ETH
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Hodge decomposition on compact Kähler manifolds

Bachelor Thesis

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Abstract

In this thesis we first introduce complex manifolds and hermitian metrics on them, starting from the point of view of smooth manifolds. We then generalize the Laplacian to differential forms and, using the theory of elliptic operators, introduce harmonic form. Afterwards, we show a regularity theorem for periodic elliptic operators and deduce the Hodge decomposition for differential forms from it. We use these harmonic forms to prove the Hodge decomposition theorem for compact Kähler manifolds. We end by applying this theorem to the so called Hodge diamond, whose structure we discuss.

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I want to thank my advisor Prof. Ana Cannas da Silva for recommending this topic, providing me with resources and taking the time to go over my questions and reviewing my drafts.

This is a presentation of material covered in the references, that are listed in the bibliography, none of the ideas are my own with the exception of some very minor adjustments.

This thesis is written in LaTeXand is based on a template provided by CADMO at ETH. The diagrams were created using the TikZ package.

Contents

Introduction

Developed in the 1930's by William V. D. Hodge, Hodge theory encompasses some classic theorems and is of fundamental importance to many fields in geometry, such as Riemannian geometry, complex geometry and, despite it being a analytic theory, algebraic geometry. In this thesis we present two important theorems in Hodge theory, namely the Hodge decomposition theorem for differential forms and the Hodge decomposition theorem for compact Kähler manifolds, which is a consequence of the former.

In Chapter 1, starting from the point of view of smooth manifolds, we introduce complex manifolds, followed by some basic structures on them, such as hermitian metrics and almost complex structures, and the cohomology of their smooth differential forms. Afterwards, we introduce the Hodge $*$-operator and use it to give the space of differential forms some structure. Basic knowledge about complexified manifolds, such as its tangent and cotangent bundle and the integration of forms is assumed. This chapter is mainly based on [Wel08, Chapter 1 and 5], [GH94, Chapter 0], [Huy05, Chapter 1 and 2] and [Voi02, Chapter 2 and 3]. Additional references are mentioned in the parts they are relevant in.

In Chapter 2, we start by giving a generalization of the Laplacian operator on $\mathbb{R}^n$ to arbitrary differential forms on a compact manifold. After that, we show the Hodge decomposition theorem for differential forms. To that end, we first introduce elliptic operators and their generaliztions, elliptic complexes. We then proceed to use the theory of Fourier series to show a regularity theorem for periodic elliptic operators on $\mathbb{R}^n$, which we then use to deduce a regularity theorem for the generalized Laplacians, which in turn leads to our desired decomposition theorem. Basic knowledge of Fourier series is assumed, as is some basic knowledge about Sobolev spaces in the setting of Fourier series, although all properties we use are listed and a proof of these properties is referenced. The first two sections of this chapter are mainly based on [Wel08, Chapter 4] and [Voi02, Chapter 5] and [Gui05], while the last two chapters will follow [War83, Chapter 6] very closely. Again any additional references are mentioned in the parts where they are relevant.

In Chapter 3, we first show a relation betweed some different Laplacians and then use the decomposition theorem from the second chapter to deduce the Hodge decomposition theorem for compact Kähler manifolds. We end this thesis by taking a look at the Hodge diamonds for such manifolds and investigate their structure. This chapter is mainly based on [Wel08, Chapter 5], [Voi02, Chapter 6], [Huy05, Chapter 3] and [GH94, Chapter 0], plus some additional references for individual parts towards the end.

Chapter 1

Preliminaries

We begin by going over some preliminaries about complex manifolds. In Section 1.1, we introduce complex manifolds and revise some basic notions about vector bundles. In Section 1.2, we introduce almost complex structures with a focus on those induced by an actual complex structure. Section 1.3 revises some basic notions about the de Rham complex and introduces the Dolbeault complexes. In Section 1.4, we take a look at hermitian metrics and ,in Section 1.5, we introduce the Hodge $*$-operator and use it to define an inner product on the space of differential forms.

1.1 Complex manifolds and vector bundles

Recall that a smooth function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is called holomorphic if it satisfies the Cauchy-Riemann equations. Namely if we write $z = x + iy$ and split $f = u(x, y) + iv(x, y)$ into its real and imaginary parts, then we require

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We define the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Taking a look at the real and imaginary parts, we see that the Cauchy-Riemann equations are equivalent to $\frac{\partial f}{\partial \overline{z}} = 0$ in which case the ordinary complex derivative is just given by $\frac{\partial f}{\partial z}$. With that in mind, we say that a smooth function $f = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic if all its components are holomorphic in each variable, namely

$$\frac{\partial f_j}{\partial \overline{z}_k} = 0$$

for all $j = 1, \ldots, m$ and $k = 1, \ldots, n$.

Definition 1.1.1 *A complex manifold is a smooth manifold such that there exist an open cover $\{U_i\}_{i \in I}$ together with charts $\varphi_i : U_i \to \mathbb{C}^n$, such that U_i is homeomorphic to $\varphi_i(U_i)$ for all $i \in I$ and the transition functions $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are holomorphic for all $i, j \in I$. The pairs (U_i, φ_i) are called* ***holomorphic charts*** *and the set $\{(U_i, \varphi_i)_{i \in I}\}$ of all these holomorphic charts is called a* ***holomorphic atlas****. A function $f : U \to \mathbb{C}$ on an open subset U is holomorphic if it is holomorphic in all holomorphic charts, meaning that $f \circ \varphi_i^{-1} : \varphi_i(U \cap U_i) \to \mathbb{C}$ is holomorphic for every $i \in I$.*

Definition 1.1.2 *A* ***complex submanifolds*** *S of a complex manifold X is a subset such that around every $x \in S$ there is a holomorphic chart (U, φ) of X such that $\varphi(U \cap S) = \varphi(U) \cap \mathbb{C}^k \times \{0\}$.*

Example 1.1.3 *We define $\mathbb{C}\mathbb{P}^n$ as the set of complex lines in $\mathbb{C}^{n+1}$. Explicitly*

$$\mathbb{C}\mathbb{P}^n = \{[z]_\sim \mid z = (z_0, \ldots, z_n) \neq 0\},$$

where $z \sim w$ if $z = \lambda w$ for some $\lambda \in \mathbb{C}$ and we write $[z_0 : \ldots : z_n]$ for $[(z_0, \ldots, z_n)]_\sim$. We cover $\mathbb{C}\mathbb{P}^n$ with $U_i = \{[z_o : \ldots : z_n] \mid z_i \neq 0\}$ and define charts $\varphi_i : U_i \to \mathbb{C}^n$

$$\varphi_i([z_0 : \ldots : z_n]) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i} \right).$$

These charts indeed define a complex structure since, assuming without loss of generality that $i > j$, on $\varphi_i(U_i \cap U_j) \subseteq \mathbb{C}^n$ we have that

$$\varphi_j \circ \varphi_i^{-1}(z_1, \ldots, z_n) = \left(\frac{z_1}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \frac{z_{i+1}}{z_j}, \ldots, \frac{z_n}{z_j} \right),$$

which is holomorphic.

Although complex manifolds and smooth manifolds are defined similarly, they have some differences. For example, by Whitney's theorem (see [Lee12, Theorem 6.15]), every smooth manifold of dimension n can be smoothly embedded into $\mathbb{R}^{2n+1}$. For complex manifolds there may be no holomorphic embedding into $\mathbb{C}^m$ for any m.

Proposition 1.1.4 *Every holomorphic map from a compact complex manifold X to $\mathbb{C}$ is constant.*

Proof Let $f : X \to \mathbb{C}$ be a holomorphic function. Since holomorphic functions are continuous and X is assumed to be compact, $|f|$ has a maximum. Let x_0 be a point where this maximum is assumed. Let $\varphi : U \to \mathbb{C}^n$ be a coordinate chart around x_0 that sends x_0 to 0 and let $B_\varepsilon(0)$ be an open ball of radius ε around $0 \in \mathbb{C}^n$ contained in the image of φ. For any $z \in B_\varepsilon(0) \subseteq \mathbb{C}^n$ define the holomorphic function

$$g_z : B_1(0) \to \mathbb{C},\ w \mapsto f \circ \varphi^{-1}(wz).$$

Since $|g_z|$ is maximal at $w = 0$, g_z, and hence f, are constant by the maximum principle (see [Hor73, Corollary 1.2.12]). □

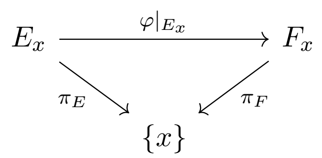
We now turn our attention to vector bundles.

Definition 1.1.5 *A **smooth vector bundle** of rank k over a smooth manifold X is a smooth manifold E together with a smooth surjective map $\pi : E \to X$, such that for each $x \in X$, its fiber $E_x \coloneqq \pi^{-1}(x)$ has the structure of a k dimensional real vector space and such that for each $x \in X$ there is a open neighbourhood U of x, such that there is a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, called a local trivialization, such that $\varphi|_x : E_x \to \{x\} \times \mathbb{R}^k$ is a linear map.*

The best known examples for vector bundles are the tangent bundle $T(X)$ and cotangent bundle $T^*(X)$.

Definition 1.1.6 *A complex vector bundle is a smooth vector bundle $\pi : E \to X$, such that every fiber E_x has the structure of a complex vector space and the local trivializations $\pi^{-1}(U) \to U \times \mathbb{C}^k \cong U \times \mathbb{R}^{2k}$ are complex linear. A holomorphic vector bundle is a complex vector bundle $\pi : E \to X$ such that E and X are complex manifolds and π is a holomorphic map.*

Definition 1.1.7 *A vector bundle homomorphism between smooth vector bundles E and F over X is a map $\varphi : E \to F$ such that $\varphi(E_x) \subseteq F_x$ for all $x \in X$ and such that $\varphi|_{E_x} : E_x \to F_x$ is linear. If φ is a diffeomorphism and all $\varphi|_{E_x}$ are linear isomorphisms we call φ a bundle isomorphism.*



For a vector bundle homomorphism $\varphi : E \to F$ we write

$$\ker \varphi = \bigcup_{x \in X} \ker \varphi|_{E_x} \text{ and } \operatorname{im} \varphi = \bigcup_{x \in X} \operatorname{im} \varphi|_{E_x}$$

Definition 1.1.8 *Let $\pi : E \to X$ be a smooth vector bundle. A section of E over an open set U is a map $\xi : U \mapsto \pi^{-1}(U)$ such that $\pi \circ \xi = \mathrm{id}$. We denote the set of smooth sections $U \to E$ by $\mathcal{E}(U, E)$.*

We denote exterior powers of the cotangent bundle as $\bigwedge^p T^*(X)$ and write $\Omega^k(X) = \mathcal{E}(X, \bigwedge^p T^*(X))$. Also note that a smooth vector bundle isomorphism induces an isomorphism of the sections.

Definition 1.1.9 *Let $f : X \to Y$ be a smooth map and $\pi_Y : E \to Y$ a smooth vector bundle, then we define the pullback bundle $\pi_X : f^*E \to X$ as*

$$f^*E = \{(x, e) \in X \times E \mid f(x) = \pi(e)\}$$

with $\pi_X(x, e) = x$.

Intuitively speaking, the pullback bundle attaches to $x \in X$ the fiber $E_{f(x)}$.

Definition 1.1.10 *Let $U \subseteq X$ be open. A local frame of a smooth vector bundle $\pi : E \to X$ of degree k over U is, if one even exists, a collection of k everywhere linear independent smooth sections.*

Note that if $\xi_1, \dots, \xi_k$ is a local frame we can write every section as $g_1\xi_1 + \cdots + g_k\xi_k$ for smooth functions g_i, giving us a identification $\mathcal{E}(U, E) \cong (\mathcal{C}^\infty(U, \mathbb{R}))^k$.

1.2 Almost complex structures

Definition 1.2.1 *Let V be a finite dimensional real vector space. A **linear complex structure** on V is a linear map $J : V \to V$ such that $J^2 = -\operatorname{id}$.*

If a real vector space admits a linear complex structure, it is necessarily even dimensional since

$$0 \leq (\det(J))^2 = \det(J^2) = \det(-\operatorname{id}) = (-1)^{\dim_{\mathbb{R}}(V)}.$$

Two linear complex structures, J_1 on V_1 and J_2 on V_2, are said to be isomorphic if there exist a linear isomorphism $\varphi : V_1 \to V_2$ such that $\varphi J_1 = J_2\varphi$. The complex linear structure enables us to view V as a complex vector space by defining $(a + bi)v = av + bJv$. In the other, direction we get a complex linear structure on the underlying real vector space of a complex vector space from multiplication with i.

Proposition 1.2.2 *Every linear complex structure is isomorphic to $\mathbb{R}^{2n}$ with*

$$J_0 = \begin{bmatrix} 0 & -\operatorname{id}_n \\ \operatorname{id}_n & 0 \end{bmatrix}$$

Proof View V as a complex vector space and let $(v_1, \dots, v_n)$ be a complex basis, then $(v_1, \dots, v_n, Jv_1, \dots, Jv_n)$ is a real basis for which J is represented by J_0. □

Denote by $V_{\mathbb{C}} \coloneqq V \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of V. Extending the linear complex structure J to $V_{\mathbb{C}}$ by $J(v \otimes z) = J(v) \otimes z$ we obtain a complex linear automorphism that still satisfies $J^2 = -\operatorname{id}$, thus having minimal polynomial $t^2 - 1$. Therefore its eigenvalues are i and $-i$. We denote the eigenspaces in $V_{\mathbb{C}}$ for i by

$$V^{1,0} = \{v \otimes z \in V_{\mathbb{C}} \mid J(v \otimes z) = v \otimes iz\}$$

and for $-i$ by

$$V^{0,1} = \{v \otimes z \in V_{\mathbb{C}} \mid J(v \otimes z) = v \otimes -iz\}.$$

We thus get a decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. Defined as such, $V_{\mathbb{C}}$ carries two distinct linear complex structures, namely J and multiplication by i, the latter of which shall simply be referred to as the complex structure of $V_{\mathbb{C}}$. The decomposition of $v \in V_{\mathbb{C}}$ into its components in $V^{1,0}$ and $V^{0,1}$ is given by

$$v = \underbrace{\frac{1}{2}(v - iJv)}_{\in V^{1,0}} + \underbrace{\frac{1}{2}(v + iJv)}_{\in V^{0,1}}.$$

Note that, viewed as complex vector spaces, V and $V^{1,0}$ are isomorphic under the projection map

$$v \mapsto \frac{1}{2}(v - iJv).$$

We can now define complex conjugation on $V_{\mathbb{C}}$ by $\overline{v \otimes z} = v \otimes \overline{z}$. Note that complex conjugation on $V_{\mathbb{C}}$ is a $\mathbb{R}$-linear isomorphism between $V^{1,0}$ and $V^{0,1}$, as $J(v \otimes z) = v \otimes iz$ implies $J(\overline{v \otimes z}) = J(v) \otimes \overline{z} = \overline{J(v \otimes z)} = \overline{v \otimes iz} = -i\overline{v \otimes z}$. These are also complex subspaces of $V_{\mathbb{C}}$. For the complex vector space $V_{\mathbb{C}}$ with $\dim_{\mathbb{C}}(V_{\mathbb{C}}) = \dim_{\mathbb{R}}(V) = 2n$ let

$$\bigwedge V_{\mathbb{C}} = \bigoplus_{p=0}^{2n} \bigwedge^p V_{\mathbb{C}}$$

denote its exterior algebra. For $V_{\mathbb{C}}$ both $\bigwedge V^{1,0}$ and $\bigwedge V^{0,1}$ are complex subspaces of $\bigwedge V_{\mathbb{C}}$ with trivial intersection. Define

$$\bigwedge^{p,q} V \coloneqq \bigwedge^p V^{1,0} \otimes_{\mathbb{C}} \bigwedge^q V^{0,1}$$

which we identify with its image in $\bigwedge^{p+q} V_{\mathbb{C}}$. We thus get a decomposition

$$\bigwedge V_{\mathbb{C}} = \bigwedge V^{1,0} \oplus V^{0,1} = \bigoplus_{p+q=0}^{2n} \bigwedge^p V^{0,1} \otimes_{\mathbb{C}} \bigwedge^q V^{0,1} = \bigoplus_{k=0}^{2n} \left(\bigoplus_{p+q=k} \bigwedge^{p,q} V \right).$$

Notice that, since $\dim_{\mathbb{C}}(V^{1,0}) = \dim_{\mathbb{C}}(V^{0,1}) = \dim_{\mathbb{C}}(V) = n$, $\bigwedge^{p,q} V = 0$ whenever either $p > n$ or $q > n$. Now let W be the real dual space of V. J induces a linear complex structure on W through $Jw = w \circ J$. We can now do the same procedure for W to obtain $W_{\mathbb{C}}$ and a decomposition of its exterior powers. Note that complex linear extensions of elements in W^* generate $W^*_{\mathbb{C}}$ and that $W^{1,0}$ is the dual space of $V^{1,0}$ as for a basis element $v_j \in V$ with dual basis vector $w_j \in W$ such that $w_j(v_j) = 1$ we have that $(w_j - iJw_j)\left(\frac{1}{2}(v_j - iJv_j)\right) = 1$. Similarly $W^{0,1}$ is the dual space of $V^{0,1}$.
We now turn our attention to complex and almost complex manifolds.

Definition 1.2.3 *Let X be a differentiable manifold. An* ***almost complex structure*** *on X is a vector bundle isomorphism $J : T(X) \to T(X)$ such that $J_x : T_xX \to T_xX$ is a linear complex structure for every $x \in X$.*

Observe that giving X an almost complex structure is the same as giving $T(X)$ the structure of a complex vector bundle. A complex structure on X induces an almost complex structure on X as follows. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates, then $(x_1 = \Re z_1, y_1 = \Im z_1, \ldots, x_n = \Re z_n, y_n = \Im z_n)$ are real local coordinates and we can define J as the bundle map satisfying

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \text{ and } J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j},$$

which is clearly smooth. Now let $\zeta_1, \ldots, \zeta_n$ be other holomorphic coordinates with $\Re\zeta_j = \xi_j$ and $\Im\zeta_j = \eta_j$. Let $f = u+iv$ be the transition function between z and ζ. The components u and v induce a real change of coordinates, namely $\xi = u(x, y)$ and $\eta = v(x, y)$. Let J be the almost complex structure induced by z and J' be the almost complex structure induced by ζ, then, using the Cauchy-Riemann equations, we compute

$$\begin{aligned} J'\left(\frac{\partial}{\partial x_k}\right) &= J'\left(\sum_{j=1}^{n} \frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial \xi_j} + \frac{\partial v_j}{\partial x_k}\frac{\partial}{\partial \eta_j}\right) \\ &= \sum_{j=1}^{n} \frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial \eta_j} - \frac{\partial v_j}{\partial x_k}\frac{\partial}{\partial \xi_j} \\ &= \sum_{j=1}^{n} \frac{\partial v_j}{\partial y_k}\frac{\partial}{\partial \eta_j} + \frac{\partial u_j}{\partial y_k}\frac{\partial}{\partial \xi_j} \\ &= \frac{\partial}{\partial y_k}. \end{aligned}$$

Since now $J'(\frac{\partial}{\partial y_k}) = J'(J'(\frac{\partial}{\partial x_j})) = J'^2(\frac{\partial}{\partial x_j}) = -\frac{\partial}{\partial x_j}$, J and J' coincide, showing that the induced almost complex structure is independent of the choice of holomorphic coordinates and thus well-defined.

