Lagrangian Submanifolds of Symplectic Toric Manifolds

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Abstract

Symplectic Geometry is a branch of Mathematics that developed from the Hamiltonian formulation of Classical Mechanics in Physics. Conservation principles, *e.g.* conservation of energy, is generalized to the concept of a moment map $\mu : M \to \mathfrak{g}^*$ on a manifold M.

In this thesis, we study the connections between affine subspaces in the image of the moment map and certain lagrangian subsets of the complex toric manifold $(\mathbb{C}^2, \omega_0, \mathbb{T}^n, \mu)$. More precisely, we ask wether a lagrangian can be immersed or embedded into the preimage of a given affine subspace in the moment image.

Here, we show the existence of embedded lagrangians for so-called Delzant affine spaced and of immersed lagrangians for affine subspaces with rational slope. Furthermore, we formulate a conjecture that claims the existence of an embedded lagrangian if and only if the corresponding affine subspace is Delzant. Our results demonstrate how properties of $\mathbb{T}^n$ and their closed subgroups can be translated to lagrangians in $(\mathbb{C}^2, \omega_0)$.

Complete knowledge about all lagrangians of a symplectic manifold is enough to understand the manifold ifself. However, the set of lagrangians of a given manifold is very complicated and does not admit a clear classification. The hope of our work is to contribute to the goal of finding a subset of lagrangians on which one is also allowed to know everything about the symplectic manifold itself.

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Contents

1 Introduction

This chapter is devoted to the basics of Symplectic Geometry and provides all needed notions for later results proven in Chapter 3. Furthermore, we fix a notation. At the end we included a notation index in order to specify what we mean by certain symbols. Mainly, we follow the work of Ana Cannas da Silva as in [dS01] and [AdSL00], part 2. As prerequisites we assume basic knowledge of Differential Topology, Differential Geometry, Algebra and Representation Theory. For example [dC92] is a good reference for Differential Topology and Geometry. For the algebraic part any standard textbooks about algebra and Lie groups can be considered.

1.1 Symplectic vector spaces and manifolds

In short, the theory of symplectic manifolds starts on vector spaces and is then moved to manifolds by requiring the same properties on every tangent space, which is indeed a vector space.

Definition 1.1 - Let V be a vector space. A bilinear map $\omega : V \times V \to \mathbb{R}$ is called a **symplectic form** if it is

(a) non-degenerate, that is, for every non-zero vector $v \in V \setminus \{0\}$ there exists another vector $w \in V$ such that $\omega(v, w) \neq 0$ and

(b) skew-symmetric, that is, for all vectors $v, w \in V$ it holds that $\omega(v, w) = -\omega(w, v)$.

The pair (V, ω) is called a **symplectic vector space**.

Lemma 1.2 - *Let (V, ω) be a symplectic vector space. Then, there exists a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that for all $i, j \in \{1, \ldots, n\}$ we have*

$$\omega(e_i, e_j) = \omega(f_i, f_j) = \delta_{ij} \quad \text{and} \quad \omega(e_i, f_j) = 0.$$

*This basis is called a **symplectic basis** of (M, ω).*

Proof - A proof can be found in [dS01] on page 4. ■

Corollary 1.3 - *Every symplectic vector space (V, ω) has an even dimension, i.e. $\dim V = 2n$ for some $n \in \mathbb{N}$.*

Definition 1.4 - Let M be a differentiable manifold and $\omega \in \Omega^2(M)$ a 2-form on M. Then, the pair (M, ω) is called a **sympletic manifold** if ω is closed, that is, $\mathrm{d}\omega = 0$ holds, and if for every $p \in M$ the restriction $\omega_p \in \Omega^2(T_pM)$ is a symplectic form on T_pM. Also, ω is called a **symplectic form** on M.

Theorem 1.5 - *Every symplectic manifold (M, ω) has an even dimension, i.e. $\dim M = 2n$ for some $n \in \mathbb{N}$.*

Proof - This follows immediately from Corrollary 1.3 together with the fact that for every point $p \in M$ we have $\dim M = \dim T_pM = 2n$ for some $n \in \mathbb{N}$. This completes the proof. ■

The above Lemma 1.2 also has its analogue on symplectic manifolds.

