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A Detailed Study of the Classification of Platonic Solids

Bachelor Thesis

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Abstract

In this thesis, platonic solids and their classification are studied. We give a rigorous proof that there are exactly five platonic solids, which are uniquely determined by their Schläfli symbol. The three milestones in this proof are Euclid's foundational contributions in "Elements", Euler's formula linking the number of faces, edges and vertices, and Cauchy's rigidity theorem. A particular approach in this thesis is to associate a unique planar graph to platonic solids with the same Schläfli symbol. The thesis concludes with a complete list of the five platonic solids, their fundamental properties and provides explicit constructions.

Contents

Acknowledgement

I would like to thank my thesis supervisor Prof. Dr. Ana Cannas da Silva for her great guidance during the course of this work. She suggested a topic which was perfectly aligned with my interests. She has always been incredibly supportive answering my questions and providing ideas and inputs to proceed with my own approaches to tackle the topic. It has been very delightful working with her.

Chapter 1

Introduction

The study of platonic solids has its roots in the time of Euclid more than two thousand three hundred years ago, and still fascinates after all this time. Starting with Euclid's work, there are three milestones leading to the complete classification of platonic solids.

In his work *Elements* Euclid assembled the work of his predecessors. The main topics are geometry, proportion and number theory. Euclid arranged these known results in a logical manner, so as to demonstrate (admittedly, not always with the rigour demanded by modern mathematics) that they necessarily follow from five simple axioms. He is credited with devising a number of particularly ingenious proofs of previously discovered theorems. The *Elements* consists of thirteen books and only in the last one, Book 13, he investigates the platonic solids. With his work Euclid created a book that soon became the standard for geometry and which even today is in the syllabus of a typical geometry course in school.

The second milestone was in the eighteenth century when Euler presented his formula that connected the numbers of polygonal faces, edges and vertices of convex polyhedra. The third milestone that contributed to a complete classification of the five platonic solids is Cauchy's rigidity theorem, Cauchy's first mathematical accomplishment.

In this thesis, we aim to give the complete classification of platonic solids using Euler's formula and Cauchy's rigidity theorem. We start by giving the definition for platonic solids together with some basic geometrical definitions and useful properties.

In the second chapter, we introduce Euler's formula and give its proof. We outline the different important steps to complete the classification of platonic solids. We also give the definitions of the Schläfli symbol, named after the Swiss mathematician Ludwig Schläfli, and the corresponding Eulerian triple.

We demonstrate their connection to platonic solids by showing that for those exactly five Schläfli symbols are possible.

In the third chapter, we show that one can associate an essentially unique planar graph to a platonic solid, which is solely characterized by its Schläfli symbol. This is a personal approach which connects platonic solids with the same Schläfli symbol to the hypothesis in Cauchy's rigidity theorem.

In the fourth chapter, we prove Cauchy's rigidity theorem using a result by Steinitz. Steinitz found a gap in the original proof and could repair it one hundred years later. Cauchy's rigidity theorem will then be used to complete the classification of platonic solids.

In the last chapter, we list all five platonic solids and state some of their basic properties, e.g., all vertices of a platonic solid lie on a sphere. Furthermore, a construction of each of the five figures will be given.

This thesis is aimed at undergraduate students with interest in exploring geometry in a rigorous manner. It closely follows the book *Geometry: Euclid and Beyond* by Hartshorne [4]. Additional material stems from the classical references *Proofs from THE BOOK* by Aigner and Ziegler [1] and *Regular Polytopes* by Coxeter [2]. Coxeter's book and particularly Chapter 8 in [4] are recommended for further reading. The historically inclined reader will appreciate Euclid's work *Elements* and its translation in *Euclid's Elements of Geometry* by Fitzpatrick [3].

Chapter 2

Definition of Platonic Solids

The goal of this chapter is to give a formal definition of a platonic solid and to provide several illustrative examples. First, we need to introduce some basic and general definitions of figures in two- and three-dimensional space.

2.1 Basic Definitions and Properties

Definition 2.1 *A* ***regular polygon*** *in the plane is an equilateral and equiangular polygon, i.e., a polygon where every side has the same length and all angles between two sides are equal.*

Definition 2.2 *Two sets of points are* ***congruent*** *if there exists a combination of translations, rotations and reflections that maps one set onto the other.*

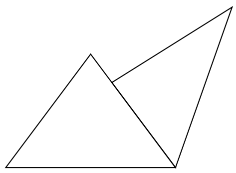
The sets of points are said to be congruent **up to a scaling factor** if they are congruent after first scaling them, thus, not taking their specific size into account. The first proposition is a general statement about regular polygons and their properties.

Proposition 2.3 *In the plane, for any $n \geq 3$, there exists a regular polygon of n sides having a given segment as a side. Any two regular n-polygons are congruent up to a scale factor. The vertices of a regular n-polygon lie on a circle.*

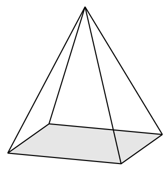
Proof Existence: Consider a circle and place n evenly spaced points on its circumference. These points will create angles of $\frac{2\pi}{n}$ at the circle's center. By adjusting the scale factor, you can match the side length to any specified segment.

Congruence, lying on a circle: Take a regular n-polygon with a side labeled $\overline{AB}$. Split the equal angles between the two edges meeting at A and between the two edges meeting at B in half and let these angle bisectors intersect at point O. This ensures that point O is equidistant from both A and B. By repeating this method for all other vertices, it becomes clear that O is at an equal distance from all of them. As a result, all these vertices lie on a circle centered at O. Therefore, any two regular n-polygons sharing a side will be congruent. Thus any two regular n-polygons are congruent up to a scale factor. $\square$

Definition 2.4 *A* ***polyhedron*** *is the surface of a solid figure in three-dimensional space bounded by plane polygons. When two polygons meet in more than one point, they must have an entire edge in common. These plane polygons are called the* ***faces*** *of the polyhedron, their edges are called the* ***edges*** *of the polyhedron and their vertices are called the* ***vertices*** *of the polyhedron.*



alignment of two faces which is not allowed



a general polyhedron

Thus for any polyhedron P we define the three sets

$$\begin{aligned}
\mathcal{F}(P) &:= \{F_1, \dots, F_f\}, \text{ the set of faces of } P,\\
\mathcal{E}(P) &:= \{E_1, \dots, E_e\}, \text{ the set of edges of } P,\\
\mathcal{V}(P) &:= \{V_1, \dots, V_v\}, \text{ the set of vertices of } P.
\end{aligned}$$

The cardinality of these sets yields the triple (f, e, v) defined as:

$$\begin{aligned}
f &= |\mathcal{F}(P)| = \text{ number of faces of } P,\\
e &= |\mathcal{E}(P)| = \text{ number of edges of } P,\\
v &= |\mathcal{V}(P)| = \text{ number of vertices of } P.
\end{aligned}$$

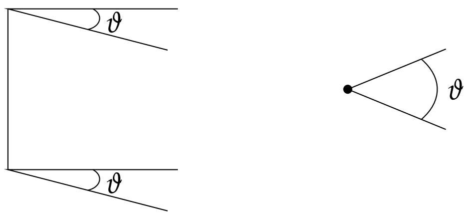
Definition 2.5 *A polyhedron P is* ***convex*** *if for any two points on P, the line segment between them is entirely contained in the solid figure bounded by the polyhedron.*

Now, we are in the position to give a definition of a platonic solid.

Definition 2.6 *A* ***platonic solid*** *is a convex polyhedron whose faces are all equal regular polygons and having the same number of faces meeting at each vertex.*

If the polyhedron satisfies all properties of a platonic solid except for convexity, then it is called a **regular polyhedron**.
The next definition introduces a further notion for polyhedra which is particularly important when stating properties of platonic solids.

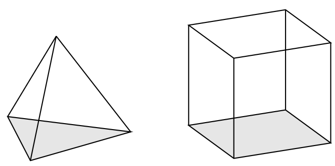
Definition 2.7 *A* ***dihedral angle*** ϑ *of two faces of a polyhedron is the angle between the two faces that meet in one edge.*



the dihedral angle ϑ of two faces
in elevation and plane view

2.2 Two Examples of Platonic Solids

The first example of a platonic solid is the cube, a figure that the reader will have most likely already seen. The cube is composed of six equal squares and every vertex connects three squares. The second example is the tetrahedron, a triangular pyramid formed of four regular triangles and where every vertex connects three triangles. By symmetry of the figure, the dihe-



Tetrahedron and Cube

dral angle of the tetrahedron is equal for any two faces. For the cube the dihedral angle is the same as well and because any two faces sharing an edge are perpendicular the dihedral angle is ninety degrees.

Chapter 3

Euler's Formula

In this chapter, we introduce Euler's formula and give a complete proof. We outline the important steps in the proof of the classification of platonic solids.

3.1 Euler's Formula

Euler's formula is an important, yet simple equation, which relates the number of faces, edges and vertices of a convex polyhedron.

Theorem 3.1 (Euler's formula.) *Let P be a convex polyhedron with the number of faces, edges and vertices denoted as f, e and v respectively. Then the following equation holds*

$$f - e + v = 2. \tag{3.1}$$

Proof Let P be a convex polyhedron. We will prove this theorem in three steps, following closely Hartshorne's proof in [4].

Step 1. The first step is to project P onto the plane. Because the polyhedron is convex it is possible to look through the center of one face and see all the other faces with no overlap. Taking a step back from the polyhedron one can also see the edges of the face one is looking through. This image can be projected onto the plane to obtain a plane figure with vertices and edges. The projection does not preserve the angles and distances, but the edges remain straight and no edges intersect. The faces of P correspond to the plane polygons restricted by the edges in the plane except for the one face one is looking through. This specific face corresponds to the area on the plane outside the plane figure.