Proposition 1.2.4 *A smooth map f between complex manifolds X_1 and X_2 is holomorphic if and only if $f_*J_1 = J_2 f_*$, where J_1 and J_2 are the induced almost complex structures and f_* denotes the induced map on the tangent spaces.*

Proof Let $x \in X_1$ and let $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_m$ and $\zeta_1 = \xi_1 + i\eta_1, \ldots, \zeta_m = \xi_m + i\eta_m$ be local coordinates around x and $f(x)$. Let $f_j = u_j + iv_j$ denote the real and imaginary parts if the components of f. The expression

$$\begin{aligned}
(J_2 f_* - f_* J_1)\left(\frac{\partial}{\partial x_i}\right) =& J_2\left(\sum_{j=1}^{n}\frac{\partial u_j}{\partial x_i}\frac{\partial}{\partial \xi_j}+\frac{\partial v_j}{\partial x_i}\frac{\partial}{\partial \eta_j}\right) - f_*\left(\frac{\partial}{\partial y_i}\right)\\
=& \sum_{j=1}^{n}\frac{\partial u_j}{\partial x_i}\frac{\partial}{\partial \eta_j}-\frac{\partial v_j}{\partial x_i}\frac{\partial}{\partial \xi_j}-\sum_{j=1}^{n}\frac{\partial u_j}{\partial y_i}\frac{\partial}{\partial \xi_j}+\frac{\partial v_j}{\partial y_i}\frac{\partial}{\partial \eta_j}\\
=& \sum_{j=1}^{n}\left(\frac{\partial u_j}{\partial x_i}-\frac{\partial v_j}{\partial y_i}\right)\frac{\partial}{\partial \eta_j}-\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial u_j}{\partial y_i}\right)\frac{\partial}{\partial \xi_j}
\end{aligned}$$

is 0 if and only if f fulfills the Cauchy-Riemann equations. □

For a complex manifold X, we can complexify its tangent and cotangent spaces fiberwise to

$$T(x)_{\mathbb{C}} = T(X)\otimes_{\mathbb{R}}\mathbb{C} \text{ and } T^*(x)_{\mathbb{C}} = T^*(X)\otimes_{\mathbb{R}}\mathbb{C}\,.$$

We represent sections of theses spaces just as sums $\alpha + i\beta$ where α and β are either in $\mathcal{E}(X, T(X))$ or $\Omega^1(X)$ respectively. Using the unique induced almost complex structure, we obtain just as in the vector space case decompositions

$$T(X)_{\mathbb{C}} = T(X)^{1,0}\oplus T(X)^{0,1} \text{ and } T^*(X) = T^*(X)^{1,0}\oplus T^*(X)^{0,1},$$

where $T(X)^{1,0}$ is the kernel of the bundle homomorphism $\mathrm{id} - iJ\,\mathrm{id} : T(X) \to T(X)$ and $T(X)^{0,1}$ is the kernel of the bundle homomorphism $\mathrm{id} + iJ\,\mathrm{id}$ and similarly for $T^*(X)$.

Let X_1 be an n-dimensional complex manifold and X_2 be an m-dimensional complex manifold with coordinates z and ζ as above. Let $f : X_1 \to X_2$ be smooth. Note that, if f is holomorphic, then f_* maps $T(X_1)^{1,0}$ to $T(X_2)^{0,1}$, as for $v \in T_p(X)^{1,0}$ we have $J_2 f_* v = f_* J_1 v = f_* iv = i f_* v$, and similarly, f_* also maps $T(X_1)^{0,1}$ to $T(X_2)^{0,1}$. The real Jacobian $J_{\mathbb{R}}$ of f representing f_* is given by the $2m \times 2n$ matrix

$$J_{\mathbb{R}} = \begin{bmatrix} \left(\frac{\partial u_j}{\partial x_k}\right)_{j,k} & \left(\frac{\partial u_j}{\partial y_k}\right)_{j,k} \\ \left(\frac{\partial v_j}{\partial x_k}\right)_{j,k} & \left(\frac{\partial v_j}{\partial y_k}\right)_{j,k} \end{bmatrix}$$

where $1 \le j \le m$ and $1 \le k \le n$. We now extend f_* by complex linearity to a map between the complexified tangent spaces. For the basis $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}$ and $\frac{\partial}{\partial \zeta_1}, \ldots, \frac{\partial}{\partial \zeta_n}, \frac{\partial}{\partial \overline{\zeta}_1}, \ldots, \frac{\partial}{\partial \overline{\zeta}_n}$, the Jacobian $J_{\mathbb{C}}$ representing f_* is now given by

$$J_{\mathbb{C}} = \begin{bmatrix} \left(\frac{\partial f_j}{\partial z_k}\right)_{j,k} & \left(\frac{\partial f_j}{\partial \overline{z}_k}\right)_{j,k} \\ \left(\frac{\partial \overline{f}_j}{\partial z_k}\right)_{j,k} & \left(\frac{\partial \overline{f}_j}{\partial \overline{z}_k}\right)_{j,k} \end{bmatrix}$$

Written out explicitly this just means

$$f_*\frac{\partial}{\partial z_k} = \sum_{j=1}^{m} \frac{\partial f_j}{\partial z_k}\frac{\partial}{\partial \zeta_j} + \frac{\partial \overline{f}_j}{\partial z_k}\frac{\partial}{\partial \overline{\zeta}_j}$$
$$f_*\frac{\partial}{\partial \overline{z}_k} = \sum_{j=1}^{m} \frac{\partial f_j}{\partial \overline{z}_k}\frac{\partial}{\partial \zeta_j} + \frac{\partial \overline{f}_j}{\partial \overline{z}_k}\frac{\partial}{\partial \overline{\zeta}_j}$$

Note that $\overline{\frac{\partial f}{\partial z}} = \frac{\partial \overline{f}}{\partial \overline{z}}$ and therefore if f is holomorphic we get

$$J_{\mathbb{C}} = \begin{bmatrix} \left(\frac{\partial f_j}{\partial z_k}\right)_{j,k} & 0 \\ 0 & \overline{\left(\frac{\partial f_j}{\partial z_k}\right)_{j,k}} \end{bmatrix}$$

Proposition 1.2.5 *Every complex manifold is orientable.*

Proof Since all transition functions are by assumption holomorphic, we get

$$\begin{aligned}\det(J_{\mathbb{R}}) = \det(J_{\mathbb{C}}) &= \det \begin{bmatrix} \left(\frac{\partial \zeta_j}{\partial z_k}\right)_{j,k} & 0 \\ 0 & \overline{\left(\frac{\partial \zeta_j}{\partial z_k}\right)_{j,k}} \end{bmatrix} \\ &= \det\left(\frac{\partial \zeta_j}{\partial z_k}\right) \overline{\det\left(\frac{\partial \zeta_j}{\partial z_k}\right)} = \left|\det\left(\frac{\partial \zeta_j}{\partial z_k}\right)\right|^2 > 0\end{aligned}$$

Therefore any holomorphic atlas induces an orientation. □

1.3 Complex differential forms

Definition 1.3.1 *The exterior derivative on a smooth manifold is the map* $d : \Omega^p(X) \mapsto \Omega^{p+1}(X)$ *given in local coordinates by*

$$d \sum_{|I|=p} \alpha_I dx_I = \sum_{|I|=p} \sum_{i=1}^{n} \frac{\partial \alpha_I}{\partial x_i} dx_i \wedge dx_I,$$

where we used multi-index notation, meaning for $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ *we write*

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_m},$$

such that $i_\ell < i_k$ *for* $\ell < k$.

An important property of the exterior derivative is that it commutes with pullbacks. For a proof see [Lee12, Proposition 14.26]. The exterior derivative yields the **de Rham complex**.

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots$$

The de Rham complex forms a chain complex, since

$$\begin{aligned} d(d\alpha) &= d \sum_{|I|=p} \sum_{i=1}^{n} \frac{\partial \alpha_I}{\partial x_i} dx_i \wedge dx_I \\ &= \sum_{|I|=p} \sum_{i,j=1}^{n} \frac{\partial^2 \alpha_I}{\partial x_i \partial x_j} dx_j \wedge dx_i \wedge dx_I \\ &= \sum_{|I|=p} \left(\sum_{i<j} \frac{\partial^2 \alpha_I}{\partial x_i \partial x_j} dx_j \wedge dx_i \wedge dx_I + \frac{\partial^2 \alpha_I}{\partial x_j \partial x_i} dx_i \wedge dx_j \wedge dx_I \right) \\ &= 0. \end{aligned}$$

We now define the complex-valued differential forms on $U \subseteq X$ as smooth sections of the complexified contangent bundle, namely as elements of $\mathcal{E}(U, \bigwedge^k T^*(X)_{\mathbb{C}})$ which we denote by $\Omega^k_{\mathbb{C}}(U)$. Extending d by complex linearity we obtain the **complexified de Rham complex**

$$0 \longrightarrow \Omega^0_{\mathbb{C}}(X) \xrightarrow{d} \Omega^1_{\mathbb{C}}(X) \xrightarrow{d} \Omega^2_{\mathbb{C}}(X) \xrightarrow{d} \cdots ,$$

which again forms a chain complex, as $d^2(\alpha+i\beta) = d(d\alpha+id\beta) = d^2\alpha+id^2\beta = 0$. The de Rham cohmology groups with real coefficients are now defined as

$$H^k(X,\mathbb{R}) = {\{\ker d : \Omega^k(X) \to \Omega^{k+1}(X)\}}\big/{\{\operatorname{im} d : \Omega^{k-1}(X) \to \Omega^k(X)\}}.$$

And, similarly, we have the de Rham cohomology groups with complex coefficients

$$H^k(X,\mathbb{C}) = {\{\ker d : \Omega^k_{\mathbb{C}}(X) \to \Omega^{k+1}_{\mathbb{C}}(X)\}}\big/{\{\operatorname{im} d : \Omega^{k-1}_{\mathbb{C}}(X) \to \Omega^k_{\mathbb{C}}(X)\}}.$$

Lemma 1.3.2 *For the complexified de Rahm complex we have that*

$$H^k(X,\mathbb{R}) \otimes \mathbb{C} \cong H^k(X,\mathbb{C}).$$

Proof First note that, by complex linearity of d, a complex-valued differential form $\gamma = \alpha + i\beta$ is closed if and only if α and β are closed. Using this, we define the maps

$$\begin{aligned} H^k(X,\mathbb{R}) \otimes \mathbb{C} &\cong H^k(X,\mathbb{C}) \\ [\varphi] \otimes u + iv &\mapsto [(u+iv)\varphi] \\ [\alpha] \otimes 1 + [\beta] \otimes i &\leftarrow [\alpha + i\beta] \end{aligned}$$

which are inverse to each other. □

Let X be an almost complex manifold and, for $k = p+q$, let $\pi_{p,q}$ denote the projection $\bigwedge^k T^*(X)_{\mathbb{C}} \to \bigwedge^{p,q} T^*(X)_{\mathbb{C}}$. On $\Omega^{p,q}_{\mathbb{C}}(X)$ we define

$$\partial = \pi_{p+1,q} \circ d$$
$$\overline{\partial} = \pi_{p,q+1} \circ d$$

For an arbitrary almost complex manifolds we only have

$$d = \sum_{r+s=p+q+1} \pi_{r,s} \circ d = \partial + \overline{\partial} + \text{rest}.$$

Definition 1.3.3 *We call an almost complex structure **integrable** if $d = \partial + \overline{\partial}$.*

The significance of this definition is that, if $d = \partial + \overline{\partial}$, then for $\alpha \in \Omega^{p,q}(X)$

$$0 = d^2\alpha = (\partial + \overline{\partial})^2\alpha = \underbrace{\partial^2\alpha}_{\in\Omega^{p+2,q}(X)} + \underbrace{\partial\,\overline{\partial}\,\alpha + \overline{\partial}\,\partial\,\alpha}_{\in\Omega^{p+1,q+1}(X)} + \underbrace{\overline{\partial}^2\alpha}_{\in\Omega^{p,q+2}(X)}.$$

Therefore, for an integrable almost complex structure, we have $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\,\overline{\partial} = -\overline{\partial}\,\partial$.

Proposition 1.3.4 *Let X be a complex manifold, then the induced almost complex structure is integrable.*

Proof For local holomorphic coordinates $z = (z_1, \dots, z_n)$, we have for $\alpha \in \Omega^{p,q}$

$$\begin{aligned} d\sum_{I,J} \alpha_{I,J} dz_I \wedge d\overline{z}_J &= \sum_{I,J} d\alpha_{I,J} \wedge dz_I \wedge d\overline{z}_J \\ &= \sum_{I,J}\sum_{k=1}^{n} \left(\frac{\partial \alpha_{I,J}}{\partial x_k} dx_k + \frac{\partial a_{I,J}}{\partial y_k} dy_k \right) \wedge dz_I \wedge d\overline{z}_J \\ &= \sum_{I,J}\sum_{k=1}^{n} \left(\frac{\partial \alpha_{I,J}}{\partial z_k} dz_k + \frac{\partial a_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \right) \wedge dz_I \wedge d\overline{z}_J \\ &= \partial \sum_{I,J} a_{I,J} dz_I \wedge d\overline{z}_J + \overline{\partial} \sum_{I,J} a_{I,J} dz_I \wedge d\overline{z}_J \end{aligned}$$

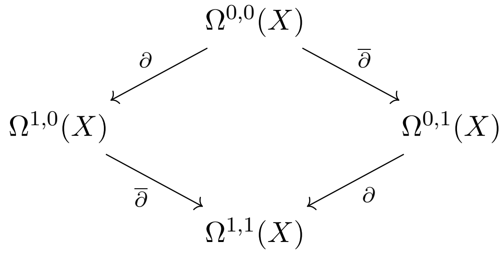
Thus any induced almost complex structure is integrable. □

Remark 1.3.5 *The above proof basically boils down to the fact that an induced almost complex structure also comes with certain coordinates for which it takes a specific form. It turns out that the converse is also true, namely that every integrable almost complex structure is induced by a complex structure. This is the so called Newlander-Nierenberg theorem. For a proof see [Hor73, Theorem 5.7.4].*

Example 1.3.6 *Any almost complex structure on a manifold X of dimension 2 is integrable. Since the complex fiber dimensions of $T^*(X)^{1,0}$ and $T^*(X)^{0,1}$ are 1, we have that $\Omega^{2,0}(X) = \Omega^{0,2}(X) = 0$ and therefore*

$$\begin{aligned} \Omega^0_{\mathbb{C}}(X) &= \Omega^{0,0}(X) \\ \Omega^1_{\mathbb{C}}(X) &= \Omega^{1,0}(X) \oplus \Omega^{0,1}(X) \\ \Omega^2_{\mathbb{C}}(X) &= \Omega^{1,1}(X). \end{aligned}$$

Hence, ∂ and $\overline{\partial}$ are the only components d can even split into.



Proposition 1.3.7 *On a complex manifold both ∂ and $\overline{\partial}$ follow the Leibniz rule*

$$\begin{aligned} \partial(\alpha \wedge \beta) &= \partial\,\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \partial\,\beta \\ \overline{\partial}(\alpha \wedge \beta) &= \overline{\partial}\,\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \overline{\partial}\,\beta \end{aligned}$$

where $\alpha \in \Omega^{p,q}(X)$ and $\beta \in \Omega^{r,s}(X)$.

Proof Using $d = \partial + \overline{\partial}$ we compute

$$\begin{aligned} d(\alpha \wedge \beta) &= d\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge d\beta \\ \partial(\alpha \wedge \beta) + \overline{\partial}(\alpha \wedge \beta) &= \underbrace{\partial\,\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \partial\,\beta}_{\in \Omega^{p+r+1,q+s}(X)} + \underbrace{\overline{\partial}\,\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \overline{\partial}\,\beta}_{\in \Omega^{p+r,q+s+1}(X)} \end{aligned}$$

The statement now follows from the fact that all terms involving ∂ have bidegree different from the terms involving $\overline{\partial}$. □

Definition 1.3.8 *The **Dolbeault complexes** of a complex manifold X are the chain complexes*

$$0 \longrightarrow \Omega^{p,0}(X) \xrightarrow{\overline{\partial}} \Omega^{p,1}(X) \xrightarrow{\overline{\partial}} \Omega^{p,2}(X) \xrightarrow{\overline{\partial}} \cdots$$

The Dolbeault cohomology groups are now defined as

$$H^{p,q}(X) = {\{\ker \overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)\}}\big/{\{\operatorname{im} d : \Omega^{p,q-1}(X) \to \Omega^{p,q}(X)\}}.$$

These groups, although they are similarly defined as the de Rham groups, can differ widely from them. Take for example $\mathbb{C}$. We have $\Omega^0_{\mathbb{C}}(X) = \Omega^{0,0}(X)$ for all complex manifolds X. Since for any connected manifold X of real dimension n and for $f \in \Omega^0_{\mathbb{C}}(X)$, we have

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j,$$

$df = 0$ implies that f is constant. On the other hand, for a connected complex manifold X of complex dimension n, we have

$$\overline{\partial} f = \sum_{i=1}^{n} \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i,$$

$\overline{\partial} f = 0$ implies that f is holomorphic. Although, by Proposition 1.1.4, $H^0(X, \mathbb{C}) = H^{0,0}(X)$ for any compact complex manifold X, on $\mathbb{C}$ this is certainly not the case, as there is a plethora of holomorphic functions on $\mathbb{C}$.

Definition 1.3.9 *The* ***Betti numbers*** *of a real manifold X are $b_k = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$ and the* ***Hodge numbers*** *of a complex manifold are $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X)$.*

Thus, for every connected manifold we have $b_0 = 1$, as every constant function is a multiple of the function constant 1. For compact complex manifolds we also have $h^{0,0} = 1$, by the same reasoning as before, but on $\mathbb{C}$ we have $h^{0,0} = \infty$.

Definition 1.3.10 *For a complex manifold X, we call elements in $\ker \overline{\partial} : \Omega^{p,0}(X) \to \Omega^{p,1}(X)$ holomorphic forms.*

Note that

$$\overline{\partial} \sum_{|J|=p} f_J dz_J = \sum_{|J|=p} \sum_{k=1}^{n} \frac{\partial f_J}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_J = 0$$

implies, that a form is holomorphic if and only if its coefficients in local holomorphic coordinates are holomorphic functions.

Remark 1.3.11 *The cohomology of the de Rham complex defined here is isomorphic to the singular cohomology with complex coefficients (see [Lee12, Theorem 18.14]). Furthermore, these de Rham and Dolbeault cohomology groups can be shown to be isomorphic to the sheaf cohomology groups of $\Omega^0_{\mathbb{C}}(X)$ and $\Omega^{0,q}(X)$ (see [Wel08, Chapter II]).*

1.4 Hermitian manifolds

Recall that, on a complex vector space V, a hermitian inner product is a map $h : V \times V \to \mathbb{C}$, such that h is complex linear in the first argument, conjugate symmetric in the second, meaning $h(u,v) = \overline{h(v,u)}$ for $u, v \in V$, and positive definite, meaning $h(v,v) > 0$ for all $v \in V \setminus \{0\}$. Now let V be a complex vector space viewed as a real vector space with its induced linear complex structure J. Let $\{x_1, \dots, x_n\}$ be a complex basis of V, then $\{x_1, y_1 = Jx_1, \dots, x_n, y_n = Jx_n\}$ is a real basis of V. The hermitian inner product h is a real bilinear map on V with the additional property that if we write $h_{j,k} := h(x_j, x_k)$, then $h(x_j, y_k) = -ih_{j,k}$, $h(y_j, x_k) = ih_{j,k}$ and $h(y_j, y_k) = h_{j,k}$. We can now extend h to $V_{\mathbb{C}}$ by complex bilinearity and see that, if $u, v \in V \subseteq V_{\mathbb{C}}$, with $u = u^{1,0} + u^{0,1}$ and $v = v^{1,0} + v^{0,1}$ their splittings into their components in $V^{1,0}$ and $V^{0,1}$, then

$$h(u^{1,0}, v^{0,1}) = \frac{1}{4}(h(u,v) + ih(u,Jv) - ih(Ju,v) + h(Ju,Jv)) = h(u,v)$$

Thus, writing $z_j = x_j + iy_j$ and $\overline{z}_j = x_j - iy_j$, we have $h(z_j, \overline{z}_k) = h(x_j^{1,0}, x_k^{0,1}) = h(x_j, x_k) = h_{j,k}$. Therefore, we can write

$$h = \sum_{i,j=1}^{n} h_{j,k} z^j \otimes \overline{z}^k.$$

Taking real and imaginary parts of h we obtain $h = g - i\omega$, where $g = \Re h$ and $\omega = -\Im h$ are real bilinear forms on V that extend to complex bilinear forms on $V_{\mathbb{C}}$. Expanding the expression for h we get

$$\sum_{i,j=1}^{n} h_{j,k} z^j \otimes \overline{z}^k = \sum_{i,j=1}^{n} h_{j,k}(x^j \otimes x^k + y^j \otimes y^k) - i \sum_{i,j=1}^{n} h_{j,k}(\underbrace{x^j \otimes y^k - y^j \otimes x^k}_{x^j \wedge y^k})$$

Therefore, using $z^j \wedge z^k = (x^j + iy^j) \wedge (x^k - iy^k) = -2ix^j \wedge y^k$, we obtain

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} z^j \wedge \overline{z}^k.$$

Remark 1.4.1 *There are different conventions in use for the wedge product. The one we use defines for vectors $v_1, \dots, v_n$ and covectors $w^1, \dots, w^n$ the pairing as $w^1 \wedge \cdots \wedge w^n(v_1, \dots, v_n) = \det(w^i(v_j)_{i,j})$. The alternative definition defines $w^1 \wedge \cdots \wedge w^n(v_1, \dots, v_n) = \frac{1}{n!}\det(w^i(v_j)_{i,j})$, in which case one has to set $h = g - 2i\omega$ so that both expressions for ω coincide. For more details on these conventions see [War83, Definition 2.09 and the remarks in 2.10].*

Definition 1.4.2 *We call ω the **fundamental form** of h.*

Lemma 1.4.3 *$g = \Re h$ is a scalar product on V viewed as a real vector space that satisfies $g(\cdot,\cdot) = g(J\cdot, J\cdot)$.*

Proof Let $u, v \neq 0 \in V$. First note that since h is non-degenerate, $h(v,v) > 0$ and thus $g(v,v) > 0$. Symmetry follows from $g(u,v) = \Re h(u,v) = \Re\overline{h(v,u)} = g(v,u)$. Lastly, $h(J\cdot, J\cdot) = -i^2 h(\cdot,\cdot) = h(\cdot,\cdot)$ implies $g(\cdot,\cdot) = g(J\cdot, J\cdot)$. □

Lemma 1.4.4 *The fundamental form ω of h is a non-degenerate alternating bilinear map that satisfies $\omega(\cdot,\cdot) = g(\cdot, J\cdot) = -g(J\cdot,\cdot)$.*

Proof Let $u, v \neq 0 \in V$. First of all note that $\Im h(u,v) = \Im\overline{h(v,u)} = -\Im h(v,u)$, showing that ω is indeed alternating. Now note that $g(u, Jv) = \Re h(u, Jv) = \Re - ih(u,v) = \Im h(u,v) = \omega(u,v)$. Non-degeneracy now follows from the fact that g is non-degenerate. □

If $(V, \langle\cdot,\cdot\rangle)$ is a euclidean vector space the inner product gives rise to a hermitian inner product on $V_{\mathbb{C}}$ by setting $h(u \otimes z, v \otimes w) = z\overline{w}\langle u, v\rangle$

Proposition 1.4.5 *On $V^{1,0}$, we have that $g = 2h$.*

Proof Let $u, v \in V$, then

$$\begin{aligned} g(u - iJu, v - iJv) =& g(u,v) + ig(u,Jv) - ig(Ju,v) + g(Ju,Jv) \\ =& 2g(u,v) + 2ig(u,Jv) \\ =& 2h(u,v) \end{aligned}$$

□

Now turning over to manifolds we define a hermitian metric fiberwise as in the vector space case such that the coefficients vary smoothly.