Theorem 1.6 (Darboux) **-** *Let (M, ω) be a symplectic manifold of dimension $2n$ and $p \in M$ be a point. Then, there exists a coordinate chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on U we have*

$$\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

*Such a chart is called a **Darboux chart**.*

Proof - A proof can be found in [dS01], Chapter 8, page 46. ■

Example 1.7 - In this thesis there is one main example of a symplectic manifold, which is presented in greater detail. For an $n \in \mathbb{N}$ consider the cartesian product of n copies of the complex line $\mathbb{C}$, *i.e.* $\mathbb{C}^n := \mathbb{C} \times \ldots \times \mathbb{C}$. Now, we define a symplectic form on $\mathbb{C}^n$ and check its properties. For that purpose, we introduce coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$ and define the 2-form

$$\omega_0 := \sum_{i=1}^{n} \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_n \in \Omega^2(\mathbb{C}^n).$$

We check the properties of Definition 1.4:

(a) Clearly ω_0 is closed, since ω_0 is exact with respect to the 1-form $\sum_{i=1}^{n} z_i \,\mathrm{d}\overline{z}_i \in \Omega^1(\mathbb{C}^n)$ and every exact form is automatically closed.

Pick a point $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and the associated basis $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}$ of $T_z\mathbb{C}^n$. Then, it is sufficient to check non-degeneracy and skew-symmetry on this basis by linearity.

(b) For the non-degeneracy let $v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial z_i} + v_i' \frac{\partial}{\partial \overline{z}_i} \in T_z\mathbb{C}^n$ be a non-zero tangent vector at z, with $v_i, v_i' \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$. By plugging into ω_0 we get

$$\iota_v \omega_0 = \omega_0(v, \cdot) = \sum_{i=1}^{n} v_i \mathrm{d}\overline{z}_n - v_i' \mathrm{d}z_i.$$

So by taking the tangent vector $w := \sum_i (-v_i') \frac{\partial}{\partial z_i} + v_i \frac{\partial}{\partial \overline{z}_i}$ we get

$$\omega_0(v, w) = \left(\sum_{i=1}^{n} v_i \mathrm{d}\overline{z}_i - v_i' \mathrm{d}z_i \right)(w) = \sum_{i=1}^{n} v_i^2 + (v_i')^2 > 0,$$

which proves the non-degeneracy of ω_0.

(c) The skew-symmetry follows immediately form the properties of the wedge product by the following computation. Let $v, w \in T_z\mathbb{C}^n$ be two tangent vectors and compute

$$\begin{aligned}\omega_0(v,w) &= \sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i(v,w)\\ &= \sum_{i=1}^n \mathrm{d}z_i(v)\mathrm{d}\overline{z}_i(w) - \mathrm{d}z_i(w)\mathrm{d}\overline{z}_i(v)\\ &= -\sum_{i=1}^n \mathrm{d}z_i(w)\mathrm{d}\overline{z}_i(v) - \mathrm{d}z_i(v)\mathrm{d}\overline{z}_i(w)\\ &= -\sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i(w,v)\\ &= -\omega_0(w,v),\end{aligned}$$

which indeed proofs the skew-symmetry.

The 2-form $\omega_0 \in \Omega^2(\mathbb{C}^n)$ is called the **standard symplectic form** of $\mathbb{C}^n$ and is of central interest in this thesis.

Definition 1.8 - Let (M,ω) and (N,ω') be two symplectic manifolds. A diffeomorphism $\varphi \in \mathrm{Diff}(M,N)$ is called a **symplectomorphism** if $\varphi^*\omega' = \omega$ holds. That is, for every $p \in M$ and for all tangent vectors $u,v \in T_pM$ we have

$$\omega(u,v) = (\varphi^*\omega')_p(u,v) = \omega'_{\varphi(p)}(\mathrm{d}\varphi_p(u), \mathrm{d}\varphi_p(v)).$$

We denote by

$$\mathrm{Symp}(M,\omega) := \{\varphi \in C^\infty(M,M) \mid \varphi^*\omega = \omega\}$$

the set of all **symplectomorphisms** of (M,ω) onto itself.

Theorem 1.9 - *Let (M,ω) be a symplectic manifold. The set of its symplectomorphisms* $\mathrm{Symp}(M,\omega)$ *forms together with the composition of functions a group.*

Proof - A proof can be found in any basic textbook about Symplectic Geometry. ■

1.2 Immersed vs. embedded lagrangians

The existence of a symplectic form ω on a manifold M equips it with a structure that allows us to distinguish certain subspaces. Among all of these, the lagrangians play a central role in Symplectic Geometry. This statement is supplemented by Alan Weinstein's famous creed

everything is a lagrangian submanifold

in [Wei81]. We follow this approach and consider lagrangians as the central objects of this thesis. Therefore, we give a detailed introduction into these spaces. As in the previous section we begin at the level of vector spaces and then lift all the properties to a manifold by consideration at every tangent space.

