Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Mathematics of Origami

Bachelor Thesis

Cécile Kunz

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Advisor: Prof. Dr. A. Cannas da Silva
Department of Mathematics, ETH Zürich

Abstract

This thesis approaches the art of paper folding from a mathematical perspective. The focus lies on flat vertex folds, i.e. folded papers whose crease pattern has only one vertex and which end up in a two-dimensional state after folding. One goal of this thesis is to count the number of possible ways to fold a given flat vertex fold. Theorems about necessary and sufficient conditions for flatfoldability are given after introducing the setup and notation. Finally, these theorems are used to count the number of ways to fold a given single-vertex fold flat.

Preface

Origami is the art of paper folding. Apart from being a beautiful handicraft, origami can be used as a mathematical tool, similar to a straightedge and compass. For example, to divide a segment into equal parts ([4], p. 13). Interestingly, we can trisect an angle using origami techniques ([4], p. 16), whereas this is not possible using only a straightedge and compass ([5]).
Origami techniques can also be found in mechanical engineering. For example, the astrophysicist Koryo Miura invented the Miura map fold to send large solar panels into space ([4], pp. 2, 3).[1]
Apart from geometric constructions, origami can be used in many other areas of mathematics. In algebra, for example, there are methods of solving equations by folding paper. This is based on the idea that the paper can be seen as a coordinate system and the crease lines represent linear equations. For second degree equations, the main idea is based on the following observation. When folding a point on a line, the resulting crease line is a tangent to the parabola that has the point as its focus and the line as its directrix ([4], p. 31). We can convince ourselves of this by taking a piece of paper and folding a point onto a line and unfolding it again. If the same point is folded repeatedly at different points on the line, after a few folds the shape of the parabola can be seen. Figure 1 provides an example, with the image of the point/focus from each fold marked on the directrix.

So far, we have seen examples of origami being used as a tool to solve mathematical problems. On the other hand, paper folding also raises many combinatorial questions about the number of certain foldable origamis. In particular, we could ask for the number of possible ways to fold a given crease pattern flat. In this bachelor thesis, we will pave the way to answering this question for a basic setup of a single-vertex crease pattern. Interestingly,

[1] The crease pattern in Figure 2.3 and Figure 3.1 is a local subset of this Miura map fold ([2], p. 4).

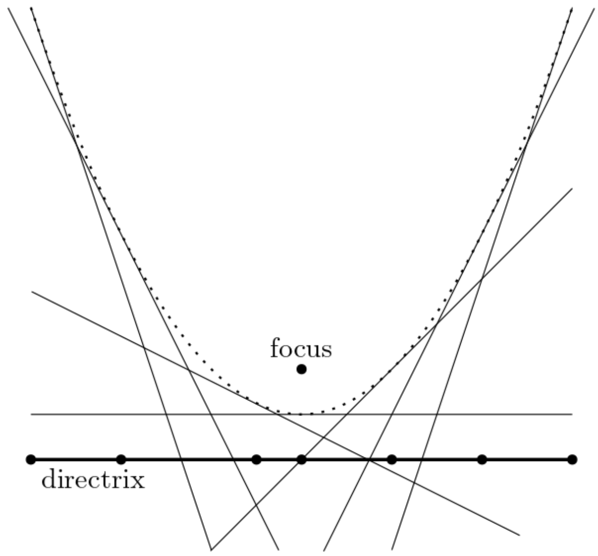


Figure 1: Folding a parabola

not all the results we will see for this single-vertex setup will extend to multiple-vertex cases, making the multiple-vertex case quite complex ([4], p. 107).

We proceed as follows.
Chapter 1 introduces a mathematical setup for paper folding. After providing general definitions, the focus of this thesis will be on flat origami.
Chapter 2 generalises the setup to cones, still restricted to flat origami. Theorems on necessary and sufficient conditions for a given crease pattern to be flatfoldable will be encountered.
Chapter 3 explains how to count the number of possible ways to fold a given crease pattern, if the latter is of a basic type.
Chapter 4 sums up what has been covered in this thesis and gives a glimpse of what lies ahead.

The thesis targets individuals with a mathematical background who possess the ability to comprehend proofs and enjoy applying intuition to mathematical problems. The statements presented in the thesis are often directly applicable by taking a piece of paper and folding it.

Unless another reference is given, this bachelor thesis follows Chapter 5 of [4]. I do not claim to be the founder of any ideas mentioned, except for minor modifications and extensions. During the research, I also consulted the book [1].

Acknowledgement

I would like to express my gratitude to my supervisor, Prof. Dr. Ana Cannas da Silva, for her guidance and support throughout the process. It was motivating to learn and work together. The support and suggestions provided during our regular meetings will continue to be useful to me in the future.

This work is based on the book [4]. None of the material presented is the product of my own ideas, except for minor modifications and extensions.

I typed this thesis in LaTeX using the template recommended by the CADMO at ETH. The figures are drawn with Ipe, version 7.2.28, and are inspired by the main reference [4].

Contents

Chapter 1

Introduction

In this chapter, we provide the setup for the rest of the thesis. We will follow the main reference [4].

1.1 Origami

In this thesis, unless otherwise noted, we will use the following notations and assumptions.
Let R denote a region in $\mathbb{R}^2$ with the properties that it is bounded, simply connected and has a non-empty interior. This we can imagine as a piece of paper. To be consistent throughout, we will refer to R simply as *paper* and we assume it to have the following properties. In particular, the paper is not stretchable, not self-intersecting and has a thickness of zero. Furthermore, R inherits an orientation from $\mathbb{R}^2$, which we may think of as distinguishing the two sides of the paper differently, say side A and side B. Unless otherwise noted, we are always looking at the same side of the paper.
When the paper is folded, crease lines are created. We will also refer to them as *edges*. We assume that the creases are straight lines or segments and have no width.

With these assumptions in mind, we state some definitions.

Definition 1.1 *Given a piece of paper $R \subset \mathbb{R}^2$, a* ***crease pattern*** *on R is a plane graph $G = (V, E)$, where V denotes the set of (discrete) vertices in R and E denotes the set of edges. Further, each edge $e \in E$ is in the interior of R, except possibly its endpoints. Vertices on the boundary of R are called* ***boundary vertices****, and vertices in the interior of R are called* ***interior vertices****.*
The ***faces*** *of the crease pattern G are the connected components of $R \setminus (V \cup E)$[1]. In Figure 1.1, an example is given.*

[1]For simplicity, we use this abuse of notation. The correct version is $R \setminus (V \cup (\cup_{e \in E} e))$, as E is a set of subsets of $\mathbb{R}^2$, but $E \not\subseteq \mathbb{R}^2$

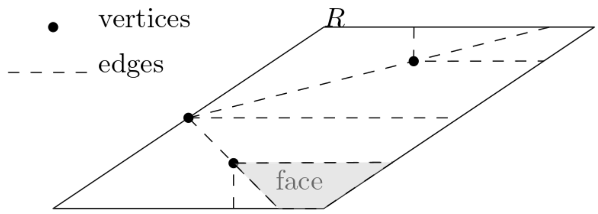


Figure 1.1: Crease pattern

Definition 1.2 *Given a crease pattern $G = (V, E)$ on R, an* ***origami*** *on G is a continuous, one-to-one mapping $\sigma : R \to \mathbb{R}^3$ such that σ is smooth (differentiable, say C^∞ for purposes of simplicity) everywhere except along the creases E.*

Definition 1.3 *Let f_1, f_2 be two adjacent faces in the crease pattern G and let σ be an origami on G such that $\sigma|_{f_1}, \sigma|_{f_2}$ are isometries. The* ***folding angle*** *of the crease e between these faces is the signed angle of displacement from a flat plane exhibited by f_1 or f_2 under σ.*
If the folding angle is positive, we say the crease between f_1 and f_2 is a ***valley crease****, and if the folding angle is negative, we say the crease is a* ***mountain crease****. See Figure 1.2 for an illustration.*

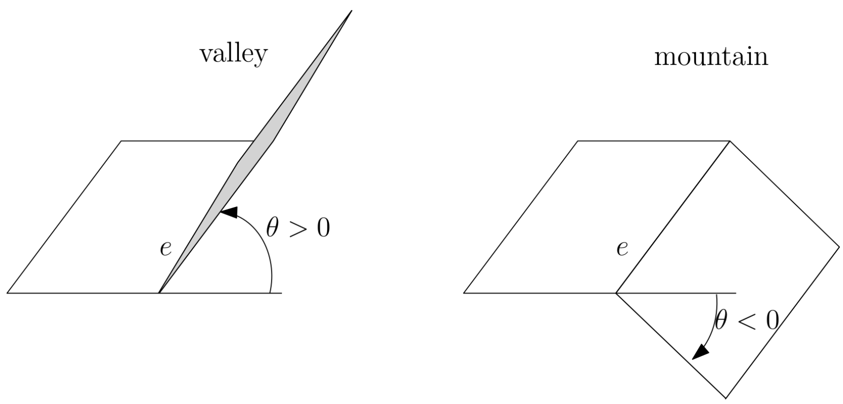


Figure 1.2: Folding angle

1.2 Flat Origami

In this thesis, we will focus on a limit case of origami that are two-dimensional after folding, i.e. lie flat in the plane. A formal definition of flat origami is given as follows.

