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Floer Homology in Symplectic Geometry

Master Thesis

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Abstract

The aim of this thesis is to present a proof of a version the Arnold conjecture in the case of a closed, symplectically aspherical manifold. In the first chapter we collect some basic material related to symplectic and almost complex manifolds. The only more advanced results are those related to the topological properties of the linear symplectic group which are needed for the definition of the Conley-Zehner index. In the second chapter we describe the construction of the Floer homology groups in full detail in the case where the coefficients are taken over $\mathbb{Z}_2$. This construction is done by considering the chain complex generated by the non-degenerate critical points of a Hamiltonian action functional on the space 1-periodic, smooth loops in M. It is graded by a type of Maslov index. The boundary operator is defined by counting the unparameterised solutions of the pseudo-gradient flow associated to the action functional on this loop space connecting two critical points. We conclude by demonstrating the equivalence of Morse and Floer homology which completes the proof of the Arnold conjecture under the above assumptions.

Contents

Introduction

In symplectic geometry one studies smooth manifolds M that are additionally endowed with a symplectic structure ω. That means that ω is a closed, non-degenerate, alternating two-form. To learn more about such pairs (M, ω) one can consider the kind of structure and dynamics they can support. Associated to every vector field $X \in \Gamma(TM)$ is a (local) flow ψ defined by the equation

$$\frac{d}{dt}\psi = X(\psi)$$

and the initial condition $\psi_0 = \mathrm{Id}_M$. This flow is guaranteed to be global if we assume compactness of M. Diffeomorphisms of this type are of particular interest when they are induced from a smooth function $H \in C^\infty(M)$ or more generally $H \in C^\infty(\mathbb{R} \times M)$. In this case the Hamiltonian vector field X_H defined by $\omega(\cdot, dH)$ through the symplectic form ω gives rise to symplectic diffeomorphism. Restricting now to the symplectic flow coming from a 1-periodic Hamiltonian $H \in C^\infty(S^1 \times M)$ one may especially inquire about the number of periodic orbits of this flow and how this number might depend on H. If H it time-independent one obtains an easy lower bound on this number since then every critical point of H leads to a constant and thus periodic orbit.

One can also rephrase this question about the number of fixed points in terms of Lagrangian intersections. A symplectic manifold M embeds diagonally as a Lagrangian submanifold in the product $M \times M$ as the diagonal. Then the number of fixed points of the 1-flow map ψ_1 coincides with the intersection number of M with the graph of ψ_1 in $M \times M$.

In the context of Lagrangian intersection Arnold formulated in [1] the conjecture that this number of fixed points satisfies the inequality

$$|fix(\psi_1)| \geq \sum_i \dim H_i(M, \mathbb{Q}).$$

Its verification has been a fruitful source of mathematics.

The 1-periodic solutions of the above flow equation can also be understood as the critical points of a suitable action functional $\mathcal{A}_H$ that is defined on the free loop space of M or more generally on a suitable covering of that space. Floer's approach to proving the Arnold conjecture was then to develop an infinite-dimensional variant of classical Morse theory for this action functional. One constructs a chain complex generated by the critical points of the action and defines a corresponding boundary operator by counting the pseudo-gradient flow lines connecting to such points. This gives rise to a homology theory from which the conjecture immediately follows after one has shown that it degenerates to regular Morse homology in a certain limiting case.

Historically this conjecture was first proved in a number of special cases before Floer's approach provided a new impact which eventually led to a proof in full generality. In [5] Eliashberg verified it in the case of Riemannian surfaces and Conley and Zehnder in the case of the $2n$-torus in [3]. Departing from these specific types of manifolds one can consider the following hierarchy of more general classes of manifolds. The simplest of these is the case when $\pi_2(M) = 0$ which is called aspherical. This is also the setting in which Floer originally established the conjecture in [7],[8] and [9]. More generally one can then study manifolds that satisfy the condition that

$$\int_{S^2} v^* c_1(M) = \tau \int_{S^2} v^* \omega$$

for every smooth map $v \in C^\infty(S^2, M)$ for a fixed real number τ. Here it is important to distinguish between the cases where $\tau > 0$, $\tau = 0$ and $\tau < 0$. The case where $\tau = 0$ is called symplectically aspherical. This is the assumption under which the Arnold conjecture will treated in this thesis. The case $\tau > 0$ is called monotone and was also resolved by Floer.

Going more generally still one can consider the weakly monotone case which means that there is no $v \in \pi_2(M)$ such that

$$\int_{S^2} v^* \omega > 0 \text{ and } 3 - n \leq \int_{S^2} v^* c_1(M) < 0$$

where n is half the dimension of M. For this case it was confirmed by Hofer, Salamon and Ono in [12] and [18].

Beyond proving the Arnold conjecture there is also a version of Floer theory that is associated to pairs of Lagrangian submanifolds of a given symplectic manifold. As described above this can be considered as a more general version of the Floer homology associated to Hamiltonian fixed points.

Finally, Floer's idea to use a type of infinite-dimensional version of Morse theory applied to an action functional has also been fruitful in areas outside of symplectic geometry. In the context of low-dimensional topology one can develop a Floer theory for Donaldson's instanton theory and following the develop of the Seiberg-Witten approach to the monopole equations in gauge theory a corresponding Seiberg-Witten-Floer theory. Both the Donaldson and Seiberg-Witten invariants are defined by moduli spaces of certain geometric partial differential equations. These can be understood as critical points of suitable action functionals, namely the Yang-Mills and Chern-Simons-Dirac actional functional, respectively.

Chapter 1

Basics on Symplectic Geometry

In this first chapter we present some basic material from the theory of symplectic and almost complex manifolds. The main references for this part were [4] and [13].

1.1 Linear Theory

1.1.1 Symplectic Vector Spaces

We begin by discussing linear symplectic geometry. This will be globalised to manifolds in the next section by way of the tangent bundle. Let $V \in \mathsf{Vect}_{\mathbb{R}}$ be a vector space. For a bilinear form $b \in V^* \otimes V^*$ we denote by $\widetilde{b}$ the induced map in $\mathsf{Hom}(V, V^*,)$ given by $\widetilde{b}(v) = b(v, \cdot)$.

Definition 1.1 *Let $V \in \mathsf{Vect}_{\mathbb{R}}$ and $\omega \in \Lambda^2(V^*)$. Then ω is called* symplectic *if $\widetilde{\omega} \in \mathsf{Mono}(V, V^*)$. In this case, we say that (V, ω) is a* symplectic vector space. *The set of all such pairs is denoted by* $\mathsf{SympVect}$.

The additional structure of a symplectic vector space allows one to distinguish the following classes of linear subspaces U of V.

Definition 1.2 *Let $(V, \omega) \in \mathsf{SympVect}$ and let $U \subset V$ be a linear subspace. Then we make the following definitions.*

1. *Denote by $U^\omega = \mathrm{Ker}(\widetilde{\omega})$ the ω-*orthogonal complement *of U.*
2. *The space U is called* isotropic *if $U \subset U^\omega$. The set of all isotropic subspaces of (V, ω) is denoted by* $\mathrm{Isotrop}(V, \omega)$.
3. *The space U is called* coisotropic *if $U^\omega \subset U$. The set of all coisotropic subspaces of (V, ω) is denoted by* $\mathrm{Coisotrop}(V, \omega)$.
4. *The space U is called* lagrangian *if it is isotropic and coisotropic. The set of all lagrangian subspaces of (V, ω) is denoted by* $\mathrm{Lag}(V, \omega)$.

In a symplectic vector space there are some bases that are singled out by the symplectic form. With respect to these bases the matrix representing the symplectic form is diagonal.

Proposition 1.1 *Let* $(V, \omega) \in \mathsf{SympVect}$ *such that* $\dim V = 2n$*. Then there exists a basis* $u_1, \ldots, u_n, v_1, \ldots, v_n$ *such that*

1. $\omega(u_j, u_k) = \omega(v_j, v_k) = 0$.
2. $\omega(u_j, v_k) = \delta_{jk}$.

Such a basis is called a symplectic basis*.*

Proof:
$\langle 1 \rangle 1.$ If $n = 1$ the statement holds.
  Proof:
  $\langle 2 \rangle 1.$ Since $\omega \in \mathsf{Mono}(V, V^*)$ there exist $u_1, v_1 \in V$ such that $\omega(u_1, v_1) = 1$.
$\langle 1 \rangle 2.$ If the statement holds for a $n - 1$ then it holds for n
  Proof:
  $\langle 2 \rangle 1.$ Let: Let $u_1, v_1 \in V$ be as above and $W = \mathrm{span}\{u_1, v_1\}^{\omega}$.
  $\langle 2 \rangle 2.$ $\dim W = 2(n - 1)$.
  $\langle 2 \rangle 3.$ By assumption there exist $u_2, \ldots, u_n, v_2, \ldots, v_n \in W$ satisfying item 1 and item 2.
  $\langle 2 \rangle 4.$ The vectors $u_1, \ldots, u_n, v_1, \ldots, v_n$ satisfy item 1 and item 2.
$\langle 1 \rangle 3.$ Q.E.D.

1.1.2 The Linear Symplectic Group

The addition of a symplectic structure to a vector space warrants the introduction of a corresponding symmetry group.

Definition 1.3 *Let* $(V, \omega) \in$ *SympVect. We define*

$$\mathsf{Sp}(V, \omega) = \{A \in \mathsf{End}(V) \ : \ \omega = A^* \omega\}. \tag{1.1}$$

to be the symplectic linear group *of linear maps that preserve the symplectic form. If* $V = \mathbb{R}^{2n}$ *and is endowed with the standard symplectic structure we write* $\mathrm{Sp}(2n)$ *instead.*

According to Proposition 1.1 we can always choose a basis in which the symplectic form is represented by the matrix

$$J = \begin{bmatrix} 0 & \mathrm{Id}_{\mathbb{R}^n} \\ -\mathrm{Id}_{\mathbb{R}^n} & 0 \end{bmatrix}.$$

Then the condition in Equation 1.1 becomes $A^T J A = J$. Hence, we can focus our study of the symplectic group on the model space $(\mathbb{R}^{2n}, J)$. Let us investigate how $\mathrm{Sp}(2n)$ relates to the other standard matrix groups. Recall the following definitions.

Definition 1.4

$$\mathrm{O}(n) = \{A \in \mathbb{R}^{n\times n} \ : \ A^T A = \mathrm{Id}_{\mathbb{R}^n}\} \tag{1.2}$$
$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \ : \ \det(A) = 1\} \tag{1.3}$$
$$\mathrm{U}(n) = \{A \in \mathbb{C}^{n\times n} \ : \ A^* A = \mathrm{Id}_{\mathbb{C}^n}\} \tag{1.4}$$
$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \ : \ \det(A) = 1\} \tag{1.5}$$

Assume that $\mathbb{R}^{2n}$ is endowed with the standard coordinates denoted by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. This can be identified with $\mathbb{C}^n$ by sending $z = x + iy$ to (x, y). Multiplication by i in $\mathbb{C}^n$ corresponds to the map $(x, y) \mapsto (-y, x)$ in $\mathbb{R}^{2n}$ under this identification. Hence, we see that this map corresponds to the matrix

$$J_0 = \begin{bmatrix} 0 & -\mathrm{Id}_{\mathbb{R}^n} \\ \mathrm{Id}_{\mathbb{R}^n} & 0 \end{bmatrix}.$$

A complex matrix $A \in \mathbb{C}^{n\times n}$ can be thought of as a real matrix in $\mathbb{R}^{2n\times 2n}$ that is compatible with the complex structure. This means that A as a real matrix under the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ satisfies $AJ_0 = J_0 A$. Hence, we can consider all the groups $\mathrm{O}(2n), \mathrm{U}(n), \mathrm{GL}(n, \mathbb{C})$ and $\mathrm{Sp}(2n)$ as subgroups of $\mathbb{R}^{2n\times 2n}$.

Lemma 1.1 *$A \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$ if and only if there exist $X, Y \in \mathbb{R}^{n\times n}$ such that*

1. $A = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \in \mathrm{GL}(2n)$.
2. $X^T Y = Y^T X$ *and* $X^T X + Y^T Y = \mathrm{Id}_{\mathbb{R}^n}$.

PROOF:
⟨1⟩1. LET: $A \in \mathbb{R}^{2n}$ such that item 1 and item 2 hold.
⟨1⟩2. $A \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$.
  PROOF:
  ⟨2⟩1. $A^T J A = \begin{bmatrix} X^T & Y^T \\ -Y^T & X^T \end{bmatrix} \begin{bmatrix} 0 & \mathrm{Id}_{\mathbb{R}^n} \\ -\mathrm{Id}_{\mathbb{R}^n} & 0 \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} = J$ using item 2.
  ⟨2⟩2. $A^T A = \begin{bmatrix} X^T & Y^T \\ -Y^T & X^T \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} = \mathrm{Id}_{\mathbb{R}^{2n}}$ using item 2.
⟨1⟩3. LET: $A \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$.
⟨1⟩4. A satisfies item 1 and item 2.
  PROOF:
  ⟨2⟩1. LET: $A = \begin{bmatrix} X & Y' \\ Y & X' \end{bmatrix}$.
  ⟨2⟩2. $A^T J A = J \iff -JA^T J = A^{-1} \implies A^{-1} = \begin{bmatrix} (X')^T & -(Y')^T \\ Y^T & X^T \end{bmatrix}$.

⟨2⟩3. $A \in O(2n) \implies A^T = A^{-1}$.
⟨2⟩4. $X = X'$ and $Y = -Y'$.
⟨2⟩5. $A \in Sp(2n) \implies$ item 2.
⟨1⟩5. Q.E.D.

Proposition 1.2 *The following equalities between the above matrix groups hold.*

$$Sp(2n) \cap O(2n) = Sp(2n) \cap GL(n, \mathbb{C}) = O(2n) \cap GL(n, \mathbb{C}) = U(n) \quad (1.6)$$

Proof:
⟨1⟩1. Let: $A \in \mathbb{R}^{2n \times 2n}$.
⟨1⟩2. $Sp(2n) \cap O(2n) = Sp(2n) \cap GL(n, \mathbb{C})$
Proof:
⟨2⟩1. $A^T J A = J$ and $A^T A = Id_{\mathbb{R}^{2n}}$ imply $A J_0 = J_0 A$.
⟨2⟩2. $A^T J A = J$ and $A J_0 = J_0 A$ imply $A^T A = Id_{\mathbb{R}^{2n}}$.
⟨1⟩3. $Sp(2n) \cap GL(n, \mathbb{C}) = O(2n) \cap GL(n, \mathbb{C})$.
Proof:
⟨2⟩1. $A^T J A = J$ and $A J_0 = J_0 A$ imply $A^T A = Id_{\mathbb{R}^{2n}}$.
⟨2⟩2. $A^T A = Id_{\mathbb{R}^{2n}}$ and $A J_0 = J_0 A = J$ imply $A^T J A$.
⟨1⟩4. $Sp(2n) \cap O(2n) = U(n)$.
Proof:
⟨2⟩1. Let: $Z = X + iY \in \mathbb{C}^{n \times n}$.
⟨2⟩2. $Z \in U(n) \iff X^T Y - Y^T X = 0$ and $X^T X + Y^T Y = Id_{\mathbb{R}^n}$.
⟨2⟩3. $Z \in U(n) \iff A = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \in Sp(2n) \cap O(2n)$ by Lemma 1.1.
⟨1⟩5. Q.E.D.

Proposition 1.3 *There is a deformation retract $r : Sp(2n) \times [0,1] \to U(n)$.*

Proof:
⟨1⟩1. Suffices: $Sp(2n)/U(n) \simeq *$.
Proof:
⟨2⟩1. Let: $A \in Sp(2n)$.
⟨2⟩2. $\exists! P \in Sym(2n), \exists! Q \in O(2n) \ : \ A = PQ$.
⟨2⟩3. $P = (AA^T)^{1/2}$.
⟨2⟩4. Let: $r : Sp(2n) \times [0,1] \to Sp(2n)$ be defined by $r(t, A) = (AA^T)^{-t/2} A$.
⟨1⟩2. Q.E.D.

This result implies in particular that $Sp(2n)$ and $U(n)$ have the same fundamental group.

Definition 1.5 *We define the following subsets of* $Sp(2n)$.

1. $Sp^*(2n) = \{A \in Sp(2n) \ : \ \det(A - Id_{\mathbb{R}^{2n}}) \neq 0\}$.
2. $Sp^+(2n) = \{A \in Sp(2n) \ : \ \det(A - Id_{\mathbb{R}^{2n}}) > 0\}$.
3. $Sp^-(2n) = \{A \in Sp(2n) \ : \ \det(A - Id_{\mathbb{R}^{2n}}) < 0\}$.

Let us establish some basic results about these sets that will be used throughout the remainder of this text. We begin by discussing the spectral theory of symplectic matrices.

Proposition 1.4 *Let $A \in \mathrm{Sp}(2n)$. Then all eigenvalues of A appear either as pairs $\lambda, 1/\lambda \in \mathbb{R}$, $\lambda, \overline{\lambda} \in S^1$ or quadruples $\lambda, \overline{\lambda}, 1/\lambda, 1/\overline{\lambda} \in \mathbb{C}$.*

PROOF:
⟨1⟩1. See section 2.2 in[13].
⟨1⟩2. Q.E.D.

For matrix $A \in \mathbb{R}^{m\times m}$ or $\mathbb{C}^{m\times m}$ we denote by $\sigma(A)$ its set of eigenvalues, and for $\lambda \in \sigma(A)$ we denote the corresponding eigenspace by $E_\lambda(A)$ or E_λ for short.

Lemma 1.2 *Let $A \in \mathrm{Sp}^*(2n)$ and $\lambda \in \sigma(A)$ with a multiplicity larger than 1. Then there exists a path $\gamma : [0,1] \to \mathrm{Sp}^*(2n)$ such that $\gamma(1)$ has $\lambda \in \sigma(\gamma(1))$ with multiplicity decreased by 1 and such that no other eigenvalue of A appears with an increased multiplicity in $\sigma(\gamma(1))$.*

PROOF:
⟨1⟩1. CASE: $\lambda \in \mathbb{C}\backslash(S^1 \cup \mathbb{R})$.
PROOF:
⟨2⟩1. $1/\lambda \in \sigma(A)$.
⟨2⟩2. LET: $v_\lambda \in E_\lambda$ and $v_{1/\lambda} \in E_{1/\lambda}$ such that $\omega_0(v_\lambda, v_{1/\lambda}) = 1$.
⟨2⟩3. LET: $V = \mathrm{span}\{v_\lambda, v_{1/\lambda}, \overline{v_\lambda}, \overline{v_{1/\lambda}}\} \subset \mathbb{C}^{2n}$.
⟨2⟩4. The set $\{v_\lambda, v_{1/\lambda}, \overline{v_\lambda}, \overline{v_{1/\lambda}}\}$ is a symplectic basis for V.
⟨2⟩5. LET: $\beta \in C^\infty([0,\varepsilon], \mathbb{C}^*)$ with $\beta(0) = 1$ and $\beta(s)\lambda \in \mathbb{C}\backslash(S^1 \cup \mathbb{R})$.
⟨2⟩6. DEFINE: $\gamma : [0,\varepsilon] \to \mathbb{C}^{2n\times 2n} = V \oplus V^\omega$ by setting

$$\begin{aligned}
\gamma(s)(v_\lambda) &= \beta(s)\lambda v_\lambda \\
\gamma(s)(v_{1/\lambda}) &= \frac{1}{\beta(s)\lambda} v_{1/\lambda} \\
\gamma(s)(\overline{v_\lambda}) &= \overline{\beta(s)}\overline{\lambda}\overline{v_\lambda} \\
\gamma(s)(\overline{v_{1/\lambda}}) &= \frac{1}{\overline{\beta(s)}\overline{\lambda}}\overline{v_{1/\lambda}} \\
\gamma(s)|_{V^\omega} &= A|_{V^\omega}
\end{aligned}$$

⟨2⟩7. $\forall s \in [0,\varepsilon]\ :\ \gamma(s) \in \mathrm{Sp}(2n)$.
⟨2⟩8. For $\mu \in \sigma(A)$

$$\begin{aligned}
\gamma(s)|_{G_\mu} &= A|_{G_\mu} \text{ if } \mu \notin \{\lambda, 1/\lambda, \overline{\lambda}, \overline{1/\lambda}\} \\
\gamma(s)|_{G_\mu \cap V^\omega} &= A|_{G_\mu \cap V^\omega} \text{ if } \mu \in \{\lambda, 1/\lambda, \overline{\lambda}, \overline{1/\lambda}\}.
\end{aligned}$$

⟨1⟩2. CASE: $\lambda \in S^1\backslash\mathbb{R}$
PROOF:
⟨2⟩1. LET: $v_\lambda \in E_\lambda$.
⟨2⟩2. LET: $V = \mathrm{span}\{v_\lambda, \overline{v_\lambda}\}$.

⟨2⟩3. Case: $\omega(v_\lambda, \overline{v_\lambda}) \neq 0$.
Proof:
⟨3⟩1. Define: $\gamma : [0,\varepsilon] \to \mathbb{C}^{2n\times 2n} = V \oplus V^\omega$ by setting
$$\gamma(s)(v_\lambda) = \beta(s)\lambda v_\lambda$$
$$\gamma(s)(\overline{v_\lambda}) = \overline{\beta(s)\lambda v_\lambda}$$
$$\gamma(s)|_{V^\omega} = A|_{V^\omega}$$
⟨2⟩4. Case: $\omega(v_\lambda, \overline{v_\lambda}) = 0$.
Proof:
⟨3⟩1. $\exists w_\lambda \in G_\lambda(A) \ : \ \omega(v_\lambda, \overline{w_\lambda}) = 1$ and $\omega(w_\lambda, \overline{w_\lambda}) = 0$.
Proof:
⟨4⟩1. ω is symplectic on $G_\lambda \oplus G_{\overline{\lambda}}$ and.
⟨3⟩2. Define: $\gamma : [0,\varepsilon] \to \mathbb{C}^{2n\times 2n} = V \oplus V^\omega$ by setting
$$\gamma(s)(v_\lambda) = \beta(s)\lambda v_\lambda$$
$$\gamma(s)(w_\lambda) = \frac{1}{\beta(s)} w_\lambda$$
$$\gamma(s)(\overline{v_\lambda}) = \overline{\beta(s)\lambda v_\lambda}$$
$$\gamma(s)(\overline{w_\lambda}) = \overline{\frac{1}{\beta(s)} w_\lambda}$$
$$\gamma(s)|_{V^\omega} = A|_{V^\omega}$$
⟨1⟩3. Case: $\lambda \in \mathbb{R}\backslash\{-1\}$
Proof:
⟨2⟩1. $1/\lambda \in \sigma(A)$.
⟨2⟩2. Let: $v_\lambda \in E_\lambda(A) \cap \mathbb{R}^{2n}$.
⟨2⟩3. $\exists v_{1/\lambda} \in G_{1/\lambda}(A) \cap \mathbb{R}^{2n} \ : \ \omega(v_\lambda, v_{1/\lambda}) = 1$.
⟨2⟩4. Let: $V = \operatorname{span}\{v_\lambda, v_{1/\lambda}\}$.
⟨2⟩5. Define: $\gamma : [0,\varepsilon] \to \mathbb{C}^{2n\times 2n} = V \oplus V^\omega$ by setting
$$\gamma(s)(v_\lambda) = \beta(s)\lambda v_\lambda$$
$$\gamma(s)(v_{1/\lambda}) = \frac{1}{\beta(s)} A(v_{1/\lambda})$$
$$\gamma(s)|_{V^\omega} = A|_{V^\omega}$$
⟨1⟩4. Case: $\lambda = -1$
Proof:
⟨2⟩1. $m_A(\lambda) \geq 2$.
⟨2⟩2. $\exists v_\lambda^1, v_\lambda^2 \in G_{-1}(A) \ : \ \omega(v_\lambda^1, v_\lambda^2) = 1$.
⟨2⟩3. Let: $V = \operatorname{span}\{v_\lambda^1, v_\lambda^2\}$.
⟨2⟩4. Define: $\gamma : [0,\varepsilon] \to \mathbb{C}^{2n\times 2n} = V \oplus V^\omega$ by setting
$$\gamma(s)(v_\lambda^1) = \beta(s) A(v_\lambda^1)$$
$$\gamma(s)(v_\lambda^2) = \frac{1}{\beta(s)} A(v_\lambda^2)$$
$$\gamma(s)|_{V^\omega} = A|_{V^\omega}$$
⟨1⟩5. Q.E.D.

Lemma 1.3 *Let $A \in \mathrm{Sp}^*(2n)$. Then there exists a path $\gamma : [0,1] \to \mathrm{Sp}(2n)^*$ such that*

1. $|\sigma(\gamma(1))| = 2n$.
2. *If $A \in \mathrm{Sp}(2n)^+$ then* $|\{\lambda \in \sigma(\gamma(1)) \ : \ \lambda \in \mathbb{R}_{>0}\}| = 0$.
3. *If $A \in \mathrm{Sp}(2n)^-$ then* $|\{\lambda \in \sigma(\gamma(1)) \ : \ \lambda \in \mathbb{R}_{>0}\}| = 2$.

PROOF:
⟨1⟩1. LET: $\gamma : [0,1] \to \mathrm{Sp}^*(2n)$ be a path such that $|\sigma(\gamma(1))| = 2n$.
  PROOF:
  ⟨2⟩1. Repeatedly apply Lemma 1.2.
⟨1⟩2. item 2 holds.
⟨1⟩3. Q.E.D.

We also have the following result.

Proposition 1.5 *The following statements hold.*

1. $\mathrm{Sp}^*(2n)$ *is open in* $\mathrm{Sp}(2n)$.
2. $\mathrm{Sp}^*(2n)$ *has two connected components which are given by* $\mathrm{Sp}^+(2n)$ *and* $\mathrm{Sp}^-(2n)$.
3. *The inclusions $\iota_\pm : \mathrm{Sp}^\pm(2n) \to \mathrm{Sp}(2n)$ induce the zero map $\pi_1(\mathrm{Sp}^\pm) \to \pi_1(\mathrm{Sp}(2n))$.*

Finally, we want to discuss how symplectic and symmetric matrices are related. We denote the set of all real symmetric matrices of size m by $\mathrm{Sym}(\mathbb{R}^m)$. Namely, we will show that symmetric matrices can be considered as integrals of symplectic matrices in a sense.

Proposition 1.6 *Let $S \in C^0([0,1], \mathrm{Sym}(\mathbb{R}^{2n}))$. Consider the initial value problem*

$$R'(t) = J_0 S(t) R(t) \ \ \textit{with} \ \ R(0) = \mathrm{Id}_{\mathbb{R}^{2n}}. \tag{1.7}$$

Then the function $R \in C^1([0,1], \mathbb{R}^{2n\times 2n})$ that solves this has the property that $R(t) \in \mathrm{Sp}(2n)$ for all $t \in [0,1]$. Conversely, assume that $R \in C^1([0,1], \mathrm{Sp}(2n))$. Then for all $t \in [0,1]$ the matrix

$$S(t) = -J_0 R'(t) R(t)^{-1} \tag{1.8}$$

is symmetric.

PROOF:
⟨1⟩1. The first statement holds, i.e. $R(t) \in \mathrm{Sp}(2n)$ for all $t \in [0,1]$.
  PROOF:
  ⟨2⟩1. $(R'(t))^T = -(R(t))^T S(t) J_0$.

⟨2⟩2. $\forall t \in [0,1] \ : \ (R(t))^T J_0 R(t) = J_0$.
Proof:
⟨3⟩1. $((R(t))^T J_0 R(t))' = 0$ and $R(0) = \mathrm{Id}_{\mathbb{R}^{2n}}$.
⟨1⟩2. The second statement holds, i.e. $S(t) = -J_0 R'(t) R(t)^{-1}$ for all $t \in [0,1]$.
Proof:
⟨2⟩1. $(S(t))' = -(J_0 R'(t)(R(t))^{-1})^T = \cdots = S$.
⟨1⟩3. Q.E.D.

1.1.3 Complex Vector Spaces

We have seen that every symplectic vector space must be even-dimensional. Consider the canonical case of $\mathbb{R}^{2n}$ with the standard symplectic structure given by $\omega = \sum_{k=1}^{n} dx^k \wedge dy^k$. The space $\mathbb{R}^{2n}$ can be considered as a complex vector space via the identification $(x_k, y_k) \mapsto z_k = x_k + iy_k$. This can be generalised to the case of an arbitrary vector space by providing a suitable automorphism J that plays the role of the imaginary unit. More explicitly, note that i induces a $\mathbb{C}$-linear map on C^n by acting by multiplication. Under the above identification this map can be described by the matrix

$$J_0 = \begin{bmatrix} 0 & -\mathrm{Id}_{\mathbb{R}^n} \\ \mathrm{Id}_{\mathbb{R}^n} & 0 \end{bmatrix}$$

This leads to the following definition.

Definition 1.6 *Let $V \in \mathsf{Vect}_{\mathbb{R}}$. A* complex structure *on V is an endomorphism $J \in \mathsf{End} V$ with the property that $J^2 = -\mathrm{Id}_V$. If such a J exists, the pair (V, J) is called a* complex vector space.

Proposition 1.7 *Let (V, J) be a complex vector space. Then $\dim_{\mathbb{R}}(V)$ is even.*

Proof:
⟨1⟩1. Let: $E_1 = \{v + Jv \ : \ v \in V\}$ and $E_2 = \{v - Jv \ : \ v \in V\}$.
⟨1⟩2. $E_1 \cong E_2$.
⟨1⟩3. $V = E_1 \oplus E_2$.
⟨1⟩4. Q.E.D.

Complex structures are closely related to symplectic structures. We are only interested in such complex structures that are compatible in the following way.

Definition 1.7 *Let $(V, \omega) \in \mathsf{SympVect}$ and $J \in \mathsf{End}(V)$ a complex structure satisfying the following additional properties.*

1. *$\forall u, v \in V : \ \omega(Ju, Jv) = \omega(u, v)$.*
2. *$\forall u \in V \backslash \{0\} : \ \omega(u, Ju) > 0$.*

*Then J is called ω-*compatible*. The space of all ω-*compatible *complex structures is denoted by $\mathcal{J}(V,\omega)$.*

Every symplectic vector space admits a compatible complex structure.

Proposition 1.8 *Let $(V,\omega) \in \mathsf{SympVect}$. Then $\mathcal{J}(V,\omega) \neq \emptyset$.*

PROOF:
$\langle 1\rangle$1. LET: $g \in V^* \otimes V^*$ be symmetric, positive-definite and nondegenerate.
$\langle 1\rangle$2. $\exists A \in \mathsf{End}(A) \ : \ \forall (u,v) \in V^2 \ : \ \omega(u,v) = g(Au,v)$.
  PROOF:
  $\langle 2\rangle$1. LET: $A = \widetilde{g}^{-1} \circ \widetilde{\omega}$.
$\langle 1\rangle$3. $A^T = -A$.
  PROOF:
  $\langle 2\rangle$1. $g(A^T u,v) = g(u,Av) = g(Av,u) = \omega(v,u) = -\omega(u,v) = g(-Au,v)$.
$\langle 1\rangle$4. AA^T is diagonalisable and $\lambda > 0$ for all $\lambda \in \sigma(AA^T)$.
$\langle 1\rangle$5. LET: $J = (AA^T)^{-1/2}A$.
$\langle 1\rangle$6. $J^2 = -\mathrm{Id}_V$.
  PROOF:
  $\langle 2\rangle$1. $J^T = A^T(AA^T)^{-1/2} = -A(AA^T)^{-1/2} = -(AA^T)^{-1/2}A = -J$.
  $\langle 2\rangle$2. $JJ^T = -J^2 = A^T(AA^T)^{-1/2}(AA^T)^{-1/2}A = \mathrm{Id}_V$.
$\langle 1\rangle$7. $J \in \mathcal{J}(V,\omega)$.
  PROOF:
  $\langle 2\rangle$1. $\forall (u,v) \in V^2 \ : \ \omega(Ju,Jv) = g(AJu,Jv) = g(J^TJAu,v) = g(u,v) = \omega(u,v)$.
  $\langle 2\rangle$2. $\forall u \in V\backslash\{0\} \ : \ \omega(u,Ju) = g(Au,Ju) = g(-JAu,u)$
  $= g((AA^T)^{1/2}u,u) > 0$ since $(AA^T)^{1/2}$ is symmetric and positive-definite.
$\langle 1\rangle$8. Q.E.D.

In fact, much more can be said about the space $\mathcal{J}(V,\omega)$.

Proposition 1.9 *The space $\mathcal{J}(V,\omega)$ is contractible.*

PROOF:
$\langle 1\rangle$1. LET: $Met = \{A \in \mathrm{Sym}(2n) \cap \mathrm{Sp}(2n) \ : \ A$ is positive-definite.$\}$.
$\langle 1\rangle$2. SUFFICES: $\mathcal{J}(V,\omega) \cong Met$
$\langle 1\rangle$3. SUFFICES ASSUME: $(V,\omega) = (\mathbb{R}^{2n},\omega_0)$ where ω_0 is the standard symplectic structure defined by J.
$\langle 1\rangle$4. LET: $J' \in \mathbb{R}^{2n\times 2n}$.
$\langle 1\rangle$5. $J' \in \mathcal{J}(V,\omega) \iff (J')^2 = -\mathrm{Id}_{\mathbb{R}^{2n\times 2n}}, (J')^TJJ' = J$ and $\omega_0(v,J'v) > 0$ for all $v \in V\backslash\{0\}$.
$\langle 1\rangle$6. DEFINE: $f : \mathcal{J}(V,\omega) \to Met$ by $f(J') = JJ'$.
  PROOF:
  $\langle 2\rangle$1. $J' \in \mathcal{J}(V,\omega) \implies JJ' \in \mathrm{Sym}(\mathbb{R}^{2n})$.
    PROOF:

$\langle 3\rangle 1.$ $(JJ')^T = -(J')^T J = (J')^T J (J')^2 = JJ'$.
$\langle 1\rangle 7.$ Define: $g : Met \to \mathcal{J}(V,\omega)$ by $g(P) = -J^{-1}P$.
Proof:
$\langle 2\rangle 1.$ $P \in Met \implies -J^{-1}P \in \mathcal{J}(V,\omega)$.
$\langle 1\rangle 8.$ $f \circ g = \mathrm{Id}_{Met}$ and $g \circ f = \mathrm{Id}_{\mathcal{J}(V,\omega)}$.
$\langle 1\rangle 9.$ Q.E.D.

1.2 Symplectic Manifolds

We can now extend the concept of a symplectic structure to manifolds.

Definition 1.8 *Let $M \in \mathsf{Man}^\infty$ and let $\omega \in \Omega^2(M)$. Then (M,ω) is called a* symplectic manifold *if it satisfies the following conditions.*

1. $d\omega = 0$.
2. $\forall p \in M : (T_pM, \omega_p) \in \mathsf{SympVect}$.

The map ω is called a *symplectic form*. The set of all symplectic forms on M will be denoted by $\Omega^2_{\mathsf{Symp}}(M)$. The set of all symplectic manifolds will be denoted by $\mathsf{SympMan}^\infty$. It consists of pairs (M,ω) where $M \in \mathsf{Man}^\infty$ and $\omega \in \Omega^2_{\mathsf{Symp}}(M)$.

1.2.1 Notions of Equivalence

We have introduced symplectic manifolds as a smooth manifold that is endowed with additional structure, namely its symplectic form. Hence, it makes sense to consider meaningful notions of equivalence that take into account this additional piece of data. The most obvious way to do this is to take the notion of a diffeomorphism and also require to preserve the symplectic form. This leads to the following definition.

Definition 1.9 *Let $(M_1,\omega_1),(M_2,\omega_2) \in \mathsf{SympMan}^\infty$. We say that (M_1,ω_1) and (M_2,ω_2) are* symplectomorphic *if there exists a $f \in \mathsf{Iso}(M_1,M_2)$ such that $\omega_1 = f^*\omega_2$. In this case we also call the corresponding symplectic forms ω_1 and ω_2 symplectomorphic. The set of all symplectomorphisms will be denoted by $\mathsf{Symp}(M_1,\omega_1,M_2,\omega_2)$. If $M_1 = M_2$ and $\omega_1 = \omega_2$ we simply write $\mathsf{Symp}(M_1,\omega_1)$.*

It is often useful to have a stronger notion of equivalence. For this we introduce the following.

Definition 1.10 *Let $M \in \mathsf{Man}^\infty$ and $\rho \in C^\infty(\mathbb{R} \times M, M)$. Then ρ is called an* isotopy *if it satisfies that*

1. $\forall t \in \mathbb{R} : \rho_t \in \mathsf{Aut}(M)$.
2. $\rho_0 = \mathrm{Id}_M$.

The set of all isotopies on M will be denoted by $\mathsf{Isotop(M)}$.

Isotopies are related to time-dependent vector fields on M in the following way. Starting out with an isotopy $\rho \in \mathsf{Isotop(M)}$ we obtain a time-dependent vector $X \in \Gamma(TM \to \mathbb{R} \times M)$ by differentiating along the flow lines of ρ. More precisely, this means that X is defined by

$$\frac{d}{dt}\rho_t = X_t \circ \rho_t. \tag{1.9}$$

Conversely, assume now that $X \in \Gamma(TM \to \mathbb{R} \times M)$ is given. Make the additional assumption that M is compact. Then we can solve Equation 1.9 for all $t \in \mathbb{R}$ and hence obtain $\rho \in \mathsf{Isotop(M)}$.

Proposition 1.10 *A time-dependent vector field on a compact manifold generates an isotopy.*

1.2.2 Local Theory

Theorem 1.1 *Let $M \in \mathsf{Man}^{\infty}_{cpt}$ and $\omega_0, \omega_1 \in \Omega^2_{\mathrm{Symp}}(M)$. Furthermore, assume that*

1. $[\omega_0] = [\omega_1] \in H^2(M, \mathbb{R})$.
2. $\forall t \in [0,1] \ : \ \omega_t = (1-t)\omega_0 + t\omega_1 \in \Omega^2_{\mathrm{Symp}}(M)$.

Then there exists a $\rho \in \mathsf{Isotop(M)}$ with the property that for all $t \in [0,1]$ it holds that $\rho_t^*\omega_t = \omega_0$.

PROOF:
⟨1⟩1. If there exists a smooth time-dependent vector field v_t on M such that

$$\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt} = 0 \tag{1.10}$$

then there exists a $\rho \in \mathsf{Isotop(M)}$ satisfying item 2.
PROOF:
⟨2⟩1. LET: $\rho \in \mathsf{Isotop(M)}$ be the isotopy obtained by integrating v.
⟨2⟩2. Equation 1.10 is equivalent to

$$\frac{d}{dt}(\rho_t^*\omega_t) = 0. \tag{1.11}$$

⟨2⟩3. $\rho_t^*\omega_t = \omega_0$.
⟨1⟩2. Equation 1.10 can be solved for v_t.
PROOF:
⟨2⟩1. $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$ by item 2.
⟨2⟩2. There exists a $\mu \in \Omega^1(M)$ such that $\omega_1 - \omega_0 = d\mu$.
⟨2⟩3. Equation 1.10 is equivalent to

$$d\iota_{v_t}\omega_t + d\mu = 0.$$

PROOF:

$\langle 3\rangle 1.$ $\mathcal{L}_{v_t}\omega_t = d\iota_{v_t}\omega_t + \iota_{v_t}d\omega_t$.
$\langle 3\rangle 2.$ $\frac{d\omega_t}{dt} = d\mu$.
$\langle 2\rangle 4.$ Equation 1.2.2 can be solved pointwise.
$\langle 1\rangle 3.$ Q.E.D.

Theorem 1.2 *Let $M \in \mathsf{Man}^\infty$ and $(N, i) \in \mathsf{SubM}(\mathrm{M})$. Furthermore, let $\omega_0, \omega_1 \in \Omega^2_{\mathsf{Symp}}(M)$. Assume that $i^*\omega_0 = i^*\omega_1$. Then there exist $U_0, U_1 \in \mathsf{Open}(M)$ and $\varphi \in \mathsf{Iso}(U_0, U_1)$ such that*

1. $\varphi \circ i = i$
2. $\varphi^*\omega_1 = \omega_0$.

Proof:
$\langle 1\rangle 1.$ Let: $U_0 \in \mathsf{Open}(M)$ be a tubular neighbourhood of N.
$\langle 1\rangle 2.$ There exists a $\mu \in \Omega^1(U_0)$ such that $\omega_1 - \omega_0 = d\mu$ and $i^*\mu = 0$.
Proof:
$\langle 2\rangle 1.$ Since U_0 is contractible and $d(\omega_1 - \omega_0) = 0$ such a μ exists.
$\langle 1\rangle 3.$ Let: $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + td\mu$.
$\langle 1\rangle 4.$ For all $t \in [0,1]$ the form ω_t is closed.
$\langle 1\rangle 5.$ There exists a $U_0' \in \mathsf{Open}(U_0)$ such that $i(N) \subset U_0'$ and for all $t \in [0,1]$ it holds that $\omega_t \in \Omega^2_{\mathsf{Symp}}(U_0')$.
$\langle 1\rangle 6.$ Let: v_t be the solution to the equation $\iota_{v_t}\omega_t = -\mu$.
$\langle 1\rangle 7.$ $v_t|_N = 0$.
$\langle 1\rangle 8.$ Let: $\rho \in \mathsf{Isotop}(\mathrm{M})$ be the isotopy generated by v.
$\langle 1\rangle 9.$ $\forall t \in [0,1] \,:\, \rho_t^*\omega_t = \omega_0$ and $\rho_t|_N = \mathrm{Id}_N$.
$\langle 1\rangle 10.$ Define: $\varphi = \rho_1$ and $U_1 = \rho_1(U_0)$.
$\langle 1\rangle 11.$ Q.E.D.

The previous result implies in particular that locally all symplectic manifolds look the same.

Theorem 1.3 (Darboux) *Let $(M, \omega) \in \mathsf{SympMan}^\infty$ and $p \in M$. Then there exists a coordinate chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ around p such that*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Proof:
$\langle 1\rangle 1.$ Let: $N = \{p\}$ with the inclusion $i : \{p\} \to M$.
$\langle 1\rangle 2.$ Let: $(x_1', \ldots, x_n', y_1', \ldots, y_n')$ be coordinates at p such that $\omega_p = \sum dx_i' \wedge dy_i'$.
$\langle 1\rangle 3.$ Apply Theorem 1.2 to see that $\omega_0 = \omega$ and $\omega_1 = \sum dx_i' \wedge dy_i'$ are sympleticmorphic around p.
$\langle 1\rangle 4.$ Q.E.D.

1.2.3 Examples of Symplectic Manifolds

We have the trivial example of a symplectic vector space considered as a manifold.

Proposition 1.11 *Let* $(V, \omega) \in \mathsf{SympVect}$. *If* V *is endowed with its canonical smooth structure as a vector space, then* $(V, \omega) \in \mathsf{SympMan}^\infty$.

Cotangent Bundles

The canonical example of a symplectic manifold is the cotangent bundle of a smooth manifold. Let $M \in \mathsf{Man}^\infty$ and $\pi : T^*M \to M$ its cotangent bundle. The projection induces a linear map on the level of tangent spaces given by its differential $D\pi : T(T^*M) \to TM$. At each point $p = (x, \xi) \in T^*M$ we can consider the dual $D\pi_p^*$ of this map. This allows us to define the following canonical 1-form on T^*M.

$$\alpha_p = D\pi_p^*(\xi) \tag{1.12}$$

Proposition 1.12 *Let* $\alpha : T^*M \to T^*(T^*M)$ *be the map defined in Equation 1.12. Then* $\alpha \in \Omega^1(T^*M)$.

Proof:
⟨1⟩1. Let: $(U, x^1, \ldots, x^n)$ be a coordinate chart around $x \in M$.
⟨1⟩2. Let: $(\pi^{-1}(U), x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$ be the induced coordinate chart on T^*M.
⟨1⟩3. In this coordinate chart α is given by

$$\alpha = \sum_{i=1}^n \xi_i dx^i.$$

Proof:
⟨2⟩1. Let: $p = (x, \xi) \in \pi^{-1}(U)$.
⟨2⟩2. Assume: $\xi = \sum_{i=1}^n \xi_i dx^i$.
⟨2⟩3. $\alpha(\frac{\partial}{\partial x^i}) = \xi\left(D\pi_p(\frac{\partial}{\partial x^i})\right) = \xi_i$.
⟨2⟩4. $\alpha(\frac{\partial}{\partial \xi^i}) = \xi\left(D\pi_p(\frac{\partial}{\partial \xi^i})\right) = 0$.
⟨1⟩4. Q.E.D.
Proof: The map α is smooth in local coordinates around every point.

Using the canonical 1-form α we obtain our symplectic form by taking the derivative. We define

$$\omega = -d\alpha. \tag{1.13}$$

Proposition 1.13 *Let* $M \in \mathsf{Man}^\infty$ *and* ω *as defined in Equation 1.13. Then* $(T^*M, \omega) \in \mathsf{SympMan}^\infty$.

Proof:
⟨1⟩1. ω is closed.
⟨1⟩2. $\forall p \in T^*M \; : \; \omega_p \in \mathsf{Mono}(T(T^*M)_p,)$
Proof:
⟨2⟩1. Let: $(\pi^{-1}(U), x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$ be local coordinates around p.
⟨2⟩2. In this coordinate chart ω is given by

$$\omega = \sum_{i=1}^{n} dx^i \wedge d\xi^i$$

⟨1⟩3. Q.E.D.

Orientable Surfaces

Recall that a manifold $M \in \mathsf{Man}^\infty$ is orientable if and only if the top exterior power bundle $\Lambda^{\dim(M)}(T^*M)$ admits a nowhere-vanishing section. Hence, if $\dim(M) = 2$ and M is orientable it admits a symplectic structure since every 2-form on a 2-dimensional manifold is automatically closed.

1.3 Almost Complex Manifolds

Just like the notion of a symplectic structure can be extended from the linear setting to the global setting of a manifold the same can be done for complex structures.

Definition 1.11 *Let $M \in \mathsf{Man}^\infty$ and $J \in \mathsf{End}(TM)$. Then J is called a* almost complex structure *if $J^2 = -\mathrm{Id}_{TM}$.*

Definition 1.12 *Let $M \in \mathsf{Man}^\infty$, $\omega \in \Omega^2(M)$ and J an almost complex structure on M. Then J is called* compatible *with ω if $\omega = J^*\omega$ and $\omega(u, Ju) > 0$ for all $p \in M$ and $u \in T_uM \backslash \{0\}$. The space of all ω-compatible almost complex structures will be denoted by $\mathcal{J}(M, \omega)$.*

Just as it is the case in the linear setting that every symplectic manifold admits a compatible complex structure it is also true that every symplectic manifold admits a compatible almost complex structure.

Proposition 1.14 *Let $(M, \omega) \in \mathsf{SympMan}^\infty$. Then $\mathcal{J}(M, \omega) \neq \emptyset$ and $\mathcal{J}(M, \omega) \simeq *$.*

Proof:
⟨1⟩1. Apply the corresponding linear results fibrewise.
⟨1⟩2. Q.E.D.

