ETH Zürich

An Example of a Lagrangian 1-to-1 Transverse Lifting

Master Thesis

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Abstract. We construct a Lagrangian Submanifold L of $\mathbb{C}P^2$ as an orbit of an $SU(2)$-action. We show that L is a 1-to-1 transverse lifting of a curve in $\mathbb{C}P^1$.

Contents

This paper was written in the framework of my Master Thesis at ETH Zurich. It is the result of a research that mainly took place in the Spring Semester 2014, at the beginning of which I approached symplectic geometry for the first time. For this first approach I have primarily used [2], [3], [4] and [5].

After the introductory chapter – where we briefly familiarize the reader with the language of Lagrangian correspondences (motivated by [7] and [8]) and where we define Lagrangian m-to-1 transverse liftings and mention a few example with $m > 1$ for $\mathbb{C}P^n$ – in chapter 2 we introduce an example of a 1-to-1 transverse lifting. The path towards this example was inspired by River Chiang's paper [6]: we have considered an analogous $SU(2)$-action in one dimension lower and we have found that one orbit of this action is a Lagrangian submanifold of $\mathbb{C}P^2$ and a 1-to-1 lifting of a curve in $\mathbb{C}P^1$. The suggestion was given by a work in progress that my supervisor – Dr. Ana Cannas da Silva – is conducting in collaboration with Dr. Meike Akveld.

In the last chapter we see (almost) the same $SU(2)$-action on the second symmetric power of $(\mathbb{C}^2)^*$, and we recover (almost) the same Lagrangian submanifold as orbit of a Hamiltonian $SU(2)$-action.

For her advice and her help, I wish to express my sincere gratitude to Dr. Ana Cannas da Silva.

1. Introduction

1.1. Embedded Compositions of Lagrangian Correspondences. Let (M,ω) be a symplectic manifold. A submanifold $i : L \hookrightarrow M$ is *Lagrangian* if $\omega_p\big|_{T_pM} \equiv 0$ for every $p \in M$ (or equivalently if $i^*\omega = 0$) and if $\dim L = \frac{1}{2}\dim M$.

Let (M_j, ω_j), $j = 0,1,2$ be symplectic manifolds and consider the Lagrangian submanifolds

$$L_{01} \subseteq M_0^- \times M_1,$$
$$L_{12} \subseteq M_1^- \times M_2,$$

where $M_j^- \times M_k$ denotes the product manifold $M_j \times M_k$ equipped with the symplectic form $-\omega_j + \omega_k$. Lagrangian submanifolds of this particular type are called *Lagrangian correspondences.* Notice that a Lagrangian submanifold L of M can be seen as a Lagrangian correpondence of the manifold $\{pt\}^- \times M$.

Given a Lagrangian correspondence $L_{01} \subseteq M_0^- \times M_1$ we can construct a *transpose* Lagrangian correspondence in $M_1^- \times M_0$:

$$(L_{01})^T = \left\{(m_1, m_0) \;\middle|\; (m_0, m_1) \in L_{01}\right\}$$

We will also consider a composition operation of Lagrangian correspondences:

$$L_{01} \circ L_{12} = \left\{(p_0, p_2) \in M_0^- \times M_2 \middle| \; \exists p \in M_1 \text{ such that } (p_0,p_1) \in L_{01},\ (p_1,p_2) \in L_{12}\right\}$$

Let Δ_{M_1} be the diagonal in $M_1 \times M_1$ and denote the projections on the single factors of $M_0 \times M_1 \times M_1 \times M_2$ by

$$\pi_0 : M_0 \times M_1 \times M_1 \times M_2 \to M_0,\quad \pi_1 : M_0 \times M_1 \times M_1 \times M_2 \to M_1,\quad \pi_1' : M_0 \times M_1 \times M_1 \times M_2 \to M_1,\quad \pi_2 : M_0 \times M_1 \times M_1 \times M_2 \to M_2$$

Then we can write:

$$L_{01} \circ L_{12} = (\pi_0, \pi_2)\left((L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)\right).$$

In general the composition may not even be an immersed submanifold. We will restrict our attention to the following case:

Definition. *A composition $L_{01} \circ L_{12}$ is said to be* embedded *if the following two conditions are satisfied:*

- $(L_{01} \times L_{12}) \pitchfork (M_0 \times \Delta_{M_1} \times M_2)$ *in* $M_0 \times M_1 \times M_1 \times M_2$,
- $(\pi_0, \pi_2) : (L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2) \to L_{01} \circ L_{12}$ *is an embedding.*

In this case, $L_{01} \circ L_{12}$ is a Lagrangian correspondence in $M_0^- \times M_2$. Notice that the transversal intersection condition is equivalent to requiring $\pi(L_{01}) \times \pi'(L_{12}) \pitchfork \Delta_{M_1}$ in $M_1 \times M_1$.

We will make use of the following result, which is proven in [7] and [8].

Lemma 1. *If the Lagrangian correspondences $L_{01} \subseteq M_0^- \times M_1$ and $L_{12} \subseteq M_1^- \times M_2$ are such that $(L_{01} \times L_{12}) \pitchfork (M_0 \times \Delta_{M_1} \times M_2)$ in $M_0 \times M_1 \times M_1 \times M_2$, then (π_0, π_2) : $(L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2) \to L_{01} \circ L_{12}$ is an immersion.*

In particular, if L_{01} and L_{12} are compact, satisfy the transverse intersection condition and (π_0, π_1) is injective, then $L_{01} \circ L_{12}$ is an embedded composition.

Since we are only going to work with compact correspondences, we restate the previous definition:

Definition. *A composition of compact correspondences $L_{01} \circ L_{12}$ is said to be* embedded *if the following two conditions are satisfied:*

- $\pi(L_{01}) \times \pi'(L_{12}) \pitchfork \Delta_{M_1}$ *in* $M_1 \times M_1$,
- $(\pi_0, \pi_2) : (L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2) \to L_{01} \circ L_{12}$ *is injective.*

In this case, $L_{01} \circ L_{12}$ is a Lagrangian correspondence in $M_0^- \times M_2$.

1.2. **Relation Between Lagrangian Submanifolds of M and of M_a.** Let (M, ω) be a compact symplectic manifold on which $\mathbb{T}^n$ acts in a Hamiltonian and effective way with moment map $\mu : M \to \mathbb{R}^n$. Let $a \in \mathbb{R}^n$ be a regular value for μ and denote by (M_a, ω_a) the reduced space. Recall that $M_a = \mu^{-1}(a)/\mathbb{T}^n$ and for

$$\begin{array}{ccc} \mu^{-1}(a) & \xhookrightarrow{\ i\ } & M \\ \big\downarrow{\scriptstyle \pi} & & \\ M_a & & \end{array}$$

where i is the inclusion and π is a smooth submersion, the symplectic form ω_a satisfies $\pi^*\omega_a = i^*\omega$.

In this context it is clear that

$$\begin{aligned} (\pi, i) : \mu^{-1}(a) &\to M_a^- \times M \\ p &\mapsto ([p], p) \end{aligned}$$

is a Lagrangian correspondence and an embedding. We will use the following notation:

$$\begin{aligned} L_\mu &= (\pi, i)\left(\mu^{-1}(a)\right) \subseteq M_a^- \times M, \\ L_\mu^T &= (i, \pi)\left(\mu^{-1}(a)\right) \subseteq M^- \times M_a. \end{aligned}$$

Let ℓ be a Lagrangian submanifold of M_a, which we can see as a Lagrangian correspondence in $\{pt\}^- \times M_a$. The composition

$$\ell \circ L_\mu = \left\{ p \in M \;\middle|\; \pi(p) \in \ell \right\} = \pi^{-1}(\ell) \subseteq \{pt\}^- \times M$$

is obviously embedded.