Definition 1.4 *Given a crease pattern $G = (V, E)$ on R, a* ***flat origami*** *on G is an infinite sequence of origami $\{\sigma_n\}_{n=1}^{\infty}$ on G such that*

- *for each face f of G, the images $\{\sigma_n(f)\}_{n=1}^{\infty}$ uniformly converge to a planar polygon congruent to f and*
- *for each crease $\ell \in E$, the folding angles of the images $\{\sigma_n(\ell)\}_{n=1}^{\infty}$ converge to either π or $-\pi$.*

*If there exists a flat origami on a given crease pattern G, then we say that G is **flatfoldable** and that G **folds flat**.*

One difficulty with this definition is that in Definition 1.2 we defined origami as a one-to-one mapping, but a flat origami, as defined above, is not necessarily injective. Therefore, we will not use the formal definition of flat origami, but we will settle for the informal idea of **flat origami**: after folding, we can press it into a book without adding new creases.

Definition 1.5 *A **mountain-valley assignment** (or MV assignment) for a crease pattern $G = (V, E)$ is a function $\mu : E \to \{-1, 1\}$ that assigns folding angles of $\mu(c)\pi$ to each crease $c \in E$. (So, -1 indicates a mountain and 1 a valley.)*
*We let **M** denote the number of mountains and **V** the number of valleys of an MV assignment.*
*An MV assignment is called **valid** if it can be realized by a flat origami on the crease pattern.*

Observation 1.6 *We observe that by the orientation of our paper R (as introduced in Section 1.1) each MV assignment has a symmetric other MV assignment in the following sense. A mountain on side A of R is a valley on side B of R and vice versa. We can think of this symmetry by multiplying the MV assignment at each crease by -1.*

We now extend our intuitive idea of flatfoldability. For this purpose, we introduce the following notation. Let v be an interior vertex in the crease pattern G of a (flat) origami. Let $\mathbf{R}_\varepsilon$ be a circle of radius ε centred at v, where ε is taken small enough so that the only creases of G that intersect the boundary of R_ε are those adjacent to v. We denote by G' the subset of the crease pattern G intersecting R_ε. In Figure 1.3, an example is given. With this notation, we introduce the following definition.

Definition 1.7 *We call a crease pattern G **globally flatfoldable**, if there is a valid MV assignment. If there exists a valid MV assignment for the subset G' of the crease pattern G on R_ε, then we call G **locally flatfoldable** at v.*

We observe that there are MV assignments that locally fold flat but do not fold flat globally ([2], p.3). Further, note that for a crease pattern having only one interior vertex, the local and global properties coincide.

Definition 1.8 *We call a crease pattern $G = (V, E)$ with only one interior vertex a **single-vertex fold**. Moreover, if G folds flat, we call it a **flat vertex fold**.*

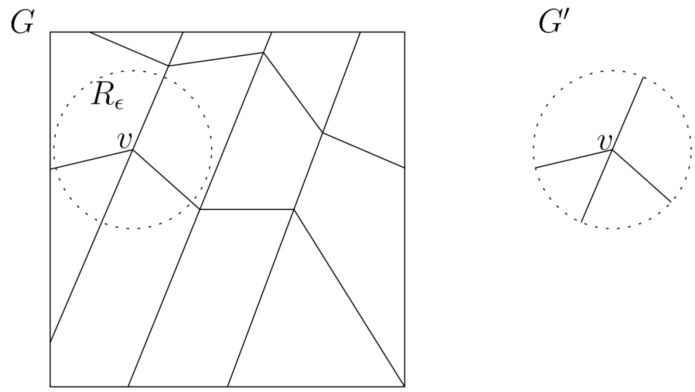


Figure 1.3: Example of a local analysis

The following notation, illustrated in Figure 1.4, will be used for single-vertex folds. We denote the edges by $\ell_0, \ell_1, \ldots$ in counterclockwise order around the vertex and by α_i the angle between the creases ℓ_i and ℓ_{i+1}.

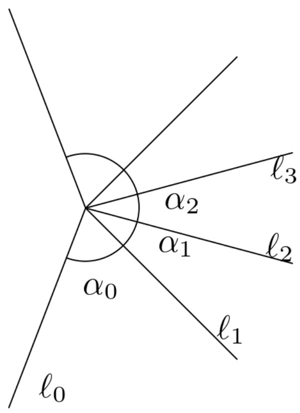


Figure 1.4: Notation for single-vertex folds

Given an MV assignment, it is not easy to see if it is valid. Of course, we can take a piece of paper and try to fold the given crease pattern flat. But how can we be sure that a given MV assignment is not valid, if we can't find a way to fold it? We may also wonder whether more than one MV assignment is valid for a given crease pattern, and if so, how many? In Chapter 2 we will encounter theorems about when a given single-vertex fold is flatfoldable, and in Chapter 3 we will see a way to count the number of valid MV assignments for some flat vertex folds.

1.3 Two-colourability of flat origami

We end this chapter with a theorem to show that origami can be a nice tool for some constructive proofs.
In graph theory, colouring vertices, edges or faces is a common problem, where one goal is to colour the vertices, edges or faces in k colours in such a way that no two adjacent ones are coloured in the same colour. From Definition 1.1 we know that the crease pattern of origami is a plane graph, so we can apply this problem to origami.

Theorem 1.9 *The faces of a flat origami crease pattern are two-colourable.*

Proof When looking at the crease pattern, fix the side of the paper facing, say side A. Then fold the paper flat according to a valid MV assignment, lay it flat in front of you such that there is an up/down orientation and keep track of the faces of side A. Colour the faces of side A in colour 1 if they are facing up and in colour 2 if they are facing down. Any two adjacent faces are separated by one crease, along which will be folded. Thus, these faces point in different directions, and therefore we have indeed a proper two-colouring of the faces. □

In Figure 1.5, we see an example of a two-colouring obtained by the method described in the proof.

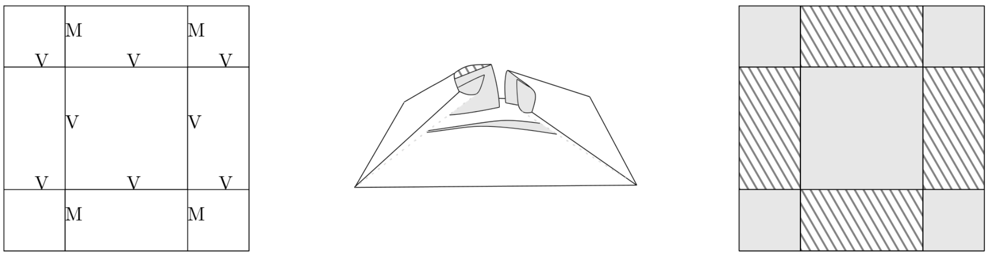


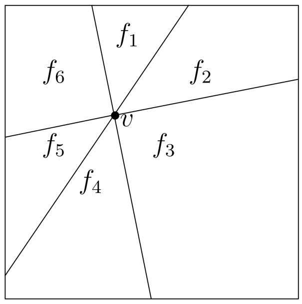
Figure 1.5: Example two-colouring

For the following corollary of Theorem 1.9, we need the definition of the degree of a vertex.