Definition 1.4.6 *A hermitian metric on a complex vector bundle $E \to X$ is a family of hermitian inner products $\langle\cdot,\cdot\rangle_x$ on the fibers E_x that depend smoothly on x, meaning that for any two sections $\xi, \eta \in \mathcal{E}(U, E)$ on an open set U we have, that $x \mapsto \langle \xi(x), \eta(x)\rangle_x$ is a smooth complex valued function on U.*

Proposition 1.4.7 *Every complex vector bundle $\pi : E \to X$ admits a hermitian metric.*

Proof Choose an open cover $\{U_\alpha\}_\alpha$ of X such that E is trivializable over all U_α and a smooth partition of unity ρ subordinate to $\{U_\alpha\}_\alpha$ (for a proof that such a ρ exists we refer to [Lee12, Theorem 2.23]). Let $\varphi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{C}^k$ be local trivializations, then we define for $\xi, \eta \in \mathcal{E}(U, E)$

$$h(\xi, \eta) := \sum_\alpha \rho_\alpha \langle \varphi_\alpha(\xi(x)), \varphi_\alpha(\eta(x))\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the standard hermitian inner product on $\mathbb{C}^k$. Since linear combinations of hermitian inner products with positive weights are again hermitian inner products this indeed defines a hermitian metric on E. □

Definition 1.4.8 *A **hermitian manifold** is a complex manifold with a hermitian metric on its real tangent space, viewed as complex vector bundle through the induced almost complex structure.*

We can as before then extend the hermitian metric to a complex bilinear map on the complexified tangent space of the form

$$h = \sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\overline{z}_k$$

Example 1.4.9 *On $\mathbb{C}^n$ we have the standard metric*

$$\sum_{j=1}^{n} dz_j \otimes d\overline{z}_j = \sum_{j=1}^{n} dx_j \otimes dx_j + dy_j \otimes dy_j - i \underbrace{(dx_j \otimes dy_j - dy_j \otimes dx_j)}_{dx_j \wedge dy_j}.$$

Thus the real part is just the standard inner product of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and the fundamental form the standard symplectic form.

Example 1.4.10 *We can make $\mathbb{C}\,\mathbb{P}^n$ into a Kähler manifold with the so called **Fubini-Study form**, which, on the open sets U_i of Example 1.1.3 with coordinates $(w_1, \dots, w_n) = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right)$, is given by*

$$\omega = \frac{i}{2} \partial \overline{\partial} \log \left(\sum_{k=1}^{n} |w_k|^2 + 1 \right).$$

We now want to show that this is indeed a well-defined global form. Note that

$$\sum_{k=1}^{n} |w_k|^2 + 1 = \sum_{\ell=0}^{n} \left| \frac{z_\ell}{z_i} \right|^2.$$

Thus, using

$$\log \left(\sum_{\ell=0}^{n} \left| \frac{z_\ell}{z_i} \right|^2 \right) = \log \left(\sum_{\ell=0}^{n} \left| \frac{z_\ell}{z_j} \right|^2 \right) + \log \left(\left| \frac{z_j}{z_i} \right|^2 \right),$$

and that, assuming without loss of generality that $i > j$, $\frac{z_j}{z_i} = w_i$ on U_i, we only need to verify $\partial \overline{\partial} \log(|w_i|^2) = 0$. Indeed,

$$\partial \overline{\partial} \log(|w_i|^2) = \partial \frac{1}{w_i \overline{w_i}} \overline{\partial}\, w_i \overline{w_i} = \partial \frac{d\overline{w_i}}{\overline{w_i}} = 0.$$

Computing ω explicitly,

$$\begin{aligned} \frac{i}{2} \partial \overline{\partial} \log \left(\sum_{k=1}^{n} |w_k|^2 + 1 \right) &= \frac{i}{2} \partial \sum_{\ell=1}^{n} \frac{w_\ell}{\sum_{k=1}^{n} |w_k|^2 + 1} d\overline{w}_\ell \\ &= \frac{i}{2} \sum_{j,\ell=1}^{n} \frac{(\sum_{k=1}^{n} |w_k|^2 + 1)\delta_{j\ell} - w_\ell \overline{w}_j}{(\sum_{k=1}^{n} |w_k|^2 + 1)^2} dw_j \wedge d\overline{w}_\ell \end{aligned}$$

we see that the hermitian metric associated to the Fubini-Study form, called the Fubini-Study metric, is given by

$$h = \sum_{j,\ell=1}^{n} \frac{(\sum_{k=1}^{n} |w_k|^2 + 1)\delta_{j\ell} - w_\ell \overline{w}_j}{(\sum_{k=1}^{n} |w_k|^2 + 1)^2} dw_j \otimes d\overline{w}_\ell$$

The Fubini-study form is closed since

$$\begin{aligned} d\omega =& (\partial + \overline{\partial}) \frac{i}{2} \partial \overline{\partial} \log \left(\sum_{k=1}^{n} |w_k|^2 + 1 \right) \\ =& \frac{i}{2} \partial^2 \overline{\partial} \log \left(\sum_{k=1}^{n} |w_k|^2 + 1 \right) - \frac{i}{2} \partial \overline{\partial}^2 \log \left(\sum_{k=1}^{n} |w_k|^2 + 1 \right) \\ =& 0 \end{aligned}$$

and real since

$$\overline{\omega} = \overline{\frac{i}{2} \partial \overline{\partial} \log \left(\sum_{k=1}^{n} |w_k|^2 + 1 \right)} = -\frac{i}{2} \overline{\partial} \partial \log \left(\sum_{k=1}^{n} |w_k|^2 + 1 \right) = \omega.$$

To show that the Fubini-Study metric is a genuine metric we only need to show that it is positive definite. This follows from the fact that for all $u, w \in \mathbb{C}^n \setminus \{0\}$

$$\sum_{j,\ell=1}^{n} (|w|^2 + 1)\delta_{j,\ell} u_j \overline{u}_\ell - w_\ell \overline{w}_j u_j \overline{u}_\ell = |u|^2 + |u|^2 |w|^2 - \overline{\langle w, u \rangle} \langle w, u \rangle \geq |u|^2,$$

where we used the Cauchy-Schwarz inequality.

Definition 1.4.11 *A* ***complex projective manifold*** *is a compact complex manifold that can be embedded as a complex submanifold of $\mathbb{C}\,\mathbb{P}^n$ for some n.*

Definition 1.4.12 *A Kähler manifold is a complex manifold X with a hermitian metric $h = g - i\omega$ such that ω is closed.*

Proposition 1.4.13 *A complex submanifold M of a Kähler manifold (X, h) is again Kähler.*

Proof Let $\iota : M \to X$ be the inclusion map. Since $h|_{TM}$ is a hermitian metric on M, the statement follows from $d(\omega|_{TM}) = d\iota^*\omega = \iota^* d\omega = 0$. □

Corollary 1.4.14 *Every complex projective manifold is Kähler.*

Proposition 1.4.15 *A hermitian metric h on a complex manifold X is a Kähler metric if and only if it osculates to order 2 with the standard metric on $\mathbb{C}^n$, meaning that around every point $x_0 \in X$ there exist holomorphic coordinates $z_1, \dots, z_n$, such that*

$$h = \sum_{i,j=1}^{n} \delta_{i,j} + O\left(|z|^2\right) dz_i \otimes d\overline{z}_j.$$

Proof First note that, if h is of said form, $d\omega|_{x_0} = 0$ is immediate. For the other direction, choose coordinates $w_1, \dots, w_n$, such that $h_{i,j}(x_0) = h(\frac{\partial}{\partial w_i}|_{x_0}, \frac{\partial}{\partial \overline{w}_j}|_{x_0}) = \delta_{i,j}$. Such coordinates always exist, as we can choose any coordinates that map x_0 to 0 and then compose with a linear map from $\mathbb{C}^n$ to $\mathbb{C}^n$, that maps $\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_n}$ to a orthonormal basis of h. In these coordinates, ω writes as

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} \left(\delta_{i,j} + \sum_{k=1}^{n} \alpha_{i,j,k} w_k + \beta_{i,j,k} \overline{w}_k + O\left(|w|^2\right) \right) dw_i \wedge d\overline{w}_j,$$

where the $\alpha_{i,j,k} = \frac{\partial h_{i,j}}{\partial z_k}(x_0)$ and $\frac{\partial h_{i,j}}{\partial \overline{z}_k}(x_0)$. Note that $h_{i,j} = \overline{h_{j,i}}$ implies $\alpha_{i,j,k} = \overline{\beta_{j,i,k}}$ and that $d\omega(x_0) = 0$ implies $\alpha_{i,j,k} = \alpha_{k,j,i}$ and $\beta_{i,j,k} = \beta_{i,k,j}$ as

$$d\omega|_{x_0} = \frac{i}{2} \sum_{i,j,k=1}^{n} \alpha_{i,j,k} dw_k \wedge dw_i \wedge d\overline{w}_j + \beta_{i,j,k} d\overline{w}_k \wedge dw_i \wedge d\overline{w}_k.$$

We now define the coordinates $z_1, \dots, z_n$ as

$$z_k = w_k + \frac{1}{2} \sum_{\ell,m=1}^{n} \alpha_{m,k,\ell} w_\ell w_m.$$

Then

$$dz_k = dw_k + \sum_{\ell,m}^{n} \alpha_{m,k,\ell} w_\ell dw_m.$$

And verify

$$\begin{aligned}
\frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k =& \frac{i}{2} \sum_{k=1}^{n} dw_k \wedge d\overline{w}_k + \frac{i}{2} \sum_{k,\ell,m=1}^{n} \alpha_{m,k,\ell} w_\ell \wedge d\overline{w}_k \\
&+ \frac{i}{2} \sum_{k,\ell,m} \beta_{k,\ell,m} \overline{w}_\ell dw_k \wedge d\overline{w}_m + \frac{i}{2} \sum_{\ell,m=1}^{n} O(|w|^2) dw_\ell \wedge d\overline{w}_m \\
=& \frac{i}{2} \sum_{i,j=1}^{n} \delta_{i,j} dw_i \wedge d\overline{w}_j + \frac{i}{2} \sum_{i,j,k=1}^{n} \alpha_{i,j,k} w_k dw_i \wedge d\overline{w}_j \\
&+ \frac{i}{2} \sum_{i,j,k} \beta_{i,j,k} \overline{w}_k dw_i \wedge d\overline{w}_j + \sum_{i,j=1}^{n} O(|w|^2) dw_i \wedge d\overline{w}_j \\
=& \omega + \sum_{i,j} O(|z|^2) dw_i \wedge d\overline{w}_j.
\end{aligned}$$

We conclude by noting that, by definition, $O(|w|^2) = O(|z|^2)$. □

1.5 Hodge $*$-operator

Let V be a finite dimensional real vector space with inner product $\langle\cdot,\cdot\rangle$. Let $\{e_1,\dots,e_n\}$ be an ordered orthonormal basis of V giving it an orientation. Extending the inner product to $\bigwedge V$ by setting

$$\langle \alpha_I e_I, \beta_J e_J\rangle = \begin{cases} \alpha_I\beta_J & \text{if } I = J \\ 0 & \text{else} \end{cases}$$

making $\{e_I\}_{I\subseteq\{1,\dots,n\}}$ an orthonormal basis and we define $\mathrm{vol} = e_1\wedge\cdots\wedge e_n$.

Definition 1.5.1 *The Hodge $*$-operator on an euclidean vector space V of real dimension n, is the linear map $* : \bigwedge^d V \to \bigwedge^{n-d} V$ given by $e_I \mapsto \operatorname{sgn}(I)e_{I^c}$, where I^c is an ordered complement and $\operatorname{sgn}(I)$ is such that $\operatorname{sgn}(I)e_I\wedge e_{I^c} = \mathrm{vol}$.*

For $\alpha,\beta\in\bigwedge^d$ we compute

$$\alpha\wedge*\beta = \sum_{|I|=d}\alpha_I e_I \wedge \sum_{|J|=d}\beta_J\operatorname{sgn}(J)e_{J^c} = \sum_{|I|,|J|=d}\alpha_I\beta_J\operatorname{sgn}(J)e_I\wedge e_{J^c}$$

$e_I\wedge e_{J^c}\neq 0$ if and only if $I = J$, in which case $e_I\wedge e_{I^c} = \operatorname{sgn}(I)\,\mathrm{vol}$. We get that

$$\alpha\wedge*\beta = \sum_{|I|=d}\alpha_I\beta_I\,\mathrm{vol} = \langle\alpha,\beta\rangle\,\mathrm{vol}$$

Extending the inner product to a hermitian inner product $\bigwedge V_{\mathbb{C}}$, we get that

$$\alpha\wedge*\beta = \langle\alpha,\beta\rangle\,\mathrm{vol}\,.$$

Note that $*1 = \mathrm{vol}$.

Remark 1.5.2 *Defined like this, $*$ has been conjugate linearly extended. In the literature one often finds that $*$ gets extended complex linearly, in which case our version $*$ would be denoted $\overline{*}$, meaning that $*$ gets composed with complex conjugation.*

Lemma 1.5.3 *On $\bigwedge^k V_{\mathbb{C}}$ we have that*

$$*^2 = (-1)^{k(n-k)}\,\mathrm{id}\,.$$

Proof By definition $*^2 e_I = \operatorname{sgn}(I)\operatorname{sgn}(I^c)e_I$. Thus $*^2 = \varepsilon\,\mathrm{id}$ for $\varepsilon\in\{-1,+1\}$. Using that for any $u\in\bigwedge^p V_{\mathbb{C}}$ and $v\in\bigwedge^q V_{\mathbb{C}}$ we have that $u\wedge v = (-1)^{pq}v\wedge u$ we compute

$$\begin{aligned} e_I\wedge *e_I &= \mathrm{vol} \\ &= *\, e_I\wedge *^2 e_I \\ &= \varepsilon * e_I\wedge e_I \\ &= \varepsilon(-1)^{k(n-k)}e_I\wedge *e_I. \end{aligned}$$

Thus $\varepsilon = (-1)^{k(n-k)}$. □

Let now X be a compact orientable riemannian manifold and fix an orientation. Then we can define the $*$-operator fiberwise on $\bigwedge^k T^*(X)$ by requiring that at $x_0 \in X$, if $\frac{\partial}{\partial x_1}|_{x_0}, \ldots, \frac{\partial}{\partial x_n}|_{x_0}$ is an ordered orthonormal basis, then $\{dx_I\}_{|I|=k}$ forms an orthonormal basis. Note that this definition actually needs orientability, since otherwise we couldn't define an ordered basis in a consistent way.

We extend the riemannian metric g by sesquilinearity to a hermitian metric h on $T^*_{\mathbb{C}}(X)$ and set $d\mu = *1$ as its canonical volume form. With this, we now define an inner product on the space of differential forms by setting

$$(\alpha, \beta) = \int_X \alpha \wedge *\beta = \int_X h(\alpha_x, \beta_x)_x d\mu.$$

If X isn't compact, we instead require the differential forms to be compactly supported. For a hermitian manifold we use its associated riemannian metric which yields the following formula.

Proposition 1.5.4 *On a hermitian manifold we have that* $d\mu = \frac{\omega^n}{n!}$.

Proof Since this is a fiberwise statement it suffices to prove it in the vector space case. Let $x_1, \ldots, x_n$ be a complex orthonormal basis, meaning $h_{i,j} = \delta_{i,j}$, such that $x_1, y_1 = Jx_1, \ldots, x_n, y_n = jx_n$ is a real basis, then we saw that

$$g = \sum_{j=1}^{n} x^j \otimes x^j + y^j \otimes y^j$$

and therefore $vol = x^1 \wedge y^1 \wedge \cdots \wedge x^n \wedge y^n$. On the other hand we have

$$\omega = \sum_{j=1}^{n} x^j \wedge y^j$$

. Thus the statement follows from

$$\omega^n = n! x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n. \qquad \square$$

Chapter 2

Harmonic forms

The main goal of this chapter is to establish the Hodge decomposition theorem for differential forms. In Section 2.1, we first generalize the Laplacian operator to the de Rham and the Dolbeault complexes. Elements of the kernels of these Laplacians are called harmonic forms. We then show in Section 2.2, that these operators are elliptic. Afterwards, in Section 2.3, we use the theory of Fourier series to prove a regularity theorem for periodic elliptic operators in $\mathbb{R}^n$. In Section 2.4, we deduce the Hodge decomposition theorem for forms. This decomposition allows us to identify the spaces of harmonic forms with the cohomology classes of the de Rham and Dolbeault complexes, which are of great importance in the Chapter 3.

2.1 Laplacian operators

In this section, we define harmonic forms by generalizing the Laplacian on $\mathbb{R}^n$ to differential forms of arbitrary degree on compact orientable riemannian manifolds.

Proposition 2.1.1 *Let (X, h) be a riemannian manifold with $\dim_{\mathbb{R}}(X) = n$. The adjoint operator of $d : \Omega^k_{\mathbb{C}}(X) \to \Omega^{k+1}_{\mathbb{C}}(X)$ is given by*

$$d^* : \Omega^k_{\mathbb{C}}(X) \to \Omega^{k-1}_{\mathbb{C}}(X),\ d^* = (-1)^{nk+n+1} * d*$$

Proof Let $\alpha \in \Omega^{k-1}_{\mathbb{C}}(X)$ and $\beta \in \Omega^k_{\mathbb{C}}(X)$. Using the Leibniz rule $d(\alpha \wedge *\beta) =$

$d\alpha \wedge *\beta + (-1)^{k-1}\alpha \wedge d * \beta$, we compute

$$\begin{aligned}
(d\alpha, \beta) &= \int_X d\alpha \wedge *\beta \\
&= \underbrace{\int_X d(\alpha \wedge *\beta)}_{=0} - \int_X (-1)^{k-1}\alpha \wedge d * \beta \\
&= \int_X (-1)^k \alpha \wedge (-1)^{(n-k+1)(k-1)} * (*d * \beta) \\
&= \int_X \alpha \wedge *(-1)^{nk+n+1}(*d * \beta) \\
&= (\alpha, d^* \beta),
\end{aligned}$$

where we used that $d * \beta$ is a $(n-k+1)$-form. □

We now show the analogous statement for the Dolbeault operators.