For the other direction we will need lemma 1. Let L be a Lagrangian submanifold in M, which we view as Lagrangian correspondence in $\{pt\}^- \times M$. The composition

$$L \circ L_\mu^T = \pi(L \cap \mu^{-1}(a)) \subseteq \{pt\}^- \times M_a$$

is embedded if:

- $L \pitchfork \mu^{-1}(a)$ in M, [1]
- $\pi\big|_{L \cap \mu^{-1}(a)}$ is injective, or, equivalently, L intersects each "reduction-orbit" at most once.

[1]This is because $\pi_1(L) \cong L$ and $\pi_1'(L_\mu^T) \cong \mu^{-1}(a)$ and because, in general, $S_1 \pitchfork S_2$ in $Q \iff S_1 \times S_2 \pitchfork \Delta_Q$ in $Q \times Q$.

We summarize our considerations:

Theorem 2.

$$\{ \text{ Lagrangian submanifolds in } M_a \text{ } \} \longrightarrow \{ \text{ Lagrangian submanifolds in } M \text{ } \}$$
$$\ell \longmapsto \ell \circ L_\mu = \pi^{-1}(\ell)$$

is always an embedded composition. While

$$\{ \text{ Lagrangian submanifolds in } M \text{ } \} \longrightarrow \{ \text{ Lagrangian submanifolds in } M_a \text{ } \}$$
$$L \longmapsto L \circ L_\mu^T = \pi(L \cap \mu^{-1}(a))$$

is an embedded composition if $L \pitchfork \mu^{-1}(a)$ in M and L intersects each "reduction-orbit" at most once.

1.3. m**-to-**1 **Transverse Liftings.** If L satisfies the transversality condition but intersects the "reduction-orbits" more than once, say L intersects (where it does) each orbit m times, we will say that L is a *m-to-1 transverse lifting* of $\pi(L \cap \mu^{-1}(a))$.

In the following chapter we will introduce an example of a 1-to-1 transverse lifting where the compact symplectic manifold M is $\mathbb{C}P^2$; but, before that, we want to mention a few known examples of m-to-1 transverse liftings with $m > 1$, where M is a complex projective space of arbitrary dimension.

Consider $(\mathbb{C}P^n, \omega_{FS})$, where ω_{FS} is the Fubini-Study form. The action of $\mathbb{T}^n$ given by

$$(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot [z_0 : z_1 : \dots : z_n] = \left[z_0 : e^{i\theta_1} z_1 : \dots : e^{i\theta_n} z_n\right]$$

is Hamiltonian with moment map

$$\mu : \mathbb{C}P^n \to \mathbb{R}^n$$
$$[z_0 : z_1 : \dots : z_n] \mapsto \left(\frac{-(n+1)\,|z_1|^2}{|z|^2} + 1, \dots, \frac{-(n+1)\,|z_n|^2}{|z|^2} + 1 \right).$$

We write $\mu_k([z_0 : z_1 : \dots : z_n]) = \frac{-(n+1)|z_k|^2}{|z|^2} + 1$. The Delzant polytope of this action is the n-dimensional simplex with vertices $(-n, 1, \dots, 1), (1, -n, 1, \dots, 1), \dots, (1, 1, \dots, 1, -n)$ and $(1, \dots, 1)$.[2] We will use the abbreviation

$$/e^{i\theta_k} \ : \qquad\qquad \text{modulo the } k\text{th circle action}$$

Example 1. We take $a \in (-n, 1)$ (i.e. we are away from the boundary of the Delzant polytope) and consider the reduction $\pi : \mu_n^{-1}(a) \to \mu_n^{-1}(a)/e^{i\theta_n} \cong \mathbb{C}P^{n-1}$. The Lagrangian submanifold $L = \mathbb{R}P^n \subseteq \mathbb{C}P^n$ intersects $\mu_n^{-1}(a)$ transversally, but it intersects orbits twice. Hence $\mathbb{R}P^n$ is a 2-to-1 transverse lifting of the Lagrangian $\pi(\mathbb{R}P^n \cap \mu_n^{-1}(a)) = \mathbb{R}P^{n-1} \subseteq \mathbb{C}P^{n-1}$.

[2] This particular choice of scaling of the moment map is not standard. It is preferred here because the barycenter of the corresponding Delzant polytope is in the origin. The alteration by a constant of the Fubini-Study form, caused by this choice, does not influence the study of the Lagrangians that we are about to expose. We will ignore this issue in the rest of the paper and we will not keep track of this constant.

Example 2. An iteration of the process of the previous example lets us conclude that $\mathbb{R}P^n \subseteq \mathbb{C}P^n$ is a 2^k-to-1 transverse lifting of $\mathbb{R}P^{n-k} \subseteq \mathbb{C}P^{n-k}$.

Example 3. For $(a_1, \dots, a_n)$ in the interior of the Delzant polytope, we consider the reductions

$$p : (\mu_1, \dots, \mu_{n-1})^{-1}(a_1, \dots, a_{n-1}) \to (\mu_1, \dots, \mu_{n-1})^{-1}(a_1, \dots, a_{n-1})/e^{i\theta_1} \dots e^{i\theta_{n-1}} \cong \mathbb{C}P^1$$

and

$$\pi : \mu_n^{-1}(a_n) \to \mu_n^{-1}(a_n)/e^{i\theta_n} = M_{a_n} \cong \mathbb{C}P^{n-1}.$$

From the $\mathbb{R}P^1 \subseteq \mathbb{C}P^1$, we obtain a Lagrangian submanifold $L = p^{-1}(\mathbb{R}P^1) \subseteq \mathbb{C}P^n$. Then L intersects $\mu_n^{-1}(a_n)$ transversally, but it intersects the orbits of the last circle action twice. Notice that $\ell = \pi(p^{-1}(\mathbb{R}P^1) \cap \mu_n^{-1}(a_n))$ is a regular $\mathbb{T}^{n-1}$ orbit in M_{a_n}. We have hence that $p^{-1}(\mathbb{R}P^1)$ is a 2-to-1 transverse lifting of ℓ.

Example 4. The previous example is generalized by taking

$$p : (\mu_1, \dots, \mu_{n-k})^{-1}(a_1, \dots, a_{n-k}) \to (\mu_1, \dots, \mu_{n-k})^{-1}(a_1, \dots, a_{n-k})/e^{i\theta_1} \dots e^{i\theta_{n-1}} \cong \mathbb{C}P^k$$

and

$$\pi : (\mu_{n-k+1}, \dots, \mu_n^{-1})(a_{n-k}, \dots, a_n) \to (\mu_{n-k+1}, \dots, \mu_n^{-1})(a_{n-k}, \dots, a_n)/e^{i\theta_{n-k+1}} \dots e^{i\theta_n} = M_{red}.$$

One obtains that $L = p^{-1}(\mathbb{R}P^k)$ is a 2^k-to-1 transverse lifting of $\ell = \pi(p^{-1}(\mathbb{R}P^k) \cap (\mu_{n-k+1}, \dots, \mu_n^{-1})(a_{n-k}, \dots, a_n))$, which is a regular $\mathbb{T}^{n-k}$-orbit in $M_{red} \cong \mathbb{C}P^{n-k}$.

2. The Example

We consider the $SU(2)$-action on $\mathbb{C}P^2$ defined by $A \cdot [z_0 : z_1 : z_2] =$

$$= \left[z_0\bar{\alpha}^2 + z_1\bar{\alpha}\bar{\beta} + z_2\bar{\beta}^2 : -2z_0\bar{\alpha}\beta + 2z_2\alpha\bar{\beta} + z_1(|\alpha|^2 - |\beta|^2) : z_0\beta^2 - z_1\alpha\beta + z_2\alpha^2\right],$$

where $A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)$, i.e. $\det(A) = |\alpha|^2 + |\beta|^2 = 1$.