Definition 1.10 - Let (V, ω) be a symplectic vector space and $W \subseteq V$ be a subspace. The set

$$W^\omega := \left\{ u \in V \,\middle|\, \forall v \in W : \omega(u,v) = 0 \right\}$$

is called the **symplectic orthogonal complement** of W inside of (V, ω).

Definition 1.11 - Let (V, ω) be a symplectic vector space. A subspace $W \subseteq V$ is called

(a) **isotropic** if $\omega\big|_W \equiv 0$ holds, that is, for all vectors $u, v \in W$ we have $\omega(u,v) = 0$, *i.e.* if $W \subset W^\omega$,

(b) **coisotropic** if W^ω is isotropic, *i.e.* $W^\omega \subseteq W$,

(c) **symplectic** if $(W, \omega\big|_W)$ is itself a symplectic vector space,

(d) **lagrangian** if it is isotropic and coisotropic at the same time, *i.e.* if $W = W^\omega$.

For the case of $(\mathbb{C}^n, \omega_0)$ as in Example 1.7 we have the following lemma that helps us to determine if a given subset is lagrangian or not.

Lemma 1.12 - *Let $(\mathbb{C}^n, \omega_0)$ be the standard complex symplectic manifold. A real subspace L of $\mathbb{C}^n$ is lagrangian if and only if $L^\perp = \mathrm{i}\, L$, where $\perp$ is meant with respect to standard euclidean product on $\mathbb{C}^n = \mathbb{R}^{2n}$.*

Proof - Denote by $h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ with $h(z_1, z_2) = z_1^* z_2$ the standard hermitian structure of $\mathbb{C}^n$. By giving real coordinates one can show that

$$h(z_1, z_2) = \langle z_1, z_2 \rangle + \mathrm{i}\, \omega_0(z_1, z_2) \tag{1.1}$$

holds for all $z_1, z_2 \in \mathbb{C}^n$. From this we compute

$$\begin{aligned}\omega(z_1, z_2) = 0 &\Leftrightarrow \operatorname{Im} h(z_1, z_2) = 0 \\ &\Leftrightarrow \operatorname{Re} h(z_1, \mathrm{i}\, z_2) = 0 \\ &\Leftrightarrow \langle z_1, \mathrm{i}\, z_2 \rangle = 0.\end{aligned}$$

This completes the proof. ■

Lemma 1.13 - *Let (V, ω) be a symplectic vector space of dimension $2n$ and $L \subset (V, \omega)$ a lagrangian subspace. Then* $\dim L = \frac{1}{2} \dim V = n$.

Definition 1.14 - Let (M, ω) be a symplectic manifold of dimension $2n$. A subset $L \subset M$ is called an **immersed lagrangian** if there exists an immersion $i : L \to M$ such that $i^*\omega = 0$ and $\dim L = \frac{1}{2} \dim M$. Furthermore, if i also an embedding, we call L an **embedded lagrangian** or a **lagrangian submanifold**.

1.3 Lie groups

As a preparation for the concept of moment maps and the later definition of toric lagrangians we give a short introduction into Lie groups and its subgroups. We follow the introduction given by Dietmar Salamon as in the first parts of [Sal13].

Definition 1.15 - A **Lie group** G is a finite-dimensional differentiable manifold, which admits a group structure and the operations of multiplication and inversion are smooth, *i.e.* the maps $\cdot : G \times G \to G$, $(g, h) \mapsto g \cdot h$ and ${}^{-1} : G \to G$, $g \mapsto g^{-1}$ are smooth.

Definition 1.16 - Let G be a Lie group. A **Lie subgroup** $H \subseteq G$ is a closed subset such that H together with the inversion and multiplication of G is again a Lie group.

Definition 1.17 - Let G be a Lie group and denote by $e \in G$ the identity element. The tangent space T_eG equipped with the standard Lie bracket of vector fields is called the **Lie algebra** of G and is denoted by $\mathfrak{g} := T_eG$.