Chapter 2

Hamiltonian Fixed Point Floer Homology

2.1 Introduction

Assume we are given a closed, connected, symplectic manifold $(M, \omega) \in \mathsf{SympMan}^{\infty}$. Every smooth function $H \in C^{\infty}(\mathbb{R} \times M)$ defines a vector field X_{H_t} on M via the equation

$$dH_t = \omega(\cdot, X_{H_t}) \tag{2.1}$$

where dH_t is the usual exterior derivative. Hence, we can consider the associated flow given by

$$\frac{d}{dt}\psi = X_{H_t}(\psi) \tag{2.2}$$

where $\psi \in C^{\infty}(\mathbb{R} \times M, M)$. Since we are assuming M to be compact this equation can be integrated to a global flow ψ. We want to study this dynamical system consisting of the pair (M, ψ). Let us restrict to the case where H_t is 1-periodic so that we may consider it as an element in $C^{\infty}(S^1 \times M)$, thinking of S^1 as the quotient $\mathbb{R}/\mathbb{Z}$. A reasonable first step to learn more about this system is to study the orbits of ψ. In particular, we are interested in the 1-periodic orbits. These correspond precisely to the fixed points of the 1-flow map ψ_1. The most basic statement kinds of statement one can consider concern the number of such fixed points. Hence, in this chapter we want to prove the following result which gives a lower bound on this number.

Theorem 2.1 (Arnold conjecture) *If every fixed point of ψ_1 is non-degenerate then we have the following inequality*

$$|fix(\psi_1)| \geq \sum_i \dim H_i(M; \mathbb{Z}/2). \tag{2.3}$$

The goal of this chapter is to prove Theorem 2.1 under some additional assumption on M. Namely, we will assume that for every smooth map $w : S^2 \to M$ the integral of $w^*\omega$ over S^2 vanishes:

$$\int_{S^2} w^*\omega = 0.$$

This can also be expressed as $\omega|_{\pi_2(M)} = 0$. One of the main reasons this assumption is made is to significantly simplify the proof of the compactness of the Floer Moduli spaces as we will see later. Additionally, we will also assume that the first Chern class $c_1(M)$ vanishes on $\pi_2(M)$. In other words, that

$$\int_{S^2} w^*c_1(M) = 0.$$

From the theory of characteristic classes we then know in particular that the tangent bundle TM can be trivialised over spheres. When the above two properties hold we say that M is *symplectically aspherical*. For the remainder of this text we will assume that M is a closed symplectic manifold satisfying these two assumptions.

The approach to prove the inequality Theorem 2.1 is as follows. We construct a homology theory from a chain complex that is generated by the non-degenerate 1-periodic orbits of the Hamiltonian flow. Then this equality holds trivially if we replace the singular homology groups on the right-hand side of Equation 2.3 with these new homology groups. Then one shows that these new homology groups are isomorphic to the Morse homology groups which are known to coincide with singular homology.

The construction of these Floer homology groups is inspired by Morse theory. The 1-periodic orbits of the flow defined in Equation 2.2 can be interpreted as the critical points of a certain action functional $\mathcal{A}_H$ on a particular infinite-dimensional manifold. The chain complex generated by these orbits is graded by introducing a suitable index map. Then one can formally consider the gradient of $\mathcal{A}_H$ with respect to an L^2-metric coming from a choice of compatible almost-complex structure. Using this gradient one can write down the equation that formally defines the flow of this gradient and consider the space of such solutions that connect to critical orbits γ^- and γ^+ whose indices differ by one. To define the value of the boundary operator ∂ on γ^- one counts the number of such solutions. In order for this to make sense it is necessary to verify that the space of solutions is a 0-dimensional compact manifold. A priori this construction depends on the choice of Hamiltonian H and compatible almost complex structure J. However, we will show that actually any two such choices that allow for the definition for Floer homology yield isomorphic groups. Thus, Floer homology is in fact an invariant of the symplectic manifold (M, ω).

A major source for the material in this chapter was [2]. Furthermore, [15] and [17] were used. Other sources are referenced when used.

2.2 Action Functional

To motivate the definition of the action functional let us first consider the case of $\mathbb{R}^{2n}$ thought of as phase space with standard coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and Hamiltonian H. Recall that in this setting the classical action functional as known from physics is given by

$$\mathcal{A}_H(\gamma) = \int_0^1 \left(\langle p(t), q'(t) \rangle - H_t(q(t), p(t)) \right) dt. \tag{2.4}$$

where $\gamma(t) = (q(t), p(t)) = (q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ is a smooth path $[0,1] \to \mathbb{R}^{2n}$ Letting $\alpha_0 = \sum_{i=1}^{n} p_i dq^i$ the integral in 2.4 can be written as

$$\int_{[0,1]} \gamma^* \alpha_0 - \int_0^1 H_t(q,p) dt.$$

More generally, we can replace $\mathbb{R}^{2n}$ with the cotangent bundle T^*M of a smooth manifold $M \in \mathsf{Man}^\infty$ and let α be the canonical 1-form defined in Equation 1.12. Let us now restrict our attention to smooth *loops* $\gamma : S^1 \to T^*M$. Then Equation 2.4 becomes

$$\mathcal{A}_H(\gamma) = \int_{S^1} \gamma^* \alpha - \int_0^1 H_t(\gamma(t)) dt. \tag{2.5}$$

Assume that there is a smooth map $w : B_1(0) \to T^*M$ such that $w|_{S^1} = \gamma$ where $B_1(0) = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ denotes the unit disk. We call such a map w a *capping* of γ. Then by Stokes' theorem $\int_{S^1} \gamma^* \alpha = -\int_{B_1(0)} w^* \omega$ where $\omega = -d\alpha$ denotes the standard symplectic form on T^*M defined in Equation 1.13. In the case of a general symplectic manifold the symplectic form is not necessarily exact which means that the expression in Equation 2.5 does not generalise. However, when we define the action functional not just for loops but rather for pairs of loops and cappings we can use the above to avoid this issues. Let us introduce some notation.

Definition 2.1 *Let $M \in \mathsf{Man}^\infty$. We denote the space of all smooth loops in M by*

$$\mathcal{L}(M) = C^\infty(S^1, M). \tag{2.6}$$

We will mainly focus on the connected component of this space containing the constant loop. Hence, we denote the space of all contractible smooth loops by

$$\mathcal{L}_0(M) = \{\gamma \in \mathcal{L}(M) : \ \gamma \in 0 \in \pi_1(M)\}. \tag{2.7}$$

We also introduce the universal covering space of $\mathcal{L}_0(M)$.

$$\widetilde{\mathcal{L}(M)} = \{[\gamma, w] : \ \gamma \in \mathcal{L}_0(M) \text{ and } w \text{ is a capping of } \gamma\}.$$

The notation $[\gamma, w]$ indicates homotopy classes of such maps.

This leads us to define the action functional as follows in the general case.

Definition 2.2 *Let $(M, \omega) \in \mathsf{SympMan}^\infty$, $H \in C^\infty(S^1 \times M)$ and $[\gamma, w] \in \widetilde{\mathcal{L}(M)}$. Then the* action functional *is defined to be*

$$\mathcal{A}_H([\gamma, w]) = -\int_{B_1(0)} w^*\omega - \int_0^1 H_t(\gamma(t))dt. \tag{2.8}$$

Under suitable assumptions on the topology of M this functional is in fact defined on $\mathcal{L}_0(M)$ itself. More specifically, assume that M satisfies that for all $v \in C^0(S^2, M)$ we have that $\int_{S^2} v^*\omega = 0$. Then if w and w' are two different choices of cappings for a loop γ we have that $\int_{B_1(0)} w^*\omega - \int_{B_1(0)} w'^*\omega = \int_{S^2} v^*\omega = 0$.

Having defined the action functional $\mathcal{A}_H$ it makes sense to consider its critical points. In other words, these points are the loops $\gamma : S^1 \to M$ where the differential $D\mathcal{A}_H$ vanishes.

Definition 2.3 *For a Hamiltonian $H \in C^\infty(S^1 \times M)$ we denote the critical points of its associated action function $\mathcal{A}_H$ by*

$$\mathrm{Crit}(\mathcal{A}_H) = \{\gamma \in \mathcal{L}_0(M) : \ D\mathcal{A}_H(\gamma) = 0\}. \tag{2.9}$$

To proceed further we first compute the differential. The differential of $\mathcal{A}_H$ at a point $\gamma \in \mathcal{L}(M)$ acts on tangent vectors at that point. Since $\gamma \in C^\infty(S^1, M)$ such a tangent vector can be identified with a section of the pull back bundle γ^*TM. In other words, a tangent vector is given by a vector field along γ.

Proposition 2.1 *The differential of the action functional $\mathcal{A}_H$ from Equation 2.8 at a point $\gamma \in \mathcal{L}(M)$ is given by*

$$D\mathcal{A}_H(\gamma)(\xi) = \int_{S^1} \omega(\gamma'(t) - X_{H_t}(\gamma(t)), \xi(t))dt. \tag{2.10}$$

Proof:
$\langle 1 \rangle 1.$ Let: $(\gamma_s)_{s \in (-\varepsilon, \varepsilon)} \subset \mathcal{L}(M)$ be a path such that $\gamma_0 = \gamma$ and $\xi = \partial_s|_{s=0}\gamma_s$.

⟨1⟩2. LET: $(w_s)_{s\in(-\varepsilon,\varepsilon)}$ be a corresponding family of cappings such that $w_0|_{S^1} = \gamma$.
⟨1⟩3. $D\mathcal{A}_H(\gamma)(\xi) = \partial_s|_{s=0}\mathcal{A}_H(\gamma) = -\partial_s|_{s=0}\int_{B_1(0)} w^*\omega - \partial_s|_{s=0}\int_{S^1} H_t(\gamma_s(t))dt.$
⟨1⟩4. $\partial_s|_{s=0}\int_{B_1(0)} w_s^*\omega = \int_{B_1(0)} w_0^*(di_{\xi_0}\omega) = \int_{S^1}\gamma^*(i_{\xi_0}\omega) = \int_{S^1}\omega(\xi(t),\gamma'(t))dt.$
⟨1⟩5. $\partial_s|_{s=0}\int_{S^1} H_t(\gamma_s(t))dt = \int_{S^1} DH_t(\gamma(t))(\xi(t))dt = -\int_{S^1}\omega(X_{H_t}(\gamma(t)),\xi(t))dt.$
⟨1⟩6. Q.E.D.

Since the symplectic form ω is by definition non-degenerate we see that the points where $D\mathcal{A}_H$ vanishes are precisely those points that satisfy the differential equation

$$\gamma'(t) = X_{H_t}(\gamma(t)). \tag{2.11}$$

These are precisely the 1-periodic integral curves of the Hamiltonian vector field X_{H_t}. Since M is compact there is an associated global flow to Equation 2.2 which we denote by ψ_t. The critical points of $\mathcal{A}_H$ then correspond to the fixed points of the 1-flow map ψ_1.

In the following we think of the circle S^1 as the quotient $\mathbb{R}/\mathbb{Z}$.

Definition 2.4 *Let* $(M,\omega) \in \mathsf{SympMan}^\infty$ *and* $H \in C^\infty(S^1 \times M)$. *The space of* 1*-periodic Hamiltonian orbits associated to these data is given by*

$$\mathcal{P}(H) = \{\gamma \in C^\infty(S^1, M) : \frac{d}{dt}\gamma(t) = X_{H_t}(\gamma(t))\}. \tag{2.12}$$

A loop $\gamma \in \mathcal{P}(H)$ *is called* non-degenerate *if*

$$\det\left(\mathrm{Id}_{T_{\gamma(0)}M} - D\psi_1(\gamma(0))\right) \neq 0. \tag{2.13}$$

In general, it is not true that all 1-periodic orbits are non-degenerate. However, we will later see that for a generic choice of Hamiltonian H this always holds. The following result is one important consequence when non-degeneracy holds for all orbits.

Proposition 2.2 *If H is chosen so that all* $\gamma \in \mathcal{P}(H)$ *are non-degenerate, then this set contains only finitely many points.*

PROOF:
⟨1⟩1. LET: $M \subset M \times M$ be embedded diagonally.
⟨1⟩2. $\gamma \in \mathcal{P}(H)$ is non-degenerate if and only if $(M, \psi_1(M))$ intersects M transversely in $M \times M$ at $\gamma(0) = \gamma(1)$.
⟨1⟩3. All points of intersection are isolated.
⟨1⟩4. Since M is compact there are only finitely many.
⟨1⟩5. Q.E.D.

We want to study the gradient-flow lines of the functional $\mathcal{A}_H$ connecting two critical points. Assume that we are given a smooth family of compatible almost-complex structures $J_t \in C^\infty(S^1 \times M, \mathcal{J}(M,\omega))$. This defines a family of Riemannian metrics via

$$g_t = \omega \circ (\mathrm{Id}_{TM} \times J_t).$$

Using this we obtain an inner product

$$\langle \xi_1, \xi_2 \rangle_J = \int_0^1 g_t(\xi_1(t), \xi_2(t))dt$$

on the tangent space $T_\gamma \mathcal{L}_0(M)$ for each $\gamma \in \mathcal{L}_0(M)$. The differential $D\mathcal{A}_H(\gamma)$ can be thought of as a 1-form on the tangent space. Hence, we compute the gradient of $\mathcal{A}_H$ with respect to this inner product by defining $D\mathcal{A}_H(\gamma)(\xi) = \langle \xi, \operatorname{grad} \mathcal{A}_H(\gamma) \rangle_J$. Then we obtain the following expression for $\operatorname{grad} \mathcal{A}_H(\gamma)$.

Proposition 2.3 *The gradient of the action functional $\mathcal{A}_H$ with respect to the metric $\langle \cdot, \cdot \rangle_J$ is given by*

$$\operatorname{grad} \mathcal{A}_H(\gamma) = J_t(\gamma)\left(\gamma'(t)\right) + \operatorname{grad} H_t.$$

Proof:
$\langle 1 \rangle$1. $D\mathcal{A}_H(\gamma)(\xi) = \int_{S^1} \omega(\gamma'(t) - X_{H_t}(\gamma(t)), \xi(t))dt.$
$\langle 1 \rangle$2. $\omega(\gamma'(t) - X_{H_t}(\gamma(t)), \xi(t)) = \omega(J_t(\gamma)(\gamma'(t) - X_{H_t}(\gamma(t))), J_t(\gamma)\xi(t)) = g_t(J_t(\gamma)(\gamma'(t) - X_{H_t}(\gamma(t))), \xi(t))).$
$\langle 1 \rangle$3. $\operatorname{grad} \mathcal{A}_H(\gamma) = J_t(\gamma)(\gamma'(t) - X_{H_t}(\gamma(t)) = J_t(\gamma)(\gamma'(t)) + \operatorname{grad} H_t.$
$\langle 1 \rangle$4. Q.E.D.

Using the expression for $\operatorname{grad} \mathcal{A}_H$ we can formally write down the equation for the corresponding (negative) gradient flow on $\mathcal{L}_0(M)$.

$$\partial_s u = -\operatorname{grad} \mathcal{A}_H(u).$$

Recasting this in terms of the underlying manifold M we obtain the following PDE.

Definition 2.5 *Let $u \in C^\infty(\mathbb{R} \times S^1, M)$. Consider the equation*

$$\bar{\partial}_{H,J}(u) = \partial_s u(s,t) + J_t(u(s,t))\partial_t(u(s,t)) + \operatorname{grad} H_t(u(s,t)) = 0. \tag{2.14}$$

The set of all solutions to 2.14 will be denoted by $\mathcal{M}(H,J)$.

We are only interested in solutions to 2.14 that connect two elements of $\mathcal{L}_0(M)$. This motivates the following definition

Definition 2.6 *Let $\gamma^-, \gamma^+ \in \mathcal{P}(H)$. Then define the space of connecting flow lines to be*

$$\mathcal{M}(\gamma^-, \gamma^+; H, J) = \{u \in \mathcal{M}(H, J) : \lim_{s \to \pm\infty} u(s,t) = \gamma^{\pm}(t)\} \tag{2.15}$$

An important number that can be associated to a solution u of Equation 2.14 is its energy.

Definition 2.7 *Let $u \in \mathcal{M}(H, J)$. Then we define its* energy *to be*

$$E_{H,J}(u) = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 |\partial_s u|_J^2 + |\partial_t u - X_{H_t}(u)|_J^2 dtds. \tag{2.16}$$

This definition of the energy is motivated in the following way. The energy of a flow line satisfying the Floer equation should be the total loss in potential energy along this line. Here, the potential energy is defined by the action functional $\mathcal{A}_H$ on the loop space. This would lead to defining

$$E(u) = -\int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}_H(u) ds.$$

We can check that this leads precisely to the definition given above.

$$\begin{aligned}
-\int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}_H(u) ds &= -\int_{-\infty}^{\infty} D\mathcal{A}_H(u)(\partial_s u) ds \\
&= -\int_{-\infty}^{\infty} \langle \operatorname{grad} \mathcal{A}_H(u), \partial_s u \rangle ds \\
&= \int_{-\infty}^{\infty} \int_{S^1} |\partial_s u|_J^2 dtds \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 |\partial_s u|_J^2 + |\partial_t u - X_{H_t}(u)|_J^2 dtds.
\end{aligned}$$

As a consequence we have in particular that if $u \in \mathcal{M}(\gamma^-, \gamma^+, H, J)$ connects two critical points, then its energy is simply the difference of the values of $\mathcal{A}_H$ at the endpoints and hence in particular finite.

Proposition 2.4 *If $u \in \mathcal{M}(\gamma^-, \gamma^+, H, J)$, then $E(u) = \mathcal{A}_H(\gamma^-) - \mathcal{A}_H(\gamma^+) < \infty$.*

More interestingly, the converse of the above proposition is also true. That is, if $u \in \mathcal{M}(H, J)$ has finite energy, then there must be critical points that are connected by u. The full proof of this is relegated to a later section because it relies on a compactness result for the space of all finite energy solutions to Equation 2.14. However, we can start with the following result.

Proposition 2.5 *Let $u \in \mathcal{M}(H,J)$ such that $E(u) < \infty$. Then there exist $\gamma^{\pm} \in \mathrm{Crit}(\mathcal{A}_H)$ such that*

$$\lim_{s\to\pm\infty} \mathcal{A}_H(u_s) = \mathcal{A}_H(\gamma^{\pm}).$$

Proof:
⟨1⟩1. $\exists (s_k)_{k\in\mathbb{N}} \subset \mathbb{R} : \exists \gamma \in C^0(S^1, M) : \lim_{k\to\infty} u_{s_k} = \gamma$ uniformly.
  Proof:
  ⟨2⟩1. $\exists (s_k)_{k\in\mathbb{N}} \subset \mathbb{R} : \lim_{k\to\infty} s_k = \infty$ and $\lim_{k\to\infty} \|\partial_t u(s_k,t) - X_{H_t}(u(s_k,t))\|_{L^2(S^1)} = 0$.
    Proof:
    ⟨3⟩1. By assumption $E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 |\frac{\partial u}{\partial s}|_J^2 + |\frac{\partial u}{\partial t} - X_{H_t}(u)|_J^2 dtds < \infty$.
  ⟨2⟩2. Let: $u_k = u_{s_k}$.
  ⟨2⟩3. The family $(u_k)_{k\in\mathbb{N}}$ is equicontinuous.
    Proof:
    ⟨3⟩1. $\exists C \in \mathbb{R} : \|\partial_t u_k\|_{L^2(S^1)} \le C$.
    ⟨3⟩2. $\forall t_1, t_2 \in S^1 : d(u_k(t_1), u_k(t_0)) \le C\sqrt{t_1 - t_0}$.
  ⟨2⟩4. Apply the Arzela-Ascoli theorem.
⟨1⟩2. $\gamma \in \mathrm{Crit}(\mathcal{A}_H)$.
  Proof:
  ⟨2⟩1. $\gamma \in C^1(S^1, M)$.
  ⟨2⟩2. $\gamma' = X_{H_t}(\gamma)$.
  ⟨2⟩3. Now use ⟨1⟩1 to bootstrap γ to a map in $C^\infty(S^1, M)$.
⟨1⟩3. $\lim_{k\to\pm\infty} \mathcal{A}_H(u_{s_k}) = \mathcal{A}_H(\gamma^{\pm})$.
  Proof:
  ⟨2⟩1. $\lim_{k\to\pm\infty} \int_0^1 H_t(u_{s_k})dt = \int_0^1 H_t(\gamma^{\pm})$.
    Proof:
    ⟨3⟩1. By ⟨1⟩1.
  ⟨2⟩2. Let: $w_k, w_{\pm} : B_1(0) \to M$ be cappings of u_{s_k} and $\gamma^{\pm}$ respectively.
  ⟨2⟩3. $\lim_{k\to\pm\infty} \int_{B_1(0)} w_k^*\omega = \int_{B_1(0)} w_{\pm}^*\omega$.
    Proof:
    ⟨3⟩1. Let: $U \in \mathsf{Open}(M)$ such that $\mathrm{Im}\,(\gamma^{\pm}) \subset U$ and such that U retracts to $\mathrm{Im}\,(\gamma^{\pm})$.
    ⟨3⟩2. $\omega|_U$ is exact: $\exists \lambda \in \Omega^1(M) : d\lambda|_U = \omega|_U$.
    ⟨3⟩3. $\exists N \in \mathbb{N} : \forall k > N : \mathrm{Im}\,(u_{s_k}) \subset U$.
    ⟨3⟩4. Think of S^2 as a cylinder C of the form $B_1(0) \times [0,1]$.
    ⟨3⟩5. Let: $v : C \to U$ be the map from the cylinder given by a homotopy between the discs w_k and $w_{\pm}$ that is contained in U.
    ⟨3⟩6. $\int_C v^*\omega = \int_{B_1(0)} w_k^*\omega - \int_{B_1(0)} w_{\pm}^*\omega = \int_{B_1(0)} w_k^*\omega|_U - \int_{B_1(0)} w_{\pm}^*\omega|_U$.

⟨3⟩7. Using ⟨3⟩2:

$$\int_{B_1(0)} w_k^*\omega - \int_{B_1(0)} w^*\omega_\pm = \int_{S^1} u_{s_k}^*\lambda - \int_{S^1} (\gamma^\pm)^*\lambda$$
$$= \int_0^1 (\lambda(\partial_t u_{s_k}) - \lambda(\partial_t \gamma^\pm))dt$$
$$= \int_0^1 \lambda(\partial_t u_{s_k} - X_{H_t}(u_{s_k}))dt$$
$$+ \int_0^1 \lambda(X_{H_r}(u_{s_k}) - X_{H_t}(\gamma^\pm))dt.$$

⟨3⟩8. $\lim_{k\to\pm\infty} \int_0^1 \lambda(\partial_t u_{s_k} - X_{H_t}(u_{s_k}))dt = 0$.

PROOF:

⟨4⟩1. $|\int_0^1 (\lambda(\partial_t u_{s_k}) - X_{H_t}(u_{s_k}))dt| \leq \sup_{S^1} \|\lambda\| \|\partial_t u_{s_k} - X_{H_t}(u_{s_k})\|_{L^1(S^1)} \leq C \sup_{S^1} \|\lambda\| \|\partial_t u_{s_k} - X_{H_t}(u_{s_k})\|_{L^2(S^1)}$.

⟨4⟩2. Now use [Stepref error 1.2].

⟨3⟩9. $\lim_{k\to\pm\infty} \int_0^1 \lambda(X_{H_r}(u_{s_k}) - X_{H_t}(\gamma^\pm))dt = 0$.

PROOF:

⟨4⟩1. By ⟨1⟩1.

⟨1⟩4. Q.E.D.

PROOF:

⟨2⟩1. The result follows from ⟨1⟩1,⟨1⟩2 and ⟨1⟩3.

2.3 Floer Chain Complex

We want to define homology groups on a compact symplectic manifold $(M,\omega) \in \mathsf{SympMan}^\infty$. This follows the usual pattern familiar from other homology theories. The first step is to define a $\mathbb{Z}$-graded vector space associated to M. This leads to a complex $CF_*(H,J)$. The second step is to define a boundary operator $\partial : CF_*(H,J) \to CF_{*-1}(H,J)$ and verify that it satisfies $\partial^2 = 0$. Then the corresponding homology groups can be defined as usual by

$$HF_*(H,J) = \frac{\mathrm{Ker}\,(\partial)}{\mathrm{Im}\,(\partial)}. \tag{2.17}$$

In our setting the complex $CF_*(H,J)$ is obtained as the free construction over the non-degenerate, contractible, 1-periodic Hamiltonian orbits in $\mathcal{P}(H)$. We will see that for a generic choice of Hamiltonian H all elements in $\mathcal{P}(H)$ are in fact non-degenerate. The grading of this complex is defined by associating a suitable index to loops in $\mathcal{L}_0(M)$. Thus, if μ_{CZ} denotes this index which we introduce in the next section then the Floer chain groups are defined as

Definition 2.8

$$CF_k(H,J) = \bigoplus_{\substack{\gamma \in \mathrm{Crit}(\mathcal{A}_H) \\ \mu_{CZ}(\gamma)=k}} \mathbb{Z}_2 \gamma.$$

In principle one can consider the chain complex generated over any principal ideal domain. However, in this text we are working over $\mathbb{Z}_2$ since this leads to a number of simplifications. The main luxury afforded by this is that $1 = -1$ so that one does not have to worry about orientations. Moreover, to show that something vanishes it suffices that it is even. To work over a different coefficient ring one has to think about how to coherently orient the moduli spaces associated to the Floer equation so that one can appropriately define the boundary operator ∂ in such a way that it still satisfies $\partial^2 = 0$. For this we refer to [11].

2.4 Conley-Zehnder Index

Our goal here is to define an index associated to non-degenerate loops $\gamma \in \mathrm{Crit}(\mathcal{A}_H)$. This will be done in two steps. Let $[\gamma, w] \in \widetilde{\mathcal{L}(M)}$ such that γ is non-degenerate. The first step is to associate to this element a path $\gamma : t \mapsto A(t)$ where $A(t) \in \mathrm{Sp}(2n)$ such that $A(0) = \mathrm{Id}_{\mathbb{R}^{2n}}$ and $1 \notin \sigma(A(1))$. The second step is to associate to this path γ a number $\mu_{CZ}(\gamma)$. A reference for the properties of μ_{CZ} is [19].

Definition 2.9 *We introduce to following set of paths of matrices.*

$$\mathcal{SP}(2n) = \{\gamma : [0,1] \to \mathrm{Sp}(2n) : \ \gamma(0) = \mathrm{Id}_{\mathbb{R}^{2n}}, \gamma(1) \in \mathrm{Sp}(2n)^*\} \tag{2.18}$$

To define the associated element in $\mathcal{SP}(2n)$ we start by choosing a symplectic trivialisation of the bundle w^*TM. This defines a path of transition matrices $A(t)$ along the boundary S^1. This can be chosen so that $A(0) = \mathrm{Id}_{T_{\gamma(0)}M}$. Moreover, $1 \notin \sigma(A(1))$ since γ is non-degenerate. This path depends on the choice of symplectic trivialisation. However, any two choices yield paths that are homotopic in $\mathcal{SP}(2n)$.

The goal now is to define a suitable index map $\mathcal{SP}(2n) \to \mathbb{Z}$. Let $\gamma \in \mathcal{SP}(2n)$. There are two possible cases. Either $\gamma(1) \in \mathrm{Sp}(2n)^+$ or $\gamma(1) \in \mathrm{Sp}(2n)^-$. By Proposition 1.5 the spaces $\mathrm{Sp}(2n)^+$ and $\mathrm{Sp}(2n)^-$ are path-connected. Hence, we can fix one matrix in each of them and extend γ to terminate at these matrices. For concreteness, we define

$$W^+ = -\mathrm{Id}_{\mathbb{R}^{2n}} \in \mathrm{Sp}(2n)^+ \tag{2.19}$$
$$W^- = \mathrm{diag}(2, -1, \dots, -1, 1/2, -1, \dots, -1) \in \mathrm{Sp}(2n)^-. \tag{2.20}$$

Now choose corresponding path extensions $\gamma_\pm : [1,2] \to \mathrm{Sp}(2n)$ with the property that

$$\gamma_\pm(1) = \gamma(1), \gamma_\pm(2) = W^\pm \text{ and} \tag{2.21}$$
$$\forall t \in [1,2] : \ \gamma_\pm(t) \in \mathrm{Sp}(2n)^\pm. \tag{2.22}$$

The particular choice of such extension makes no difference for the definition of the index.

Proposition 2.6 *Extensions of γ as in Equation 2.21 and Equation 2.22 are unique up to a homotopy that fixes the endpoints.*

PROOF:
⟨1⟩1. This follows immediately from Proposition 1.5.
⟨1⟩2. Q.E.D.

To obtain the desired map $\mathcal{SP}(2n) \to \mathbb{Z}$ we introduce an auxiliary map $\rho : \mathrm{Sp}(2n) \to S^1$. Since $\mathrm{Sp}(2n)$ is a deformation retracts to $U(n)$ their fundamental groups coincide. For $U(n)$ the map $\det : U(n) \to S^1$ induces an isomorphism of the corresponding fundamental groups. The idea then is that ρ will be an extension of det to $\mathrm{Sp}(2n)$ that still induces an isomorphism $\pi_1(\mathrm{Sp}(2n)) \to \pi_1(S^1) \cong \mathbb{Z}$.

First we give an explicit definition for ρ and then we show that it has all the properties in which we are interested. For this we need to introduce a quadratic form on $\mathbb{C}^{2n}$. We denote the real and imaginary part of a complex number by $\Re$ and $\Im$ respectively.

Definition 2.10 *Define the map $Q : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{R}$ to be*

$$Q(x,y) = \Im\omega_0(\overline{x}, y) \tag{2.23}$$

where ω_0 is the standard symplectic structure given by $z^T Jz$. We also set $Q(x) = Q(x,x)$.

Let us collect the properties of this map that are important for our purposes.

Proposition 2.7 *The map Q defined in Equation 2.23 has the following properties.*

1. *The map Q is $\mathbb{R}$-bilinear: $\forall \alpha, \beta \in \mathbb{R}, \forall x, y, u, v \in \mathbb{C}^{2n}$:*
   $Q(\alpha x + u, \beta y + v) = \alpha Q(x) + Q(u) + \beta Q(y) + Q(v)$.
2. *The map Q is symmetric: $\forall x, y \in \mathbb{C}^{2n} : \ Q(x,y) = Q(y,x)$.*
3. *The map Q is non-degenerate: $(\forall y \in \mathbb{C}^{2n} : \ Q(x,y) = 0) \implies x = 0$.*
4. $\forall x, y \in \mathbb{C}^{2n} : \ Q(ix, iy) = Q(x,y)$.
5. $\forall x, y \in \mathbb{C}^{2n} : \ Q(\overline{x}, \overline{y}) = -Q(x,y)$.

PROOF:

⟨1⟩1. The verification of all these properties is an elementary computation using the properties of the symplectic form ω_0 and basic facts about complex numbers.
⟨1⟩2. Q.E.D.

We also need the following result about the signature of Q when restricted to certain generalised eigenspaces of a symplectic matrix.

Proposition 2.8 *Let $A \in \mathrm{Sp}(2n)$ and $\lambda \in \sigma(A) \cap (\mathbb{C}\backslash S^1)$. Then $\sigma(Q|_{G_\lambda(A) \oplus G_{\overline{\lambda}^{-1}}(A)}) = 0$.*

Proof:
⟨1⟩1. $\omega_0|_{G_\lambda \oplus G_{1/\lambda}}$ is non-degenerate.
⟨1⟩2. $\exists u'_1, \ldots, u'_k, v'_1, \ldots, v'_k \in \mathbb{C}^{2n}$:
1. $(u'_1, \ldots, u'_k)$ is a basis of $G_\lambda(A)$.
2. $(v'_1, \ldots, v'_k)$ is a basis of $G_{\overline{\lambda}}(A)$.
3. $(u'_1, \ldots, u'_k, \overline{v'_1}, \ldots, \overline{v'_k})$ is a symplectic basis of $G_\lambda \oplus G_{1/\lambda}$.

⟨1⟩3. Let: $u_j = u'_j + iv'_j$ and $v_j = u'_j - iv'_j$.
⟨1⟩4. Q is positive definite on $\operatorname{span}\{u_1, \ldots, u_k\}$.
Proof:
⟨2⟩1.

$$\begin{aligned} Q(u_j, u_k) &= Q(u'_j + iv'_j, u'_k + iv'_k) \\ &= Q(u'_j, u'_k) + Q(u'_j, iv'_k) + Q(iv'_j, u'_k) + Q(iv'_j, i'v_k) \\ &= Q(u'_j, iv'_k) + Q(iv'_j, u'_k) = Q(iv'_k, u'_j) + Q(iv'_j, u'_k) \\ &= \Im\omega_0(-i\overline{v'_k}, u'_j) + \Im\omega_0(-i\overline{v'_k}, u'_k) \\ &= 2\delta_{jk}. \end{aligned}$$

⟨1⟩5. Q is negative definite on $\operatorname{span}\{v_1, \ldots, v_k\}$.
Proof:
⟨2⟩1.

$$\begin{aligned} Q(v_j, v_k) &= Q(u'_j - iv'_j, u'_k - iv'_k) \\ &= Q(u'_j, u'_k) - Q(u'_j, -iv'_k) - Q(iv'_j, u'_k) + Q(iv'_j, i'v_k) \\ &= -Q(u'_j, iv'_k) - Q(iv'_j, u'_k) = -Q(iv'_k, u'_j) - Q(iv'_j, u'_k) \\ &= -\Im\omega_0(-i\overline{v'_k}, u'_j) + -\Im\omega_0(-i\overline{v'_k}, u'_k) \\ &= -2\delta_{jk}. \end{aligned}$$

⟨1⟩6. $\operatorname{span}\{u_1, \ldots, u_k\} \perp_Q \operatorname{span}\{v_1, \ldots, v_k\}$.
Proof:
⟨2⟩1.

$$\begin{aligned} Q(u_j, v_k) &= Q(u'_j + iv'_j, u'_k - iv'_k) \\ &= Q(iv'_j, u'_k) - Q(iv'_k, u'_j) \\ &= \delta_{jk} - \delta_{jk} = 0. \end{aligned}$$

$\langle 1 \rangle 7$. Q.E.D.

After this preparation we can now turn to the definition of the map ρ. For $A \in \mathrm{Sp}(2n)$ let $\lambda \in \sigma(A) \cap (S^1 \backslash \{\pm 1\})$. The quadratic form Q allows us to decompose the corresponding generalized eigenspace $G_\lambda(A)$ as a direct sum $G_\lambda(A) = G_\lambda^+(A) \oplus G_\lambda^-(A)$ such that Q restricted to each of these summands is positive definite and negative definite, respectively. The dimensions of these subspaces will be denoted by $m_+^A(\lambda)$ and $m_-^A(\lambda)$, respectively. Consequently, the total multiplicity of the eigenvalue λ is given by $m^A(\lambda) = \dim(G_\lambda(A)) = m_+^A(\lambda) + m_-^A(\lambda)$. We also write $\sigma_A(\lambda) = m_+^A(\lambda) - m_-^A(\lambda)$ for the signature of the eigenvalue λ.

Definition 2.11 *Let $A \in \mathrm{Sp}(2n)$ and $m_0 = \sum_{\lambda \in \sigma(A) \cap \mathbb{R}_-} m_-^A(\lambda)$ be the total multiplicity of negative eigenvalues of A. Then we define $\rho : \mathrm{Sp}(2n) \to S^1$ as*

$$\rho(A) = (-1)^{m_0/2} \prod_{\lambda \in \sigma(A) \cap (S^1 \backslash \pm 1)} \lambda^{m_+^A(\lambda)}. \tag{2.24}$$

We begin by showing that ρ is continuous. Continuity means that if a sequence of matrices $(A_k)_{k \in \mathbb{N}} \subset \mathrm{Sp}(2n)$ tends to $A \in \mathrm{Sp}(2n)$ then the values $(\rho(A_k)_{k \in \mathbb{N}})$ tend to $\rho(A)$. Hence, inspecting the definition of ρ we are led to consider the dependence of eigenvalues and eigenspaces under the limiting process. It is a standard result that the eigenvalues of matrix depend continuously on its entries. We state this fact without proof in the following form.

Proposition 2.9 *Let $(A_k)_{k \in \mathbb{N}} \subset \mathbb{C}^{m \times m}$ be a sequence of matrices such that there is a matrix $A \in \mathbb{C}^{m \times m}$ such that $\lim_{k \to \infty} A_k = A$. Let $\lambda_1, \ldots, \lambda_m \in \sigma(A)$ be the eigenvalues of A, including multiple eigenvalues. Then the eigenvalues $\lambda_1^k, \ldots, \lambda_m^k$ of A_k can be ordered in such a way that*

$$\forall i \in \{1, \ldots, m\} : \lim_{k \to \infty} \lambda_i^k = \lambda_i.$$

Next we consider the continuous dependence of the corresponding generalised eigenspaces. Here, continuity is defined with respect to the topology of the Grassmannian spaces. That is, if $Gr_{\mathbb{C}}(r, m)$ denotes the spaces of r-dimensional subspaces of $\mathbb{C}^m$ then we say that a sequence of subspaces $(V_k)_{k \in \mathbb{N}} \subset Gr_{\mathbb{C}}(r, m)$ converges to $V \in Gr_{\mathbb{C}}(r, m)$ if every V_k admits a (unitary) basis B_k that converges to a basis B of V.

Then we have the following result.

Proposition 2.10 *Let $(A_k)_{k \in \mathbb{N}} \subset \mathbb{C}^{m \times m}$ be a sequence of matrices such that there is a matrix $A \in \mathbb{C}^{m \times m}$ such that $\lim_{k \to \infty} A_k = A$ and let $\lambda \in \sigma(A)$. Let $(\lambda_i^k)_{i \in \{1, \ldots, r_k\}}$ be the eigenvalues of A_k such that*

$$\forall i \in \{1, \ldots, r_k\} : \lim_{k \to \infty} \lambda_i^k = \lambda.$$

Then

$$\lim_{k\to\infty} \bigoplus_{\{\lambda_i^k\}} G_{\lambda_i^k}(A_k) = G_\lambda(A).$$

PROOF:
⟨1⟩1. LET: $V_k = \bigoplus_{\{\lambda_i^k\}} G_{\lambda_i^k}(A_k)$.
⟨1⟩2. $V_k \subset \operatorname{Ker}\left(\prod_{i=1}^{r_k}(A_k - \lambda_i^k \mathrm{Id}_{\mathbb{C}^m})^{m_{\lambda_i^k}(A_k)}\right)$
PROOF:
⟨2⟩1. $G_{\lambda_i^k}(A_k) = \bigcup_{j\in\mathbb{N}}(A_k - \lambda_i^k \mathrm{Id}_{\mathbb{C}^m})^{-j}\{0\}$.
⟨1⟩3. LET: $(v_k)_{k\in\mathbb{N}} \subset \mathbb{C}^m$ be a sequence such that $v_k \in V_k$ for all $k \in \mathbb{N}$ and such that there exists a $v \in \mathbb{C}^m$ such that $\lim_{k\to\infty} v_k = v$.
⟨1⟩4. $v \in G_\lambda(A)$.
PROOF:
⟨2⟩1. $(A - \lambda \mathrm{Id}_{\mathbb{C}^m})^{m_\lambda(A)} v = 0$.
⟨1⟩5. Q.E.D.

In order to prove that ρ is continuous it is more convenient to slightly rewrite the definition.

Lemma 2.1

$$\rho(A) = (-1)^{m_0/2} \prod_{\lambda\in\sigma(A)\cap S^1\cap\{z\in\mathbb{C}:\ \Im(z)>0\}} \lambda^{\sigma(\lambda)}. \tag{2.25}$$

PROOF:
⟨1⟩1. LET: $\lambda \in \sigma(A) \cap S^1$.
⟨1⟩2. $\overline{\lambda} \in \sigma(A)$.
⟨1⟩3. $G_\lambda(A) \cong G_{\overline{\lambda}}(A)$.
⟨1⟩4. $\lambda^{m_+^A(\lambda)} \overline{\lambda}^{-m_+^A(\overline{\lambda})} = \lambda^{\sigma(\lambda)} = \overline{\lambda}^{\sigma(\overline{\lambda})}$.
PROOF:
⟨2⟩1. $m_+^A(\overline{\lambda}) = m_-^A(\lambda)$.
⟨2⟩2. $m_-^A(\overline{\lambda}) = m_+^A(\lambda)$.
⟨2⟩3. $\sigma(\overline{\lambda}) = -\sigma(\lambda)$.
⟨1⟩5. Eigenvalues of A in S^1 appear in pairs λ and $\overline{\lambda}$.
⟨1⟩6. Q.E.D.

In other words, it is sufficient to consider eigenvalues that lie on the upper half-circle in the complex plane. When one considers the eigenvalues of the approximating matrices A_k one therefore needs to be concerned with whether the corresponding sequences of eigenvalues of A_k lie in $S^1 \cap \{z \in \mathbb{C} :\ \Im(z) > 0\}$ or not. More precisely, if $\lambda \in \sigma(A) \cap S^1 \cap \{z \in \mathbb{C} :\ \Im(z) > 0\}$ is approximated by sequences $(\lambda_i^k)_{k\in\mathbb{N}}$ with $i \in \{1, \ldots, r_k\}$ then for, say, $i \in \{1, \ldots, s_k\}$ the λ_i^k will lie in this set and for $i \in \{s_k + 1, \ldots, r_k\}$ they will not. In the latter case we note that since elements of S^1 are invariant under continuous map $z \mapsto \overline{z}^{-1}$ we also have that $\overline{\lambda_i^k}^{-1}$ tend to λ. With these preparations we can now prove the continuity of ρ.

Proposition 2.11 *The map ρ from Equation 2.24 is continuous.*

Proof:
⟨1⟩1. Let: $A \in \mathrm{Sp}(2n)$ and $(A_k)_{k\in\mathbb{N}} \subset \mathrm{Sp}(2n)$ such that $\lim_{k\to\infty} A_k = A$.
⟨1⟩2. Let: $\lambda \in \sigma(A) \cap S^1 \cap \{z \in \mathbb{C} : \Im(z) > 0\}$.
⟨1⟩3. Group the sequences of eigenvalues of A_k as follows:
1. (λ_i^k) for $i \in \{1, \dots, s_k\}$ if eventually $\lambda_i^k \in S^1 \cap \{\Im(z) > 0\}$.
2. (λ_i^k) for $i \in \{s_k + 1, \dots, r_k\}$ otherwise.

⟨1⟩4. $\lambda^{\sigma(\lambda)} = \lim_{k\to\infty} \prod_{i=1}^{s_k} (\lambda_i^k)^{\sigma(\lambda_i^k)}$.
Proof:
⟨2⟩1. Case: $s_k = r_k$.
Proof:
⟨3⟩1. $\sum_{i=1}^{s_k} \sigma(\lambda_i^k) = \sigma(\lambda)$ eventually.
⟨2⟩2. Case: $s_k < r_k$.
Proof:
⟨3⟩1. Group the eigenvalues $\lambda_{s_k+1}^k, \dots, \lambda_{r_k}^k$ into pairs of the form $(\lambda, \overline{\lambda}^{-1})$ with the same multiplicity.
⟨3⟩2. $\sigma(Q|_{G_\lambda(A)\oplus G_{\overline{\lambda}^{-1}}(A)}) = 0$ for such pairs.
Proof:
⟨4⟩1. By Proposition 2.8.
⟨3⟩3. This reduces the case to the previous case ⟨2⟩1.
⟨1⟩5. Let: $\lambda \in \sigma(A) \cap \mathbb{R}_{>0}$.
⟨1⟩6. If this $\lambda \neq 0$ then it and the corresponding λ_i^k do not enter into the definition of ρ.
⟨1⟩7. Suffices: to consider $\lambda = 1$.
⟨1⟩8. Then only those $\lambda 6k_i$ that eventually lie in $S^1 \cap \{\Im(z) > 0\}$ are relevant.
⟨1⟩9. Such λ_i^k satisfy $\lim_{k\to\infty} (\lambda^k)^{\sigma(\lambda_i^k)} = 1$.
⟨1⟩10. Let: $\lambda \in \sigma(A) \cap \mathbb{R}_{<0}$.
⟨1⟩11. Group the sequences of eigenvalues of (A_k) as follows:
1. (λ_i^k) for $i \in \{1, \dots, s_k\}$ if eventually $\lambda_i^k \in S^1 \cap \{\Im(z) > 0\}$.
2. (λ_i^k) for $i \in \{s_k + 1, \dots, t_k\}$ if eventually $\lambda_i^k \in \mathbb{R}$.
3. (λ_i^k) for $i \in \{t_k + 1, \dots, r_k\}$ otherwise.

⟨1⟩12. $(-1)^{m(\lambda)/2} = \lim_{k\to\infty} \prod_{i=1}^{s_k} (\lambda_i^k)^{\sigma(\lambda_i^k)} (-1)^{\frac{1}{2}\sum_{i=s_k+1}^{t_k} m(\lambda_i^k)}$.
Proof:
⟨2⟩1. Case: $\lambda = -1$.
Proof:
⟨3⟩1. The $(\lambda_i^k)_{i\in\{t_k+1,\dots,r_k\}}$ can either be grouped into quadruples of the form $(\mu, \overline{\mu}, 1/\mu, \overline{1/\mu})$ or they are given by the conjugates of $(\lambda_{i\in\{1,\dots,s_k\}}^k)$.

⟨3⟩2. For each such quadruple $m(\mu) + m(\overline{\mu}) + m(1/\mu) + m(\overline{1/\mu}) = 4m(\mu)$.

⟨3⟩3. For each such quadruple $\frac{1}{2}(m(\mu) + m(\overline{\mu}) + m(1/\mu) + m(\overline{1/\mu})) = 0 \mod \mathbb{Z}_2$.

⟨3⟩4. $\frac{1}{2}m(\lambda) = \sum_{i=1}^{s_k} \sigma(\lambda_i^k) + \frac{1}{2}\sum_{i=s_k+1}^{t_k} m(\lambda_i^k) \mod \mathbb{Z}_2$.

PROOF:

⟨4⟩1. Modulo $\mathbb{Z}_2$ we have the following equalities

$$\begin{aligned}\frac{1}{2}m(\lambda) &= \frac{1}{2}\sum_{i=1}^{s_k} m(\lambda_i^k) + \frac{1}{2}\sum_{i=1}^{s_k} m(\overline{\lambda_i^k}) + \frac{1}{2}\sum_{i=1}^{t_k} m(\lambda_i^k) \\ &= \sum_{i=1}^{s_k} m(\lambda_i^k) + \frac{1}{2}\sum_{i=1}^{t_k} m(\lambda_i^k) \mod \mathbb{Z}_2.\end{aligned}$$

⟨4⟩2. For any eigenvalue λ we have $\sigma(\lambda) = m(\lambda) \mod \mathbb{Z}_2$.

⟨2⟩2. CASE: $\lambda \neq -1$.

PROOF:

⟨3⟩1. $s_k = 0$.

⟨3⟩2. $\frac{1}{2}\sum_{i=1}^{r_k} m(\lambda_i^k) = \frac{1}{2}\sum_{i=1}^{t_k} m(\lambda_i^k) \mod \mathbb{Z}_2$.