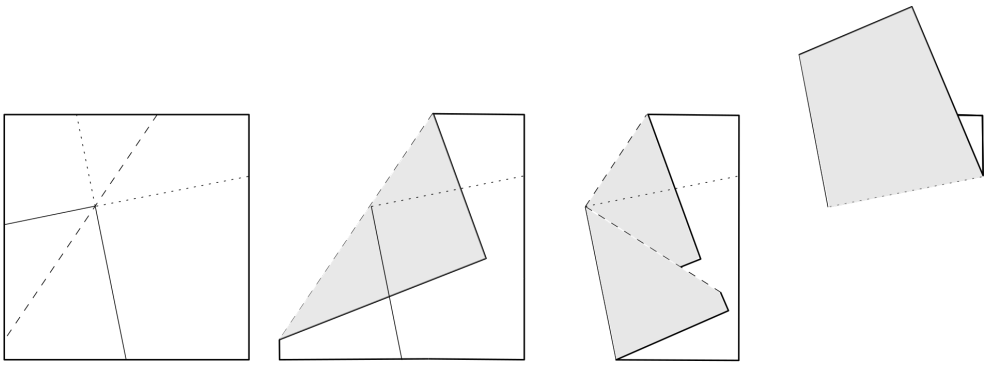
Definition 1.10 *An edge is* ***incident*** *to a vertex, if the vertex is an endpoint of the edge. The* ***degree*** *of a vertex is the number of edges incident to that vertex.*

Corollary 1.11 *The degree of the interior vertex in a flat vertex fold is even.*

Proof Let v be the interior vertex and $f_1, \ldots, f_k$ the faces in clockwise order around v (as illustrated in Figure 1.6a). The only possibility for a two-colouring is to colour the faces with even indices in one colour and the faces with odd indices in a second colour. But this gives a proper two-colouring of the faces if and only if k is even. By Theorem 1.9, such a two-colouring exists. Thus, we conclude that there must be an even number of faces, which, in the case of a single vertex, corresponds to the degree of v. $\square$



(a) Single-vertex fold



(b) Flatfoldability

Figure 1.6: Vertex degree of flat vertex fold

We observe that the crease pattern in Figure 1.6a is indeed a flat vertex fold, as in Definition 1.8. Since it has exactly one interior vertex and, as can be seen in Figure 1.6b, the crease pattern folds flat. One important thing to note in this specific example is that the angles of faces f_1 and f_5 are the same. Similarly, the angles of f_2 and f_4 as well as f_3 and f_6 are the same. This ensures that after folding along the dashed crease line, the dotted edge is mapped to the other dotted edge and the remaining two edges also have the same image after mapping.
Instead of doing a proof by picture, we could apply Kawasaki's Theorem 2.7, which will be introduced later, to see that this crease pattern is indeed flatfoldable.

So far we have learned some basic concepts and have seen a very hands-on proof using origami. In the next chapter, we will restrict ourselves to flat vertex folds. The theorems we will encounter will lead us into the area of combinatorics.

Chapter 2

Flat Vertex Folds

In this chapter, theorems on necessary and sufficient conditions for a given crease pattern to be flatfoldable will be encountered.

2.1 Cone

For now, we focus on single-vertex folds and we use the following notation, also illustrated in Figure 1.4. Given a flat vertex fold $G = (V, E)$, let $E = \{\ell_0, \ell_1, \ldots, \ell_{2n-1}\}$ be the edges meeting at the vertex, and let α_i be the angle between the edges ℓ_i and ℓ_{i+1} (and α_{2n-1} is between ℓ_{2n-1} and ℓ_0). Recall from Corollary 1.11 that the number of edges in a flat vertex fold is indeed even. To generalise things later, we need the following definition. In Figure 2.1, an illustration is given for the paper R being in a round shape.

Definition 2.1 *A single-vertex crease pattern on a* ***cone*** *is a single-vertex crease pattern G on a closed bounded region R on the surface of a cone that includes the apex and where the interior vertex of G is placed at the apex of the cone. The* ***cone angle*** *of the cone is the sum of the angles around the apex of the cone. If the cone angle equals* 2π*, then we are folding flat paper. If the cone angle is greater than* 2π*, then we are folding hyperbolic paper.*

Unless otherwise noted, we will focus on flat paper.

2.2 Mountain-Valley Parity: Maekawa's Theorem

In the last chapter, the question of the number of valid MV assignments for a given crease pattern arose. The following theorem, also called **Maekawa's Theorem**, turns out to be a useful tool to get closer to an answer to this question. This theorem is said to be a foundational result for flat vertex folds.

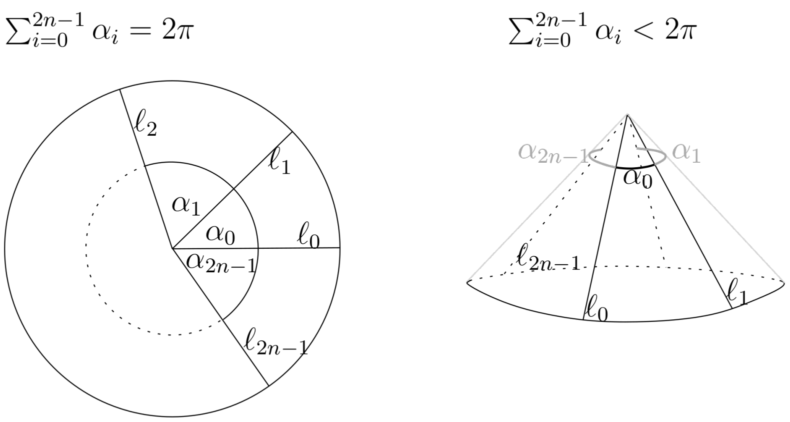


Figure 2.1: Flat paper vs. Cone

Theorem 2.2 *The difference between the number of mountain and valley creases in a flat vertex fold on a cone with cone angle $\leq 2\pi$ is 2. In other words, if μ is a valid MV assignment for a flat vertex fold on a cone with creases $\ell_0, \ldots, \ell_{2n-1}$ and cone angle $\leq 2\pi$, then $\sum_{i=0}^{2n-1} \mu(\ell_i) = \pm 2$.*

We will refer to $\sum_{i=0}^{2n-1} \mu(\ell_i) = \pm 2$ as **Maekawa's Condition**.

Observation 2.3 *We observe that Maekawa's Theorem does not hold for general multiple-vertex crease patterns. A counterexample can be found in Figure 1.5, where we have a valid MV assignment with eight valleys and four mountains.*

As we saw in the introductory chapter, some proofs can be done in a very practical way. All we have to do is fold paper, look at it, and find an argument to generalise it to any fold of that kind. The following proof will be of this type.
Recall from Definition 1.5 that M denotes the number of mountains and V the number of valleys.

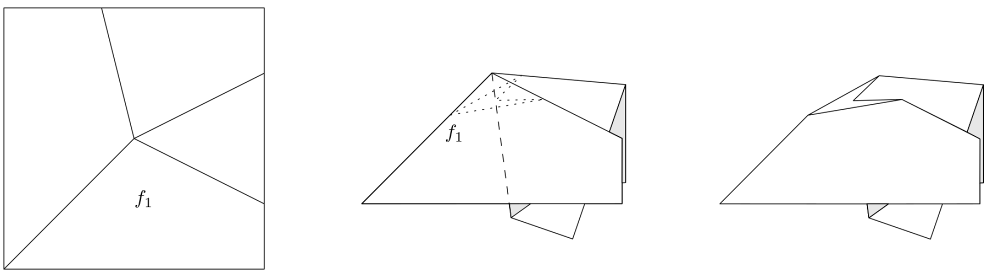


Figure 2.2: Proof of Maekawa's Theorem 2.2

Proof Let G be the crease pattern for a given flat vertex fold. We fold G flat and cut off the apex/vertex. See Figure 2.2 for an illustration. We imagine that we are walking clockwise along this created cut edge, starting at any point on it. Whenever we encounter a crease, we change direction accordingly. If the fold is a mountain, we rotate by π, and if it is a valley, we rotate by $-\pi$. When we are back at the starting point, we have made a turn of 2π. This gives us the equation πM-πV = 2π. We divide by π and find that the difference between the number of mountain and valley creases is two. By the symmetry argument of Observation 1.6 we also get the value -2. □

Maekawa's Theorem works only in one direction. There are MV assignments such that the difference between M and V is two, but the assignment is not valid. In Figure 2.3, two MV assignments are given. Both satisfy Maekawa's Condition, but are not flatfoldable. Later we will encounter Hull's Theorem 2.10 which allows us to show that they are indeed not flatfoldable.