Step 2. As a next step, we define two operations on the plane figure.

(i) removing edges: *One takes any edge that separates two faces, or that separates one face from the area outside the figure and removes that edge.* This decreases the number of edges by one but it also decreases the number of faces by one since the operation joins the two faces, which were separated by the removed edge. Thus the value of the expression $f - e + v$ stays the same.

(ii) removing vertices: *If at some point, (i) results in a vertex having only one edge connected to it, one removes the vertex as well as the edge.* This decreases the number of edges by one and this time it decreases the number of vertices by one too. The removed vertex was contained in exactly one face of the plane figure, since the vertex was only connected to one edge. Thus, the number of faces does not change. So again the value of the expression $f - e + v$ stays unchanged.

Step 3. As a last step, we apply these operations on the plane figure. We repeat operation (i) until it is no longer possible. This means that there are no loops in the remaining figure and thus there must be at least one vertex with only one edge connected to it. Use operation (ii) until there are only vertices and no edges left in the remaining figure. Note that the original figure is connected and it stays connected by performing step (i) or (ii), so it is just one vertex left in the plane. Hence

$$\begin{aligned} f = 1, \;\; e = 0, \;\; v = 1 \\ \implies f - e + v = 2 \end{aligned}$$

Since the value of the expression $f - e + v$ is unchanged by applying the operations multiple times, the original expression $f - e + v$ is equal to 2 and thereby the proof is complete. $\square$

Clearly, platonic solids have this characteristic triple (f, e, v), which satisfies Euler's formula. However, there is another pair of numbers that characterizes platonic solids. The pair of numbers follows directly from the definition of platonic solids and is defined as follows:

Definition 3.2 *Let P be a platonic solid. Then let $n \in \mathbb{N}$ be the number of vertices of one face of P and $c \in \mathbb{N}$ be the number of faces that meet at a vertex. The pair (n, c) is a special case of the* ***Schläfli symbol*** *(see [2]).*

This pair of numbers is well-defined since by definition all the faces of a platonic solid are equal regular n-polygons and at every vertex the same number c of faces are meeting. Furthermore, the number of edges meeting at each vertex is also c because any two neighbouring faces meeting in more than one point have one edge in common.

3.2 Strategy for the Proof of the Classification of Platonic Solids

In this section, we outline the proof that exactly five platonic solids exist by dividing it into four steps. We will prove the first two steps, whereas the other two steps will be proven independently in the next chapters. We will postpone the ultimate proof to Chapter 5. The outline of the important steps is as follows

1. We show that the definition of a platonic solid restricts the possible pairs of $(n, c) \in \mathbb{N} \times \mathbb{N}$.
2. For any platonic solid we connect the two sets of numbers (n, c) and (f, e, v) using the definition of two polyhedra being similarly arranged.
3. We introduce Cauchy's Rigidity Theorem.
4. Finally, we apply Cauchy's Rigidity Theorem to the special case of platonic solids.

Lemma 3.3 *For a platonic solid there are only five possible pairs (n, c) given by*

$$\mathcal{A} := \{(3,3), (3,4), (3,5), (4,3), (5,3)\}$$

Proof Let P be a convex regular polyhedron with associated pair $(n, c) \in \mathbb{N} \times \mathbb{N}$ and triple $(f, e, v) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Any regular polygon has at least three vertices and each vertex of a polyhedron must connect at least three edges to build a three dimensional figure. Hence the following inequalities hold:

$$n \geq 3 \text{ and } c \geq 3. \tag{3.2}$$

The next step is to take a closer look at counting the edges of the polyhedron P: Each edge appears in exactly two faces of P and every face is bounded by n edges. Also, each edge is connected to exactly two vertices and every vertex connects c edges. Thus the following two equations hold:

$$fn = 2e = vc. \tag{3.3}$$

Now, consider Euler's formula stating that $f - e + v = 2$ and substitute f and v to get an equation depending only on n, c and e:

$$2 = f - e + v = \frac{2e}{n} - e + \frac{2e}{c} = \left(\frac{2}{n} - 1 + \frac{2}{c}\right) e. \tag{3.4}$$

As the number of edges e is always positive, it follows that the term in the brackets on the right hand side must also be positive. This leads to the following inequalities:

$$\begin{aligned}\frac{2}{c} - 1 + \frac{2}{n} > 0 &\iff \frac{1}{c} + \frac{1}{n} > \frac{1}{2} \\ &\iff \frac{1}{c} > \frac{1}{2} - \frac{1}{n} \qquad (3.5)\end{aligned}$$

Using (3.5) and $n \geq 3$, it follows that $c < 6$. Similarly, using $c \geq 3$ it holds that $n < 6$. Together with the conditions (3.2), this results in

$$n \in \{3,4,5\}, \quad c \in \{3,4,5\}.$$

By inserting all possible values for n into the equation (3.5), we obtain the following five pairs (n, c).

$$\left.\begin{aligned} n = 3 : \tfrac{1}{c} > \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6} &\implies c \in \{3,4,5\} \\ n = 4 : \tfrac{1}{c} > \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4} &\implies c = 3 \\ n = 5 : \tfrac{1}{c} > \tfrac{1}{2} - \tfrac{1}{5} = \tfrac{3}{10} &\implies c = 3 \end{aligned}\right\} \{(3,3),(3,4),(3,5),(4,3),(5,3)\}$$

These pairs determine the five possible pairs (n, c) of a platonic solid, thus the elements of $\mathcal{A}$. This concludes the proof of the first lemma. $\square$

Lemma 3.4 *For any platonic solid there exists a map from the pair* (n, c) *onto the corresponding triplet* (f, e, v) *given by*

$$\begin{aligned} e &= \frac{2}{(\frac{2}{c} - 1 + \frac{2}{n})} \\ f &= \frac{2e}{n} \\ v &= \frac{2e}{c} \end{aligned}$$

Proof Let P be a convex regular polyhedron. We use Equations (3.4) and (3.3) in the proof of Lemma 3.3 to find an explicit formula for e depending only on n and c. This formula then yields an explicit formula for f and v.

$$\stackrel{(3.4)}{\implies} e = \frac{2}{(\frac{2}{c} - 1 + \frac{2}{n})} \qquad (3.6)$$

$$\stackrel{(3.3)}{\implies} f = \frac{2e}{n}, \quad v = \frac{2e}{c} \qquad (3.7)$$

These formulas are well-defined since $n, c \neq 0$ and $\frac{2}{c} - 1 + \frac{2}{n} \neq 0$ for all $(n, c) \in \mathcal{A}$, where $\mathcal{A}$ is denoting the set of all possible pairs (n, c) for a platonic solid, as in Lemma 3.3. To verify that $f, e, v \in \mathbb{N}$, we insert the value for every $(n, c) \in \mathcal{A}$ into the Formulas (3.6) and (3.7). For example for $(n, c) = (3, 3)$, we obtain

$$\begin{aligned} e &= \frac{2}{(\frac{2}{c} - 1 + \frac{2}{n})} = \frac{2}{(\frac{2}{3} - 1 + \frac{2}{3})} = 6 \\ f &= \frac{2e}{n} = \frac{12}{3} = 4 \\ v &= \frac{2e}{c} = \frac{12}{3} = 4 \end{aligned}$$

which shows that $f, e, v \in \mathbb{N}$ for $(n, c) = (3, 3)$. The verification for the other four pairs (n, c) is left to the reader. This concludes the proof of Lemma 3.4. □

The next definition is needed to address Step 2 above, namely to describe the relation between two convex polyhedra with the same number of faces.

Definition 3.5 *Let $P, \overline{P}$ be two polyhedra. If a bijection*

$$\varphi : \mathcal{F}(P) \to \mathcal{F}(\overline{P})$$

*exists, such that each face $F_k \in \mathcal{F}(P)$ is congruent to the face $\varphi(F_k) \in \mathcal{F}(\overline{P})$ and such that φ extends to a bijection of vertices and edges preserving all incidence relations, then P and $\overline{P}$ are called **similarly arranged**.*

If two platonic solids satisfy the hypothesis from above except for having congruent faces, then the term **combinatorially equivalent** is used (e.g., page 75 in [1]). We work with the term similarly arranged since it is used in the main reference of this thesis [4]. The notion of combinatorially equivalent is much weaker than the notion of congruence of two polyhedra. For instance two pyramids with triangular base and different heights are combinatorially equivalent but not congruent, as illustrated in Figure 3.1 below.

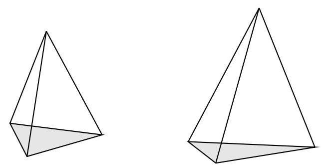


Figure 3.1: Example of two combinatorially equivalent polyhedra

A crucial result, mentioned in Step 3 above, is Cauchy's rigidity theorem. It will be proven independently in Chapter 5 below.

Theorem 3.6 (Cauchy's rigidity Theorem (CRT).) *Let $P, \overline{P}$ be two convex polyhedra made of congruent faces. Suppose that P and $\overline{P}$ are similarly arranged. Then P and $\overline{P}$ are congruent.*

To apply the CRT, as stated in Step 4 above, one needs a last lemma, which connects the assumption in CRT to the pair (n, c) of a platonic solid.

Lemma 3.7 *If two platonic solids have the same pair (n, c), then they are similarly arranged.*

We postpone the proof of the lemma into the next chapter, where we will give an explicit labeling of the faces of a platonic solid which will then be used to prove this lemma.

Chapter 4

Labeling Faces of a Platonic Solid

In this chapter, we construct an essentially unique labeling of the faces of a platonic solid. This unique labeling is used to prove that two platonic solids with the same Schläfli symbol are similarly arranged, as stated in Lemma 3.7.