PROOF:

⟨4⟩1. Group $(\lambda_i^k)_{i\in\{t_k+1,\dots,r_k\}}$ into quadruples of the form $(\mu, \overline{\mu}, 1/\mu, \overline{1/\mu})$.

⟨4⟩2. For each such quadruple $m(\mu) + m(\overline{\mu}) + m(1/\mu) + m(\overline{1/\mu}) = 4m(\mu)$.

⟨1⟩13. Q.E.D.

We will need the following properties of ρ. One can also show that a map satisfying these properties is in fact uniquely determined, but this will not be necessary for our purposes.

Theorem 2.2 *For every $n \in \mathbb{N}$ there exists a continuous map $\rho : \mathrm{Sp}(2n) \to S^1$ with the following properties.*

1. $\forall A, B \in \mathrm{Sp}(2n) : \ \rho(BAB^{-1}) = \rho(A)$.
2. $\forall A \in \mathrm{Sp}(2m), B \in \mathrm{Sp}(2n) : \ \rho\left(\begin{bmatrix} A & 0 \\ 0 & B\end{bmatrix}\right) = \rho(A)\rho(B)$.
3. $\forall A \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n) : \ \rho(A) = \det(X + iY)$ *where* $A = \begin{bmatrix} X & -Y \\ Y & X\end{bmatrix}$.
4. *Let $A \in \mathrm{Sp}(2n)$ such that $\sigma(A) \subset \mathbb{R}$ and let m_0 be the total multiplicity of all negative eigenvalues of A. Then $\rho(A) = (-1)^{m_0/2}$.*

PROOF:

⟨1⟩1. LET: ρ be the map defined in Equation 2.24.

⟨1⟩2. ρ is continuous.

PROOF:

⟨2⟩1. By Proposition 2.11.

⟨1⟩3. ρ satisfies item 1.

PROOF:
⟨2⟩1. LET: $A, B \in \mathrm{Sp}(2n)$.
⟨2⟩2. $\sigma(A) = \sigma(BAB^{-1})$ and for all $\lambda \in \sigma(A)$ it holds that $G_\lambda(BAB^{-1}) = B(G_\lambda(A))$.
⟨1⟩4. ρ satisfies item 2.
PROOF:
⟨2⟩1. $G_\lambda\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = G_\lambda(A) \oplus G_\lambda(B)$.
⟨2⟩2. $m^\pm\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = m^\pm(G_\lambda(A)) + m^\pm(G_\lambda(B))$.
⟨2⟩3. $\rho\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \rho(A)\rho(B)$.
⟨1⟩5. ρ satisfies item 3.
PROOF:
⟨2⟩1. LET: $A = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \in \mathrm{Sp}(n) \cap \mathrm{O}(2n)$.
⟨2⟩2. LET: $Z = X + iY \in \mathrm{U}(n)$.
⟨2⟩3. $\lambda \in \sigma(A) \implies \lambda \in \sigma(Z)$.
PROOF:
⟨3⟩1. LET: $\lambda \in \sigma(A)$.
⟨3⟩2. $\exists x = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{R}^{2n} : \ Ax = \lambda x$.
⟨3⟩3. $Z(u + iv) = (X + iY)(u + iv) = \lambda(u + iv)$.
⟨2⟩4. $\lambda \in \sigma(Z) \implies \lambda \in \sigma(A)$.
PROOF:
⟨3⟩1. LET: $\lambda \in \sigma(Z)$.
⟨3⟩2. $\exists z \in \mathbb{C}^n : \ Zz = \lambda z$.
⟨3⟩3. $A\begin{bmatrix} z \\ -iz \end{bmatrix} = \lambda \begin{bmatrix} z \\ -iz \end{bmatrix}$.
⟨2⟩5. $m^+_A(\lambda) = m^+_Z(\lambda)$ and $m_A(-1) = 2m_U(-1)$.
PROOF:
⟨3⟩1. $G_\lambda(A) = E_\lambda(A)$ and $G_\lambda(Z) = E_\lambda(Z)$.
PROOF:
⟨4⟩1. A and Z are diagonalisable over $\mathbb{C}$ since they are in particular normal.
⟨3⟩2. $J_0 \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$ is diagonalisable.
⟨3⟩3. $E_i(J_0) = \{\begin{bmatrix} z \\ iz \end{bmatrix} : z \in \mathbb{C}^n\}$ and $E_{-i}(J_0) = \{\begin{bmatrix} z \\ -iz \end{bmatrix} : z \in \mathbb{C}^n\}$.
⟨3⟩4. The map Q is positive definite on $E_{-i}(J_0)$ and negative definite on $E_i(J_0)$.
PROOF:
⟨4⟩1. LET: $v_+ \in E_i(J_0)$ and $v_- \in E_{-i}(J_0)$.
⟨4⟩2. $Q(v_+) = \Im(\omega(\overline{v_+}, v_+)) = \Im(\omega(\overline{v_+}, -iJ_0 v_+))$
$= -\|\Re v_+\|^2 - \|\Im v_+\|^2$.

⟨4⟩3. $Q(v_-) = \Im(\omega(\overline{v_-}, v_-)) = \Im(\omega(\overline{v_-}, iJ_0 v_-))$
$= \|\Re v_-\|^2 + \|\Im v_-\|^2$.
⟨3⟩5. $G^+_\lambda(A) = E^+_\lambda(A) = E_\lambda(A) \cap E_{-i}(J_0)$ and $G^-_\lambda(A) = E^-_\lambda(A) = E_\lambda(A) \cap E_i(J_0)$.
⟨3⟩6. $E^+_\lambda(A) \cong E_\lambda(Z)$.
Proof:
⟨4⟩1. $E^+_\lambda(A) = \{\begin{bmatrix} z \\ -iz \end{bmatrix} \in \mathbb{C}^{2n} : A \begin{bmatrix} z \\ -iz \end{bmatrix} = \lambda \begin{bmatrix} z \\ -iz \end{bmatrix}\}$.
⟨4⟩2. $E_\lambda(Z) = \{z \in \mathbb{C}^n : Zz = (X + iY)z = \lambda z\}$.
⟨2⟩6. $\det(Z) = (-1)^{m_U(-1)} \prod_{\lambda \in \sigma(Z) \setminus \mathbb{R}} \lambda^{m_U(\lambda)}$.
⟨1⟩6. ρ satisfies item 4.
Proof:
⟨2⟩1. $\sigma(A) \subset \mathbb{R} \implies \sigma(A) \cap (S^1 \setminus \mathbb{R}) = \emptyset$.
⟨2⟩2. $\rho(A) = (-1)^{m_0/2}$.
⟨1⟩7. Q.E.D.

Having the map ρ at our disposal we can give the definition of the index map $\mu : \mathcal{SP}(2n) \to \mathbb{Z}$. For a path $\gamma : [0,1] \to \mathrm{Sp}(2n)$ we can consider the composition $\rho \circ \gamma : [0,1] \to S^1$. This composition admits a lift $\alpha : [0,1] \to \mathbb{R}$ and so we can define

$$\Delta(\gamma) = \frac{\alpha(0) - \alpha(1)}{\pi}. \tag{2.26}$$

Here we regard $\pi : \mathbb{R} \to S^1$ as the universal covering space with $\pi(t) = e^{it}$ so that $\alpha(0) - \alpha(1) \in \pi\mathbb{Z}$. Now take a path extension $\gamma^\pm$ of γ extending it to end at one of the matrices $W^\pm$ depending on the sign of $\det(\gamma(1) - \mathrm{Id}_{\mathbb{R}^{2n}})$.

Definition 2.12 *Let $\gamma \in \mathcal{SP}(2n)$. We define its* Maslov index *to be the number*

$$\mu(\gamma) = \Delta(\gamma) + \Delta(\gamma^\pm). \tag{2.27}$$

This index has the following properties.

Theorem 2.3 *Let μ be as defined in Equation 2.27. This map has the following properties.*

1. $\forall \gamma \in \mathcal{SP}(2n) : \mu(\gamma) \in \mathbb{Z}$.
2. $\forall \gamma_0, \gamma_1 \in \mathcal{SP}(2n) : (\gamma_0 \simeq \gamma_1 \iff \mu(\gamma_0) = \mu(\gamma_1))$.
3. $\forall \gamma \in \mathcal{SP}(2n) : \mathrm{sign}(\det(\gamma(1) - \mathrm{Id}_{\mathbb{R}^{2n}})) = (-1)^{\mu(\gamma) - n}$.
4. $\forall S \in \mathrm{Sym}(\mathbb{R}^{2n}) \cap \mathrm{GL}(2n)$ *with* $\|S\| < 2\pi$:
   $\mu(t \mapsto \exp(tJ_0 S)) = \mu^-(S) - n$.

Proof:
⟨1⟩1. Let: $\gamma^\# = \gamma \# \gamma^\pm$ be the concatenation of γ with the extension $\gamma^\pm$ connecting it to $W^\pm$.

$\langle 1\rangle 2.$ $\mu(\gamma^{\#}) \in \mathbb{Z}$.
PROOF:
$\langle 2\rangle 1.$ $\rho(W^{\pm}) = \pm 1$.
$\langle 2\rangle 2.$ LET: $\alpha : [0,1] \to \mathbb{R}$ be a lift of $\rho \circ \gamma^{\#}$.
$\langle 2\rangle 3.$ $\alpha(1) - \alpha(0) \in \pi\mathbb{Z}$.
$\langle 1\rangle 3.$ item 2 holds.
PROOF:
$\langle 2\rangle 1.$ $\gamma_0 \simeq \gamma_1 \iff \gamma_0^{\#} \simeq \gamma_1^{\#}$.
$\langle 2\rangle 2.$ LET: $\alpha_0, \alpha_1 : [0,1] \to \mathbb{R}$ be lifts of $\rho \circ \gamma_0^{\#}$ and $\rho \circ \gamma_1^{\#}$.
$\langle 2\rangle 3.$ $\gamma_0^{\#} \simeq \gamma_1^{\#} \iff \alpha_0 \simeq \alpha_1 \iff \Delta(\alpha_0) = \Delta(\alpha_1) \iff \mu(\gamma_0) = \mu(\gamma_1)$.
$\langle 1\rangle 4.$ item 3 holds.
PROOF:
$\langle 2\rangle 1.$ CASE: $\gamma(1) \in \mathrm{Sp}(2n)^{+}$.
$\langle 2\rangle 2.$ $\gamma^{\#}(2) = W^{+}$.
$\langle 2\rangle 3.$ $\rho(\gamma^{\#}(2)) = (-1)^{n}$.
$\langle 2\rangle 4.$ CASE: $\gamma(1) \in \mathrm{Sp}(2n)^{-}$.
$\langle 2\rangle 5.$ $\gamma^{\#}(2) = W^{-}$.
$\langle 2\rangle 6.$ $\rho(\gamma^{\#}(2)) = (-1)^{n-1}$.
$\langle 1\rangle 5.$ item 4 holds.
PROOF:
$\langle 2\rangle 1.$ LET: $P \in C^{0}([0,1], \mathrm{O}^{+}(2n))$ such that $P(0) = \mathrm{Id}_{\mathbb{R}^{2n}}$ and $P(1)^{T} S P(1)$ is diagonal.
PROOF:
$\langle 3\rangle 1.$ S is symmetric.
$\langle 2\rangle 2.$ LET: $S(\tau) = P(\tau)^{T} S P(\tau)$ and $\psi_{\tau}(t) = \exp(t J_0 S(\tau))$ for $\tau \in [0,1]$.
$\langle 2\rangle 3.$ $\forall \tau \in [0,1] : \psi_{\tau} \in \mathcal{SP}(2n)$.
PROOF:
$\langle 3\rangle 1.$ $\forall \tau \in [0,1] : 1 \notin \sigma(\psi_{\tau}(1))$.
PROOF:
$\langle 4\rangle 1.$ $\|S\| < 2\pi$.
$\langle 2\rangle 4.$ $\forall \tau, \tau' \in [0,1] : \mu(\psi_{\tau}) = \mu(\psi_{\tau'})$.
$\langle 2\rangle 5.$ SUFFICES ASSUME: S is diagonal.
$\langle 2\rangle 6.$ SUFFICES ASSUME: The only eigenvalues of S are $\pm\pi$.
$\langle 2\rangle 7.$ LET: $S_1 = \mathrm{diag}(\pi, \pi), S_2 = \mathrm{diag}(\pi, -\pi)$ and $S_3 = \mathrm{diag}(-\pi, -\pi)$.
$\langle 2\rangle 8.$ LET: $R_1(t) = \exp(t J_0 S_1) = \begin{bmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix}$.
$\langle 2\rangle 9.$ $\mu(R_1(t)) = -1 = \mu^{-}(S) - 1$.
PROOF:
$\langle 3\rangle 1.$ $\rho(R_1(t)) = e^{it\pi}$.
$\langle 2\rangle 10.$ LET: $R_2(t) = \exp(t J_0 S_2) = \begin{bmatrix} \cosh(\pi t) & \sinh(\pi t) \\ \sinh(\pi t) & \cosh(\pi t) \end{bmatrix}$.
$\langle 2\rangle 11.$ $\mu(R_2(t)) = 0 = \mu^{-}(S) - 1$.
PROOF:

⟨3⟩1. $\sigma(R_2(t)) = \{e^{\pi t}, e^{-\pi t}\}$.
⟨3⟩2. $\rho(R_2(t)) = 1$.
⟨2⟩12. Let: $R_3(t) = \exp(tJ_0S_3) = \begin{bmatrix} \cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & \cos(\pi t) \end{bmatrix}$.
⟨2⟩13. $\mu(R_3(t)) = 1 = \mu^-(S) - 1$.
Proof:
⟨3⟩1. $\rho(R_1(t)) = e^{-it\pi}$.
⟨1⟩6. Q.E.D.

This concludes the definition of the index map μ for paths in $\mathcal{SP}(2n)$. Now we can simply define the Conley-Zehnder index of $[\gamma, w] \in \widetilde{\mathcal{L}_0(M)}$ as explained in the beginning.

Definition 2.13 *We set*

$$\mu_{CZ}([\gamma, w]) = \mu_{CZ}(\gamma) = \mu(A)$$

where A is the path of symplectic matrices associated to $[\gamma, w]$ as described in the beginning of the section. Note that the definition does not depend on the choice of w according to our assumptions on M.

2.5 Solutions of the Floer Equation

2.5.1 Analytic Framework

In this section we briefly set up the analytic framework which is used for the analysis of the Floer equation, its linearisation and related equations. For Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ we denote by $W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$ the usual spaces spaces of k-times weakly differentiable (equivalence classes of) functions whose derivatives up to order k are p-integrable. There are also local versions of these spaces, denoted by $W^{p,k}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ where integrability only needs to hold on every compact subset $K \subset \mathbb{R}^n$. The corresponding notion of convergence for these localised spaces is given by uniform convergence in the local $W^{k,p}(K)$-norm for every compact subset of the domain.

Similarly, we have the notion of C^∞_{loc}-convergence which by definition is uniform convergence of all derivatives on every compact subset K. There is a corresponding metric that induces this notion of convergence which we denote simply by d. In particular, whenever we work in the space $\mathcal{L}_0(M)$ this is metric we use.

Often we will need to work with Sobolev spaces of functions taking values in a compact manifold M. These can be defined with respect to some choice of local coordinates on M. Then compactness of M implies that this definition does depend on the particular choice of coordinates. The usual Sobolev embedding theorems continue to hold for the manifold-valued setting. A case that is especially relevant for our purposes is when the domain of definition if 2-dimensional such as $\mathbb{R}^2$ or $\mathbb{R}^2 \times S^1$. Then for $p > 2$ elements of $W^{1,p}_{loc}$ have unique continuous representatives so that we may consider them as being pointwise defined.

2.5.2 Elliptic Regularity

In this section we want to study the regularity properties of solutions to the Floer equation and its linearisation. The main result is that any weak solution is automatically infinitely often differentiable and that the weak and strong topologies coincide on the space of solution. These results are crucial for the proofs concerning the transversality of the moduli spaces and their compactness. We will work our way towards this regularity result through a number intermediary steps. In the following, whenever we take derivatives of maps that are not differentiable that is to be understood in the sense of distributions.

As is typical when dealing with estimates in functional analysis there is a tendency for numerous constant to proliferate coming from various bounds on embedding and linear operators. In order to maintain a reasonable degree of readability we will therefore generically accumulate all constant into the letter C which may therefore have varying values at different stages within the same proof. This is justified since we are really only interested in the existence of certain estimates and not what properties the corresponding constant has.

The backbone for all results in this section the following regularity result about solutions to a certain type of generalised 2-dimensional Laplace equation.

Proposition 2.12 *Let* $U \in \mathsf{Open}(\mathbb{R}^2)$ *and* $u, f, g, h \in L^p_{loc}(U)$ *and assume that these functions satisfy the equation*

$$\Delta u = f + \partial_s g + \partial_t h. \tag{2.28}$$

Then $u \in W^{1,p}_{loc}(U)$ *and for every relatively compact set* $V \in \mathsf{Open}(U)$ *with the property that* $\overline{V} \subset U$ *there exists a* $C > 0$ *such that*

$$\|u\|_{W^{1,p}(V)} \leq C \left(\|f\|_{L^p(U)} + \|g\|_{L^p(U)} + \|h\|_{L^p(U)} + \|u\|_{L^p(U)} \right). \tag{2.29}$$

Proof:
⟨1⟩1. Let: $V \in \mathsf{Open}(U)$ such that V is relatively compact and $\overline{V} \subset U$.
⟨1⟩2. Case: $f = g = h = 0$.
  Proof:
  ⟨2⟩1. $u \in W^{1,p}_{loc}(U)$.

Proof:
⟨3⟩1. u is harmonic and hence $u \in C^\infty(U)$.
⟨2⟩2. Let: $V \in \mathsf{Open}(U)$ such that V is relatively compact and $\overline{V} \subset U$.
⟨2⟩3. Let: $r > 0$ such that $\overline{V} \subset U_{3r} = \{x \in U : \ B_{3r}(x) \subset U\}$.
⟨2⟩4. Let: $\chi_r = \frac{1}{\pi r^2}\chi_{B_r(0)}$ and $\psi = \chi_r \star \chi_r \star \chi_r$.
⟨2⟩5. $u = \psi \star u$ in U_{3r}.
⟨2⟩6. $\|u\|_{W^{1,p}(V)} \leq C\|u\|_{L^p(V)} \leq C\|u\|_{L^p(U)}$.
Proof:
⟨3⟩1. $\|\partial_s u\|_{L^p(V)} = \|\partial_s\psi \star u\|_{L^p(V)} \leq \sup_U \|\partial_s\psi\|\|u\|_{L^p(V)} \leq C\|u\|_{L^p(V)}$.
⟨3⟩2. $\|\partial_t u\|_{L^p(V)} = \|\partial_t\psi \star u\|_{L^p(V)} \leq \sup_U \|\partial_t\psi\|\|u\|_{L^p(V)} \leq C\|u\|_{L^p(V)}$.
⟨1⟩3. Case: $f, g, h \in L^p_{loc}(U)$ are arbitrary.
Proof:
⟨2⟩1. Let: $\beta \in C_0^\infty(U)$ be a compactly supported, smooth bump function such that $\beta|_V = 1$ and $V - \text{supp}(\beta) = \{x - y : \ x \in V \text{ and } y \in \text{supp}(\beta)\}$.
⟨2⟩2. Let: $K(s,t) = \frac{1}{2\pi}\log(\sqrt{s^2+t^2})$ be the fundamental solution of the 2-dimensional Laplacian, and $K_s(s,t) = \partial_s K(s,t) = \frac{s}{2\pi(s^2+t^2)}$ $K_t(s,t) = \partial_t K(s,t) = \frac{t}{2\pi(s^2+t^2)}$ its partial derivatives.
⟨2⟩3. Let: $v = K \star \beta f + K_s \star \beta g + K_t \star \beta h$.
⟨2⟩4. $\Delta(u - v) = 0$ in V.
Proof:
⟨3⟩1. $\Delta u = f + \partial_s g + \partial_t h$.
⟨3⟩2.

$$\begin{aligned}\Delta v &= \Delta K \star (\beta f) + \Delta K_s \star (\beta g) + \Delta K_t \star (\beta h) \\ &= \delta \star (\beta f) + \partial_s\delta \star (\beta g) + \partial_t\delta \star (\beta h) \\ &= \beta f + \partial_s(\beta f) + \partial_t(\beta h),\end{aligned}$$

⟨3⟩3. $\beta|_V = 1$.
⟨2⟩5. $\|\nabla v\|_{W^{1,p}(V)} \leq C(\|f\|_{L^p(U)} + \|g\|_{L^p(U)} + \|h\|_{L^p(U)})$.
Proof:
⟨3⟩1. $\|\nabla(K \star (\beta f))\|_{L^p(U)} \leq C\|f\|_{L^p(\text{supp}\,\beta)}$.
Proof:
⟨4⟩1. $\|\nabla(K \star (\beta f))\|_{L^p(U)} \leq \|K_s \star (\beta f)\|_{L^p(U)} + \|K_t \star (\beta f)\|_{L^p(U)}$.
⟨4⟩2. $\|K_s \star (\beta f)\|_{L^p(U)} \leq \|K_s\|_{L^1(V - \text{supp}\,\beta)}\|\beta f\|_{L^p(\text{supp}\,\beta)} \leq C\|f\|_{L^p(\text{supp}\,\beta)}$.
⟨4⟩3. $\|K_t \star (\beta f)\|_{L^p(U)} \leq \|K_s\|_{L^1(V - \text{supp}\,\beta)}\|\beta f\|_{L^p(\text{supp}\,\beta)} \leq C\|f\|_{L^p(\text{supp}\,\beta)}$.
⟨3⟩2. $\|\nabla(K_s \star (\beta g))\|_{L^p(U)} \leq C\|g\|_{L^p()}$.
Proof:
⟨4⟩1. Let: $(g_n)_{n\in\mathbb{N}} \subset C_0^\infty(U) : \ L^p(U) - \lim_{n\to\infty} g_n = \beta g$.
⟨4⟩2. $\nabla(K_s \star g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(U)$.
Proof:
⟨5⟩1. $\forall n, m \in \mathbb{N} : \ \|\nabla(K_s \star g_n) - \nabla(K_s \star g_m)\|_{L^p} \leq \|\Delta(g_n - g_m)\|_{L^p}$ by the Calerón-Zygmund inequality.
⟨4⟩3. $L^p(U) - \lim_{n\to\infty} \nabla(K_s \star g_n) = \nabla(K_s \star (\beta g))$.

⟨4⟩4. $\nabla(K_s \star (\beta g)) \in L^p(U)$.
⟨4⟩5. $\|\nabla(K_s \star (\beta g))\|_{L^p(U)} \leq C\|g\|_{L^p(\operatorname{supp}\beta)}$.
⟨3⟩3. $\|\nabla(K_t \star (\beta h))\|_{L^p(U)} \leq C\|h\|_{L^p()}$.
PROOF:
⟨4⟩1. Replace s with t and g with h in the proof of ⟨3⟩2.
⟨2⟩6. $\|v\|_{W^{1,p}(V)} \leq C(\|f\|_{L^p(U)} + \|g\|_{L^p(U)} + \|h\|_{L^p(U)})$.
PROOF:
⟨3⟩1. Combine ⟨2⟩5 with the Poincaré inequality $\|v\|_{W^{1,p}(V)} \leq C\|\nabla v\|_{W^{1,p}(V)}$.
⟨1⟩4. Q.E.D.
PROOF:
⟨2⟩1. Applying ⟨1⟩2 to $u - v$ we obtain

$$\begin{aligned}\|u\|_{W^{1,p}(V)} &\leq \|u-v\|_{W^{1,p}(V)} + \|v\|_{W^{1,p}(V)} \\ &\leq C(\|f\|_{L^p(U)} + \|g\|_{L^p(U)} + \|h\|_{L^p(U)} + \|u\|_{L^p(U)}).\end{aligned}$$

Linear Results

The linear results in this subsection will be needed for the analysis of the differential of the Floer operator. As we will see later in suitable local coordinates the differential has the form $D_s = \partial_s + J_0\partial_t + S$ where S is a matrix valued function on $\mathbb{R} \times S^1$. For now we assume that $S \in C^0(\mathbb{R} \times S^1, \mathsf{End}(\mathbb{R}^{2m}))$ and that there are limit matrices $S^\pm \in C^0(S^1, \mathsf{End}(\mathbb{R}^{2n}))$ such that $\lim_{s\to\pm\infty} S(s,t) = S^\pm(t)$. We will rely on the following result about the Cauchy-Riemann operator $\bar\partial = \partial_s + J_0\partial_t$ without proof.

Proposition 2.13 *Let $p > 1$, $U \in \mathsf{Open}(\mathbb{C})$, $f \in W^{k,p}_{loc}(U)$ and $u \in L^p_{loc}(U)$ such that $\bar\partial u = f$. Then $u \in W^{k+1,p}_{loc}(U)$. Furthermore, if $V \in \mathsf{Open}(\mathbb{C})$ such that $\overline{V} \subset U$ then there exists a $C \in \mathbb{R}$ such that*

$$\|u\|_{W^{k+1,p}(V)} \leq C\left(\|f\|_{W^{k,p}(U)} + \|u\|_{L^p(U)}\right).$$

We start with the following lemma.

Lemma 2.2 *Let $p > 1$. Consider D_S as an operator $W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Then there exists a $C \in \mathbb{R}$ such that for every $\xi \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ we have*

$$\|\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq C(\|D_S\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})}).$$

PROOF:
⟨1⟩1. LET: $U \in \mathsf{Open}(\mathbb{R} \times S^1)$ and $K \subset U$ be compact.
⟨1⟩2. $\|\xi\|_{W^{1,p}(K,\mathbb{R}^{2n})} \leq C(\|D_S\xi\|_{L^p(U,\mathbb{R}^{2n})} + \|\xi\|_{L^p(U,\mathbb{R}^{2n})})$.
PROOF:
⟨2⟩1. $\|\xi\|_{W^{1,p}(K,\mathbb{R}^{2n})} \leq C(\|\bar\partial\xi\|_{L^p(U,\mathbb{R}^{2n})} + \|\xi\|_{L^p(U,\mathbb{R}^{2n})})$ by Proposition 2.13.
⟨2⟩2. $\|\bar\partial\xi\|_{L^p(K,\mathbb{R}^{2n})} \leq \|D_S\xi\|_{L^p(U,\mathbb{R}^{2n})} + \|S\xi\|_{L^p(U,\mathbb{R}^{2n})}$.

⟨1⟩3. Case: $S = S(t)$ is independent of s.
Proof:
⟨2⟩1. $\|\xi\|_{W^{1,p}([0,1]\times S^1,\mathbb{R}^{2n})} \leq C(\|D_S\xi\|_{L^p([-1,2]\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^p([-1,2]\times S^1,\mathbb{R}^{2n})})$.
⟨2⟩2. D_S is translation-invariant: $\forall s, s_0 \in \mathbb{R}, t \in S^1 : \; D_S(\xi(s+s_0,t)) = (D_S\xi)(s+s_0,t)$.
⟨2⟩3. $\forall k \in \mathbb{Z} : \; \|\xi\|_{W^{1,p}([k,k+1]\times S^1,\mathbb{R}^{2n})} \leq C(\|D_S\xi\|_{L^p([k-1,k+2]\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^p([k-1,k+2]\times S^1,\mathbb{R}^{2n})})$.
⟨2⟩4. $\|\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq C(\|D_S\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})})$.
Proof:
⟨3⟩1. $\forall k \in \mathbb{Z} : \; \|\xi\|^p_{W^{1,p}([k,k+1]\times S^1,\mathbb{R}^{2n})} \leq C(\|D_S\xi\|^p_{L^p([k-1,k+2]\times S^1,\mathbb{R}^{2n})} + \|\xi\|^p_{L^p([k-1,k+2]\times S^1,\mathbb{R}^{2n})})$ using the inequality $(a+b)^p \leq 2^p(a^p+b^p)$.
⟨3⟩2. $\|\xi\|^p_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq C(\|D_S\xi\|^p_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|\xi\|^p_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})})$
$\leq C(\|D_S\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})})^p$.
⟨1⟩4. Case: $S \in C^0(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ is arbitrary.
Proof:
⟨2⟩1. Let: $\varepsilon > 0$.
⟨2⟩2. $\exists M > 0$:
1. $\forall s > M, t \in S^1 : \; \|S(s,t) - S^+(t)\| < \varepsilon$.
2. $\forall s < -M, t \in S^1 : \; \|S(s,t) - S^-(t)\| < \varepsilon$.
⟨2⟩3. Case: $\mathrm{supp}(\xi) \subset \{s > M\}$.
Proof:
⟨3⟩1. $\|\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq C(\|D_S\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \varepsilon\|\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.
Proof:
⟨4⟩1. $\|\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq C(\|D_{S^+}\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})})$ by ⟨1⟩3.
⟨4⟩2. $\|D_{S^+}\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} = \|D_{S^+}\xi\|_{L^p([M,\infty)\times S^1,\mathbb{R}^{2n})}$ by ⟨2⟩3.
⟨4⟩3. $\|D_{S^+}\xi\|_{L^p([M,\infty)\times S^1,\mathbb{R}^{2n})} \leq \|D_S\xi\|_{L^p([M,\infty)\times S^1,\mathbb{R}^{2n})} + \varepsilon\|\xi\|_{W^{1,p}([M,\infty)\times S^1,\mathbb{R}^{2n})}$ by ⟨2⟩2.
⟨4⟩4. Now use monotonicity of the Sobolev norms with respect to the domain of integration.
⟨3⟩2. Now choose ε sufficiently small so that the desired inequality holds.
⟨2⟩4. Case: $\mathrm{supp}(\xi) \subset \{s < -M\}$.
Proof:
⟨3⟩1. Exactly analogous to ⟨2⟩3.
⟨2⟩5. Case: $\mathrm{supp}(\xi) \subset \{|s| > M\}$.
Proof:
⟨3⟩1. Let: $\xi^+ = \chi_{\{s>M\}}\xi$ and $\xi^- = \chi_{\{s<-M\}}\xi$ where χ denotes the indicator function.
⟨3⟩2. $\xi = \xi^+ + \xi^-$.
⟨3⟩3. $\|\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq C(\|D_S\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})})$.
Proof:
⟨4⟩1. Apply ⟨2⟩3 and ⟨2⟩4 to ξ^+ and ξ^- respectively and combine the

inequalities.

⟨2⟩6. Case: $\mathrm{supp}(\xi) \subset \{-M-1 \le s \le M+1\}$.

Proof:

⟨3⟩1. By ⟨1⟩2.

⟨2⟩7. Case: $\xi \in W^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ is arbitrary.

Proof:

⟨3⟩1. Let: $\beta \in C^\infty(\mathbb{R}, [0,1])$ be a smooth bump function such that:

1. $\forall |s| \ge M+1:\ \beta(s) = 0$.
2. $\forall |s| \le M:\ \beta(s) = 1$.

⟨3⟩2. $\xi = \beta\xi + (1-\beta)\xi$.

⟨3⟩3. $\|\beta\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \le C(\|D_S(\beta\xi)\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|\beta\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})})$ by ⟨2⟩6.

⟨3⟩4. $\|(1-\beta)\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \le C(\|D_S((1-\beta)\xi)\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|(1-\beta)\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})})$ by ⟨2⟩5.

⟨3⟩5. Let: $K = \sup_{\mathbb{R}} |\beta'(s)|$.

⟨3⟩6. $\|D_S(\beta\xi)\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \le \|D_S\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + K\|\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.

⟨3⟩7. $\|D_S((1-\beta)\xi)\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \le \|D_S\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + K\|\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.

⟨3⟩8. $\|\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \le \|\beta\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|(1-\beta)\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.

⟨3⟩9. Now combine ⟨3⟩3,⟨3⟩4 and ⟨3⟩8.

⟨1⟩5. Q.E.D.

This allows us to prove the main regularity result for the linear case.

Theorem 2.4 *Now assume additionally that $S \in C^\infty(\mathbb{R}\times S^1, \mathrm{End}(\mathbb{R}^{2n}))$ and that*

$$\lim_{s\to\pm\infty} \partial_s S = 0.$$

Let $\xi \in L^p(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ such that $D_S\xi = 0$. Then $\xi \in C^\infty(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ and $\xi \in W^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$. If in addition $p > 2$ then $\xi \in W^{1,q}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ for every $q > 1$.

Proof:

⟨1⟩1. $\bar\partial\xi = -S\xi \in L^p(\mathbb{R}\times S^1, \mathbb{R}^{2n})$.

⟨1⟩2. $\forall k \in \mathbb{N}:\ \xi \in W^{k,p}_{loc}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$.

Proof:

⟨2⟩1. $\xi \in W^{1,p}_{loc}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ by Proposition 2.13.

⟨2⟩2. If $\xi \in W^{k,p}_{loc}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ then $\xi \in W^{k+1,p}_{loc}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$.

Proof:

⟨3⟩1. $S\xi \in W^{k,p}_{loc}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$.

⟨3⟩2. $\xi \in W^{k+1,p}_{loc}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ by Proposition 2.13.

⟨1⟩3. $\xi \in C^\infty(\mathbb{R}\times S^1, \mathbb{R}^{2n})$.

Proof:

⟨2⟩1. By ⟨1⟩2 and the Sobolev embedding.

⟨1⟩4. $\xi \in W^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$.

Proof:
$\langle 2\rangle 1.$ By Lemma 2.2.
$\langle 1\rangle 5.$ Assume: $p > 2$.
$\langle 1\rangle 6.$ Suffices: $\xi \in L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ for every $q > 1$.
Proof:
$\langle 2\rangle 1.$ Redo the bootstrapping proof from $\langle 1\rangle 2$.
$\langle 1\rangle 7.$ Let: $f(s) = \int_0^1 \|\xi(s,t)\|^2 dt$.
$\langle 1\rangle 8.$ $\sup_{s\in\mathbb{R}} |f(s)| < \infty$.
Proof:
$\langle 2\rangle 1.$ $\int_0^1 \|\xi(s,t)\|^2 dt \le (\int_0^1 \|\xi(s,t)\|^p dt)^{2/p}$ for every $s \in \mathbb{R}$ by Hölder's inequality.
$\langle 2\rangle 2.$ $\int_{-\infty}^{\infty} \int_0^1 \|\xi(s,t)\|^p dt ds = \|\xi\|^p_{L^p(\mathbb{R}\times S^1, \mathbb{R}^{2n})}$.
$\langle 1\rangle 9.$ $\exists \delta > 0 : \ \forall s \in \mathbb{R}, t \in S^1 : \ \|\xi(s,t)\| \le e^{-\delta|s|}$.
Proof:
$\langle 2\rangle 1.$ This follows from the exponential decay property of such solutions proved later in this section.
$\langle 1\rangle 10.$ Q.E.D.
Proof:
$\langle 2\rangle 1.$ $\langle 1\rangle 9$ implies that ξ is p-integrable.

Nonlinear Results

Now we turn towards the nonlinear results. We begin with the following lemma.

Lemma 2.3 *Let $U, V \in \mathsf{Open}(\mathbb{R} \times S^1)$ be relatively compact sets such that $\overline{V} \subset U$ and $p, q, r \in \mathbb{R} \cup \{\infty\}$ such that $p > 2, r > 1$ and $1/p + 1/q = 1/r$. Furthermore, let $u \in L^p_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n}), f \in L^r_{loc}(U, \mathbb{R}^{2n})$and $J \in W^{1,p}(U, \mathcal{J}(\mathbb{R}^{2n}))$ with the property that $\|J\|_{W^{1,p}(U,\mathcal{J}(\mathbb{R}^{2n}))} \le C$ for some $C > 0$. Assume that u satisfies the equation*

$$\partial_s u + J \partial_t u = f. \tag{2.30}$$

Then, if $u \in L^q_{loc}(U, \mathbb{R}^{2n})$, the following is true.

1. *$u \in W^{1,r}_{loc}(U, \mathbb{R}^{2n})$.*
2. *$\exists K > 0 : \ \|u\|_{W^{1,r}(V,\mathbb{R}^{2n})} \le K \left(\|f\|_{L^r(U,\mathbb{R}^{2n})} + \|u\|_{L^q(U,\mathbb{R}^{2n})} \right)$.*

Proof:
$\langle 1\rangle 1.$ $\Delta u = \partial_s ((\partial_t J)u + f) + \partial_t (-(\partial_s J)u - Jf)$.
Proof:
$\langle 2\rangle 1.$ Apply $\partial_s - J\partial_t$ to Equation 2.30.
$\langle 1\rangle 2.$ Assume: $u \in L^q_{loc}(U, \mathbb{R}^{2n})$.
$\langle 1\rangle 3.$ $(\partial_t J)u + f \in L^r_{loc}(U, \mathbb{R}^{2n})$ and $-(\partial_s J)u - Jf \in L^r_{loc}(U, \mathbb{R}^{2n})$.

PROOF:
⟨2⟩1. LET: $K \subset U$ be compact.
⟨2⟩2. $\int_K \|(\partial_t J)u\|^r dtds \leq (\int_K \|\partial_t J\|^p dtds)^{r/p} (\int_K \|u\|^q dtds)^{r/q}$ by Hölder's inequality.
⟨2⟩3. $\int_K \|(\partial_s J)u\|^r dtds \leq (\int_K \|\partial_s J\|^p dtds)^{r/p} (\int_K \|u\|^q dtds)^{r/q}$ by Hölder's inequality.
⟨1⟩4. $u \in L^r_{loc}(U, \mathbb{R}^{2n})$.
PROOF:
⟨2⟩1. LET: $K \subset U$ be compact.
⟨2⟩2. $\int_K \|u\|^r dtds \leq (\int_K \|u\|^p dtds)^{2r/p} (int_K \|u\|^q dtds)^{2r/q}$ by Hölder's inequality.
⟨1⟩5. $u \in W^{1,r}_{loc}(U, \mathbb{R}^{2n})$.
PROOF:
⟨2⟩1. Apply Proposition 2.12.
⟨1⟩6. $\exists C > 0 : \|u\|_{W^{1,r}(V,\mathbb{R}^{2n})} \leq C \left(\|f\|_{L^r(U,\mathbb{R}^{2n})} + \|u\|_{L^q(U,\mathbb{R}^{2n})} \right)$
PROOF:
⟨2⟩1. $\exists C > 0 : \|u\|_{W^{1,r}(V,\mathbb{R}^{2n})} \leq C \left(\|(\partial_t J)u\|_{L^r(U)} + \|f\|_{L^r(U)} + \|(\partial_s J)u\|_{L^r(U)} + \|Jf\|_{L^r(U)} \right)$ by Proposition 2.12.
⟨2⟩2. $\|(\partial_t J)u\|_{L^r(U)} \leq \|J\|_{W^{1,p}(U)} \|u\|_{L^q(U)} \leq C\|u\|_{L^q(U)}$.
⟨2⟩3. $\|(\partial_s J)u\|_{L^r(U)} \leq \|J\|_{W^{1,p}(U)} \|u\|_{L^q(U)} \leq C\|u\|_{L^q(U)}$.
⟨2⟩4. $\|Jf\|_{L^r(U)} \leq C\|f\|_{L^r(U)}$ by the assumption on J.
⟨1⟩7. Q.E.D.

Applying the previous result to our setting we obtain the following theorem.

Theorem 2.5 *Let $k, p \in \mathbb{R}_{\geq 0}$ such that $p > 2$. Let $u \in L^p_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n}), h \in W^{k,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ and $J \in W^{\min(k,1),p}_{loc}(\mathbb{R} \times S^1, \mathcal{J}(\mathbb{R}^{2n}))$. Assume that u satisfies the equation*

$$\partial_s u + J\partial_t u + h = 0. \tag{2.31}$$

Then $u \in W^{k+1,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Moreover, if $V, U \in \mathsf{Open}(\mathbb{R} \times S^1)$ are relatively compact such that $V \subset U$ then there exists a constant $C > 0$ such that

$$\|u\|_{W^{l+1,p}(V,\mathbb{R}^{2n})} \leq C(\|h\|_{W^{l,p}(U,\mathbb{R}^{2n})} + \|u\|_{W^{l,p}(U,\mathbb{R}^{2n})}) \tag{2.32}$$

for every $l \in \{0, \ldots, \max(k-1, 0)\}$.

PROOF:
⟨1⟩1. CASE: $k = 0$.
PROOF:
⟨2⟩1. $u \in W^{1,p/2}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
PROOF:
⟨3⟩1. By Lemma 2.3 and taking $p = q$.

⟨2⟩2. Let: $r \in (1,2), q = \frac{2r}{2-r}$ and $r' = \frac{pq}{p+q} = \frac{2pr}{2p+2r-pr}$.
⟨2⟩3. If $u \in W^{1,r}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ then $u \in W^{1,r'}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
Proof:
⟨3⟩1. $u \in L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ by Rellich's embedding theorem.
⟨3⟩2. $u \in W^{1,r'}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ by Lemma 2.3 since $1/r' = 1/p + 1/q$.
⟨2⟩4. $\forall r \in (1,2] : u \in W^{1,r}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
Proof:
⟨3⟩1. Let: $r_0 \in (1, \min(2, p/2))$ and $r_{m+1} = \frac{2pr_m}{2p+2r_m-pr_m}$ for $m \in \mathbb{N}$.
⟨3⟩2. $\forall m \in \mathbb{N}_0 : r_{m+1} > r_m$.
Proof:
⟨4⟩1. $r_{m+1} = r_m \frac{2p}{2p+(2-p)r_m} > r_m$ since $p > 2$.
⟨3⟩3. $\exists m \in \mathbb{N}_0 : r_m \geq 2$.
Proof:
⟨4⟩1. Assume: $\forall m \in \mathbb{N} : r_m < 2$.
⟨4⟩2. $\exists \tilde{r} \in (1,2] : \lim_{m\to\infty} r_m = \tilde{r}$.
⟨4⟩3. $\tilde{r} = \frac{2p\tilde{r}}{2p+2\tilde{r}-p\tilde{r}}$.
⟨4⟩4. $(2-p)\tilde{r}^2 = 0$ which is a contradiction since $p > 2$.
⟨3⟩4. The statement follows using ⟨2⟩3.
⟨2⟩5. $u \in W^{1,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
Proof:
⟨3⟩1. $u \in L^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
⟨3⟩2. Now apply Lemma 2.3 with $q = \infty$.
⟨2⟩6. There exists a $C > 0$ such that Equation 2.32 holds.
⟨1⟩2. Case: $k = 1$.
Proof:
⟨2⟩1. Let: $u_s = \partial_s u$, $u_t = \partial_t u$, $g_1 = \partial_s h + (\partial_s J)(\partial_t u)$ and $g_2 = \partial_t h + (\partial_t J)(\partial_t u)$.
⟨2⟩2. $\partial_s u_s + J\partial_t u_s + g_1 = 0$.
Proof:
⟨3⟩1. Apply ∂_s to Equation 2.31.
⟨2⟩3. $\partial_s u_t + J\partial_t u_t + g_2 = 0$.
Proof:
⟨3⟩1. Apply ∂_t to Equation 2.31.
⟨2⟩4. $u_s, u_t \in W^{1,p/2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
Proof:
⟨3⟩1. $g_1, g_2 \in L^{p/2}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
Proof:
⟨4⟩1. By Hölder's inequality.
⟨3⟩2. Apply Lemma 2.3.
⟨2⟩5. $u_s, u_t \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
⟨2⟩6. $u \in W^{2,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
⟨1⟩3. If the statement holds for some k then it also holds for $k+1$.

PROOF:
⟨2⟩1. ASSUME: $J \in W^{k+1,p}_{loc}(\mathbb{R} \times S^1, \mathrm{End}(\mathbb{R}^{2n}))$ with $J^2 = -\mathrm{Id}_{\mathbb{R}^{2n}}$ and $h \in W^{1,k+1}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
⟨2⟩2. ASSUME: The statement holds for k.
⟨2⟩3. LET: $u \in L^p_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ be a solution of Equation 2.31.
⟨2⟩4. $u \in W^{k+1,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ by ⟨2⟩2.
⟨2⟩5. $u \in W^{k+2,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
PROOF:
⟨3⟩1. $(\partial_s J)(\partial_t u) \in W^{k,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
⟨3⟩2. $\partial_s u, \partial_t u \in W^{k+1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
⟨1⟩4. Q.E.D.

We can now use this theorem to show that weak solutions of the Floer equation are automatically infinitely often differentiable.

Theorem 2.6 *Let $u \in W^{1,p}_{loc}(\mathbb{R} \times S^1, M)$ such that $\bar{\partial}_{H,J}(u) = 0$. Then $u \in C^\infty(\mathbb{R} \times S^1, M)$. Moreover, The C^∞_{loc} and $W^{1,p}_{loc}$ topologies coincide on the space $\{u \in W^{1,p}_{loc} : \bar{\partial}_{H,J}(u) = 0\}$.*

PROOF:
⟨1⟩1. Since M is compact there exists a finite atlas of charts, and we may assume that u takes values in $\mathbb{R}^{2n}$.
⟨1⟩2. $\forall k \geq 1 : \; u \in W^{k,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \implies u \in W^{k+1,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
PROOF:
⟨2⟩1. By Theorem 2.5.
⟨1⟩3. $u \in \cap_{k \geq 1}^\infty W^{k,p}_{loc}(\mathbb{R} \times S^1, M) \subset C^\infty(\mathbb{R} \times S^1, M)$.
⟨1⟩4. The C^∞_{loc} and $W^{1,p}_{loc}$ topologies coincide on the space $\{u \in W^{1,p}_{loc} : \bar{\partial}_{H,J}(u) = 0\}$.
PROOF:
⟨2⟩1. LET: $(u_n)_{n \in \mathbb{N}} \subset \{u \in W^{1,p}_{loc} : \bar{\partial}_{H,J}(u) = 0\}$ such that $W^{1,p}_{loc} - \lim_{n \to \infty} u_n = u$ where $u \in W^{1,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ such that $\bar{\partial}_{H,J} u = 0$.
⟨2⟩2. LET: $J_n = J(u_n)$, $h_n = \mathrm{grad}\, H_t(u_n)$ and $h = \mathrm{grad}\, H_t(u)$.
⟨2⟩3. If $W^{k,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) - \lim_{n \to \infty} u_n = u$ then $W^{k+1,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) - \lim_{n \to \infty} u_n = u$.
PROOF:
⟨3⟩1. ASSUME: $W^{k,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) - \lim_{n \to \infty} u_n = u$.
⟨3⟩2. $W^{k,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) - \lim_{n \to \infty} J_n = J(u)$.
⟨3⟩3. $W^{k,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) - \lim_{n \to \infty} h_n = h$.
⟨3⟩4. $\partial_s(u_n - u) + J_n \partial_t (u_n - u) + h_n - h + (J(u) - J_n)\partial_t u = 0$.
PROOF:
⟨4⟩1. $\partial_s u + J(u)\partial_t u + h = 0$.
⟨4⟩2. $\partial_s u_n + J_n \partial_t u_n + h_n = 0$.