Via the 1-1-correspondence of points $[z_0 : z_1 : z_2] \in \mathbb{C}P^2$ with homogeneous polynomials of degree two in two complex variables, up to scaling, $P\begin{pmatrix} x \\ y \end{pmatrix} = z_0x^2 + z_1xy + z_2y^2$, the action defined above corresponds to

$$A \cdot P\begin{pmatrix} x \\ y \end{pmatrix} = P\left(A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

In our example of 1-to-1 transverse lifting, the Lagrangian submanifold L is the orbit of the point $[1 : 0 : 1]$ under this action. It has the form

$$L = SU(2) \cdot [1 : 0 : 1] = \left\{\left[\bar{\alpha}^2 + \bar{\beta}^2 : 2\alpha\bar{\beta} - 2\bar{\alpha}\beta : \alpha^2 + \beta^2\right] \middle| |\alpha|^2 + |\beta|^2 = 1\right\},$$

let us prove that it is in fact Lagrangian.

Proposition 3. *The $SU(2)$-orbit L of $[1 : 0 : 1] \in \mathbb{C}P^2$ is a Lagrangian submanifold of $(\mathbb{C}P^2, \omega_{FS})$.*

Proof. Let us use the abbreviations:

$$X = \Re(\alpha^2 + \beta^2),$$
$$Y = \Im(\alpha^2 + \beta^2),$$
$$Z = 2 \cdot \Im(\alpha\bar{\beta}),$$

then $|\alpha|^2 + |\beta|^2 = 1$ implies $X^2 + Y^2 + Z^2 = 1$. We can write

$$L = \left\{ [X - iY : 2iZ : X + iY] \,\middle|\, X^2 + Y^2 + Z^2 = 1 \right\}.$$

The Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is generated as a real vector space by

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

From these generators we obtain the following curves in $SU(2)$:

$$A_1(t) = Exp\left(t \cdot \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos(t) & i \cdot \sin(t) \\ i \cdot \sin(t) & \cos(t) \end{pmatrix},$$
$$A_2(t) = Exp\left(t \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix},$$
$$A_3(t) = Exp\left(t \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

From

$$A_1(t) \cdot \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha\cos(t) - i\bar{\beta}\sin(t) & \beta\cos(t) + i\bar{\alpha}\sin(t) \\ -\bar{\beta}\cos(t) + i\alpha\sin(t) & \bar{\alpha}\cos(t) + i\beta\sin(t) \end{pmatrix},$$

we obtain the curve in $\mathbb{C}P^2$ corresponding to σ_1 which passes through the point $p = [X - iY : 2iZ : X + iY] \in \mathbb{C}P^2$ at $t = 0$:[3]

$$[\cos^2(t) \cdot (X - iY) - \sin^2(t)(X + iY) + 2\sin(t)\cos(t)Z :$$
$$: 2i\cos^2(t) \cdot Z - 2i\sin^2(t) \cdot Z - 4i\sin(t)\cos(t) \cdot X :$$
$$: \cos^2(t) \cdot (X + iY) - \sin^2(t) \cdot (X - iY) + 2\sin(t)\cos(t) \cdot Z].$$

We will assume that $Z \neq 0$ and leave the other case to be checked by the reader. In this case, we can work in the chart $[w_0 : 1 : w_2] \cong (w_0, w_2) \in \mathbb{C}^2$ and we can write $p = \left[\frac{-Y}{2Z} + i\frac{-X}{2Z} : 1 : \frac{Y}{2Z} + i\frac{-X}{2Z}\right]$. The derivative at $t = 0$ of the curve corresponding to σ_1 in this chart[4] is then:

$$v_1 = \left(\frac{-XY}{Z^2} + i\frac{-X^2 - Z^2}{Z^2}, \frac{XY}{Z^2} + i\frac{-X^2 - Z^2}{Z^2}\right) =: (\mathcal{A}, -\bar{\mathcal{A}})$$

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[3] I.e. we substitute α and β with $\alpha\cos(t) - i\bar{\beta}\sin(t)$ and $\beta\cos(t) + i\bar{\alpha}\sin(t)$ in $[\bar{\alpha}^2 + \bar{\beta}^2 : 2\alpha\bar{\beta} - 2\bar{\alpha}\beta : \alpha^2 + \beta^2]$.

[4] I.e. we divide the first and last projective coordinate by the middle, noticing that in a neighbourhood of $t = 0$ the middle projective coordinate of the curve is non-zero, and differentiate with respect to t at $t = 0$.

For the curves corresponding to σ_2 and σ_3 we respectively obtain the following tangent vectors at p:

$$v_2 = \left(\frac{-Z^2 - Y^2}{Z^2} + i\frac{-XY}{Z^2}, \frac{Z^2 + Y^2}{Z^2} + i\frac{-XY}{Z^2} \right) =: (\mathcal{B}, -\bar{\mathcal{B}}),$$

$$v_3 = \left(\frac{-X}{Z} + i\frac{Y}{Z}, \frac{X}{Z} + i\frac{Y}{Z} \right) =: (\mathcal{C}, -\bar{\mathcal{C}}).$$

Since

$$\det \begin{pmatrix} \Re(\mathcal{A}) & \Re(\mathcal{B}) \\ \Im(\mathcal{A}) & \Im(\mathcal{B}) \end{pmatrix} = \frac{-1}{Z^2} \neq 0,$$

we conclude that v_1 and v_2 are independent over $\mathbb{R}$. Moreover we notice that

$$v_3 = \frac{-Y}{Z} v_1 + \frac{X}{Z} v_2.$$

Therefore we have that $\dim(L) = 2 = \frac{1}{2}\dim(\mathbb{C}P^2)$.

We still need check that the Fubini-Study form ω_{FS} vanishes on L. In this chart we can write

$$\begin{aligned} \omega_{FS}\big|_{p=[w_0:1:w_2]} &= \frac{i}{2}\partial\bar{\partial} \ln(1 + |w_0|^2 + |w_2|^2) = \\ &= \frac{i}{2(1 + |w_0|^2 + |w_2|^2)^2} \cdot \Bigg(\left(1 + |w_2|^2\right) \, dw_0 \wedge d\bar{w}_0 + \\ &\qquad + \left(1 + |w_0|^2\right) \, dw_2 \wedge d\bar{w}_2 + \\ &\qquad - \bar{w}_0 w_2 \, dw_0 \wedge d\bar{w}_2 + \\ &\qquad - w_0 \bar{w}_2 \, dw_2 \wedge d\bar{w}_0 \Bigg). \end{aligned}$$

Notice that at the chosen point p we have $|w_0|^2 = \frac{Y^2}{4Z^2} + \frac{X^2}{4Z^2} = |w_2|^2$. We compute

$$dw_0 \wedge d\bar{w}_0(v_1, v_2) + dw_2 \wedge d\bar{w}_2(v_1, v_2) = (\mathcal{A} \cdot \bar{\mathcal{B}} - \bar{\mathcal{A}} \cdot \mathcal{B}) + \big((-\bar{\mathcal{A}}) \cdot (-\mathcal{B}) - (-\mathcal{A}) \cdot (-\bar{\mathcal{B}})\big) = 0$$

$$dw_0 \wedge d\bar{w}_2(v_1, v_2) = \mathcal{A} \cdot (-\mathcal{B}) - (-\mathcal{A}) \cdot \mathcal{B} = 0$$

$$dw_2 \wedge d\bar{w}_0(v_1, v_2) = -\bar{\mathcal{A}} \cdot \bar{\mathcal{B}} - \bar{\mathcal{A}} \cdot (-\bar{\mathcal{B}}) = 0,$$

therefore $\omega_{FS}\big|_{T_pL} \equiv 0$, so L is Lagrangian. □

Recall that $\mathbb{C}P^1$ can be obtained by reducing $\mathbb{C}P^2$ with respect to the *first* circle action

$$e^{i\phi} \cdot [z_0 : z_1 : z_2] = \left[z_0 : e^{i\phi} z_1 : z_2\right]$$

or the *second* circle action

$$e^{i\phi} \cdot [z_0 : z_1 : z_2] = \left[z_0 : z_1 : e^{i\phi} z_2\right].$$

The transverse intersection condition is clearly satisfied for both reductions. This is because, for $a \in (-2, 1)$, $\mu_i^{-1}(a) \cong S^3$ $(i = 1, 2)$ and there is at least one direction in L that is not contained in the tangent space of this sphere. More details about this and a full description of the fibration of L over the image of the moment map can be found in the appendix.