4.1 Establishing Some Notions

Let P be a platonic solid with its Schläfli symbol $(n, c) \in \mathcal{A}$. Recall that the number n is determined by the number of vertices of each regular polygon, the faces of the platonic solid, and the number c is the number of faces that contain a vertex $V \in \mathcal{V}(P)$. By Lemma 3.4, this determines the triple (f, e, v), which denotes the cardinalities of the sets $\mathcal{F}(P), \mathcal{E}(P)$ and $\mathcal{V}(P)$ respectively. The elements of $\mathcal{F}(P)$ will be considered as sets of points in three-dimensional space, $F_i \subseteq \mathbb{R}^3$. The edges $E_k \in \mathcal{E}(P)$ and vertices $V_k \in \mathcal{V}(P)$ can be defined as subsets in $\mathbb{R}^3$ by

$$E_k = F_{i(k)} \cap F_{j(k)} \tag{4.1}$$

$$V_k = F_{j_1(k)} \cap \cdots \cap F_{j_c(k)} \tag{4.2}$$

Note that every edge is the intersection of two uniquely determined distinct faces $F_{i(k)}$ and $F_{j(k)}$ in $\mathcal{F}(P)$. Furthermore, every vertex is the intersection of c uniquely determined distinct faces as per definition of a platonic solid.

For each platonic solid we will show that one can construct a labeling of the faces such that the edges and vertices are determined by the labeling via (4.1) and (4.2). More specifically, for each platonic solid, characterized by its Schläfli symbol (n, c) and the corresponding triple (f, e, v), referred to as Eulerian triple, we will associate a unique plane graph with f faces, e edges and v vertices and which preserves all incidence relations of the platonic solid.

To construct such a labeling of a platonic solid P, we first choose an arbitrary face F_1 and an arbitrary vertex V_1 of F_1 as starting data. The selection of any other starting data, say F_1' and V_1' on F_1', is equivalent up to isometry, because all faces of P are congruent and thus, there is a congruence that maps F_1 onto F_1' and in addition also V_1 into V_1'. By convexity of the platonic solid P, one can also require that this congruence maps the normal vector of face F_1 pointing outwards of P onto the normal vector of face F_1' pointing outwards of P.

In the following sections, we will give such an essentially unique labeling for all faces in $\mathcal{F}(P)$. We will use this labeling to prove Lemma 3.7:

Given two platonic solids P and $\overline{P}$ with the same Schläfli symbol (n, c), we assume that all faces in $\mathcal{F}(P)$ and $\mathcal{F}(\overline{P})$ have been labeled by such a labeling. Then we can define the following map:

$$\begin{aligned} \varphi : \mathcal{F}(P) &\to \mathcal{F}(\overline{P}) \\ F_i &\mapsto \overline{F}_i \end{aligned}$$

Since the incidence relations among the faces are determined by the labeling, the domain of φ can be extended from the set of faces onto the sets of edges and vertices of the platonic solids. Equations (4.1) and (4.2) are used to give the following definition for the extension of φ:

$$\begin{aligned} \varphi(E_k) &= \varphi(F_{i(k)} \cap F_{j(k)}) := \varphi(F_{i(k)}) \cap \varphi(F_{j(k)}) \\ \varphi(V_k) &= \varphi(F_{j_1(k)} \cap \cdots \cap F_{j_c(k)}) := \varphi(F_{j_1(k)}) \cap \cdots \cap \varphi(F_{j_c(k)}) \end{aligned}$$

It follows from the extended definition of φ and the essentially unique labeling of $\mathcal{F}(P)$ that the incidence relations are preserved and thus the proof of Lemma 3.7 is complete. Lemma 3.7 can be stated again in more detail as follows

Proposition 4.1 *Let P and $\overline{P}$ be two platonic solids with the same Schläfli symbol $(n, c) \in \mathcal{A}$. Assume all faces in $\mathcal{F}(P)$ and $\mathcal{F}(\overline{P})$ have been labeled with the same labeling which leads to the same plane graph. Then, the map $\varphi : \mathcal{F}(P) \to \mathcal{F}(\overline{P}), F_i \mapsto \overline{F}_i$ can be extended to the set of edges and vertices of P and $\overline{P}$ respectively, i.e., P and $\overline{P}$ are similarly arranged.*

4.2 The First Partial Labeling

Given a platonic solid P with Schläfli symbol (n, c) and starting face F_1 and starting vertex V_1 in F_1, we will determine an essentially unique labeling of the first $n+1$ faces of P. To guarantee uniqueness, we choose the positive orientation of the face F_1 determined by the normal vector of F_1 pointing outwards of P. This allows us to make the following partial labeling of faces and their incidence relations unique.

- Starting from V_1, label all vertices of F_1, which form a regular n polygon in consecutive order counterclockwise with respect to the normal vector of F_1 such that $(V_i, V_{i+1})_{i=1,\dots,n-1}$ and (V_n, V_1) are edges of F_1.
- For $k = 2, \dots, n$ denote by F_k the unmarked face which shares the edge (V_{k-1}, V_k) with face F_1.
- Finally, let F_{n+1} denote the face, which has the common edge (V_n, V_1) with face F_1.

This is a labeling of $n+1$ faces of P. Note that consecutive faces F_{k-1} and F_k do not necessarily share an edge.

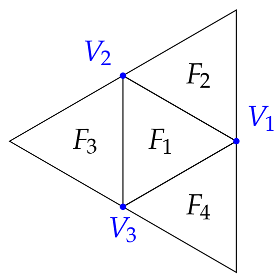


Figure 4.1: Illustration of first partial labeling for a platonic solid with Schläfli symbol $(3, c)$

Lemma 4.2 *Let P be a platonic solid with Schläfli symbol (n, c). Suppose that the first $n+1$ faces in $\mathcal{F}(P)$ are labeled according to the partial labeling above. Then the following three statements hold:*

1. *$F_1 \cap F_i$ is an edge for $i = 2, \dots, n+1$.*
2. *For every vertex V_i, three faces containing V_i have been labeled for $i = 1, \dots, n$.*
3. *If $c = 3$, then $F_{n+1} \cap F_2$ and $F_i \cap F_{i+1}$ are edges in $\mathcal{E}(P)$ for $i = 2, \dots, n$.*
4. *If $c \geq 4$, then $F_{n+1} \cap F_2$ and $F_i \cap F_{i+1}$ contain no edges of $\mathcal{E}(P)$ for $i = 2, \dots, n$.*

Proof

1. It follows from the first partial labeling of the first $n+1$ faces that $F_1 \cap F_i$ is an edge for $i = 2, \dots, n+1$.
2. Since $c \geq 3$ for platonic solids, there are at least three faces which contain V_i, $i = 2, \dots, n$. By construction, F_i has edge (V_{i-1}, V_i) and F_{i+1} has edge (V_i, V_{i+1}). Thus F_1, F_i and F_{i+1} contain V_i. The faces F_1, F_2 and F_n contain V_n.
3. As seen in proof of (2.), F_1, F_i and F_{i+1} contain V_i. If we assume that for some i, $F_i \cap F_{i+1}$ is not an edge, then (1.) implies that there exists at least one more face containing V_i. This contradicts $c = 3$, and thus $F_i \cap F_{i+1}$ is an edge in $\mathcal{E}(P)$. A similar argument shows that $F_{n+1} \cap F_2$ is an edge in $\mathcal{E}(P)$.
4. Again, we use that F_1, F_i and F_{i+1} contain V_i for $i = 2, \dots, n$. If we assume that for some i, $F_i \cap F_{i+1}$ is an edge, then (1.) implies that these three faces are the only ones containing V_i. This contradicts $c > 3$. A similar argument shows that $F_{n+1} \cap F_2$ is not an edge in $\mathcal{E}(P)$. $\square$

We will apply this first partial labeling of platonic solids to each of the five possible Schläfli symbols. In some cases the labeling is complete whereas in other cases more work is still needed to extend the labeling to all faces.

4.3 The First Two Cases

Schläfli Symbol (3,3)

The labeling of the platonic solid with Schläfli symbol $(3, 3)$ is complete with the first partial labeling in Section 4.2. To check this, consider the Eulerian triple $(f, e, v) = (4, 6, 4)$ which belongs to the Schläfli symbol $(n, c) = (3, 3)$. By construction, all $n + 1 = 4 = f$ faces of the polyhedron have been labeled. Since the first partial labeling is unique, the labeling of this platonic solid is unique and this case is complete.

To determine the corresponding planar graph of this labeling, we consider the set of vertices. By Lemma 4.2 the three vertices V_1, V_2, V_3 are contained in three faces, namely,

$$V_1 = F_1 \cap F_2 \cap F_4,$$
$$V_2 = F_1 \cap F_2 \cap F_3,$$
$$V_3 = F_1 \cap F_3 \cap F_4.$$

There is a fourth vertex $V_4 = F_2 \cap F_3 \cap F_4$. The resulting plane graph has $f = 4$ faces, $e = 6$ edges and $v = 4$ vertices, as illustrated in Figure 4.2.

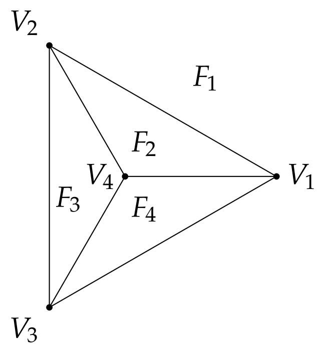


Figure 4.2: Planar graph resulting from the unique labeling for the case of a platonic solid with Schläfli symbol $(3, 3)$

Schläfli Symbol (4,3)

A platonic solid P with Schläfli symbol $(4, 3)$ and corresponding Eulerian triple $(6, 12, 8)$ has six faces. The first partial labeling of Section 4.2 enumerates $n + 1 = 5 = f - 1$ faces. The labeling is uniquely extended since there is no choice other than to label the only unmarked face of P by F_6. The label of the sixth face is uniquely determined and the first five faces are labeled by the first partial labeling, thus the labeling of a platonic solid P with Schläfli symbol $(4, 3)$ is unique and this case is complete.