⟨3⟩5. Let: $V, U \in \mathsf{Open}(\mathbb{R} \times S^1)$ be relatively compact such that $V \subset U$.
⟨3⟩6. $\exists C > 0$:

$$\|u_n - u\|_{W^{k+1,p}(V,\mathbb{R}^{2n})} \leq C(\|h_n - h\|_{W^{k,p}(U,\mathbb{R}^{2n})} + \|u_n - u\|_{W^{k,p}(U,\mathbb{R}^{2n})} + \|J_n - J(u)\|_{W^{k,p}(U,\mathbb{R}^{2n})})$$

Proof:
⟨4⟩1. By applying Theorem 2.5 to ⟨3⟩4 and using that $\sup_{n\in\mathbb{N}}\|J_n\|_{W^{k,p}(U,\mathbb{R}^{2n})} < \infty$.

⟨1⟩5. Q.E.D.
Proof:
⟨2⟩1. $\forall k \in \mathbb{N}$: $W^{k,p}_{loc}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) - \lim_{n\to\infty} u_n = u$.

Theorem 2.7 *Let $U \in \mathsf{Open}(\mathbb{R} \times S^1)$ and let $(u_n)_{n\in\mathbb{N}} \subset C^\infty(U, M)$ such that $\bar{\partial}_{H,J} u_n = 0$. Assume that $\sup_{n\in\mathbb{N}}\|\operatorname{grad} u_n\|_{L^\infty(U)} < \infty$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}} \subset (u_n)_{n\in\mathbb{N}}$ such that*

$$C^\infty_{loc}(U, M) - \lim_{k\to\infty} u_{n_k}$$

exists.

Proof:
⟨1⟩1. Suffices: $W^{1,p}_{loc}(U, M) - \lim_{k\to\infty} u_{n_k}$ exists.
Proof:
⟨2⟩1. By Theorem 2.6.
⟨1⟩2. $\exists (u_{n_k})_{k\in\mathbb{N}} \subset (u_n)_{n\in\mathbb{N}}$: $C^0_{loc}(U, M) - \lim_{k\to\infty} u_{n_k}$ exists.
Proof:
⟨2⟩1. By the Arzelà-Ascoli theorem since the u_n are equicontinuous and totally bounded on every compact subset of U by assumption.
⟨1⟩3. Suffices Assume: $C^0_{loc}(U, M) - \lim_{n\to\infty} u_n$ exists.
⟨1⟩4. Let: $V \in \mathsf{Open}(U)$ be relatively compact.
⟨1⟩5. $(u_n)_{n\in\mathbb{N}}$ is a $W^{1,p}_{loc}(V, M)$-Cauchy sequence.
Proof:
⟨2⟩1. Let: $n, m \in \mathbb{N}$ and $u_n, u_m \in (u_n)_{n\in\mathbb{N}}$.
⟨2⟩2. Suffices Assume: $M = \mathbb{R}^{2N}$.
⟨2⟩3. Let: $J_n = J(u_n)$.
⟨2⟩4. $J_n \in W^{1,p}(V, \mathbb{R}^{2N})$.
⟨2⟩5. $\sup_{n\in\mathbb{N}}\|J_n\|_{W^{1,p}(V,\mathbb{R}^{2N})} < \infty$.
Proof:
⟨3⟩1. $\|\partial_s J_n\|_{L^p(V,\mathbb{R}^{2N})} \leq \sup_{p\in M}\|DJ(p)\|\|\partial_s u_n\|_{L^p(V,\mathbb{R}^{2n})} \leq C \sup_{s\in M}\|DJ(s)\|$.
⟨3⟩2. $\|\partial_t J_n\|_{L^p(V,\mathbb{R}^{2N})} \leq \sup_{p\in M}\|DJ(p)\|\|\partial_t u_n\|_{L^p(V,\mathbb{R}^{2n})} \leq C \sup_{s\in M}\|DJ(s)\|$.
⟨3⟩3. $\|J_n\|_{L^p(V,\mathbb{R}^{2N})} \leq C \sup_{s\in M}\|DJ(s)\|$.
⟨2⟩6. Let: $h = (J_m - J_n)\partial_t u_m + \operatorname{grad} H_t(u_m) - \operatorname{grad} H_t(u_n)$.
⟨2⟩7. $h \in L^p(V, \mathbb{R}^{2N})$.
Proof:
⟨3⟩1. $h \in C^0(U, \mathbb{R}^{2N})$.

⟨3⟩2. V is relatively compact in U.
⟨2⟩8. $\partial_s(u_n - u_m) + J_n\partial_t(u_n - u_m) = h$.
⟨2⟩9. LET: $W \in \mathsf{Open}(U)$ be relatively compact such that $V \subset W$.
⟨2⟩10. $\exists C > 0 : \|u_n - u_m\|_{W^{1,p}(V,\mathbb{R}^{2N})} \leq C(\|u_n - u_m\|_{L^p(W,\mathbb{R}^{2N})} + \|h\|_{L^p(W,\mathbb{R}^{2N})}$.
PROOF:
⟨3⟩1. By Theorem 2.5.
⟨2⟩11. $\|h\|_{L^p(W,\mathbb{R}^{2N})} \leq C\|u_n - u_m\|_{L^p(W,\mathbb{R}^{2N})}$.
PROOF:
⟨3⟩1.

$$\begin{aligned}\|h\|_{L^p(W,\mathbb{R}^{2N})} &\leq \|(J_m - J_n)\partial_t u_m\|_{L^p(W,\mathbb{R}^{2N})} + \|\mathrm{grad}\, H_t(u_m) - \mathrm{grad}\, H_t(u_n)\|_{L^p(W,\mathbb{R}^{2N})} \\ &\leq C(\sup_{p\in M}\|DJ(p)\| + \sup_{S^1\times M}\|D\,\mathrm{grad}\, H\|)\|u_n - u_m\|_{L^p(W,\mathbb{R}^{2N})}.\end{aligned}$$

⟨1⟩6. Q.E.D.

2.5.3 Similarity Principle and Unique Continuation

In this section we want to establish a similarity property of solutions of a perturbed Cauchy-Riemann equation. For this material we follow the presentation in [10]. Holomorphic functions on an open set are uniquely determined by their values on a set containing a limit point. Solutions of the linearised Floer equation which is in effect just a certain perturbed Cauchy-Riemann enjoy a similar property. This will be used to prove the transversality of the differential of the Floer operator.

Let $J \in W^{1,p}(B_\varepsilon(0), \mathsf{End}(\mathbb{C}^n))$ be a map such that $J(z)^2 = -\mathrm{Id}_{\mathbb{C}^n}$ for every $z \in B_\varepsilon(0)$. We assume that $p > 2$ which guarantees that the map J has a unique continuous representative. Furthermore, let $S \in L^p(B_\varepsilon(0), \mathsf{End}(\mathbb{C}^n))$. Now consider the equation

$$\partial_s u + J\partial_t u + Su = 0 \tag{2.33}$$

where u is a function $B_\varepsilon(0) \to \mathbb{C}^n$. Such maps u satisfying the above equation behave similarly to actual holomorphic functions, i.e. those that satisfy the unperturbed Cauchy-Riemann equation $\partial_s u + J_0\partial_t u = 0$. This is made precise in the following theorem. For its proof we will use the following result about the Cauchy-Riemann operator without proof.

Theorem 2.8 *For $p > 1$ the Cauchy-Riemann operator $\bar\partial : W^{1,p}(S^2, \mathbb{C}^n) \to L^p(S^2, \Lambda^{0,1}(T^*S^2) \otimes \mathbb{C}^n)$ is surjective and its kernel consists of the constant functions.*

Now we turn to the theorem.

Theorem 2.9 *Let $u \in W^{1,p}(B_\varepsilon(0), \mathbb{C}^n)$. Assume that u solves Equation 2.33 and that $u(0) = 0$. Then there exists a $\delta \in (0, \varepsilon)$, a $\Phi \in W^{1,p}(B_\delta(0), \mathrm{GL}_\mathbb{R}(\mathbb{C}^n))$ and a holomorphic map $\sigma : B_\delta(0) \to \mathbb{C}^n$ such that*

1. $\forall z \in B_\delta(0): \; u(z) = \Phi(z)\sigma(z)$.

2. $\forall z \in B_\delta(0): \; J(z)\Phi(z) = \Phi(z)i$.

Proof:
⟨1⟩1. Let: $\Psi \in W^{1,p}(B_\delta(0), \mathrm{GL}_\mathbb{R}(\mathbb{C}^n))$ such that $J\Psi = \Psi i$.
⟨1⟩2. Let: $v(z) = \Psi^{-1}(z)u(z)$ and $\widetilde{S} = \Psi^{-1}(\partial_s \Psi + J\partial_t\Psi + S\Psi)$.
⟨1⟩3. $\partial_s v + i\partial_t v + \widetilde{S}v = 0$.
Proof:
⟨2⟩1.

$$\begin{aligned} 0 &= \partial_s u + J\partial_t u + Su \\ &= (\partial_s\Psi)v + \Psi\partial_s v + J(\partial_t\Psi)v + J\Psi\partial_t v + S\Psi v \\ &= \Psi\left(\partial_s v + i\partial_t v + \widetilde{S}v\right). \end{aligned}$$

⟨2⟩2. Ψ is invertible.
⟨1⟩4. Let: $\widetilde{S}^{\pm} = \frac{1}{2}\left(\widetilde{S} \mp i\widetilde{S}i\right)$.
⟨1⟩5. $\exists D \in L^\infty(B_\varepsilon(0), \mathsf{End}(\mathbb{C}^n)): \; \forall z \in B_\varepsilon(0):$
1. $D(z)v(z) = v(z)$
2. $\forall \lambda \in \mathbb{C}: \; D(\lambda z) = \overline{\lambda}D(z)$.

Proof:
⟨2⟩1. Let: $D(z)\zeta = |v(z)|^{-2}v(z)v(z)^T\overline{\zeta}$ for $v(z) \neq 0$ and $D(z) = 0$ for $v(z) = 0$.
⟨1⟩6. Let: $A = \widetilde{S}^+ + \widetilde{S}^- D$.
⟨1⟩7. 1. $A \in L^p(B_\varepsilon(0), \mathsf{End}(\mathbb{C}^n))$.
2. $\forall z \in B_\varepsilon(0): \; A(z)v(z) = \widetilde{S}(z)v(z)$.

⟨1⟩8. Let: $A_\delta \in L^p(S^2, \mathsf{End}(\mathbb{C}^n))$ be defined by

$$A_\delta(z) = \begin{cases} A(z) & z \in B_\delta(0) \\ 0 & z \notin B_\delta(0) \end{cases}$$

for $\delta \in (0, \varepsilon)$.
⟨1⟩9. Let: $D_\delta : W^{1,p}(S^2, \mathsf{End}(\mathbb{C}^n)) \to L^p\left(\Lambda^{(0,1)}\left(T^*S^2\right) \otimes \mathsf{End}(\mathbb{C}^n)\right)$ be given by $D_\delta\Theta = \overline{\partial}\Theta + A_\delta\Theta d\overline{z}$ for $\Theta \in W^{1,p}(S^2, \mathsf{End}(\mathbb{C}^n))$.
⟨1⟩10. The operator $\Theta \mapsto (D_\delta\Theta, \Theta(0))$ is bijective for sufficiently small $\delta > 0$.
Proof:
⟨2⟩1. The operator $\Theta \mapsto (\overline{\partial}\Theta, \Theta(0))$ is bijective.
⟨2⟩2. $\lim_{\delta\to 0}\|A_\delta\|_{L^p(S^2)} = 0$.
⟨1⟩11. For sufficiently small $\delta > 0: \exists!\Theta_\delta \in W^{1,p}(S^2, \mathsf{End}(\mathbb{C}^n)): \; D_\delta\Theta_\delta = 0$ and $\Theta_\delta(0) = \mathrm{Id}_{\mathbb{C}^n}$ because of ⟨1⟩10.
⟨1⟩12. Let: $\Phi(z) = \Psi(z)\Theta_\delta(z)$ and $\sigma(z) \in \Theta_\delta(z)^{-1}v(z)$.
⟨1⟩13. $\Psi \in W^{1,p}(S^2, \mathsf{End}(\mathbb{C}^n))$.
⟨1⟩14. $\Psi(z)\sigma(z) = u(z)$.
⟨1⟩15. σ is holomorphic in $B_\delta(0)$.
Proof:

⟨2⟩1. $\partial_s v + i\partial_t v + Av = \partial_s v + i\partial_t v + \widetilde{S}v = 0.$
⟨2⟩2. $\partial_s v + i\partial_t v + Av = \Theta_\delta \partial_s \sigma + i\Theta_\delta \partial_t \sigma + (\partial_s \Theta_\delta)\sigma + i(\partial_t \Theta_\delta)\sigma + A\Theta_\delta \sigma$
$= \Theta_\delta \partial_s \sigma + i\Theta_\delta \partial_t \sigma + (\partial_s + i\partial_t + A)v = \Theta_\delta(\partial_s \sigma + i\partial_t \sigma).$
⟨2⟩3. Θ_δ is invertible.
⟨1⟩16. $\forall z \in B_\delta(0) : \; J(z)\Phi(z) = \Phi(z)i.$
⟨1⟩17. Q.E.D.

The above similarity result can now be used to prove a unique continuation result for solutions of a certain type perturbed Cauchy-Riemann equation. We consider the equation

$$\partial_s u + J(z,u)\partial_t u + g(z,u) = 0 \tag{2.34}$$

where $J \in W^{1,p}(\mathbb{C} \times \mathbb{C}^n, \mathrm{End}(\mathbb{C}^n))$ such that $J^2 = \mathrm{Id}_{\mathbb{C}^n}$ and $g \in W^{1,p}(\mathbb{C} \times \mathbb{C}^n, \mathbb{C}^n)$. For the formulation of the result we first make the following definition.

Definition 2.14 *Let $u \in W^{1,p}(\mathbb{C}, \mathbb{C}^n)$. We say that u* vanishes to infinite order *at $z_0 \in \mathbb{C}$ if for all $k \geq 0$*

$$\lim_{r \to 0} \frac{\sup_{|z-z_0| \leq r} |u(z)|}{r^k} = 0.$$

Standard results about holomorphic functions tell us in particular that the set where they vanish to infinite order must be open and closed. We now extend this statement to solutions of Equation 2.34. The proof basically consists of showing that solutions of Equation 2.34 also satisfy an equation of the form Equation 2.33. Then one can simply apply Theorem 2.9.

Proposition 2.14 *Let $u, v \in W^{1,p}(U, \mathbb{C}^n)$ be solutions to Equation 2.34. Then the set*

$$\{z \in U : \; u - v \textit{ vanishes to infinite order at } z\}$$

is both open and closed.

PROOF:
⟨1⟩1. LET: $w = u - v$ and $J'(z) = J(z, u(z))$.
⟨1⟩2. $\partial_s w + J' \partial_t w + Sw = 0.$
PROOF:
⟨2⟩1.

$$\begin{aligned} \partial_s w + J'\partial_t w &= J(z,v)\partial_t v + g(z,v) - J(z,u)\partial_t u - g(z,u) + J'\partial_t w \\ &= (J(z,v) - J(z,u))\partial_t v + g(z,v) - g(z,u) \\ &= \left(\int_0^1 \frac{d}{d\tau} J(z, u - \tau w) d\tau \right) \partial_t v + \int_0^1 \frac{d}{d\tau} g(z, u - \tau w) d\tau. \end{aligned}$$

⟨2⟩2. The previous line defines S.

$\langle 1 \rangle 3.$ Q.E.D.

With a view towards applications in the Floer setting consider next the following special case of Equation 2.34.

$$\partial_s u + J_t(u)(\partial_t u - X_t(u)) = 0. \tag{2.35}$$

Definition 2.15 *Let $u \in C^l(B_\varepsilon(0), \mathbb{C}^n)$. Assume that u satisfies Equation 2.35. Then we denote by*

$$C(u) = \{(s,t) \in B_\varepsilon : \partial_s u(s,t) = 0\}$$

its set of critical points.

Proposition 2.15 *Let $u \in C^l(B_\varepsilon(0), \mathbb{C}^n)$. Assume that u satisfies Equation 2.35 and that there is a point $(s,t) \in B_\varepsilon$ such that $\partial_s u(s,t) \neq 0$. Then the set $C(u)$ is discrete.*

Proof:
$\langle 1 \rangle 1.$ Let: $\psi_t : \mathbb{C} \to \mathbb{C}^n$ be the local diffeomorphism generated by the vector field X.
$\langle 1 \rangle 2.$ Let: $v(s,t) = \psi_t^{-1}(u(s,t))$
$\langle 1 \rangle 3.$ $D\psi_t(v)\partial_s v + J_t(u) D\psi_t(v)\partial_t v = 0$
  Proof:
  $\langle 2 \rangle 1.$ $\partial_s u = D\psi_t(v)\partial_s v$
  $\langle 2 \rangle 2.$ $\partial_t u - X_t(u) = D\psi_t(v)\partial_s v$
$\langle 1 \rangle 4.$ $\partial_s v + \psi_t^* J_t(v)\partial_t v = 0$
$\langle 1 \rangle 5.$ The set $\{(s,t) \in B_\varepsilon : dv(s,t) = 0\}$ is discrete.
  Proof:
  $\langle 2 \rangle 1.$ v is not constant.
  $\langle 2 \rangle 2.$ By the unique continuation result for every $(s,t) \in B_\varepsilon$ such that $dv(s,t) = 0$ there exists a $\delta \in (0,\varepsilon)$ such that $dv(s,t) \neq 0$ for $|(s,t)| \in (0,\delta)$.
$\langle 1 \rangle 6.$ $dv(s,t) = 0 \iff \partial_s u(s,t) = 0.$
$\langle 1 \rangle 7.$ Q.E.D.

Next we want to study of so called *regular points* of solutions of Equation 2.35 with a view towards applications to the definition of Floer homology. For this we impose additionally the following asymptotic behaviour on u. Consider the conditions

$$\lim_{s \to \pm\infty} u(s,t) = \gamma^{\pm}(t) \text{ uniformly in } t \tag{2.36}$$

and

$$\lim_{s \to \pm\infty} \partial_s u(s,t) = \text{ uniformly in } t. \tag{2.37}$$

For such u satisfying the above we introduce the following notion.

Definition 2.16 *Let $u \in C^l(\mathbb{R}^2, M)$. Assume that u satisfies Equation 2.35, Equation 2.36 and Equation 2.37. Also assume that u is 1-periodic in t. A point $(s,t) \in \mathbb{R}^2$ is called* regular *for u if*

1. $\partial_s u(s,t) \neq 0.$
2. $u(s,t) \neq \gamma^{\pm}(t).$
3. $u(s,t) \notin u(\mathbb{R}\backslash\{s\}, t).$

The set of regular points of u will be denoted by $R(u)$.

For our purposes the most important property of $R(u)$ is that it is a open and dense set. This is mainly used to show transversality of the Floer operator by proving that at each u the image of its linearisation has a trivial annihilator.

Theorem 2.10 *Let $u \in C^l(\mathbb{R}^2, M)$ as in Definition 2.16. Then the set $R(u)$ is open and dense.*

⟨1⟩1. LET: $v(s,t) = \psi_t^{-1} u(s,t)$.
⟨1⟩2. LET: $\gamma^{\pm} = \gamma^{\pm}(t)$.
⟨1⟩3. $\partial_s v + \psi_t^* J_t(v) \partial_t v = 0$.
⟨1⟩4. The set $R(v)$ is open.
  PROOF:
  ⟨2⟩1. ASSUME: $\exists (s,t) \in R(v) : \exists (s_m, t_m)_{m\in\mathbb{N}} \notin R(v) : \lim_{m\to\infty}(s_m, t_m) = (s,t)$
  ⟨2⟩2. $\exists (s'_m)_{m\in\mathbb{N}} \in \mathbb{R} : u(s_m, t_m) = u(s'_m, t_m)$ and $s'_m \neq s_m$.
    PROOF:
    ⟨3⟩1. $\exists N \in \mathbb{N} : \forall m > N : \partial_s u(s_m, t_m) \neq 0$ and $u(s_m, t_m) \neq \gamma^{\pm}$
    ⟨3⟩2. $(s_m, t_m) \notin R(v)$.
  ⟨2⟩3. The sequence $(s'_m)_{m\in\mathbb{N}}$ is bounded.
    PROOF:
    ⟨3⟩1. ASSUME: The sequence is unbounded.
    ⟨3⟩2. SUFFICES ASSUME: $\lim_{m\to\infty} s'_m = \pm\infty$.
    ⟨3⟩3. $\lim_{m\to\infty} u(s'_m, t_m) = \gamma^{\pm}$
    ⟨3⟩4. This is a contradiction to the assumption 2.
  ⟨2⟩4. SUFFICES ASSUME: $\exists s' \in \mathbb{R} : \lim_{m\to\infty} s'_m = s'$.
  ⟨2⟩5. $u(s,t) = u(s',t)$.
  ⟨2⟩6. $s' = s$.
  ⟨2⟩7. This contradicts the assumption 3.
⟨1⟩5. The set $R(v)$ is dense.
  PROOF:
  ⟨2⟩1. SUFFICES: $R(v)$ is dense in $\mathbb{R}^2\backslash C(v)$.
  ⟨2⟩2. $\forall (s,t) \in \mathbb{R}^2\backslash C(v) : \exists (s_n)_{n\in\mathbb{N}} \subset \mathbb{R}$:
    1. $\forall n \in \mathbb{N} : (s_n, t) \notin C(v)$.
    2. $\forall n \in \mathbb{N} : v(s_n, t) \neq \gamma^{\pm}(t)$.
    3. $\lim_{n\to\infty}(s_n, t) = (s,t)$.

$\langle 2\rangle 3$. Let: $(s_0, t_0) \in \mathbb{R} \times [0,1]$ such that $\partial_s v(s_0, t_0) \neq 0$ and $v(s_0, t_0) \neq \gamma^{\pm}(t_0)$.

$\langle 2\rangle 4$. $\exists (s_n, t_n)_{n\in\mathbb{N}} \subset R(v) : \lim_{n\to\infty}(s_n, t_n) = (s_0, t_0)$.

Proof:

$\langle 3\rangle 1$. Assume: $\langle 2\rangle 4$ is false.

$\langle 3\rangle 2$. $\exists \varepsilon > 0 : B_\varepsilon(s_0, t_0) \cap R(v) = \emptyset$.

$\langle 3\rangle 3$. $\exists M \geq 0$:

1. $|t - t_0| \leq \varepsilon$ and $|s| \geq M \implies \begin{cases} v(s,t) \notin v(B_\varepsilon(s_0, t_0)) \\ \gamma^{\pm}(t) \notin v(B_\varepsilon(s_0, t_0)) \end{cases}$

2. $|t - t_0| \leq \varepsilon \implies$ the map $\begin{cases} [s_0 - \varepsilon, s_0 + \varepsilon] \to M \\ s \mapsto v(s,t) \end{cases}$ is an injective immersion.

Proof:

$\langle 4\rangle 1$. Assume: $\exists (s'_n, t'_n,)_{n\in\mathbb{N}} \subset \mathbb{R} \times [0,1]$:

1. $\lim_{n\to\infty}(s'_n, t'_n) = (\pm\infty, t_0)$.
2. $\lim_{n\to\infty} v(s'_n, t'_n) = v(s_0, t_0)$.
3. $\forall n \in \mathbb{N} : v(s'_n, t'_n) \in v(B_\varepsilon(s_0, t_0))$.

$\langle 4\rangle 2$. $\exists (s_n, t_n)_{n\in\mathbb{N}} \subset \mathbb{R} \times [0,1]$:

1. $\lim_{n\to\infty}(s_n, t_n) = (s_0, t_0)$.
2. $\forall n \in \mathbb{N} : v(s_n, t_n) = v(s'_n, t'_n)$.

$\langle 4\rangle 3$. $v(s_0, t_0) = \gamma^{\pm}(t)$ which is a contradiction to $\langle 2\rangle 3$.

$\langle 4\rangle 4$. Assume: $\exists (s_n)_{n\in\mathbb{N}}, (s'_n)_{n\in\mathbb{N}} \subset \mathbb{R}, (t_n)_{n\in N} \subset [0,1]$:

1. $\forall n \in \mathbb{N} : s_n \neq s'_n$.
2. $\lim_{n\to\infty} s_n = \lim_{n\to\mathbb{N}} s'_n = s_0$.
3. $\lim_{n\to\infty} t_n = t_0$.
4. $\forall n \in \mathbb{N} : v(s_n, t_n) = v(s'_n, t_n)$.

$\langle 4\rangle 5$. $\partial_s v(s_0, t_0) = 0$ which contradicts $\langle 2\rangle 3$.

$\langle 3\rangle 4$. $\forall (s,t) \in B_\varepsilon(s_0, t_0) : \partial_s v(s,t) \neq 0$ and $v(s,t) \neq \gamma^{\pm}(t)$.

$\langle 3\rangle 5$. The set $C(v) \cap ([-M, M] \times [0,1])$ is finite.

$\langle 3\rangle 6$. Suffices Assume: $\forall (s,t) \in C(v) \cap ([-M, M] \times [0,1]) : v(s,t) \neq v(s_0, t_0)$.

Proof:

$\langle 4\rangle 1$. v is not locally constant around (s_0, t_0) since $(s_0, t_0) \notin C(v)$.

$\langle 4\rangle 2$. (s_0, t_0) can be slightly perturbed so that the assumption holds.

$\langle 3\rangle 7$. Suffices Assume: $v(B_\varepsilon(s_0, t_0)) \cap v(C(v) \cap ([-M, M] \times [0,1])) = \emptyset$.

Proof:

$\langle 4\rangle 1$. Decrease ε if necessary.

$\langle 3\rangle 8$. $\forall (s,t) \in B_\varepsilon(s_0, t_0) : \exists s' \in \mathbb{R}\backslash\{s\}$:

1. $v(s,t) = v(s',t)$.
2. $\partial_s v(s',t) \neq 0$.
3. $|s'| \leq M$.
4. $|s' - s_0| > \varepsilon$.

$\langle 3\rangle 9$. For every $(s,t) \in B_\varepsilon(s_0,t_0)$ the set $\{(s',t) \in B_\varepsilon(s_0,t_0) : v(s',t) = v(s,t)\}$ is finite.

PROOF:

$\langle 4\rangle 1$. ASSUME: this set is infinite.

$\langle 4\rangle 2$. It has a limit point since it is bounded.

$\langle 4\rangle 3$. $\partial_s v = 0$ by the continuation principle Proposition 2.14 which is a contradiction.

$\langle 3\rangle 10$. LET: $s_1,\dots,s_N \in [-M,M]$ be all the points such that $\forall j \in \{1,\dots,N\} : v(s_0,t_0) = v(s_j,t_0)$.

$\langle 3\rangle 11$. $\forall j \in \{1,\dots,N\}, r > 0 : \exists \delta > 0 :$

$$|t-t_0|, |s-s_0| < \delta \implies \exists s' \in B_r(s_j) : v(s,t) = v(s',t).$$

$\langle 3\rangle 12$. LET: $r < \varepsilon/2$ and $\delta > 0$ as in $\langle 3\rangle 11$.

$\langle 3\rangle 13$. LET: $\Sigma_j = \{(s,t) \in \overline{B_\delta(s_0,t_0)} : v(s,t) \in v\left(\overline{B_r(s_j)} \times \{t\}\right)\}$ for $j \in \{1,\dots,N\}$.

$\langle 3\rangle 14$. $\overline{B_\delta(s_0,t_0)} = \cup_{j=1}^N \Sigma_j$.

PROOF:

$\langle 4\rangle 1$. By $\langle 3\rangle 11$.

$\langle 3\rangle 15$. $\exists j' \in \{1,\dots,N\} : \operatorname{int}\Sigma_{j'} \neq \emptyset$ by $\langle 3\rangle 14$.

$\langle 3\rangle 16$. $\exists \rho > 0, (s_*,t_*) \in \operatorname{int}(\Sigma_{j'}) : B_\rho(s_*,t_*) \subset \Sigma_{j'} \subset B_\delta(s_0,t_0) \subset B_\varepsilon(s_0,t_0)$ (shrinking δ if necessary for the last step).

$\langle 3\rangle 17$. SUFFICES ASSUME: $\forall j \in \{1,\dots,N\} : B_r(s_j) \cap B_\varepsilon(s_0) = \emptyset$.

PROOF:

$\langle 4\rangle 1$. Shrink r if necessary.

$\langle 3\rangle 18$. $\exists s'_* \in \overline{B_r(s_{j'})} \setminus \{s_*\} : v(s_*,t_*) = v(s'_*,t_*)$.

PROOF:

$\langle 4\rangle 1$. By definition of Σ_j. See $\langle 3\rangle 13$.

$\langle 3\rangle 19$. LET: $v_1(s,t) = v(s+s_*, t+t_*)$ and $v_2(s,t) = v(s+s'_*, t+t_*)$.

$\langle 3\rangle 20$. 1. For $i = 1,2$ the function v_i satisfy $\partial_s v_i + J\partial_t v_i = 0$.

2. $v_1(0,0) = v_2(0,0)$.

3. $Dv_i(0,0) \neq 0$.

$\langle 3\rangle 21$. $\forall (s,t) \in B_\rho(0,0) : \exists s' \in \overline{B_{2r}(0)} : v_1(s,t) = v_2(s',t)$.

PROOF:

$\langle 4\rangle 1$. By definition of Σ_j. See $\langle 3\rangle 13$.

$\langle 3\rangle 22$. $\forall (s,t) \in B_\rho(0,0) : v_1(s,t) = v_2(s,t)$.

PROOF:

$\langle 4\rangle 1$. By the unique continuation principle Proposition 2.14.

$\langle 3\rangle 23$. $v_2 - v_1$ satisfies the equation $\partial_s w + J_0\partial_t + Sw = 0$.

$\langle 3\rangle 24$. $v_1 = v_2$.

PROOF:

$\langle 4\rangle 1$. $v_1 = v_2$ on $B_\rho(0,0)$.

$\langle 4\rangle 2$. Now use the unique continuation principle Proposition 2.14.

$\langle 3\rangle 25$. $\forall k \in \mathbb{Z}, \forall (s,t) \in \mathbb{R}^2 : v(s,t) = v(s + k(s'_* - s_*), t)$.

PROOF:

⟨4⟩1. $v(s,t) = v(s \pm (s'_* - s_*), t)$ by ⟨3⟩24 and the definition of v_1 and v_2.

⟨4⟩2. Now use induction.

⟨3⟩26. $\forall t \in \mathbb{R}: \ v(s,t) = \gamma^{\pm}(t)$ by taking $k \to \pm\infty$.

⟨3⟩27. ⟨3⟩26 contradicts property 2 in ⟨2⟩2. Hence, the assumption ⟨3⟩1 is false.

⟨1⟩6. $R(v) = R(u)$.

Proof:

⟨2⟩1. Because ψ_t is a diffeomorphism.

⟨1⟩7. Q.E.D.

2.5.4 Exponential Decay

In this section we want to establish that solutions to the Floer equation that have finite energy decay exponentially fast . More precisely, let $u \in \mathcal{M}(H,J)$ such that $E(u) < \infty$. We can consider the linearisation of u which is a tangent field ξ along u given by $\xi = \partial_s u$. Then we want to show that for certain positive constants C and δ we have that

$$\|\xi(s,t)\| \leq Ce^{-\delta|s|} \tag{2.38}$$

for every $t \in S^1$.

This result is important for showing that the moduli spaces of solutions that connect critical orbits are a smooth manifold. This is shown by exhibiting them as the zero section of a suitable section of a certain Banach bundle. The base of this bundle will be defined as maps u that satisfy this exponential decay. As a result the base is a Banach manifold whereas without the decay property completeness would fail. However, completeness is needed to apply the infinite dimensional version of the implicit function theorem from which the manifold structure of the moduli spaces is obtained.

To prove the estimate Equation 2.38 we begin by deriving an analogous L^2-type estimate. This is done in two steps. We follow the presentation in section 12 of [2].

First we show that the energy

$$\frac{1}{2}\int_0^1 \|\xi(s,t)\|^2 dt$$

satisfies a certain second-order differential inequality. This is a straightforward computation that uses the following ingredients. The linearised solution ξ satisfies the linearised equation $D\bar{\partial}_{H,J}(u)\xi = 0$, the matrices S appearing in the linearised operator tend to symmetric matrices at infinity and finally we use that the operator $-J_0\partial_t - S$ is invertible from $W^{1,2}(S^1)$ to $L^2(S^1)$ for sufficiently large $|s|$ which allows us the obtain an upper bound on $\|\xi\|^2_{L^2(S^1)}$. Having derived this differential inequality the exponential decay for the L^2-norm follows using elementary analysis.

The second step consists of turning this L^2 estimate into a pointwise estimate. At its core, this step relies on a mean value inequality that holds for functions v solving the Laplace inequality $\Delta v \leq 0$. Analogous to the mean value property of harmonic functions this inequality states that on any ball $B_r(x)$ the value of v at x can be estimated by

$$v(x) \leq \frac{1}{\pi r^2} \int_{B_r(x)} v ds dt.$$

We will not give a proof of this result which directly follows from the divergence theorem.

Proposition 2.16 *Let $u \in \mathcal{M}(H, J)$ with $E(u) < \infty$ and $\xi = \partial_s u$. Then there exist $C, \delta > 0$ such that*

$$\int_0^1 \|\xi(s,t)\|^2 dt \leq C e^{-\delta|s|}.$$

PROOF:
⟨1⟩1. LET: $f(s) = \frac{1}{2}\int_0^1 \|\xi(s,t)\|^2 dt$.
⟨1⟩2. $\exists \delta \in \mathbb{R} : \; f'' \geq \delta^2 f$.
  PROOF:
  ⟨2⟩1. $f'(s) = \int_0^1 \langle \xi, \partial_s \xi \rangle dt$.
  ⟨2⟩2. $f''(s) = \int_0^1 \|\partial_s \xi\|^2 dt + \int_0^1 \langle \xi, \partial_s^2 \xi \rangle dt$.
  ⟨2⟩3. $\int_0^1 \langle \xi, \partial_s^2 \xi \rangle dt = -\int_0^1 \left(\langle \xi, J_0 \partial_s \partial_t \xi \rangle + \langle \xi, \partial_s S \xi \rangle + \langle S\xi, \partial_s \xi \rangle \right) dt$.
  ⟨2⟩4. $\int_0^1 \langle \xi, J_0 \partial_s \partial_t \xi \rangle dt = -\int_0^1 \|\partial_s \xi\|^2 dt - \int_0^1 \langle S\xi, \partial_s \xi \rangle dt$.
  ⟨2⟩5. $f''(s) = \int_0^1 \|\partial_s \xi\|^2 dt - \int_0^1 \langle \xi, \partial_s S \rangle + \int_0^1 \langle \xi (S^T - S) \partial \xi \rangle dt$.
  ⟨2⟩6. $\exists s_0 \in \mathbb{R} : \; \forall s > s_0 : \; \int_0^1 \langle \xi, \partial_s S \xi \rangle dt \leq \varepsilon \int_0^1 \|\xi\|^2 dt$.
  ⟨2⟩7. $|\int_0^1 \langle \xi, (S^T - S) \partial_s \xi \rangle dt| \leq \eta \left(\int_0^1 \|\xi\|^2 dt \right)^{1/2} \left(\int_0^1 \|\partial_s \xi\|^2 dt \right)^{1/2}$.
  ⟨2⟩8. LET: $A = -J_0 \partial_t - S$.
  ⟨2⟩9. $\exists s_0 \in \mathbb{R} : \; \forall s > s_0 : A : W^{1,2}(S^1) \to L^2(S^1)$ is invertible and there exists a $C > 0$ such that $\|A\xi\|_{L^2(S^1)} \geq C\|\xi\|_{L^2(S^1)}$.
    PROOF:
    ⟨3⟩1. LET: $A_0 = -J_0 \partial_t + S^{\pm}$.
    ⟨3⟩2. A_0 is invertible on L^2.
    ⟨3⟩3. $\lim_{s \to \pm\infty} S - S^{\pm} = 0$.
  ⟨2⟩10. $f''(s) \geq \|A\xi\|^2_{L^2} - \varepsilon \|\xi\|^2_{L^2} - \eta \|\xi\|_{L^2} \|A\xi\|_{L^2} \geq (C(C - \eta) - \varepsilon)\|\xi\|^2_{L^2} \geq \delta^2 f(s)$.
⟨1⟩3. $f''(s) \geq \delta f(s)$ for $s \geq s_0$ implies $f(s) \leq f(s_0) e^{-\delta(s - s_0)}$ for $s \geq s_0$.
  PROOF:
  ⟨2⟩1. LET: $g(s) = e^{-\delta s}(f'(s) + \delta f(s))$.

$\langle 2\rangle 2.$ $g'(s) \geq 0$ for $s \geq s_0$.
Proof:
$\langle 3\rangle 1.$ $g'(s) = e^{\delta s}(-\delta(f'(s) + \delta f(s)) + f''(s) + \delta f'(s)) = e^{-\delta s}(f''(s) - \delta^2 f(s))$.
$\langle 3\rangle 2.$ $f''(s) - \delta^2 f(s) \geq 0$ for $s \geq s_0$.
$\langle 2\rangle 3.$ $g|_{[s_0,\infty)}$ is increasing.
$\langle 2\rangle 4.$ Assume: $\exists s_1 \geq s_0 : \ g(s_1) > 0$.
$\langle 2\rangle 5.$ $\forall s \geq s_1 : \ f'(s) + \delta f(s) \geq e^{\delta s} g(s_1)$.
$\langle 2\rangle 6.$ Let: $h(s) = e^{\delta s} f(s)$.
$\langle 2\rangle 7.$ $\exists C \in \mathbb{R} : \ \forall s \geq s_1 : \ h(s) \geq C e^{2\delta s}$.
Proof:
$\langle 3\rangle 1.$ $\forall s \geq s_1 : \ h'(s) = e^{\delta s}(\delta f + f') \geq e^{2\delta s} g(s_1)$.
$\langle 3\rangle 2.$
$\langle 2\rangle 8.$ $\lim_{s\to\infty} f(s) = \infty$ which is a contradiction.
$\langle 2\rangle 9.$ $\forall s \geq s_0 : \ f(s) \leq e^{\delta s_0} f(s_0) e^{-\delta s}$.
Proof:
$\langle 3\rangle 1.$ $\forall s \geq s_0 : \ g(s) \leq 0$.
$\langle 3\rangle 2.$ $\forall s \geq s_0 : \ f'(s) + \delta f(s) \leq 0$.
$\langle 3\rangle 3.$ $h|_{[s_0,\infty)}$ is decreasing.
$\langle 3\rangle 4.$ $\forall s \geq s_0 : \ e^{\delta s} f(s) = h(s) \leq h(s_0) = e^{\delta s_0} f(s_0)$.
$\langle 1\rangle 4.$ Q.E.D.
Proof:
$\langle 2\rangle 1.$ Combine $\langle 1\rangle 2$ and $\langle 1\rangle 3$.

This concludes the first step outlined above. The next result is needed to the turn the L^2-estimate into a pointwise estimate.

Lemma 2.4 *Let $w \in C^2(B_r(s_0,t_0), \mathbb{R})$ such that there exists a number $b > 0$ such that $\Delta w \geq -b$. Then*

$$w(s_0,t_0) \leq \frac{br^2}{8} + \frac{1}{r^2}\int_{B_r(s_0,t_0)} w\,ds\,dt.$$

Proof:
$\langle 1\rangle 1.$ Let: $v(s,t) = w(s_0 + s, t + 0 + t) + \frac{b}{4}(s^2 + t^2)$ for $(s,t) \in B_r(0)$.
$\langle 1\rangle 2.$ $\Delta v(s,t) \geq 0$.
Proof:
$\langle 2\rangle 1.$ $\Delta v(s,t) = \Delta w(s_0 + s, t_0 + t) + b$.
$\langle 1\rangle 3.$ $v(0) \leq \frac{1}{\pi r^2}\int_{B_r(0)} v\,ds\,dt$.
Proof:
$\langle 2\rangle 1.$ By the mean value inequality since $\Delta v \geq 0$ on $B_r(0)$.
$\langle 1\rangle 4.$ Q.E.D.
Proof:
$\langle 2\rangle 1.$ $w(s_0,t_0) = v(0)$.

⟨2⟩2.

$$w(s_0, t_0) \leq \frac{1}{\pi r^2} \int_{B_r(0)} v dsdt = \frac{1}{\pi r^2} \int_{B_r(s_0,t_0)} w dsdt + \frac{1}{\pi r^2} \int_{B_r(0)} \frac{b(s^2+t^2)}{4} dsdt$$
$$= \frac{1}{\pi r^2} \int_{B_r(s_0,t_0)} w dsdt + \frac{br^2}{8}.$$

We cannot directly apply this result. Instead we have the following intermediate step.

Lemma 2.5 *Let $w \in C^2(\mathbb{R}^2)$ such that*

1. $\forall (s,t) \in \mathbb{R}^2 : \ w(s,t) > 0.$
2. $\Delta w \geq -aw.$

Then it holds that for all $(s_0, t_0) \in \mathbb{R}^2$ that

$$w(s_0, t_0) \leq \frac{8a}{\pi} \int_{B_1(s_0,t_0)} w dsdt.$$

PROOF:
⟨1⟩1. LET: $f : [0,1] \to \mathbb{R}$ be given by $f(\rho) = (1-\rho)^2 \sup_{B_\rho(s_0,t_0)} w$.
⟨1⟩2. $\exists \rho_* \in [0,1) : \ f(\rho_*) = \max_{[0,1]} f$.
  PROOF:
  ⟨2⟩1. $f \in C^0([0,1])$.
  ⟨2⟩2. 1. $f(0) = w(s_0, t_0)$.
     2. $f(1) = 0$.
⟨1⟩3. LET: $c = \sup_{B_{\rho_*}(s_0,t_0)} w$ and $\eta = \frac{1-\rho_*}{2}$.
⟨1⟩4. LET: $z_* \in B_{\rho_*}(s_0, t_0)$ such that $w(z_*) = c$.
⟨1⟩5. $\Delta w \geq -4ac$.
  PROOF:
  ⟨2⟩1. $B_\eta(z_*) \subset B_{\rho_*+\eta}(s_0,t_0) \subset B_1(s_0,t_0)$.
    PROOF:
    ⟨3⟩1. $\eta \in (0, 1/2]$ and $\rho_* + \eta \leq 1$.
  ⟨2⟩2. $c\eta^2 \leq \frac{2a}{\pi} \int_{B_1(s_0,t_0)} w dsdt$.
    PROOF:
    ⟨3⟩1. $c = w(z_*) \leq \frac{acr^2}{2} + \frac{1}{\pi r^2} \int_{B_1(s_0,t_0)} w dsdt$.
      PROOF:
      ⟨4⟩1. Apply Lemma 2.4 with $r \leq \eta$.
    ⟨3⟩2. LET: $r = \frac{\eta}{\sqrt{a}} < \eta$ in ⟨3⟩1.
    ⟨3⟩3. $c \leq \frac{c}{2} + \frac{a}{\pi\eta^2} \int_{B_1(s_0,t_0)} w$.
⟨1⟩6. Q.E.D.
  PROOF:
  ⟨2⟩1. $w(s_0,t_0) \leq f(\rho_*) = (1-\rho_*)^2 c$.
  ⟨2⟩2. $(1-\rho_*)^2 = 4\eta^2$.

⟨2⟩3. Now use the inequality for $c\eta^2$ above..

We now verify the assumption of the previous lemma in the case of the squared norm of ξ.

Lemma 2.6 *There exists a number $a > 1$ such that*

$$\Delta\|\xi\|^2 \geq -a\|\xi\|^2.$$

Proof:
⟨1⟩1. $\Delta\langle\xi,\xi\rangle = \partial_t^2\langle\xi,\xi\rangle + \partial_s^2\langle\xi,\xi\rangle = 2\left(\|\partial_s\xi\|^2 + \|\partial_t\xi\|^2 + \langle\Delta\xi,\xi\rangle\right).$
⟨1⟩2. $\langle\Delta\xi,\xi\rangle = \langle-\partial_s S\xi,\xi\rangle - \langle S\partial_s\xi,\xi\rangle + \langle J_0\partial_t S,\xi\rangle + \langle J_0 S\partial_t\xi,\xi\rangle.$
  Proof:
  ⟨2⟩1. $\Delta = (\partial_s - J_0\partial_t)(\partial_s + J_0\partial_t).$
  ⟨2⟩2. $\Delta\xi = -(\partial_s - J_0\partial_t)(S\xi).$
⟨1⟩3. $\Delta\langle\xi,\xi\rangle = 2\left(\|\partial_s\xi\|^2 - \langle S\partial_s\xi,\xi\rangle\right) + 2\left(\|\partial_t\xi\|^2 + \langle J_0 S\partial_t\xi,\xi\rangle\right) - 2\langle\partial_s S\xi,\xi\rangle + 2\langle J_0\partial_t S\xi,\xi\rangle.$
⟨1⟩4. $\|\partial_s\xi\|^2 - \langle S\partial_s\xi,\xi\rangle = \|\partial_s\xi - \frac{1}{2}S^T\xi\|^2 - \frac{1}{4}\|S^T\xi\|^2 \geq -\frac{1}{4}\|S^T\xi\|^2 \geq -c_1\|\xi\|^2.$
⟨1⟩5. $\|\partial_t\xi\|^2 + \langle J_0 S\partial_t\xi,\xi\rangle = \|\partial_t\xi + \frac{1}{2}S^T J_0^T\xi\|^2 - \frac{1}{4}\|S^T J_0^T\xi\|^2 \geq -\frac{1}{4}\|S^T J_0^T\xi\|^2 \geq -c_2\|\xi\|^2.$
⟨1⟩6. $\langle\partial_s S\xi,\xi\rangle \leq c_3\|\xi\|^2.$
⟨1⟩7. $\langle J_0\partial_t S\xi,\xi\rangle \geq -c_4\|\xi\|^2.$
⟨1⟩8. Define: $a = c_1 + c_2 + 2c_3 + 2c_4.$
⟨1⟩9. Q.E.D.

Finally we can put everything together to obtain the desired pointwise exponential decay property of solutions to the linearised Floer equation.

Theorem 2.11 *Let $\xi \in C^2(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ such that $D\bar{\partial}_{H,J}(u)(\xi) = 0$. Then if $E(u) < \infty$ there exist constants $\delta > 0$ and $C > 0$ such that*

$$\|\xi(s,t)\| \leq Ce^{-\delta|s|}$$

for every $(s,t) \in \mathbb{R}\times S^1$.

Proof:
⟨1⟩1. $\int_0^1\|\xi(s,t)\|^2 dt \leq Ce^{-\delta|s|}.$
⟨1⟩2. $\exists a > 1 \ : \ \Delta\|\xi\|^2 \geq -a\|\xi\|^2.$
⟨1⟩3. $\|\xi(s,t)\|^2 \leq C\int_{B_1(s_0,t_0)}\|\xi\|^2 dtds \leq Ce^{-\delta|s|}.$
⟨1⟩4. Q.E.D.

2.6 Differential of the Floer Operator

For the study of the Floer equation we need to investigate its linearisation in the neighbourhood of a solution. Thus we begin by computing the differential of $\bar\partial_{H,J}$. Then we want to show that it is a Fredholm map if all critical points of $\mathcal{A}_H$ non-degenerate and compute its index.