We want to investigate the intersection of L with reduction-orbits. Let

$$p = \left[\bar{\alpha}^2 + \bar{\beta}^2 : 2\alpha\bar{\beta} - 2\bar{\alpha}\beta : \alpha^2 + \beta^2\right],$$
$$q = \left[\bar{\gamma}^2 + \bar{\delta}^2 : 2\gamma\bar{\delta} - 2\bar{\gamma}\delta : \gamma^2 + \delta^2\right]$$

be two different points in L. We will restrict our attention here only to the case where all homogeneous coordinates are non-zero.

It is easy to see that two different points can be non-trivially related by the first circle action: we can just take $\gamma = -\alpha$ and $\delta = \beta$ and we obtain $p = e^{i\pi} \cdot q$.

On the other hand, let us suppose by contradiction that p and q are non-trivially related by the second circle action. This translates to (in the chart $[1 : w_1 : w_2] \cong (w_1, w_2) \in \mathbb{C}^2$), for $e^{i\phi} \neq 1$:

$$\begin{cases} \dfrac{2\gamma\bar{\delta} - 2\bar{\gamma}\delta}{\bar{\gamma}^2 + \bar{\delta}^2} = \dfrac{2\alpha\bar{\beta} - 2\bar{\alpha}\beta}{\bar{\alpha}^2 + \bar{\beta}^2}, \\ \dfrac{\gamma^2 + \delta^2}{\bar{\gamma}^2 + \bar{\delta}^2} = e^{i\phi}\dfrac{\alpha^2 + \beta^2}{\bar{\alpha}^2 + \bar{\beta}^2}. \end{cases}$$

Since

$$\frac{2\gamma\bar{\delta} - 2\bar{\gamma}\delta}{2\alpha\bar{\beta} - 2\bar{\alpha}\beta} = \frac{4i\Im(\gamma\bar{\delta})}{4i\Im(\alpha\bar{\beta})} \in \mathbb{R},$$

we obtain, from the first equation, that

$$\mathcal{X} \coloneqq \frac{\bar{\gamma}^2 + \bar{\delta}^2}{\bar{\alpha}^2 + \bar{\beta}^2} \in \mathbb{R}.$$

Therefore $\mathcal{X} = \bar{\mathcal{X}}$ and the second equation now reads

$$\mathcal{X} = e^{i\phi}\mathcal{X},$$

which implies that $e^{i\phi} = 1$ and contradicts the assumption.

Hence, L intersects reduction-orbits of the second circle action at most once and is therefore a 1-to-1 transverse lifting of the curve in $\mathbb{C}P^1 = \mu_2^{-1}(a)/e^{i\theta_2}$ given by

$$\left[\bar{\alpha}^2 + \bar{\beta}^2 : 2\alpha\bar{\beta} - 2\bar{\alpha}\beta\right],$$

where we assume that $\left[\bar{\alpha}^2 + \bar{\beta}^2 : 2\alpha\bar{\beta} - 2\bar{\alpha}\beta : \alpha^2 + \beta^2\right] \in \mu_2^{-1}(a)$.

3. Another Point of View

3.1. The Symplectic Lie Group and the Symplectic Lie Algebra. Let (V, Ω) be a symplectic vector space and consider

$$Sp(V) = \left\{g \in GL(V) \;\middle|\; \Omega(gv, gw) = \Omega(v, w) \;\forall v, w \in V\right\} \leq GL(V).$$

Recall that any $2n$-dimensional symplectic vector space is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_0)$, where

$$\Omega_0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

We will just identify $\Omega = \Omega_0$. Consider the map

$$\begin{aligned} f : GL(V) &\to End(V) \\ g &\mapsto g^T \Omega g \end{aligned}$$

and compute for $g \in GL(V)$ and $X \in T_g GL(V) \cong End(V)$:

$$(D_g f)(X) = \frac{d}{dt}\bigg|_{t=0} f(g + tX) = (D_{\mathbb{1}} f)(-\Omega g^T \Omega X).$$

Since $X \mapsto -\Omega g^T \Omega X$ is a vector space isomorphism, we conclude that f has constant rank. Since

$$(D_{\mathbb{1}} f)(X) = \Omega X + X^T \Omega,$$

we can conclude that the rank is non-zero, so that $Sp(V)$ is a Lie group (of dimension $n(2n+1)$) with Lie algebra

$$\mathfrak{sp}(V) = \ker(D_{\mathbb{1}} f) = \left\{ X \in End(V) \;\middle|\; \Omega X + X^T \Omega = 0 \right\}.$$

In particular we see that X belongs to $\mathfrak{sp}(V)$ if and only if we can write

$$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},$$

with B and C symmetric. To simplify later computation we rewrite this remark: let

$$X = \begin{pmatrix} X_{1,1} & \cdots & X_{1,n} & X_{1,n+1} & \cdots & X_{1,n+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{n,1} & \cdots & X_{n,n} & X_{n,n+1} & \cdots & X_{n,n+n} \\ X_{n+1,1} & \cdots & X_{n+1,n} & X_{n+1,n+1} & \cdots & X_{n+1,n+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{n+n,1} & \cdots & X_{n+n,n} & X_{n+n,n+1} & \cdots & X_{n+n,n+n} \end{pmatrix}.$$

Then X is in $\mathfrak{sp}(V)$ if and only if

$$\begin{aligned} X_{k,j} &= -X_{n+j,n+k}, \\ X_{n+j,k} &= X_{k,n+j}, \\ X_{j,n+k} &= X_{n+k,j}, \end{aligned}$$

for $j, k = 1, \dots, n$.

Moreover, it is easy to see that the adjoint representation is given by $Ad_g(X) = gXg^{-1}$.

3.2. **The Action of $Sp(V)$ on V.** In this section we will identify the tangent space of V at any point with V itself. The indices j, k run from 1 to n. The symplectic form is given by $\Omega = dx_j \wedge dx_{n+j}$. Notice that if we introduce the variables $z_j = x_j + ix_{j+n}$, we can also write $\Omega = \frac{i}{2} dz_j \wedge d\bar{z}_j$. We consider the natural action of $Sp(V)$ on V.