Since G carries a group structure we can perform left and right multiplication. For a fixed element $g \in G$ we denote these maps by $L_g : G \to G$ and $R_g : G \to G$, respectively. Since these operations are smooth they also admits derivatives

$$(\mathrm{d}L_g)_h : T_hG \longrightarrow T_{gh}G \quad \text{and} \quad (\mathrm{d}R_g)_h : T_hG \longrightarrow T_{hg}G$$

for $g, h \in G$. In particular, if we choose $h = e$ and a tangent vector $v \in \mathfrak{g}$ we have that $(\mathrm{d}L_g)_e(v), (\mathrm{d}R_g)_e(v) \in T_gG$. So, for a fixed tangent vector v in the Lie algebra of G, we can define two vector fields $g \mapsto (\mathrm{d}L_g)_e(v)$ and $g \mapsto (\mathrm{d}R_g)_e(v)$, called **left-invariant** and **right-invariant** vector fields generated by v, respectively. Note that both vector fields agree on $g = e$.

Since $\mathfrak{g}$ is a tangent space it is also a vector space and hence we can define a representation of G on $\mathfrak{g}$ as follows.

Definition 1.18 - Let G be a Lie group with Lie algebra $\mathfrak{g}$. The **adjoint representation** of G onto $\mathfrak{g}$ is defined by

$$\mathrm{Ad}_g := \mathrm{d}(R_{g^{-1}} \circ L_g) : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Let $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ be the natural pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$. We define the **coadjoint representation** $\mathrm{Ad}_g^*(\xi)$ for an element $\xi \in \mathfrak{g}^*$ via

$$\left\langle \mathrm{Ad}_g^*(\xi), X \right\rangle = \left\langle \xi, \mathrm{Ad}_{g^{-1}}(X) \right\rangle$$

for any $X \in \mathfrak{g}$.

Proposition 1.19 - *The map*

$$\begin{array}{rccc} \mathrm{Ad} : & G & \longrightarrow & \mathrm{Aut}(\mathfrak{g}) \\ & g & \longmapsto & \mathrm{Ad}_g \end{array}$$

is a group homomorphism and is called the ***adjoint action*** *of G onto its Lie algebra $\mathfrak{g}$. The map*

$$\begin{array}{rccc} \mathrm{Ad}^* : & G & \longrightarrow & \mathrm{Aut}(\mathfrak{g}^*) \\ & g & \longmapsto & \mathrm{Ad}_g^* \end{array}$$

is called the ***coadjoint action*** *of G onto its dual Lie algebra $\mathfrak{g}^*$ and is also a group homomorphism.*

Proof - A proof can be found in nearly any standard text book about Lie groups or Symplectic Geometry. ■

1.4 Moment maps

The following section provides the concepts that are crucial for the later discussion as in Chapter 3. Here, we introduce a certain class of maps, so-called moment maps. Later we study the properties of images of lagrangians with respect to these maps. We follow mainly [AdSL00] for this introduction.

Definition 1.20 - Let G be a Lie group and M be a differentiable manifold. An **action** ψ of G onto M is a group homomorphism

$$\begin{array}{rccc} \psi: & G & \longrightarrow & \mathrm{Diff}(M) \\ & g & \longmapsto & \psi_g. \end{array}$$

The action ψ is called **smooth** if the evaluation map

$$\begin{array}{ccc} M \times G & \longrightarrow & M \\ (p, g) & \longmapsto & \psi_g(p) \end{array}$$

is smooth.

Remark 1.21 - Throughout this thesis we write an action as $\psi : G \to \mathrm{Diff}(M)$ or as its evaluation map $\psi : G \times M \to M$ via $\psi(g, p) = \psi_g(p)$, depending on which representation is more suitable.

Since the group of symplectomorphisms is a subgroup of the group of diffeomorphisms it makes sense to introduce a further restriction of the definition of an action as follows.

Definition 1.22 - Let (M, ω) be a sympletic manifold. An action $\psi : G \to \mathrm{Diff}(M)$ is called **symplectic** if the image of ψ is a subgroup of $\mathrm{Symp}(M, \omega) \subseteq \mathrm{Diff}(M)$, *i.e.* if the map

$$\psi : G \longrightarrow \mathrm{Symp}(M, \omega)$$

is a group homomorphism.

The case of the Lie group being the real line, *i.e.* $G = \mathbb{R}$, is of special interest because G can be imagined as time. In particular, this defines a special case of symplectic actions as follows.

Definition 1.23 - Let (M, ω) be a symplectic manifold, $\psi : \mathbb{R} \to \mathrm{Symp}(M, \omega)$ be a symplectic action and X the vector field generated by the flow of ψ. Then, ψ is called a **hamiltonian action** if there exists a function $H \in C^\infty(M)$ such that

$$\mathrm{d}H = \iota_X \omega$$

holds. The function H is called a **hamiltonian function**, X the **hamiltonian vector field** of H and the triple (M, ω, H) a **hamiltonian system**.