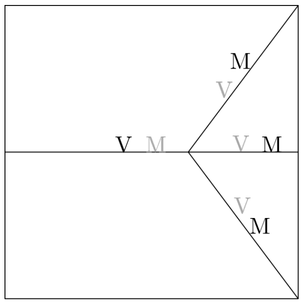


Figure 2.3: Maekawa's Theorem is not sufficient

Remark 2.4 *The sign of the total sum in Maekawa's Condition is determined by the orientation of the paper R. To demonstrate this, we will first examine an example involving two creases. In Figure 2.4, the negative side of the paper, say side A, has a tiling to distinguish it from the positive side B. The two edges in this crease pattern are either both valleys, then the total sum in Maekawa's Condition is* 2 *and the positive side of the paper is inside. Or both edges are mountains, then the sum is negative and the negative side of the paper is inside. According to the Jordan Curve Theorem, there is always an inside and an outside.*
In general, flat vertex folds on a cone exhibit the following property.

$$\sum_{i=0}^{2n-1} \mu(\ell_i) = \begin{cases} +2, & \textit{if the positive side is inside} \\ -2, & \textit{if the negative side is inside} \end{cases}$$

2.3 Flatfoldability: Kawasaki's Theorem

In the last section, we have learned about Maekawa's Theorem and how it is only a necessary condition for flatfoldability. In this section, we will encounter a theorem that is necessary and sufficient for flatfoldability. In order to get there, we need the following definition and lemma.

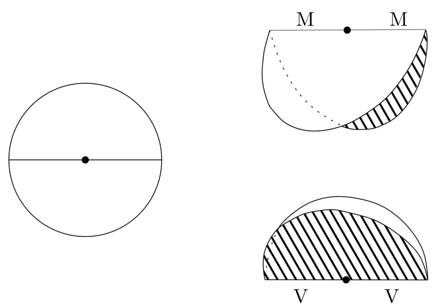


Figure 2.4: Sign determined by paper orientation

Definition 2.5 *For a finite sequence of positive real numbers $(\alpha_0, \dots, \alpha_{2n-1})$, we define the* ***partial alternating sum*** *as*

$$S(i,j) = \alpha_i - \alpha_{i+1} + \alpha_{i+2} - \dots \pm \alpha_j$$

for $0 \leq i, j \leq 2n-1$ and we take indices $\mod 2n$.

Lemma 2.6 *Let $(\alpha_0, \dots, \alpha_{2n-1})$ be a sequence of $2n$ positive real numbers such that $S(0, 2n-1) = 0$. Then there exists a k with $0 \leq k \leq 2n-1$ such that $S(k,i) \geq 0$ for every $0 \leq i \leq 2n-1$.*

Proof If $S(0,i) \geq 0 \ \forall i$, we let $k = 0$ and we're done.
Suppose $\exists i$ s.t. $S(0,i) < 0$ and we let $k-1$ be an index such that

$$S(0, k-1) = \min_{0 \leq i \leq 2n-1} \{S(0,i)\}.$$

Note that $S(0,0) = \alpha_0 > 0$ and $S(0,i) > S(0,i-1)$ for i even, by definition of the partial alternating sum and the fact that the α's are positive. So $\boldsymbol{k-1}$ **must be odd**, otherwise the alternating sum wouldn't be minimal.
By the choice of $k-1$ we have the following inequality for all $0 \leq i \leq 2n-1$.

$$\sum_{t=0}^{k-1} (-1)^t \alpha_t = S(0, k-1) \leq S(0,i) = \sum_{t=0}^{i} (-1)^t \alpha_t \tag{2.1}$$

We distinguish the following three cases for i.

- When $\boldsymbol{i \geq k}$, we subtract $\sum_{t=0}^{k-1} (-1)^t \alpha_t$ on both sides of the inequality (2.1) to get that

  $$0 \leq \sum_{t=k}^{i} (-1)^t \alpha_t.$$

  Since $k-1$ is odd, we have that k is even. Thus, the alternating sum above starts with a positive term and is therefore equal to $S(k,i)$. We conclude for this case that

  $$S(k,i) \geq 0.$$

- For $\boldsymbol{i \leq k-2}$, the following inequality is obtained by subtracting $\sum_{t=0}^{i}(-1)^t\alpha_t$ from both sides of (2.1).

$$\sum_{t=i+1}^{k-1}(-1)^t\alpha_t \leq 0 \tag{2.2}$$

  To use the inequality (2.2), we first split $S(0,2n-1)$ as follows.

$$S(0,2n-1) = \sum_{t=0}^{2n-1}(-1)^t\alpha_t = \sum_{t=0}^{i}(-1)^t\alpha_t + \sum_{t=i+1}^{k-1}(-1)^t\alpha_t + \sum_{t=k}^{2n-1}(-1)^t\alpha_t$$

  If we take the indices $\mod 2n$ and use that $S(0,2n-1)=0$, the above reduces to

$$0 = \sum_{t=i+1}^{k-1}(-1)^t\alpha_t + \sum_{t=k}^{i}(-1)^t\alpha_t.$$

  Which is equivalent to

$$\sum_{t=k}^{i}(-1)^t\alpha_t = -\sum_{t=i+1}^{k-1}(-1)^t\alpha_t.$$

  Now we use the fact that k is even and thus $\sum_{t=k}^{i}(-1)^t\alpha_t = S(k,i)$. Together with Equation (2.2) we get that $S(k,i) \geq 0$.

- When $\boldsymbol{i = k-1}$, we rearrange $S(0,2n-1)$ as follows, using the fact that k is even and that the indices are taken $\mod 2n$.

$$S(0,2n-1) = \sum_{t=0}^{2n-1}(-1)^t\alpha_t = \sum_{t=k}^{k-1}(-1)^t\alpha_t = S(k,k-1)$$

  With the assumption $S(0,2n-1)=0$ we get that $S(k,k-1)=0$.

In each of the above cases we get $S(k,i) \geq 0$. □

Now, we are ready for **Kawasaki's Theorem**.

Theorem 2.7 *Let G be a single-vertex crease pattern on a cone with cone angle $A \leq 2\pi$ and with consecutive angles between the creases $\alpha_0,\ldots,\alpha_{2n-1}$. Then G is flatfoldable if and only if $S(0,2n-1)=0$.*

The condition $S(0,2n-1)=0$ is also referred to as the **Kawasaki Condition**. As we will see, this proof of Theorem 2.7 is constructive.

Proof $\Rightarrow$ Let G be a flat vertex fold with consecutive angles $\alpha_0,\ldots,\alpha_{2n-1}$ and suppose that G is embedded on a piece of paper R that is either a circle or cone of radius 1 with the interior vertex at the circle's centre (or cone's apex). Then the angles α_i in radian measure are equal to

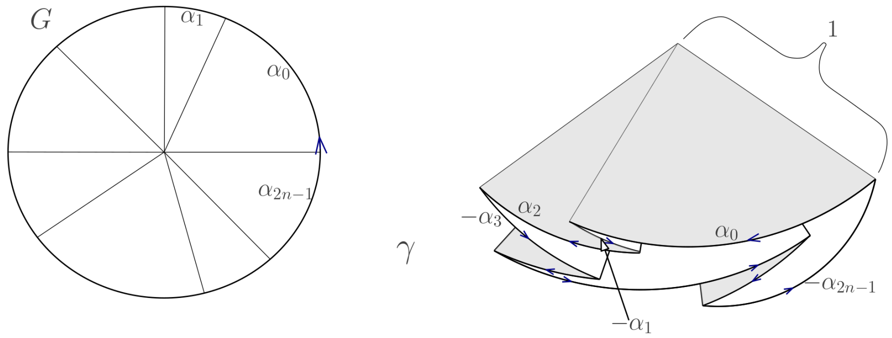


Figure 2.5: First direction of Kawasaki's Theorem

the arc-lengths. Let γ be the oriented curve on the boundary of R, and assume that γ starts at the crease between angles α_{2n-1} and α_0. Further, assume that γ travels on the arcs in the direction of the increasing indices of the angles. We fold G flat and consider the image of γ under this folding. It will follow an arc of length α_0 in one direction, then an arc of length α_1 in the opposite direction, and so on until it returns to the starting point. See Figure 2.5 for an illustration. The (oriented) distance travelled can thus be expressed as

$$0 = \alpha_0 - \alpha_1 + \alpha_2 - \cdots - \alpha_{2n-1} = S(0, 2n-1).$$

$\Leftarrow$ Assume, we are given a single-vertex fold G satisfying $S(0, 2n-1) = 0$. By Lemma 2.6, there is a k such that $S(k, i) \geq 0$ for every $0 \leq i \leq 2n-1$. As in the lemma, we will take the indices $\mod 2n$. Let ℓ_i denote the crease between the angles α_{i-1} and α_i in G, as in the first part of Figure 2.6. We define the following MV assignment for G.