Let $X \in \mathfrak{sp}(V)$, then $(\lambda_*(X))_v = \frac{d}{dt}\big|_{t=0} Exp(tX)v = Xv$. Then we see that, for $w \in V \cong T_v V$:

$$(\iota_{\lambda_*(X)} \Omega)_v(w) = \Omega(Xv, w).$$

We can write the vector field

$$\lambda_*(X) = X_{j,k}v_k + X_{j,n+k}v_{n+k}\frac{\partial}{\partial x_j} + X_{n+j,k}v_k + X_{n+j,n+k}v_{n+k}\frac{\partial}{\partial x_{n+j}}$$

and the constant vector field w as

$$w = w_j\frac{\partial}{\partial x_j} + w_{n+j}\frac{\partial}{\partial x_{n+j}}.$$

From this it follows that

$$\Omega(Xv, w) = X_{j,k}v_k w_{j+n} + X_{j,k+n}v_{k+n}w_{j+n} - X_{j+n,k}v_k w_j - X_{j+n,k+n}v_{k+n}w_j.$$

We consider also the function in v:

$$\Omega(Xv, v) = X_{j,k}v_k v_{j+n} + X_{j,k+n}v_{k+n}v_{j+n} - X_{j+n,k}v_k v_j - X_{j+n,k+n}v_{k+n}v_j,$$

and we take its exterior derivative:

$$\begin{aligned} d(\Omega(Xv, v)) =& X_{k,j}v_{k+n} - X_{j+n,k}v_k - X_{k+n,j}v_k - X_{j+n,k+n}v_{k+n}dx_j + \\ &+ X_{j,k}v_k + X_{j,k+n}v_{k+n} + X_{k,j+n}v_{k+n} - X_{k+n,j+n}v_k dx_{j+n}. \end{aligned}$$

We evaluate this 1-form on the constant vector field w:

$$\begin{aligned} (d(\Omega(Xv, v)))(w) &= v_k w_{j+n} \cdot \underbrace{\left(X_{j,k} - X_{k+n,j+n}\right)}_{=2\cdot X_{j,k}} + \\ &+ v_{k+n}w_{j+n} \cdot \underbrace{\left(X_{j,k+n} + X_{k,j+n}\right)}_{=2\cdot X_{j,k+n}} + \\ &+ v_k w_j \cdot \underbrace{\left(-X_{j+n,k} - X_{k+n,j}\right)}_{=-2\cdot X_{j+n,k}} + \\ &+ v_{k+n}w_j \cdot \underbrace{\left(X_{k,j} - X_{j+n,k+n}\right)}_{=-2\cdot X_{j+n,k+n}} = \\ &= 2 \cdot \Omega(Xv, w) = 2 \cdot (\iota_{\lambda_*(X)}\Omega)(w). \end{aligned}$$

This computation implies that the map

$$\begin{aligned} \mu : V &\to \mathfrak{sp}(V)^* \\ v &\mapsto \left(X \mapsto \frac{1}{2}\Omega(Xv, v)\right) \end{aligned}$$

is a candidate moment map. We check invariance, for $g \in Sp(V)$:

$$\begin{aligned} \mu(gv)(X) &= \frac{1}{2}\Omega(Xgv, gv) = \frac{1}{2}\Omega(g^{-1}Xgv, v) = \mu(v)(g^{-1}Xg) = \mu(v)(Ad_{g^{-1}}(X)) = \\ &= Ad_g^*(\mu(v))(X). \end{aligned}$$

Therefore we conclude:

Lemma 4. *The natural action of $Sp(V)$ on (V,Ω) is Hamiltonian with moment map*

$$\mu : V \to \mathfrak{sp}(V)^*$$
$$v \mapsto \Big(X \mapsto \frac{1}{2}\Omega(Xv, v)\Big).$$

3.3. **Symplectic Action of a Lie Group on a Symplectic Vector Space.** Suppose that a Lie group G acts on (V,Ω) in a symplectic way. We can view this action as a representation

$$\rho : G \to Sp(V),$$

where $\rho(g)(v) = g \cdot v$. Recall that [5] the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{D_e\rho} & \mathfrak{sp}(V) \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle Exp} \\ G & \xrightarrow[\rho]{} & Sp(V). \end{array}$$

For $X \in \mathfrak{g}$, the induced vector field on V is given, at $v \in V$, by

$$(\lambda_*(X))_v = \frac{d}{dt}\Big|_{t=0} exp(tX) \cdot v = \frac{d}{dt}\Big|_{t=0} \rho(exp(tX))(v) = \frac{d}{dt}\Big|_{t=0} Exp(D_e\rho(tX))(v) =$$
$$= (D_e\rho)(v).$$

We conclude generalizing lemma 4:

Lemma 5. *Let the Lie group G act on a symplectic vector space (V,Ω) in a symplectic way. Then the action is Hamiltonian with moment map*

$$\mu : V \to \mathfrak{g}^*$$
$$v \mapsto \Big(X \mapsto \frac{1}{2}\Omega((D_e\rho)(X)v, v)\Big).$$

The following will be relevant in the next section:

Corollary 6. *The result also holds if G acts in a unitary way on a complex vector space V equipped with the Hermitian metric h.*

Proof. Consider V as a real vector space and equip it with the symplectic form $\mathfrak{I}(h)$. □

3.4. **The Hamiltonian $SU(2)$-action.** Let $h(z,w) = \bar{z}^T w$ be the standard Hermitian metric on $\mathbb{C}^2$, its imaginary part is the standard symplectic form $\frac{i}{2}dz^1 \wedge d\bar{z}^1 + \frac{i}{2}dz^2 \wedge d\bar{z}^2$. The metric h induces a Hermitian metric h^* on the dual vector space $(\mathbb{C}^2)^*$ for which the elements

$$x : \mathbb{C}^2 \to \mathbb{C}, \qquad\qquad y : \mathbb{C}^2 \to \mathbb{C}$$
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto z_1 \qquad\qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto z_2$$

[5] This holds for any Lie group homomorphism.

form a unitary basis. We can extend h^* to obtain a Hermitian metric $h^{\mathcal{S}^2}$ on the second symmetric power $\mathcal{S}^2(\mathbb{C}^2)^*$. A unitary basis on $\mathcal{S}^2(\mathbb{C}^2)^*$ is then given by[6]

$$x^2 \qquad \sqrt{2}xy \qquad y^2.$$

We denote the coordinates on $\mathcal{S}^2(\mathbb{C}^2)^*$ by (u_0, u_1, u_2), i.e. an element $P \in \mathcal{S}^2(\mathbb{C}^2)^*$ is written as

$$P\begin{pmatrix} x \\ y \end{pmatrix} = u_0 x^2 + u_1\sqrt{2}xy + u_2 y^2.$$

We now want to consider again the $SU(2)$-action on $\mathbb{C}P^2$ from the previous chapter, seeing it as an $SU(2)$-action on $\mathcal{S}^2(\mathbb{C}^2)^*$, i.e. for $A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$,

$$A \cdot P\begin{pmatrix} x \\ y \end{pmatrix} = P\left(A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

It is easily seen that this action is given in this basis by

$$\begin{pmatrix} \bar{\alpha}^2 & \sqrt{2}\bar{\alpha}\bar{\beta} & \bar{\beta}^2 \\ -\sqrt{2}\bar{\alpha}\beta & \alpha\bar{\alpha} - \beta\bar{\beta} & \sqrt{2}\alpha\bar{\beta} \\ \beta^2 & -\sqrt{2}\alpha\beta & \alpha^2 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}.$$

One checks that this matrix is unitary. Therefore:

$$\begin{aligned} \rho : SU(2) &\to U(\mathcal{S}^2(\mathbb{C}^2)^*, h^{\mathcal{S}^2}) \\ A &\mapsto \left[P\begin{pmatrix} x \\ y \end{pmatrix} \mapsto P\left(A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)\right] \end{aligned}$$

(where $U(\mathcal{S}^2(\mathbb{C}^2)^*, h^{\mathcal{S}^2})$ denotes the unitary group of the second symmetric power vector space) is a unitary representation.

With respect to the basis $\{x^2, \sqrt{2}xy, y^2\}$ we get that the symplectic form obtained taking the imaginary part of the Hermitian metric is

$$\Im(h^{\mathcal{S}^2}) = \frac{i}{2} \cdot (du_0 \wedge d\bar{u}_0 + du_1 \wedge d\bar{u}_1 + du_2 \wedge d\bar{u}_2),$$

i.e. the standard symplectic stucture, which we denote Ω. If we see $\mathcal{S}^2(\mathbb{C}^2)^*$ as a 6-dimensional real vector space, which we endow with the symplectic form Ω, we can conclude that ρ is a symplectic representation, i.e. it takes values in $SP(\mathcal{S}^2(\mathbb{C}^2)^*, \Omega)$. By lemma 5, we obtain that the considered $SU(2)$-action is Hamiltonian.