Proposition 2.17 *The differential of the map $\bar\partial_{H,J}$ defined in Equation 2.14 at u is given by the map*

$$D\bar\partial_{H,J}(u) : W^{1,p}(\mathbb{R} \times S^1, u^*TM) \to L^p(\mathbb{R} \times S^1, u^*TM) \tag{2.39}$$

$$D\bar\partial_{H,J}(u)(\xi) = \nabla_s \xi + J(u)\nabla_t \xi + \nabla_\xi J(u)\partial_t u + \nabla_\xi \operatorname{grad} H_t(u) \tag{2.40}$$

PROOF:
⟨1⟩1. LET: $\xi \in W^{1,p}(\mathbb{R} \times S^1, u^*TM)$ and $u_\tau = \exp_u(\tau\xi)$ for $\tau \in (-\varepsilon, \varepsilon)$.
⟨1⟩2. $D\bar\partial_{H,J}(u)(\xi) = \nabla_\tau|_{\tau=0}\bar\partial_{H,J}(u_\tau)$.
⟨1⟩3. $\nabla_\tau|_{\tau=0}\bar\partial_{H,J}(u_\tau) = \nabla_\tau|_{\tau=0}(\partial_s u_\tau) + \nabla_\tau|_{\tau=0}(J(u_\tau)\partial_t u_\tau) + \nabla_\tau|_{\tau=0}(\operatorname{grad} H_t(u_\tau))$.
⟨1⟩4. $\nabla_\tau|_{\tau=0}(\partial_s u_\tau) = \nabla_s \xi$.
⟨1⟩5. $\nabla_\tau|_{\tau=0}(J(u_\tau)\partial_t u_\tau) = \nabla_\tau|_{\tau=0}(J(u_\tau))\partial_t u + J(u)\nabla_\tau|_{\tau=0}(\partial_t u_\tau)$.
⟨1⟩6. $\nabla_\tau|_{\tau=0}(\operatorname{grad} H_t(u_\tau)) = \nabla_\xi \operatorname{grad} H_t(u)$.
⟨1⟩7. Q.E.D.

Our first goal is to show that this map is a Fredholm operator. For this we simplify the problem by working in suitable coordinates. Choose a symplectic trivialisation of u^*TM. This is given by a smooth family of isomorphisms $\Phi_{s,t} : \mathbb{R}^{2n} \to T_{u(s,t)}TM$ such that $J \circ \Phi = \Phi \circ J_0$ where $(s,t) \in \mathbb{R} \times S^1$. Let $\xi \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Then in these coordinates the operator $D\bar\partial_{H,J}$ is given by $D_S(\xi) = \Phi^{-1} \circ D\bar\partial_{H,J} \circ \Phi(\xi)$.

Proposition 2.18 *With respect to the trivialisation above the operator $D\bar\partial_{H,J}$ from Equation 2.40 has the form*

$$D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \tag{2.41}$$

$$D_S \xi = \partial_s \xi + J_0 \partial_t \xi + S\xi. \tag{2.42}$$

where the function $S \in C^\infty(\mathbb{R} \times S^1, \mathrm{End}(\mathbb{R}^{2n}))$ is given by

$$S(s,t) = \Phi_{s,t}^{-1} \circ (\nabla_s \Phi + J(u)(\nabla_t \Phi) + \nabla_{\Phi \cdot} J(u)\partial_t u + \nabla_{\Phi \cdot} \operatorname{grad} H_t(u))) \tag{2.43}$$

PROOF:
⟨1⟩1. $D\bar\partial_{H,J}(u)(\Phi\xi) = \nabla_s(\Phi\xi) + J(u)\nabla_t(\Phi\xi) + \nabla_{\Phi\xi} J(u)\partial_t u + \nabla_{\Phi\xi} \operatorname{grad} H_t(u)$.
⟨1⟩2. $\nabla_s(\Phi\xi) = \Phi(\partial_s \xi) + (\nabla_s \Phi)\xi$.
⟨1⟩3. $J(u)\nabla_t(\Phi\xi) = \Phi J_0 \partial_t \xi + J(u)(\nabla_t \Phi)\xi$.
⟨1⟩4. $\Phi^{-1} \circ D\bar\partial_{H,J} \circ \Phi(u) = \partial_s + J_0\partial_t + S$.
⟨1⟩5. Q.E.D.

Proposition 2.19 *The matrix-valued function defined in Equation 2.43 has the property that there exist $S^\pm(t) \in C^\infty(S^1, \mathrm{Sym}(\mathbb{R}^{2n}))$ such that*

$$\lim_{s \to \pm\infty} S(s,t) = S^\pm(t)$$

uniformly in t.

⟨1⟩1. $\lim_{s\to\pm\infty} S(s,t) = \Phi_{s,t}^{-1} \circ (J(u)(\nabla_t \Phi_{s,t}) + \nabla_{\Phi\cdot} \operatorname{grad} H_t(u))$.
⟨1⟩2. Q.E.D.

2.6.1 Fredholm Property

For the proof that the operator D_S is Fredholm we consider more generally operators of the form in Equation 2.42 where S is any function $S \in C^\infty(\mathbb{R} \times S^1, \mathrm{Sym}(\mathbb{R}^{2n}))$ such that the limit matrix functions

$$S^\pm(t) = \lim_{s\to\pm\infty} S(s,t) \tag{2.44}$$

exist, are symmetric and the convergence is uniform in t. Then we can consider the ordinary differential equation

$$(R^\pm)'(t) = J_0 S^\pm(t) R^\pm(t)$$
$$R^\pm(0) = \mathrm{Id}_{\mathbb{R}^{2n}}$$

whose solution we know to be a path of symplectic matrices. We introduce the following notion.

Definition 2.17 *The function $S \in C^\infty(\mathbb{R} \times S^1, \mathrm{End}(\mathbb{R}^{2n}))$ is called* non-degenerate *if the solutions $R^\pm$ to the above ODE have the property that* $\det(R^\pm(1) - \mathrm{Id}_{\mathbb{R}^{2n\times 2n}}) \neq 0$.

The first step is to show that D_S is in fact an isomorphism if S does not depend on s. This is proved in two stages. First we take $p = 2$.

Proposition 2.20 *Assume $p = 2$. Furthermore, let $S \in C^\infty(\mathbb{R} \times S^1, \mathrm{Sym}(\mathbb{R}^{2n}))$ be non-degenerate and independent of s. Then D_S is bijective.*

Proof:
⟨1⟩1. Let: $A = J_0\partial_t + S : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$
⟨1⟩2. A is invertible.
  Proof:
  ⟨2⟩1. Suffices: Solve the equation $\eta' = J_0 S\eta - J_0\zeta$ for a given $\zeta \in L^2(S^1, \mathbb{R}^{2n})$.
  ⟨2⟩2. If $\zeta = 0$ the equation has the solution given by $\eta(t) = R(t)\eta_0$.
  ⟨2⟩3. Assume: $\zeta \neq 0$ and $\eta(t) = R(t)\eta_0(t)$ for a suitable $\eta_0 \in W^{1,2}(S^1, \mathbb{R}^{2n})$.
  ⟨2⟩4. η_0 must satisfy $\eta_0'(t) = -R^{-1}(t) J_0 \zeta(t)$.
  ⟨2⟩5. $\eta(t) = R(t)\left(\eta_0(0) - \int_0^t R^{-1}(s) J_0 \zeta(s) ds\right)$.
  ⟨2⟩6. $\eta \in W^{1,2}(S^1, \mathbb{R}^{2n})$
    Proof:
    ⟨3⟩1. $\eta(0) = \eta(1)$ because $\mathrm{Id}_{\mathbb{R}^{2n}} - R(1)$ is invertible.
⟨1⟩3. The equation $\partial_s \eta = -A\eta(s) + \zeta(s)$ has a solution for a given $\zeta \in L^2(S^1, \mathbb{R}^{2n})$.

PROOF:
⟨2⟩1. Solve the homogeneous equation

$$\partial_s \eta = -A\eta$$
$$\eta(0) = \eta_0.$$

⟨2⟩2. The operator $A : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ is closed.
⟨2⟩3. A is self-adjoint: $\forall \xi, \eta \in W^{1,2}(S^1, \mathbb{R}^{2n}) : \ \langle A\xi, \eta \rangle = \langle \xi, A\eta \rangle$.
⟨2⟩4. LET: $T = i \circ A^{-1} : L^2(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$
where $i : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ is the inclusion.
⟨2⟩5. T is continuous and compact.
⟨2⟩6. $L^2(S^1, \mathbb{R}^{2n})$ admits an orthonormal basis consisting of eigenvectors of T.
⟨2⟩7. LET: $L^2(S^1, \mathbb{R}^{2n}) = L^2_+(S^1, \mathbb{R}^{2n}) \oplus L^2_-(S^1, \mathbb{R}^{2n})$ be the decomposition with respect to the positive and negative eigenvalues, respectively. The corresponding projections are denoted by $p^\pm$.
⟨2⟩8. LET: $A^\pm = A|_{L^2_\pm(S^1, \mathbb{R}^{2n})}$.
⟨2⟩9. $A^\pm$ are monotone: $\forall \xi \in L^2(S^1, \mathbb{R}^{2n}) : \ \langle A\xi, \xi \rangle \geq 0$.
⟨2⟩10. The two differential equations

$$\eta' = -A^+\eta \text{ and } \eta' = A^-\eta$$

have solution for $s \geq 0$ which are defined by the two semi-groups of operators denoted by

$$\exp(-A^+ s) \text{ and } -\exp(A^-(-s))$$

respectively.
⟨2⟩11. LET: $K : \mathbb{R} \to \mathsf{End}(L^2(S^1, \mathbb{R}^{2n}))$ be given by

$$K(s) = \begin{cases} \exp(-A^+ s) \circ p^+ & s \geq 0 \\ -\exp(A^-(-s)) \circ p^- & s < 0 \end{cases}$$

.
⟨2⟩12. K is continuous for $s \neq 0$.
⟨2⟩13. LET: λ be the largest positive eigenvalue of T and $\mu = \lambda^{-1}$.
⟨2⟩14. $\|K(s)\eta\|_{L^2(S^1, \mathbb{R}^{2n})} = \|\exp(-A^+ s) \circ p^+ \eta\|_{L^2(S^1, \mathbb{R}^{2n})} \leq e^{-\mu s} \|\eta\|_{L^2(S^1, \mathbb{R}^{2n})}$
⟨2⟩15. $\exists \delta > 0 : \ \|K(s)\|_{\mathsf{End}(L^2(S^2, \mathbb{R}^{2n}))} \leq e^{-\delta |s|}$
⟨2⟩16. LET: $Q : L^2(\mathbb{R}, L^2(S^1, \mathbb{R}^{2n})) \to W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ be given by

$$(Q\zeta)(s,t) = \int_{-\infty}^{\infty} K(-\sigma)\zeta(s+\sigma, t) d\sigma$$

⟨2⟩17. $\forall \zeta \in L^2(\mathbb{R}, L^2(S^1, \mathbb{R}^{2n})) : \ Q\zeta \in L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$
PROOF:
⟨3⟩1.

$$\begin{aligned} \int_{\mathbb{R}} \|K(-\sigma)\zeta(s+\sigma)\|_{L^2} d\sigma &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \|K(-\sigma)\zeta(s+\sigma)\|^2_{L^2(S^1)} ds \right)^{1/2} d\sigma \\ &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-2\delta|\sigma|} \|\zeta(s+\sigma)\|^2_{L^2(S^1)} ds \right)^{1/2} d\sigma \\ &\leq \int_{\mathbb{R}} e^{-\delta|\sigma|} \|\zeta\|_{L^2} d\sigma < \infty. \end{aligned}$$

⟨2⟩18. If $\eta = Q\zeta$ then $\zeta = \partial_s \eta + A\eta$.

Proof:
⟨3⟩1. Let: $\zeta^{\pm} = p^{\pm}(\zeta)$.
⟨3⟩2.

$$(Q\zeta)(s,t) = \int_{-\infty}^{s} \exp(-A^{+}(s-\sigma))\zeta^{+}(\sigma,t)d\sigma - \int_{s}^{\infty} \exp(A^{-}(\sigma-s))\zeta^{-}(\sigma,t)d\sigma.$$

⟨3⟩3. Let: $\eta^{+} = \int_{-\infty}^{s} \exp(-A^{+}(s-\sigma))\zeta^{+}(\sigma,t)d\sigma$ and
$\eta^{-} = -\int_{s}^{\infty} \exp(A^{-}(\sigma-s))\zeta^{-}(\sigma,t)d\sigma$.
⟨3⟩4. $\partial_s \eta^{\pm} + A^{\pm}\eta^{\pm} = \zeta^{\pm}$.
⟨3⟩5. $\zeta = \partial_s \eta + A\eta$.
⟨2⟩19. $D \circ Q = \mathrm{Id}_{L^2(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.
Proof:
⟨3⟩1. $D(Q\zeta) = \zeta$ by ⟨2⟩18.
⟨2⟩20. $Q \circ D = \mathrm{Id}_{W^{1,2}(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.
Proof:
⟨3⟩1. Let: $\eta^{\pm} = p^{\pm}(\eta)$.
⟨3⟩2.

$$\begin{aligned} Q(A\eta) &= \int_{-\infty}^{s} \exp(-A^{+}(s-\sigma))A^{+}\eta^{+}(\sigma)d\sigma - \int_{s}^{\infty} \exp(A^{-}(\sigma-s))A^{-}\eta^{-}(\sigma)d\sigma \\ &= \int_{-\infty}^{s} A^{+}\exp(-A^{+}(s-\sigma))\eta^{+}(\sigma)d\sigma - \int_{s}^{\infty} A^{-}\exp(A^{-}(\sigma-s))\eta^{-}(\sigma)d\sigma \end{aligned}$$

⟨3⟩3.

$$\begin{aligned} &\int_{-\infty}^{s} A^{+}\exp(-A^{+}(s-\sigma))\eta^{+}(\sigma)d\sigma \\ =&\int_{-\infty}^{s} -\partial_s\left(\exp(-A^{+}(s-\sigma))\eta^{+}(\sigma)\right)d\sigma \\ =&-\partial_s\int_{-\infty}^{0} \exp(-A^{+}(-\sigma)\eta^{+}(s+\sigma))d\sigma + \eta^{+}(s) \\ =&-\int_{-\infty}^{0} \exp(-A^{+}(-\sigma))\partial_s\eta^{+}(s+\sigma)d\sigma + \eta^{+}(s) \end{aligned}$$

⟨3⟩4.

$$\begin{aligned} &-\int_{s}^{\infty} A^{-}\exp(A^{-}(\sigma-s))\eta^{-}(\sigma)d\sigma \\ =&-\int_{s}^{\infty} \partial_s\left(\exp(A^{-}(\sigma-s))\eta^{-}(\sigma)\right)d\sigma \\ =&\partial_s\int_{-\infty}^{0} \exp(A^{-}(-\sigma))\eta^{-}(s+\sigma)d\sigma + \eta^{-}(s) \\ =&\int_{0}^{\infty} \exp(A^{-}(-\sigma))\partial_s\eta^{-}(s+\sigma)d\sigma + \eta^{-}(s) \end{aligned}$$

⟨3⟩5. $Q(A\eta) = -Q(\partial_s\eta) + \eta$.
Proof:
⟨4⟩1. Add ⟨2⟩3 and ⟨2⟩4.
⟨1⟩4. Q.E.D.

Next we prove the general case. This is split into the two cases $p > 2$

and $1 < p < 2$. The first case follows by combining the usual Sobolev embeddings with the elliptic regularity results for operators of the type D_S. For the second case a typical duality argument is used where one considers the adjoint operator D_S^* where results from the first case apply since D_S^* is of the same form as D_S.

Proposition 2.21 *Let $p > 1$. Furthermore, let $S \in C^\infty(\mathbb{R} \times S^1, \mathrm{Sym}(\mathbb{R}^{2n}))$ be non-degenerate and independent of s. Then D_S is bijective.*

PROOF:
⟨1⟩1. $\exists C > 0 : \forall k \in R, \forall \xi \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) :$
$$\|\xi\|_{W^{1,p}([k,k+1]\times S^1,\mathbb{R}^{2n})} \leq C \left(\|D_S\xi\|_{L^p([k-1/2,k+3/2]\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^p([k-1/2,k+3/2]\times S^1\ \mathbb{R}^{2n})} \right).$$
PROOF:
⟨2⟩1. ASSUME: $k = 0$.
⟨2⟩2. In this case the result follows from the elliptic regularity results earlier.
⟨2⟩3. ASSUME: $k \neq 0$.
⟨2⟩4. LET: $\zeta(s,t) = \xi(s-k,t)$.
⟨2⟩5. $D_S\zeta(s,t) = (D_S\xi)(s-k,t)$.
⟨2⟩6. Apply the case $k = 0$ to ζ.
⟨1⟩2. ASSUME: $p > 2$.
⟨1⟩3. $\exists C_1 > 0 : \forall k \in \mathbb{R}, \forall \xi \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) :$
$$\|\xi\|_{W^{1,p}([k,k+1]\times S^1)} \leq C \left(\|D_S\xi\|_{L^p([k-1,k+2]\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^2([k-1,k+2]\times S^1\ \mathbb{R}^{2n})} \right).$$
PROOF:
⟨2⟩1. SUFFICES ASSUME: $k = 0$.
PROOF:
⟨3⟩1. D_S is invariant under translation in the s-direction.
⟨2⟩2. The inclusion $L^p([-1,2] \times S^1, \mathbb{R}^{2n}) \to L^2([-1,2] \times S^1, \mathbb{R}^{2n})$ is continuous.
⟨2⟩3. The inclusion $W^{1,p}([-1,2] \times S^1, \mathbb{R}^{2n}) \to W^{1,2}([-1,2] \times S^1, \mathbb{R}^{2n})$ is continuous.
⟨2⟩4. The inclusion $W^{1,p}([-1/2,3/2] \times S^1, \mathbb{R}^{2n}) \to W^{1,2}([-1/2,3/2] \times S^1, \mathbb{R}^{2n})$ is continuous.
⟨2⟩5. The inclusion $W^{1,2}([-1/2,3/2] \times S^1, \mathbb{R}^{2n}) \to L^p([-1/2,3/] \times S^1, \mathbb{R}^{2n})$ is continuous.
⟨2⟩6. $\|\xi\|_{W^{1,p}([0,1])} \leq C \left(\|D_S\xi\|_{L^p([-1/2,3/2]\times S^1,\mathbb{R}^{2n})} + \|\xi\|_{L^p([-1/2,3/2]\times S^1\ \mathbb{R}^{2n})} \right)$.
PROOF:
⟨3⟩1. By ⟨1⟩1.
⟨2⟩7. $\|\xi\|_{L^p([-1/2,3/2])} \leq C' \left(\|D_S\xi\|_{L^2([-1,2])} + \|\xi\|_{L^2([-1,2])} \right)$.
PROOF:
⟨3⟩1. By ⟨1⟩1 and ⟨2⟩5.
⟨2⟩8. $\|D_S\xi\|_{L^p([-1/2,3/2]\times S^1,\mathbb{R}^{2n})} + \|D_S\xi\|_{L^2([-1,2])} \leq C''\|D_S\xi\|_{L^p([-1,2]\times S^1,\mathbb{R}^{2n})}$.
⟨2⟩9. Q.E.D.

⟨1⟩4. $\exists C > 0$:
$$\xi \in W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n}), D_S\xi \in L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$
$$\implies \xi \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \text{ and } \|\xi\| \leq C\|D_S\xi\|.$$
Proof:
⟨2⟩1. $\forall$ compact $K \subset \mathbb{R} \times S^1$: $\xi \in W^{1,p}(K, \mathbb{R}^{2n})$.
Proof:
⟨3⟩1. $\|\xi\|^p_{W^{1,p}([k,k+1])} \leq 2^p C_1 \left(\|D_S\xi\|^p_{L^p([k-1,k+2])} + \|\xi\|^p_{L^2([k-1,k+2])} \right)$
⟨3⟩2. $\|\xi\|^p_{L^2([k,k+1])} \leq C_2 \left(\|D_S\xi\|^2_{L^p([k-1,k+2])} + \int_{k+1}^{k+2} \|\xi(s,\cdot)\|^p_{L^2} ds \right)$
⟨3⟩3. $\|\xi\|^p_{W^{1,p}(\mathbb{R}\times S^1)} \leq C_3 \left(\|D_S\xi\|^p_{L^p(\mathbb{R}\times S^1)} + \|\xi\|^p_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \right)$
⟨3⟩4. $\|\xi\|_{L^2(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq C\|D_S\xi\|_{L^2(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.
Proof:
⟨4⟩1. Let: $Q =$.
⟨4⟩2. $\|\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} = \|QD_S\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.
⟨4⟩3. $\|Q\zeta\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq C\|\zeta\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.
⟨1⟩5. D_S is bijective for $p > 2$.
Proof:
⟨2⟩1. D_S is injective.
Proof:
⟨3⟩1. $\forall \xi \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$: $\|\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq C\|D_S\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.
⟨2⟩2. D_S is surjective.
Proof:
⟨3⟩1. Suffices: Im (D_S) is dense.
⟨3⟩2. Let: $\zeta \in L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \cap L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.
⟨3⟩3. $\exists \xi \in W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$: $D_S\xi = \zeta$.
⟨3⟩4. $\xi \in W^{1,p}(\mathbb{R} \times S^1)$.
⟨1⟩6. D_S is bijective for $1 < p < 2$.
Proof:
⟨2⟩1. $\exists C > 0$: $\|\xi\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq C\|D_S\xi\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.
Proof:
⟨3⟩1. Let: $q \in \mathbb{R}$ such that $1/q + 1/p = 1$.
⟨3⟩2. Let: $D_S^* : W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ be defined by
$$D_S^* = -\partial_s + J_0\partial_t + S.$$
⟨3⟩3. $\forall \eta \in W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n}), \forall \xi \in C_0^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$:
$$\langle D_S^*\eta, \xi\rangle = \langle \eta, D_S\xi\rangle.$$
⟨3⟩4. D_S^* is bijective.
Proof:
⟨4⟩1. By ⟨1⟩5 since $q > 2$.
⟨3⟩5. $\|\xi\|_{L^p} \leq C\|D_S\xi\|_{L^p}$.
Proof:
⟨4⟩1. $\|\xi\|_{L^p} = \sup_{\zeta\in L^q, \|\zeta\|_{L^q}=1} |\langle \zeta, \eta\rangle|$.
⟨4⟩2. $\|\xi\|_{L^p} \leq \|D_S\xi\|_{L^p} \sup_{\eta\in W^{1,q}, \|D_S^*\eta\|_{L^q}=1} \|\eta\|_{L^q}$.

Proof:
⟨5⟩1. By ⟨3⟩4 for every $\zeta \in L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ there exists a $\eta \in W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ such that $D_S^*\eta = \zeta$.
⟨5⟩2. Now use ⟨3⟩3.
⟨4⟩3. $\|\eta\|_{L^q} \leq C\|D_S^*\eta\|_{L^q}$.
⟨3⟩6. $\|\partial_s\xi\|_{L^p} \leq C\|D_S\xi\|_{L^p}$.
Proof:
⟨4⟩1. $\|\partial_s\xi\|_{L^p} = \sup_{\zeta \in L^q, \|\zeta\|_{L^q}=1} |\langle \zeta, \partial_s\xi\rangle| = \sup_{\eta \in W^{1,q}, \|D_S^*\eta\|_{L^q}=1} |\langle D_S^*\eta, \partial_s\xi\rangle|$.
⟨4⟩2. $D_S^*\partial_s = \partial_s D_S^*$.
⟨4⟩3. $|\langle D_S^*\eta, \partial_s\xi\rangle| = |\langle \partial_s\eta, D_S\xi\rangle|$.
⟨4⟩4. $\|\eta\|_{W^{1,q}} \leq C\|D_S^*\eta\|_{L^q}$.
⟨4⟩5. $\|\partial_s\xi\|_{L^p} \leq \|D_S\xi\| \sup_{\eta \in W^{1,q}, \|D_S^*\eta\|_{L^q}=1} \|\partial_s\eta\| \leq C\|D_S\xi\|$.
⟨3⟩7. $\|\partial_t\xi\|_{L^p} \leq C\|D_S\xi\|_{L^p}$.
Proof:
⟨4⟩1. $\|\partial_t\xi\|_{L^p} = \sup_{\zeta \in L^q, \|\zeta\|_{L^q}=1} |\langle \zeta, \partial_t\xi\rangle| = \sup_{\eta \in W^{1,q}, \|D_S^*\eta\|_{L^q}=1} |\langle D_S^*\eta, \partial_t\xi\rangle|$.
⟨4⟩2. $|\langle D_S^*\eta, \partial_t\xi\rangle| = |\langle \partial_t\eta, D_S\xi\rangle + \langle \partial_t S\eta, \xi\rangle|$.
Proof:
⟨5⟩1. $\langle D_S^*\eta, \partial_t\xi\rangle = \langle \partial_t D_S^*\eta, \xi\rangle$.
⟨5⟩2. $\partial_t D_S^* = D_S^*\partial_t + \partial_t S$.
⟨4⟩3. Let: $C = \sup_{t \in S^1} \|\partial_t S\|$.
⟨4⟩4. $\|\partial_t\xi\|_{L^p} \leq \|D_S\xi\|_{L^q} \sup_{\eta \in W^{1,q}, \|D_S^*\eta\|_{L^q}=1} \|\eta\|_{W^{1,q}} + C\|\xi\|_{L^p} \sup_{\eta \in W^{1,q}, \|D_S^*\eta\|_{L^q}=1} \|\eta\|_{L^q}$
Proof:
⟨5⟩1. Combine the previous three steps.
⟨4⟩5. $\|\eta\|_{L^q} \leq \|\eta\|_{W^{1,q}} \leq C'\|D_S^*\eta\|_{L^q}$.
⟨4⟩6. Q.E.D.
Proof:
⟨5⟩1. Combine ⟨4⟩4 with ⟨4⟩5 and ⟨3⟩5.
⟨3⟩8. Q.E.D.
Proof:
⟨4⟩1. $\|\xi\|_{W^{1,p}} = \|\xi\|_{L^p} + \|\partial_s\xi\|_{L^p} + \|\partial_t\xi\|_{L^p}$.
⟨4⟩2. Now use ⟨3⟩5, ⟨3⟩6 and ⟨3⟩7.
⟨2⟩2. D_S is injective and has closed image.
Proof:
⟨3⟩1. By ⟨2⟩1.
⟨2⟩3. D_S^* is injective.
Proof:
⟨3⟩1. By duality.
⟨1⟩7. Q.E.D.

We can now use the special case where S is a path of symmetric matrices independent of s to prove the Fredholm property in the general case. That is we take $S \in C^\infty(\mathbb{R} \times S^1, \mathrm{End}(\mathbb{R}^{2n}))$ such that $\lim_{s\to\pm\infty} S(s,t)$ exists and is uniform in t.

Theorem 2.12 *Let $S \in C^\infty(\mathbb{R} \times S^1, \mathrm{End}(\mathbb{R}^{2n}))$ be non-degenerate. Then the operator $D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is Fredholm for every $p > 1$.*

Proof:

$\langle 1\rangle 1.$ $\exists B > 0 : \ \forall \eta \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) : \ \|\eta\|_{W^{1,p}} \le B \|D_{S^\pm}\eta\|_{L^p}$.

Proof:

$\langle 2\rangle 1.$ $D_{S^\pm}$ is bijective by Proposition 2.21.

$\langle 2\rangle 2.$ $D_{S^\pm}$ is continuous by the open mapping theorem.

$\langle 1\rangle 2.$ $\exists M, C > 0 : \ \forall \eta \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$:

$$(\forall |s| \le M - 1 : \ \eta(s,t) = 0) \implies \|\eta\|_{W^{1,p}} \le C \|D_S \eta\|_{L^p}.$$

Proof:

$\langle 2\rangle 1.$ Since $\lim_{s\to\infty} S(s,t) = S^\pm(t)$ for all $t \in S^1$.

$\langle 1\rangle 3.$ Let: $\beta \in C^\infty(\mathbb{R}, [0,1])$ be a cut-off function such that

$$\beta(t) = \begin{cases} 0 & |s| \ge M \\ 1 & |s| \le M - 1 \end{cases}.$$

$\langle 1\rangle 4.$ $\|\eta\|_{W^{1,p}} \le C_1 \left(\|D_S(\beta\eta)\|_{L^p} + \|\beta\eta\|_{L^p} + \|D_S(1-\beta)\eta\|_{L^p}\right)$.

Proof:

$\langle 2\rangle 1.$ $\eta = \beta\eta + (1-\beta)\eta$.

$\langle 2\rangle 2.$ Now use the product rule.

$\langle 1\rangle 5.$ $D_S(\beta\eta) = (\partial_s + J_0\partial_t + S)(\beta\eta) = \beta D_S(\eta) + \beta'(s)\eta$.

Proof:

$\langle 2\rangle 1.$ By the product rule.

$\langle 1\rangle 6.$ $\|D_S(\beta\eta)\|_{L^p} \le \|D_S(\eta)\|_{L^p} + K\|\eta\|_{L^p([-M,M])}$.

Proof:

$\langle 2\rangle 1.$ $\exists K > 0 : \ \forall s \in [-M, M] : \ |\beta'(s)| \le K$.

$\langle 2\rangle 2.$ $\forall |s| \ge M : \ \beta'(s) = 0$.

$\langle 1\rangle 7.$ $\|\eta\|_{W^{1,p}} \le C_2 \Big(\|\eta\|_{L^p([-M,M])} + \|D_S(\eta)\|_{L^p}\Big)$.

$\langle 1\rangle 8.$ The restriction map $W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p([-M,M] \times S^1, \mathbb{R}^{2n})$ is compact.

$\langle 1\rangle 9.$ D_S has finite-dimensional kernel and closed image.

Proof:

$\langle 2\rangle 1.$ Let: $q \in \mathbb{R}$ be such that $1/p + 1/q = 1$ and

$D_S^* = -\partial_s + J_0\partial_t + S^T : W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.

$\langle 2\rangle 2.$ Let: $F = \{\eta \in L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n}) : \ \langle \mathrm{Im}\,(D_S)\,, \eta\rangle\}$.

$\langle 2\rangle 3.$ $\forall \eta \in F : \ \eta \in W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.

Proof:

$\langle 3\rangle 1.$ $D_S^*\eta = 0$ in the weak sense.

$\langle 3\rangle 2.$ Now use elliptic regularity to boost η.

$\langle 2\rangle 4.$ D_S^* has finite-dimensional cokernel.

$\langle 1\rangle 10.$ D_S has finite-dimensional cokernel.

Proof:

$\langle 2\rangle 1.$ Let: $E = \{\varphi \in \mathrm{Hom}(L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}), \mathbb{R}) : \ \varphi|_{\mathrm{Im}(D_S)} = 0\}$.

$\langle 2\rangle 2.$ $\forall \varphi \in E : \ \exists \eta \in L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n}) : \ \varphi = \langle \eta, \cdot\rangle$ by the Riesz representation theorem.

⟨1⟩11. Q.E.D.

2.6.2 Index Computation

We have seen in the previous section that the differential of the Floer map is a Fredholm map. Hence, it makes sense to consider its index at a solution $u \in \mathcal{M}(H,J)$ given as usual by $\dim \operatorname{Ker}\left(D\bar{\partial}_{J,H}(u)\right) - \dim \operatorname{Coker}\left(D\bar{\partial}_{J,H}(u)\right)$. This index is the tentative dimension of the moduli space around u assuming it admits a manifold structure in the first place.

Assuming now that u is a solution connecting two non-degenerate, critical loops of $\mathcal{A}_H$ we want to express the index in terms of the indices of these bounding loops. Recall that the operator $D\bar{\partial}_{H,J}(u)$ is of the form

$$D\bar{\partial}_{H,J}(u) = \partial_s + J_0\partial_t + S(s,t) =: D_S$$

after choosing suitable coordinates. Here, $S \in C^\infty(\mathbb{R} \times S^1, \mathsf{End}(\mathbb{R}^{2n}))$ is such that the limits $\lim_{s\to\pm\infty} S(s,t)$ exist and are given by symmetric matrix valued functions $S^\pm \in C^\infty(S^1, \mathrm{Sym}(\mathbb{R}^{2n}))$, and such that the convergence is uniform in t. From now on we will denote the set of such matrix-valued functions by $C^\infty_{as}(\mathbb{R} \times S^1, \mathsf{End}(\mathbb{R}^{2n}))$ (for asymptotically symmetric). Computing the index of D_S directly is not easy. Instead we modify the function S in a two-step process that leaves the Fredholm index of D_S invariant. We first introduce these operations we will apply to S. After that we show that they do not affect the index.

We begin with the first modification. Let $S \in C^\infty_{as}(\mathbb{R} \times S^1, \mathsf{End}(\mathbb{R}^{2n}))$. Fix a number $\sigma > 0$ and let $\chi_\sigma : C^\infty(\mathbb{R}, [0,1])$ be a smooth bump function such that

1. $\forall s \in [-\sigma, \sigma] :\ \chi_\sigma(s) = 1$ and
2. $\forall s > 2\sigma :\ \chi_\sigma(s) = 0$.

Then denote by $\mathrm{Op}_1(S,\sigma) \in C^\infty_{as}(\mathbb{R} \times S^1, \mathsf{End}(\mathbb{R}^{2n}))$ the function

$$\mathrm{Op}_1(S,\sigma) = \begin{cases} \chi_\sigma(s)S(s,t) + (1-\chi_\sigma(s))S^-(t) & s \le -\sigma \\ S(s,t) & |s| \le \sigma \\ \chi_\sigma(s)S(s,t) + (1-\chi_\sigma(s))S^+(t) & s \ge \sigma \end{cases}.$$

In other words, the modification Op_1 produces a matrix-valued function that is constantly equal to its limits in the $\pm s$-direction outside a compact interval.

Proposition 2.22 *Let $S \in C^\infty_{as}(\mathbb{R} \times S^1, \mathsf{End}(\mathbb{R}^{2n}))$. Fix $s_0 > 0$ and let $S' = \mathrm{Op}_1(S, s_0) \in C^\infty_{as}(\mathbb{R} \times S^1, \mathsf{End}(\mathbb{R}^{2n}))$. Then $\mathrm{Ind}(D_S) = \mathrm{Ind}(D_{S'})$.*

Proof:
⟨1⟩1. Let: $\varepsilon > 0$.
⟨1⟩2. $\exists \sigma_0 > 0$:
1. $\forall s \in (-\infty, -\sigma_0], \forall t \in S^1 : \ \|S(s,t) - S^-(t)\| < \varepsilon$.
2. $\forall s \in [\sigma_0, \infty), \forall t \in S^1 : \ \|S(s,t) - S^+(t)\| < \varepsilon$.

⟨1⟩3. Let: $\widetilde{S}(s,t) = \mathrm{Op}_1(S, \sigma_0)$.
⟨1⟩4. Let:

$$C(s,t) = \begin{cases} (1-\chi(s))(S(s,t) - S^-(t)) & s \le -\sigma_0 \\ 0 & |s| \le \sigma_0 \\ (1-\chi(s))(S(s,t) - S^+(t)) & s \ge \sigma_0 \end{cases}$$

⟨1⟩5. $D_S - D_{\widetilde{S}} = C$.
⟨1⟩6. $\forall (s,t) \in \mathbb{R} \times S^1 : \ \|C\| < \varepsilon$.
⟨1⟩7. $D_{\widetilde{S}}$ is Fredholm and $\mathrm{Ind} D_S = \mathrm{Ind} D_{\widetilde{S}}$.
Proof:
⟨2⟩1. Choose $\varepsilon > 0$ sufficiently small. Then this follows from the standard invariance properties of the Fredholm index.

⟨1⟩8. Let: $\widetilde{S}_0(s,t) = \mathrm{Op}_1(S, s_0)$.
⟨1⟩9. Q.E.D.
Proof:
⟨2⟩1. $D_{\widetilde{S}}$ and $D_{\widetilde{S}_0}$ and lie in the same path-component.

Now we describe the second modification which will produce a matrix that is diagonal and t-independent. For this we first introduce the following diagonal matrices. We let

$$I_\varepsilon = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix}$$

and then make the following definition.

Definition 2.18 *We define matrices $S_k \in \mathrm{Sym}(\mathbb{R}^{2n})$ for $k \in \{1, \ldots, n\}$:*

$$S_k(\varepsilon) = \begin{cases} \mathrm{diag}(-\pi \mathrm{Id}_{\mathbb{R}^2}, \ldots, -\pi \mathrm{Id}_{\mathbb{R}^2}, (n-k-1)\mathrm{Id}_{\mathbb{R}^2}) & \text{if } n = k \mod 2 \\ \mathrm{diag}(-\pi \mathrm{Id}_{\mathbb{R}^2}, \ldots, -\pi \mathrm{Id}_{\mathbb{R}^2}, I_\varepsilon, (n-k-2)\mathrm{Id}_{\mathbb{R}^2}) & \text{if } n-1 = k \mod 2 \end{cases} \tag{2.45}$$

Let $S \in C^\infty_{as}(\mathbb{R} \times S^1, \mathsf{End}(\mathbb{R}^{2n}))$. The limit matrices $S^\pm(t) = \lim_{s \to \pm\infty} S(s,t)$ generate paths of symplectic matrices $R^\pm(t) = e^{tJS^\pm(t)}$. Denote by $k^\pm = \mu(R^\pm(t))$ the corresponding Maslov index of these paths of symplectic matrices. Now consider the matrices $S_{k^\pm}(1)$ which also define paths of symplectic matrices $R_{k^\pm}(t) = e^{tJS_{k^\pm}(1)}$ in an analogous manner. These induced paths have the same index, i.e. $\mu(R^\pm(t)) = \mu(R_{k^\pm}(t))$.

From our knowledge of the topology of the space $\mathcal{SP}(2n)$ we can therefore conclude that the two paths are homotopic. Let $\psi^\pm \in C^1([0,1]^2, \mathrm{Sp}(2n))$ be such a homotopy between $R_{k^\pm}$ and $R^\pm$. In fact, we can find such a homotopy that is 1-periodic in t so that $\psi^\pm \in C^1([0,1] \times S^1, \mathrm{Sp}(2n))$. Correspondingly there is a homotopy on the level of generating paths of symmetric matrices.

Proposition 2.23 *Let $S^\pm : [0,1]^2 \to \mathrm{Sym}(R^{2n})$ be the function given by*

$$S^\pm_r(t) = -J(\psi^\pm_r)'(t)(\psi^\pm_r)^{-1}.$$

Then

1. $S^\pm : [0,1] \times S^1 \to \mathrm{Sym}(R^{2n})$*, i.e.* $S^\pm$ *is* 1*-periodic in the t-variable.*
2. $S^\pm_0(t)$ *generates* $R^\pm(t)$*.*
3. $S^\pm_1(t) = S_{k^\pm}(1)$ *and generates* $R_{k^\pm}(t)$*.*

After this preparation we can now define the second type of modification that makes S diagonal.

Definition 2.19 *Let $S \in C^\infty_{as}(\mathbb{R} \times S^1, \mathsf{End}(\mathbb{R}^{2n}))$. Then denote by $\mathrm{Op}_2(S, r, \sigma)$ the function*

$$\mathrm{Op}_2(S, r, \sigma) = \begin{cases} S(s,t) & r = 0 \\ S^+_r(t) & s \geq \sigma \\ S^-_r(t) & s \leq -\sigma \\ S(s) & r = 1 \end{cases}$$

Proposition 2.24 *Let $S \in C^\infty_{as}(\mathbb{R} \times S^1, \mathsf{End}(\mathbb{R}^{2n}))$. Then $\mathrm{Ind} D_S = \mathrm{Ind} D_{\mathrm{Op}_2(S,r,\sigma)}$ for all $r \in [0,1]$. In particular $\mathrm{Ind} D_S = \mathrm{Ind} D_{\mathrm{Op}(S,1,\sigma)}$.*

PROOF:
⟨1⟩1. S and $\mathrm{Op}_2(S, r, \sigma)$ are homotopic so that the corresponding operators lie in the same path-component.
⟨1⟩2. Q.E.D.

Finally, we can now compute the index of D_S by computing $\mathrm{Ind} D_{\mathrm{Op}_2(S,1,\sigma)}$. This can actually be further simplified. In principle we need to compute the dimension of the kernel and cokernel of D_S where S is obtained by applying the two modification steps above. Hence, S is diagonal and asymptotically constant with limit matrices given by $S_{k^\pm}(1)$ as in Definition 2.18. We can reduce the computation to the case of 2×2-blocks. Furthermore, instead of computing the cokernel of D_S we can compute the kernel of the adjoint D^*_S which is given by $-\partial_s + J_0\partial_t + S^T$. In the following proof we will on occasion identify $\mathbb{R}^2$ with $\mathbb{C}$ in the obvious way.

Proposition 2.25 *Assume that S is of the form*

$$\begin{bmatrix} d_1(s) & 0 \\ 0 & d_2(s) \end{bmatrix}$$

where

$$d_i(s) = \begin{cases} d_i^- & s \leq -s_0 \\ d_i^+ & s \geq s_0 \end{cases}$$

for $i \in \{1,2\}$. *Also assume that* $d_i^\pm \notin 2\pi\mathbb{Z}$ *for* $i \in \{1,2\}$. *Then the following holds.*

1. *If* $d_1(s) = d_2(s) =: d(s)$ *then*

$$\dim \operatorname{Ker}(D_S) = 2|\{l \in \mathbb{Z} : d^- < 2\pi l < d^+\}|$$
$$\dim \operatorname{Ker}(D_S^*) = 2|\{l \in Z : d^+ < 2\pi l < d^-\}|.$$

2. *If* $\sup_{\mathbb{R}} \|S(s)\| < 1$ *then*

$$\dim \operatorname{Ker}(D_S) = |\{i \in \{1,2\} : d_i^- < 0 \text{ and } 0 < d_i^+\}|$$
$$\dim \operatorname{Ker}(D_S^*) = |\{i \in \{1,2\} : d_i^+ < 0 \text{ and } 0 < d_i^-\}|.$$

Proof:
⟨1⟩1. Case: 1.
Proof:
⟨2⟩1. Let: $\eta = (\eta_1, \eta_2) \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^2)$ such that $D_S\eta = 0$.
⟨2⟩2. Let: $\bar{\partial} = \partial_s + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \partial_t$.
⟨2⟩3. $\bar{\partial}\eta + S\eta = 0$.
⟨2⟩4. $\exists B \in C^\infty(\mathbb{R}, \mathrm{GL}(2,\mathbb{C})) : \bar{\partial}B + SB = 0$.
Proof:
⟨3⟩1. Let: $D(s) = \int_0^s d(\tau)d\tau$.
⟨3⟩2. Let: $b(s) = \exp(-\int_0^s D(s)$.
⟨3⟩3. Let: $B = \begin{bmatrix} b(s) & 0 \\ 0 & b(s) \end{bmatrix}$.
⟨2⟩5. Let: $\widetilde{\eta} = B^{-1}\eta$.
⟨2⟩6. $\bar{\partial}\widetilde{\eta} = 0$.
Proof:
⟨3⟩1. $\partial_s(B^{-1}\eta) = (\partial_s B^{-1})\eta + B^{-1}\partial_s\eta$.
⟨3⟩2. $J_0\partial_t(B^{-1}\eta) = B^{-1}J_0\partial_t\eta = B^{-1}J_0\partial_t\eta$.
⟨3⟩3. $\bar{\partial}\widetilde{\eta} = B^{-1}\bar{\partial}\eta + (\partial_s B^{-1})\eta = -B^{-1}S\eta + (\partial_s B^{-1})\eta$.
⟨3⟩4. $\partial_s B^{-1} = -B^{-1}(\partial_s B)B^{-1}$.
⟨3⟩5. $\bar{\partial}\widetilde{\eta} = -B^{-1}(S + (\partial_s B)B^{-1})\eta = -B^{-1}(SB + (\partial_s B))B^{-1}\eta = 0$.
Proof:
⟨4⟩1. $\partial_s B = \bar{\partial}B$.
⟨4⟩2. $\bar{\partial}B + SB = 0$.
⟨2⟩7. $\widetilde{\eta} \in C^\infty(\mathbb{R} \times S^1, \mathbb{R}^2)$.
Proof:

⟨3⟩1. By elliptic regularity.

⟨2⟩8. There is a complex Fourier expansions $\widetilde{\eta}(s+it) = \sum_{l\in\mathbb{Z}} c_l e^{(s+it)2\pi l}$ that converges pointwise for $(s,t) \neq (0,0)$.

⟨2⟩9. $\eta(s,t) = \sum_{l\in\mathbb{Z}} e^{2\pi sl}\left(a_l \begin{bmatrix} e^{-D(s)}\cos(2\pi lt) \\ e^{-D(s)}\sin(2\pi lt)\end{bmatrix} + b_l \begin{bmatrix} -e^{-D(s)}\sin(2\pi lt) \\ e^{-A(s)}\cos(2\pi lt)\end{bmatrix}\right)$.

PROOF:

⟨3⟩1. $\widetilde{\eta}(s,t) = \sum_{l\in\mathbb{Z}} e^{2\pi sl}\left(a_l \begin{bmatrix} \cos(2\pi lt) \\ \sin(2\pi lt)\end{bmatrix} + b_l \begin{bmatrix} -\sin(2\pi lt) \\ \cos(2\pi lt)\end{bmatrix}\right)$.

⟨2⟩10. For $s \leq -s_0$ we have $\eta(s,t) = \sum_{l\in\mathbb{Z}} \begin{bmatrix} e^{(2\pi l-d^-)s+K_1}(a_l\cos(2\pi lt) - b_l\sin(2\pi lt)) \\ e^{(2\pi l-d^-)s+K_2}(a_l\sin(2\pi lt) + b_l\cos(2\pi lt))\end{bmatrix}$.

⟨2⟩11. For $s \geq s_0$ we have $\eta(s,t) = \sum_{l\in\mathbb{Z}} \begin{bmatrix} e^{(2\pi l-d^+)s+L_1}(a_l\cos(2\pi lt) - b_l\sin(2\pi lt)) \\ e^{(2\pi l-d^+)s+L_2}(a_l\sin(2\pi lt) + b_l\cos(2\pi lt))\end{bmatrix}$.

⟨2⟩12. $\eta \in W^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^2) \iff \forall l \in \mathbb{Z}: \lim_{s\to\pm\infty} e^{(2\pi l-d^\pm)s} = 0$.

⟨2⟩13. $\forall l \in \mathbb{Z}: \lim_{s\to\pm\infty} e^{(2\pi l-d^\pm)s} = 0 \iff$

1. $a_l = b_l = 0$ or $d^- < 2\pi < d^+$ if $l \neq 0$.
2. $a_0 = 0$ or $d^- < 0 < d^+$ and $b_0 = 0$ or $d^- < 0 < d^+$ if $l = 0$.

⟨2⟩14. $\dim \mathrm{Ker}\,(D_S) = 2|\{l \in \mathbb{Z}: d^- < 2\pi l < d^+\}|$.

⟨1⟩2. CASE: 2.

PROOF:

⟨2⟩1. ASSUME: $\sup_{s\in\mathbb{R}}\|S(s)\| < 1$.

⟨2⟩2. Every $\xi \in \mathrm{Ker}\,(D_S)$ and every $\eta \in \mathrm{Ker}\,(D_S^*)$ is t-independent.