$$\mu(\ell_{k+i}) = \begin{cases} -1 & \text{for } i \text{ odd} \\ \ \ 1 & \text{for } i \text{ even, } i \neq 0 \\ -1 & \text{for } i = 0 \end{cases}$$

By showing that this assignment is valid, we prove flatfoldability of G. In order to do this, we imagine that we cut along the crease ℓ_k and fold the other creases according to the MV assignment defined above. This results in a zigzag shape, as can be seen in the second part of Figure 2.6. Firstly, we observe that this zigzag shape folds flat. Second, it has no self-intersections and since $S(k, i) \geq 0$ for all $0 \leq i \leq 2n-1$ there are no layers between the two ends of the cut ℓ_k. Finally, since $S(k, k-1) = S(0, 2n-1) = 0$, both ends of the cut end at the same location. The third part of the figure illustrates this. So in this zigzag shape, the cut can be glued back together, making ℓ_k a mountain crease and proving that the MV assignment is valid. $\square$

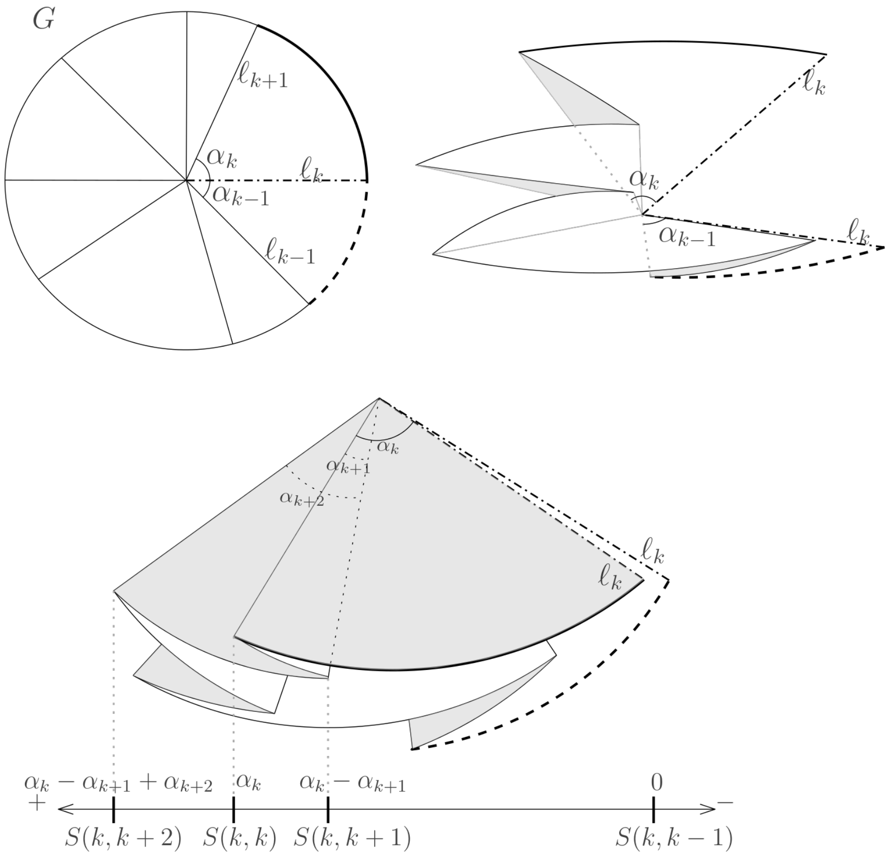


Figure 2.6: Other direction of Kawasaki's Theorem

We recall from Section 2.1 that the cone angle A is defined as $A = \sum_{i=0}^{2n-1} \alpha_i$. This is used in the following corollary to Kawasaki's Theorem.

Corollary 2.8 *Let G be a single-vertex crease pattern on a cone with cone angle $A \leq 2\pi$ and with consecutive angles between the creases $\alpha_0, \ldots, \alpha_{2n-1}$. Then G is flatfoldable if and only if $\alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1} = \alpha_0 + \alpha_2 + \cdots + \alpha_{2n-2} = \frac{A}{2}$.*

Proof We recall the Kawasaki Condition $S(0, 2n-1) = 0$. Adding the cone angle A to both side of this equation leads to

$$\sum_{i=0}^{n-1} 2\alpha_{2i} = A.$$

Dividing by two results in the second part of the corollary.
If we first multiply the Kawasaki Condition by -1 and then add the cone angle A, we get that

$$\sum_{i=1}^{n} 2\alpha_{2i-1} = A.$$

Again, dividing by two completes the proof. □

2.4 Big-Little-Big Lemma

So far we have seen two important theorems: Maekawa's Theorem, a statement on the mountain-valley assignment, and Kawasaki's Theorem, a statement on the angle sequence. In this section, we come across a lemma that proves certain MV assignments to be invalid by looking at the angle sequence. The lemma is called the **big-little-big lemma**, where the name refers to the (relative) size of consecutive angles.

Lemma 2.9 *Let G be a flat vertex fold with angle sequence* $(\alpha_0, \ldots, \alpha_{2n-1})$ *and a valid MV assignment* μ. *If* $\alpha_{i-1} > \alpha_i < \alpha_{i+1}$ *for some i, then* $\mu(\ell_i) \neq \mu(\ell_{i+1})$. *(That is,* $\mu(\ell_i) + \mu(\ell_{i+1}) = 0$.*)*

We illustrate this lemma in Figure 2.7, where we have $\alpha_0 > \alpha_1 < \alpha_2$. If we assign $\mu(\ell_1) = \mu(\ell_2)$ then, two angles try to cover a smaller angle on the same side of the paper, which cannot be folded flat without creating a new crease or forcing self-intersection of the paper.

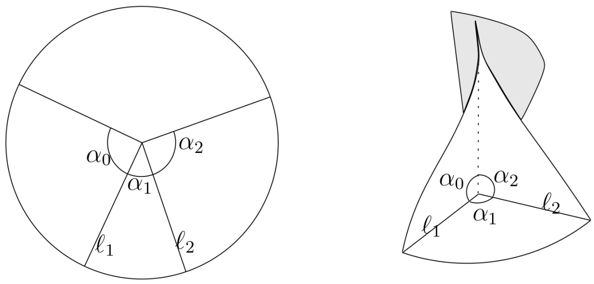


Figure 2.7: Big-Little-Big

As the name of the big-little-big lemma suggests, it is about *one* (little) angle between two bigger angles. The next section is about the case where we have more than one little angle, but all equal, enclosed by bigger angles as illustrated in Figure 2.8. It will turn out that the Big-Little-Big Lemma is a special case of the more general theorem stated in the next section. Therefore, we do not prove Lemma 2.9 separately.

2.5 Generalisation: Hull's Theorem

In this section, we consider crease patterns as in Figure 2.8. We follow the theorem and the proof from [3], but with a variation of the indices.

Theorem 2.10 *Let G be a flat vertex fold with angle sequence* $(\alpha_0, \ldots, \alpha_{2n-1})$, *and suppose that we have* $\alpha_1 = \alpha_2 = \cdots = \alpha_k < \alpha_0, \alpha_{k+1}$ *(where the indices are taken* $\mod 2n$*). Then an MV assignment* μ *for G will be valid among the creases*

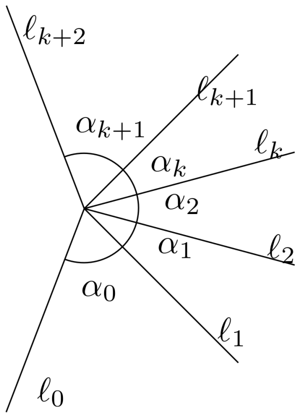


Figure 2.8: $\alpha_0 > \alpha_1 = \cdots = \alpha_k < \alpha_{k+1}$

$\ell_1, \ldots, \ell_{k+1}$ *if and only if*

$$\sum_{j=1}^{k+1} \mu(\ell_j) = \begin{cases} 0, & \textit{if } k \textit{ is odd.} \\ \pm 1, & \textit{if } k \textit{ is even.} \end{cases}$$

Observation 2.11 *For $k = 1$, Theorem 2.10 reduces to the Big-Little-Big Lemma 2.9 from the last section.*

Proof $\Rightarrow$ To prove the first direction, we assume that μ is a valid MV assignment among the creases $\ell_1, \ldots, \ell_{k+1}$. In the flatfolded state we consider the part corresponding to the angles $\alpha_0, \ldots, \alpha_{k+1}$ and we add a section of paper P_β with angle β so that it is still flatfolded. We use the notation $\mu(P_\beta)$ to denote the sum of the MV values of the creases on the added paper section P_β. Now, we apply Maekawa's Theorem 2.2 to get

$$\sum_{i=1}^{k+1} \mu(\ell_i) + \mu(P_\beta) = \pm 2. \tag{2.3}$$

We consider the two cases separately and recall from Remark 2.4 that the sign of the sum depends on the side of the paper that lies inside.