Proposition 2.1.2 *Let (X, h) be a hermitian manifold with $\dim_{\mathbb{C}}(X) = n$. The adjoints of $\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$ and $\overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$ are given by $\partial^* = - * \partial *$ and $\overline{\partial}^* = - * \overline{\partial} *$.*

Proof We first show $\overline{\partial}^* = -*\overline{\partial}*$. Again using linearity we consider $\alpha \in \Omega^{p,q-1}$ and $\beta \in \Omega^{p,q}$. We compute

$$\begin{aligned}
(\overline{\partial}\,\alpha, \beta) &= \int_X \overline{\partial}\,\alpha \wedge *\beta \\
&= \int_X \overline{\partial}(\alpha \wedge *\beta) - (-1)^{p+q-1}\alpha \wedge \overline{\partial} *\beta.
\end{aligned}$$

Note that $\alpha \wedge *\beta \in \Omega^{n,n-1}(X)$ where $d = \overline{\partial}$ so we can proceed

$$\begin{aligned}
&= \underbrace{\int_X d(\alpha \wedge *\beta)}_{=0} + \int_X (-1)^{p+q}\alpha \wedge (-1)^{p+q+1} * (*\,\overline{\partial} *\beta) \\
&= \int_X \alpha \wedge *(- * \overline{\partial} *\beta) \\
&= (\alpha, \overline{\partial}^*\, \beta).
\end{aligned}$$

Since for manifolds with even real dimension, such as complex manifolds, $d^* = - * d*$, we can use $d = \partial + \overline{\partial}$ and Proposition 2.1.1 to compute

$$\partial^* + \overline{\partial}^* = d^* = - * d* = - * \partial * - * \overline{\partial} *.$$

Thus we get $\partial^* = - * \partial *$. □

Note that we also could have directly used $d = \partial + \overline{\partial}$ with Proposition 2.1.1 and compare bidegrees to obtain the same result.

Definition 2.1.3 *For the operators d, ∂ and $\overline{\partial}$ we define their Laplacians as $\Delta_d = d^*d + dd^*$, $\Delta_\partial = \partial^*\partial + \partial\partial^*$ and $\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\,\overline{\partial}^*$.*

Notice that all these Laplacians are self-adjoint.

Corollary 2.1.4 *On a compact riemannian manifold, $\Delta_d\alpha = 0$ if and only if $d\alpha = 0$ and $d^*\alpha = 0$. The same holds for Δ_∂ and $\Delta_{\overline{\partial}}$.*

Proof If $\alpha \in \ker d \cap \ker d^*$ then $\Delta_d\alpha = dd^*\alpha + d^*d\alpha = 0$. On the other hand, if $\alpha \in \ker \Delta_d$ we compute

$$0 = (\alpha, \Delta_d\alpha) = (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha) = \|d\alpha\|^2 + \|d^*\alpha\|^2$$

implying $d\alpha = 0$ and $d^*\alpha = 0$. The proofs for Δ_∂ and $\Delta_{\overline{\partial}}$ are similar. □

Let L be either d, ∂ or $\overline{\partial}$ and let E^p be either $\Omega^p_{\mathbb{C}}(X)$ or $\Omega^{q,p}(X)$, depending on which operator L represents. We now concern ourselves with finding solutions to the equation $\Delta_L\omega = \eta$. Observe that for all $\varphi \in E^p$, a solution ω satisfies

$$(\Delta_L\omega, \varphi) = (\eta, \varphi) = (\omega, \Delta_L\varphi).$$

This allows us to rephrase our problem. Namely, any solution ω induces a bounded linear form $\ell : E^p \to \mathbb{C}$ by defining $\ell(\varphi) = (\omega, \varphi)$. This linear form satisfies $\ell(\Delta_L\varphi) = (\eta, \varphi)$. We call any linear form ℓ satisfying this condition a weak solution of $\Delta_L\omega = \eta$. Every solution $\omega \in E^p$ of course induces such a weak solution. The remainder of this chapter is devoted to showing that the converse also holds. Namely, that every weak solution $\ell(\cdot)$ is given by $(\omega, \cdot)$, for some $\omega \in E^p$.

2.2 Elliptic operators

In this section we introduce elliptic operators and show that the Laplacians Δ_d and $\Delta_{\overline{\partial}}$ are elliptic operators. For some preparation, let E and F be vector bundles over X and denote by $T'(X)$ the real cotangent bundle $T^*(X)$ without the zero section. Let $\pi : T'(X) \to X$ be the restriction of the projection and let π^*E and π^*F denote the pullback bundles of E and F.

Definition 2.2.1 *A differential operator of order k between smooth complex vector bundles E and F over X is a map $L : \mathcal{E}(X, E) \to \mathcal{E}(X, F)$, such that for any choice of local coordinates and any local frames $f_1, \ldots, f_n \in \mathcal{E}(U, E)$ and $g_1, \ldots, g_m$ over any, for both bundles trivializable, open set U, can be written as*

$$\left(L\left(\sum_{k=1}^{n} \xi_k f_k\right)\right)_i = \sum_{\substack{j=1 \\ |\alpha|\le p}}^{n} a_J^{i,j} \frac{\partial^{|\alpha|}\,\xi_j}{\partial\, x_1^{\alpha_1} \cdots \partial\, x_n^{\alpha_n}} g_j$$

for $a_J^{i,j}$ smooth complex valued functions on U. The space of all differential operators from E to F of order k is denoted by $\mathrm{Diff}_k(E, F)$.

We now define elliptic operators and their generalisations, elliptic complexes. To that end, we start with some motivation for the subsequent definitions. Let $U \subseteq \mathbb{R}^n$ be open and consider the space of complex valued functions on U viewed as sections in $\mathcal{E}(U, U \times \mathbb{C})$. Let L be a differential operator $\mathcal{E}(U, U \times \mathbb{C}) \to \mathcal{E}(U, U \times \mathbb{C})$ of order k, which we can write as

$$L = \sum_{|J| \leq k} a_J D^J$$

where $a_J \in \mathcal{E}(U, U \times \mathbb{C})$, $J = (j_1, \ldots, j_n)$, $|J| = j_1 + \cdots + j_n$ and

$$D^J = \left(\frac{1}{i}\right)^{|J|} \frac{\partial^{|J|}}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}},$$

such that at least one $a_J \neq 0$ with $|J| = k$. Identify $T^*(U) \cong U \times \mathbb{R}^n$ with coordinates $(x, \xi) = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$. The symbol of L is now defined as

$$\sigma(L)(x, \xi) = \sum_{|J|=k} a_J \xi^J,$$

where $\xi^J = \xi^{j_1} \ldots \xi^{j_n}$. We now call such a differential operator L elliptic if $\sigma(L)(x, \xi) \neq 0$ for all $(x, \xi) \in T'U$. We can equivalently view $\sigma(L)$ as a bundle map $\pi^*(U \times \mathbb{C}) \to \pi^*(U \times \mathbb{C})$. Since the fiber over (x, ξ) is isomorphic to $\mathbb{C}$, $\sigma(L)(x, \xi)$ is given by the linear map $z \in \mathbb{C} \mapsto \sigma(L)(x, \xi)z \in \mathbb{C}$. In this context a differential operator is elliptic, if its symbol induces a bundle isomorphism. Similarly, for a differential operator $\mathcal{E}(U, U \times \mathbb{C}^n) \to \mathcal{E}(U, U \times \mathbb{C}^m)$ let $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathcal{E}(U, U \times \mathbb{C}^n)$

$$(L(\varphi))_i = \sum_{\ell=1}^{m} \sum_{|J| \leq k} a_J^{i,\ell} D^J \varphi_\ell.$$

Again, only taking components of degree k, we define the symbol of L as the bundle map $\pi^*(U \times \mathbb{C}^n) \to \pi^*(U \times \mathbb{C}^m)$ that, for (x, ξ), is given by the $n \times m$ matrix

$$\sigma(L)(x, \xi) = \left(\sum_{|J|=k} a_J^{i,\ell} \xi^J \right)_{i,\ell}.$$

We say that the differential operator is elliptic if its symbol is a bundle isomorphism, meaning for all (x, ξ) the matrix representing $\sigma(L)(x, \xi)$ is invertible.

Definition 2.2.2 *Let $L : \mathcal{E}(X, E) \to \mathcal{E}(X, F)$ be a differential operator of order k. The **symbol** $\sigma(L)$ of L is the bundle homomorphism $\pi^*E \to \pi^*F$ obtained by assigning each $(x, \xi) \in T'(X)$ the linear map represented in local coordinates by the matrix of the top degree part of L with all $\frac{\partial}{\partial x_i}$ replaced by ξ_i. The operator L is called an **elliptic operator**, if its symbol is a bundle isomorphism $\pi^*E \to \pi^*F$.*

Note that this requires E and F to have the same fiber dimension. By the chain rule we immediately see that $\sigma(L \circ L') = \sigma(L)\sigma(L')$, given in local coordinates by matrix multiplication. That the symbol is well defined boils down to the following, informally speaking. Let x and y be local coordinates and let

$$\sum_{i=1}^{n} \xi_i dx_i = \sum_{i=1}^{n} \eta_i dy_i.$$

Therefore

$$\xi_i = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \eta_j.$$

Hence

$$\begin{aligned}
\xi_{i_1} \dots \xi_{i_k} &= \left(\sum_{j_1=1}^{n} \frac{\partial y_{j_1}}{\partial x_{i_1}} \eta_{j_1} \right) \cdots \left(\sum_{j_k=1}^{n} \frac{\partial y_{j_k}}{\partial x_{i_k}} \eta_{j_k} \right) \\
&= \sum_{j_1,\dots,j_k=1}^{n} \frac{\partial y_{j_1}}{\partial x_{i_1}} \cdots \frac{\partial y_{j_k}}{\partial x_{i_k}} \eta_{j_1} \dots \eta j_k.
\end{aligned}$$

On the other hand, the Leibniz rule implies

$$\frac{\partial}{\partial y_i}\left(f \frac{\partial}{\partial y_j} \right) = f \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} + \underbrace{\frac{\partial f}{\partial y_i} \frac{\partial}{\partial y_j}}_{\substack{\text{of lower} \\ \text{order}}},$$

and thus

$$\begin{aligned}
\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} &= \left(\sum_{j_1=1}^{n} \frac{\partial y_{j_1}}{\partial x_{i_1}} \frac{\partial}{\partial y_{j_1}} \right) \cdots \left(\sum_{j_k=1}^{n} \frac{\partial y_{j_k}}{\partial x_{i_k}} \frac{\partial}{\partial y_{j_k}} \right) \\
&= \sum_{j_1,\dots,j_k=1}^{n} \frac{\partial y_{j_1}}{\partial x_{i_1}} \cdots \frac{\partial y_{j_k}}{\partial x_{i_k}} \frac{\partial}{\partial y_{i_1}} \cdots \frac{\partial}{\partial y_{i_k}} \\
&\quad + \text{lower order terms.}
\end{aligned}$$

Therefore, only taking the top order parts makes $\frac{\partial}{\partial x_{i_1}} \dots \frac{\partial}{\partial x_{i_k}}$ and $\xi_{i_1} \dots \xi_{i_k}$ transform alike.

Definition 2.2.3 *Let E be a sequence of vector bundles $E_1, \dots, E_n$ over a compact orientable manifold X with differential operators $L_i : \mathcal{E}(X, E_i) \to \mathcal{E}(X; E_{i+1})$ of a fixed order k between them, such that $L_i \circ L_{i+1} = 0$. Then*

$$\mathcal{E}(X, E_1) \xrightarrow{L_1} \mathcal{E}(X, E_2) \xrightarrow{L_2} \dots \xrightarrow{L_{n-1}} \mathcal{E}(X, E_n)$$

is called an ***elliptic complex****, if the sequence of symbols*

$$0 \longrightarrow \pi^*E_1 \xrightarrow{\sigma(L_1)} \pi^*E_2 \xrightarrow{\sigma(L_2)} \dots \xrightarrow{\sigma(L_{n-1})} \pi^*E_n \longrightarrow 0$$

is exact. For such a complex we define its cohomology as

$$H^i(E) = {}^{\ker L_i}\!\big/_{\operatorname{im} L_{i-1}}.$$

For an elliptic complex E, we can equip each E_i with a hermitian metric and thus equip $\mathcal{E}(X, E_i)$ with an inner product

$$(\alpha, \beta)_{E_i} = \int_X \langle \alpha(x), \beta(x) \rangle_{E_i} d\mu$$

This now allows us to define the adjoint operators $L_i^* : \mathcal{E}(X, E_{i+1}) \to \mathcal{E}(X; E_i)$ of L_i by demanding $(L_i\alpha, \beta)_{E_{i+1}} = (\alpha, L_i^*\beta)_{E_i}$.

Remark 2.2.4 *We can extend the notion of elliptic complexes to non-compact orientable manifolds, in which case we to restrict ourselves to compactly supported sections, in order to have well defined hermitian products.*

Lemma 2.2.5 *Let $L : \mathcal{E}(X, E) \to \mathcal{E}(X, F)$ be a differential operator of order k between vector bundles E and F over a manifold X, both equipped with a hermitian metric. Then the adjoint of L exists and is again a differential operator $L^* : \mathcal{E}(X, F) \to \mathcal{E}(X, E)$.*

Proof We closely follow [Wel08, Proposition 2.8]. Using a partition of unity it suffices to consider sections that are compactly supported in a small enough open subset of X, over which E and F are trivializable. Choosing local frames, we can write

$$(\xi, \eta)_E = \int_{\mathbb{R}^n} \overline{\eta}^T h_E \xi \varphi dx,$$

where φdx a representation of the volume form $d\mu$ in local coordinates. We can now write the differential operator L as a matrix

$$\sum_{|\alpha| \le k} A_\alpha(x) D^\alpha$$

where the A_α are matrices of smooth functions on X. Writing everything out we obtain

$$(L\xi, \eta)_F = \int_{\mathbb{R}^n} \sum_{|\alpha| \le k} \overline{\eta}^T h_F A_\alpha D^\alpha(\xi) \varphi dx$$

Integrating $|\alpha|$ times by parts, we now obtain

$$\begin{aligned}
(L\xi,\eta) &= \int_{\mathbb{R}^n} \sum_{|\alpha|\leq k} \overline{D^\alpha(\eta^T \overline{h_F A_\alpha}\varphi)}\xi dx \\
&= \int_{\mathbb{R}^n} \left(\sum_{|\alpha|\leq k} \overline{D^\alpha(\eta^T \overline{h_F A_\alpha}\varphi)} h_E^{-1}\varphi^{-1} \right) h_E \xi \varphi dx \\
&= \int_{\mathbb{R}^n} \overline{\left(\sum_{|\alpha|\leq k} D^\alpha \left(\eta \overline{h_F^T A_\alpha^T} \varphi \right) \left(\overline{h_E}^{-1} \right)^T \varphi^{-1} \right)}^T h_E \xi \varphi dx.
\end{aligned}$$

Demanding that

$$\sum_{|\alpha|\leq k} B_\alpha D^\alpha \eta = \sum_{|\alpha|\leq k} D^\alpha \left(\eta \overline{h_F^T A_\alpha^T} \varphi \right) \left(\overline{h_E}^{-1} \right)^T \varphi^{-1},$$

implicitly defines the matrices B_α. Moreover, the symbol of L^* consists only of the terms in which η alone gets differentiated, as all other terms have lower order derivatives of η. Thus

$$\sigma(L^*)(x,\tau) = \sum_{|\alpha|=k} \tau^\alpha \overline{h_F A_\alpha h_E^{-1}}^T.$$

We verify

$$\begin{aligned}
(\xi, \sigma(L^*)(x,\tau)\eta)_{F_x} =& \overline{\xi}^T h_E \sum_{|\alpha|=k} \tau^\alpha \overline{h_F A_\alpha h_E^{-1}}^T \eta \\
=& \sum_{|\alpha|=k} \tau^\alpha \overline{\xi}^T \overline{A_\alpha}^T h_F \eta \\
=& \sum_{|\alpha|=k} \left(\overline{\tau^\alpha A_\alpha \xi} \right)^T h_F \eta \\
=& (\sigma(L)(x,\tau)\xi, \eta)
\end{aligned}$$

Therefore, we see that $\sigma(L^*)(x,\tau) = \sigma(L)(x,\tau)^*$. □

Example 2.2.6 *For compactly supported $\mathbb{C}^m$-valued functions on $\mathbb{R}^n$ and a differential operator*

$$L\xi = \sum_{|\alpha|\leq k} a_\alpha D^\alpha \xi,$$

the formula above simplifies to

$$L^*\xi = \sum_{|\alpha|\leq k} D^\alpha (\overline{a_\alpha}^T \xi).$$

So, for the exterior derivative d on 0-forms, which we identify with complex valued functions, d is just the $n \times 1$ matrix

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} = i \begin{pmatrix} D^1 \\ \vdots \\ D^n \end{pmatrix}$$

and thus, d^* is given by the $1 \times n$ matrix

$$-i \begin{pmatrix} D^1 & \cdots & D^n \end{pmatrix} = - \begin{pmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{pmatrix}.$$

We compute

$$\Delta_d f = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f = -\Delta f.$$

Thus the Laplacian Δ_d is just the negative of the ordinary Laplacian Δ on $\mathbb{R}^n$. This actually also holds for all other forms. We claim that $\Delta_d f_I dx_I = (-\Delta f) dx_I$. Note that since for we can always find a permutation sending I to $\{1, \ldots, m\}$ and absorb a potential minus sign into f, it suffices to show the claim for forms of the form $f dx_1 \wedge \cdots \wedge dx_m$. We compute

$$\begin{aligned}
df dx_1 \wedge \cdots \wedge dx_n &= \\
&\sum_{k>m} \frac{\partial f}{\partial x_k} (-1)^m dx_1 \wedge \cdots \wedge dx_m \wedge dx_k \\
*df dx_1 \wedge \cdots \wedge dx_n &= \\
&\sum_{k>m} \overline{\frac{\partial f}{\partial x_k}} (-1)^{k-1} dx_{m+1} \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n \\
d * df dx_1 \wedge \cdots \wedge dx_n &= \\
&\sum_{\substack{k>m \\ \ell \le m}} \overline{\frac{\partial^2 f}{\partial x_k \partial x_\ell}} (-1)^{k-1} dx_\ell \wedge dx_{m+1} \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n \\
&+ \sum_{k>m} \overline{\frac{\partial^2 f}{\partial x_k^2}} (-1)^m dx_{m+1} \wedge \cdots \wedge dx_n \\
*d * df dx_1 \wedge \cdots \wedge dx_n &= \\
&\sum_{\substack{k>m \\ \ell \le m}} \frac{\partial^2 f}{\partial x_k \partial x_\ell} (-1)^{nm-m+\ell+1} dx_1 \wedge \cdots \wedge dx_{\ell-1} \wedge dx_{\ell+1} \wedge \cdots \wedge dx_m \wedge dx_k \\
&+ \sum_{k>m} \frac{\partial^2 f}{\partial x_k^2} (-1)^{mn} dx_1 \wedge \cdots \wedge dx_m
\end{aligned}$$

and, similarly,

$$
\begin{aligned}
&*f dx_1 \wedge \cdots \wedge dx_n = \overline{f} dx_{m+1} \wedge \cdots \wedge dx_n \\
&d{*}f dx_1 \wedge \cdots \wedge dx_n = \\
&\quad \sum_{\ell \leq m} \overline{\frac{\partial f}{\partial x_\ell}} dx_\ell \wedge dx_{m+1} \wedge \cdots \wedge dx_n \\
&{*}d{*}f dx_1 \wedge \cdots \wedge dx_n = \\
&\quad \sum_{\ell \leq m} \frac{\partial f}{\partial x_\ell} (-1)^{nm-n+\ell+1} dx_1 \wedge \cdots \wedge dx_{\ell-1} \wedge dx_{\ell+1} \wedge dx_m \\
&d * d{*}f dx_1 \wedge \cdots \wedge dx_n = \\
&\quad \sum_{\substack{\ell \leq m \\ k > m}} \frac{\partial^2 f}{\partial x_\ell \partial x_k} (-1)^{nm-n+\ell+m} dx_1 \wedge \cdots \wedge dx_{\ell-1} \wedge dx_{\ell+1} \wedge dx_m \wedge dx_k \\
&\quad + \sum_{\ell \leq m} \frac{\partial^2 f}{\partial x_\ell^2} (-1)^{nm-n} dx_1 \wedge \cdots \wedge dx_m.
\end{aligned}
$$