⟨2⟩3. LET: $\xi \in \mathrm{Ker}\,(D_S)$.

⟨2⟩4. $\partial_s\xi_1 = -d_1(s)\xi_1$ and $\partial_s\xi_2 = -d_2(s)\xi_2$.

⟨2⟩5. $\xi_i \in W^{1,p}(\mathbb{R}) \iff d_i^- < 0 < d_i^+$ for $i \in \{1,2\}$.

⟨2⟩6. The computation for $\dim \mathrm{Ker}\,(D_S^*)$ is analogue.

⟨1⟩3. Q.E.D.

Next, we can the above result to compute the indices of D_S where S consists of 2×2 blocks. Here we have to distinguish the different cases arising from the relative parity of the indices of the limiting matrices and the dimension.

Proposition 2.26 *Let S be obtained by applying the operation $\mathrm{Op}_2(\cdot, 1, \sigma)$. Then*

$$\mathrm{Ind}(D_S) = k^- - k^+.$$

PROOF:

⟨1⟩1. CASE: $k^+ - n = k^- - n = 0 \mod \mathbb{Z}_2$.

PROOF:

⟨2⟩1. $\dim \mathrm{Ker}\,(D_S) = 2|\{l \in \mathbb{Z}: n-1-k^- < 2l < n-1-k^+\}|$

$$= \begin{cases} k^- - k^+ & k^- > k^+ \\ 0 & \text{else.}\end{cases}$$

⟨2⟩2. $\dim \mathrm{Ker}\,(D_S^*) = 2|\{l \in \mathbb{Z}: k^- - n + 1 < 2l < k^+ - n + 1\}|$

$$= \begin{cases} k^- - k^+ & k^- < k^+ \\ 0 & \text{else.}\end{cases}$$

⟨2⟩3. $\mathrm{Ind}(D_S) = k^- - k^+$.
⟨1⟩2. Case: $k^+ - n = 1 \mod \mathbb{Z}_2$ and $k^- - n = 0 \mod \mathbb{Z}_2$.
Proof:
⟨2⟩1. Suffices Assume: $S_{k^-} = S_{k^-}(\varepsilon)$ and consequently $\sup_{s\in\mathbb{R}} \|S_{k^-}\| < 1$.
⟨2⟩2. $\dim \mathrm{Ker}\,(D_S) = 2|\{l \in \mathbb{Z} :\ n-1-k^- < 2l < n-2-k^+\}| + 1$
$$= \begin{cases} k^- - k^+ & k^- > k^+ \\ 1 & k^- < k^+. \end{cases}.$$
⟨2⟩3. $\dim \mathrm{Ker}\,(D_S^*) = 2|\{l \in \mathbb{Z} :\ k^- - n + 1 < 2l < k^+ - n + 2\}|$
$$= \begin{cases} k^- - k^+ + 1 & k^- < k^+ \\ 0 & \text{else}. \end{cases}.$$
⟨2⟩4. $\mathrm{Ind}(D_S) = k^- - k^+$.
⟨1⟩3. Case: $k^+ - n = 0 \mod \mathbb{Z}_2$ and $k^- - n = 1 \mod \mathbb{Z}_2$.
Proof:
⟨2⟩1. $\dim \mathrm{Ker}\,(D_S) = 2|\{l \in \mathbb{Z} :\ k^+ - n + 1 < 2l < k^- - n + 2\}|$
$$= \begin{cases} k^- - k^+ + 1 & k^- > k^+ \\ 0 & \text{else}. \end{cases}.$$
⟨2⟩2. $\dim \mathrm{Ker}\,(D_S^*) = 2|\{l \in \mathbb{Z} :\ n-1-k^+ < 2l < n-2-k^-\}| + 1$
$$= \begin{cases} k^+ - k^- & k^- < k^+ \\ 1 & k^- < k^+. \end{cases}.$$
⟨2⟩3. $\mathrm{Ind}(D_S) = k^- - k^+$.
⟨1⟩4. Case: $k^+ - n = k^- - n = 1 \mod \mathbb{Z}_2$.
Proof:
⟨2⟩1. $\dim \mathrm{Ker}\,(D_S) = \begin{cases} k^- - k^+ & k^- < k^+ \\ 0 & \text{else}. \end{cases}.$
⟨2⟩2. $\dim \mathrm{Ker}\,(D_S^*) = \begin{cases} k^- - k^+ & k^- > k^+ \\ 0 & \text{else}. \end{cases}.$
⟨2⟩3. $\mathrm{Ind}(D_S) = k^- - k^+$.
⟨1⟩5. Q.E.D.

Finally, we can compute the index of our original operator D_S.

Theorem 2.13 *Let $D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ be the linearisation of the Floer operator along a solution $u \in \mathcal{M}(\gamma^-, \gamma^+, H, J)$ where $\gamma^\pm$ are non-degenerate. Then its Fredholm index is given by*

$$\mathrm{Ind}(D_s) = \mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+).$$

Proof:
⟨1⟩1. The operators D_S and $D_{\mathrm{Op}_2(S,1,\sigma)}$ lie in the same path-component.
⟨1⟩2. Q.E.D.

2.7 Moduli Spaces

2.7.1 Transversality

In this section we want to discuss under which conditions the space of solutions to the Floer equation that connect two distinct, non-degenerate, critical points $\gamma^{\pm}$ has the structure of a smooth manifold. This relies on an application of an infinite-dimensional version of the implicit function theorem. For this we consider the operator $\overline{\partial}_{H,J}$ as a section of a suitable vector bundle so that its intersection with the zero section is precisely the set of solutions of that equation. To conclude that this set is a manifold we need to verify that $\overline{\partial}_{J,H}$ intersects transversally. We allow the almost complex structure to depend on t which is not directly needed for the analysis of the Floer equation itself but is relevant to the later treatment of invariance properties of Floer homology. We follow the presentation in [10].

We begin by introducing the spaces we are going to work with. For the base space we take a subset of $W^{1,p}(\mathbb{R} \times S^1, M)$ whose elements satisfy an exponential decay condition. All zeroes of $\overline{\partial}_{J,H}$ satisfy this condition so this is not a restriction. Furthermore, from now on we need to assume that $p > 2$ which ensures that elements of $W^{1,p}(\mathbb{R} \times S^1, M)$ have a unique continuous representative.

Definition 2.20 *Let $\gamma^{\pm} \in \mathcal{P}(H)$. An element $u \in W^{1,p}(\mathbb{R} \times S^1, M)$ is called* asymptotically constant *if there exists an $s_0 \in \mathbb{R}$ and vector fields*

$$\xi^{-} \in W^{1,p}((-\infty, -s_0] \times S^1, (\gamma^{-})^*TM) \text{ and}$$
$$\xi^{-} \in W^{1,p}([s_0, \infty) \times S^1, (\gamma^{+})^*TM)$$

such that for $s \in (-\infty, -s_0] \cup [s_0, \infty)$ u is given by $u(s,t) = \exp_{\gamma^{\pm}(t)}(\xi^{\pm}(s,t))$. Then we define

$$\mathcal{B}^{1,p}(\gamma^{-}, \gamma^{+}) = \{u \in W^{1,p}(\mathbb{R} \times S^1, M) : \text{ u is asymptotically constant}\}. \tag{2.46}$$

As the fibre over $u \in \mathcal{B}^{1,p}(\gamma^{-}, \gamma^{+})$ we take the following space.

Definition 2.21 *Let $u \in \mathcal{B}^{1,p}(\gamma^{-}, \gamma^{+})$. Define*

$$\mathcal{F}^p_u = L^p(\mathbb{R} \times S^1, u^*TM). \tag{2.47}$$

We collect these fibres to form a vector bundle.

Definition 2.22 *Define the Banach bundle $\pi : \mathcal{E}^p \to \mathcal{B}^{1,p}(\gamma^{-}, \gamma^{+})$ by setting $\mathcal{E}^p_u = \mathcal{F}^p_u$.*

Hence, we consider the Banach bundle $\pi : \mathcal{E}^p \to \mathcal{B}^{1,p}(\gamma^-, \gamma^+)$, and $\overline{\partial}_{J,H}$ as a section of this bundle. At points $u \in \mathcal{B}^{1,p}(\gamma^-, \gamma^+)$ such that $\overline{\partial}_{J,H}(u) = 0$ the tangent space canonically splits into a vertical and horizontal part. The vertical part is given by the kernel of the projection, i.e. $\operatorname{Ker}(D\pi_u)$. This can be identified with $\mathcal{F}^p_u$. The horizontal part is defined by identifying $T_u\mathcal{B}^{1,p}(\gamma^-, \gamma^+)$ via the inclusion of the zero section $i : \mathcal{B}^{1,p}(\gamma^-, \gamma^+) \to \mathcal{E}^p$ with a subspace of $T_{u,0}\mathcal{E}^p$. Following the differential $D\overline{\partial}_{H,J}(u)$ by the projection onto the vertical part gives the vertical derivative still denoted by the same expression.

We will use the following general result.

Theorem 2.14 *Let $\pi : \mathcal{E} \to \mathcal{B}$ be a Banach manifold and $s \in \Gamma(\mathcal{E})$. Assume that $Ds(x)$ is a Fredholm operator that is surjective and admits a right inverse for every $x \in s^{-1}(0)$. Then $s^{-1}(0)$ admits the structure of a submanifold whose dimension around x is given by* $\operatorname{Ind} Ds(x)$.

We are interested in the following spaces of Hamiltonians and compatible almost complex structures.

Definition 2.23 *Let $(M, \omega) \in \mathsf{SympMan}^\infty$.*

$$\mathcal{HJ}_{reg}(M,\omega) = \{(H,J) \in C^\infty(M) \times \mathcal{J}(M,\omega) : \ \forall \gamma \in \mathrm{Crit}(\mathcal{A}_H) : \ \gamma \text{ is non-degenerate and } \forall \gamma^\pm, \forall u \in \mathcal{M}(\gamma^-,\gamma^+,H,J) : \ D\overline{\partial}_{H,J}(u) \text{ is surjective.}\}$$

Now let $H, H_0 \in C^\infty(M)$ and $J \in \mathcal{J}(M,\omega)$. We set

$$\mathcal{J}_{reg}(M,\omega,H) = \{J \in \mathcal{J}(M,\omega) : \ (H,J) \in \mathcal{HJ}_{reg}(M,\omega)\} \text{ and}$$
$$\mathcal{H}_{reg}(M,\omega,J) = \{H \in C^\infty(M,H_0) : \ (H,J) \in \mathcal{HJ}_{reg}(M,\omega)\}.$$

We want to apply the above result to our setting. We have already verified that $D\overline{\partial}_{J,H}$ is a Fredholm operator at every solution to the Floer equation and computed its index. It remains to show that it is also surjective at each such point. This is in fact not true for every choice of H and J. However, it is true for a choice that lies in a dense, open subspace of all possible choices.

We first deal with the almost complex structure. To show this we first consider the new bundle $\mathcal{E}^p \to \mathcal{B}^{1,p} \times \mathcal{J}^l$ where $\mathcal{J}^l$ is the C^l-completion of $\mathcal{J}(M,\omega)$. Then for a fixed H the operator $\overline{\partial}_{J,H}$ can also be considered as a section of this bundle. We denote this map by $\sigma_H(u,J) = \overline{\partial}_{H,J}(u)$ which is a map $\mathcal{B}^{1,p} \times \mathcal{J}^l \to \mathcal{E}^p$. We will show that this section is transverse to the zero section for each $u \in \mathcal{M}(\gamma^-,\gamma^+,H,J)$ and then use the following Sard-Smale theorem followed by projecting $\mathcal{M}(\gamma^-,\gamma^+,H,J) \subset \mathcal{B}^{1,p} \times \mathcal{J}^l \to \mathcal{J}^l$.

Theorem 2.15 (Sard-Smale) *Let $M, N \in \mathsf{Man}^\infty_{Ban}$ and $f \in C^l(M,N)$. Assume that M and N are separable and that f is a Fredholm map such that $\operatorname{Ind}(f) = k$. If $l \geq \max(\{k+1, 1\})$ then the set of regular values of f is residual in N.*

To verify the transversality of σ_H we need to consider its linearisation. If $(V,\omega) \in \mathsf{SympVect}$ and $J_t \in \mathcal{J}(V,\omega)$ for $t \in [0,1]$ the tangent space at such a family of complex structures is given by maps $Y \in C^l(\mathbb{R} \times TM)$ with the property that $J_t Y_t + Y_t J_t = 0$, $\omega(Y_t v, w) + \omega(v, Y_t w) = 0$ for all $t \in \mathbb{R}$ and $v, w \in T_pM$ for all $p \in M$ and that Y is 1-periodic in t.

To discuss transversality we need the differential of the section σ_H which is provided by the following proposition.

Proposition 2.27 *The differential of the map* $\sigma_H : \mathcal{B}^{1,p} \times \mathcal{J}^l \to \mathcal{E}^p$ *is given by*

$$
\begin{aligned}
&D\sigma_H(u,J) : T_u\mathcal{B}^{1,p} \times T_J\mathcal{J}^l \to \mathcal{F}^p_u \\
&D\sigma_H(u,J)(\xi,Y) = D\bar{\partial}_{J,H}(u)\xi + Y_t(\partial_t u - X_{H_t}(u))
\end{aligned}
$$

Here is then the main result concerning transversality with respect to the almost complex structure.

Proposition 2.28 *For every* (u,J) *with* $J \in \mathcal{J}^l$ *and* $u \in \mathcal{M}(\gamma^-,\gamma^+,H,J)$ *the operator* $D\sigma_H(u,J)$ *is surjective.*

PROOF:
⟨1⟩1. LET: $q \in (1,\infty)$ such that $1/q + 1/p = 1$.
⟨1⟩2. SUFFICES: If $\eta \in L^q(u^*TM)$ such that

$$\forall (\xi,Y) \in T_u\mathcal{B}^{1,p} \times T_J\mathcal{J}^l : \langle \eta, D\sigma_H(u,J)(\xi,Y)\rangle = 0 \quad (2.48)$$

then $\eta = 0$.
⟨1⟩3. LET: $\eta \in L^q(u^*TM)$ such that Equation 2.48 holds.
⟨1⟩4. $\forall \xi \in T_u\mathcal{B}^{1,p} : \int_{-\infty}^{\infty}\int_0^1 \langle \eta, D\bar{\partial}_{H,J}(u)\xi\rangle dtds = 0$.
  PROOF:
  ⟨2⟩1. Choose $Y = 0$ in Equation 2.48 and use the formula for $D\sigma_H$.
⟨1⟩5. LET: $(D\bar{\partial}_{H,J}(u))^*$ be the formal adjoint of $D\bar{\partial}_{H,J}(u)$.
⟨1⟩6. $\eta \in C^l(u^*TM)$ and η is a strong solution of $(D\bar{\partial}_{H,J}(u))^*\eta = 0$.
  PROOF:
  ⟨2⟩1. η is a weak solution of $(D\bar{\partial}_{H,J}(u))^*\eta = 0$ by ⟨1⟩4.
  ⟨2⟩2. Apply elliptic regularity results for the operator $(D\bar{\partial}_{H,J}(u))^*$.
⟨1⟩7. If $(s,t) \in R(u)$ then $\eta(s,t) = 0$.
  PROOF:
  ⟨2⟩1. $\int_{-\infty}^{\infty}\int_0^1 \langle \eta, Y_t(u)\partial_t u\rangle dtds = 0$.
  ⟨2⟩2. ASSUME: $\exists (s_0,t_0) \in R(u) : \eta(s_0,t_0) \neq 0$.
  ⟨2⟩3. $\exists A \in \mathsf{End}(T_{u(s_0,t_0)}M, J_{t_0}, \omega) : \langle \eta(s_0,t_0), A\partial_s u(s_0,t_0)\rangle > 0$.
⟨1⟩8. $\eta = 0$.
  PROOF:
  ⟨2⟩1. SUFFICES: $\exists U \in \mathsf{Open}(\mathbb{R}^2) : \eta|_U = 0$.
    PROOF:
    ⟨3⟩1. Because of ⟨1⟩6 the continuation result from [] applies to η.
  ⟨2⟩2. DEFINE: $U = R(u)$
    which is open.

$\langle 2\rangle 3.$ Apply $\langle 1\rangle 7$.
$\langle 1\rangle 9.$ Q.E.D.
Proof:
$\langle 2\rangle 1.$ We have shown that $D\sigma_H(u, J)$ has dense range.
$\langle 2\rangle 2.$ $D\sigma_H(u, J)$ is Fredholm.

Combining the above with the Sard-Smale theorem we obtain the generic transversality of the Floer operator by way of perturbing the almost complex structure.

Theorem 2.16 *Let $H \in C^\infty(M)$. Assume that every $\gamma \in \mathrm{Crit}(\mathcal{A}_H)$ is non-degenerate. Then $\mathcal{J}_{reg}(M, \omega, H)$ is of second category in $\mathcal{J}(M, \omega)$.*

Proof:
$\langle 1\rangle 1.$ $\mathcal{J}^l_{reg}(\gamma^-, \gamma^+) \subset \mathcal{J}^l$ is of second category.
Proof:
$\langle 2\rangle 1.$ $\mathcal{M}(\gamma^-, \gamma^+, \mathcal{J}^l)$ is a Banach manifold.
$\langle 2\rangle 2.$ The projection $\mathcal{M}(\gamma^-, \gamma^+, \mathcal{J}^l) \to \mathcal{J}^l$ is Fredholm.
$\langle 2\rangle 3.$ Now apply the Sard-Smale theorem.
$\langle 1\rangle 2.$ Let: $\mathcal{J}_{reg,K} = \{J \in \mathcal{J} : D\bar{\partial}_{H,J}$ is surjective if $u \in \mathcal{M}(\gamma^-, \gamma^+, J)$ satisfies $|\partial_s u| \leq K\}$ for $K \in \mathbb{R}$.
$\langle 1\rangle 3.$ $\mathcal{J}_{reg,K}$ is open in $\mathcal{J}$ with respect to the C^∞-topology.
Proof:
$\langle 2\rangle 1.$ Suffices: $\mathcal{J} \setminus \mathcal{J}_{reg,K}$ is closed.
$\langle 2\rangle 2.$ Let: $(J_n)_{n\in\mathbb{N}} \subset \mathcal{J} \setminus \mathcal{J}_{reg,K}$ such that $C^\infty - \lim_{n\to\infty} J_n = J \in \mathcal{J}$.
$\langle 2\rangle 3.$ Let: $u_n \in \mathcal{M}(\gamma^-, \gamma^+, H, J_n)$ for $n \in \mathbb{N}$.
$\langle 2\rangle 4.$ Suffices Assume: $(u_n)_{n\in\mathbb{N}}$ converge weakly to a broken trajectory $(v_1, \ldots, v_k)$.
$\langle 2\rangle 5.$ This would imply that $D\sigma_H(u_n, J_n)$ is surjective for sufficiently large n which contradicts the assumption.
$\langle 1\rangle 4.$ $\mathcal{J}_{reg,K}$ is dense in $\mathcal{J}$ with respect to the C^∞-topology.
Proof:
$\langle 2\rangle 1.$ $\mathcal{J}^l_{reg,K}$ is dense in $\mathcal{J}^l$ with respect to the C^l-topology.
$\langle 2\rangle 2.$ $\mathcal{J}_{reg,K}$ is dense in $\mathcal{J}^l$ with respect to the C^l-topology.
Proof:
$\langle 3\rangle 1.$ $\mathcal{J}_{reg,K} = \mathcal{J}^l_{reg,K} \cap \mathcal{J}$.
$\langle 2\rangle 3.$ $\mathcal{J}_{reg,K}$ is dense in $\mathcal{J}$ with respect to the C^l-topology for every l.
$\langle 1\rangle 5.$ Q.E.D.
Proof:
$\langle 2\rangle 1.$ $\mathcal{J}_{reg} = \bigcap_{K>0} \mathcal{J}_{reg,K}$.

Next we deal with the Hamiltonian. The approach is essentially the same as for the almost complex structure. Fix $H_0 \in C^\infty(M)$ such that all 1-periodic orbits of its flow are non-degenerate and let $\mathcal{H}^l = C^l(M, H_0)$ be the set $C^l(M)$-functions that agree to second order with H_0.. We introduce the bundle $\mathcal{E}^p \to \mathcal{B}^{1,p} \times \mathcal{H}^l$ and consider the section $\sigma_J : \mathcal{B}^{1,p} \times \mathcal{H}^l \to \mathcal{E}^p$ given by

$$\sigma_J(u,H) = \bar{\partial}_{H,J}(u).$$

Theorem 2.17 *Let $J \in \mathcal{J}(M,\omega)$ and $H_0 \in C^\infty(M)$. Assume that every $\gamma \in \mathrm{Crit}(\mathcal{A}_{H_0})$ is non-degenerate. Then $\mathcal{H}_{reg}(M,\omega,H_0,J)$ is of second category in $C^\infty(M,H_0)$.*

PROOF:
⟨1⟩1. $D\sigma_J(u,H)(\xi,h) = D\bar{\partial}_{H,J}(u)\xi - \operatorname{grad} h_t(u)$.
⟨1⟩2. SUFFICES: If $\eta \in L^q(u^*TM)$ such that
$$\forall(\xi,h) \in T_u\mathcal{B}^{1,p} \times \mathcal{H}^l : \langle \eta, D\sigma_H(u,H)(\xi,h)\rangle = 0 \qquad (2.49)$$
then $\eta = 0$.
⟨1⟩3. $\forall(s,t) \in \mathbb{R}^2 : \{\eta(s,t), \partial_s\eta(s,t)\}$ is linearly independent.
PROOF:
⟨2⟩1. ASSUME: $\exists(s_0,t_0) \in \mathbb{R}^2 : \{\eta(s,t), \partial_s\eta(s,t)\}$ is linearly dependent.
⟨2⟩2. SUFFICES ASSUME: $t_0 \in (0,1)$ and $(s_0,t_0) \in R(u)$.
⟨2⟩3. $\exists U_0 \in \mathrm{Open}([0,1] \times M) : (t_0, u(s_0,t_0)) \in U_0$ and $V_0 = \{(s,t) \in \mathbb{R}^2 : (t,u(s,t)) \in U_0\}$ is a neighbourhood of (s_0,t_0).
⟨2⟩4. For sufficiently small $\varepsilon > 0$ and t close to t_0 there is an embedding $B_\varepsilon(s_0,0) \to U_0$ given by $\exp_{u(s,t)}(r\eta(s,t))$.
⟨2⟩5. $\exists h \in C^\infty(\mathbb{R} \times M) : \operatorname{supp} h \subset U_0$ and $h_t(g_t(r,s)) = r\beta(r)\beta(s - s_0)\beta(t-t_0)$ for a suitable cutoff function $\beta : \mathbb{R} \to [0,1]$.
⟨2⟩6. This leads to a function for which Equation 2.49 does not hold.
⟨1⟩4. ASSUME: $\eta \in L^q(u^*TM)$ satisfies Equation 2.49 and is of class C^l.
⟨1⟩5. $\exists!\lambda \in C^l(\mathbb{R}^2\backslash C(u), \mathbb{R}) : \forall(s,t) \in \mathbb{R}^2\backslash C(u) : \eta(s,t) = \lambda(s,t)\partial_s u(s,t)$.
⟨1⟩6. $\forall(s,t) \in \mathbb{R}^2\backslash C(u) : \partial_s\lambda(s,t) = 0$.
PROOF:
⟨2⟩1. ASSUME: $\exists(s_0,t_0) \in \mathbb{R}^2\backslash C(u) : \partial_s\lambda(s,t) \neq 0$.
⟨2⟩2. SUFFICES ASSUME: $(s_0,t_0) \in R(u)$ since $R(u)$ is dense.
⟨2⟩3. LET: $V_0 \in \mathrm{Open}(\mathbb{R}^2)$ such that $(s_0,t_0) \in V_0$ and $\rho \in C^\infty(\mathbb{R}^2,[0,1])$ such that $\mathrm{supp}(\rho) \subset V_0$ and $\int_{V_0} \lambda\partial_s\rho \neq 0$.
⟨2⟩4. This leads to a function h for which Equation 2.49 is not fulfilled.
⟨1⟩7. $\lambda = \lambda(t)$ is s-independent and extends to $\mathbb{R}^2$.
⟨1⟩8. $\eta = 0$.
⟨1⟩9. ASSUME: $\eta \neq 0$.
⟨1⟩10. SUFFICES ASSUME: $\forall t \in \mathbb{R} : \lambda(t) > 0$.
PROOF:
⟨2⟩1. $\{(s,t) \in \mathbb{R}^2 : \eta(s,t) \neq 0\}$ is discrete.
⟨1⟩11. $\frac{d}{ds}\int_0^1 \langle \eta, \partial_s u\rangle = 0$.
PROOF:
⟨2⟩1. $\frac{d}{ds}\int_0^1 \langle \eta, \partial_s u\rangle = \int_0^1 (\langle \eta, \nabla_s\partial_s u\rangle + \langle \nabla_s\eta, \partial_s u\rangle)ds = \int_0^1 (\langle \eta, D\bar{\partial}_{H,J}(u)\partial_s u\rangle + \langle D\bar{\partial}^*_{H,J}(u)\eta, \partial_s u\rangle)ds = 0$.

⟨1⟩12. $\int_0^1 \langle \eta, \partial_s u \rangle = \int_0^1 \lambda(t) |\partial_S u(s,t)|^2 dt > 0$.
⟨1⟩13. Q.E.D.
Proof:
⟨2⟩1. ⟨1⟩11 and ⟨1⟩12 would imply that $\int_{-\infty}^{\infty} \int_0^1 \langle \eta, \partial_s u \rangle = \infty$ which is a contradiction.
⟨2⟩2. Finally we can boost the result from a C^l-result to a C^∞-result as in the above proof.

2.7.2 Compactness

In this section we discuss compactness properties of the solution spaces to the Floer equation. This is needed for two reasons. To define the boundary operator in the chain complex used in the definition of the Floer homology groups we want to count the number of solutions connecting two generating critical loops whose indices differ by 1. Secondly, to verify the chain complex condition $\partial^2 = 0$ one also needs to study the solution spaces connecting two loops whose indices differ by 2.

We begin by proving a technical result about finite energy solutions to the Floer equation.

Lemma 2.7 *There exists a constant* $C > 0$ *such that*

$$\forall u \in \mathcal{M}^b(H,J), \forall (s,t) \in \mathbb{R} \times S^1 : \; \|\mathrm{grad}\, u(s,t)\| \leq C. \tag{2.50}$$

Proof:
⟨1⟩1. Assume: There exists no $C > 0$ satisfying Equation 2.50.
⟨1⟩2. $\exists (s_k, t_k)_{k \in \mathbb{N}} \subset \mathbb{R} \times S^1 : \; \lim_{k \to \infty} \|\mathrm{grad}\, u_k(s_k, t_k)\| = \infty$.
⟨1⟩3. $\exists (\varepsilon_k)_{k \in \mathbb{N}} \subset \mathbb{R}_{>0} : \; \lim_{k \to \infty} \varepsilon_k = 0$ and $\lim_{k \to \infty} \varepsilon_k \|\mathrm{grad}\, u_k(s_k, t_k)\| = \infty$.
⟨1⟩4. $\exists (s'_k, t'_k)_{k \in \mathbb{N}} \subset \mathbb{R} \times S^1, \exists (\varepsilon'_k)_{k \in \mathbb{N}} \subset \mathbb{R}_{>0}$:
$$\lim_{k \to \infty} \varepsilon'_k \|\mathrm{grad}\, u_k(s'_k, t'_k)\| = \infty$$
$$\forall (s,t) \in B((s'_k, t'_k), \varepsilon'_k) : \; 2\|\mathrm{grad}\, u_k(s'_k, t'_k)\| \geq \|\mathrm{grad}\, u_k(s,t)\|$$
⟨1⟩5. Let: $R_K = \|\mathrm{grad}\, u_k(s'_k, t'_k)\|$ and $v_k(s,t) = u_k\left(\frac{(s,t)}{R_k} + (s'_k, t'_k)\right)$.
⟨1⟩6. $\mathrm{grad}\, v_k(s,t) = \frac{1}{R_k} \mathrm{grad}\, u_k(\frac{(s,t)}{R_k} + (s'_k, t'_k))$.
⟨1⟩7. $\|\mathrm{grad}\, v_k(0,0)\| = 1$.
⟨1⟩8. $\forall (s,t) \in B(0, \varepsilon'_k R_k) : \; \|\mathrm{grad}\, v_k(s,t)\| \leq 2$.
⟨1⟩9. The v_k satisfy
$$\partial_s v_k + J(v_k)\partial_t v_k + \frac{1}{R_k} \mathrm{grad}\, H_{t_k + t/R_k}(v_k) = 0.$$
Proof:
⟨2⟩1. The u_k satisfy $\bar{\partial}_{J,H} u_k = 0$.
⟨1⟩10. There exists a subsequence $(v_{K_l})_{l \in \mathbb{N}}$ of $(v_K)_{k \in \mathbb{N}}$ that converges in the $C^\infty_{loc}(\mathbb{R} \times S^1, M)$-topology to a function $v \in C^\infty(\mathbb{R} \times S^1, M)$.
Proof:

⟨2⟩1. By elliptic regularity Theorem 2.7.

⟨1⟩11. v has the following properties:

1. $\|\operatorname{grad} v(0,0)\| = 1$.
2. $\|\operatorname{grad} v(s,t)\| \leq 2$.
3. $\partial_s v + J(v)\partial_t v = 0$.

⟨1⟩12. $E(v) < \infty$.

PROOF:

⟨2⟩1. LET: $B_k = B((s'_k, t'_k), \varepsilon'_k)$.

⟨2⟩2. $\int_{B_k} \|\operatorname{grad} v_{k_l}\|^2 \leq 3 \sup_{u \in \mathcal{M}^b(J,H)} E(u) + 2\int_{B_k} \|X_{H_t}\|^2 dtds$.

PROOF:

⟨3⟩1.

$$\int_{B(0,\varepsilon'_{k_l} R_{k_l})} \|\operatorname{grad} v_{k_l}\|^2 = \int_{B_{k_l}} \|\operatorname{grad} u_{k_l}\|^2 dtds = \int_{B_{k_l}} \left(\|\partial_s u_{k_l}\|^2 + \|\partial_t u_{k_l}\|^2\right) dtds$$
$$\leq \int_{B_{k_l}} (\|\partial_s u_{k_l}\|^2 + \|\partial_t u_{k_l} - X_{H_t}(u_{k_l})\|^2$$
$$+ \|X_{H_t}(u_{k_l})\|^2 + 2\|\partial_t u_{k_l} - X_{H_t}(u_{k_l})\|\|X_{H_t}(u_{k_l})\|)dtds$$
$$\leq \int_{B_{k_l}} \left(\|\partial_s u_{k_l}\|^2 + 2\|\partial_t u_{k_l} - X_{H_t}(u_{k_l})\|^2 + 2\|X_{H_t}(u_{k_l})\|\right) dtds$$
$$\leq 3E(u_{k_l}) + 2\int_{B_{k_l}} \|X_{H_t}\|^2 dtds$$
$$\leq 3 \sup_{u \in \mathcal{M}^b(J,H)} E(u) + 2\int_{B_{k_l}} \|X_{H_t}\|^2 dtds.$$

⟨2⟩3. $\exists C > 0 : \sup_{u \in \mathcal{M}^b(J,H)} E(u) < C$.

PROOF:

⟨3⟩1. $\forall u \in \mathcal{M}^b(J,H) : \exists \gamma^\pm : \operatorname{Crit}(\mathcal{A}_H) : E(u) = \mathcal{A}_H(\gamma^-) - \mathcal{A}_H(\gamma^+)$ since $E(u) < \infty$.

⟨3⟩2. $|\operatorname{Crit}(\mathcal{A}_H)| < \infty$.

⟨2⟩4. $\lim_{l\to\infty} \int_{B_{k_l}} \|X_{H_t}\|^2 dtds = 0$.

⟨1⟩13. $0 < \int_{\mathbb{R}^2} v^*\omega < \infty$.

PROOF:

⟨2⟩1.

$$\int_{\mathbb{R}^2} v^*\omega = \int_{\mathbb{R}^2} \omega \circ Dv = \int_{\mathbb{R}^2} \omega(\partial_s v, \partial_t v) dsdt$$
$$= \int_{\mathbb{R}^2} \omega(-J(v)\partial_t v, \partial_t v) dsdt$$
$$= \int_{\mathbb{R}^2} \omega(\partial_t v, J(v)\partial_t v) dsdt$$
$$= \int_{\mathbb{R}^2} \|\partial_t v\|^2 dsdt < \infty$$

⟨1⟩14. $\exists (r_k)_{k\in\mathbb{N}} \subset \mathbb{R}_{>0} : \lim_{k\to\infty} r_k = \infty$ and $\lim_{k\to\infty} l(v(\partial B_{r_k}(0))) = 0$ where l denotes the length of the boundary.

PROOF:

⟨2⟩1. Omitted.
⟨1⟩15. $\exists N \in \mathbb{N}: \; k > N: \; v(\partial B_{r_k}(0))$ is contained in a Darboux chart U.
⟨1⟩16. $\omega|_U$ is closed.
⟨1⟩17. $\exists \lambda \in \Omega^1(M): \; d\lambda|_U = \omega|_U$ since $U \cong \mathbb{R}^{2n}$.
⟨1⟩18. There is a ball $D_r \subset U$ whose boundary is $v(\partial B_r(0))$ and such that $D_r \cup v(B_r) =: S_r^2 \cong S^2$.
⟨1⟩19. $\int_{S_r^2} \omega = 0$ by assumption on M.
⟨1⟩20. Q.E.D.
Proof:
⟨2⟩1. $|\int_{D_r} \omega| \leq |\int_{v(\partial B_r(0))} \lambda| \leq l(v(\partial B_r(0))) \sup_U \|\lambda\|$.
⟨2⟩2. $\lim_{r\to\infty} l(v(\partial B_r(0))) = 0$ by ⟨1⟩14.
⟨2⟩3. $\lim_{r\to\infty} \int_{v(B_r(0))} \omega = \int_{\mathbb{R}^2} \omega > 0$.
⟨2⟩4. This contradicts ⟨1⟩19.

With this bound we can now easily prove the compactness result by first obtaining a continuous candidate for the limit of a subsequence by the Arzelà-Ascoli theorem. Then we use elliptic regularity to boost it and the convergence to C^∞.

Theorem 2.18 *The space $\mathcal{M}^b(H,J) = \{u \in \mathcal{M}(H,J): \; E(u) < \infty\}$ is compact in the $C^\infty_{loc}(\mathbb{R} \times S^1, M)$ topology assuming that $\omega|_{\pi_2(M)} = 0$.*

Proof:
⟨1⟩1. Let: $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}(H,J)$ be a sequence.
⟨1⟩2. $\exists (u_{n_k})_{k\in\mathbb{N}} \subset (u_n)_{n\in\mathbb{N}}: \; \exists u_0 \in \mathcal{M}(H,J): \; C^0_{loc} - \lim_{k\to\infty} u_{n_k} = u_0$.
Proof:
⟨2⟩1. Let: $K_m = [-m,m] \times S^1$ for $m \in \mathbb{N}_0$.
⟨2⟩2. Suffices: $\exists (u_{n_k})_{k\in\mathbb{N}} \subset (u_n)_{n\in\mathbb{N}}: \; \exists u_0 \in \mathcal{M}(H,J): \; \forall m \in \mathbb{N}: \; C^0(K_m) - \lim_{k\to\infty} u_{n_k} = u_0$.
⟨2⟩3. $\exists (u^{(0)}_{n_k})_{k\in\mathbb{N}} \subset (u_n)_{n\in\mathbb{N}}: \; \exists u_0 \in \mathcal{M}(H,J): \; C^0(K_0) - \lim_{k\to\infty} u^{(0)}_{n_k} = u$.
Proof:
⟨3⟩1. By the Arzelà-Ascoli theorem since (u_n) is totally bounded and equicontinuous.
⟨2⟩4. $\forall m \in \mathbb{N}: \; \exists (u^{(m)}_{n_k})_{k\in\mathbb{N}} \subset (u^{(m-1)}_{n_k})_{k\in\mathbb{N}}: \; \exists u_0^m \in \mathcal{M}(H,J): \; C^0(K_m) - \lim_{k\to\infty} u_{n_k} = u_0^{(m)}$.
⟨2⟩5. $\forall m \in \mathbb{N}: \; u_0^{(m)} = u_0$.
⟨2⟩6. Let: $u_{n_k} = u^{(k)}_{n_k}$.
⟨1⟩3. $u_0 \in C^\infty(\mathbb{R} \times S^1, M)$.
Proof:
⟨2⟩1. By Theorem 2.6.
⟨1⟩4. $C^\infty_{loc} - \lim_{k\to\infty} u_{n_k} = u_0$.
Proof:
⟨2⟩1. By Theorem 2.6.

⟨1⟩5. Q.E.D.

The compactness of the space $\mathcal{M}^b(H,J)$ implies in particular that trajectories with finite energy tend to critical points.

Proposition 2.29 *Assume that H is chosen so that all critical points are non-degenerate. Let $u \in \mathcal{M}^b(H,J)$. Then there exist critical points $\gamma^\pm \in \mathrm{Crit}(\mathcal{A}_H)$ such that*

$$\lim_{s\to\pm\infty} u = \gamma^\pm.$$

PROOF:
⟨1⟩1. SUFFICES ASSUME: $s \to \infty$.
⟨1⟩2. $\exists\varepsilon > 0 : \bigcap_{\gamma\in\mathrm{Crit}(\mathcal{A}_H)} B_\varepsilon(\gamma) = \emptyset$.
⟨1⟩3. LET: $U_\varepsilon = \bigcup_{\gamma\in\mathrm{Crit}(\mathcal{A}_H)} B_\varepsilon(\gamma)$.
⟨1⟩4. $\exists s_\varepsilon \in \mathbb{R} : u([s_\varepsilon,\infty) \times S^1) \subset U_\varepsilon$.
  PROOF:
  ⟨2⟩1. ASSUME: $\exists\varepsilon_0 > 0 : \exists(s_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ with $\lim_{k\to\infty} s_k = \infty : \forall k \in \mathbb{N} : u_{s_k} \notin U_{\varepsilon_0}$.
  ⟨2⟩2. $\exists(s_{k'})_{k'\in\mathbb{N}} \subset (s_k)_{k\in\mathbb{N}}$ with $\lim_{k'\to\infty} s_{k'} = \infty : \exists\gamma' \in \mathrm{Crit}(\mathcal{A}_H) : \lim_{k'\to\infty} u_{s_{k'}} = \gamma'$.
    PROOF:
    ⟨3⟩1. By Proposition 2.5.
  ⟨2⟩3. This is a contradiction.
⟨1⟩5. $\exists\gamma^+ \in \mathrm{Crit}(\mathcal{A}_H) : u([s_\varepsilon,\infty) \times S^1) \subset B_\varepsilon(\gamma^+)$.
⟨1⟩6. $\lim_{s\to\infty} u_s = \gamma^+$.
⟨1⟩7. Q.E.D.

To define the boundary operator of the Floer complex we want to count the number of flow lines connecting two critical points whose indices differ by 1. However, each solution to the Floer equation admits a free $\mathbb{R}$-action by translating the s-parameter. More precisely, if $u \in \mathcal{M}(H,J)$ we define for each $r \in \mathbb{R}$ the group action $\mathrm{sh}(r)u$ by setting

$$\mathrm{sh}(r)u(s,t) = u(s+r,t).$$

Thus, to have any hope of obtaining a finite number we need to consider the quotient of the solution spaces under this action.

Definition 2.24 *Let $\gamma^\pm \in \mathrm{Crit}(\mathcal{A}_H)$. We define the space of connecting, unparameterised flow lines to be*

$$\underline{\mathcal{M}}(\gamma^-,\gamma^+,H,J) = \mathcal{M}(\gamma^-,\gamma^+,H,J)/\,\mathrm{sh}\,.$$

First we should note that the space $\underline{\mathcal{M}}(\gamma^-,\gamma^+,H,J)$ endowed with the quotient topology remains Hausdorff. Unpacking the definition of Hausdorff for the quotient we find that we have to show the following. If $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}(\gamma^-,\gamma^+,H,J)$ is a sequence such that there are two sequences of real numbers $(r_n^1)_{n\in\mathbb{N}}$ and $(r^2_{n\in\mathbb{N}})$ such that both $\mathrm{sh}(r_n^1)u_n$ and $\mathrm{sh}(r_n^2)u_n$ converge then their limits coincide in the quotient. This is verified in the following proposition.

Proposition 2.30 *Let $\gamma^-,\gamma^+ \in \mathrm{Crit}(\mathcal{A}_H)$ such that $\gamma^- \neq \gamma^+$. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}(\gamma^-,\gamma^+,H,J)$ such that the following holds.*

1. $\exists (r_n^1)_{n\in\mathbb{N}}, (r_n^2)_{n\in\mathbb{N}} \subset \mathbb{R}:$
2. $\exists \gamma^1,\gamma^2 \in \mathrm{Crit}(\mathcal{A}_H)\backslash\{\gamma^-\} : \exists u^1 \in \mathcal{M}(\gamma^-,\gamma^1,H,J), u^2 \in \mathcal{M}(\gamma^-,\gamma^2,H,J):$
3. $\lim_{n\to\infty}\mathrm{sh}(r_n^1)u_n = u^1$ *and* $\lim_{n\to\infty}\mathrm{sh}(r_n^2)u_n = u^2$.

Then $\gamma^1 = \gamma^2$ and there exists a $r \in \mathbb{R}$ such that $u^1 = \mathrm{sh}(r)u^2$.

Proof:
⟨1⟩1. $\mathcal{A}_H(\gamma^-) > \mathcal{A}_H(\gamma^+), \mathcal{A}_H(\gamma^1), \mathcal{A}_H(\gamma^2)$.
⟨1⟩2. $\exists \alpha \in \mathbb{R}: \mathcal{A}_H(\gamma^-) > \alpha > \max(\mathcal{A}_H(\gamma^+), \mathcal{A}_H(\gamma^1), \mathcal{A}_H(\gamma^2))$.
⟨1⟩3. Let: $\varepsilon > 0$ such that $\mathcal{A}_H(\gamma^-) - \varepsilon > \alpha$.
⟨1⟩4. The sequence $(r_n^1 - r_n^2)_{n\in\mathbb{N}}$ is bounded.
  Proof:
  ⟨2⟩1. Assume: $(r_n^1 - r_n^2)_{n\in\mathbb{N}}$ is unbounded.
  ⟨2⟩2. Suffices Assume: $\forall n \in \mathbb{N}: r_n^1 = 0$ and $\lim_{n\to\infty} r_n^2 = \infty$.
  ⟨2⟩3. $\exists r' \in \mathbb{R}: \forall s \le r': \mathcal{A}_H(u^2(s,\cdot)) > \mathcal{A}_H(\gamma^-) - \varepsilon$.
  ⟨2⟩4. $\exists N_1 \in \mathbb{N}: \forall n > N_1: \mathcal{A}_H(u_n(r' + r_n^2,\cdot)) > \mathcal{A}_H(\gamma^-) - \varepsilon$.
  ⟨2⟩5. $\exists r'' \in \mathbb{R}: \forall s \ge r'': \mathcal{A}_H(u(s,\cdot)) < \alpha$.
  ⟨2⟩6. $\exists N_2 \in \mathbb{N}: \forall n > N_2: \mathcal{A}_H(u_n(r'',\cdot)) \le \alpha$.
  ⟨2⟩7. $r' + r_n^2 < r''$ which is a contradiction because of ⟨2⟩4.
⟨1⟩5. $(r_n^1 - r_n^2)$ admits a converging subsequence.
⟨1⟩6. Suffices Assume: $\lim_{n\to\infty}(r_n^1 - r_n^2) = r_*$.
⟨1⟩7. Q.E.D.
  Proof:
  ⟨2⟩1. $\lim_{n\to\infty}\mathrm{sh}(r_n^2)u_n = u^2$.
  ⟨2⟩2. $\lim_{n\to\infty}\mathrm{sh}(r_n^2)u_n = \mathrm{sh}(r_*)u^1$.

We will now study the compactness properties of the space $\underline{\mathcal{M}}(\gamma^-,\gamma^+,H,J)$. Here, the difference of the indices of γ^- and γ^+ will play a crucial role. If $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) = 1$ this space is indeed compact. However, if this difference is larger than 1 this will not be the case anymore. What this means is that if we take a sequence $([u_n])_{n\in\mathbb{N}} \subset \underline{\mathcal{M}}(\gamma^-,\gamma^+,H,J)$ then it may not admit a subsequence that converges to an element in $\underline{\mathcal{M}}(\gamma^-,\gamma^+,H,J)$. Thus, we are led to compactify $\underline{\mathcal{M}}(\gamma^-,\gamma^+,H,J)$ by adding enough points so that every sequence does have a subsequence that converges to a point in this larger space. These additional points have a very direct geometric interpretation. Namely, they come from so-called *broken trajectories*. We begin their discussion with the following definition.

Definition 2.25 *Let* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}(\gamma^-, \gamma^+, H, J)$ *be a sequence. We say that* (u_n) converges up to l-fold breaking *if the following holds.*

1. $\exists \gamma^- = \gamma_0, \gamma_1, \ldots, \gamma_l = \gamma^+ \in \mathrm{Crit}(H)$:
2. $\exists u^i \in \mathcal{M}(\gamma^{i-1}, \gamma^i, H, J)$ *for* $i \in \{1, \ldots, l\}$:
3. $\exists (r_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}$ *for* $i \in \{1, \ldots, l\}$:
4. $\forall i \in \{1, \ldots, l\}$: $\lim_{n\to\infty} \mathrm{sh}(r_n^i) u_n = u^i$.

Adding these broken trajectories makes the space $\mathcal{M}(\gamma^-, \gamma^+, H, J)$ compact.