To determine the corresponding graph of the labeling of a platonic solid with Schläfli symbol $(4, 3)$, we consider the set of edges. By Lemma 4.2, there are four edges incident with F_1, four edges are of the form $(F_k \cap F_{k+1})_{k=2,3,4}$ and $(F_5 \cap F_2)$. Furthermore, there are four edges incident with F_6. This and the labeling of the first n vertices results in the corresponding unique graph which has $f = 6$ faces, $e = 12$ edges and $v = 8$ vertices as can be read off from Figure 4.3.

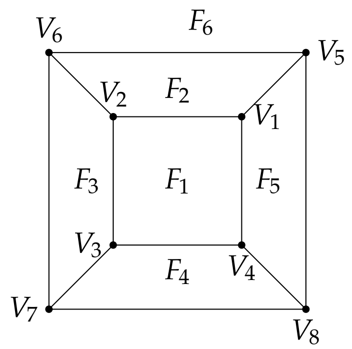


Figure 4.3: Illustration of the unique plane graph of a platonic solid with Schläfli symbol $(4, 3)$

4.4 The Second Partial Labeling

The next partial labeling of faces of a platonic solid P is for the cases where the number of faces c is larger than three. Let $\mathcal{W} \subseteq \mathcal{V}(P)$ denote the set of vertices, for which every vertex in $\mathcal{W}$ is contained in exactly three faces, which have already been labeled. In this next partial labeling we want to label the remaining unlabeled faces containing vertices $V_i \in \mathcal{W}$.
It follows from Lemma 4.2 that for each vertex $V_i \in \{V_1, \ldots, V_n\}$, there exist exactly three labeled faces containing V_i and thus $\mathcal{W} = \{V_1, \ldots, V_n\}$. This implies that the number of unlabeled faces containing V_i is equal to $(c-3)$. Thus the total number of unlabeled faces containing the vertices of $\mathcal{W}$ is at most $n(c-3)$.

We construct the second partial labeling as follows: For $i = 1, \ldots, n$, we label the remaining unlabeled $(c-3)$ faces containing V_i in consecutive order counterclockwise with respect to the normal vector of F_1 pointing outwards of P.
Case $\mathbf{c} = \mathbf{4}$. The n newly labeled faces $F_{n+2}, \ldots, F_{2n+1}$ must satisfy the incidence relations

$$
\begin{aligned}
V_1 &= F_1 \cap F_2 \cap F_{n+1} \cap F_{n+2}, \\
V_i &= F_1 \cap F_i \cap F_{i+1} \cap F_{(n+1)+i}, \quad i = 2, \ldots, n
\end{aligned}
$$

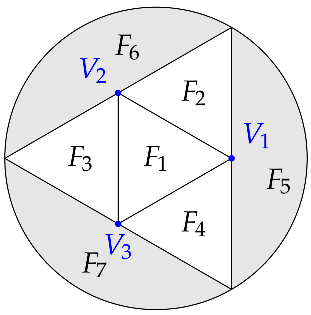


Figure 4.4: Case $c = 4$

Case $\mathbf{c} = \mathbf{5}$. The $2n$ newly labeled faces $F_{n+2}, \ldots, F_{3n+1}$ must satisfy the incidence relations

$$
\begin{aligned}
V_1 &= F_1 \cap F_2 \cap F_{n+1} \cap F_{n+2} \cap F_{n+3}, \\
V_i &= F_1 \cap F_i \cap F_{i+1} \cap F_{(n+1)+(2i-1)} \cap F_{(n+1)+(2i)}, \quad i = 2, \ldots, n
\end{aligned}
$$

Our next step is to show case by case that this labeling process did not label any face twice. The first partial labeling yields that $F_1, \ldots, F_{n+1}$ are distinct.

It is known from Chapter 3 that $(n, 4) = (3, 4)$ and $(n, 5) = (3, 5)$ are the only possible Schläfli symbols. For these two cases, the second partial labeling is illustrated by Figure 4.4 and Figure 4.5.

Case $\mathbf{c = 4}$. It holds that all faces containing the same vertex are distinct. Otherwise, we would get $c < 4$, a contradiction. This implies that $F_5 \notin \{F_1, F_2, F_4\}$, $F_6 \notin \{F_1, F_2, F_3\}$ and $F_7 \notin \{F_1, F_3, F_4\}$. Furthermore, F_5 contains V_1 but F_3, F_6 and F_7 do not and hence $F_5 \notin \{F_3, F_6, F_7\}$. F_6 contains V_2 but F_4, F_5 and F_7 do not, therefore, it holds It follows that $F_6 \notin \{F_4, F_5, F_7\}$. F_7 contains V_3 but F_2, F_5 and F_6 do not, and thus we conclude that $F_5 \notin \{F_2, F_5, F_6\}$. This implies that all faces $F_1, \ldots, F_7$ are distinct.

Case $\mathbf{c = 5}$. Again it holds that all faces containing the same vertex are distinct. Otherwise, we would get $c < 5$, a contradiction. This implies that $F_5 \notin \{F_1, F_2, F_4, F_6\}$. Furthermore, F_5 contains V_1 but F_3, F_7, F_8, F_9 and F_{10} do not and thus $F_5 \notin \{F_3, F_7, F_8, F_9, F_{10}\}$. Similarly, we can show that each face $F_6, \ldots, F_{10}$ is distinct from all other faces. This implies that all faces $F_1, \ldots, F_{10}$ are distinct. By construction of the second labeling, the labeling of all faces $F_1, F_2 \ldots, F_{(n+1)+n(c-3)}$ is unique and the incidence relations are determined by the unique planar graphs of Figure 4.4 and Figure 4.5.

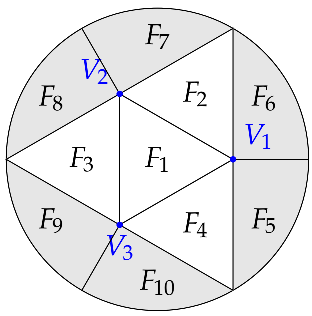


Figure 4.5: Case $c = 5$

4.5 The Other Cases

Before we look at the next case of the Schläfli symbols, we state the following lemma where $f - 1$ faces of a platonic solid P have been labeled by a unique labeling, yielding a unique labeling for all faces of P.

Lemma 4.3 *Let P be a platonic solid with Schläfli symbol (n, c). Assume that $f - 1$ faces have already been labeled by a unique labeling. Then the labeling of all f faces in $\mathcal{F}(P)$ is unique.*

Proof The last unlabeled face is uniquely determined and can be denoted by F_f. This last step of labeling the faces of P is unique. Since the labeling of the other $f-1$ faces is unique too, the statement follows. □

Schläfli Symbol (3,4)

For a platonic solid P with Schläfli symbol $(3,4)$ and Eulerian triple $(8,12,6)$, we see that $n+1 = 4 = \frac{1}{2}f$ faces are enumerated in the first partial labeling.

It follows from Lemma 4.2 that for every vertex $V_1, \ldots, V_n$, three faces containing these vertices are already labeled. Thus we can apply the second partial labeling as described in Section 4.4. This labels n additional faces and thus we will have labeled $2n+1 = 7 = f-1$ faces in total. Using Lemma 4.3, the last face F_8 can be labeled uniquely and the labeling is complete.

The associated plane graph has $f = 8$ faces, $e = 12$ edges and $v = 6$ vertices. The vertices are each uniquely determined by the four faces containing it.

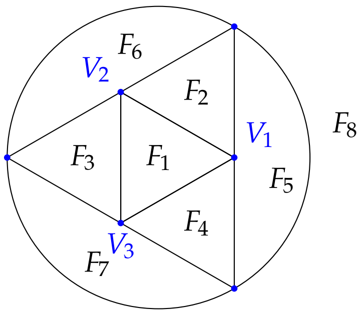


Figure 4.6: Illustration of the unique planar graph associated to a platonic solid with Schläfli symbol $(3,4)$

Schläfli Symbol (5,3)

As in case $(3,4)$, the first partial labeling labeled exactly half of the faces of a platonic solid P with Eulerian triple $(f,e,v) = (12,30,20)$. It follows from Lemma 4.2 that all faces containing $V_1, \ldots, V_n$ have already been labeled since $c = 3$. As $c = 3$, the second partial labeling cannot be applied and another construction is needed.

The first partial labeling determines the faces $F_1, \ldots, F_6$ and vertices $V_1, \ldots, V_5$ as illustrated in Figure 4.7. By Lemma 4.2, each of the faces $F_2, \ldots, F_6$ contains exactly two of the vertices $V_1, \ldots, V_5$ whereas another three vertices have not been labeled yet. The next step is to label these vertices.

Denote the four vertices contained in two labeled faces, F_i and F_{i+1}, $i = 2, \ldots, n$, by V_{n+i} and let V_{2n} be the vertex contained in F_2 and F_{n+1}. Denote the vertices contained in only one labeled face F_i, $i = 2, \ldots, n+1$, by V_{i+9}. Per definition, the vertices $V_6, \ldots, V_{10}$ are contained in two labeled faces, and since $c = 3$ for all vertices $V_i \in \{V_6, \ldots, V_{10}\}$, there is exactly one unlabeled face that contains V_i. Denote this uniquely determined face by $F_{i+1}, i = 6, \ldots, 10$.