Finally, we can compute

$$
\begin{aligned}
\Delta_d f dx_1 \wedge dx_m &= d^* df dx_1 \wedge dx_m + dd^* f dx_1 \wedge dx_m \\
&= (-1)^{nm+1} * d * df dx_1 \wedge dx_m + (-1)^{nm+n+1} d * d * f dx_1 \wedge dx_m \\
&= \sum_{\substack{k > m \\ \ell \leq m}} \frac{\partial^2 f}{\partial x_k \partial x_\ell} (-1)^{m+\ell} dx_1 \wedge \cdots \wedge dx_{\ell-1} \wedge dx_{\ell+1} \wedge \cdots \wedge dx_m \wedge dx_k \\
&\quad - \sum_{k > m} \frac{\partial^2 f}{\partial x_k^2} dx_1 \wedge \cdots \wedge dx_m \\
&\quad + \sum_{\substack{\ell \leq m \\ k > m}} \frac{\partial^2 f}{\partial x_\ell \partial x_k} (-1)^{\ell+m+1} dx_1 \wedge \cdots \wedge dx_{\ell-1} \wedge dx_{\ell+1} \wedge dx_m \wedge dx_k \\
&\quad - \sum_{\ell \leq m} \frac{\partial^2 f}{\partial x_\ell^2} dx_1 \wedge \cdots \wedge dx_m \\
&= - \sum_{k=0}^{n} \frac{\partial^2 f}{\partial x_k^2} dx_1 \wedge \cdots \wedge dx_m \\
&= (-\Delta f) dx_1 \wedge \cdots \wedge dx_m.
\end{aligned}
$$

Proposition 2.2.7 *The complexified de Rham complex is elliptic.*

Proof Let $(x, \xi = \xi_1 dx_1 + \cdots + \xi_n dx_n) \in T'(X)$ be fixed and let us calculate its symbol at (x, ξ) for forms of degree p, which is a linear map between finite dimensional vector spaces, $\sigma(d)(x, \xi) : \bigwedge^p T_x^* X \otimes \mathbb{C} \to \bigwedge^{p+1} T_x^* X \otimes \mathbb{C}$. Since the exterior derivative of a p-form α is given by

$$d \sum_{|J|=p} \alpha_J dx_J = \sum_{|J|=p} \sum_{j=1}^{n} \left(i \left(\frac{1}{i} \frac{\partial}{\partial x_j} \alpha_J \right) \right) dx_j \wedge dx_J,$$

we get for its symbol the linear map

$$\sum_{|J|=p} \alpha_J dx_J \mapsto i \sum_{|J|=p} \alpha_J \sum_{j=1}^{n} \xi_j dx_j \wedge dx_J,$$

which can be written in short as $\alpha \mapsto i\xi \wedge \alpha$. That this sequence is exact follows from the following linear algebra argument. Let V be a finite dimensional complex vector space with an ordered basis $\{e_1, \ldots, e_n\}$. We want to show that the sequence $L_p : \bigwedge^p V \to \bigwedge^{p+1} V$, given by $v \mapsto u \wedge v$ for a fixed $u \in v$, is exact. Since we can extend any element of V to a basis, we can assume that, without loss of generality, $u = e_1$. It is clear that $L_{p+1} \circ L_p = 0$, so we only need to show that $L_{p+1} v = 0$ means, that $v \in \operatorname{im} L_p$ for $v \neq 0$. But this is clear since, $L_{p+1} v = 0$ means that $v = \sum_{|J|=p} v_J e_J$ only has $v_J \neq 0$ if $1 \in J$. Removing the e_1 from all e_J now gives an element that maps to v. □

Proposition 2.2.8 *The Dolbeault complexes are elliptic.*

Proof Similarly to the case of the de Rham complex, let $(x, \xi) \in T'X$ be fixed and let $\xi = \xi^{1,0} + \xi^{0,1}$ be the decomposition of ξ into its components under the embedding $T'X \hookrightarrow T^*X \otimes \mathbb{C}$. Then a similar computation as before shows that $\sigma(x, \xi)(\overline{\partial})(\alpha) = i\xi^{0,1} \wedge \alpha$, which, by the same reasoning as before, is exact. □

Note that since $\partial\, \alpha = \overline{\overline{\partial}\, \overline{\alpha}}$, $\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$ also forms an ellitpic complex with symbol $\alpha \mapsto i\xi^{1,0} \wedge \alpha$.

Definition 2.2.9 *Let E be an elliptic complex. For all j we define a Laplacian $\Delta_{L_j} : \mathcal{E}(X, E_j) \to \mathcal{E}(X, E_j)$ by $\Delta_{L_j} = L_j^* L_j + L_{j-1} L_{j-1}^*$.*

Proposition 2.2.10 *The Laplacians of an elliptic complex are elliptic operators.*

Proof Since we have that

$$\sigma(x, \xi)(\Delta_L) = \sigma(x, \xi)(L)\sigma(x, \xi)(L)^* + \sigma(x, \xi)(L)^* \sigma(x, \xi)(L)$$

which is a composition of linear maps between the finite dimensional inner product space $(E_j)_x$, the statement follows from the following linear algebra argument. Consider the following sequence of finite dimensional inner product spaces which is assumed to be exact at V.

$$U \xrightarrow{A} V \xrightarrow{B} W$$

We now want to show that $AA^* + B^*B$ is an automorphism of V. Since V is finite dimensional it suffices to show that $(AA^* + B^*B)v \neq 0$ for every $v \neq 0$. We compute

$$\langle (AA^* + B^*B)v, v\rangle_V = \|A^*v\|_V^2 + \|Bv\|_V^2$$

If $Bv \neq 0$ we are done. Otherwise assume there is a $v \neq 0$ such that $Bv = 0$. Since the sequence is exact at V, there is a $u \in U$ such that $Au = v$. We now have that

$$\|v\|^2 = (Au, v) = (u, A^*v) > 0$$

and thus $A^*v \neq 0$. □

Corollary 2.2.11 *On any compact orientable riemannian manifold, Δ_d is an elliptic operator and on any compact complex manifold, both Δ_∂ and $\Delta_{\overline{\partial}}$ are elliptic.*

In the special case of $\mathbb{R}^n$ with the standard metric and only looking at compactly supported forms we can calculate the symbol for Δ_d explicitly. To that end we start with a useful a identity that will also be useful later on.

Lemma 2.2.12 *On $\mathbb{R}^n$ with standard metric, only looking at compactly supported forms, we have the operator that maps a form α to $dx_i \wedge \alpha$, which we shall denote as $dx_i\wedge$, is adjoint to contracting with $\frac{\partial}{\partial x_i}$, meaning*

$$(dx_i\wedge)^* = \iota_{\frac{\partial}{\partial x_i}}.$$

Proof A direct computation shows for $\alpha \in \Omega_{\mathbb{C}}^{k-1}(X)$

$$\begin{aligned}(dx_i \wedge \alpha, \beta) =& \int_X dx_i \wedge \alpha \wedge *\beta \\ =&(-1)^{k-1} \int_X \alpha \wedge *(*^{-1} dx_i \wedge *)\beta \\ =&(-1)^{k-1} \int_X \alpha * \left((-1)^{(k-1)(n-k+1)} * dx_i \wedge *\right) \beta \\ =&(\alpha, (-1)^{n(k+1)} * dx_i \wedge *\beta).\end{aligned}$$

It thus remains to show that $(-1)^{n(k+1)} * dx_i \wedge * = \iota_{\frac{\partial}{\partial x_i}}$. Since, as in Example 2.2.6, for every $I \subseteq \{1, \ldots, n\}$ with $|I| = k$ we can, up to a sign, find a permutation mapping I to $\{1, \ldots, k\}$, it suffices to show the statement for

$dx_1 \wedge \cdots \wedge dx_k$. We compute

$$\begin{aligned}(-1)^{n(k+1)} &* dx_i \wedge * dx_1 \wedge \cdots \wedge dx_k \\ &=(-1)^{n(k-1)} * dx_i \wedge dx_{k+1} \wedge \cdots \wedge dx_n \\ &=(-1)^{i-1} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k \\ &=(-1)^{i-1}(\iota_{\frac{\partial}{\partial x_i}} dx_i) \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k \\ &=(-1)^{i-1}\iota_{\frac{\partial}{\partial x_i}} dx_i \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k \\ &=\iota_{\frac{\partial}{\partial x_i}} dx_1 \wedge \cdots \wedge dx_k,\end{aligned}$$

where we used that

$$\iota_Y(\alpha \wedge \beta) = \iota_Y(\alpha) \wedge \beta + (-1)^k \alpha \wedge \iota_Y(\beta). \qquad \square$$

From this proof we immediately see that $((\xi_1 dx_1 + \cdots + \xi_n dx_n)\wedge)^* = \iota_{\overline{\xi_1}\frac{\partial}{\partial x_1}} + \cdots + \iota_{\overline{\xi_n}\frac{\partial}{\partial x_n}}$.

Proposition 2.2.13 *The symbol of $\mathbb{R}^n$ with the standard metric, when only considering compactly supported forms, then we have $\sigma(\Delta_d)(x,\xi) = -\|\xi\|^2 \operatorname{id}$ and for $\mathbb{C}^n$ we have that $\sigma(\Delta_\partial)(x,\xi) = \sigma(\Delta_{\overline{\partial}})(x,\xi) = -\frac{1}{2}\|\xi\|^2 \operatorname{id}$.*

Proof For $\xi = \xi_1 dx_1 + \cdots + \xi_n dx_n$, write $\eta(\xi) = \overline{\xi_1}\frac{\partial}{\partial x_1} + \cdots + \overline{\xi_n}\frac{\partial}{\partial x_n}$. For any differential form α in the complexified de Rham complex, we have

$$\begin{aligned}\sigma(\delta_d)(x,\xi) &= i\iota_{\eta(\xi)} i\xi \wedge \alpha + i\xi \wedge (i\iota_{\eta(\xi)}\alpha) \\ &= -(\iota_{\eta(\xi)}\xi) \wedge \alpha \\ &= -\|\xi\|^2 \alpha.\end{aligned}$$

The statement for Δ_∂ and $\Delta_{\overline{\partial}}$ follow by the same argument and the fact that by Proposition 1.4.5, we have that $\|\xi^{1,0}\|^2 = \|\xi^{0,1}\|^2 = \frac{1}{2}\|\xi\|^2$. $\square$

2.3 Regularity for periodic elliptic operators

In this section, we show a regularity theorem for periodic elliptic operators on $\mathbb{R}^n$. Throughout, we follow [War83, Chapter 6] very closely. Fix n and m and let $\mathcal{P}$ denote the space of smooth 2π periodic functions $\mathbb{R}^n \to \mathbb{C}^m$. Then remember that, for a function $u \in \mathcal{P}$, its Fourier series is given by

$$\sum_{\xi \in \mathbb{Z}^n} u_\xi e^{i\langle x,\xi\rangle}$$

with

$$u_\xi = \frac{1}{(2\pi)^n}\int_{(0,2\pi)^n} u(x) e^{-i\langle x,\xi\rangle} dx$$

and converges absolutely and uniformly to u.

Definition 2.3.1 *For m and n fixed let $\mathcal{S}$ denote the set of all functions $u : \mathbb{Z}^n \to \mathbb{C}^m$, where we write u_ξ for $u(\xi)$. For any $s \in \mathbb{Z}$ we now define the* ***Sobolev space*** *W^s as*

$$W^s = \left\{ u \in \mathcal{S} \mid \sum_{\xi \in \mathbb{Z}^n} \left(1 + |\xi|^2\right)^s |u_\xi|^2 < \infty \right\}.$$

We henceforth identify $u \in \mathcal{P}$ with the coefficients of its Fourier series viewed as the function $\xi \mapsto u_\xi \in \mathcal{S}$. This embeds $\mathcal{P}$ into $\mathcal{S}$. Since for $u \in \mathcal{P}$ we have that $(\frac{\partial}{\partial x_j} u)_\xi = -i\xi_j u_\xi$, we can extend the differential operators D^α to $\mathcal{S}$ by setting $(D^\alpha u)_\xi = \xi^\alpha u_\xi$. For $u, v \in W^s$, we define their scalar product in W^s as

$$\langle u, v \rangle_s = \sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^s \langle u_\xi, v_\xi \rangle,$$

which is well defined since, by the Cauchy-Schwarz inequality, we have that

$$\left| \sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^s \langle u_\xi, v_\xi \rangle \right|^2 \leq \left(\sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^s |u_\xi|^2 \right) \left(\sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^s |v_\xi|^2 \right).$$

Therfore, defining the Sobolev s-norm as $\| \cdot \|_s = \langle \cdot, \cdot \rangle_s$, we see that W^s consists of all $u \in \mathcal{S}$ with $\|u\|_s < \infty$. The following Proposition summarizes some properties of these Sobolev spaces that we will need later and is part of [War83, Theorem 6.18], where a direct proof can be found.

Proposition 2.3.2 *The Sobolev spaces W^s have the following properties.*

I *$\|u\|_t \leq \|u\|_s$ for $t < s$ and $u \in \mathcal{S}$, giving us inclusions*

$$\cdots \supseteq W^{s-1} \supseteq W^s \supseteq W^{s+1} \supseteq \ldots.$$

We can therefore define $W^{-\infty} = \bigcup_{s \in \mathbb{Z}} W^s$. For $u \in \mathcal{P}$ we also have Parseval's identity

$$\|u\|_0 = \|u\| := \frac{1}{(2\pi)^n} \int_{(2\pi)^n} |u|^2 dx.$$

Therefore, W^0 is the completion of $\mathcal{P}$ with respect to $|\cdot|$.

II *$\mathcal{P}$ is dense in W^s for all s.*

III *For $r < s < t$ and any constant $c_1 > 0$ there is a constant c_2, depending c_1, such that*

$$\|u\|_s^2 \leq c_1 \|u\|_t^2 + c_2 \|u\|_r^2.$$

IV *D^α is a bounded linear map $W^{s+|\alpha|} \to W^s$ with $\|D^\alpha u\|_s \leq \|u\|_{s+|\alpha|}$.*

V *For any smooth periodic function $f : \mathbb{R}^n \to \mathbb{C}$, there exist constants $c_1, c_2 > 0$ such that for any $u \in \mathcal{P}$ we have that*

$$\|fu\|_s \leq c_1 \sup|f|\|u\|_s + c_2\|u\|_{s-1}.$$

This especially implies, that there is a constant $c > 0$ such that

$$\|fu\|_s \leq c\|u\|_s.$$

We also have the following two fundamental lemmas.

Lemma 2.3.3 *(Sobolev) Let $s \geq \lfloor \frac{n}{2} \rfloor + 1 + m$ and $u \in W^s$. Then u corresponds to a $\mathcal{C}^k$ function.*

Lemma 2.3.4 *(Rellich) For $s > r$ the inclusion $W^s \subseteq W^r$ is a compact operator, meaning it maps bounded sets to sets whose closure is compact.*

For a proof see [War83, Theorem 6.22] for Sobolev's lemma and [War83, Theorem 6.23] for Rellich's lemma. Note that as a direct consequence of Sobolev's lemma we get

$$\bigcap_{s \in \mathbb{Z}} W^s = \mathcal{P}\,.$$

We will henceforth call an operator periodic, if all its coefficients are in $\mathcal{P}$. Let us now take a look at how periodic differential operators interact with the Sobolev norms.

Lemma 2.3.5 *Let L be a periodic operator of order k and fix $s \in \mathbb{Z}$. Then there exist constants $c_1, c_2 > 0$ such that*

$$\|Lu\|_s \leq c_1 M \|u\|_{s+k} + c_2\|u\|_{s+k-1}$$

where M is an upper bound to the absolute values of the coefficients of the highest order order part of L.

Proof In the case $m = 1$, we have that

$$\begin{aligned}
\|Lu\|_s &= \left\| \sum_{|J| \leq k} a_J D^J u \right\|_s \\
&\leq \sum_{|J| \leq k} \|a_J D^J u\|_s \\
&\leq \sum_{|J| \leq k} c_{J,1} \sup|a_J| \|D^J u\|_s + c_{J,2}\|D^J u\|_{s-1} \\
&\leq \sum_{|J| \leq k} c_{J,1} \sup|a_J| \|u\|_{s+|J|} + c_{J,2}\|u\|_{s+|J|-1} \\
&\leq \sum_{|J| = k} c_{J,1} M \|u\|_{s+k} + c_2\|u\|_{s+k-1} \\
&\leq c_1 M \|u\|_{s+k} + c_2\|u\|_{s+k-1},
\end{aligned}$$

where we used Proposition 2.3.2 **V**. The general case now follows since by

$$\begin{aligned}
\|Lu\|_s^2 &= \sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^s |(Lu)_\xi)|^2 \\
&= \sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^s \sum_{i=1}^m |((Lu)_i)_\xi)|^2 \\
&= \sum_{i=1}^m \|(Lu)_i\|_s^2 \\
&= \sum_{i=1}^m \left\| \sum_{j=1}^m L_{ij}u_j \right\|_s^2 \\
&\leq \sum_{i=1}^m \left(\sum_{j=1}^m \|L_{ij}u_j\|_s \right)^2 \\
&\leq \sum_{i,j=1}^m m\|L_{ij}u_j\|_s^2,
\end{aligned}$$

we have, that

$$\begin{aligned}
\|Lu\|_s \leq& \sqrt{m} \sum_{i,j=1}^m \|L_{ij}u_j\|_s \\
\leq& \sqrt{m} \sum_{i,j=1}^m c_{i,j,1}M\|u\|_{s+k} + c_{i,j,2}\|u\|_{s+k-1} \\
\leq& c_1 M\|u\|_{s+k} + \|u\|_{s+k-1}.
\end{aligned}$$

□

Proposition 2.3.6 *Let L be a periodic elliptic operator of order k. Then for every s there exists a constant $c > 0$ such that*

$$\|u\|_{s+k} \leq c(\|Lu\|_s + \|u\|_s)$$

for every $u \in W^{s+k}$.

Proof Our proof will be partitioned into three steps. First we will show, that the statement holds for periodic elliptic operators, that only consist of their top order part with all its coefficients constant. In a second step we will show, that the statement holds locally around every point and in the third and last step, we deduce that it holds globally. Note that, due to $\mathcal{P}$ being dense in W^s for all s, it suffices to show the statement for $u \in \mathcal{P}$.
Now for the first step, let L be a periodic elliptic operator of order k with constant coefficients, that only consists of its order k part. Let $\sigma(L)(x,\xi)$

be the symbol of L. Note that $(Lu)_\xi = \sigma(L)(x,\xi)u_\xi$. Since $\sigma(L)(x,\xi)$ is a invertible matrix, there exists $c > 0$, such that

$$|\sigma(L)(x,\xi)p| \geq c$$

for all $p \in \mathbb{C}^n$ with $|p| = 1$ and $|\xi| = 1$. Thus we can compute

$$\begin{aligned}
(\|Lu\|_s + \|u\|_s)^2 \geq & \|Lu\|_s^2 + \|u\|_s^2 \\
\geq & \sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^s |\sigma(L)(x,\xi)u_\xi|^2 + \sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^s |u_\xi|^2 \\
\geq & \sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^s |u_\xi|^2 (c|\xi|^{2k}+1) \\
\geq & c' \sum_{\xi\in\mathbb{Z}^n} (1+|\xi|^2)^{s+k} |u_\xi|^2 \\
= & c'\|u\|_{s+k}^2,
\end{aligned}$$

where c' is such that

$$1 + c|\xi|^{2l} \geq c'(1+|\xi|)^k.$$

Such a c' exists since the condition is equivalent to

$$\frac{1}{c'} - 1 + |\xi|^{2k}\left(\frac{c}{c'} - 1\right) \geq \sum_{\ell=1}^{k-1} |\xi|^{2\ell},$$

which holds for c' small enough. Now for the second step choose $p \in \mathbb{R}^n$. Let L be any periodic elliptic operator and let L_0 be the constant coefficient operator consisting of the top order part of L at p. Thus, for any $\varepsilon > 0$ there exists a small open neighbourhood U of p, such that there exists a periodic operator $\tilde{L}$ that agrees with $L_0 - L$ on U and whose highest order coefficients have absolute value smaller or equal ε on U. Note that since all operators involved are periodic, this actually holds for all periodic translates of U. Let $u \in \mathcal{P}$ be supported in the union of U with its periodic translates. Using the first step and Lemma 2.3.5 we compute

$$\begin{aligned}
\|u\|_{s+k} \leq & c_0(\|L_0u\|_s + \|u\|_s) \\
\leq & c_0(\|Lu\|_s + \|\underbrace{(L_0 - L)u}_{=\tilde{L}u}\|_s + \|u\|_s) \\
\leq & c_0(\|Lu\|_s + c_1\varepsilon\|u\|_{s+k} + c_2\|u\|_{s+k-1} + \|u\|_s) \\
\leq & c_0(\|Lu\|_s + c_1\varepsilon\|u\|_{s+k} + \frac{1}{2}\|u\|_{s+k} + c_2c_3\|u\|_s + \|u\|_s),
\end{aligned}$$

where c_0 is the constant from the statement for step 1, c_1 and c_2 come from Lemma 2.3.5 and c_3 come from Proposition 2.3.2 **III**. The statements now follows by choosing U, such that $\varepsilon < \frac{1}{2c_0c_1}$.