[6]The factor $\sqrt{2}$ is included to make the second basis element of norm 1. In the general nth symmetric power the monomial $x^k y^{n-k}$ is multiplied by $\sqrt{\binom{n}{k}}$: it is a matter of counting permutations in the symmetric power.

3.5. **The Moment Map of the $SU(2)$-action.** In this section we want to compute the moment map of the $SU(2)$-action. In order to achieve this, we will again look at lemma 5. Recall the generators of the Lie algebra $\mathfrak{su}(2)$:

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

First we need compute $(D_e\rho)(\sigma_j)$:

$$(D_e\rho)(\sigma_1) = \frac{d}{dt}\bigg|_{t=0} \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\cdot\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) = \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} 1 & -\sqrt{2}ti & -t^2 \\ -\sqrt{2}ti & 1-t^2 & -\sqrt{2}ti \\ -t^2 & -\sqrt{2}ti & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ -\sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & -\sqrt{2}i & 0 \end{pmatrix}.$$

Similarly:

$$(D_e\rho)(\sigma_2) = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix},$$
$$(D_e\rho)(\sigma_3) = \begin{pmatrix} -2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2i \end{pmatrix}.$$

Therefore at the point $u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}$ the moment map is given at the generators by:

$$\mu(u)(\sigma_1) = \frac{1}{2}\Omega\left(\begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ -\sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & -\sqrt{2}i & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}\right) =$$
$$= \frac{1}{2}\Omega\left(\begin{pmatrix} -\sqrt{2}iu_1 \\ -\sqrt{2}iu_0 - \sqrt{2}iu_2 \\ -\sqrt{2}iu_1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}\right) =$$
$$= \frac{\sqrt{2}}{2}\left(\bar{u}_0 u_1 + u_0\bar{u}_1 + u_2\bar{u}_1 + \bar{u}_2 u_1\right),$$

and

$$\mu(u)(\sigma_2) = \frac{i\sqrt{2}}{2}\left(u_0\bar{u}_1 - \bar{u}_0 u_1 - u_2\bar{u}_1 + \bar{u}_2 u_1\right),$$
$$\mu(u)(\sigma_3) = |u_2|^2 - |u_0|^2.$$

3.6. **The Lagrangian Submanifold $\tilde{L}$ in $\mathbb{C}P^2$.** Now we look at the point $u = \begin{pmatrix}1\\0\\1\end{pmatrix}$ and we notice that $\mu(u) \equiv 0$. We claim that the orbit $SU(2)\cdot u$ is isotropic[7]. In fact, if for $X, Y \in \mathfrak{g}$ we denote by $X^{\#}, Y^{\#}$ their induced vector fields on $\mathcal{S}^2(\mathbb{C}^2)^*$, we obtain that $\Omega_u(X_u^{\#}, Y_u^{\#}) = d(\mu(u)(X))(Y_u^{\#}) = 0$.

Points in the stabilizer of u must satisfy:

$$\bar{\alpha}\beta = \alpha\bar{\beta} \qquad \text{and} \qquad \bar{\alpha}^2 + \bar{\beta}^2 = \alpha^2 + \beta^2 = 1.$$

From this one easily computes that $Stab_u = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in SU(2) \;\middle|\; \alpha, \beta \in \mathbb{R} \right\} \cong S^1$. Topologically the orbit $SU(2)\cdot u$ is an S^2.

We notice that the $SU(2)$-action commutes with the diagonal circle action (which is also Hamiltonian). Therefore, by reducing with respect to the diagonal circle action, we obtain a Hamiltonian $SU(2)$-action on the projective space of $\mathcal{S}^2(\mathbb{C}^2)^*$, which corresponds to $\mathbb{C}P^2$. The isotropic orbit $SU(2)\cdot u$ descends to the isotropic orbit

$$\tilde{L} = \left\{ \left[\bar{\alpha}^2 + \bar{\beta}^2 : \sqrt{2}(\alpha\bar{\beta} - \bar{\alpha}\beta) : \alpha^2 + \beta^2 \right] \;\middle|\; |\alpha|^2 + |\beta|^2 = 1 \right\} \subseteq \mathbb{C}P^2,$$

which is of dimension 2. Therefore $\tilde{L}$ is Lagrangian. Unfortunately, we are off by a factor from the L that was considered in the previous chapter. This however does not compromise the computations exposed earlier.

Appendix

Reparametrization of L. Recall that the $\mathbb{T}^2$-action on $\mathbb{C}P^2$ given by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] = [z_0 : z_1 e^{i\theta_1} : z_2 e^{i\theta_2}]$ is Hamiltonian with moment map

$$\mu : \mathbb{C}P^2 \to \mathbb{R}^2$$
$$[z_0 : z_1 : z_2] \mapsto \left(\frac{-3|z_1|^2}{|z|^2} + 1, \frac{-3|z_2|^2}{|z|^2} + 1\right).$$

Its image - the Delzant polytope - is the isosceles triangle Δ with vertices $(-2,1), (1,-2)$ and $(1,1)$. We see that, in L, it holds:

$$|z_0|^2 = |z_2|^2 = |\alpha|^4 + |\beta|^4 + (\alpha\bar{\beta})^2 + (\bar{\alpha}\beta)^2$$
$$|z_1|^2 = 8|\alpha|^2|\beta|^2 - 4(\alpha\bar{\beta})^2 - 4(\bar{\alpha}\beta)^2$$

and we compute

$$\frac{\frac{-3|z_2|^2}{|z|^2} + 1}{\frac{-3|z_1|^2}{|z|^2} + 1} = \frac{-1}{2}.$$

So the image of L under μ is contained in the line segment $\mathcal{L}$ obtained by intersectig Δ with the line $\{\mu_2 = \frac{-1}{2}\mu_1\}$. The endpoints of this segment are $(-2,1)$ and $(1, \frac{-1}{2})$. When $\alpha = \cos(t)$ and $\beta = \sin(t)$ or when $\alpha = i\cos(t)$ and $\beta = i\sin(t)$, we obtain the point

[7] I.e. the symplectic form vanishes on the orbit.

$[1:0:1] \in L$, which is mapped to $(1, \frac{-1}{2})$ under μ. When $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = \frac{-i}{\sqrt{2}}$, we obtain the point $[0:2i:0] = [0:1:0] \in L$, which is mapped to $(-2,1)$. Since L is connected, we conclude that $\mu(L) = \mathcal{L}$.

The question that arises is: how does the fiber of a point $p \in \ell$ with respect to the $\mathbb{T}^2$-action intersect L? Let $p = (p_1, p_2) \in \mathcal{L}$, hence $p_2 = \frac{-1}{2}p_1$. One easily computes that

$$\mu^{-1}(p) = \left\{\left[\sqrt{\frac{p_1}{6} + \frac{1}{3}} : \sqrt{\frac{1-p_1}{3}}e^{i\theta_1} : \sqrt{\frac{p_1}{6} + \frac{1}{3}}e^{i\theta_2}\right]\right\} \subseteq \mathbb{C}P^2$$

In all of the following α and β are assumed to satisfy $|\alpha|^2 + |\beta|^2 = 1$.

$p = (-2,1)$: We have $\mu^{-1}((-2,1)) = \left\{\left[0 : e^{i\theta_1} : 0\right]\right\} \cong \{pt\}$. We have already seen that, with $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = \frac{-i}{\sqrt{2}}$, we obtain this point.