Per construction of the first labeling, we know that the faces $F_1, \ldots, F_6$ are distinct. It remains to show that the faces $F_7, \ldots, F_{10}$ are distinct. First, for $i = 6, \ldots, 10$, the face F_i contains the vertex V_{i-1} but the other four faces do not. Second, the three faces containing V_i, $i = 6, \ldots, 10$, must be distinct too since otherwise $c < 3$. This means that all faces $F_1, \ldots, F_{11}$ are distinct. At this point, we have uniquely labeled $11 = f - 1$ faces of P. Lemma 4.3 implies that there exists a unique labeling of the last face F_{12} and this case is complete. The corresponding unique plane graph of the platonic solid with $f = 12$ faces, $e = 30$ edges and $v = 20$ vertices is shown in Figure 4.7.

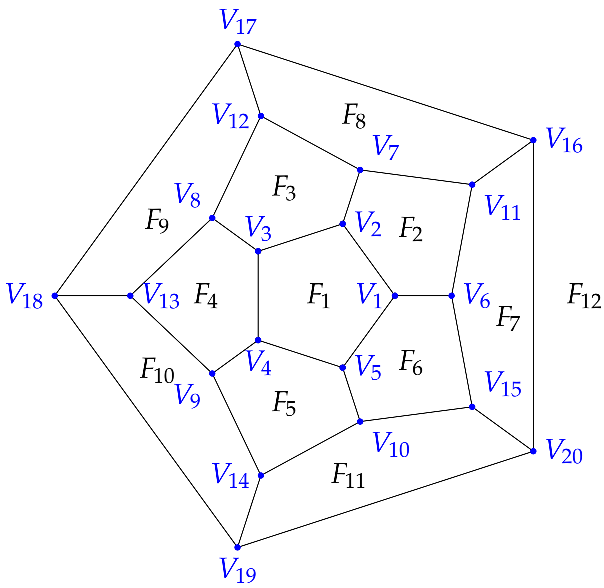


Figure 4.7: Illustration of the unique plane graph of a platonic solid with Schläfli symbol $(5, 3)$

Schläfli Symbol (3,5)

Once more, we start with the first partial labeling which labels $n + 1$ faces of a platonic solid P with Schläfli symbol $(3, 5)$. By Lemma 4.2, every vertex V_1, V_2, V_3 is contained in three labeled faces. Since $n = 3$ and $c = 5$, we can enumerate the next $(c-3)n = 6$ faces with the second partial labeling as described in detail in Section 4.4. As a result, ten faces $F_1, \ldots, F_{10}$ of P have been labeled as illustrated in Figure 4.8.

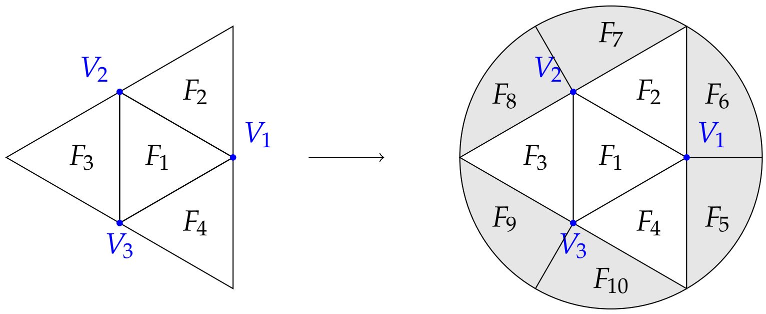


Figure 4.8: Planar graph of the labeled faces of P with Schläfli symbol $(3, 5)$: From the labeled faces of the first partial labeling to the labeled faces of the second partial labeling

By construction of the first partial labeling, each face of $F_2, \ldots, F_{n+1}$ contains two labeled vertices and one vertex of each triangular face $F_2, \ldots, F_{n+1}$ has not been labeled up to this point. If we assume that it is the same vertex for all n faces, we would get a platonic solid with Schläfli symbol $(3, 3)$ which is a contradiction. If we assume that there are only two vertices, the number c of each vertex would not be the same for all which is a contradiction to the definition of a platonic solid. Thus, we can label the remaining three unlabeled vertices contained in one of the faces F_i by V_{i+2}, $i = 2, \ldots, n+1$, as shown in Figure 4.9.

These newly labeled vertices V_4, V_5, V_6 are each contained in three labeled faces and thus the vertices satisfy the assumptions of the second partial labeling. By applying the process of the second partial labeling on the set of vertices $\{V_4, V_5, V_6\}$, six more faces $F_{11}, \ldots, F_{16}$ are obtained. The next step is to show that the newly labeled faces $F_{11}, \ldots, F_{16}$ are distinct from each other and also from the other ten faces $F_1, \ldots, F_{10}$.
First, the five faces containing the vertex V_i, $i = 4, 5, 6$, are distinct because otherwise $c < 5$. It follows that F_{11} and F_{12} are not equal to the faces F_2, F_6, F_7, as well as $F_{11} \neq F_{12}$. The same holds for the faces containing V_5 and V_6: $F_{13}, F_{14} \notin \{F_3, F_8, F_9\}$, $F_{13} \neq F_{14}$, also F_{15} and F_{16} are not in the set of faces $\{F_4, F_5, F_{10}\}$, and F_{15} is not equal to F_{16}. Second, F_{11} and F_{12} contain V_4 but all faces in the set $\{F_1, F_3, F_4, F_5, F_8, F_9, F_{10}, F_{13}, \ldots, F_{16}\}$ do not. Hence F_{11} and F_{12} cannot be in this set. Together with the first reasoning above, it follows that $F_{11}, F_{12} \notin \{F_1, \ldots, F_{10}, F_{13}, \ldots, F_{16}\}$. Similarly, we can show that the pair of faces F_{13}, F_{14} and F_{15}, F_{16} are distinct from all the other labeled faces. Thus, we have uniquely labeled the sixteen faces $F_1, \ldots, F_{16}$.

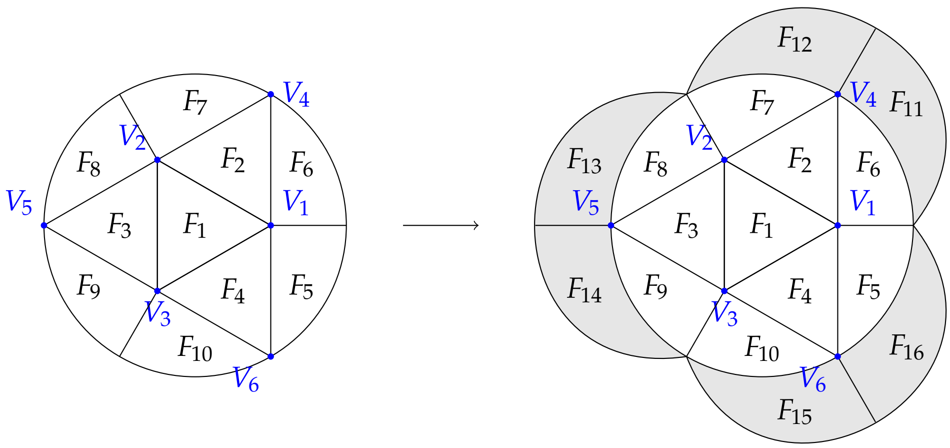


Figure 4.9: Labeling three new vertices V_4, V_5, V_6
and applying the second partial labeling again

The next step is to label all vertices that are contained by four labeled faces each. As illustrated by the right graph in Figure 4.9, there are three such vertices. We will denote the vertex contained in the labeled faces F_5, F_6, F_{11} and F_{16} by V_7. Going counterclockwise, the next vertex contained in four labeled faces, namely F_7, F_8, F_{12} and F_{13}, is denoted by V_8. The last vertex contained in F_9, F_{10}, F_{14} and F_{15} is denoted by V_9. The three newly labeled vertices V_7, V_8, V_9 are illustrated in the upper graph in Figure 4.10.
By construction, each of the newly labeled vertices has exactly one unlabeled face. We denote this face containing V_i by F_{i+10}, $i = 7, 8, 9$, as illustrated in the lower graph in Figure 4.10.
To show that $F_{17} \notin \{F_1, \ldots, F_{16}, F_{18}, F_{19}\}$, we argue similarly as in the previous step, namely, we consider the four other faces F_5, F_6, F_{11}, F_{16} containing V_7, and the fourteen faces not containing V_7. A similar argument applies to the faces F_{18} and F_{19}. Hence, we can conclude that all $19 = f - 1$ faces $F_1, \ldots, F_{19}$ are distinct. By applying Lemma 4.3, there is a unique labeling for the remaining face F_{20} and this last case is complete. The resulting planar graph has $f = 20$ faces, $e = 30$ edges and $v = 12$ vertices.