Now for the third and final step we again let L be any periodic elliptic operator. From the second step we know that for every $p \in \mathbb{R}^n$ there is a open neighbourhood U_p, such that the statement holds for all $u \in \mathcal{P}$, that are supported in the union of U with its periodic translates. Denote $\mathbb{T}^n = \mathbb{R}^n / 2\pi\,\mathbb{Z}^n$ the n-Torus, which is a compact manifold. Since, by the definition of the quotient topology, these U_p descend to an open cover of $\mathbb{T}^n$, there are $p_1, \dots, p_t \in \mathbb{R}^n$, such that the unions of U_p with their periodic translates cover all of $\mathbb{R}^n$. The idea now is to use a periodic partition of unity $\rho_1, \dots, \rho_t$, such that ρ_i is supported to the union of U_{p_i} with their periodic translates, where by the second step the statement holds, to deduce the statement globally. We compute

$$\begin{aligned}
\|Lu\|_{s+\ell} \leq& \sum_{i=1}^{t} \|\rho_i u\|_{s+\ell} \\
\leq& c_1 \sum_{i=1}^{t} (\|L\rho_i u\|_s + \|\rho_i u\|_s) \\
\leq& c_1 c_2 t \|Lu\|_s + c_1 \sum_{i=1}^{t} \| \underbrace{(L\rho_i - \rho_i L)}_{:=Q_i} u\|_s + c_1 c_3 t \|u\|_s \\
\leq& c_1 c_2 t \|Lu\|_s + c_1 c_4 t \|u\|_{s+k-1} + c_1 c_3 t \|u\|_s \\
\leq& c_1 c_2 t \|Lu\|_s + \frac{1}{2}\|u\|_{s+k} + c_5 \|u\|_s + c_1 c_3 t \|u\|_s
\end{aligned}$$

from which the statement follows. Here c_1 is the maximum of all constant that arise from applying the second step, c_2 and c_3 come from Proposition 2.3.2 **V**, where again c_2 is the maximum of all occurring constants, c_4 comes from Lemma 2.3.5 and the fact that all the Q_i are, due to that Leibniz rule, periodic differential operators of order at most $k-1$ and c_5 comes from Proposition 2.3.2 **III**. □

We now turn our attention to **difference quotients**. Since for any a function $u \in \mathcal{P}$ and any $h \in \mathbb{R}^n$

$$(u(x+h))_\xi = e^{i\langle h,\xi\rangle} u_\xi,$$

we define for all $u \in \mathcal{S}$

$$(T_h(u))_\xi = e^{i\langle h,\xi\rangle} u_\xi.$$

Note that $\|T_h(u)\|_s = \|u\|_s$ for all $s \in \mathbb{Z}$. The difference quotient is now defined as

$$(u^h)_\xi = \left(\frac{T_h(u) - u}{|h|}\right)_\xi = \left(\frac{e^{i\langle h,\xi\rangle} - 1}{|h|}\right) u_\xi.$$

The importance of these difference quotients stems from the following proposition.

Proposition 2.3.7 *Assume that $u \in W^s$ and that there is a constant c such that $\|u^h\|_s \leq c$ for all $h \in \mathbb{R}^n$. Then $u \in W^{s+1}$.*

Proof Let e_i be the i-th standard basis vector of $\mathbb{R}^n$, then

$$\lim_{t\to\infty} \left| \frac{e^{i\langle te_i,\xi\rangle} - 1}{t} \right|^2 = |\xi|^2.$$

By assumption, we have for any $N > 0$, that

$$c^2 \geq \lim_{t\to\infty} \sum_{|\xi|\leq N} (1+|\xi|^2)^s |(u^{te_i})_\xi|^2 = \sum_{|\xi|\leq N} (1+|\xi|^2)^s |u_\xi|^2 |\xi_i|^2.$$

Thus,

$$\sum_{|\xi|\leq N} (1+|\xi|^2)^{s+1} |u_\xi|^2 \leq \|u\|_s^2 + nc^2.$$

Since this expression is independent of N we have that $\|u\|_{s+1} < \infty$. □

We are now ready to prove the regularity theorem.

Theorem 2.3.8 *Let L be a periodic elliptic operator of order k and let $Lu = v$ for $v \in W^s$ and some $u \in W^{-\infty}$. Then $u \in W^{s+k}$.*

Proof We only need to show that if $u \in W^s$ and $v \in W^{s-k+1}$, then $u \in W^{s+1}$, since the statement follows then by repeated application of the aforementioned. Fix $h \in \mathbb{R}^n$ and let L^h denote the operator where the coefficients of L were replaced by their difference quotients. For any $u \in \mathcal{P}$ and hence for any element in $W^{-\infty}$, we get that

$$\begin{aligned} L(u^h) = &\sum_{|J|\leq k} \frac{a_J(x) D^J (u(x+h) - u(x))}{|h|} \\ = &\sum_{|J|\leq k} \frac{a_J(x+h) D^J u(x+h) - a_J(x) D^J u(x)}{|h|} \\ - &\sum_{|J|\leq k} \frac{(a_J(x+h) - a_J(x)) D^J u(x+h)}{|h|} \\ = &(Lu)^h - L^h(T_h u). \end{aligned}$$

It follows from Proposition 2.3.6, that there exists a constant c such that

$$\begin{aligned} \|u^h\|_s \leq &c\|L(u^h)\|_{s-k} + c\|u^h\|_{s-k} \\ \leq &c\|L(u^h)\|_{s-k} + c\|L^h(T_h u)\|_{s-k} + c\|u^h\|_{s-k} \\ \leq &c\|Lu\|_{s-k+1} + c\|u\|_s. \end{aligned}$$

Since this expression is independent of h, Theorem 2.3.7 implies that $u \in W^{s+1}$. □

2.4 Hodge decomposition for differential forms

In this section, we deduce the Hodge decomposition theorem for differential forms from the regularity theorem from the previous section. This section again follows [War83, Chapter 6] very closely. Throughout this section we let L denote either d, ∂ or $\overline{\partial}$ and let E^p denote either $\Omega^p(X)$ for d or $\Omega^{q,p}(X)$ for ∂ and $\overline{\partial}$, where X is assumed to be a compact oriented manifold that is a complex manifold in the case of ∂ and $\overline{\partial}$.

Lemma 2.4.1 *Let $p \in \mathbb{R}^n$ and P an elliptic differential operator of order 2. Denote by ℓ a weak solution to $Pu = v$. Then there is a neighbourhood W_p of p and $u_p \in \mathcal{P}$, such that $\ell(f) = \langle u_p, f\rangle$ for all $f \in \mathcal{C}_c^\infty(W_p, \mathbb{C}^m)$.*

Proof Let Q' be a translate of the open 2π-cube Q such that $p \in Q'$ and choose an open set V such that $p \in V \subseteq \overline{V} \subseteq Q'$. Let $\tilde{\ell}$ be the restriciton of ℓ to $\mathcal{C}_c^\infty(V, \mathbb{C}^m)$, which is canonically embedded in $\mathcal{P}$ by extending the functions periodically outside Q'. Since $\tilde{\ell}$ is bounded it extends uniquely to the Hilbert space W^0. Thus, there exists a $\tilde{u} \in W^0$ such that $\tilde{\ell}(f) = \langle \tilde{u}, f\rangle_0$ for all $t \in W^0$, as the completion of $\mathcal{C}_c^\infty(V, \mathbb{C}^m)$, with its ordinary inner product from Proposition 2.3.2 **I**, is a subspace of W^0. We now choose open neighbourhoods U and U_0 of p such that $\overline{U} \subseteq U_0 \subseteq \overline{U}_0 \subseteq V$ and such that there exists an periodic elliptic operator $\tilde{P}$ that coincides with P on U_0. We now inductively choose a sequence of open neighbourhoods U_k of p such that $\overline{U} \subseteq U_k$ and $\overline{U_k} \subseteq U_{k-1}$ together with functions $\rho_k : \mathbb{R}^n \to [0,1]$ that are supported in U_{k-1} and identically 1 on U_k. We now set

$$v_1 = \rho_1 \tilde{u} \in W^0$$

and

$$M_1 = \tilde{P}\rho_1 - \rho_1 \tilde{P},$$

so that we can write

$$\tilde{P}v_1 = \rho_1 \tilde{P}\tilde{u} + M_1 \tilde{u}.$$

We want to show that $\tilde{u}$ is represented by a smooth function in $\mathcal{P}$. Due to the Sobolev lemma 2.3.3 is suffices to show that $\tilde{u} \in W^s$ for all s. It holds that

$$\rho_1 \tilde{P}\tilde{u} = \rho_1 v \in \mathcal{C}_c^\infty(U_0) \subseteq \mathcal{P},$$

as

$$\begin{aligned}\langle \rho_1 \tilde{P}\tilde{u}, \varphi\rangle_0 - \langle \rho_1 v, \varphi\rangle_0 =& \langle \tilde{u}, \tilde{P}^* \rho_1 \varphi\rangle_0 - \langle v, \rho_1 \varphi_1\rangle_0 \\ =& \langle \tilde{u}, P^* \rho_1 \varphi\rangle_0 - \langle v, \rho_1 \varphi_1\rangle_0 \\ =& \tilde{\ell}(P^* \rho_1 \varphi) - \tilde{\ell}(P^* \rho_1 \varphi) \\ =& 0\end{aligned}$$

for all $\varphi \in \mathcal{P}$, where we used that $\rho_1\varphi \in \mathcal{C}_c^\infty(U_0, \mathbb{C}^m)$. Note that, by the Leibniz rule, M_1 is a differential operator of order 1, thus $M_1\tilde{u} \in W^{-1}$ and therefore $\tilde{P}v_1 \in W^{-1}$. By Proposition 2.3.7 $v_1 \in W^1$. Iterating the same argument with

$$v_k = \rho_k\tilde{u}$$

and

$$M_k = \tilde{P}\rho_k - \rho_k\tilde{P}$$

we see that $v_k \in W^k$. We conclude by choosing any open neighbourhood W_p of p and a function $\rho : \mathbb{R}^n \to [0,1]$ such that $\overline{W_p} \Subset U$ and such that ρ is compactly supported in U and 1 on W_p. Since now $\rho\rho_k\tilde{u} = \rho\tilde{u}$ on U for all k we see that $\rho\tilde{u} \in W^k$ for all k. Thus $\rho\tilde{u}$ is represented by a smooth function $u_p \in \mathcal{P}$ and thus, for $f \in \mathcal{C}_c^\infty(W_p, \mathbb{C}^m)$, we have that $\ell(f) = \langle u_p, f\rangle$. □

Theorem 2.4.2 *Let $\alpha \in E^p$ and let ℓ be a weak solution to $\Delta_L\omega = \alpha$. Then there exists $\omega \in E^p$ such that ℓ is given by $(\omega, \cdot)$.*

Proof We begin by covering our manifold with finitely many open coordinate patches of the form (B, φ) such that $\varphi(B) = \mathbb{R}^n$. Sections that are compactly supported in B now induce $\mathbb{C}^m$-valued smooth functions on $\mathbb{R}^n$ for m the fiber dimension of E^p. Denote by $(\cdot,\cdot)'$ the inner product on $\mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{C}^m)$ induced by the inner product on E^p. Denote by Δ the differential operator on $\mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{C}^m)$ induced by Δ_L and let Δ^* be its formal adjoint with respect to the standard norm $(\cdot,\cdot)$ on $\mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{C}^m)$. Note that the adjoint of Δ for $(\cdot,\cdot)'$ is by definition just Δ itself again. For every $x \in \mathbb{R}^n$ there exists a hermitian positive definite matrix A such that

$$\langle\xi, \eta\rangle_x' = \langle\xi, A\eta\rangle_x.$$

These matrices vary smoothly, allowing us to write $(\xi, \eta)' = (\xi, A\eta)$. We can now compute an expression of Δ^* on $\mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{C}^m)$, namely

$$\begin{aligned}(\Delta^*\xi, \eta) &= (\xi, \Delta\eta)\\ &= (\xi, A^{-1}\Delta\eta)'\\ &= (\Delta A^{-1}\xi, \eta)'\\ &= (A\Delta A^{-1}\xi, \eta).\end{aligned}$$

Thus we see that $\Delta^* = A\Delta A^{-1}$. For a linear form ℓ' induced from a linear form on E^p, we now define a new linear form ℓ on $\mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{C}^m)$ by $\ell(\xi) = \ell'(A^{-1}\xi)$. Lemma 2.4.1 implies that for each $p \in \mathbb{R}^n$ there is a neighbourhood W_p and an element $u_p \in \mathcal{P}$ such that $\ell(f) = (u_p, f)$ for all $f \in \mathcal{C}_c^\infty(W_p, \mathbb{C}^m)$. These u_p now piece together to a function $u \in \mathcal{C}^\infty$ which represents ℓ on all of $\mathbb{R}^n$. We get for each differential form, that is compactly supported in one of the coordinate patches, a form representing a weak solution for the restriction to said coordinate patch, as

$$\ell'(\xi) = \ell(A\xi) = (u, A\xi) = (u, \xi)'.$$

Using a partition of unity on the manifold, these now piece together to a global form representing ℓ. □

Theorem 2.4.3 *Let $(\alpha_n)_{n\geq 0}$ be a sequence in E^p such that both $\|\alpha_n\|$ and $\|\Delta_L \alpha_n\|$ are bounded sequences. Then $(\alpha_n)_{n\geq 0}$ has a Cauchy subsequence.*

Proof We continue with the setups of the proofs of Lemma 2.4.1 and Theorem 2.4.2. Choosing a partition of unity and the fact that our manifold is assumed to be compact, it suffices to show the theorem for the sequence $(\varphi\alpha_n)_{n\geq 0}$, for $\varphi : M \to [0,1]$ compactly supported in a small neighbourhood around a point x. Consider again a chart around x to V with φ supported U_0 and identify each $\varphi\alpha_n$ with its corresponding function in $\mathcal{C}_c^\infty(U_0, \mathbb{C}^m)$. The norm $\|\cdot\|$ is equal to $\|\cdot\|_0$ and by paracompactness of U_0 equivalent to $\|\cdot\|'$. Thus by Rellich's lemma 2.3.4 it suffices to show that the sequence is bounded in the $\|\cdot\|_1$ norm. Using the fact that the differential operator Δ, that is induced by Δ_L, is of order 2 and that there is a periodic operator that is equal to Δ on $\mathcal{C}_c^\infty(U_0, \mathbb{C}^m)$, we can, using Proposition 2.3.6, estimate

$$\begin{aligned}\|\varphi\alpha_n\|_1 &\leq c_1(\|\Delta\varphi\alpha_n\|_{-1} + \|\varphi\alpha_n\|_{-1}) \\ &\leq c_1(\|\varphi\Delta\alpha_n\|_{-1} + \|\varphi\alpha_n\|_{-1} + \|(\Delta\varphi - \varphi\Delta)\alpha_n\|_{-1})\end{aligned}$$

for some $c_1 > 0$. There also exist $c_2, c_3 > 0$ such that

$$\begin{aligned}\|\varphi\Delta\alpha_n\|_{-1} \leq \|\varphi\Delta\alpha_n\| &\leq c_2\|\Delta\alpha_n\|' \\ \|\varphi\alpha_n\|_{-1} \leq \|\varphi\alpha_n\| &\leq c_3\|\alpha_n\|'.\end{aligned}$$

Now let $\rho = \mathbb{R}^n \to [0,1]$ be a function, that is compactly supported on V and that is 1 on U_0. We now compute

$$\begin{aligned}\|(\Delta\varphi - \varphi\Delta)\alpha_n\|_{-1} \leq& \|(\Delta\varphi - \varphi\Delta)\rho\alpha_n\|_{-1} \\ \leq& c_4\|\rho\alpha_n\| \\ \leq& c_5\|\alpha_n\|',\end{aligned}$$

for some $c_4, c_5 > 0$, where we used, that $\Delta\varphi - \varphi\Delta$ is a differential operator of order at most 1. Putting everything together we see that there is a $\tilde{c} > 0$, such that

$$\|\varphi\alpha\|_1 \leq \tilde{c}(\|\Delta\alpha_n\|' + \|\alpha_n\|')$$

which is, by assumption, bounded. □

Proposition 2.4.4 *There exists a constant $c > 0$ such that $\|\alpha\| \leq c\|\Delta_L\alpha\|$ for all $\alpha \in (\mathcal{H}_L^p)^\perp$.*

Proof Suppose by contradiction that there exists a sequence $(\alpha_i)_{i\geq 0}$ in $(\mathcal{H}_L^p)^\perp$, such that $\|\alpha_i\| = 1$ and $\|\Delta_L \alpha_i\| \to 0$ as $i \to \infty$. Then, by Theorem 2.4.3, there exists a Cauchy subsequence $(\alpha_{i_k})_{k\geq 0}$. We now define a bounded linear form ℓ on E^p by

$$\ell(\varphi) = \lim_{k\to\infty} (\alpha_{i_k}, \varphi).$$

The linear form ℓ is a weak solution of $\Delta_L \alpha = 0$, since

$$\ell(\Delta_L \varphi) = \lim_{k\to\infty} (\alpha_{i_k}, \Delta_L \varphi) = \lim_{k\to\infty} (\Delta_L \alpha_{i_k}, \varphi) = 0.$$

Theorem 2.4.2 now implies that there exists a $\alpha \in E^p$, such that $\ell(\varphi) = (\alpha, \varphi)$, thus we get $\alpha_{i_k} \to \alpha$. Therefore, $\|\alpha\| = 1$ and $\alpha \in (\mathcal{H}_L^p)^\perp$, since $(\mathcal{H}_L^p)^\perp$ is a closed subspace. But, by definition, $\Delta_L \alpha = 0$ and thus $\alpha \in \mathcal{H}_L^p$, which gives us the desired contradiction. □

Theorem 2.4.5 *E^p has a decomposition*

$$E^p = \Delta_L (E^p) \oplus \mathcal{H}_L^p = L\left(E^{p-1}\right) \oplus L^*\left(E^{p+1}\right) \oplus \mathcal{H}_L^p$$

where $\mathcal{H}_L^p$ is finite-dimensional.