$\Rightarrow \boldsymbol{L \cap \mu^{-1}((-2,1)) = \mu^{-1}((-2,1)) \cong \{pt\}}$

Notice that if we take $\alpha = \pm i\beta$, we always end up in this case.

$p = (1, \frac{-1}{2})$: We have $\mu^{-1}((1, \frac{-1}{2})) = \left\{\left[\sqrt{\frac{1}{2}} : 0 : \sqrt{\frac{1}{2}}e^{i\theta_2}\right]\right\} = \left\{\left[1 : 0 : e^{i\theta_2}\right]\right\} \cong S^1$. Taking $\alpha = e^{i\frac{\theta_2}{4}}$ and $\beta = 0$ we obtain

$$\left[\bar{\alpha}^2 + \bar{\beta}^2 : 2(\alpha\bar{\beta} - \bar{\alpha}\beta) : \alpha^2 + \beta^2\right] = \left[e^{-i\frac{\theta_2}{2}} : 0 : e^{i\frac{\theta_2}{2}}\right] = \left[1 : 0 : e^{i\theta_2}\right],$$

so any point is taken with an appropriate choice of α and β.

$\Rightarrow \boldsymbol{L \cap \mu_\Delta^{-1}((1, \frac{-1}{2})) = \mu^{-1}((1, \frac{-1}{2})) \cong S^1}$

Notice that we end up in this case if we choose one of α or β to be zero, if we take them both real or both purely imaginary and if $\alpha = \pm\beta$.

Other cases: Since on all of $\mathcal{L}$ the preimages with respect to μ must satisfy $|z_0|^2 = |z_2|^2$, we introduce the variable $\zeta = \frac{|z_0|^2}{|z_1|^2} = \frac{|z_2|^2}{|z_1|^2} \in [0, \infty]$. The cases $\zeta = 0, \infty$ have already been considered. Then we can write, for $\zeta \in (0, \infty)$:

$$\mathcal{L} \setminus \left\{(-2,1), (1, \frac{-1}{2})\right\} \ni p = (p_1, \frac{-1}{2}p_1) = (\frac{-3|z_1|^2}{|z|^2} + 1, \frac{-3|z_2|^2}{|z|^2} + 1) = (\frac{2\zeta - 2}{2\zeta + 1}, \frac{-\zeta + 1}{2\zeta + 1})$$

and

$$\mu^{-1}(p) = \left\{\left[\sqrt{\frac{\zeta}{2\zeta + 1}} : \sqrt{\frac{1}{2\zeta + 1}}e^{i\theta_1} : \sqrt{\frac{\zeta}{2\zeta + 1}}e^{i\theta_2}\right]\right\} = \left\{\left[\sqrt{\zeta} : e^{i\theta_1} : \sqrt{\zeta}e^{i\theta_2}\right]\right\}.$$

We have seen that $\mu(L) = \mathcal{L}$, so we know that there exist some α and β such that

$$\left\{\left[\bar{\alpha}^2 + \bar{\beta}^2 : 2(\alpha\bar{\beta} - \bar{\alpha}\beta) : \alpha^2 + \beta^2\right]\right\} = \left\{\left[\sqrt{\zeta} : e^{i\theta_1} : \sqrt{\zeta}e^{i\theta_2}\right]\right\},$$

therefore

$$\frac{\alpha^2 + \beta^2}{\bar{\alpha}^2 + \bar{\beta}^2} = \frac{\sqrt{\zeta}e^{i\theta_2}}{\sqrt{\zeta}} = e^{i\theta_2} \qquad \text{and} \qquad \frac{2\alpha\bar{\beta} - 2\bar{\alpha}\beta}{\bar{\alpha}^2 + \bar{\beta}^2} = \frac{1}{\sqrt{\zeta}}e^{i\theta_1}.$$

From the first equality it follows, for some $r>0$ and $k\in\mathbb{Z}$, that $\alpha^2+\beta^2 = re^{i\frac{\theta_2}{2}+i\pi k}$. Consequently, from the second

$$2\alpha\bar{\beta}-2\bar{\alpha}\beta = \frac{r}{\sqrt{\zeta}}e^{i(\theta_1-\frac{\theta_2}{2}+\pi k)}$$

Since $2\alpha\bar{\beta}-2\bar{\alpha}\beta = 4i\Im(\alpha\bar{\beta})$ is purely imaginary, we conclude that

$$\pm\frac{\pi}{2} = \theta_1-\frac{\theta_2}{2}+\pi k \qquad\Longleftrightarrow\qquad \theta_1 = \frac{\pi}{2}+\frac{\theta_2}{2}+\pi k,$$

i.e. the choice of $\theta_2\in[0,2\pi)$ restricts the choice of θ_1 to only two possibilities in $[0,2\pi)$. Better: the choice of $\theta_2\in[0,4\pi)$ restricts the choice of θ_1 to a unique possibility $\theta_1 = \frac{\pi}{2}+\frac{\theta_2}{2}$.

However we can choose θ_2 freely. In fact, let $\alpha' := e^{i\frac{\tau}{4}}\alpha$ and $\beta' := e^{i\frac{\tau}{4}}\beta$. We have

$$\frac{\alpha'^2+\beta'^2}{\bar{\alpha}'^2+\bar{\beta}'^2} = e^{i\tau}e^{i\theta_2} \qquad\text{and}\qquad \frac{2\alpha'\bar{\beta}'-2\bar{\alpha}'\beta'}{\bar{\alpha}'^2+\bar{\beta}'^2} = e^{i\frac{\tau}{2}}\frac{1}{\sqrt{\zeta}}e^{i\theta_1}.$$

Therefore

$$\left\{\left[\bar{\alpha}'^2+\bar{\beta}'^2 : 2(\alpha'\bar{\beta}'-\bar{\alpha}'\beta') : \alpha'^2+\beta'^2\right]\right\} = \left\{\left[\sqrt{\zeta} : e^{i(\theta_1+\frac{\tau}{2})} : \sqrt{\zeta}e^{i(\theta_2+\tau)}\right]\right\}.$$

So letting τ vary in $[0,4\pi)$ we obtain a different point than $\left\{\left[\sqrt{\zeta} : e^{i\theta_1} : \sqrt{\zeta}e^{i\theta_2}\right]\right\}$, however $\mu\left(\left[\sqrt{\zeta} : e^{i(\theta_1+\frac{\tau}{2})} : \sqrt{\zeta}e^{i(\theta_2+\tau)}\right]\right) = p$.

$$\boldsymbol{\Rightarrow L\cap\mu^{-1}(p) = \left\{\left[\sqrt{\zeta} : e^{i(\frac{\pi}{2}+\frac{\theta_2}{2})} : \sqrt{\zeta}e^{i\theta_2}\right]\middle|\theta_2\in[0,4\pi)\right\}\cong S^1}$$

The Example. Let $c\in(\frac{-1}{2},1)$ and suppose that $\mu_2([z_0:z_1:z_2]) = c$. This implies

$$|z_2|^2 = \frac{1-c}{2+c}\cdot\left(|z_0|^2+|z_1|^2\right) > 0,$$

and hence

$$\mu_2^{-1}(c) = \left\{[z_0:z_1:z_2] = [\frac{z_0}{z_2}:\frac{z_1}{z_2}:1] \;\middle|\; \left|\frac{z_0}{z_2}\right|^2+\left|\frac{z_1}{z_2}\right|^2 = \frac{2+c}{1-c} > 0\right\}\cong S^3.$$

If $[\frac{z_0}{z_2}:\frac{z_1}{z_2}:1]\in\mu_2^{-1}(c)$ we obtain

$$x := \mu_1([\frac{z_0}{z_2}:\frac{z_1}{z_2}:1]) = (c-1)\cdot\frac{|z_1|^2}{|z_2|^2}+1\in[-1-c,1],$$

from which we deduce that

$$\frac{|z_1|^2}{|z_2|^2} = \frac{1-x}{1-c} \qquad\text{and}\qquad \frac{|z_0|^2}{|z_2|^2} = \frac{1+x+c}{1-c}.$$

Therefore

$$\mu_2^{-1}(c) = \left\{\left[\sqrt{\frac{1+x+c}{1-c}}e^{i\delta_0} : \sqrt{\frac{1-x}{1-c}}e^{i\delta_1} : 1\right] \;\middle|\; x\in[-1-c,1]\;\delta_0,\delta_1\in[0,2\pi)\right\}.$$

Without loss of generality we choose $c=0$ in what follows.