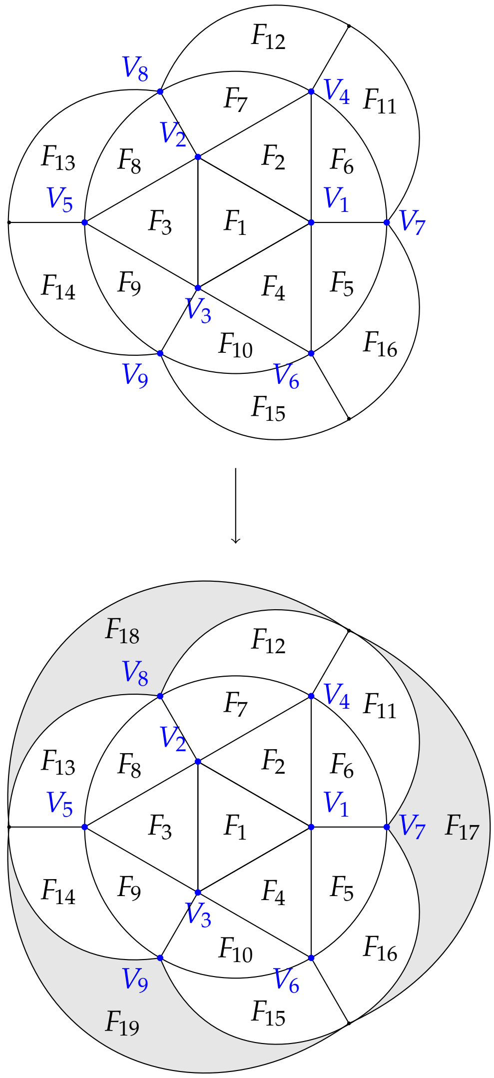


Figure 4.10: Labeling three vertices V_7, V_8, V_9
and three faces F_{17}, F_{18}, F_{19}

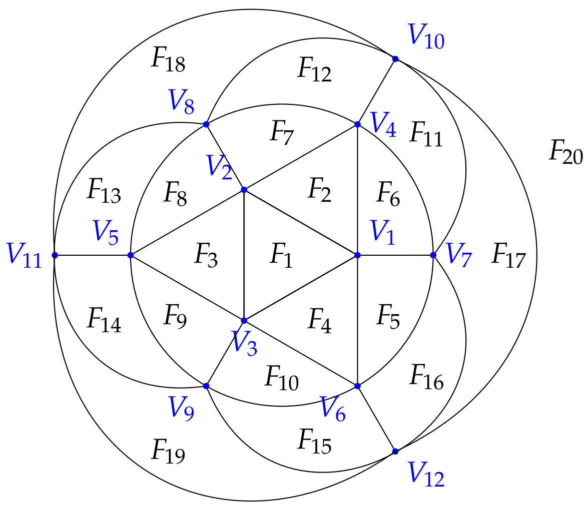


Figure 4.11: Unique planar graph associated to a platonic solid with Schläfli symbols $(3, 5)$

Chapter 5

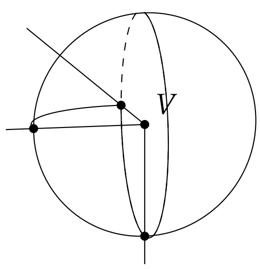
Euclid's Classification of Platonic Solids

5.1 Auxiliary Results

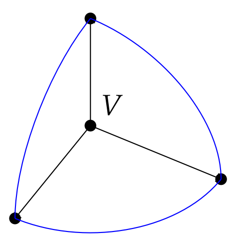
This chapter accounts for the most recent efforts to come up with a rigorous proof of Euclid's classification of platonic solids which was finally achieved in the nineteenth and twentieth century. It covers important results of Steinitz and Cauchy.

Definition 5.1 *Let V be a vertex of a given polyhedron P. Intersecting the faces in $\mathcal{F}(P)$ containing V with a small sphere centered on the vertex results in a spherical polygon, called* ***vertex figure*** *at the vertex V.*

Note that the interior angles of the vertex figure match the dihedral angles of the initial polyhedron.



(a) Small sphere with center at vertex V intersecting the polyhedron



(b) Resulting spherical polygon at vertex V

Visualization of a vertex figure of a cube

Lemma 5.2 (Steinitz.) *Let $p = V_1 \cdots V_n$ and $q = W_1 \cdots W_n$ be two polygons in the plane, where V_i and W_i denote the vertices of the corresponding polygons, $i = 1, \ldots, n$. Suppose that all sides in both polygons are equal except the last, i.e*

$$l(V_i V_{i+1}) = l(W_i W_{i+1}), \quad i = 1, \ldots n-1$$

where $l(V_i V_{i+1})$ denotes the length of the edge in p connecting V_i and V_{i+1} and analogously for the length of edges in q. Suppose also that the angles of the first polygon are less than or equal to the angles of the second one, i.e.,

$$\angle V_i \leq \angle W_i, \quad i = 2, \ldots, n-1,$$

with at least one strict inequality. We denote the angle between two edges meeting at a vertex V with $\angle V$. Then

$$l(V_n V_1) < l(W_n W_1).$$

Proof We prove this lemma by induction over n. Let p and q be two convex polygons with the same number of vertices n.

Case I: n = 3 *This is Proposition 24 in Euclid's First Book.*
Assume that two sides of the two triangles p and q are equal but one of the included angle is greater than the other one. This implies that the base of p is also greater than the base of q.

Case II: $n \geq 4; \exists i : \angle V_i = \angle W_i$
The two triangles $V_{i-1}V_iV_{i+1}$ and $W_{i-1}W_iW_{i+1}$ are congruent. Because $l(V_{i-1}V_i) = l(W_{i-1}W_i)$ as well as $l(V_iV_{i+1}) = l(W_iW_{i+1})$, it follows that

$$l(V_{i-1}V_{i+1}) = l(W_{i-1}W_{i+1}).$$

Look at the polygons omitting the vertices V_i and W_i:

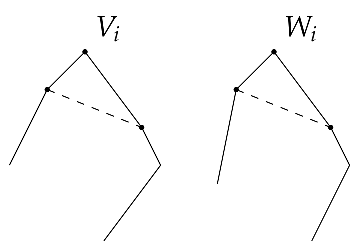


Illustration of two polygons where $\angle V_i = \angle W_i$

These new polygons have $n-1$ edges and satisfy the assumptions of this lemma, so we apply the induction hypothesis and the result follows.

Case III: $n \geq 4; \forall i : \angle V_i < \angle W_i$
The idea of this case is to construct a new point V_1' such that

$$\begin{aligned} l(V_1'V_2) &= l(V_1V_2) \\ \angle V_1'V_2V_3 &= \angle W_2 \end{aligned}$$

where $\angle V_1'V_2V_3$ is the angle at V_2 between the new edge $V_1'V_2$ and V_2V_3.

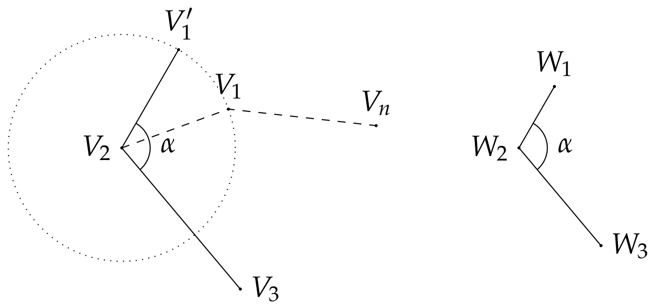


Illustration of the construction of V_1'

First, we compare the polygon $V_1 \cdots V_n$ to $V_1'V_2 \cdots V_n$. It holds that

$$\angle V_3 = \angle V_3 \overset{\text{Case II}}{\Longrightarrow} l(V_nV_1) < l(V_nV_1').$$

Second, we compare the polygon $V_1'V_2 \cdots V_n$ to $W_1 \cdots W_n$. It holds that

$$\angle V_1'V_2V_3 = \angle W_2 \overset{\text{Case II}}{\Longrightarrow} l(V_nV_1) < l(V_nV_1').$$

Case III only holds if the new polygon $V_1'V_2 \cdots V_n$ is convex since otherwise the cases before cannot be applied to this new polygon. Thus, we need to consider the following last case.

Case IV: $\mathbf{V_1'V_2 \cdots V_n}$ is not convex.
We choose a new point V_1^* which lies between V_1 and V_1' such that V_1', V_n and V_{n-1} are collinear and such that

$$l(V_1^*V_2) = l(V_1V_2).$$

Such a point exists since the polygon $V_1'V_2 \cdots V_n$ is not convex. Since V_1', V_n and V_{n-1} are collinear, it holds that

$$l(V_1^*V_n) = l(V_1^*V_{n-1}) - l(V_{n-1}V_n). \tag{5.1}$$

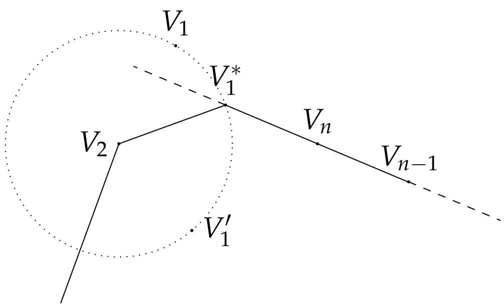


Illustration of the construction of V_1^* where $V_1'V_2 \cdots V_n$ is not convex

First, we compare $V_1 \cdots V_n$ to $V_1^* V_2 \cdots V_n$. We see that V_1^* changes the value of at most three angles. Since $n \geq 4$, there exist one pair of corresponding vertices in each polygon where the angles are the same. Thus, *Case II* can be applied again and we get

$$l(V_n V_1) < l(V_n V_1^*). \tag{5.2}$$

Second, we compare $V_1^* V_2 \cdots V_{n-1}$ to $W_1 \cdots W_{n-1}$ and by the induction hypothesis it follows that

$$l(V_{n-1} V_1^*) < l(W_{n-1} W_1). \tag{5.3}$$

Finally,

$$\begin{aligned} l(V_1 V_n) &\overset{(5.2)}{<} l(V_1^* V_n) \\ &\overset{(5.1)}{=} l(V_1^* V_{n-1}) - l(V_{n-1} V_n) \\ &\overset{(5.3)}{<} l(W_1 W_{n-1}) - l(W_{n-1} W_n) \\ &\leq l(W_1 W_n) \end{aligned}$$

where the last inequality is the triangle inequality. □

The following result is the analogy of Steinitz' result for polygons on the surface of a sphere. We are going to prove it only for polygons in the plane. The proof for spherical polygons uses results from the first part of Euclid's Book I [3].

Lemma 5.3 *Let $p, \overline{p}$ be two convex polygons in the plane or on the sphere with corresponding sides being equal. Mark each vertex in p as follows*

$$\begin{cases} + & \text{if } \angle V < \angle \overline{V} \\ - & \text{if } \angle V > \angle \overline{V} \\ = & \text{if } \angle V = \angle \overline{V} \end{cases}$$

Then, either all vertices are marked with $=$ or, as we make a circuit across all vertices, ignoring the ones marked with $=$, the sign must change at least four times.