If ***k* is even**, the extra section P_β has exactly one crease. See Figure 2.9 for an example. If the positive side of the paper is on the inside, then P_β is adding a valley. Together with Remark 2.4, Equation (2.3) reduces to the following.

$$\sum_{i=1}^{k+1} \mu(\ell_i) + 1 = +2 \Rightarrow \sum_{i=1}^{k+1} \mu(\ell_i) = +1$$

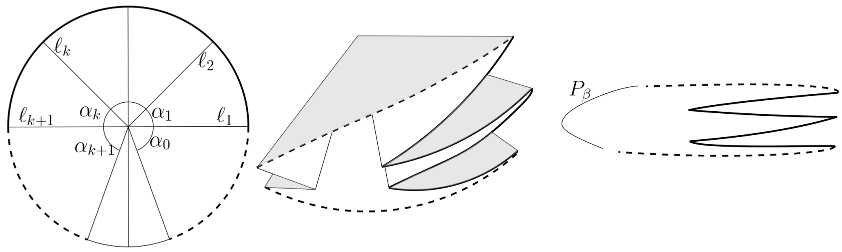


Figure 2.9: Case k even

Similarly, if the negative side is inside, a mountain crease is added.

$$\sum_{i=1}^{k+1} \mu(\ell_i) - 1 = -2 \Rightarrow \sum_{i=1}^{k+1} \mu(\ell_i) = -1$$

For ***k* odd**, there are always two alternative ways to add the extra section P_β. In one case, the negative side is inside, in the other case, the positive side is inside. The extra section P_β is adding two creases of the same MV parity, as illustrated in Figure 2.10. If the positive side is inside, then P_β is adding two valleys. Together with Remark 2.4, Equation (2.3) reduces to the following.

$$\sum_{i=1}^{k+1} \mu(\ell_i) + 2 = 2$$

Similarly, if the negative side is inside, two mountains are added, giving the following equation.

$$\sum_{i=1}^{k+1} \mu(\ell_i) - 2 = -2$$

In both cases we get

$$\sum_{i=1}^{k+1} \mu(\ell_i) = 0.$$

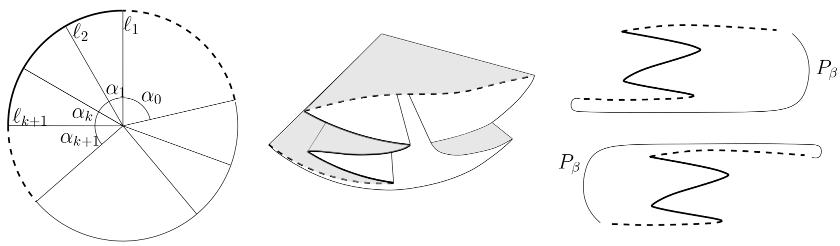


Figure 2.10: Case k odd

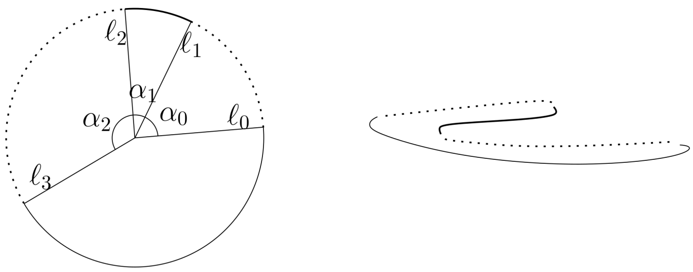


Figure 2.11: Induction base k=1

$\Leftarrow$ We prove the other direction by induction on k. For $k = 1$ we have $\sum_{j=1}^{2} \mu(\ell_j) = 0$. Hence, the two neighbouring creases to the little angle have opposite MV parity. As it can be seen in Figure 2.11 this is flat-foldable and thus the MV assignment is valid among the creases ℓ_1, ℓ_2. In the case $k = 2$, we have $\sum_{j=1}^{3} \mu(\ell_j) = \pm 1$. So we have either two mountains and one valley or one mountain and two valleys among ℓ_1, ℓ_2, ℓ_3. We show that all three possible assignments of two mountains and one valley among these creases are valid. This can be checked by folding the creases ℓ_1, ℓ_2, ℓ_3 accordingly and making sure they end up flat. See Figure 2.12 for an illustration. The other three cases follow by the symmetry property of Observation 1.6.

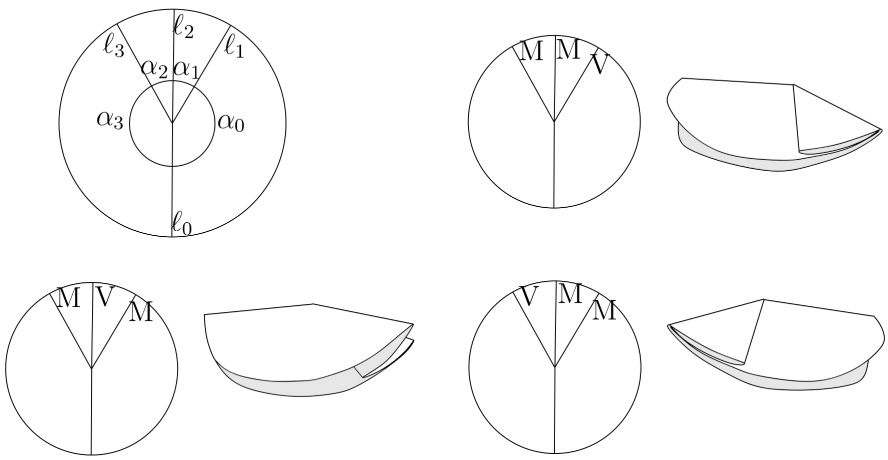


Figure 2.12: Induction base k=2

For an arbitrary k, there always exist two neighbouring creases of opposite MV parity, say ℓ_i, ℓ_{i+1} with $1 \leq i \leq k$. We fold these two creases and imagine that the sections of angles α_i and α_{i+1} have been fused to the other layers. See Figure 2.13 for an illustration. This results in a fold with angle sequence $(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+2}, \ldots, \alpha_{2n-1})$. Since the creases ℓ_i and ℓ_{i+1} have been of opposite MV parity, the sum takes the same value as before.

$$\sum_{j=1, j\neq i,i+1}^{k+1} \mu(\ell_j) = \sum_{j=1}^{k+1} \mu(\ell_j)$$

We apply the induction hypothesis to get that the MV assignment is valid among the creases $\ell_1, \ldots, \ell_{k+1}$. □

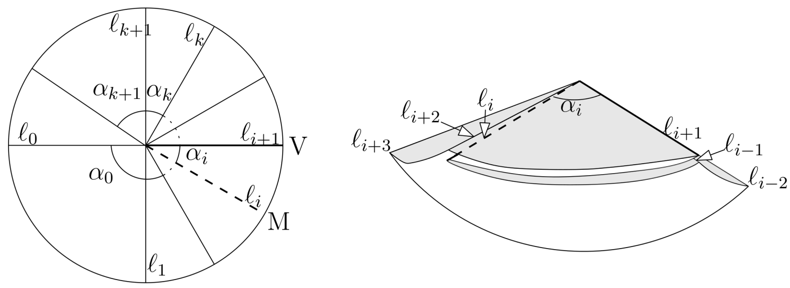


Figure 2.13: Fusion of layers

Chapter 3

Counting valid MV assignments

In the last chapter, we have learned tools to find out whether a given MV assignment is valid or not. In this chapter, we want to count the number of valid MV assignments. Recall from Definition 1.5 that an MV assignment is valid, if it can be realized by a flat origami. We begin this chapter with an example.

3.1 Example

We consider the crease pattern in Figure 3.1 and note that we have already seen it in Section 2.2. Now, we want to count the number of valid MV assignments for this crease pattern.