The $\mathbb{T}^2$-action on $\mathbb{C}P^2$ induces, via the inclusion

$$\begin{aligned} i_2 : S^1 &\hookrightarrow \mathbb{T}^2 \\ e^{i\phi} &\mapsto (1, e^{i\phi}), \end{aligned}$$

a circle action that is given by

$$e^{i\phi} \cdot [z_0 : z_1 : z_2] = [z_0 : z_1 : z_2 e^{i\phi}] = [z_0 e^{-i\phi} : z_1 e^{-i\phi} : z_2],$$

which has moment map $\mu_2 : \mathbb{C}P^2 \to \mathbb{R}$. The reduced space at 0 with respect to this circle action is $\mu_2^{-1}(0)/S^1 \cong \mathbb{C}P^1$. First we want to check that $L \pitchfork \mu_2^{-1}(0)$. This intersection happens above the point $(0,0)$ in the Delzant polytope, where $\zeta = \frac{|z_0|^2}{|z_1|^2} = \frac{|z_2|^2}{|z_1|^2} = 1$ and where $x = 0$. In a neighbourhood of $x = 0$:

$$\mu_2^{-1}(0) = \left\{ \left[\sqrt{1+x} e^{i\delta_0} : \sqrt{1-x} e^{i\delta_1} : 1 \right] \;\middle|\; \delta_0, \delta_1 \in [0, 2\pi) \right\},$$

and from the reparametrization obtained in the previous section we can write, in a neighbourhood of $\zeta = 0$:

$$L = \left\{ \left[\sqrt{\zeta} : e^{i(\frac{\pi}{2} + \frac{\theta_2}{2})} : \sqrt{\zeta} e^{i\theta_2} \right] \;\middle|\; \theta_2 \in [0, 4\pi) \right\} = \left\{ \left[e^{-i\theta_2} : \frac{1}{\sqrt{\zeta}} e^{i(\frac{\pi}{2} - \frac{\theta_2}{2})} : 1 \right] \;\middle|\; \theta_2 \in [0, 4\pi) \right\}.$$

Looking at this description in the chart $[w_0 : w_1 : 1] \cong (w_0, w_1) \in \mathbb{C}^2$ it is clear that this intersection is transversal.

It remains to check that L intersects the "reduction-orbits" just once. We have

$$L \cap \mu^{-1}((0,0)) = \left\{ \left[e^{-i\theta_2} : e^{i(\frac{\pi}{2} - \frac{\theta_2}{2})} : 1 \right] \;\middle|\; \theta_2 \in [0, 4\pi) \right\}.$$

and clearly, for $\phi \in [0, 2\pi)$,

$$\left[e^{-i\theta_2 - i\phi} : e^{i(\frac{\pi}{2} - \frac{\theta_2}{2}) - i\phi} : 1 \right] \in L \quad \Longleftrightarrow \quad \frac{\pi}{2} - \frac{\theta_2}{2} - \phi = \frac{\pi}{2} - \frac{\theta_2 + \phi}{2} \quad \Longleftrightarrow \quad \phi = 0.$$

Hence, L intersects orbits only once. If we denote by π_2 the projection $\mu_2^{-1}(0) \to \mu_2^{-1}(0)/S^1 \cong \mathbb{C}P^1$, we conclude that L is a 1-to-1 transverse lifting of the Lagrangian curve

$$\begin{aligned} \pi_2(L \cap \mu_2^{-1}(0)) &= \left\{ \left[e^{-i\theta_2} : e^{i(\frac{\pi}{2} - \frac{\theta_2}{2})} \right] \;\middle|\; \theta_2 \in [0, 4\pi) \right\} = \left\{ \left[1 : e^{i(\frac{\pi}{2} + \frac{\theta_2}{2})} \right] \;\middle|\; \theta_2 \in [0, 4\pi) \right\} \cong \\ &\cong \left\{ \left[1 : e^{\delta} \right] \;\middle|\; \delta \in [0, 2\pi) \right\} \subseteq \mathbb{C}P^1. \end{aligned}$$

The Non-Example. The example of the previous section was obtained by reducing with respect to the circle action induced by the inclusion of the second circle factor of $\mathbb{T}^2$. The reduction with respect to the other inclusion,

$$\begin{aligned} i_1 : S^1 &\hookrightarrow \mathbb{T}^2 \\ e^{i\phi} &\mapsto (e^{i\phi}, 1), \end{aligned}$$

does not provide an example of a 1-to-1 transverse lifting, but solely a 2-to-1 transverse lifting. Following the same procedure as earlier we obtain:

$$\mu_1^{-1}(0) = \left\{ \left[\sqrt{1+x}e^{i\delta_0} : 1 : \sqrt{1-x}e^{i\delta_2} \;\middle|\; x \in [-1,1] \right] \delta_0, \delta_2 \in [0, 2\pi) \right\},$$

and in a neighbourhood of $\zeta = 0$:[8]

$$L = \left\{ \left[\sqrt{\zeta} : e^{i(\frac{\pi}{2}+\frac{\theta_2}{2})} : \sqrt{\zeta}e^{i\theta_2} \right] \;\middle|\; \theta_2 \in [0, 4\pi) \right\} = \left\{ \left[\sqrt{\zeta}e^{-i(\frac{\theta_2}{2}+\frac{\pi}{2})} : 1 : \sqrt{\zeta}e^{i(\frac{\theta_2}{2}-\frac{\pi}{2})} \right] \;\middle|\; \theta_2 \in [0, 4\pi) \right\},$$

so we can again conclude that $L \pitchfork \mu_1^{-1}(0)$.

However,

$$L \cap \mu^{-1}((0,0)) = \left\{ \left[e^{i(\frac{-\theta_2}{2}-\frac{\pi}{2})} : 1 : e^{i(\frac{\theta_2}{2}-\frac{\pi}{2})} \right] \;\middle|\; \theta_2 \in [0, 4\pi) \right\},$$

and, for $\phi \in [0, 2\pi)$,

$$\left[e^{i(\frac{-\theta_2}{2}-\frac{\pi}{2}-\phi)} : 1 : e^{i(\frac{\theta_2}{2}-\frac{\pi}{2}-\phi)} \right] \in L \iff \theta_2 + 2\phi = \theta_2 - 2\phi \iff \phi = 0 \text{ or } \phi = \pi.$$

So, in this case L intersects orbits twice. If we use the notation $\pi_1 : \mu_1^{-1}(0) \to \mu_1^{-1}(0)/S^1 \cong \mathbb{C}P^1$, we obtain that L is a 2-to-1 transverse lifting of

$$\pi_1(L \cap \mu_1^{-1}(0)) = \left\{ \left[e^{i(\frac{-\theta_2}{2}-\frac{\pi}{2})} : e^{i(\frac{\theta_2}{2}-\frac{\pi}{2})} \right] \;\middle|\; \theta_2 \in [0, 4\pi) \right\} = \left\{ \left[1 : e^{i\theta_2} \right] \;\middle|\; \theta_2 \in [0, 4\pi) \right\} \subseteq \mathbb{C}P^1.$$

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[8]The intersection happens again above $(0,0)$.