Thus, there is an implicit one-to-one map φ that maps each vertex of p onto a vertex of $\overline{p}$.

Proof The number of changes of sign must be even since it is a closed circuit of vertices. We assume there are exactly two changes of sign. We start by taking the diagonal V_iV_j of the polygon p such that p is cut into two convex polygons. One of these new smaller polygons contains only the $+$ vertices, denoted by p_+ and the other contains only the $-$ vertices, denoted by p_-.

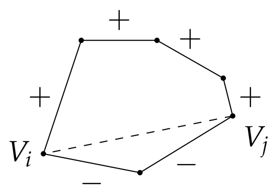


Illustration of a polygon cut into two smaller convex polygons

Applying Steinitz's Lemma 5.2 on both p_+, p_- yields:

$$\text{Lemma 5.2 for } p_+ : V_iV_j < W_iW_j$$
$$\text{Lemma 5.2 for } p_- : V_iV_j > W_iW_j$$

This is a contradiction. □

The marking system of vertices in the plane used in Lemma 5.3 yields a similar marking system for two polyhedra. Instead of marking a vertex, mark each edge of a polyhedron according to the relation of its dihedral angle to the corresponding edge and its dihedral angle of the other polyhedron. This leads to a new bounded connected surface, called a net:

Definition 5.4 *Mark each edge of a polyhedron P with $+, -$ or $=$ according as its dihedral angle is less than, greater than or equal to the corresponding dihedral*

angle of another polyhedron $\overline{P}$. Consider only edges marked with $+$ and $-$ and the vertices that belong to those edges. Together they build a so-called ***net****.*
Also, any maximal union of faces of the polyhedron that are not separated by edges of the net is called a ***net-face****.*

Notice, that a net-face is no longer a plane polygon, instead, it is a connected surface bounded by net-edges.

5.2 Cauchy's Rigidity Theorem

In this section we will state and prove Cauchy's Rigidity Theorem.

Theorem 5.5 (Cauchy's rigidity theorem.) *Let $P, \overline{P}$ be two convex polyhedra made of congruent faces. Suppose that P and $\overline{P}$ are similarly arranged with a bijection φ. Then P and $\overline{P}$ are congruent.*

Proof This proof is divided in three steps.

Step 1: Marking edges and vertices.
Mark each edge of P with $+, -$ or $=$ according to its dihedral angle being less than, greater than or equal to the corresponding dihedral angle of $\overline{P}$. Also, for every vertex of P look at its vertex figure. The spherical polygon is convex since it results from a convex solid. Its vertices inherit the markings from the edges, which by construction correspond to the increase or decrease of this polygon angle as compared to the vertex figure of the second polygon (see [4]).

Step 2: Applying Steinitz.
By applying Steinitz on the surface of a sphere (Lemma 5.3), we deduce that by going step by step through the edges which intersect at that vertex, either they are all marked with a $=$ or there are at least four changes of sign.

Step 3: Counting the total number of changes of signs.

- Case 1: First assume that all dihedral angles are changing, i.e all edges are marked with either $+$ or $-$. Again by Lemma 5.3, it holds that

  $$t := \sum_{\text{all vertices}} \text{number of changes of sign of edges at one vertex } \geq 4v,$$

  where v is the number of vertices. To get the other inequality for t, look at the change of sign of a single face. On a triangular face, at least two adjacent edges must have the same sign. Hence, that face can contribute at most two changes of sign to its three vertices. It follows that a face of n sides can contribute to at most n changes of sign if n is even, or $n-1$ if n is odd. This leads to the second inequality

$$2f_3 + \sum_{n\geq 4} nf_n \geq t,$$

where f_n denotes the number of faces of n sides. Putting the two inequalities together, we get

$$2f_3 + n\sum_{n\geq 4} f_n \geq 4v \overset{(3.1)}{=} 4(e-f+2) \overset{(3.3)}{=} 2nf - 4f + 8 = (2n-4)\sum f_n + 8$$

For the second equality, we use Euler's formula (3.1). For the third equality, we need Equation (3.3) from the proof of Lemma 3.3. Also, it holds that $f = \sum f_n$. Thus, it follows that

$$0 \geq \sum_{n\geq 4} (n-4)f_n + 8,$$

which is impossible, since all terms in the sum are non-negative.

- Case 2: Second, assume that there exist some edges marked with $=$ and some marked with $+$ and $-$. If there were no edges marked with $+$ or $-$, the two polyhedra would be congruent, and the proof would be complete. We will apply the same idea as in the proof of *Case 1* but using only the $+$ and $-$ marked edges and corresponding vertices, hence the net (Definition 5.4) of the polyhedron. The argumentation works as in *Case 1* with the exception that we cannot apply Euler's formula directly since a net is not necessarily a polyhedron. Instead, we apply the proof of Euler's formula (Theorem 3.1) to the net. We cannot assume that the plane figure, the projection of the net onto the plane, is connected anymore. The assumption of a connected plane figure is only used in the last step of the proof. Hence, with a net, there could be more than one vertex left after applying the two operations (i) and (ii) to the net as in the proof of Euler's formula. This means that we get the inequality

$$f - e + v \geq 2.$$

The argument of *Case 1* still works with this inequality and we thus get a contradiction.

The only possible case left is that all dihedral angles of the two polyhedra are equal. This implies that it is possible to build the two polyhedra step by step into congruent figures, which completes the proof. □

5.3 Complete Classification of Platonic Solids

Theorem 5.6 (Euclid's classification.) *Up to congruence (and a scale factor), there are exactly five platonic solids, which are uniquely determined by the five possible pairs* $(n, c) \in \mathcal{A}$.

Proof By Lemma 3.3, there are only five possible pairs (n, c) for a platonic solid. The explicit construction of a platonic solid with one of the five Schläfli symbols stated in Lemma 3.3 will be given in Chapter 6.
Assume there exist two platonic solids possibly not congruent with the same pair (n, c). Then Lemma 3.7 implies that these two polyhedra are similarly arranged. Finally, we conclude with Cauchy's rigidity Theorem 3.6 that the two polyhedra must be congruent. $\square$

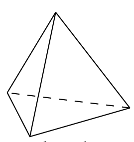
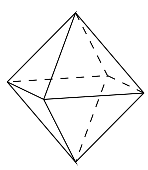
Chapter 6

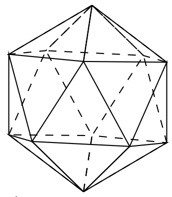
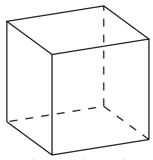
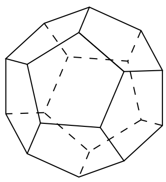
The Five Platonic Solids

In the previous chapters, we stated a precise definition of a platonic solid and we proved that up to congruence exactly five platonic solids exist. In this chapter, we give explicit illustrations of these five platonic solids together with their characteristic attributes. Furthermore, we state some basic properties of the platonic solids.

6.1 Complete List of All Platonic Solids

This section will focus on giving a complete table of all platonic solids and their characteristic parameters such as the Schläfli symbol (n, c) and the Eulerian triple (f, e, v). Recall that f, e and v denote the number of faces, edges and vertices, respectively. Furthermore, the Schläfli symbol (n, c) specifies that the corresponding polyhedron has regular n-polygons as faces, where c of them are meeting in each vertex.

	Schläfli symbol (n, c)	Eulerian triple (f, e, v)	dihedral angle [1] ϑ
Tetrahedron 	(3,3)	(4,6,4)	70.52°
Octahedron 	(3,4)	(8,12,6)	109.48°

	Schläfli symbol (n, c)	Eulerian triple (f, e, v)	dihedral angle [1] ϑ
Icosahedron 	(3,5)	(20,30,12)	138.18°
Cube 	(4,3)	(6,12,8)	90°
Dodecahedron 	(5,3)	(12,30,20)	116.56°

6.2 The Origin of the Concept of Platonic Solids

We give some historical background on Euclid's work on Platonic Solids. In contrast to the approach of this thesis where we have not relied on specific three-dimensional models of platonic solids, Euclid had another approach.

He starts by giving explicit three-dimensional models inscribed in a sphere for the tetrahedron, cube, octahedron, icosahedron, and dodecahedron by characterizing them by the number and type of faces they have. A pyramid is a solid figure formed by joining a point to each of the vertices of a polygon in a plane not containing the point. Euclid does not use the word

[1] See *Table I, (i)* in [2]. The dihedral angle, denoted by ϑ, can be calculated using the following formula, see *Chapter 4* in [5]:

$$\sin\left(\frac{\vartheta}{2}\right) = \frac{\cos(\frac{\pi}{c})}{\sin(\frac{\pi}{n})}$$

where n, c are the known numbers of the Schläfli symbol.

tetrahedron, as we did, defining it to be a triangular pyramid formed of four equilateral triangles. Euclid defines a cube as a solid figure contained by six equal squares, the octahedron and icosahedron as solid figures bounded by 8 (resp. 20) equilateral triangles, and a dodecahedron as a figure bounded by 12 regular pentagons. Euclid then states in his work *Elements*, translated by Fitzpatrick [3]: *So, I say that, beside the five aforementioned figures, no other (solid) figure can be constructed (which is) contained by equilateral and equiangular (planes), equal to one another.* Euclid argues as follows: If we use equilateral triangles, then we can put together three, four, or five of them at one vertex, but six triangles would lie flat. If we use squares, we can put three at one vertex, but no more. If we use regular pentagons, we can again put three at a vertex. If we try to use hexagons, three of them would lie flat, so for a stronger reason, we cannot use regular polygons of more sides. These five cases, he says, correspond to the tetrahedron, octahedron, icosahedron, cube, and dodecahedron, respectively; hence, there are no others.