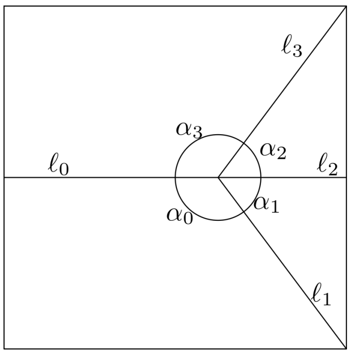


Figure 3.1: $\alpha_1 = \alpha_2 < \alpha_0, \alpha_3$

First, we apply Maekawa's Theorem 2.2 to this four-edge crease pattern. Therefore, we must have either three mountains and one valley or three valleys and one mountain crease. There are in total $2 \cdot \binom{4}{1} = 8$ possibilities for this. Hence, we have an upper bound on the number of valid MV assignments.

Observe that we have $\alpha_1 = \alpha_2 < \alpha_0, \alpha_3$. We use Theorem 2.10 for k even to get $\sum_{j=1}^{3} \mu(\ell_j) = \pm 1$. In particular, the two assignments where the edges ℓ_1, ℓ_2, ℓ_3 have the same MV parity (see Figure 2.3) are invalid.

For the remaining six MV assignments, we can check that they are valid by folding them and seeing that they end in a flat state. Figure 3.2 shows three results of the different MV assignments. The other three are the symmetrical cases where the mountains and valleys are reversed.

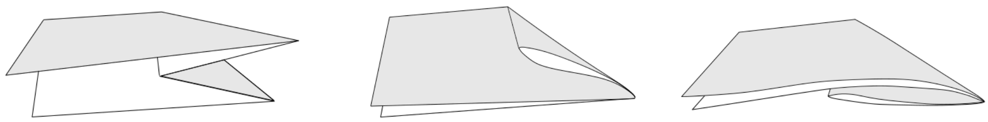


Figure 3.2: Valid MV assignments

3.2 Bounds

As already seen in the example before, we can use Maekawa's Theorem for an upper bound on the number of valid MV assignments for a given crease pattern. Before we generalise this in a theorem, we introduce the following notation. The settings are single-vertex folds of the form $G = (V, E)$ with $E = \{\ell_0, \ell_1, \dots, \ell_{2n-1}\}$ and α_i the angle between ℓ_i and ℓ_{i+1}. We denote by $C(\alpha_0, \dots, \alpha_{2n-1})$ the number of valid MV assignments for G.
The following lemma treats a special case where all angles are equal.

Lemma 3.1 *Any MV assignment, that satisfies Maekawa's Condition [1] for a single-vertex fold on a cone with cone angle $A \leq 2\pi$ with all angles equal, is valid.*

Proof We prove this lemma by induction on the number of edges.
For $2n = 2$, there are exactly two possibilities for an MV assignment that satisfies Maekawa's condition; either both edges are mountains or both are valleys. They are clearly flatfoldable, so the two MV assignments are valid.
Now, suppose the claim holds for $2n \geq 2$ and consider a crease pattern with $2n + 2$ edges and all angles being equal. Consider an MV assignment that satisfies Maekawa's Theorem. Then, there exists an angle α_k with $\mu(\ell_k) \neq \mu(\ell_{k+1})$. Folding these two edges according to μ gives a new single-vertex crease pattern on $2n$ edges with angles $(\alpha_0, \dots, \alpha_{k-1}, \alpha_{k+2}, \dots, \alpha_{2n-1})$ on a cone with cone angle $A - 2\alpha_k$. See Figure 3.3 for an illustration.
Since $\mu(\ell_k) \neq \mu(\ell_{k+1})$ is equivalent to $\mu(\ell_k) + \mu(\ell_{k+1}) = 0$, the creases $\ell_0, \dots, \ell_{k-1}, \ell_{k+2}, \dots, \ell_{2n-1}$ of the new crease pattern still satisfy Maekawa's condition. Finally, we use induction to show that the MV assignment is valid. $\square$

[1] $\sum_{j=0}^{2n-1} \mu(\ell_j) = \pm 2$

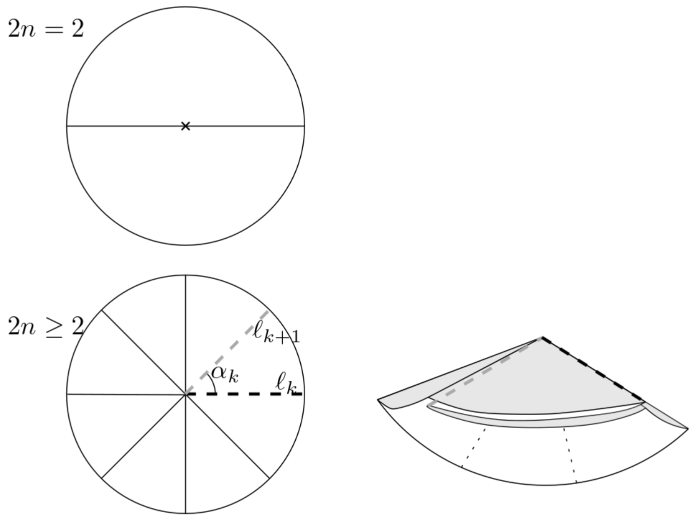


Figure 3.3: Induction on number of edges

In the following theorem, we give bounds on the number of valid MV assignments for G.

Theorem 3.2 *Let G be a flat vertex fold with angle sequence $(\alpha_0, \ldots, \alpha_{2n-1})$. Then $2^n \leq C(\alpha_0, \ldots, \alpha_{2n-1}) \leq 2 \cdot \binom{2n}{n-1}$ are sharp bounds.*

Proof We start by proving the **upper bound**. Recall from Maekawa's Theorem 2.2 that for flat vertex folds we have $\sum_{j=0}^{2n-1} \mu(\ell_j) = \pm 2$. In other words, $n-1$ out of the $2n$ edges have equal MV parity and the remaining $n+1$ edges have the opposite MV parity. This gives us the upper bound on the number of valid MV assignments $C(\alpha_0, \ldots, \alpha_{2n-1}) \leq 2 \cdot \binom{2n}{n-1}$.
From Lemma 3.1 we know that if all the angles are equal, any MV assignment satisfying Maekawa's condition is valid. Therefore, we get that this upper bound is sharp.

We prove the **lower bound** by induction on the number of edges. The proof follows a similar idea to the one illustrated in Figure 3.3. The induction basis is given by $2n = 2$, where the lower bound holds with equality. Now for $2n > 2$, we consider a $2n$-edge flat vertex fold on a cone with cone angle A. Among all angles, we choose a smallest α_i. If $\alpha_i < \alpha_{i-1}, \alpha_{i+1}$, then from Lemma 2.9 we immediately have that $\mu(\ell_i) \neq \mu(\ell_{i+1})$. If α_i has a neighbour of the same size, we apply Theorem 2.10 to get that not all the edges that are incident to these smallest angles can have the same MV parity. So we can assume without loss of generality that $\mu(\ell_i) \neq \mu(\ell_{i+1})$ (else we would rename the edges). Note that we have two possibilities for this, either $\mu(\ell_i) = 1$ or $\mu(\ell_i) = -1$ and $\mu(\ell_{i+1})$ is determined by $\mu(\ell_i)$.
We fold the edges ℓ_i, ℓ_{i+1} according to one of these MV assignments. We obtain a new cone with cone angle $A - 2\alpha_i$ and angle sequence

$(\alpha_0, \ldots, \alpha_{i-2}, \alpha_{i-1} - \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_{2n-1})$. Note that by the choice of α_i, we have $\alpha_{i-1} - \alpha_i + \alpha_{i+1} > 0$. We use the induction hypothesis on this new flat vertex fold with $2n-2$ edges, which gives us 2^{n-1} as a lower bound. Finally, the lower bound of 2^n is obtained by multiplying 2^{n-1} by the number of possible values of $\mu(\ell_i)$, which is two as discussed above. $\square$

Although Theorem 3.2 gives us sharp bounds on the number of valid MV assignments, it does not tell us how to compute the exact number, if not all angles are equal.

3.3 Recursion

The following theorem gives a recursion to count the number of valid MV assignments for crease patterns of the form shown in Figure 2.8.

Theorem 3.3 *Let G be a flat vertex fold with angle sequence $(\alpha_0, \ldots, \alpha_{2n-1})$, and suppose that we have $\alpha_1 = \alpha_2 = \cdots = \alpha_k < \alpha_0, \alpha_{k+1}$ for some k. Then*

$$C(\alpha_0, \ldots, \alpha_{2n-1}) = \begin{cases} \binom{k+1}{\frac{k+1}{2}} \cdot C(\alpha_0 - \alpha_1 + \alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_{2n-1}), & \text{if } k \text{ is odd.} \\ \binom{k+1}{\frac{k}{2}} \cdot C(\alpha_0, \alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_{2n-1}), & \text{if } k \text{ is even.} \end{cases}$$

Before proving this theorem, let us first consider examples of how we apply this recursion to a given flat vertex fold.