Proof The second equality follows from the fact that the images of L and L^* are orthogonal as $(L\alpha, L^*\beta) = (LL\alpha, \beta) = 0$ and the assertion that $\mathcal{H}_L^p$ is finite dimensional follows from the fact that otherwise it would have an infinite orthononal sequence which would contradict Theorem 2.4.3. To show the theorem we now have show that $(\mathcal{H}_L^p)^\perp = \Delta_L(E^p)$. Note that $\Delta_L(E^p) \subseteq (\mathcal{H}_L^p)^\perp$ follows directly from the fact that

$$(\Delta_L \omega, \alpha) = (\omega, \Delta_L \alpha) = 0$$

for all $\omega \in E^p$ and $\alpha \in \mathcal{H}_L^p$. Now for the other inclusion, let $\alpha \in (\mathcal{H}_L^p)^\perp$ and define a bounded linear functional ℓ on $\Delta_L E^p$ by

$$\ell(\Delta_L \varphi) = (\alpha, \varphi).$$

ℓ is well defined since for $\Delta_L \varphi = \Delta_L \psi$, we have that $\varphi - \psi \in \mathcal{H}_L^p$. Let therefore $\psi = \varphi - \pi\varphi$, where π denotes the projection onto $\mathcal{H}_L^p$. By Proposition 2.4.4, ℓ is bounded as

$$|\ell(\Delta_L \psi)| \leq |(\alpha, \psi)| \leq \|\alpha\|\|\psi\| \leq c\|\alpha\|\|\Delta_L \psi\|.$$

By the Hahn-Banach theorem we can extend ℓ to all of E^p which gives us a weak solution to $\Delta_L \omega = \alpha$. Thus, by Theorem 2.4.2, there exists a $\omega \in E^p$ such that $\Delta_L \omega = \alpha$ and thus $(\mathcal{H}_L^p)^\perp \subseteq \Delta_L(E^p)$. □

This means, explicitly written out, that for all $\Omega^k_{\mathbb{C}}(X)$ in the de Rham complex, we have the decomposition

$$\Omega^k(X) = \Delta_d\left(\Omega^k_{\mathbb{C}}(X)\right) \oplus \mathcal{H}^k_d = d\left(\Omega^{k-1}_{\mathbb{C}}(X)\right) \oplus d^*\left(\Omega^{k+1}_{\mathbb{C}}(X)\right) \oplus \mathcal{H}^k_d$$

and for all $\Omega^{p,q}(X)$ in the Dolbeault complex, we have that

$$\Omega^{p,q}(X) = \Delta_{\overline{\partial}}\left(\Omega^{p,q}(X)\right) \oplus \mathcal{H}^{p,q}_{\overline{\partial}} = \overline{\partial}\left(\Omega^{p,q-1}(X)\right) \oplus \overline{\partial}^*\left(\Omega^{p,q+1}(X)\right) \oplus \mathcal{H}^{p,q}_{\overline{\partial}}$$

and, similarly, for the ∂ operator, we have that

$$\Omega^{p,q}(X) = \Delta_{\partial}\left(\Omega^{p,q}(X)\right) \oplus \mathcal{H}^{p,q}_{\partial} = \partial\left(\Omega^{p+1,q}(X)\right) \oplus \partial^*\left(\Omega^{p+1,q}(X)\right) \oplus \mathcal{H}^{p,q}_{\partial},$$

where the spaces $\mathcal{H}^k_d$, $\mathcal{H}^{p,q}_{\partial}$ and $\mathcal{H}^{p,q}_{\overline{\partial}}$ are finite dimensional.

Corollary 2.4.6 *For $\alpha \in E^p$ the equation $\Delta_L\omega = \alpha$ has a solution if and only if α is orthogonal to $\mathcal{H}^p_L$.*

Definition 2.4.7 *We define the **Green's operator** $G_L : E^p \to E^p$ to be the map sending α to the unique solution of $\Delta_L\omega = \alpha - \pi\alpha$ in $(\mathcal{H}^p_L)^\perp$, where π denotes the projection onto $\mathcal{H}^p_L$.*

Proposition 2.4.8 *L and G_L commute.*

Proof First note that $L\Delta_L = \Delta_L L$. Also note that $L\pi = \pi L = 0$ as the image of π is $\mathcal{H}^{p+1}$ and $(L\alpha, \beta) = (\alpha, L^*\beta) = 0$ for all $\beta \in \mathcal{H}^{p+1}_L$. Notice that this especially means that, for all $\alpha \in E^p$, we have that $L\alpha \in (\mathcal{H}^{p+1}_L)^\perp$. G_L sends an element $\alpha \in E^p$ to its unique solution ω of $\Delta_L\omega = \alpha - \pi\alpha$ in $(\mathcal{H}^p_L)^\perp$ and it sends $L\alpha$ to its unique solution ω' of $\Delta_L\omega' = L\alpha - \pi L\alpha = L(\alpha - \pi\alpha)$ in $(\mathcal{H}^{p+1})^\perp$. Since now $L(\alpha - \pi\alpha) = L\Delta_L\omega = \Delta_L L\omega$ and $L\omega \in (\mathcal{H}^{p+1}_L)^\perp$ as seen above we get that $\omega' = L\omega$ and thus $LG_L\alpha = L\omega = \omega' = G_L L\alpha$. □

Theorem 2.4.9 *Each cohomology class of E^p has a unique harmonic representative.*

Proof We argue based on [Roe99, Theorem 6.2]. We show that the inclusion of $\mathcal{H}^p_L$ into E^p is a chain homotopy equivalence and, therefore, induces an isomorphism in cohomology. For more informations see [Bre93, Chapter IV, Proposition 15.2]. Since L restricted to $\mathcal{H}^p_L$ is just the zero map, the cohomology of the complex restricted to the $\mathcal{H}^p_L$ is just given by $\mathcal{H}^p_L$ itself.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathcal{H}^{p-1}_L & \xrightarrow{\;0\;} & \mathcal{H}^p_L & \xrightarrow{\;0\;} & \mathcal{H}^{p+1}_L & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \\
\cdots & \longrightarrow & E^{p-1} & \xrightarrow{\;L\;} & E^p & \xrightarrow{\;L\;} & E^{p+1} & \longrightarrow & \cdots
\end{array}$$

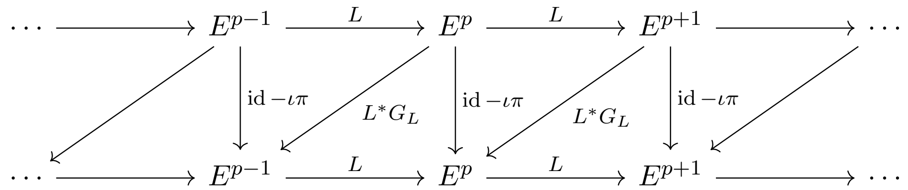
We claim that π is a chain homotopy inverse for ι. Since $\pi\iota = \mathrm{id}$, we only need to show that there exists a map $F : E^p \to E^{p-1}$, such that $\mathrm{id} - \iota\pi = LF + FL$. Remember that by the definition of the Green's operator and Theorem 2.4.5, we get a decomposition

$$\alpha = LL^*G_L\alpha + L^*LG_L\alpha + \pi\alpha$$

for every $\alpha \in E^p$. Since, by Proposition 2.4.8, G_L commutes with L this can be rewritten as

$$\alpha = L(L^*G_L)\alpha + (L^*G_L)L\alpha + \pi\alpha.$$

As $(\mathrm{id} - \iota\pi)\alpha = \alpha - \pi\alpha$, we see that $F = L^*G_L$ can be chosen as our chain homotopy.



Thus the inclusion map ι induces an isomorphism in cohomology. □

Corollary 2.4.10 *For every compact orientable manifold, all Betti numbers are finite and for every compact complex manifold all Hodge numbers are finite.*

Remark 2.4.11 *Observe that since all the operators used for Theorem 2.4.5 are real, the Hodge decomposition theorem for differential forms also holds for the real de Rham complex.*

Chapter 3

The Hodge decomposition theorem and Hodge diamond

In this chapter we prove the Hodge decomposition theorem for complex de Rham cohomology groups of a compact Kähler manifold. To that end, in Section 3.1 we start by showing some commutator identities on compact Kähler manifolds that lead us to the relation $\Delta_{\overline{\partial}} = \Delta_{\partial} = 2\Delta_d$. Together with the fact that all de Rahm and Dolbeault cohomology classes have a unique harmonic representative, we then obtain the decomposition theorem in Section 3.2. This leads us then, in Section 3.3, to a structure called the Hodge diamond, whose properties, namely its symmetries and the so called Lefschetz decomposition, are discussed in the end of the chapter.

3.1 Kähler identities

The Kähler identities are a collection of commutator relations on Kähler manifolds. In this section, we first prove some of them, that are needed to prove $\Delta_{\overline{\partial}} = \Delta_{\partial} = 2\Delta_d$, which is the main result of this section.

Definition 3.1.1 *On a Kähler manifold (X, h) with fundamental form ω, we define the Lefschetz operator as*

$$L : \bigwedge^{p,q} X \to \bigwedge^{p+1,q+1} X,\ \alpha \mapsto \omega \wedge \alpha$$

Lemma 3.1.2 *The adjoint operator*

$$L^* = \bigwedge^{p,q} X \to \bigwedge^{p-1,q-1}$$

of L is given by $L^* = (-1)^{p+q} * L *$.

Proof We compute

$$\begin{aligned}(L\alpha,\beta) &= \int_X L\alpha \wedge *\beta \\ &= \int_X \omega \wedge \alpha \wedge *\beta \\ &= \int_X \alpha \wedge \omega \wedge *\beta \\ &= \int_X \alpha \wedge L * \beta \\ &= \int_X \alpha \wedge *(*^{-1}L*)\beta \\ &= (\alpha, (-1)^{p+q} * L * \beta).\end{aligned}$$

□

Let $[\cdot,\cdot]$ denote the commutator $[A,B] = AB - BA$ of two operators A and B. Using the Lefschetz operator and the fact that on a Kähler manifold $d\omega = 0$, we now proceed to show some commutator relations on Kähler manifolds.

Proposition 3.1.3 *On any Kähler manifold, we have that* $[L,\partial] = [L,\overline{\partial}] = [L,d] = 0$.

Proof Using $d\omega = 0$ and thus $\partial\,\omega = 0$, we compute

$$[L,\partial]\alpha = \omega \wedge \partial\,\alpha - \partial(\omega \wedge \alpha) = \omega \wedge \partial\,\alpha - \partial\,\omega \wedge \alpha - \omega \wedge \partial\,\alpha = 0.$$

The computation for $[L,\overline{\partial}]$ is similar. As $d = \partial + \overline{\partial}$ we get $[L,d] = [L,\partial] + [L,\overline{\partial}] = 0$. □

Corollary 3.1.4 *Taking the adjoints of the above relations, we obtain that on a Kähler manifold we have* $[L^*,\partial^*] = [L^*,\overline{\partial}^*] = [L^*,d^*] = 0$.

Proposition 3.1.5 *On a Kähler manifold, we have that* $[L,\partial^*] = i\,\overline{\partial}$, $[L,\overline{\partial}^*] = -i\,\partial$, $[L^*,\partial] = i\,\overline{\partial}^*$ *and* $[L^*,\overline{\partial}] = -i\,\partial$.

Proof We first proof the first equality $[L,\overline{\partial}^*] = -i\,\partial$, where we follow [Dem12, Example 3.12 of Chapter VI]. We use Proposition 1.4.15, which states that there are coordinates that osculate up to order 2 with the standard metric of $\mathbb{C}^n$ within a chart and use that all computations only involve at most first order partial derivatives. Let $x_0 \in X$ and let $(z_1,\ldots,z_n)$ be the coordinate system constructed in Proposition 1.4.15 around x_0. For all differential forms ξ,η that are compactly supported in a small enough neighbourhood of x_0, the inner product in these coordinates is given by

$$(\xi,\eta) = \int_{\mathbb{C}^n} \sum_{I,J} \xi_{I,J}\overline{\eta}_{I,J} + O(|z|^2)d\mu.$$

With that, we can directly compute an expression for $\overline{\partial}^*$ at x_0. For two such forms ξ and η we get

$$\begin{aligned}
(\overline{\partial}^* \xi, \eta) =& (\xi, \overline{\partial}\, \eta) \\
=& \left(\xi, \sum_{I,J,k} \frac{\partial \eta_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J \right) \\
=& \sum_k \left(2\iota_{\frac{\partial}{\partial \overline{z}_k}} \xi, \sum_{I,J} \frac{\partial \eta_{I,J}}{\partial \overline{z}_k} dz_I \wedge d\overline{z}_J \right) \\
=& 2 \sum_k \int_{\mathbb{C}^n} \sum_{I,J} (\iota_{\frac{\partial}{\partial \overline{z}_k}} \xi)_{I,J} \overline{\frac{\partial \eta_{I,J}}{\partial \overline{z}_k}} + O(|z|^2) d\mu \\
=& -2 \sum_k \int_{\mathbb{C}^n} \sum_{I,J} \frac{\partial (\iota_{\frac{\partial}{\partial \overline{z}_k}} \xi)_{I,J}}{\partial\, z_k} \overline{\eta_{I,J}} + O(|z|) d\mu \\
=& \left(-2 \sum_{I,J,k} \frac{\partial\, \xi_{I,J}}{\partial\, z_k} \iota_{\frac{\partial}{\partial \overline{z}_k}} dz_I \wedge d\overline{z}_J + O(|z|), \eta \right),
\end{aligned}$$

where we used that at x_0, Lemma 2.2.12 gives us

$$(d\overline{z}_k \wedge)^* = (dx_k \wedge)^* + (idy_k \wedge)^* = \iota_{\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}} = 2\iota_{\frac{\partial}{\partial \overline{z}_k}}.$$

Thus at x_0, the adjoint of $\overline{\partial}$ is given by

$$\overline{\partial}^* \xi = -2 \sum_{I,J,k} \frac{\partial \xi_{I,J}}{\partial z_k} \iota_{\frac{\partial}{\partial \overline{z}_k}} dz_I \wedge d\overline{z}_J + O(|z|).$$

We now compute

$$\begin{aligned}
[L, \overline{\partial}^*]\xi =& -2 \sum_{I,J,k} \frac{\partial \xi_{I,J}}{\partial z_k} \omega \wedge \iota_{\frac{\partial}{\partial \overline{z}_k}} (dz_I \wedge dz_J) + O(|z|) \\
& + 2 \sum_{I,J,k} \frac{\partial \xi_{I,J} + O(|z|^2)}{\partial z_k} \iota_{\frac{\partial}{\partial \overline{z}_k}} (\omega \wedge dz_I \wedge dz_J) \\
=& 2 \sum_{I,J,k} \frac{\partial \xi_{I,J}}{\partial z_k} \underbrace{(\iota_{\frac{\partial}{\partial \overline{z}_k}} \omega)}_{= -\frac{1}{2} i dz_k} \wedge dz_I \wedge dz_J + O(|z|) \\
=& -i\, \partial\, \xi + O(|z|).
\end{aligned}$$

Thus, the statement holds for x_0 and, since x_0 was arbitrary, it holds for all of X. The second equation now follows through complex conjugation and the last two by taking adjoints. □

Proposition 3.1.6 *On a riemannian manifold X with $\dim_{\mathbb{R}} X = n$, we have that $*\Delta_d = \Delta_d *$.*

Proof For $\alpha \in \Omega^p_{\mathbb{C}}(X)$ we compute

$$\begin{aligned}
\Delta_d * =& dd^* * + d^* d* \\
=& (-1)^{n(n-p+1)+1} d * d * * + (-1)^{n(n-p+2)+1} \\
=& (-1)^{n(p+1)+1} * d * d * + (-1)^{n(p+2)+1} * * d * d \\
=& * \, dd^* + * d^* d \\
=& * \, \Delta_d .
\end{aligned}$$

□

Proposition 3.1.7 *On a Kähler manifold we have that $[\Delta_d, L] = [\Delta_\partial, L] = [\Delta_{\overline{\partial}}, L] = 0$.*

Proof Since the Laplacians are proportional to each other it suffices to show $[\Delta_\partial, L] = 0$. We compute

$$\begin{aligned}
[\Delta_\partial, L] =& \partial^* \partial L + \partial \partial^* L - L \partial^* \partial - L \partial \partial^* \\
=& \partial^* \partial L + \partial \partial^* L - \partial^* \partial L - i \overline{\partial} \partial - \partial \partial^* L - i \partial \overline{\partial} \\
=& 0.
\end{aligned}$$

□

Lemma 3.1.8 *On a Kähler manifold, we have that $\partial \overline{\partial}^* + \overline{\partial}^* \partial = 0$ and $\overline{\partial} \partial^* + \partial^* \overline{\partial} = 0$.*

Proof

$$\begin{aligned}
\partial \overline{\partial}^* + \overline{\partial}^* \partial &= -i(\partial \, i \, \overline{\partial}^* + i \, \overline{\partial}^* \partial) \\
&= -i(\partial [L^*, \partial] + [L^*, \partial] \partial) \\
&= -i(\partial L^* \partial - \partial^2 L^* + L^* \partial^2 - \partial L^* \partial) \\
&= 0
\end{aligned}$$

The second equality now follows through complex conjugation. □

With these preparations, we are now able to prove the main result of this section.

Proposition 3.1.9 *On a Kähler manifold, we have the following relation between the Laplacians*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}.$$

Proof Using Proposition 3.1.5, we first show that $\Delta_\partial = \Delta_{\overline{\partial}}$.

$$\begin{aligned}
\Delta_\partial &= \partial\,\partial^* + \partial^*\,\partial \\
&= i(\partial[L^*,\overline{\partial}] + [L^*,\overline{\partial}]\,\partial) \\
&= i(\partial\,L^*\,\overline{\partial} - \partial\,\overline{\partial}\,L^* + L^*\,\overline{\partial}\,\partial - \overline{\partial}\,L^*\,\partial) \\
&= -i(\overline{\partial}\,L^*\,\partial - \overline{\partial}\,\partial\,L^* + L^*\,\partial\,\overline{\partial} - \partial\,L^*\,\overline{\partial}) \\
&= -i(\overline{\partial}[L^*,\partial] + [L^*,\partial]\,\overline{\partial}) \\
&= \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} \\
&= \Delta_{\overline{\partial}}
\end{aligned}$$

And finally, using Lemma 3.1.8, we get

$$\begin{aligned}
\Delta_d &= dd^* + d^*d \\
&= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\
&= \partial\,\partial^* + \partial\,\overline{\partial}^* + \overline{\partial}\,\partial^* + \overline{\partial}\,\overline{\partial}^* + \partial^*\,\partial + \partial^*\,\overline{\partial} + \overline{\partial}^*\,\partial + \overline{\partial}^*\,\overline{\partial} \\
&= \Delta_\partial + \Delta_{\overline{\partial}} + \underbrace{(\partial\,\overline{\partial}^* + \overline{\partial}^*\,\partial)}_{=0} + \underbrace{(\overline{\partial}\,\partial^* + \partial^*\,\overline{\partial})}_{=0} \\
&= 2\Delta_{\overline{\partial}}.
\end{aligned}$$

□

3.2 The Hodge decomposition theorem

In this section, we deduce the Hodge decomposition theorem for compact Kähler manifolds and proof a useful statement for compact Kähler manifolds called the $\partial\,\overline{\partial}$-lemma.

Theorem 3.2.1 *Let X be a compact complex manifold of Kähler type. Then there is a direct sum decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

with $H^{p,q}(X) = \overline{H^{q,p}}(X)$.

Proof According to Theorem 2.4.5, we only need to show that

$$\mathcal{H}^k_d(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\overline{\partial}}(X)$$

For $p+q=k$ let $\alpha \in \mathcal{H}^k_d$ and let $\alpha^{p,q}$ be its components in $\Omega^{p,q}(X)$. Since $\Delta_{\overline{\partial}}$ preserves the bidegree, we get that

$$\Delta_d\alpha = \sum_{p+q=k} \Delta_d\alpha^{p,q} = \sum_{p+q=k} 2\Delta_{\overline{\partial}}\alpha^{p,q} = 0$$

Since the decomposition $\Omega^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X)$ is direct, we get that $\Delta_{\overline{\partial}}\alpha^{p,q} = 0$ and conversely, that if all components $\alpha^{p,q}$ are in $\mathcal{H}^{p,q}_{\overline{\partial}}(X)$, then $\alpha \in \mathcal{H}^k_d(X)$. $H^{p,q}(X) = \overline{H^{q,p}}(X)$ follows immediately from the fact that $\Omega^{p,q}(X) = \overline{\Omega^{q,p}}(X)$. □

Although the harmonic representative of a class depends on the choice Kähler metric, this decomposition does not.