Unfortunately, Euclid's conclusion is not correct because of some missing hypotheses, nor is his proof of the corrected result complete. The next two subsections will examine the importance of the missing hypotheses in more detail.

6.2.1 Convexity of the Platonic Solids

One important requirement for the five platonic solids to be unique (up to congruence) is that they have to be convex. Without this additional requirement there is a figure such as the *punched-in icosahedron*. Consider one vertex A of an icosahedron, and let $BCDEF$ be the pentagon formed by the five adjacent vertices. Take off the pentagonal pyramid made by $ABCDEF$, and replace it by the pentagonal pyramid $A'BCDEF$, where A' is the reflection of the point A in the plane of $BCDEF$. The point A' is then inside the original icosahedron, so that the new figure is like an icosahedron elsewhere, but has a concavity at A'. This is a figure bounded by 20 equal equilateral triangles, but it is not congruent to the icosahedron or any other of the platonic solids listed above.

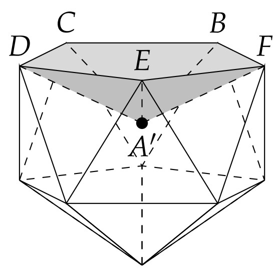


Illustration of the punched-in icosahedron

6.2.2 The Number of Faces Containing One Vertex

Another requirement for platonic solids is that the number c in the Schläfli symbol has to be the same at each vertex of the platonic solid. Otherwise, a *triangular dipyramid* could be considered as another platonic solid. If we think of two equal tetrahedra, glued together along one face, we get a convex polyhedron whose faces are 6 equilateral triangles, which is not contained in the list above of the five platonic solids. The inconsistency is that we have three faces meeting at the top and bottom vertices, but four faces meeting at each of the vertices along the glued face.

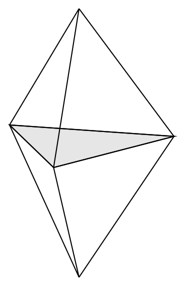


Illustration of a triangular dipyramid

6.3 Basic Properties of Platonic Solids

In this section we state three properties on the geometrical nature of platonic solids.

Theorem 6.1

1. *In each platonic solid, all dihedral angles are equal.*
2. *All vertices of a platonic solid lie on a sphere.*
3. *For any two vertices of a platonic solid, there is a rigid motion of the figures taking one vertex to the other.*

Proof We prove these properties for each platonic solid individually and present a corresponding construction simultaneously.

Tetrahedron

- Construction: We start with an equilateral triangle with side length one. We denote the line starting at the center of the triangle and going upwards while being perpendicular to the plane the triangle lies in by h. Then there exists a point H on h such that H has distance one to some vertex of the triangle. By symmetry of the construction, H will have distance one to the other two vertices as well. We connect H to the vertices of the triangle and get the tetrahedron with side length one.

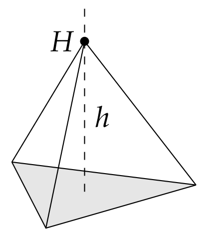


Illustration of the construction of the tetrahedron

- Equal dihedral angles: By the symmetry of the figure, all dihedral angles must be equal.
- Being inscribed in a sphere: We denote the intersection point of all four axes going through one vertex and through the center of the opposite face by $\mathcal{O}$. It follows from the symmetry of the figure that the vertices of the tetrahedron are all equidistant to the center $\mathcal{O}$ and thus lie on a sphere with center $\mathcal{O}$.
- Existence of a rigid motion: A rotation about an axis passing through one vertex and through the center of the opposite face will map any vertex onto another one.

Octahedron

- Construction: We start with the sphere of radius one and draw three mutually perpendicular lines going all through the center of the sphere denoted by $\mathcal{O}$. The vertices of the octahedron are the six intersection points of these lines with the sphere. Connecting these points results in the figure of an octahedron.

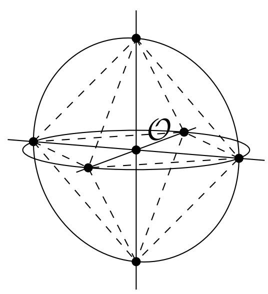


Illustration of the construction of the octahedron

- Being inscribed in a sphere: The fact that the octahedron is inscribed in a sphere follows directly from its construction.
- Existence of a rigid motion: Any two adjacent vertices can be mapped onto any other two by repeatedly rotating around the three axes of the sphere.
- Equal dihedral angles: Two adjacent vertices are connected by an edge, which is the intersection of two faces. These two faces have a dihedral angle according to Definition 2.7. It is possible to map two adjacent vertices onto any other adjacent pair. This implies that the dihedral angle must be preserved by the rigid motion. It follows that all dihedral angles must be equal.

Icosahedron

- Construction: We begin with a regular pentagon of unit length in a plane, denoted by $BCDEF$. Proposition 2.3 states that the pentagon lies on a circle. Similarly to the construction of a tetrahedron, we take the perpendicular line to the plane of $BCDEF$ through the center of the circle and find a point on the line, denoted by A, with distance one to B. By the symmetry of construction, A will have distance one to all the points of the pentagon $BCDEF$. We connect A to these points and get a pentagonal pyramid with five equilateral triangles as its upper faces. By symmetry, the dihedral angle between any two adjacent triangles has to be equal. This finishes the first step.
  We repeat this step to construct the first half of the icosahedron. We make another such congruent pentagonal pyramid with a regular pentagon $B'A'D'GH$ and a top vertex, denoted by C'. The dihedral angles in the new pyramid are the same as in the first pentagon. Hence, if we glue the triangle $A'B'C'$ onto the triangle ABC, the points D and D' will coincide. This gives us a convex figure made of eight equilateral triangles. We repeat this once more to get a convex figure made of ten equilateral triangles where all dihedral angles are equal.

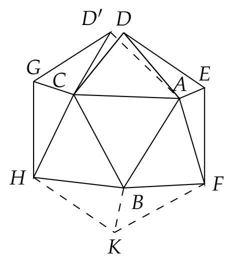


Illustration of the construction of the first half of the icosahedron

As we go around the six edges that form the outer bound of this figure, the angle between any two consecutively edges is equal to the interior angle of a regular pentagon.
After making another figure of ten equilateral triangles, we glue the two congruent figures together and get the icosahedron with all dihedral angles being the same. The boundary of the two figures fit together since all edge angles and all dihedral angles are the same.

- Equal dihedral angles: The dihedral angles are all the same as argued in the construction of the icosahedron.
- Being inscribed in a sphere: We take the two perpendicular lines of two adjacent faces through their center and denote the intersection point by O. Since all dihedral angles are equal, the intersection point O will be the same for any two adjacent faces. By construction, all four points of the two adjacent points are equidistant to O, thus all vertices of the solid are equidistant to O. Therefore, the icosahedron is inscribed in a sphere with center O.
- Existence of a rigid motion: The described construction is symmetric under a rotation of the initial triangle ABC onto itself. Since all the dihedral angles are the same, the initial triangle can be chosen arbitrarily. It follows that under one rotation, any vertex can be mapped onto an adjacent vertex. Thus, using different rotations consecutively will map any vertex onto any other vertex.

Cube

- Construction: We start with a sphere of radius one. As in the construction of the octahedron, we take three mutually perpendicular lines x, y, z going through the center of the sphere, denoted by $\mathcal{O}$. There are six intersection points of the axes x, y, z with the sphere. We denote them by $F_1, \ldots, F_6$. We take a square of side two and put its center on one of the six intersection points $F_1, \ldots, F_6$ such that the square lies tangent to the sphere. This square is one of the six faces of the cube. Repeating this step for each intersection point completes the construction of the cube.
- Equal dihedral angles: By construction, every face of the cube is perpendicular to one of the axes x, y, z and any two faces sharing an edge cannot be perpendicular to the same axis. Thus, any two faces sharing an edge are perpendicular to each other since the axes are perpendicular to one another. It follows that all dihedral angles are equal.
- Being inscribed in a sphere: By construction, the center of each face is equidistant to the center $\mathcal{O}$. By symmetry of the face, it follows that

every vertex of one face is also equidistant to $\mathcal{O}$ and by the symmetry of the cube all vertices are equidistant to $\mathcal{O}$. Thus, all vertices of the cube lie on a sphere with center $\mathcal{O}$.

- Existence of a rigid motion: Similar to the octahedron, every vertex can be mapped onto any other by rotating about the three axes of the small sphere used in the construction.

Dodecahedron

- Construction: To construct a dodecahedron, we use the knowledge of how to construct an icosahedron. We take an icosahedron and for each five triangles meeting at the same vertex V, we make a regular plane pentagon by joining the five midpoints of the triangles containing V. This pentagon is the first face of the dodecahedron. We do this for each of the twelve vertices of the icosahedron. The result is the dodecahedron inscribed in the original icosahedron. By construction, the icosahedron and dodecahedron are *dual* to each other. By applying a similar construction to the dodecahedron, one obtains a (scaled) icosahedron again.

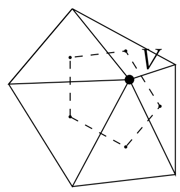


Illustration of the construction of the first face of the dodecahedron

- Equal dihedral angles: All relations between two faces of the dodecahedron are the same since the icosahedron is symmetric and thus all the dihedral angles are equal.
- Being inscribed in a sphere: We know that the faces of the icosahedron are equidistant to their center $\mathcal{O}$. Because the vertices of the dodecahedron are in the center of these faces, they are also equidistant to $\mathcal{O}$. So, the dodecahedron is inscribed in a sphere with center $\mathcal{O}$.
- Existence of a rigid motion: A triangle of the icosahedron is sent to an adjacent triangle through a rotation about an axis passing through two opposite vertices. This means that a vertex of the dodecahedron is mapped onto an adjacent vertex. Therefore, applying consecutive rotations will send any vertex to any other. $\square$

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