Remark 1.24 - From the computation

$$\mathrm{d}H(X) = \iota_X \omega(X) = \omega(X, X) = -\omega(X, X) = 0$$

we see that H is constant along the flow lines of X and this shows that H could be seen as the total energy in a mechanical system. For further details on the connections of Symplectic Geometry and Physics consider the excellent works [AMRC80] and [Arn89].

The notion of a hamiltonian can be generalized to other Lie groups as in the following definition.

Definition 1.25 - Let (M, ω) be a symplectic manifold, G a Lie group with Lie algebra $\mathfrak{g}$ and denote by $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. A symplectic action ψ is called a **hamiltonian action** if there exists a map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

satisfying the following properties:

(a) For every tangent vector $X \in \mathfrak{g}$, define a map $\mu^X : M \to \mathbb{R}$ via $\mu^X(p) := \langle \mu(p), X \rangle$ and denote by $X^{\#}$ the vector field on M that is generated by the one-parameter subgroup $\{\exp(tX) \,|\, t \in \mathbb{R}\} \subseteq G$. Then, we have

$$\mathrm{d}\mu^X = \iota_{X^{\#}}\omega,$$

i.e. the function μ^X is a hamiltonian function for the vector field $X^{\#}$.

(b) The map μ is equivariant with respect to the action ψ and the coadjoint action Ad^* of G on $\mathfrak{g}^*$, that is, for all $g \in G$ we have

$$\mu \circ \psi_g = \mathrm{Ad}^*_g \circ \mu.$$

The quadruple (M, ω, G, μ) is called a **hamiltonian G-space**, μ a **moment map** and $\mu(M) \subset \mathfrak{g}^*$ its **moment image**.

Later on we are interested in actions on subspaces that come from a Lie subgroup of G. For this purpose the following proposition is useful.

Proposition 1.26 - *Let G be a compact Lie group, $H \subseteq G$ a Lie subgroup and (M, ω, G, μ) a hamiltonian G-space. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ their corresponding Lie algebras and by $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ the projection dual to the natural inclusion $i : \mathfrak{h} \to \mathfrak{g}$. Then, the restriction of the G-action on H is hamiltonian with moment map*

$$i^* \circ \mu : M \to \mathfrak{h}^*.$$

1.5 Symplectic reduction

As a final ingredient of this chapter we present the principle of **symplectic reduction**. In short, it is the mathematical formulation of the fact that the number of coordinates in phase space of a physical system can be reduced by $2k$ if there are k conserved quantities.

Theorem 1.27 (Marsden-Weinstein-Meyer [MW74, Mey71]) **-** *Let* (M, ω, G, μ) *be a hamiltonian* G*-space and* G *be compact. Write* $i : \mu^{-1}(0) \hookrightarrow M$ *for the inclusion map and also assume that* G *acts freely on* $\mu^{-1}(0)$*. Then,*

(i) the quotient $M_{\mathrm{red}} := \mu^{-1}(0) \big/ G$ *is a manifold,*

(ii) the map $\pi : \mu^{-1}(0) \to M_{\mathrm{red}}$ *defines a principal* G*-bundle and*

(iii) there exists a symplectic form $\omega_{\mathrm{red}} \in \Omega^2(M_{\mathrm{red}})$ *with* $i^*\omega = \pi^*\omega_{\mathrm{red}}$*.*

The pair $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ *is called the* ***reduced space*** *of* (M, ω) *with respect to* μ *and* G*. Furthermore, the dimensions of* M*,* M_{red} *and* G *are related via*

$$\dim M = \dim M_{\mathrm{red}} + 2 \dim G. \tag{1.2}$$

Proof - A proof can be found in [MS17], Section 5.4 on page 224. ■

Proposition 1.28 - *Let* $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ *be the reduced space of a hamiltonian* G*-space* (M, ω, G, μ) *with projection map* $\pi : M \to M_{\mathrm{red}}$ *and* $L \subset (M_{\mathrm{red}}, \omega_{\mathrm{red}})$ *be a lagrangian submainfold. Then, the pre-image* $\pi^{-1}(L) \subset (M, \omega)$ *is a lagrangian submanifold.*

Proof - Let $L \subset (M_{\mathrm{red}}, \omega_{\mathrm{red}})$ be a lagrangian submanifold and denote by $j : L \hookrightarrow M_{\mathrm{red}}$ the inclusion map. Furthermore, let $k : \pi^{-1}(L) \hookrightarrow \mu^{-1}(0)$ be the inclusion map of the pre-image of L. Since L is a lagrangian submanifold we know that $\dim L = \frac{1}{2} \dim M_{\mathrm{red}}$ and $j^*\omega_{\mathrm{red}} = 0$. In the same way, we need to show $\dim(\pi^{-1}(L)) = \frac{1}{2} \dim M$ and $(i \circ k)^*\omega = 0$.