Example 3.4 *Let us repeat the example from Section 3.1, now using Theorem 3.3. We have the angle sequence $(127°, 53°, 53°, 127°)$ (see Figure 3.1). We use the recursion for $k = 2$ to get*

$$C(127°, 53°, 53°, 127°) = 3 \cdot C(127°, 127°).$$

From Lemma 3.1 we have that all MV assignments satisfying Maekawa's Condition are valid for the crease pattern with angle sequence $(127°, 127°)$. There are two such MV assignments, namely either both edges are mountains or both are valleys. Therefore, we have

$$C(127°, 127°) = 2.$$

As we already know from Section 3.1 the number of valid MV assignments for G is indeed

$$C(127°, 53°, 53°, 127°) = 3 \cdot C(127°, 127°) = 3 \cdot 2 = 6.$$

In the next example, we also look at what happens geometrically in each recursion step. This will give us the intuition we need to prove the theorem.

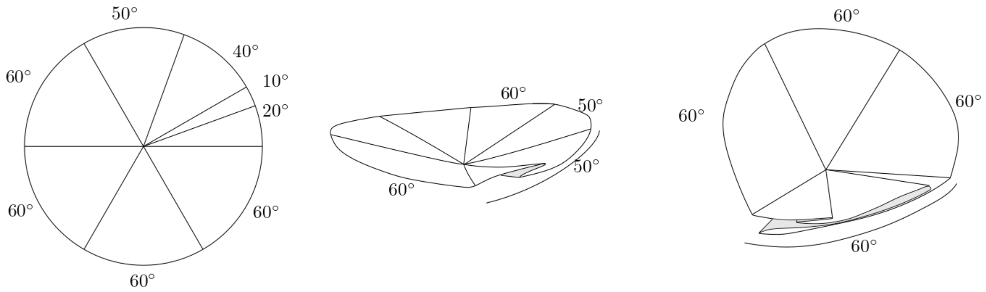


Figure 3.4: Example 3.5, Recursion

Example 3.5 *Consider the angle sequence*

$$(\alpha_0, \alpha_1, \ldots, \alpha_{2n-1}) = (20, 10, 40, 50, 60, 60, 60, 60),$$

that is also illustrated in Figure 3.4. (The angles are measured in degrees, for simplicity we omit the ° sign.) Since we have eight angles, $2n - 1 = 7$ yields $n = 4$. We use Theorem 3.3 to compute the number of valid MV assignments.
In the first step, we have $k = 1$ and

$$\alpha_0 = 20 > 10 = \alpha_1 = \alpha_k < 40 = \alpha_2.$$

Applying the theorem, we get that

$$\binom{k+1}{\frac{k+1}{2}} \cdot C(\alpha_0 - \alpha_1 + \alpha_{k+1}, \alpha_{k+2}, \ldots) = \binom{2}{1} \cdot C(50, 50, 60, 60, 60, 60).$$

*In the second part of Figure 3.4, we can see **geometrically** how, in the case of **k odd**, the angles $\alpha_0, \ldots, \alpha_{k+1}$ are replaced by a new angle of measure $\alpha_0 - \alpha_1 + \alpha_{k+1}$, where the part consisting of the equal angles is folded flat and fused to one of the two larger angles α_0 or α_{k+1}. If we had $k > 1$ and k odd, then the equal angles would be layers upon each other, like the cross-section in Figure 2.10.*
We continue to count the number of valid MV assignments for the new sequence. Since the angle sequence is cyclic, we can reorder the new sequence to

$$(\alpha_0, \alpha_1, \ldots, \alpha_5) = (60, 50, 50, 60, 60, 60).$$

We apply the theorem again, now for $k = 2$ and

$$\alpha_0 = 60 > 50 = \alpha_1 = \alpha_2 < 60 = \alpha_3,$$

to get that

$$C(60, 50, 50, 60, 60, 60) = \binom{k+1}{\frac{k}{2}} \cdot C(\alpha_0, \alpha_{k+1}, \ldots) = \binom{3}{1} \cdot C(60, 60, 60, 60).$$

*In the last part of Figure 3.4 we see what happens **geometrically when k is even**. The part consisting of the equal angles is folded flat and fused to one of the two larger*

angles α_0 or α_{k+1}. This larger angle completely covers all smaller angles. Hence, the smaller angles are ignored in this recursion step.
For the last computation, we use Lemma 3.1 because all the angles are equal. Together with the upper bound from Theorem 3.2 we get $C(60, 60, 60, 60) = 2 \cdot \binom{4}{1}$.
In the following, we add up all the calculation steps.

$$\begin{aligned} C(20, 10, 40, 50, 60, 60, 60, 60) &= \binom{2}{1} \cdot C(40, 50, 60, 60, 60, 60) \\ &= \binom{2}{1} \cdot \binom{3}{1} \cdot C(60, 60, 60, 60) \\ &= \binom{2}{1} \cdot \binom{3}{1} \cdot 2 \cdot \binom{4}{1} \\ &= 48 \end{aligned}$$

With a first intuition from the above examples, we are ready to prove Theorem 3.3.

Proof We prove both cases separately.
If ***k* is odd**, then Theorem 2.10 gives us $\sum_{j=1}^{k+1} \mu(\ell_j) = 0$. Since all the angles $\alpha_1, \ldots, \alpha_k$ are equal, any $\frac{k+1}{2}$ of among the $k+1$ creases $\ell_1, \ldots, \ell_{k+1}$ can be valleys, and the others are mountains. We fix one of these possibilities and then fold the creases $\ell_1, \ldots, \ell_{k+1}$ accordingly. When we fuse the layers of paper around these angles, the angles $\alpha_0, \ldots, \alpha_{k+1}$ are replaced by an angle with the measure $\alpha_0 - \alpha_1 + \alpha_{k+1}$. This gives us the stated recursion. See the first and second image of Figure 3.4 for an illustration.
If ***k* is even**, we have by Theorem 2.10 that $\sum_{j=1}^{k+1} \mu(\ell_j) = \pm 1$.
If $\sum_{j=1}^{k+1} \mu(\ell_j) = 1$, we have $\frac{k}{2} + 1$ valleys and $\frac{k}{2}$ mountains.
Similarly, if $\sum_{j=1}^{k+1} \mu(\ell_j) = -1$, we have $\frac{k}{2}$ valleys and $\frac{k}{2} + 1$ mountains.
In both cases, there are $\binom{k+1}{\frac{k}{2}}$ possible ways to choose which of the creases $\ell_1, \ldots, \ell_{k+1}$ are valleys and which are mountains. This gives the factor in the recursion.
Now we consider such an MV assignment for the creases $\ell_1, \ldots, \ell_{k+1}$ and we fold them accordingly. Then the angles α_0 or α_{k+1} will absorb the folded layers. Hence, we can omit the angles $\alpha_1, \ldots, \alpha_k$ in the sequence for the next recursion step. The third image of Figure 3.4 gives an illustration. $\square$

Chapter 4

Conclusion

A main goal of this thesis was to count the number of valid MV assignments for flat vertex folds. We ended up with a recursion formula to compute the number of valid MV assignments, specifically designed for flat vertex folds of certain types, where larger angles enclose one or more smaller angles (Theorem 3.3). Additionally, using Lemma 3.1 and Theorem 3.2, we are able to calculate the number of valid MV assignments for flat vertex folds having uniformly equal angles.

Throughout this thesis, we encountered various theorems. Kawasaki's Theorem 2.7, for instance, serves as a certificate for flatfoldability, while Hull's Theorem 2.10 aids in identifying situations where MV assignments, despite satisfying Maekawa's Condition, are invalid.

A central theorem in this thesis is Maekawa's Theorem 2.2. As can be seen in the proof of Theorem 3.2, Maekawa's Theorem establishes an upper bound on the number of valid MV assignments for flat vertex folds. Further, in this thesis, we used Maekawa's Theorem to prove several other statements. However, Observation 2.3 highlights its limitation in the context of multiple-vertex folds.

Looking ahead, a possible extension of this work involves addressing the enumeration of multiple-vertex flat folds, a subject treated in the Chapters 6 and 7 of [4].

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Eidgenössische Technische Hochschule Zürich
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[1] E.g. ChatGPT, DALL E 2, Google Bard
[2] E.g. ChatGPT, DALL E 2, Google Bard
[3] E.g. ChatGPT, DALL E 2, Google Bard