfrac{1}{2i}\begin{pmatrix}\overline{Z} & Id\end{pmatrix} J \begin{pmatrix}Z\\Id\end{pmatrix} > 0$$

Therefore, we get $-\frac{1}{2i}(\overline{E}F - \overline{F}E) > 0$. Now we can prove that F is invertible. Assume v is a solution of $Fv = 0$. That means $\overline{vF} = 0$, therefore $\overline{v}(\overline{E}F - \overline{F}E)v = 0$, so $v = 0$, i.e. F is invertible. Now we can write $\phi(Z) = EF^{-1}$. Since we have seen that $E^T F = F^T E$ we get that EF^{-1} is symmetric. Lastly, we have

$$-\tfrac{1}{2i}\overline{F}(\overline{F^{-1}E} - EF^{-1})F > 0$$

i.e.

$$-\tfrac{1}{2i}(EF^{-1} - \overline{EF^{-1}}) > 0$$

hence, $\mathrm{Im}(\phi(Z)) = \mathrm{Im}(EF^{-1}) > 0$. We have proved that $\phi(Z)$ is symmetric and its imaginary part is positive definite, which means it is in $\mathcal{S}_n$. Now we want to prove that given $\phi, \psi \in Sp(\mathbb{R}^{2n})$ we have $\phi(\psi(Z)) = (\phi\psi)(Z)$. Let $\phi = \begin{pmatrix}A & B\\C & D\end{pmatrix}$ and $\psi = \begin{pmatrix}E & F\\G & H\end{pmatrix}$, then we have

$$\phi(\psi(Z)) = \phi(EZ+F)(GZ+H)^{-1})$$

$$= (A(EZ+F)(GZ+H)^{-1} + B)(C(EZ+F)(GZ+H)^{-1} + D)^{-1}$$

$$= (A(EZ+F)(GZ+H)^{-1} + B)(GZ+H)(GZ+H)^{-1}(C(EZ+F)(GZ+H)^{-1} + D)^{-1}$$

$$= (A(EZ+F) + B(GZ+H))(C(EZ+F) + D(GZ+H))^{-1}$$

$$= ((AE+BG)Z + (AF+BH))((CE+DG)Z + (CF+DH))^{-1}$$

$$= (\phi\psi)(Z) \qquad .$$

Therefore we have proved that this is actually a well-defined group action.

Finally, we prove that this action is indeed transitive and that the stabiliser of $(i \cdot Id)$ is the unitary group. Given any $Z \in \mathcal{S}_n$, where $Z = X + iY$, since Y is positive definite we can define the following matrix

$$\phi = \begin{pmatrix}Y^{1/2} & XY^{-1/2}\\ 0 & Y^{-1/2}\end{pmatrix}$$

It is then easy to check that $\phi \in Sp(\mathbb{R}^{2n})$ and $\phi(i \cdot Id) = Z$. This means that any $Z \in \mathcal{S}_n$ is in the orbit of $(i \cdot Id)$, i.e. the action is transitive. Now we take the matrix $(i \cdot Id)$ and $\phi = \begin{pmatrix}A & B\\C & D\end{pmatrix} \in Sp(\mathbb{R}^{2n})$ such that $\phi(i \cdot Id) = i \cdot Id$. That means

$$(AiId + B) \cdot (CiId + D)^{-1} = i \cdot Id$$

and this is if and only if

$$(iA + B) = iD - C$$

i.e.,

$$A = D \text{ and } B = -C$$

Therefore ϕ is in the stabiliser of $(i \cdot Id)$ if and only if

$$\phi = \begin{pmatrix}A & -B\\B & A\end{pmatrix}$$

and since it is symplectic it satisfies $AB^T = BA^T$ and $AA^T + BB^T = Id_n$. This is precisely the definition of the unitary group given in Section 2.1, therefore we can write $\mathcal{S}_n \cong Sp(\mathbb{R}^{2n})/U(n)$. $\square$

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