Let us start with the dimension. From property *(ii)* of Theorem 1.27 we know that $\pi : \mu^{-1}(0) \to M_{\mathrm{red}}$ defines a principal G-bundle. Therefore, we have

$$\begin{aligned}\dim\left(\pi^{-1}(L)\right) &= \dim L + \dim G\\ &= \frac{1}{2}\dim M_{\mathrm{red}} + \dim G\\ &\overset{(1.2)}{=} \frac{1}{2}\dim M - \dim G + \dim G\\ &= \frac{1}{2}\dim M.\end{aligned}$$

For the coisotropy use the assumption of $j^*\omega_{\mathrm{red}} = 0$ and apply π^* on both sides. Thus, we compute

$$\begin{aligned}0 &= \pi^*(0) = \pi^*(j^*\omega_{\mathrm{red}}) = (j\circ\pi)^*\omega_{\mathrm{red}}\\ &= (\pi\circ k)^*\omega_{\mathrm{red}} = k^*(\pi^*\omega_{\mathrm{red}}) = k^*(i^*\omega)\\ &= (i\circ k)^*\omega,\end{aligned}$$

where we have used the equation of property *(iii)* of Theorem 1.27 and the commutativity (denoted by the $\square$) in the first square of the following diagram.

$$\begin{tikzcd}
\pi^{-1}(L) \arrow[r, hook, "k"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & \mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d, "\pi"'] & M\\
L \arrow[r, hook, "j"] & M_{\mathrm{red}} &
\end{tikzcd}$$

Therefore, $\pi^{-1}(L) \subset (M,\omega)$ is lagrangian.

As a last step we show that $\pi^{-1}(L)$ is embedded. The submanifold $L \subset M_{\mathrm{red}}$ has dimension is $\dim(L) = \frac{1}{2}\dim M_{\mathrm{red}} \leqslant \dim M_{\mathrm{red}}$. Then, there exist some adapted charts $(U_\alpha, \varphi_\alpha)_{\alpha\in I}$ of M_{red} such that $(\varphi_\alpha)_i = 0$ holds for all $i > \dim L$. Since $\pi^{-1}(L) \to L$ is a principal bundle we have local triviality. Therefore, the charts φ_α can be extended locally be the fiber dimensions. This again is then an adapted chart of L inside of M and hence L is a submanifold. This completes the proof. ■

2 Torus actions and symplectic toric manifolds

Building on the first chapter we are now able to create the setting in which the later theorems of Chapter 3 are formulated. In short, if M has the dimension $2n$, we consider an action of the n-dimensional torus on M. We follow [AdSL00], part B, section I.1.4.

2.1 Torus as a Lie group

Definition 2.1 - Consider the complex space $\mathbb{C}^n$ and define the subset

$$\mathbb{T}^n := \left\{ (\mathrm{e}^{\mathrm{i}\,\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\,\theta_n}) \in \mathbb{C}^n \,\middle|\, \theta_1, \ldots, \theta_n \in [0, 2\pi) \right\} \subset \mathbb{C}^n, \tag{2.1}$$

which is called an **n-dimensional torus** (embedded in $\mathbb{C}^n$), or **n-torus**.

Remark 2.2 - In some parts we simply write $(\theta_1, \ldots, \theta_n)$ for the coordinates of $\mathbb{T}^n$ where $\theta_i \in [0, 2\pi)$ for all $i \in \{1, \ldots, n\}$. However, the above embedding of $\mathbb{T}^n$ into $\mathbb{C}^n$ is useful to define an action of $\mathbb{T}^n$ onto $\mathbb{C}^n$ and is also used.

Lemma 2.3 - *The n-torus $\mathbb{T}^n$ is diffeomorphic to a cartesian product of n circles, i.e.*

$$\mathbb{T}^n \simeq \underbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}_{n\text{-times}} .$$

Furthermore, $\mathbb{T}^n$ is a Lie group and its coadjoint action is trivial. Therefore, the corresponding Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ can be identified with $\mathbb{R}^n$, i.e. $\mathfrak{g} \simeq \mathfrak{g}^* \simeq \mathbb{R}^n$.