Theorem 2.19 *Let* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}(\gamma^-, \gamma^+, H, J)$ *be a sequence. Then* $(u_n)_{n\in\mathbb{N}}$ *has a subsequence that converges up to l-fold breaking.*

⟨1⟩1. LET: $\varepsilon > 0 : \cap_{\gamma\in\mathrm{Crit}(\mathcal{A}_H)} B_\varepsilon(\gamma) = \emptyset$.
⟨1⟩2. LET: $r_n^1 = \inf\{r \in \mathbb{R} : \forall t \in S^1 : d(u_n(r,t), \gamma^-(t)) > \varepsilon)$.
⟨1⟩3. SUFFICES ASSUME: $\exists u^1 \in \mathcal{M}^b(H, J) : C^\infty_{loc} - \lim_{n\to\infty} \mathrm{sh}(r_n^1) u_n = u^1$.
  ⟨2⟩1. $\mathcal{M}^b(H, J)$ is compact in C^∞_{loc}.
  ⟨2⟩2. $\mathrm{sh}(r_n^1) u_n$ has a converging subsequence which we can after relabelling assume to be $\mathrm{sh}(r_n^1) u_n$ itself.
⟨1⟩4. $\forall t \in S^1 : \forall s < 0 : u^1(s,t) \in \overline{B_\varepsilon}(\gamma^-)$ and $\forall t \in S^1 : u^1(0,t) \in \partial B_\varepsilon(\gamma^-)$.
⟨1⟩5. $\exists \gamma \in \mathrm{Crit}(\mathcal{A}_H) \backslash \{\gamma^-\} : u^1 \in \mathcal{M}(\gamma^-, \gamma, H, J)$.
⟨1⟩6. CASE: $\gamma = \gamma^+$.
  PROOF:
  ⟨2⟩1. The proof is finished with $l = 0$.
⟨1⟩7. CASE: $\gamma \neq \gamma^+$.
  PROOF:
  ⟨2⟩1. ASSUME: 1. $\exists (r_n^0)_{n\in\mathbb{N}}, \ldots, (r_n^k)_{n\in\mathbb{N}} \subset \mathbb{R}$:
    2. $\exists \gamma^i \in \mathrm{Crit}(\mathcal{A}_H)$ for $i \in \{1, \ldots, k\}$ with $\gamma^+ \neq \gamma^i$:
    3. $\exists u^i \in \mathcal{M}(\gamma^{i-1}, \gamma^i, H, J)$:
    4. $\forall i \in \{1, \ldots, k\} : \lim_{n\to\infty} \mathrm{sh}(r_n^i) u_n = u^i$.
  ⟨2⟩2. $\exists s^* \in \mathbb{R} : \forall s \geq s^* : u^k(r, \cdot) \in B_\varepsilon(\gamma^k)$.
  ⟨2⟩3. $\exists N \in \mathbb{N} : \forall n \geq N : u_n(r_n^k + s^*, \cdot) \in B_\varepsilon(\gamma^k)$.
  ⟨2⟩4. LET: $r_n^{k+1} = \sup\{r \geq r_n^k + s^* : \forall s' \in [r_n^K + r^*, s], : (u_n)_{s'} \in B_\varepsilon(\gamma^k)\}$
  ⟨2⟩5. SUFFICES ASSUME: $\exists u^{k+1} \in \mathcal{M}^b(H, J) : C^\infty_{loc} - \lim_{n\to\infty} \mathrm{sh}(r_n^{k+1}) u_n = u^{k+1}$.
  ⟨2⟩6. $\exists \gamma^{k+1} \in \mathrm{Crit}(\mathcal{A}_H) \backslash \{\gamma^k\} : u^{k+1} \in \mathcal{M}(\gamma^k, \gamma^{k+1}, H, J)$.
    PROOF:
    ⟨3⟩1. $\lim_{n\to\infty} r_n^{k+1} - r_n^k = \infty$.
      PROOF:
      ⟨4⟩1. ASSUME: $\sup_{n\in\mathbb{N}} |r_n^{k+1} - r_n^k| < \infty$.
      ⟨4⟩2. $\exists M \in \mathbb{R} : \forall n \in \mathbb{N} : [s^*, r_n^{k+1} - r_n^k] \subset [-M, M]$ by ⟨2⟩4.

⟨4⟩3. $C^0([s^*, r_n^{k+1} - r_n^k]) - \lim_{n\to\infty} \mathrm{sh}(r_n^k)u_n = u^k$.
⟨4⟩4. $\forall s \in [s^*, r_n^{k+1} - r_n^k] : u_n(r_n^{k+1} + s, \cdot) \in B_\varepsilon(\gamma^k)$ which is a contradiction since $u_n(r_n^{k+1}) \in \partial B_\varepsilon(\gamma^k$.
⟨3⟩2. Let: $r \in \mathbb{R}_{<0}$.
⟨3⟩3. $\exists N \in \mathbb{N} : \forall n > N : r_n^k + s^* < r_n^{k+1} + r < r_n^{k+1}$.
⟨3⟩4. $\forall n > N : \forall s \in \mathbb{R}_{<0} : (\mathrm{sh}(r_n^{k+1})u_n)_s \in B_\varepsilon(\gamma^k)$.
⟨3⟩5. $u^{k+1}((-\infty, 0), \cdot) \subset \overline{B_\varepsilon(\gamma^k)}$ by letting $n \to \infty$.
⟨3⟩6. $\forall n \in \mathbb{N} : u_n(r_n^{k+1}) \in \partial B_\varepsilon(\gamma^k)$.
⟨3⟩7. $u^{k+1}(0, \cdot) \in \partial B_\varepsilon(\gamma^k)$.
⟨3⟩8. Hence, u^{k+1} eventually exists $B_\varepsilon(\gamma^k)$ which implies the result.
⟨1⟩8. Q.E.D.

We know that if $\gamma^- \neq \gamma^+$ then $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+)$ can only happen if $\mathcal{M}(\gamma^-, \gamma^+, H, J) = \emptyset$. Consequently, we must have that $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) \geq 1$ if $\mathcal{M}(\gamma^-, \gamma^+, H, J) \neq \emptyset$. Thus, we obtain the following two corollaries of Theorem 2.19.

Corollary 2.1 *Let $\gamma^-, \gamma^+ \in \mathrm{Crit}(\mathcal{A}_H)$ such that $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) = 1$. Then $\underline{\mathcal{M}}(\gamma^-, \gamma^+, H, J)$ is compact.*

For the case $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) = 2$ we first make the following definition.

Definition 2.26 *We define the compactification of $\underline{\mathcal{M}}(\gamma^-, \gamma^+, H, J)$ to be*

$$\begin{aligned}\underline{\mathcal{M}}^+(\gamma^-, \gamma^+, H, J) = &\underline{\mathcal{M}}(\gamma^-, \gamma^+, H, J) \\ &\cup \bigcup_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_H) \\ \mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^0) = 1}} \underline{\mathcal{M}}(\gamma^-, \gamma^0, H, J) \times \underline{\mathcal{M}}(\gamma^0, \gamma^+, H, J).\end{aligned}$$

Corollary 2.2 *Let $\gamma^-, \gamma^+ \in \mathrm{Crit}(\mathcal{A}_H)$ such that $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) = 2$. Then $\underline{\mathcal{M}}^+(\gamma^-, \gamma^+, H, J)$ is compact.*

2.8 Boundary Map

We have now established all the necessary results to formally define the boundary operator $\partial : CF_*(H, J) \to CF_{*-1}(H, J)$. Let $(H, J) \in \mathcal{HJ}_{reg}$ be a generic choice of Hamiltonian and almost complex structure such that all critical points of $\mathcal{A}_H$ are non-degenerate and such that at every solution $u \in \mathcal{M}(H, J)$ the differential $D\bar{\partial}_{H,J}(u)$ is surjective. Let $\gamma^- \in CF_{k+1}(H, J)$ and $\gamma^+ \in CF_k(H, J)$. Then it follows from the results in the previous sections that the space $\underline{\mathcal{M}}(\gamma^-, \gamma^+, H, J)$ is a compact, 0-dimensional manifold. Hence, the following definition makes sense.

Definition 2.27 *We denote by $n_2(\gamma^-, \gamma^+, H, J)$ the (finite) number of elements in the space $\underline{\mathcal{M}}(\gamma^-, \gamma^+, H, J)$ modulo $\mathbb{Z}_2$.*

With this we can now define ∂ by defining it on the generators of $CF_*(H, J)$.

Definition 2.28 *For $\gamma \in CF_k(H, J)$ set*

$$\partial\gamma = \sum_{\gamma' \in CF_{k-1}(H,J)} n_2(\gamma, \gamma')\gamma'.$$

The key property that needs to be established is that $\partial^2 = 0$. To show this it is clearly sufficient to verify that

$$\sum_{\gamma^0}\sum_{\gamma^+} n_2(\gamma^-, \gamma^0) n_2(\gamma^0, \gamma^+) = 0.$$

We will do this by proving that $\sum_{\gamma^0} n(\gamma^-, \gamma^0) n(\gamma^0, \gamma^+)$ is even which will imply the result since we are working over $\mathbb{Z}_2$. In order to do this we will exhibit this product as the number of boundary points of a compact 1-dimensional smooth manifold for which we know that its boundary consists of an even number of points. More specifically, we have to study the compactification of $\underline{\mathcal{M}}(\gamma^-, \gamma^+, H, J)$ where $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) = 2$.

2.9 Gluing

We know that $\underline{\mathcal{M}}(\gamma^-, \gamma^+, H, J)$ is a smooth, 1-dimensional manifold if $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) = 2$. As explained in the previous section, in order to show that $\partial^2 = 0$ we need to study the compactified space $\underline{\mathcal{M}}^+(\gamma^-, \gamma^+, H, J)$ and count its boundary points. However, it is not at all clear that this space is still a manifold after the points corresponding to broken trajectories have been added. We now want to show that this is indeed the case and that its boundary is precisely given by the points obtained from broken trajectories, that is

$$\partial\underline{\mathcal{M}}^+(\gamma^-, \gamma^+, H, J) = \bigcup_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_H) \\ \mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^0) = 1}} \underline{\mathcal{M}}(\gamma^-, \gamma^0, H, J) \times \underline{\mathcal{M}}(\gamma^0, \gamma^+, H, J).$$

To accomplish this we need to study the space $\underline{\mathcal{M}}^+(\gamma^-, \gamma^+, H, J)$ in the neighbourhood of its boundary. For this let $\gamma^0 \in \mathrm{Crit}(\mathcal{A}_H)$ such that $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^0) = \mu_{CZ}(\gamma^0) - \mu_{CZ}(\gamma^+) = 1$ and let $([u_1], [u_2]) \in \underline{\mathcal{M}}(\gamma^-, \gamma^0, H, J) \times \underline{\mathcal{M}}(\gamma^0, \gamma^+, H, J)$ be an unparameterised broken trajectory. We construct a differentiable map $\psi : [r_0, \infty) \to \mathcal{M}(\gamma^-, \gamma^+, H, J)$ for some r_0 such that it descends to an embedding $\pi \circ \psi : [r_0, \infty) \to \underline{\mathcal{M}}(\gamma^-, \gamma^+, H, J)$ with the property that $\lim_{r \to \infty} (\pi \circ \psi)(r) = ([u_1], [u_2])$. In addition we require that any sequence in $\underline{\mathcal{M}}^+(\gamma^-, \gamma^+, H, J)$ that tends to $([u_1], [u_2])$ eventually lies in the image of $\pi \circ \psi$. From this we can then conclude that around $([u_1], [u_2])$ the space $\underline{\mathcal{M}}^+(\gamma^-, \gamma^+, H, J)$ looks like a half-closed interval whose one-sided boundary point is given by $([u_1],[u_2])$. This is exactly what it means to be a boundary point of a 1-dimensional manifold.

The construction of ψ is split into two steps. In the first step we construct a map w_r that is a smooth family of approximations to the broken trajectory (u_1, u_2), but is deficient in the sense that these approximations are not themselves solutions to the Floer equation. In the second step we construct the actual map ψ by taking the exponential of a suitable vector field along w_r. This vector field is obtained through an iterative process that ensures that ψ actually gives true solutions to the Floer equation. We will not give full proofs for all the results in this section to keep this text at a reasonable length. The treatment of the gluing results follows section 9 of [2].

2.9.1 Construction of w_r

For w_r we will use the exponential map $\exp_p : T_pM \to M$ with $p \in M$ to smooth out the broken trajectory (u_1, u_2) around the critical point γ^0. In order for this to work we need to pick $r_0 \in \mathbb{R}$ large enough so that $u_1(s+r,t))$ and $u_2(s-r,t)$. fall into the injectivity radius of M so that the exponential maps defines a diffeomorphism.

Now fix two smooth step-functions $\beta^\pm : \mathbb{R} \to [0,1]$ and $\varepsilon \in (0,1)$ such that

$$\beta^-(s) = \begin{cases} 1 & s \leq -1 \\ 0 & s \geq -\varepsilon \end{cases}$$

and

$$\beta^+(s) = \begin{cases} 1 & s \geq 1 \\ 0 & s \leq \varepsilon \end{cases} .$$

With this we can define w_r. Outside a compact set w_r is given by truncating the trajectories u_1 and u_2 and inside we use the exponential map and the smooth step-functions $\beta^\pm$ to interpolate between the two. As r becomes larger the approximation w_r becomes closer to u_1 and u_2 away from γ^0.

There are some simple properties of w_r that we can immediately read off from the definition.

Definition 2.29 *Define the map* $w : [r_0, \infty) \times \mathbb{R} \times S^1 \to M$ *to be*

$$w_r(s,t) = \begin{cases} u_1(s+r,t) & s \leq -1 \\ \exp_{\gamma^0(t)}\left(\beta^-(s)\exp^{-1}_{\gamma^0(t)}(u_1(s+r,t)) + \beta^+(s)\exp^{-1}_{\gamma^0(t)}(u_2(s-r,t))\right) & s \in [-1,1] \\ u_2(s-r,t) & s \geq 1 \end{cases} . \tag{2.51}$$

Proposition 2.31 *The map w defined in Equation 2.51 has the following properties.*

1. $w_r \in C^\infty(\mathbb{R} \times S^1, M)$ *with* $\lim_{s\to\pm\infty} w_r(s,\cdot) = \gamma^\pm$.
2. $\forall t \in S^1 : \forall s \in [-\varepsilon, \varepsilon] : w_r(s,t) = \gamma^0(t)$.
3. $\forall s \in (-\infty, r-1] : w_r(s-r,t) = u_1(s,t)$.
4. $\forall s \in [1-r, \infty) : w_r(s+r,t) = u_2(s,t)$.
5. *w is differentiable with respect to r.*
6. $\lim_{r\to\infty} w_r(s,t) = \gamma^0(t)$.

PROOF:
⟨1⟩1. item 1 holds.
⟨2⟩1. Smoothness follows from the smoothness of the involved functions. At the boundary of the different domains of definition it follows from the properties of $\beta^\pm$.
⟨2⟩2. This follows from the corresponding properties of u_1 and u_2.
⟨1⟩2. item 2 holds.
⟨2⟩1. $\forall s \in [-\varepsilon, \varepsilon] : \beta^-(s)\exp^{-1}_{\gamma^0(t)}(u_1(s+r,t)) + \beta^+(s)\exp^{-1}_{\gamma^0(t)}(u_2(s-r,t)) = 0$.
⟨2⟩2. $\forall t \in S^1 : \exp_{\gamma^0(t)}(0) = \gamma^0(t)$.
⟨1⟩3. item 3 and item 4 hold.
⟨2⟩1. This is true by definition of w_r.
⟨1⟩4. item 5 holds.
⟨2⟩1. This is true because with respect to r the map w_r is a composition of differentiable functions.
⟨1⟩5. item 6 holds.
⟨2⟩1. This is true because $\lim_{r\to\infty} u_1(s+r,t) = \lim_{r\to\infty} u_2(s-r,t) = \gamma^0(t)$.
⟨1⟩6. Q.E.D.

The next step is to use w_r to construct a similar family of actual solutions ψ_r.

2.9.2 Construction of ψ

To construct ψ_r we take a suitable vector field $\xi \in W^{1,p}(w_r^* TM)$ along w_r and use the exponential map to "flow" along it to obtain a true solution of the Floer equation. In other word, we want ψ_r to be of the form

$$\psi_r = \exp_{w_r} \xi(r) \tag{2.52}$$

such that $\bar{\partial}_{H,J}(\psi_r) = 0$. This can be rephrased as saying that we are looking for a solution ξ of the equation $(\bar{\partial}_{H,J} \circ \exp_{w_r})(\xi) = 0$. Hence, we are led work with this composed operator.

Definition 2.30 *Denote this composed operator by* $\bar{\partial}^r_{H,J} := \bar{\partial}_{H,J} \circ \exp_{w_r}$.

To solve Equation 2.52 we will employ an iterative process, namely a general version of the well-known Newton-Picard method. We begin by discussing this method in general and then apply it to our situation. In the classical version of the Newton-Picard method tries to find the zero of a function f by using its derivative to linearly approximate at some starting point. One then iteratively computes the zero of that approximation and takes that value as the new starting point. More generally, the derivative is a linear map. In order for the method to work it needs to be invertible, at least from one side. We have the following general result.

Proposition 2.32 *Let X, Y be Banach spaces and $F : X \to Y$ be a differentiable map. Let $R(x) = F(x) - F(0) + DF(0)(x)$. Assume that there exists continuous $G \in \mathrm{Hom}(Y, X)$ such that:*

1. $DF(0) \circ G = \mathrm{Id}_Y$.
2. $\forall x, y \in B_r(0) : \; \|GR(x) - GR(y)\| \leq C(\|x\| + \|y\|)\|x - y\|$.
3. $\|GF(0)\| \leq \frac{1}{2}\min(r, \frac{1}{5}C)$.

Then there is a unique $z \in \mathrm{Im}\,(G) \cap B_\varepsilon(0)$ that satisfies $F(z) = 0$. In addition, $\|z\| \leq 2\|GF(0)\|$.

To apply this result to our setting we must determine the differential of $\overline{\partial}^r_{H,J}$.

Proposition 2.33 *In suitable local coordinates the differential at zero $D\overline{\partial}^r_{H,J}(0) : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is given by*

$$D\overline{\partial}^r_{H,J}(0)(\xi) = \partial_s \xi + J_0 \partial_t \xi + S_r \xi$$

where $S_r : \mathbb{R} \times S^1 \to \mathbb{R}^{2n \times 2n}$ has the property that

$$S_r(s,t) = \begin{cases} S_1(s + r, t) & s \leq -1 \\ S_2(s - r, t) & s \geq 1 \end{cases}$$

where S_1 and S_2 are the matrix-valued functions associated to the coordinate expressions of $D\overline{\partial}_{H,J}(u_1)$ and $D\overline{\partial}_{H,J}(u_2)$.

Proposition 2.34 *The map $D\overline{\partial}^r_{H,J}(0)$ is a Fredholm operator with $\mathrm{Ind}(D\overline{\partial}^r_{H,J}(0)) = 2$.*

The next thing thing we need to is a right-inverse for $D\overline{\partial}^r_{H,J}(0)$. A continuous right-inverse corresponds to splitting $W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) = E^1_r \oplus E^2_r$ into closed subspaces such that $D\overline{\partial}^r_{H,J}(0)$ restricted to E^2_r is bijective. This space is going to be constructed as the L^2-complement of a certain subspace which is spanned by approximated solutions of the linearised Floer equation along the broken trajectory (u_1, u_2). We now describe this in more detail. For $\xi \in W^{1,p}(u_1^* TM)$ and $\eta \in W^{1,p}(u_2^* TM)$ we define their concatenation as follows.

Definition 2.31

$$\xi\#_r\eta(s,t) := \begin{cases} \xi(s+r,t) \\ D\exp_{\gamma^0(t)}\left(\beta^-(s)D\exp^{-1}_{\gamma^0(t)}(\xi(s+r,t)) + \beta^+(s)D\exp^{-1}_{\gamma^0(t)}(\eta(s-r,t))\right) \\ \eta(s-r,t) \end{cases}$$

where the three cases are $s < -1$ $s \in [-1,1]$ and $s > 1$.

Then we use this to define the spaces E^1_r and E^2_r.

Definition 2.32

$$E^1_r = \{\xi\#_r\eta :\ \xi \in \operatorname{Ker}\left(D\overline{\partial}_{H,J}(u_1)\right), \eta \in \operatorname{Ker}\left(D\overline{\partial}_{H,J}(u_2)\right)\}$$

$$E^2_r = \{\zeta \in W^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n}) :\ \forall \xi\#_r\eta \in E^1_r :\ \int_{\mathbb{R}\times S^1} \langle \zeta, \xi\#_r\eta\rangle = 0\}$$

The space E^1_r is finite-dimensional since the differential $D\overline{\partial}_{H,J}$ is Fredholm at solutions connecting non-degenerate critical loops. Consequently, we have that $W^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n}) = E^1_r \oplus (E^1_r)^\perp = E^1_r \oplus E^2_r$. We will not go further into the technical details of verifying that this way we can obtain a splitting that has all the necessary properties. We conclude by remarking that this way a suitable right-inverse can be constructed so that the above general lemma can be applied to define the gluing map ψ.

2.9.3 Properties of ψ

Having constructed the map ψ it remains to verify that it has the necessary properties to ensure that the compactified space $\underline{\mathcal{M}}(\gamma^-,\gamma^+,H,J)$ is a smooth manifold whose boundary consists of points coming from broken trajectories. For this one needs to show that close to one of the points $([u_1],[u_2]) \in \partial\underline{\mathcal{M}}^+(\gamma^-,\gamma^+,H,J)$ the space looks like a half-closed interval. These properties are summarised in the following two theorems.

Theorem 2.20 *The map $\psi : [r_0,\infty) \to \mathcal{M}(\gamma^-,\gamma^+,H,J)$ is differentiable and the induced map $\widehat{\psi} = \pi \circ \psi : [r_0,\infty) \to \underline{\mathcal{M}}(\gamma^-,\gamma^+,H,J)$ is an immersion with the property that*

$$\lim_{r\to\infty} \widehat{\psi}(r) = ([u_1],[u_2]) \in \underline{\mathcal{M}}^+(\gamma^-,\gamma^+,H,J).$$

Theorem 2.21 *Let $([u_n])_{n\in\mathbb{N}} \subset \underline{\mathcal{M}}(\gamma^-,\gamma^+,H,J)$ be a sequence such that $\lim_{n\to\infty}[u_n] = ([u_1],[u_2])$. Then there exists a $N \in \mathbb{N}$ such that for $n > N$ it holds that $[u_n] \in \operatorname{Im}(\widehat{\psi})$.*

We will omit their proofs because they would go beyond the scope of this text.

It remains to verify the chain map property ∂^2 using what we have just concluded.

Theorem 2.22 *The map $\partial : CF_*(H,J) \to CF_{*-1}(H,J)$ satisfies $\partial^2 = 0$.*

Proof:

⟨1⟩1. As explained in earlier this follows because the sum appearing in the expression for ∂^2 is exactly the number of points in the boundary $\partial\underline{\mathcal{M}}^+(\gamma^-,\gamma^+,H,J)$ modulo $\mathbb{Z}_2$.

⟨1⟩2. Q.E.D.

2.10 Invariance

Up to this point our definition of the Floer homology groups depends on a choice of a pair of Hamiltonian and compatible almost complex structure $(H,J) \in \mathcal{HJ}_{reg}$. In this section we want to verify that the homology groups $HF_*(H,J)$ are in fact all isomorphic and so do not actually depend on this choice. This will be done in the following way. For any two choices $(H^\pm, J^\pm)$ we can consider a path χ connecting them. From this path we will construct a corresponding chain map $\Phi^\#_\chi$ from $CF_*(H^-,J^-)$ to $CF_*(H^+,J^+)$. Thus, we obtain an induced map Φ_χ on the level of homology, going from $HF_*(H^-,J^-)$ to $HF_*(H^+,J^+)$. This map is independent of the choice of path χ. To prove this we will construct a chain homotopy between the chain maps $\Phi^\#_0$ and $\Phi^\#_1$ coming from two paths χ_0 and χ_1 connecting $(H^\pm,J^\pm)$. Finally, to see that Φ_χ is an isomorphism we will show that it is functorial in the following sense. If χ is the constant path connecting (H,J) to itself then $\Phi_\chi = \mathrm{Id}_{HF_*(H,J)}$. Moreover, given three pairs $(H^\pm,J^\pm),(H^0,J^0) \in \mathcal{HJ}_{reg}$ and corresponding paths χ_0 and χ_1 connecting (H^-,J^-) to (H^0,J^0) and (H^0,J^0) to (H^+,J^+), respectively, then the induced maps satisfy

$$\Phi_\chi = \Phi_{\chi_1} \circ \Phi_{\chi_0}.$$

From this it immediately follows that the induced maps are isomorphism by considering the case that $(H^+,J^+) = (H^-,J^-)$ and $\chi = const$.

This section is divided into two parts. The first part deals with the construction of the chain maps $\Phi^\#$. The second part deals with the verification of the functorial properties of the construction that lead to the isomorphism as described above.

2.10.1 Definition of the Chain Map $\Phi^\#$

The chain map $\Phi^\#_\chi$ is associated to a path χ connecting two regular pairs $(H^\pm,J^\pm)$. For technical reasons we restrict the paths to be *asymptotically* *constant* as per the following definition.

Definition 2.33 *Let* $(H^{\pm}, J^{\pm}) \in \mathcal{HJ}_{reg}$ *and let* $\chi \in C^{\infty}(\mathbb{R}, C^{\infty}(S^1 \times M) \times \mathcal{J}(M, \omega))$ *be a smooth path. Then* χ *is called an* asymptotically constant path connecting (H^-, J^-) to (H^+, J^+) *if there exists a* $T > 0$ *such that*

$$\chi(s) = (H^s, J^s) = \begin{cases} (H^-, J^-) & s \leq -T \\ (H^+, J^+) & s \geq T \end{cases}.$$

The set of all such paths will be denoted by $P(H^-, J^-, H^+, J^+)$.

The advantage of working with asymptotically constant paths is that at the two pieces of $\mathbb{R}$ going to $\pm\infty$ one can reuse the analysis of the ordinary Floer moduli spaces.

We introduce the Floer differential operator associated to a path χ. This is analogous to the operator $\overline{\partial}_{H,J}$ we have considered up until this point, except that H and J now depend on the s-parameter according to the path χ.

Definition 2.34 *Let* $\chi \in P(H^-, J^-, H^+, J^+)$. *Define the section* $\overline{\partial}_{\chi} : \mathcal{B}^{1,p} \to \mathcal{E}^p$ *via*

$$\overline{\partial}_{\chi}(u) = \partial_s u + J^s(u)\partial_t u - J^s(u) X_{H^s_t}(u).$$

We will define the chain map $\Phi^{\#}_{\chi}$ by counting the zeroes of the section $\overline{\partial}_{\chi}$. This works very much in the same way as for the definition of the Floer boundary operator. Hence, we are led to essentially repeat most of the analysis of the moduli spaces of the Floer equation for the map $\overline{\partial}_{\chi}$. Let us introduce the corresponding space of solutions.

Definition 2.35 *Let* $\chi \in P(H^-, J^-, H^+, J^+)$ *and* $\gamma^{\pm} \in \mathrm{Crit}(\mathcal{A}_{H^{\pm}})$.

$$\mathcal{N}(\gamma^-, \gamma^+, \chi) = \{u \in C^{\infty}(\mathbb{R} \times S^1, M) : \overline{\partial}_{\chi}(u) = 0 \text{ and } \lim_{s \to \pm\infty} u_s = \gamma^{\pm}\}.$$

First we address the question of when $\mathcal{N}(\gamma^-, \gamma^+, \chi)$ is a smooth manifold. This goes along the same lines as before. We verify the Fredholm property of the vertical derivative $D\overline{\partial}_{\chi}$ at a solution $u \in \mathcal{N}(\gamma^-, \gamma^+, \chi)$ and compute the index.

Theorem 2.23 *Let* $\chi \in P(H^-, J^-, H^+, J^+)$. *Then for every* $u \in \mathcal{N}(\gamma^-, \gamma^+, \chi)$ *the map* $D\overline{\partial}_{\chi}(u)$. *is a Fredholm operator with index* $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+)$.

Next we need to consider when the section $\overline{\partial}_{\chi}$ is transverse to the zero section. As before this is not the case for every path $\chi \in P(H^-, J^-, H^+, J^+)$. Paths for which this holds are called *regular*.

Definition 2.36 *Let $\chi \in P(H^-, J^-, H^+, J^+)$. Then χ is called* regular *if for all $\gamma^\pm \in \mathrm{Crit}(\mathcal{A}_{H^\pm})$ and for all $u \in \mathcal{N}(\gamma^-, \gamma^+, \chi)$ the map $D\bar{\partial}_\chi(u)$ is surjective. The set of all regular asymptotically constant paths connecting (H^-, J^-) to (H^+, J^+) will be denoted by $P_{reg}(H^-, J^-, H^+, J^+)$.*

Hence, for a regular path χ the implicit function theorem immediately implies that $\mathcal{N}(\gamma^-, \gamma^+, \chi)$ is a smooth manifold of dimension $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+)$ for any pair $\gamma^\pm \in \mathrm{Crit}(\mathcal{A}_{H^\pm})$. Not every path χ is regular. However, every path can be arbitrarily well approximated by one.

Theorem 2.24 *Let $(H^\pm, J^\pm) \in \mathcal{HJ}_{reg}$, $\chi \in P(H^-, J^-, H^+, J^+)$ and $\varepsilon > 0$. Then there exists a $\chi' \in P_{reg}(H^-, J^-, H^+, J^+)$ such that $\|H^s - H'^s\|_{C^\infty(S^1 \times M)} < \varepsilon$ for all $s \in \mathbb{R}$.*

We now know that the space $\mathcal{N}(\gamma^-, \gamma^+, \chi)$ is a smooth manifold for a generic χ. We want to count the number of points of this set modulo $\mathbb{Z}_2$ to define Φ_χ for the case that $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+)$. Note that this is contrary to the definition of the boundary operator where the difference of the indices was 1. This is due to the fact that we no longer have a free $\mathbb{R}$-action on $\mathcal{N}(\gamma^-, \gamma^+, \chi)$ since H and J now depend on s and thus are not invariant under translation anymore.

In the case $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+)$ the manifold is $\mathcal{N}(\gamma^-, \gamma^+, \chi)$ 0-dimensional. Consequently, in order to count the number of its points we still need to verify compactness of this space. We start again with C^∞_{loc}-compactness. For this define an analogous energy function.

Definition 2.37 *Let $u \in C^\infty(\mathbb{R} \times S^1, M)$ such that $\bar{\partial}_\chi(u) = 0$. Then we define the energy of u to be*

$$E(u) = \int_{-\infty}^{\infty} \|\partial_s u\|_J^2 ds.$$

The space of all such finite energy solutions will be denoted by

$$\mathcal{N}^b(\chi) := \{u \in \mathcal{N}(\chi) : \ E(u) < \infty\}.$$

To prove compactness of this space we can proceed analogously and first prove a global bound on the gradient of finite energy solutions.

Lemma 2.8 *There exists a constant $C > 0$ such that for all $u \in \mathcal{N}^b(\chi)$ we have $E(u) < C$.*

Proof:

⟨1⟩1. The proof of this is analogous to the corresponding proof of the existence of such a constant in the case of $\mathcal{M}^b(H, J)$.

⟨1⟩2. Q.E.D.

This estimate immediately yields the compactness result similar to before.

Theorem 2.25 *Let $\chi \in P(H^-, J^-, H^+, J^+)$. Then the space $\mathcal{N}^b(\chi)$ is compact in the $C^\infty_{loc}(\mathbb{R} \times S^1, M)$-topology.*

PROOF:
⟨1⟩1. The proof is completely analogous to the proof of the corresponding result about $\mathcal{M}^b(\gamma^-, \gamma^+, H, J)$ using the analogue of the above estimate.
⟨1⟩2. Q.E.D.

Next we have the corresponding result about convergence to broken trajectories which is the first step towards compactness.

Theorem 2.26 *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}(\gamma^-, \gamma^+, \chi)$ where $\chi \in P_{reg}(H^-, J^-, H^+, J^+)$. Then there exists a subsequence still denoted by $(u_n)_{n\in\mathbb{N}}$ such that*

1. $\exists \gamma_0^-, \dots, \gamma_{l^-}^- \in \mathrm{Crit}(\mathcal{A}_{H^-})$.
2. $\exists \gamma_0^+, \dots, \gamma_{l^+}^+ \in \mathrm{Crit}(\mathcal{A}_{H^+})$.
3. $\exists (a_n^i)_{n\in\mathbb{N}} \subset \mathbb{R} : \lim_{n\to\infty} a_n^i = -\infty$ *for* $i \in \{0, \dots, l^- - 1\}$.
4. $\exists (b_n^j)_{n\in\mathbb{N}} \subset \mathbb{R} : \lim_{n\to\infty} a_n^j = \infty$ *for* $j \in \{0, \dots, l^+ - 1\}$.
5. $\exists v^i \in \mathcal{M}(\gamma_i^-, \gamma_{i+1}^-, H^-, J^-)$ *for* $i \in \{0, \dots, l^- - 1\}$.
6. $\exists w^j \in \mathcal{M}(\gamma_j^+, \gamma_{j+1}^+, H^+, J^+)$ *for* $j \in \{0, \dots, l^+ - 1\}$.
7. $\exists u_* \in \mathcal{N}(\gamma^-, \gamma^+, \chi)$:
8. $\lim_{n\to\infty} \mathrm{sh}(a_n^i) u_n = v^i$ *for* $i \in \{0, \dots, l^- - 1\}$.
9. $\lim_{n\to\infty} \mathrm{sh}(b_n^j) u_n = w^j$ *for* $j \in \{0 \dots, l^+ - 1\}$.
10. $\lim_{n\to\infty} u_n = u_*$.
11. $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) \geq l^- + l^+$.

PROOF:
⟨1⟩1. Outside a compact parameter set in s the path χ is constant so that the proof is essentially analogous to the previous broken trajectories result.
⟨1⟩2. Q.E.D.

Summarising the results so far we know that $\mathcal{N}(\gamma^-, \gamma^+, \chi)$ is a smooth, compact 0-dimensional manifold in the case $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+)$. Thus, we can make the following definition.

Definition 2.38 *Let $\chi \in P_{reg}(H^-, J^-, H^+, J^+)$. For $\gamma^\pm \in \mathrm{Crit}(\mathcal{A}_{H^\pm})$ with $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+)$ denote by $n_2^\chi(\gamma^-, \gamma^+)$ the number of elements of $\mathcal{N}(\gamma^-, \gamma^+, \chi)$*

modulo $\mathbb{Z}_2$. Then we define the map $\Phi^{\#}_{\chi} : CF_*(H^-, J^-) \to CF_*(H^+, J^+)$ by setting its value on a generator $\gamma^- \in CF_k(H^-, J^-)$ to be

$$\Phi^{\#}_{\chi}(\gamma^-) = \sum_{\substack{\gamma^+ \in \mathrm{Crit}(\mathcal{A}_{H^+}) \\ \mu_{CZ}(\gamma^+)=k}} n_2^{\chi}(\gamma^-, \gamma^+)\gamma^+.$$

2.10.2 Properties of $\Phi^{\#}$ and Φ

We will now show that $\Phi^{\#}$ is chain map so that it descends to a map Φ on the level of homology. Let us spell out what this property means. Fix a $\gamma^- \in CF_k(H^-, J^-)$. Then

$$\begin{aligned}(\partial \circ \Phi^{\#}_{\chi})(\gamma^-) &= \partial\Big(\sum_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^0}) \\ \mu_{CZ}(\gamma^0)=k}} n_2^{\chi}(\gamma^-, \gamma^0)\gamma^0\Big) \\ &= \sum_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^0}) \\ \mu_{CZ}(\gamma^0)=k}} n_2^{\chi}(\gamma^-, \gamma^0)\partial(\gamma^0) \\ &= \sum_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^+}) \\ \mu_{CZ}(\gamma^0)=k}} \sum_{\substack{\gamma^+ \in \mathrm{Crit}(\mathcal{A}_{H^+}) \\ \mu_{CZ}(\gamma^+)=k-1}} n_2^{\chi}(\gamma^-, \gamma^0) n_2(\gamma^0, \gamma^+)\gamma^+\end{aligned}$$

and similarly

$$(\Phi^{\#}_{\chi} \circ \partial)(\gamma^-) = \sum_{\substack{\gamma^+ \in \mathrm{Crit}(\mathcal{A}_{H^+}) \\ \mu_{CZ}(\gamma^+)=k-1}} \sum_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^-}) \\ \mu_{CZ}(\gamma^0)=k-1}} n_2(\gamma^-, \gamma^0) n_2^{\chi}(\gamma^0, \gamma^+)\gamma^+.$$

Thus, in order to show that $\partial \circ \Phi^{\#}_{\chi} = \Phi^{\#}_{\chi} \circ \partial$ it suffices to show that

$$\sum_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^+}) \\ \mu_{CZ}(\gamma^0)=k}} n_2^{\chi}(\gamma^-, \gamma^0) n_2(\gamma^0, \gamma^+) - \sum_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^-}) \\ \mu_{CZ}(\gamma^0)=k-1}} n_2(\gamma^-, \gamma^0) n_2^{\chi}(\gamma^0, \gamma^+) = 0 \tag{2.53}$$

for every $\gamma^+ \in CF_{k-1}(H^+, J^+)$. This is done in an analogous way to the earlier proof that $\partial^2 = 0$. First note that the minus sign in Equation 2.53 can be replaced by a plus sign since we are working over $\mathbb{Z}_2$. Now to show that the resulting sum vanishes we exhibit it as the number of boundary points of a compact, 1-dimensional manifold which is even. The manifold that will serve this purpose is of course the compactification of $\mathcal{N}(\gamma^-, \gamma^+, \chi)$. For this we will use the broken trajectories result Theorem 2.26 for the case where the indices of the critical points differ by 1, followed by a gluing result to verify that the space obtained by adding the broken trajectories is still a manifold.

Theorem 2.27 *Let $\gamma^- \in \mathcal{A}_{\mathrm{Crit}(H^-)}$ and $\gamma^0, \gamma^+ \in \mathrm{Crit}(\mathcal{A}_{H^+})$ such that $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^0) = \mu_{CZ}(\gamma^+) + 1$. Furthermore, let $(u^1, [u^2]) \in \mathcal{N}(\gamma^-, \gamma^0, \chi) \times \underline{\mathcal{M}}(\gamma^0, \gamma^+, H^+, J^+)$. Then there exists an $r_0 > 0$ such that there exists an embedding $\psi : [r_0, \infty) \to \mathcal{N}(\gamma^-, \gamma^+, \chi)$ with the property that*

$$\lim_{r \to \infty} \psi(r) = (u^1, [u^2]).$$

Moreover, if $(u_n)_{n \in \mathbb{N}} \subset \mathcal{N}(\gamma^-, \gamma^+, \chi)$ is a sequence such that $\lim_{n \to \infty} u_n = (u^1, [u^2])$ then there exists an $N \in \mathbb{N}$ such that for $n > N$ we have that $u_n \in$ Im (ψ).

Theorem 2.28 *Let $\gamma^\pm \in \mathrm{Crit}(\mathcal{A}_{H^\pm})$ such that $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+) + 1$. Then the compactified space $\mathcal{N}^+(\gamma^-, \gamma^+, \chi)$ is a 1-dimensional manifold and its boundary can be identified as*

$$\begin{aligned}\partial \mathcal{N}^+(\gamma^-, \gamma^+, \chi) = & \bigcup_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^-}) \\ \mu_{CZ}(\gamma^0) = \mu_{CZ}(\gamma^-) - 1}} \underline{\mathcal{M}}(\gamma^-, \gamma^0, H^-, J^-) \times \mathcal{N}(\gamma^0, \gamma^+, \chi) \\ & \cup \bigcup_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^+}) \\ \mu_{CZ}(\gamma^0) = \mu_{CZ}(\gamma^-)}} \mathcal{N}(\gamma^-, \gamma^0, \chi) \times \underline{\mathcal{M}}(\gamma^0, \gamma^+, H^+, J^+).\end{aligned}$$

PROOF:
⟨1⟩1. This follows from the broken trajectories result, gluing and transversality.
⟨1⟩2. Q.E.D.

Thus we obtain the chain map property.

Proposition 2.35 *Let $(H^\pm, J^\pm) \in \mathcal{HJ}_{reg}$ and $\chi \in P_{reg}(H^-, J^-, H^+, J^+)$. Then the map $\Phi^\#_\chi : CF^-_*(H^-, J^-) \to CF^+_*(H^+, J^+)$ is a chain map. In more detail this means that*

$$\partial \circ \Phi^\#_\chi = \Phi^\#_\chi \circ \partial. \tag{2.54}$$

Note that the two boundary operators above correspond to different chain complexes. However, we will not indicate this in the notation.

PROOF:
⟨1⟩1. $|\partial \mathcal{N}^+(\gamma^-, \gamma^+, \chi)| = 0 \mod \mathbb{Z}_2$.
PROOF:
⟨2⟩1. $\partial \mathcal{N}^+(\gamma^-, \gamma^+, \chi)$ is a compact 1-dimensional manifold.
⟨1⟩2.

$$\begin{aligned}|\partial \mathcal{N}^+(\gamma^-, \gamma^+, \chi)| = & \sum_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^+}) \\ \mu_{CZ}(\gamma^0) = k}} n_2^\chi(\gamma^-, \gamma^0) n_2(\gamma^0, \gamma^+) \\ & + \sum_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^-}) \\ \mu_{CZ}(\gamma^0) = k-1}} n_2(\gamma^-, \gamma^0) n_2^\chi(\gamma^0, \gamma^+) \mod \mathbb{Z}_2.\end{aligned}$$

⟨1⟩3. Q.E.D.

Next we want to show that the induced chain map does not depend on the path connecting (H^-, J^-) to (H^+, J^+). For this let $\chi_0, \chi_1 \in P_{reg}(H^-, J^-, H^+, J^+)$. Then there is a homotopy h^r with $r \in [0,1]$ that connects these two paths. For every such homotopy we will construct a chain homotopy between the induced maps $\Phi^{\#}_{\chi_0}$ and $\Phi^{\#}_{\chi_1}$. That is, a map $P_h : CF_*(H^-, J^-) \to CF_{*+1}(H^+, J^+)$ such that

$$\Phi^{\#}_{\chi_1} - \Phi^{\#}_{\chi_0} = P_\chi \circ \partial + \partial \circ P_\chi.$$

It is elementary homological algebra to see that chain homotopic maps induce the same map on the level homology. The above identify immediately implies that restricted to cycles the difference $\Phi^{\#}_{\chi_1} - \Phi^{\#}_{\chi_0}$ is a boundary. Hence, on the level of homology the maps $\Phi^{\#}_{\chi_1}$ and $\Phi^{\#}_{\chi_0}$ coincide.

To define the chain homotopy P_h we introduce a suitable parameterised moduli space.

Definition 2.39 *Let $\gamma^\pm \in \mathrm{Crit}(\mathcal{A}_{H^\pm})$. We define the following parameterised moduli space associated to the homotopy χ.*

$$\mathcal{O}(\gamma^-, \gamma^+, h) := \{(u, r) : \; u \in \mathcal{N}(\gamma^-, \gamma^+, h^r)\}.$$

We also denote by

$$\mathcal{O}(h) := \bigcup_{\substack{\gamma^- \in \mathrm{Crit}(\mathcal{A}_{H^-}) \\ \gamma^+ \in \mathrm{Crit}(\mathcal{A}_{H^+})}} \mathcal{O}(\gamma^-, \gamma^+, h)$$

the total moduli space.

The definition of P_h involves counting the elements of $\mathcal{O}(\gamma^-, \gamma^+, h)$. Thus, we once again need go through the usual analysis to verify transversality and compactness results. We will not include proofs for these statements because they are all essentially the same as for the case of $\mathcal{M}$ and $\mathcal{N}$. More details can be found in section 11.3. of [2].

Theorem 2.29 *If $h = (H, J)$ is a homotopy between two asymptotically constant regular paths χ_0 and χ_1 connecting (H^-, J^-) to (H^+, J^+) then for any $\varepsilon > 0$ there exists a perturbed homotopy $h' = (H', J')$ such it still connects χ_0 and χ_1, it satisfies $\|H - H'\|_{C^\infty(S^1 \times M)} < \varepsilon$ for every $(r, s) \in [0,1] \times \mathbb{R}$ and such that $\mathcal{O}(\gamma^-, \gamma^+, h)$ is a smooth manifold of dimension $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) + 1$.*

Next we address the C^∞_{loc}-compactness. The proof goes along the same lines as earlier. One first establishes an global energy bound for all elements in $\mathcal{O}(h)$. Here the energy is defined analogously with respect to the Riemannian metric coming from the symplectic form ω and two parameter dependent compatible almost complex structure J_s^r that is provided by the homotopy h. Next this is used to provide a uniform bound on the gradient of elements $u \in \mathcal{O}(h)$ which implies equicontinuity. Consequently, as before we can use the Arzelà-Ascoli theorem to obtain a continuous candidate whose regularity we can then boost using elliptic regularity. We omit the explicit details. Here is the statement.

Theorem 2.30 *The space $\mathcal{O}(h)$ is compact with respect to the C^∞_{loc}-topology.*

After this we deal with an analogous broken trajectory result. This serves a similar purpose to before. In the case $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) + 1 = 0$ it tells us that there are no broken trajectories so that we can simply count the elements of $\mathcal{O}(\gamma^-, \gamma^+, h)$. This allows us to define P_h. Then in the case $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) + 1 = 1$ it tells us which types of broken trajectories we need to add to obtain compactness which is needed to verify the chain homotopy property. Its proof is analogous to earler broken trajectories results and is omitted.

Theorem 2.31 *Let $(u_n, r_n)_{n\in\mathbb{N}} \subset \mathcal{O}(\gamma^-, \gamma^+, h)$. Then there exists a subsequence still denoted by $(u_n, r_n)_{n\in\mathbb{N}}$ such that*

1. $\exists \gamma_0^-, \ldots, \gamma_{l^-}^- \in \mathrm{Crit}(\mathcal{A}_{H^-})$.
2. $\exists \gamma_0^+, \ldots, \gamma_{l^+}^+ \in \mathrm{Crit}(\mathcal{A}_{H^+})$.
3. $\exists (a_n^i)_{n\in\mathbb{N}} \subset \mathbb{R} : \lim_{n\to\infty} a_n^i = -\infty$ *for* $i \in \{0, \ldots, l^- - 1\}$.
4. $\exists (b_n^j)_{n\in\mathbb{N}} \subset \mathbb{R} : \lim_{n\to\infty} a_n^j = \infty$ *for* $j \in \{0, \ldots, l^+ - 1\}$.
5. $\exists v^i \in \mathcal{M}(\gamma_i^-, \gamma_{i+1}^-, H^-, J^-)$ *for* $i \in \{0, \ldots, l^- - 1\}$.
6. $\exists w^j \in \mathcal{M}(\gamma_j^+, \gamma_{j+1}^+, H^+, J^+)$ *for* $j \in \{0, \ldots, l^+ - 1\}$.
7. $\exists (u_*, r_*) \in \mathcal{O}(\gamma^-, \gamma^+, h)$:
8. $\lim_{n\to\infty} \mathrm{sh}(a_n^i) u_n = v^i$ *for* $i \in \{0, \ldots, l^- - 1\}$.
9. $\lim_{n\to\infty} \mathrm{sh}(b_n^j) u_n = w^j$ *for* $j \in \{0 \ldots, l^+ - 1\}$.
10. $\lim_{n\to\infty} (u_n, r_n) = (u_*, r_*)$.
11. $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) + 1 \geq l^- + l^+$.

As explained above here is the first special case of Theorem 2.31.

Corollary 2.3 *If $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) + 1 = 0$ then $\mathcal{O}(\gamma^-, \gamma^+, h)$ is a compact 0-dimensional manifold.*

This allows us to define the operator P_h. Denote by $n_2^h(\gamma^-, \gamma^+)$ the parity of the set $\mathcal{O}(\gamma^-, \gamma^+, h)$.