Proposition 3.2.2 *The Hodge decomposition is independent of the choice of a Kähler metric.*

Proof Let γ be any closed (p,q)-form that represents an element of $H^{p,q}$. Choosing an arbitrary Kähler metric, we can uniquely write $\gamma = \alpha + \Delta_d\beta = \alpha + dd^*\beta + d^*d\beta$ where α and β must be of type (p,q). Since γ is closed, both α and $\Delta_d\beta$ are closed and thus $d^*d\beta = 0$ leaving $\gamma = \alpha + dd^*\beta$ meaning $[\gamma] = [\alpha]$. Since the harmonic form α came from an arbitrary choice of Kähler metric, the statement follows. □

Lemma 3.2.3 ($\partial\,\overline{\partial}$-lemma) *Let X be a compact Kähler manifold and let $\alpha \in \Omega^{p,q}$ with $d\alpha = 0$, then the following are equivalent:*

- ***I*** *$\alpha = d\beta$ for $\beta \in \Omega^{p+q-1}$.*
- ***II*** *$\alpha = \partial\,\gamma$ for $\gamma \in \Omega^{p-1,q}$.*
- ***III*** *$\alpha = \overline{\partial}\,\gamma'$ for $\gamma' \in \Omega^{p,q-1}$.*
- ***IV*** *$\alpha = \partial\,\overline{\partial}\,\delta$ for $\delta \in \Omega^{p-1,q-1}$.*
- ***V*** *α is orthogonal to $\mathcal{H}^{p,q}_{\overline{\partial}}(X)$.*

Proof Note that **I** to **IV** all imply **V** as the harmonic forms for all three operators coincide. Also **IV** immediately implies **II** and **III**. To see that **IV** implies **I** note that

$$d\left(\overline{\partial}\,\frac{\delta}{2} - \partial\,\frac{\delta}{2}\right) = \partial\,\overline{\partial}\,\delta = \alpha.$$

Thus, it remains to show that **V** implies **IV**. Suppose that α is orthogonal to $\mathcal{H}^{p,q}_{\partial}(X)$, then Theorem 2.4.5 implies that $\alpha \in \partial\,\Omega^{p-1,q}(X) \oplus \partial^*\,\Omega^{p,q+1}(X)$. Let $\eta = \partial^*\,\gamma$ be any element in the image of ∂^*, then

$$(\alpha, \eta) = (\alpha, \partial^*\,\gamma) = (\partial\,\alpha, \gamma) = 0,$$

as by assumption $d\alpha = 0$ and therefore $\partial\,\alpha = 0$. Thus $\alpha = \partial\,\gamma$. Let $\gamma = \gamma_0 + \gamma_1 + \gamma_2$ be the decomposition of γ with respect to the $\overline{\partial}$ operator, where γ_0 is harmonic, $\gamma_1 = \overline{\partial}\,\beta_1$ is in the image of $\overline{\partial}$ and $\gamma_2 = \overline{\partial}^*\,\beta_2$ is in the image of $\overline{\partial}^*$. Since γ_0 is also harmonic for $\overline{\partial}$ we would have $\overline{\partial}\,\gamma_0 = 0$ and thus $\overline{\partial}(\gamma_1 + \gamma_2) = \alpha$. We can thus omit γ_0. We now have $\alpha = \partial\,\overline{\partial}\,\beta_1 + \partial\,\overline{\partial}^*\,\beta_2$ and since $\overline{\partial}\,\alpha = 0$ we have

$$\overline{\partial}\,\partial\,\overline{\partial}^*\,\beta_2 = -\,\overline{\partial}\,\partial\,\overline{\partial}\,\beta_1 = \overline{\partial}^2\,\partial\,\beta_1 = 0.$$

Using Lemma 3.1.8 we now get

$$\| \partial \overline{\partial}^* \beta_2 \|^2 = (\partial \overline{\partial}^* \beta_2, \partial \overline{\partial}^* \beta_2) = (\beta_2, \overline{\partial} \partial^* \partial \overline{\partial}^* \beta_2) = (\beta_2, -\partial^* \overline{\partial} \partial \overline{\partial}^* \beta_2) = 0.$$

Thus, $\alpha = \partial \overline{\partial} \beta_1$. □

3.3 The Hodge diamond

In this section, we present the Hodge diamond and some of its symmetries that stem from duality theorems of the de Rham and Dolbeault groups and the so called Lefschetz decomposition. The Hodge diamond is a way to present the Hodge numbers in a diamond shape.

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & \iddots & & \vdots & & \ddots & \\
h^{n,0} & \cdots & & & & \cdots & h^{0,n} \\
 & \ddots & & \vdots & & \iddots & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & & & h^{n,n} & & &
\end{array}$$

Although this way of writing down the Hodge numbers is certainly possible for any complex manifold, where some entries might not be finite, this diamond has some nice properties in the Kähler case. First of all, note that $H^{p,q}(X) = \overline{H^{q,p}}(X)$ immediately implies that $h^{p,q} = h^{q,p}$. By the Hodge Decomposition Theorem 3.2.1 the sum of the rows are equal to the Betti numbers. This diamond also has some symmetry properties such as being mirror symmetric around the middle column. Another symmetry is given by a half-turn rotation around the middle which also reflects a symmetry of the Betti numbers given by Poincaré duality as the following two theorems show.

Theorem 3.3.1 *(Poincaré duality) Let (X, g) be a compact orientable riemannian manifold with $\dim_{\mathbb{R}} X = n$ and riemannian metric g. Then $H^k(X, \mathbb{C}) \cong H^{n-k}(X, \mathbb{C})^*$.*

Proof Since the isomorphism $* : \Omega^r_{\mathbb{C}}(X) \to \Omega^{n-r}_{\mathbb{C}}(X)$ commutes with Δ_d it sends $\mathcal{H}^k_d(X)$ to $\mathcal{H}^{n-k}_d(X)$. We now get a dual pairing $\mathcal{H}^k_d(X) \times \mathcal{H}^{n-k}_d(X) \to \mathbb{C}$

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta.$$

which is indeed a sesquilinear map. This pairing is non-degenerate as $\langle \alpha, *\alpha \rangle = \|\alpha\|^2 > 0$ for any $\alpha \neq 0$ in $\mathcal{H}^k_d$. This pairing now descendes to cohomology and induces an isomorphism $H^r(X) \cong H^{n-r}(X)^*$. □

Corollary 3.3.2 *For any compact orientable manifold X with $\dim_{\mathbb{R}} X = n$, we immediately obtain that $b_k = b_{n-k}$ which especially implies that for a compact complex manifold X with $\dim_{\mathbb{C}} X = n$, we have that $b_k = b_{2n-k}$.*

In the case of a compact complex manifold we can even refine this theorem.

Theorem 3.3.3 (Serre duality) *Let X be a compact complex manifold. Then $H^{p,q}(X) \cong H^{n-p,n-q}(X)^*$.*

Proof Just as for Poincaré duality, $* : \Omega^{p,q}(X) \to \Omega^{n-p,n-q}(X)$ commutes with $\Delta_{\overline{\partial}}$ and thus sends $\mathcal{H}^{p,q}(X)$ isomophically onto $\mathcal{H}^{n-p,n-q}(X)$. We again define a dual pairing $\mathcal{H}^{p,q}_{\overline{\partial}}(X) \times \mathcal{H}^{n-p,n-q}_{\overline{\partial}}(X) \to \mathbb{C}$

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta.$$

This pairing is again non degenerate and therefore induces an isomorphism $H^{p,q}(X) \cong H^{n-p,n-q}(X)^*$. □

Corollary 3.3.4 *For every compact Kähler manifold we have that $h^{q,p} = h^{p,q} = h^{n-p,n-q} = h^{n-q,n-p}$.*

We can also say something about the middle column of the Hodge diamond.

Proposition 3.3.5 *For every compact Kähler manifold X of complex dimension n, the groups $H^{p,p}(X)$ are non-trivial.*

Proof We show that the closed (p,p)-form ω^p is not exact. Assume by contradiction that $\omega^p = d\alpha$. Then

$$\int_X \omega^n = \int_X d\alpha \wedge \omega^{n-p} = \int_X d(\alpha \wedge \omega^n) - \int_X \alpha \wedge d\omega^{p-n} = 0,$$

contradicting the fact that ω^n is a volume form. □

Corollary 3.3.6 *On a compact Kähler manifold $h^{p,p} > 0$ for $0 \leq p \leq n$ and $b_k > 0$ for all $0 \leq k \leq n$ even.*

Notice that a form α in $\Omega^{p,0}(X)$ is harmonic if and only if $\overline{\partial}\alpha = 0$, meaning α is a holomorphic form. From Proposition 1.1.4 we know that a 0-form α, meaning a function on X, is holomorphic if and only if α is constant, thus $h^{0,0} = b_0 = 1$ and therefore also $h^{n,n} = b_{2n} = 1$. If generally, for any $0 \leq k \leq n$, $\bigwedge^k T^*_{\mathbb{C}}(X)$ is a trivial holomorphic bundle, then $h^{k,0} = \binom{n}{k}$ as every constant section defines a harmonic form. For the case of $k = n$ we call $\bigwedge^n T^*_{\mathbb{C}}(X)$ its **canonical bundle** and compact Kähler manifolds, whose canonical bundle is a trivial holomorphic bundle, called Calabi-Yau manifolds, have $h^{n,0} = 1$.

Lemma 3.3.7 *Let X be a hermitian manifold and let $k \leq n = \dim_{\mathbb{C}}(X)$. Then the operator*

$$L^{n-k} : \Omega^k_{\mathbb{C}}(X) \to \Omega^{2n-k}_{\mathbb{C}}(X)$$

is an isomorphism.

Proof We base this proof on [BGG02, Proposition 1.1]. Since L^{n-k} is also a bundle homomorphism between bundles of the same dimension, it suffices to show injectivity. We argue by induction and begin with the base case $k = 0$. In this case, since ω^n is a volume form and $f \mapsto f\omega^n$ is only zero if the function f is. Now for the inductive step. Assume the statement holds up to $k-1$ and suppose there exists $\alpha \in \Omega^k_{\mathbb{C}}(X)$ such that

$$\omega^{n-k} \wedge \alpha = 0.$$

This implies that

$$\omega^{n-k+1} \wedge \alpha = 0$$

and thus for every vector field Y, we have that

$$0 = \iota_Y(\omega^{n-k+1} \wedge \alpha) = (n-k+1)(\iota_Y\omega) \wedge \underbrace{\omega^{n-k} \wedge \alpha}_{=0} + \omega^{n-k+1} \wedge (\iota_Y\alpha).$$

But since $\iota_Y\alpha \in \Omega^{k-1}_{\mathbb{C}}(X)$ our induction hypothesis implies that $\iota_Y\alpha = 0$ for all vector fields Y, implying $\alpha = 0$. □

Note that for $p+q = k \leq n$, L^{n-k} maps $\Omega^{p,q}(X)$ to $\Omega^{n-q,n-p}$. Therefore $L^{n-p-q} : \Omega^{p,q}(X) \to \Omega^{n-p,n-q}(X)$ is also an isomorphism. The important first observation now is that for $k < n$, L up to L^{n-k} are injective.

Theorem 3.3.8 (Hard Lefschetz theorem) *Let X be a compact Kähler manifold. Then for $k \leq n$ the map $L^{n-k} : H^k(X) \to H^{2n-k}(X)$ is an isomorphism.*

Proof Since by Proposition 3.1.7 L commutes with Δ_d, for any harmonic form $\alpha \in \mathcal{H}^k_d(X)$, $\Delta_d L^m\alpha = L\Delta_d\alpha = 0$. Thus, L^m maps harmonic forms to harmonic forms for all possible m. Therefore $L^{n-k} : \mathcal{H}^k_d(X) \to \mathcal{H}^{2n-k}_d(X)$ is an injective homomorphism. Since these spaces have the same dimension, which is finite, L^{n-k} is an isomorphism. □

Note that the same also holds for the Dolbeault groups as L is the bigraded.

Definition 3.3.9 *For a compact Kähler manifold, we call a cohomology class $[\alpha] \in H^{p,q}(x)$, for $p+q = k \leq n$, **primitive** if $L^{n-k+1}[\alpha] = 0$. We denote the space of primitive classes in $H^{p,q}(X)$ as $PH^{p,q}(X)$, and similarly the space of primitive classes in $H^k(X)$ as $PH^k(X)$ and the spaces of representing primitive harmonic forms are denoted by $\mathcal{P}\,\mathcal{H}^{p,q}_{\bar\partial}(X)$ and $\mathcal{P}\,\mathcal{H}^k_d(X)$ respectively.*

Theorem 3.3.10 (Lefschetz decomposition) *Let X be a compact Kähler manifold, then the cohomology groups decompose into images of primitive groups under iterations of L.*

Proof The statement follows inductively if we show that for $p+q = k \leq n$ we have that

$$\mathcal{H}^{p,q}_{\bar\partial}(X) = L\,\mathcal{H}^{p-1,q-1}_{\bar\partial}(X) \oplus \mathcal{P}\,\mathcal{H}^{p,q}_{\bar\partial}(X).$$

Since L^{n-k} is injective on $L\,\mathcal{H}^{p-1,q-1}_{\overline{\partial}}$, $L\,\mathcal{H}^{p-1,q-1}_{\overline{\partial}}(X)$ and $\mathcal{P}\,\mathcal{H}^{p,q}_{\overline{\partial}}(X) = \ker L^{n-k}$ have indeed trivial intersection. Now let $\alpha \in \mathcal{H}^{p,q}_{\overline{\partial}}(X)$ and assume $L^{n-k+1}\alpha = \beta \neq 0$. Since $\beta \in \mathcal{H}^{n-q+1,n-p+1}_{\overline{\partial}}(X)$, the Hard Lefschetz Theorem 3.3.8 implies that $\beta = L^{n-k+2}\gamma$ for some $\gamma \in \mathcal{H}^{p-1,q-1}_{\overline{\partial}}(X)$. Therefore $\alpha - L\gamma \in \mathcal{P}\,\mathcal{H}^{p,q}_{\overline{\partial}}(X)$. Thus these two spaces do indeed span all of $\mathcal{H}^{p,q}_{\overline{\partial}}(X)$.□

The Lefschetz decomposition basically says that the map L carries the cohomology classes, starting with the primitive ones, down along the columns down where they get embedded into all groups between them and their eventual destination, which is the group isomorphic under the hard Lefschetz theorem of the space where they have a primitive component, where said component ends its journey. The following diagram illustrates the middle column for a compact Kähler manifold X with $\dim_{\mathbb{C}}(X) = n = 2p$.

$$
\begin{array}{ccccccc}
\mathcal{H}^{0,0}_{\overline{\partial}}(X) & = & \mathcal{P}\,\mathcal{H}^{0,0}_{\overline{\partial}}(X) & & & & \\
 & & \downarrow L & & & & \\
\mathcal{H}^{1,1}_{\overline{\partial}}(X) & = & L\,\mathcal{P}\,\mathcal{H}^{0,0}_{\overline{\partial}}(X) & \bigoplus & \mathcal{P}\,\mathcal{H}^{1,1}_{\overline{\partial}}(X) & & \\
 & & \downarrow L & & \downarrow L & & \\
 & & \vdots & & \vdots & & \\
 & & \downarrow L & & \downarrow L & & \\
\mathcal{H}^{1,1}_{\overline{\partial}}(X) & = & L^{p}\,\mathcal{P}\,\mathcal{H}^{0,0}_{\overline{\partial}}(X) & \bigoplus & L^{p-1}\,\mathcal{P}\,\mathcal{H}^{1,1}_{\overline{\partial}}(X) & \bigoplus \cdots & \bigoplus \mathcal{P}\,\mathcal{H}^{p,p}_{\overline{\partial}}(X) \\
 & & \downarrow L & & \downarrow L & & \\
 & & \vdots & & \vdots & & \\
 & & \downarrow L & & \downarrow L & & \\
\mathcal{H}^{n-1,n-1}_{\overline{\partial}}(X) & = & L^{n-1}\,\mathcal{P}\,\mathcal{H}^{0,0}_{\overline{\partial}}(X) & \bigoplus & L^{n-2}\,\mathcal{P}\,\mathcal{H}^{1,1}_{\overline{\partial}}(X) & & \\
 & & \downarrow L & & & & \\
\mathcal{H}^{n,n}_{\overline{\partial}}(X) & = & L^{n}\,\mathcal{P}\,\mathcal{H}^{0,0}_{\overline{\partial}}(X) & & & &
\end{array}
$$

Corollary 3.3.11 *On a compact Kähler manifold we have for $p+q = k \leq n$ that $h^{p,q} \geq h^{p-1,q-1}$ and therefore $b_k \geq b_{k-2}$.*

Example 3.3.12 *Let $\mathbb{T}^n_{\mathbb{C}} = \mathbb{C}^n/_{\Lambda}$ where Λ is a lattice formed by $2n$ $\mathbb{R}$-linear independent vectors. For example, we can just take the lattice $\Lambda = \operatorname{span}_{\mathbb{Z}}\{e_1, ie_1, \ldots, e_n, ie_n\}$. The coordinates $dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z}_n$ on $\mathbb{C}^n$ induce global coordinates on $\bigwedge^{p,q} \mathbb{T}^n_{\mathbb{C}}$ for all p, q and the standard metric $h = \sum_{i,j=0}^{n} dz_i \otimes d\overline{z}_j$ on $\mathbb{C}^n$ descends to $\mathbb{T}^n_{\mathbb{C}}$ giving it a Kähler structure.*

We claim that the harmonic forms are of the form $\alpha_{I,J} dz_I \wedge d\bar{z}_J$ for $\alpha_{I,J}$ constant. As such a form is indeed harmonic, it suffices to show that for any harmonic $\alpha \in \Omega^{p,q}(\mathbb{T}^n_{\mathbb{C}})$ has constant coefficients. Since being harmonic is a local property, any harmonic form on $\mathbb{T}^n_{\mathbb{C}}$ lifts to a harmonic form on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. We saw in Example 2.2.6 that the coefficients of such a form are harmonic functions. Since these functions lift to periodic, and thus bounded, harmonic functions on $\mathbb{C}^n$, they must be constant by Liouville's Theorem (see [ABR01, Theorem 2.1.]).
We therefore see that the Hodge numbers $h^{p,q}$ are equal to the complex fiber dimension of the vector bundles $T^*_{\mathbb{C}}(\mathbb{T}^n_{\mathbb{C}})^{p,q}$. Namely, we need to count in how many ways we can choose p holomorphic coordinates out of n and in how many ways we can choose q anti-holomorphic coordinates out of n. Thus

$$h^{p,q} = \dim_{\mathbb{C}} \Omega^{p,q} = \binom{n}{p}\binom{n}{q}.$$

We see that we alawys have $h^{0,0} = h^{n,n} = 1$ which holds for all Kähler manifolds. The second row and the second to last row are always n and for the left and right most entries we have $h^{n,0} = h^{0,n} = 1$. For $n = 1$ we obtain the Hodge diamond

$$\begin{array}{ccc} & 1 & \\ 2 & & 2 \\ & 1 & \end{array}$$

and for $n = 2$ we obtain

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 2 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

The Hodge diamond of $\mathbb{T}^n_{\mathbb{C}}$ for an arbitrary n is of the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & n & & n & & \\ & \binom{n}{2}\binom{n}{0} & & \binom{n}{1}\binom{n}{1} & & \binom{n}{0}\binom{n}{2} & \\ \iddots & & & \vdots & & & \ddots \\ \binom{n}{n}\binom{n}{0} & \cdots & & & & \cdots & \binom{n}{0}\binom{n}{n} \\ \ddots & & & \vdots & & & \iddots \\ & \binom{n}{n-2}\binom{n}{n} & & \binom{n}{n-1}\binom{n}{n-1} & & \binom{n}{n}\binom{n}{n-2} & \\ & & n & & n & & \\ & & & 1 & & & \end{array}$$

Using Vandermonde's identity we can now calculate the Betti numbers.

$$b_r = \sum_{p=0}^{r} h^{p,r-p} = \sum_{p=0}^{r} \binom{n}{p}\binom{n}{r-p} = \binom{2n}{r}$$

Corollary 3.3.13 *Since $b_1 = h^{1,0} + h^{0,1} = 2h^{1,0}$, we see that a Kähler manifold must always have an even first Betti number and repeating the argument for all odd rows we see that they even must have all odd Betti numbers be even.*

We can use 3.2.1 also in the other direction to obtain some Hodge numbers from Betti numbers.

Example 3.3.14 *Every compact orientable surface is diffeomorphic to the Riemann surface Σ_g for some $g \geq 0$, called their genus, see [Lee11, Chapter 6] and [CM16, Chapter 1]. Using common methods from algebraic topology one can see that $H^k(\Sigma_g) \cong \mathbb{C}^{2g}$. Since Σ_g can be embedded within $\mathbb{R}^3$, they carry an almost complex structure which must be integrable by 1.3.6. For example Let $\nu(x)$ be a normal vector field, then for $v \in T_x(\Sigma_g)$ we can define an almost complex structure by*

$$J_x(v) = \nu(x) \times v,$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. Thus all these Σ_g are Kähler manifolds and must necessarily have the Hodge diamond

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

Example 3.3.15 *It is a well known fact from algebraic topology (for example using a cell decomposition) that the Betti numbers of $\mathbb{C}\mathbb{P}^n$ are $b_k = 1$ for k even and $b_k = 0$ for k odd. Since we know that $h^{p,p} > 0$ for all p by Proposition 3.3.5, we see that these are all non-zero Hodge numbers.*

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & \cdot^{\cdot^{\cdot}} & & & \vdots & & & \ddots & \\ 0 & & \cdots & & & & \cdots & & 0 \\ & \ddots & & & \vdots & & & \cdot^{\cdot^{\cdot}} & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Thus, the only non-zero primitive cohomology group is $PH^{0,0}(X) = H^{0,0}(X)$, which is generated by the class [1] and $H^{p,p}$ is generated by $[\omega^p]$. Note that this directly implies that $\mathbb{C}\mathbb{P}^n$ has no non-trivial holomorphic forms other than constant function.

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