Proof - The claim of being diffeomorphic to the product of n circles and being a Lie group follows immediately from Equation 2.1. The triviality of the coadjoint action follows from the fact that $\mathbb{T}^n$ is an abelian Lie group. This completes the proof. ■

Proposition 2.4 - *Let $H \subseteq \mathbb{T}^n$ be a Lie subgroup of the n-dimensional torus. Then, there exists two natural numbers $r, m \in \{0, \ldots, n\}$ with $r + m \leqslant n$ and possibly some $k_1, \ldots, k_r \in \mathbb{N}$ such that*

$$H \simeq \underbrace{\mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_r}}_{r\text{-times}} \times \underbrace{\mathbb{S}_1 \times \ldots \times \mathbb{S}_1}_{m\text{-times}} .$$

Proof - A proof can be found in textbooks about compact connected abelian Lie groups. ■

Remark 2.5 - Note that the cases $r = 0$ and $m = 0$ are also valid and for example $r = 0$ means that $H \simeq \mathbb{T}^m$ for some $m \leqslant n$.

2.2 Torus actions and symplectic toric manifolds

Definition 2.6 - Let (M, ω) be a symplectic manifold and $\mathbb{T}^n$ the n-dimensional torus. A symplectic action $\psi : G \to \mathrm{Symp}(M)$ is called an **n-torus action** if G diffeomorphic to the n-torus, *i.e.* $G \simeq \mathbb{T}^n$.

Actions of tori on symplecic manifolds are very common in Physics and are also mathematically of general interest. See [AdSL00] for further details. One of their main advantages is that the abelian structure makes the action easier to handle. Furthermore, the following theorem yields a connection of their moment images (with respect to a given moment map) and polytopes in $\mathbb{R}^n$.

Theorem 2.7 (Atiyah [Ati82], Guillemin-Sternberg [GS]) **-** *Let (M, ω) be a compact connected symplectic manifold. Suppose that $\psi : \mathbb{T}^n \to \mathrm{Symp}(M, \omega)$ is a hamiltonian torus action with moment map $\mu : M \to \mathbb{R}^n$. Then,*

(i) the levelsets of μ are connected, that is, for every $a \in \mathbb{R}^n$ the set $\mu^{-1}(a)$ is connected,

(ii) the image of μ is convex, that is, $\mu(M) \subset \mathbb{R}^n$ is convex and

(iii) the image of μ is the convex hull of the images of the fixed points of the action ψ.

The moment image $\mu(M)$ in this case is called a ***moment polytope****.*

Proof - A proof can be found in [MS17], Section 5.5, page 237. ■

Note that in the above definition we did not specify any requirements to the dimension of M and no relation to the dimension of the torus. However, in the following we work with the this definition.

Definition 2.8 - A **symplectic toric manifold** is a connected symplectic manifold (M, ω) together with an effective n-torus action, where $n = \frac{1}{2} \dim M$ and with a corresponding moment map $\mu : M \to \mathbb{R}^n$, *i.e.* a hamiltonian $\mathbb{T}^n$-space $(M^{2n}, \omega, \mathbb{T}^n, \mu)$.

Remark 2.9 - Note that we did not assume the manifold to be compact, which is a common definition in the literature. Therefore, Theorem 2.7 is only true for compact symplectic toric manifolds. However, it was shown that similar results can be extended also to non-compact symplectic toric manifolds. For details consider [KL09]. Here, we take this definition because we want to consider $(\mathbb{C}^n, \omega_0, \mathbb{T}^n, \mu)$, as in Example 1.7, as a symplectic toric manifold.

Lemma 2.10 - *Let $(M, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold and for $a_1, \ldots, a_n \in \mathbb{N}$ consider the diagonal inclusion*

$$\begin{array}{ccc} \mathbb{S}^1 & \longrightarrow & \mathbb{T}^n \\ \mathrm{e}^{\mathrm{i}\,\theta} & \longmapsto & \left(\mathrm{e}^{\mathrm{i}\,a_1\theta}, \ldots, \mathrm{e}^{\mathrm{i}\,a_n\theta}\right). \end{array}$$

Then, a corresponding moment map $\mu' : M \to \mathbb{R}$ is given via

$$\mu'(p) = a_1\mu_1(p) + \ldots + a_n\mu_n(p).$$

Proof - For the above inclusion the dual projection is given by the map $i^* : \mathbb{R}^n \to \mathbb{R}$ with $(x_1, \ldots, x_n) \mapsto a_1x_1 + \ldots + a_nx_n$. Then, the claim follows directly from Proposition 1.26. This completes the proof. ■

Definition 2.11 - Let $(M_1, \omega_1, \mathbb{T}_1^n, \mu_1)$ and $(M_2, \omega_2, \mathbb{T}_2^n, \mu_2)$ be two symplectic toric manifolds. Then, they are called **equivalent** to each other if there exists an isomorphism $\lambda : \mathbb{T}_1^n \to \mathbb{T}_2^n$ and a λ-equivariant symplectomorphism $\varphi : M_1 \to M_2$ such that $\mu_1 = \mu_2 \circ \varphi$.