Definition 2.40

$$P_h(\gamma^-) = \sum_{\substack{\gamma^+\in\mathrm{Crit}(\mathcal{A}_{H^+})\\ \mu_{CZ}(\gamma^+)=k+1}} n_2^h(\gamma^-,\gamma^+)\gamma^+.$$

After having now defined the operator P_h we still need to verify that it is a chain homotopy between $\Phi_0^\#$ and $\Phi_1^\#$, that is the property in Equation 2.54. Let us unravel what we need to show by applying Equation 2.54 to an element $\gamma^- \in CF_k(H^-,J^-)$. We have that

$$\begin{aligned}
(P_h \circ \partial)(\gamma^-) &= P_h\Big(\sum_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^-})\\ \mu_{CZ}(\gamma^+)=k-1}} n_2(\gamma^-,\gamma^0)\gamma^0\Big)\\
&= \sum_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^-})\\ \mu_{CZ}(\gamma^0)=k-1}} n_2(\gamma^-,\gamma^0)P_h(\gamma^0)\\
&= \sum_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^-})\\ \mu_{CZ}(\gamma^0)=k-1}} \sum_{\substack{\gamma^+\in\mathrm{Crit}(\mathcal{A}_{H^+})\\ \mu_{CZ}(\gamma^+)=k}} n_2(\gamma^-,\gamma^0)n_2^h(\gamma^0,\gamma^+)\gamma^+.
\end{aligned}$$

Similarly,

$$(\partial \circ P_h)(\gamma^-) = \sum_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^+})\\ \mu_{CZ}(\gamma^0)=k+1}} \sum_{\substack{\gamma^+\in\mathrm{Crit}(\mathcal{A}_{H^+})\\ \mu_{CZ}(\gamma^+)=k}} n_2(\gamma^0,\gamma^+)n_2^h(\gamma^-,\gamma^0)\gamma^+.$$

Hence,

$$\begin{aligned}
(P_h\circ\partial - \partial\circ P_h)(\gamma^-) = \sum_{\substack{\gamma^+\in\mathrm{Crit}(\mathcal{A}_{H^+})\\ \mu_{CZ}(\gamma^+)=k}} \Bigg(&\sum_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^-})\\ \mu_{CZ}(\gamma^0)=k-1}} n_2(\gamma^-,\gamma^0)n_2^h(\gamma^0,\gamma^+)\\
&+ \sum_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^+})\\ \mu_{CZ}(\gamma^0)=k+1}} n_2(\gamma^0,\gamma^+)n_2^h(\gamma^-,\gamma^0)\Bigg)\gamma^+.
\end{aligned}$$

On the other hand, we see that

$$(\Phi_0^\# - \Phi_1^\#)(\gamma^-) = \sum_{\substack{\gamma^+\in\mathrm{Crit}(\mathcal{A}_{H^+})\\ \mu_{CZ}(\gamma^+)=k}} (n_2^{\chi_0}(\gamma^-,\gamma^+) - n_2^{\chi_1}(\gamma^-,\gamma^+))\gamma^+.$$

To prove that both expressions coincide we need to verify that the coefficients of γ^+ in the two sums have the same parity since we are working over $\mathbb{Z}_2$. For this it is sufficient to show that the sum of these coefficients is even. This is accomplished by exposing it as the number of boundary points of a 1-dimensional compact manifold. This manifold is going to be the compactification of $\mathcal{O}(\gamma^-,\gamma^+,h)$ where $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+)$. We already know that $\mathcal{O}(\gamma^-,\gamma^+,h)$ itself is a 1-dimensional manifold. The broken trajectory result Theorem 2.31 tells what we need to add to this set to compactify. The condition $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+)$ implies (trivially) that $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) + 1 = 1$. Thus, according to the broken trajectories result Theorem 2.31 above there can only be two types of broken trajectories. This is, in the notation of that theorem, due to the condition $1 \geq l^- + l^+ \geq 0$ so that either $l^- = 1$ and $l^+ = 0$ or $l^- = 0$ and $l^+ = 1$. These two types are of the form $([u_1],(u_2,r))$ and $((u_1',r'),[u_2'])$, respectively, where $[u_1] \in \underline{\mathcal{M}}(\gamma^-,\gamma^0,H^-,J^-)$, $[u_2] \in \underline{\mathcal{M}}(\gamma'^0,\gamma^+,H^+,J^+)$, $(u_2,r) \in \mathcal{O}(\gamma'^0,\gamma^+,h)$ and $(u_2',r') \in \mathcal{O}(\gamma^-,\gamma^0,h)$ for critical points $\gamma^-,\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^-})$ and $\gamma'^0,\gamma^+ \in \mathrm{Crit}(\mathcal{A}_{H^+})$. Consequently, we define the compactification of $\mathcal{O}(\gamma^-,\gamma^+,h)$ as follows.

Definition 2.41

$$\mathcal{O}^+(\gamma^-,\gamma^+,h) = \mathcal{O}(\gamma^-,\gamma^+,h) \cup \bigcup_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^-})\\ \mu_{CZ}(\gamma^0)=\mu_{CZ}(\gamma^-)-1}} \underline{\mathcal{M}}(\gamma^-,\gamma^0,H^-,J^-) \times \mathcal{O}(\gamma^0,\gamma^+,h)$$
$$\cup \bigcup_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^+})\\ \mu_{CZ}(\gamma^0)=\mu_{CZ}(\gamma^+)+1}} \mathcal{O}(\gamma^-,\gamma^0,h) \times \underline{\mathcal{M}}(\gamma^0,\gamma^+,H^+,J^+).$$

It remains to show that this is still a manifold. This is done by a gluing result, similar to what we have seen before. According to the two types of broken trajectories there are two different but completely analogous gluing results.

Theorem 2.32 *Let $\gamma^- \in \mathrm{Crit}(\mathcal{A}_{H^-})$ and $\gamma^0,\gamma^+ \in \mathrm{Crit}(\mathcal{A}_{H^+})$. Assume that $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+) = \mu_{CZ}(\gamma^0) - 1$. Furthermore, let $(u^1,r) \in \mathcal{O}(\gamma^-,\gamma^0,h)$ and $[u^2] \in \underline{\mathcal{M}}(\gamma^0,\gamma^+,H^+,J^+)$. Then there exists a $r_0 > 0$ such that there exists an embedding $\psi : [r_o,\infty) \to \mathcal{O}(\gamma^-,\gamma^+,h)$ with the property that*

$$\lim_{\lambda\to\infty} \psi(\lambda) = ((u^1,r),[u^2]).$$

Moreover, if $((u_n,r_n))_{n\in\mathbb{N}} \subset \mathcal{O}(\gamma^-,\gamma^+,h)$ is a sequence such that $\lim_{n\to\infty}(u_n,r_n) = ((u^1,r),[u^2])$ then there exists a $N \in \mathbb{N}$ such that for $n > N$ we have that $(u_n,r_n) \in \mathrm{Im}\,(\psi)$.

Finally, we can put everything together in the following theorem.

Theorem 2.33 *Let h be a homotopy of regular asymptotically constant paths between $\chi_0, \chi_1 \in P_{reg}(H^-, J^-, H^+, J^+)$ and $\gamma^\pm \in \mathrm{Crit}(\mathcal{A}_{H^\pm})$. Assume that $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+)$. Then the compactification $\mathcal{O}^+(\gamma^-, \gamma^+, h)$ is a 1-dimensional manifold and its boundary can be identified as*

$$\begin{aligned}\partial\mathcal{O}^+(\gamma^-,\gamma^+,h) =& \mathcal{N}(\gamma^-,\gamma^+,\chi_0)\times\{0\}\cup\mathcal{N}(\gamma^-,\gamma^+,\chi_1)\times\{1\}\\ &\cup \bigcup_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^-})\\ \mu_{CZ}(\gamma^0)=\mu_{CZ}(\gamma^-)-1}} \underline{\mathcal{M}}(\gamma^-,\gamma^0,H^-,J^-)\times\mathcal{O}(\gamma^0,\gamma^+,h)\\ &\cup \bigcup_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^+})\\ \mu_{CZ}(\gamma^0)=\mu_{CZ}(\gamma^+)+1}} \mathcal{O}(\gamma^-,\gamma^0,h)\times\underline{\mathcal{M}}(\gamma^0,\gamma^+,H^+,J^+).\end{aligned}$$

Using this result we can complete the proof that P_h is a chain homotopy as discussed above.

Theorem 2.34 *Let h be a homotopy of regular asymptotically constant paths between $\chi_0, \chi_1 \in P_{reg}(H^-, J^-, H^+, J^+)$. Then the operator $P_h : CF_*(H^-, J^-) \to CF_*(H^+, J^+)$ defined in Definition 2.40 is a chain homotopy.*

Proof:
$\langle 1\rangle 1.$ $|\partial\mathcal{O}^+(\gamma^-,\gamma^+,h)| = 0 \mod \mathbb{Z}_2$.
  Proof:
  $\langle 2\rangle 1.$ $\mathcal{O}^+(\gamma^-,\gamma^+,h)$ is a 1-dimensional compact manifold.
$\langle 1\rangle 2.$

$$\begin{aligned}|\partial\mathcal{O}^+(\gamma^-,\gamma^+,h)| =& (n_2^{\chi_0}(\gamma^-,\gamma^+) + n_2^{\chi_1}(\gamma^-,\gamma^+))\\ &+ \sum_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^-})\\ \mu_{CZ}(\gamma^0)=k-1}} n_2(\gamma^-,\gamma^0)n_2^h(\gamma^0,\gamma^+)\\ &+ \sum_{\substack{\gamma^0\in\mathrm{Crit}(\mathcal{A}_{H^+})\\ \mu_{CZ}(\gamma^0)=k+1}} n_2(\gamma^0,\gamma^+)n_2^h(\gamma^-,\gamma^0).\end{aligned}$$

$\langle 1\rangle 3.$ $n_2^{\chi_0}(\gamma^-,\gamma^+) + n_2^{\chi_1}(\gamma^-,\gamma^+) = n_2^{\chi_0}(\gamma^-,\gamma^+) - n_2^{\chi_1}(\gamma^-,\gamma^+)$.
$\langle 1\rangle 4.$ Q.E.D.

Hence, we obtain in particular the invariance of the induced map on the level of homology.

Corollary 2.4 *For any two asymptotically constant regular paths $\chi_0, \chi_1 \in P_{reg}(H^-, J^-, H^+, J^+)$ we have that $\Phi_{\chi_0} = \Phi_{\chi_1}$.*

Finally, we turn towards the functoriality properties which will give us our desired isomorphism. For this, let $(H^\pm, J^\pm), (H^0, J^0) \in \mathcal{HJ}_{reg}$, $\chi_0 \in P_{reg}(H^-, J^-, H^0, J^0)$ and $\chi_1 \in P_{reg}(H^0, J^0, H^+, J^+)$. Furthermore, let $\chi \in P_{reg}(H^-, J^-, H^+, J^+)$. Then our goal is to show that

$$\Phi_\chi = \Phi_{\chi_1} \circ \Phi_{\chi_0}. \tag{2.55}$$

If we had an equality on the chain level of the form $\Phi^{\#}_{\chi} = \Phi^{\#}_{\chi_1} \circ \Phi^{\#}_{\chi_0}$ that would obviously be sufficient. However, in general we cannot hope for such a result. Instead, we can construct a particular path $\chi_2 \in P_{reg}(H^-, J^-, H^+, J^+)$ for which equality does hold on the chain level. Then Equation 2.55 follows by observing that on the level of homology $\Phi_{\chi} = \Phi_{\chi_2}$ according to Corollary 2.4.

This identity on the level of chains amounts to showing the equality

$$n_2^{\chi_2}(\gamma^-, \gamma^+) = \sum_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^+}) \\ \mu_{CZ}(\gamma^0) = \mu_{CZ}(\gamma^+)}} n_2^{\chi_0}(\gamma^-, \gamma^0) n_2^{\chi_1}(\gamma^0, \gamma^+)$$

for every $\gamma^- \in \mathrm{Crit}(\mathcal{A}_{H^-})$ and $\gamma^+ \in \mathrm{Crit}(\mathcal{A}_{H^+})$ with $\mu_{CZ}(\gamma^-) = \mu_{CZ}(\gamma^+)$. To do this we will show that the sets $\mathcal{N}(\gamma^-, \gamma^+, \chi_2)$ and

$$\bigcup_{\substack{\gamma^0 \in \mathrm{Crit}(\mathcal{A}_{H^+}) \\ \mu_{CZ}(\gamma^0) = \mu_{CZ}(\gamma^+)}} \mathcal{N}(\gamma^-, \gamma^0, \chi_0) \times \mathcal{N}(\gamma^0, \gamma^+, \chi_1)$$

have the same number of elements, or in other words are bijective.

The particular path χ_2 is obtained by concatenating the paths χ_0 and χ_1 followed by a small perturbation if necessary, to ensure that it is still regular.

We thus conclude with stating the following proposition which completes the proof of the invariance of the Floer homology groups.

Proposition 2.36 *There exists a regular asymptotically constant path χ_2 such that the morphisms $\Phi^{\#}_{\chi_2}$ and $\Phi^{\#}_{\chi_1} \circ \Phi^{\#}_{\chi_0}$ coincide on the chain level.*

2.11 Computing the Floer Homology Groups

2.11.1 Morse Theory Summary

In this section we very briefly summarise the definitions and result from Morse theory that are needed to define the Morse homology groups. This material is assumed to be known and consequently all proofs will be omitted. They can be found for example in [16] or the first part of [2]. Purpose of including it here is merely to provide the necessary context for the next section where we will show that for suitable choices of Hamiltonian and compatible almost complex structure the Floer and Morse homology groups coincide. This will be the last crucial step to finish the proof of the Arnold conjecture.

Definition 2.42 *Let $x \in M$ and $f \in C^\infty(M)$. Then x is called a* critical point *of f if $df(x) = 0$. The set of all critical points of f is denoted by* $\mathrm{Crit}(f)$.

Definition 2.43 *Let $f \in C^\infty(M)$ and $x \in \mathrm{Crit}(f)$. The Hessian* $\mathrm{Hess}_x(f)$ *of f at x is defined to be the bilinear form*

$$\mathrm{Hess}_x(f)(\xi, \eta) = \xi(\tilde{\eta} f)(x)$$

where $\tilde{\eta}$ is an extension of η around the point x. If $\mathrm{Hess}_x(f)$ *is non-degenerate as a bilinear form on T_xM then x is called* non-degenerate.

Definition 2.44 *Let $f \in C^\infty(M)$. Then f is called a* Morse function *if every $x \in \mathrm{Crit}(f)$ is non-degenerate.*

In the neighbourhood of a non-degenerate critical point a manifold admits a special kind of coordinate chart in which the function f coincide with its second order expansion.

Theorem 2.35 *Let $f \in C^\infty(M)$ and $x \in \mathrm{Crit}(f)$ a non-degenerate critical point. Then there exists a coordinate chart (U, φ) with $x \in U \in \mathsf{Open}(M)$ and $\varphi \in \mathsf{Iso}(U, \mathbb{R}^n)$ such that $\varphi(x) = 0$ such that*

$$f \circ \varphi^{-1}(x_1, \ldots, x_n) = f(x) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2.$$

Such a coordinate chart is called a Morse chart.

Definition 2.45 *Let $x \in \mathrm{Crit}(f)$ a non-degenerate critical point. Then its* Morse index *is defined to be the dimension of largest subspace of T_xM on which* $\mathrm{Hess}_x(f)$ *is negative-definite. It is denoted by* $\mathrm{Ind}(x)$.

The Morse complex is the free vector space over $\mathbb{Z}_2$ generated by the critical points of a Morse function f. It is graded over $\mathbb{Z}$ by the Morse index.

Definition 2.46

$$CM_k(f) = \{ \sum_{\substack{x \in \mathrm{Crit}(f) \\ \mathrm{Ind}(x) = k}} \lambda_x x : \ \lambda_x \in \mathbb{Z}_2 \}$$

To define the boundary operator that will turn the vector spaces $CM_k(f)$ into a chain complex we want to count the number of certain flow lines connecting two critical points whose indices differ by 1. We begin by introducing the vector fields that are suitable to define the corresponding flow equation.

Definition 2.47 *Let $f \in C^\infty(M)$ be a Morse function and $\xi \in \Gamma(TM)$. Then ξ is called a* pseudo-gradient field adapted to f *if*

1. *$\forall x \in M : \ df(x)(\xi(x)) \leq 0$ and $df(x)(\xi(x)) = 0 \iff x \in \mathrm{Crit}(f)$.*
2. *For every $x \in \mathrm{Crit}(f)$ there is Morse chart such that in this chart $\xi = \mathrm{grad}\, f$.*

One special case of the above definition is when ξ is actually the gradient with respect to some fixed Riemannian metric of f.

Definition 2.48 *Let $f \in C^\infty(M)$ be a Morse function and $x \in \mathrm{Crit}(f)$. Let $\psi \in C^\infty(\mathbb{R} \times M)$ be the flow generated by a pseudo-gradient vector field adapted to f. Then we define the associated stable and unstable manifolds to be*

$$W^s(x) = \{y \in M : \lim_{s\to\infty} \psi^s(y) = x\}$$

and

$$W^u(x) = \{y \in M : \lim_{s\to-\infty} \psi^s(y) = x\}$$

respectively.

One can show that the stable and unstable manifolds are indeed manifolds.

Proposition 2.37 *The sets $W^s(x)$ and $W^u(x)$ are submanifolds of M that are diffeomorphic to $B_k(0)$ for some k. Moreover, we have $\dim W^u(x) = \operatorname{codim} W^s(x) = \mathrm{Ind}(x)$ for every $x \in \mathrm{Crit}(f)$.*

Definition 2.49 *Let $f \in C^\infty(M)$ be a Morse function and $\xi \in \Gamma(TM)$ a pseudo-gradient field adapted to f. Then we say that ξ satisfies the* Smale condition *if for all $x^-, x^+ \in \mathrm{Crit}(f)$ the manifolds $W^s(x^-)$ and $W^u(x^+)$ intersect transversally. In this case the pair (f, ξ) is called a* Morse-Smale pair*.*

Definition 2.50

$$\mathcal{M}(x^-, x^+, f, \xi) = \{y \in M : \lim_{s\to-\infty} \psi_s(y) = x^- \text{ and } \lim_{s\to\infty} \psi_s(y) = x^+\}.$$

The space $\mathcal{M}(x^-, x^+, f, \xi)$ carries a free $\mathbb{R}$-action given by translating along the s-variable. As in the Floer case we denote this action by sh. Hence, it makes sense to consider the quotient under this action.

Definition 2.51

$$\underline{\mathcal{M}}(x^-, x^+, f, \xi) = \mathcal{M}(x^-, x^+, f, \xi) / \operatorname{sh}.$$

Theorem 2.36 *If $(f\,xi)$ is a Morse-Smale pair then $\mathcal{M}(x^-, x^+, f, \xi)$ is a smooth manifold of dimension $\mathrm{Ind}(x^-) - \mathrm{Ind}(x^+)$. Consequently, $\underline{\mathcal{M}}(x^-, x^+, f, \xi)$ is a smooth manifold of dimension $\mathrm{Ind}(x^-) - \mathrm{Ind}(x^+) - 1$.*

Definition 2.52 *Let $x \in CM_k(f)$. Then we define the boundary operator $\partial : CM_k(f) \to CM_{k-1}(f)$ to be*

$$\partial(x) = \sum_{y \in CM_{k-1}(f)} n_2(x, y) y.$$

One can then show that $(CM_*(f), \partial)$ is indeed a chain complex, i.e. that $\partial^2 = 0$. Thus, we can define the corresponding Morse homology groups.

Definition 2.53 *Let (f, ξ) be a Morse-Smale pair on M. Then we define*

$$HM_*(f, \xi) = \frac{\text{Ker}\,(\partial)}{\text{Im}\,(\partial)}.$$

These homology groups do not depend on the choice of Morse function and pseudo-gradient field. Hence, they are an invariant of the manifold. For compact, orientable manifolds Morse homology and singular homology coincide. Since symplectic manifolds are in particular because ω^n defines a nowhere vanishing section of the top exterior algebra this applies in to our case.

Theorem 2.37 *Let (f, ξ) be a Morse-Smale pair on $M \in \mathsf{Man}^{\infty}_{cpt}$ where M is orientable. Then $H_*(M) = HM_*(f, \xi)$.*

2.11.2 Linking Floer Homology to Morse Homology

In this final section we show that Floer homology and Morse homology coincide given our assumptions on the underlying symplectic manifold (M, ω). This is accomplished by choosing a particular time-independent Hamiltonian and compatible almost complex structure. These have to be chosen so that both Floer homology and Morse homology are defined and are isomorphic. In fact, we will see that there is an equality on the level chain complexes already. Let us elaborate how this works.

The Floer chain complex $(CF_*(H, J), \partial)$ is generated by the critical points of the associated action functional $\mathcal{A}_H$ on the loop space. First we show that for a Hamiltonian that is sufficiently small in the C^2-topology all 1-periodic orbits are actually constant. This means that they coincide with the critical points of the Hamiltonian which shows that the total vector spaces $CF_*(H, J)$ and $CM_*(H)$ are the same.

Proposition 2.38 *There exists an $\varepsilon > 0$ such that if $H \in C^\infty(M)$ has the property that $\|H\|_{C^2(M)} < \varepsilon$ then every $\gamma \in \text{Crit}(\mathcal{A}_H)$ is constant.*

Proof:

⟨1⟩1. There is an $\varepsilon > 0$ such that for every $\gamma \in \text{Crit}(\mathcal{A}_H)$ there exists a Darboux chart $(U_\gamma, \varphi_\gamma)$ such that
   1. γ is contained in this chart
   2. in each chart $\|DX_H\| < 2\pi$ and
   3. the total number of these charts is finite.

⟨1⟩2. Suffices Assume: $H \in C^\infty(\mathbb{R}^{2n})$ and $M = \mathbb{R}^{2n}$.

⟨1⟩3. If $\|DX_H\| < 2\pi$ then every $\gamma \in \text{Crit}(\mathcal{A}_H)$ is constant.

PROOF:
⟨2⟩1. LET: $\gamma \in \mathrm{Crit}(\mathcal{A}_H)$.
⟨2⟩2. LET: 1. $\gamma(t) = \sum_{n\in\mathbb{Z}} c_n e^{2in\pi t}$
2. $\gamma'(t) = \sum_{n\in\mathbb{Z}} 2in\pi c_n e^{2in\pi t}$
3. $\gamma''(t) = -\sum_{n\in\mathbb{Z}} 4n^2\pi^2 c_n e^{2in\pi t}$
be the Fourier expansions of γ, γ' and γ''.
⟨2⟩3. $\|\gamma''\|^2_{L^2} \geq 4\pi^2\|\gamma'\|^2_{L^2}$.
PROOF:
⟨3⟩1. $\|\gamma''\|^2_{L^2} = \sum_{n\neq 0} 16n^4\pi^4|c_n|^2$.
⟨3⟩2. $\|\gamma'\|^2_{L^2} \sum_{n\neq 0} 4n^2\pi^2|c_n|^2$.
⟨2⟩4. If $\gamma' \neq 0$ then $\|\gamma''\|_{L^2} < 2\pi\|\gamma'\|_{L^2}$.
PROOF:
⟨3⟩1. $\gamma'' = DX_H\gamma'$.
⟨3⟩2. $\|\gamma''\|_{L^2} < 2\pi\|\gamma'\|_{L^2}$.
⟨2⟩5. Q.E.D.
PROOF:
⟨3⟩1. ASSUME: $\gamma \neq 0$.
⟨3⟩2. $2\pi\|\gamma'\|_{L^2} > \|\gamma''\|_{L^2} \geq 2\pi\|\gamma'\|_{L^2}$ which is a contradiction.

Next we need to consider the grading. Critical orbits are graded by the Conley-Zehnder which is defined by considering the Maslov index of the path of symplectic matrices obtained by trivialising the pullback of the tangent bundle over a disk whose boundary is the orbit. This path is generated by the Hessian of H according to

$$\exp(tJ_0 \,\mathrm{Hess}\, H).$$

Since we take H to be a Morse function it follows that $\mathrm{Hess}_x H$ has no eigenvalues in $2\pi\mathbb{Z}$ for $x \in \mathrm{Crit}(H)$. In particular $\|\mathrm{Hess}_x H\| < 2\pi$ so that Theorem 2.3 gives us the following relationship between the Conley-Zehnder index and the Morse index:

$$\mu_{CZ}(\gamma) = \mathrm{Ind}(\gamma) - n.$$

Thus we can conclude the two graded vector spaces $CM_*(H)$ and $CF_*(H, J)$ coincide up to a shift of n in the grading.

The other part we need to consider is the boundary operator ∂ for the two complexes. The Floer boundary operator is defined by counting the unparameterised solutions to the equation $\overline{\partial}_{H,J}(u) = 0$ that connect two critical points whose indices differ by 1. We want to show that these flow lines are the same as the flow lines defined by the pseudo-gradient flow equation that is used to define the boundary operator for Morse homology. For this one needs to choose a pseudo-gradient field X adapted to H that satisfies the Smale condition. The relevant flow equation for the Morse setting is given by

$$\frac{d}{ds}u + X(u) = 0. \tag{2.56}$$

Contrast this with the Floer equation

$$\partial_s u + J(u)\partial_t u + \operatorname{grad} H(u) = 0. \tag{2.57}$$

To relate solutions of these two equations to each other we want to choose X such that it is the gradient of H with respect to the metric defined by a compatible almost complex structure $J \in \mathcal{J}(M, \omega)$. Specifically this means that X will be given by $-JX_H$. Moreover, we will show that solutions to Equation 2.57 are actually independent of t for sufficiently small H so that indeed the equations Equation 2.56 and Equation 2.57 coincide.

We are restricted in our choice of J. In order for Morse homology to be defined the vector field X needs to satisfy the Smale condition. On the other hand, Floer homology is only defined when our chosen J leads to transversality of $\bar{\partial}_{H,J}$. The flow equation Equation 2.56 defines an operator $C^\infty(\mathbb{R}, M) \to C^\infty(\mathbb{R}, TM)$.

Definition 2.54 *We define*

$$\bar{\partial}_X(u) = \frac{d}{ds}u + X(u)$$

to be this associated operator.

We want to express the Morse-Smale condition in terms of $\bar{\partial}_X$. For this we need to study its linearisation $D\bar{\partial}_X(u)$ along a solution. As in the Floer case the analysis of this operator is simplified by working in coordinates.

$$D\bar{\partial}_X(u)(Y) = \frac{d}{ds}Y + AY$$

where A has the property that $\lim_{s\to\pm\infty} A(s) = \operatorname{Hess}_{x^\pm} H$.

Definition 2.55

$$\mathcal{M}(f, X) = \{Y \in C^\infty(\mathbb{R}, TM) : \; D\bar{\partial}_X(u)(Y) = 0\}.$$

Proposition 2.39 *Assume that H is a Morse function and let $x^\pm \in \operatorname{Crit}(\mathcal{A}_H)$. Moreover, let $u \in C^\infty(\mathbb{R}, M)$ be such that $D\bar{\partial}_X(u) = 0$ and $\lim_{s\to\pm\infty} u(s) = x^\pm$. Then $D\bar{\partial}_X(u)$ is a Fredholm operator.*

Proof:
$\langle 1 \rangle$1. $\exists T > 0 : \; \exists C > 0 : \; \forall Y \in \mathcal{M}(H, X) :$

$$\|Y\|_{W^{1,2}(\mathbb{R},\mathbb{R}^{2n})} \le C(\|D\bar{\partial}_X(u)(Y)\|_{L^2(\mathbb{R},\mathbb{R}^{2n})} + \|Y\|_{L^2([-T,T],\mathbb{R}^{2n})}).$$

Proof:

⟨2⟩1.
⟨1⟩2. $\forall B \in \mathrm{Sym}(\mathbb{R}^{2n}) \cap \mathrm{GL}(2n) : \exists C > 0 : \forall Y \in W^{1,2}(\mathbb{R}, \mathbb{R}^{2n}) :$

$$\|Y\|_{W^{1,2}(\mathbb{R},\mathbb{R}^{2n})} \leq C \|\frac{d}{ds}Y + BY\|_{L^2(\mathbb{R},\mathbb{R}^{2n})}.$$

Proof:
⟨2⟩1. Let: $Z = \frac{d}{ds}Y + BY$.
⟨2⟩2. Let: $\widehat{Y}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi s}Y(s)ds$ be the Fourier transform of Y.
⟨2⟩3. $\forall \xi \in \mathbb{R} : \widehat{Z}(\xi) = (i\xi + B)\widehat{Y}(\xi)$.
⟨2⟩4. $\|Z\|_{L^2(\mathbb{R},\mathbb{R}^{2n})} = \|\widehat{Z}\|_{L^2(\mathbb{R},\mathbb{R}^{2n})}$ by the Plancherel equality.
⟨2⟩5.
⟨1⟩3. $\exists C > 0 :$

$$\int_{-M}^{M} (\|Y\|^2 + \|\frac{d}{ds}Y\|^2)ds \leq C \int_{-M}^{M} (\|Y\|^2 + \|\frac{d}{ds}Y + AY\|^2)ds.$$

Proof:
⟨2⟩1. $\int_{-M}^{M}(\|\frac{d}{ds}Y + AY\|^2)ds \geq \int_{-M}^{M}(\frac{1}{2}\|\frac{d}{ds}Y\|^2 - \|AY\|^2)ds$.
⟨2⟩2. $\int_{-M}^{M}(\frac{1}{2}\|\frac{d}{ds}Y\|^2)ds \leq C\int_{-M}^{M}\|Y\|^2 ds + \int_{-M}^{M}\|D\overline{\partial}_X(u)Y\|^2 ds$.
⟨2⟩3. $\int_{-M}^{M}\|AY\|^2 ds \leq C\int_{-M}^{M}\|Y\|^2 ds$.
⟨2⟩4.
⟨1⟩4. $\mathrm{Ker}\left(D\overline{\partial}_X(u)\right)$ is finite-dimensional.
Proof:
⟨2⟩1. The inclusion followed by restriction $W^{1,2}(\mathbb{R}, \mathbb{R}^{2n}) \to L^2(\mathbb{R}, \mathbb{R}^{2n}) \to L^2([-T, T], \mathbb{R}^{2n})$ is compact by Rellich's embedding theorem.
⟨1⟩5. $\mathrm{Im}\left(D\overline{\partial}_X(u)\right)$ is closed.
Proof:
⟨2⟩1. Same as ⟨1⟩4.
⟨1⟩6. $\mathrm{Coker}\left(D\overline{\partial}_X(u)\right)$ is finite-dimensional.
Proof:
⟨2⟩1. Let: $D_X^*(u) : W^{1,2}(\mathbb{R}, \mathbb{R}^{2n}) \to L^2(\mathbb{R}, \mathbb{R}^{2n})$ be given by

$$D_X^*(u) = -\frac{d}{ds} + A^T.$$

⟨2⟩2. $\mathrm{Coker}\left(D\overline{\partial}_X(u)\right) = \mathrm{Ker}\,(D_X^*)$.
Proof:
⟨3⟩1. $\mathrm{Coker}\left(D\overline{\partial}_X(u)\right) \cong \{Z \in L^2(\mathbb{R}, \mathbb{R}^{2n}) : \forall Y \in W^{1,2}(\mathbb{R}, \mathbb{R}^{2n}) : \int_{-\infty}^{\infty}\langle D\overline{\partial}_X(u)Y, Z\rangle = 0\}$.
⟨3⟩2. $\mathrm{Coker}\left(D\overline{\partial}_X(u)\right) \subset \mathrm{Ker}\,(D_X^*)$.
Proof:
⟨4⟩1. Let: $Z \in \mathrm{Coker}\left(D\overline{\partial}_X(u)\right)$.
⟨4⟩2. $D_X^*(u)(Z) = 0$ in the sense of distributions.
⟨4⟩3. $\int_{-\infty}^{\infty}\langle D\overline{\partial}_X(u)(Y), Z\rangle = -\int_{-\infty}^{\infty}(\langle \frac{d}{ds}Z, Y\rangle - \langle A^T Z, Y\rangle)ds$.
⟨4⟩4. $Z \in \mathrm{Ker}\,(D_X^*)$.
⟨3⟩3. $\mathrm{Coker}\left(D\overline{\partial}_X(u)\right) \supset \mathrm{Ker}\,(D_X^*)$.

$\langle 1\rangle 7.$ Q.E.D.

Having verified the Fredholm property we now want to compute the index of $D\bar{\partial}_X(u)$.

Proposition 2.40 *The index of the Fredholm operator $D\bar{\partial}_X(u)$ is given by* $\mathrm{Ind}(D\bar{\partial}_X(u)) = \mathrm{Ind}(x^-) - \mathrm{Ind}(x^+)$.

PROOF:

$\langle 1\rangle 1.$ LET: $\Phi_{(\sigma,s)} : T_{u(\sigma)}M \to T_{u(s)}M$ be the map sending $Y \in T_{u(\sigma)}M$ to the value of s of the unique solution Z of $D\bar{\partial}_X(u)(Z) = 0$ that satisfies $Z(\sigma) = Y$ where $\sigma, s \in \mathbb{R}$.

$\langle 1\rangle 2.$ LET: 1. $E^u(\sigma) = \{Y \in T_{u(\sigma)}M : \lim_{s\to-\infty} \Phi_{(\sigma,s)}Y = 0\}$ and
2. $E^s(\sigma) = \{Y \in T_{u(\sigma)}M : \lim_{s\to\infty} \Phi_{(\sigma,s)}Y = 0\}$.

$\langle 1\rangle 3.$ $\forall\sigma \in \mathbb{R} :\ E^u(\sigma) \cong T_{u(\sigma)}W^u(x^-)$ and $E^s(\sigma) \cong T_{u(\sigma)}W^s(x^+)$.

$\langle 1\rangle 4.$ $\forall\sigma \in \mathbb{R} :\ \mathrm{Ker}\left(D\bar{\partial}_X(u)\right) \cong T_{u(\sigma)}W^u(x^-) \cap T_{u(\sigma)}W^s(x^+)$.

$\langle 1\rangle 5.$ $\forall\sigma \in \mathbb{R} :\ \mathrm{Coker}\left(D\bar{\partial}_X(u)\right) \cong \mathrm{Ker}\left(D\bar{\partial}^*_X(u)\right) \cong (T_{u(\sigma)}W^u(x^-) \cap T_{u(\sigma)}W^s(x^+))^\perp$.

PROOF:

$\langle 2\rangle 1.$ LET: $\Psi_{(\sigma,s)} : T_{u(\sigma)}M \to T_{u(s)}M$ be the map sending $Z \in T_{u(\sigma)}M$ to the value of s of the unique solution Y of $D\bar{\partial}^*_X(u)(Y) = 0$ that satisfies $Y(\sigma) = Z$ where $\sigma, s \in \mathbb{R}$.

$\langle 2\rangle 2.$ $\Psi_{(\sigma,s)} = \Phi^*_{(\sigma,s)}$.

PROOF:

$\langle 3\rangle 1.$ LET: $Z, Y \in \mathbb{R}^{2n}$ and$Z(s) = \Psi_{(\sigma,s)}$.

$\langle 3\rangle 2.$ $\langle \Psi_{(\sigma,s)}Z, \Phi_{(\sigma,s)}Y\rangle = \langle Z, Y\rangle$.

$\langle 3\rangle 3.$ $\langle \frac{d}{ds}Z - A^T Z, Y\rangle = 0$.

PROOF:

$\langle 4\rangle 1.$ $\langle \frac{d}{ds}, \Psi_{(\sigma,s)}Y\rangle + \langle \Psi_{(\sigma,s)}Z, \frac{d}{ds}Y\rangle$ by differentiating $\langle 3\rangle 2$.

$\langle 4\rangle 2.$ Now use $Y(s) = \Phi_{(\sigma,s)}Y$ and $\frac{d}{ds}Y = -AY$.

$\langle 3\rangle 4.$ $D\bar{\partial}^*_X(u)(Z) = 0$ which proves the statement.

$\langle 2\rangle 3.$ $D\bar{\partial}^*_X(u)(Y) = 0 \iff Y(\sigma) \perp E^u(\sigma)$ and $Y(\sigma) \perp E^s(\sigma)$.

$\langle 1\rangle 6.$ $\mathrm{Ind}(D\bar{\partial}_X(u)) = \mathrm{Ind}(x^-) - \mathrm{Ind}(x^+)$.

PROOF:

$\langle 2\rangle 1.$ $\dim\mathrm{Ker}\left(D\bar{\partial}_X(u)\right) = \dim(T_{u(\sigma)}W^u(x^-) \cap T_{u(\sigma)}W^s(x^+))$
$= \dim W^u(x^-) + \dim W^s(x^+) - \dim(T_{u(\sigma)}W^u(x^-) + T_{u(\sigma)}W^s(x^+))$.

$\langle 2\rangle 2.$ $\dim\mathrm{Ker}\left(D\bar{\partial}^*_X(u)\right) = \dim(T_{u(\sigma)}W^u(x^-) \cap T_{u(\sigma)}W^s(x^+))^\perp$
$= 2n - \dim(T_{u(\sigma)}W^u(x^-) + T_{u(\sigma)}W^s(x^+))^\perp$.

$\langle 2\rangle 3.$ $\mathrm{Ind}(D\bar{\partial}_X(u)) = \dim\mathrm{Ker}\left(D\bar{\partial}_X(u)\right) - \dim\mathrm{Ker}\left(D\bar{\partial}^*_X(u)\right)$
$= \dim W^u(x^-) + \dim W^s(x^+) - 2n = \mathrm{Ind}(x^-) + 2n - \mathrm{Ind}(x^+) - 2n$.

$\langle 1\rangle 7.$ Q.E.D.

As mentioned above we now reformulate the Morse-Smale condition by showing that it is equivalent to surjectivity of $D\bar{\partial}_X(u)$.

Proposition 2.41 *The pair (H, X) is Morse-Smale if and only if $D\bar{\partial}_X(u)$ is surjective for every $u \in \mathcal{M}(H, X)$.*

PROOF:
⟨1⟩1. $D\bar{\partial}_X(u)$ is surjective if and only if $D\bar{\partial}_X^*(u)$ is injective.
⟨1⟩2. $D\bar{\partial}_X^*(u)$ is injective if and only if $T_{u(\sigma)}W^u(x^-) + T_{u(\sigma)}W^s(x^+) = T_{u(\sigma)}M$ which is exactly the transversality condition needed for the Smale property.
⟨1⟩3. Q.E.D.

Thus, whenever we can choose J so that for $X = -JX_H$ the operator $D\bar{\partial}_X$ is surjective at every solution u Morse homology is well-defined. Now we need to look on the Floer side of things. In order for Floer homology to be defined we need $D\bar{\partial}_{H,J}$ to be surjective at every solution of the Floer equation. For this we first have the following result.

Proposition 2.42 *If H is sufficiently small in the $C^2(M)$-norm then* $\operatorname{Ker}\left(D\bar{\partial}_{H,J}(u)\right) = \operatorname{Ker}\left(D\bar{\partial}_X(u)\right)$.

PROOF:
⟨1⟩1. $\operatorname{Ker}\left(D\bar{\partial}_{H,J}(u)\right) \supset \operatorname{Ker}\left(D\bar{\partial}_X(u)\right)$.
  PROOF:
  ⟨2⟩1. Elements in $\operatorname{Ker}\left(D\bar{\partial}_X(u)\right)$ are t-independent.
⟨1⟩2. $\operatorname{Ker}\left(D\bar{\partial}_{H,J}(u)\right) \subset \operatorname{Ker}\left(D\bar{\partial}_X(u)\right)$.
  PROOF:
  ⟨2⟩1. LET: $\xi \in \operatorname{Ker}\left(D\bar{\partial}_{H,J}(u)\right)$.
  ⟨2⟩2. $\int_0^1 J(u)\partial_t\xi dt = 0$ since ξ is 1-periodic in t.
  ⟨2⟩3. $\int_0^1 \xi(s,t)dt \in \operatorname{Ker}\left(D\bar{\partial}_X(u)\right)$ using ⟨2⟩2 and rearranging Equation 2.57.
  ⟨2⟩4. SUFFICES ASSUME: $\int_0^1 \xi(s,t)dt = 0$.
  ⟨2⟩5. $\int_{-\infty}^{\infty}\int_0^1 \|\xi(s,t)\|^2 dtds \le \int_{-\infty}^{\infty} \|\partial_t\xi(s,t)\|^2 dtds$.
    PROOF:
    ⟨3⟩1. $\forall t_1, t_0 \in [0,1] : \ \xi(s,t_1) - \xi(s,t_0) = \int_{t_0}^{t_1} \partial_t\xi(s,t)dt$.
    ⟨3⟩2. $\xi(s,t_1) = \int_0^1 \int_{t_0}^{t_1} \partial_t\xi(s,t)dtdt_0$ since $\int_0^1 \xi(s,t)dt = 0$.
    ⟨3⟩3. $\int_0^1 \|\xi(s,t_1)\|^2 dt_1 \le \int_0^1\int_0^1\int_{t_0}^{t_1} \|\partial_t\xi(s,t)\|^2 dtdt_0dt_1 \le \int_0^1 \|\partial_t\xi(s,t)\|^2 dt$.
    ⟨3⟩4. Now integrate with respect to s over $\mathbb{R}$.
  ⟨2⟩6. $\|\operatorname{grad}\xi\|^2_{L^2(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \le \sup_{s\in\mathbb{R}} \|S(s)\|^2 \|\xi\|^2_{L^2(\mathbb{R}\times S^1,\mathbb{R}^{2n})}$.
    PROOF:
    ⟨3⟩1. $\Delta = (\partial_s - J\partial_t)(\partial_s + J\partial_t)$.
    ⟨3⟩2. $\|\partial_s\xi\|^2_{L^2(\mathbb{R}\times S^1,\mathbb{R}^{2n})} + \|\partial_t\xi\|^2_{L^2(\mathbb{R}\times S^1,\mathbb{R}^{2n})} = -\langle\xi, \Delta\xi\rangle$.

⟨3⟩3. $-\langle \xi, \Delta\xi \rangle = -\langle \xi, (\partial_s - J\partial_t)(\partial_s + J\partial_t)\xi \rangle = \|\partial_s\xi + J\partial_t\xi\|^2_{L^2(\mathbb{R},\mathbb{R}^{2n})}$.
⟨3⟩4. $\|\partial_s\xi + J\partial_t\xi\|^2_{L^2(\mathbb{R},\mathbb{R}^{2n})} = \|S(s)\xi\|^2_{L^2(\mathbb{R},\mathbb{R}^{2n})} \le \sup_{s\in\mathbb{R}}\|S(s)\|^2\|\xi\|^2_{L^2(\mathbb{R},\mathbb{R}^{2n})}$.
⟨2⟩7. $\xi = 0$.
Proof:
⟨3⟩1. $\exists \varepsilon > 0 : \ \|H\|_{C^2(M)} < \varepsilon \implies \sup_{s\in\mathbb{R}}\|S(s)\|^2 < 1$.
⟨3⟩2. Now combine ⟨2⟩5 and ⟨2⟩6.
⟨1⟩3. Q.E.D.

This implies in particular that $D\bar{\partial}_{H,J}$ is surjective along every flow line of the Morse equation if X satisfies the Smale condition because the Fredholm indices and kernels of $D\bar{\partial}_{H,J}$ and $D\bar{\partial}_X$ agree.

Proposition 2.43 *If H is a Morse function and $X = -JX_H$ has the Smale property then for a sufficiently large $k \in \mathbb{N}$ the elements of $\mathcal{M}(H_k, J, \gamma^-, \gamma^+)$ are independent of t where $H_k = H/k$ and $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) \le 2$.*

Proof:
⟨1⟩1. Assume: this is false.
⟨1⟩2. $\exists (n_k)_{k\in\mathbb{N}} \subset \mathbb{N} : \ \lim_{k\to\infty} n_k = \infty$ and $\exists (u_{n_k})_{k\in\mathbb{N}} : \ u_{n_k} \in \mathcal{M}(H_{n_k}, J, \gamma^-, \gamma^+)$.
⟨1⟩3. Let: $v_{n_k}(s,t) = u_{n_k}(n_k s, n_k t) \in \mathcal{M}(H, J, \gamma^-, \gamma^+)$.
⟨1⟩4. Case: $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) = 1$.
Proof:
⟨2⟩1. Suffices Assume: $\exists v \in \mathcal{M}(H, J, \gamma^-, \gamma^+) : \ \lim_{k\to\infty} v_{n_k} = v$ where the v_{n_k} do not depend on t.
⟨2⟩2. v is t-independent.
Proof:
⟨3⟩1. Let: $r, s \in \mathbb{R}$ and $t \in [0,1]$.
⟨3⟩2. $v_{n_k}(s,t) = v_{n_k}(s, t + \frac{[rn_k]}{n_k})$ since v_{n_k} is $1/n_k$-periodic in t.
⟨3⟩3. $v(s,t) = v(s, t+r)$.
Proof:
⟨4⟩1. Let $k \to \infty$ in ⟨3⟩2.
⟨2⟩3. v_{n_k} is t-independent for sufficiently large k which contradicts our assumption.
Proof:
⟨3⟩1. Since $D\bar{\partial}_{H,J}(u)$ is surjective for sufficiently large k we would have $v_{n_k}(s,t) = v(s+\sigma_k, t) = v(s+\sigma_k)$ where the sequence $(\sigma_k)_{k\in\mathbb{N}}$ is obtained from the result about the convergence of broken trajectories.
⟨1⟩5. Case: $\mu_{CZ}(\gamma^-) - \mu_{CZ}(\gamma^+) = 2$.
Proof:
⟨2⟩1. The proof of this case is omitted because it depends on the details of the proof of the existence of almost complex compatible structure J for which $(H, -JX_H)$ is Morse-Smale as in Theorem 2.39.
⟨1⟩6. Q.E.D.

We still need to verify that we can actually find almost complex structures J for which the operators $D\bar{\partial}_X(u)$ are surjective so that consequently both Morse and Floer homology are defined and coincide for a sufficiently small t-independent Hamiltonian H. This is done in the following theorem.

Theorem 2.38 *Let* $(M,\omega) \in \mathsf{SympMan}^\infty$ *and* $H \in C^\infty(M)$ *a Morse function. Then there exists a dense set of* $\mathcal{J}(M,\omega)$ *of almost complex structures* J *compatible with* ω *such that* $(H, -JX_H)$ *is Morse-Smale.*

We do not discuss the proof of this result further and refer to section 10.3 of [2].

With this we can now finally finish the proof of the Arnold conjecture as announced at the beginning of this chapter.

Theorem 2.39 *Let* $(M,\omega) \in \mathsf{SympMan}^\infty_{cpt}$ *be a closed, symplectically aspherical manifold. Assume that* $H \in C^\infty(S^1 \times M)$ *is non-degenerate. Then the number of* 1*-periodic orbits of the associated Hamiltonian flow* ψ *satisfies the inequality*

$$|fix(\psi_1)| \geq \sum_k \dim H_k(M, \mathbb{Z}_2).$$

Proof:
$\langle 1 \rangle 1.$ $|fix(\psi_1)| = \dim CF_*(H,J) \geq \dim HF_*(H,J) = \sum_k \dim HF_k(H,J)$.
$\langle 1 \rangle 2.$ $HF_k(H,J) \cong HM_{n+k}(H, -JX_H)$.
$\langle 1 \rangle 3.$ $HM_{n+k}(H,J) \cong H_{n+k}(M, \mathbb{Z}_2)$.
$\langle 1 \rangle 4.$ Q.E.D.

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