2.3 Examples

2.3.1 The sphere as a compact symplectic toric manifold

As a first example of a symplectic toric manifold we consider the two-dimensional sphere $\mathbb{S}^2$. To do so, we view the sphere as the set

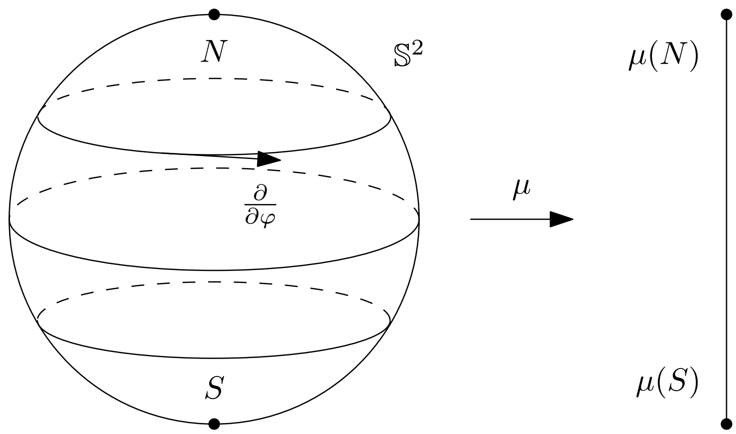
$$\mathbb{S}^2 = \left\{ \frac{1}{\sqrt{1+h^2}} (\cos\varphi, \sin\varphi, h) \in \mathbb{R}^3 \,\middle|\, \varphi \in [0, 2\pi),\ h \in (-1, 1) \right\} \cup \{N, S\},$$

where $N = (0, 0, 1)$ is the north pole and $S = (0, 0, -1)$ the south pole. We define the symplectic form $\omega_0 = \mathrm{d}\varphi \wedge \mathrm{d}h$ on it. A hamiltonian action $\psi : \mathbb{S}^1 \to \mathrm{Symp}(\mathbb{S}^2, \omega_0)$ of the circle $\mathbb{S}^1$ on S^2 is given via

$$\psi(\theta, (\varphi, h)) = (\varphi + \theta, h).$$

Note that the flowlines of ψ are given via the integral curves of $\frac{\partial}{\partial\varphi}$. It is easy to verify that the map $\mu : \mathbb{S}^2 \to \mathbb{R}$ defined via $\mu(\varphi, h) = h$ is a moment map for the above action ψ. Hence, the quadruple $(\mathbb{S}^2, \omega_0, \mathbb{S}^1, \mu)$ is indeed a compact symplectic toric manifold. Furthermore, we want to give it as an illustration of Theorem 2.7.

The fixed points of ψ are the north pole N and south pole S. Also, consider the following image which represents the action and the moment image.



From there it can be seen that $\mu(\mathbb{S}^2) = [\mu(S), \mu(N)] = [-1, 1] \subset \mathbb{R}$ is a convex subset in $\mathbb{R}$ and the moment image is given by the convex combination of the images of the fixed points.

2.3.2 The complex space as a symplectic toric manifold

Here, we build upon the description of the complex space $(\mathbb{C}^n, \omega_0)$ as symplectic manifold as in Example 1.7. An n-torus action $\psi : \mathbb{T}^n \to (\mathbb{C}^n, \omega_0)$ can be defined via

$$\psi\left(\left(\mathrm{e}^{\mathrm{i}\,\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\,\theta_n}\right), (z_1, \ldots, z_n)\right) := \left(\mathrm{e}^{\mathrm{i}\,\theta_1} z_1, \ldots, \mathrm{e}^{\mathrm{i}\,\theta_n} z_n\right),$$

i.e. by performing rotations of θ_i in the arguments of every z_i. We show that a compatible moment map is given by

$$\begin{array}{rccc} \mu : & \mathbb{C}^n & \longrightarrow & \mathbb{R}^n \\ & (z_1, \ldots, z_n) & \longmapsto & -\frac{1}{2}\left(|z_1|^2, \ldots, |z_n|^2\right) \end{array}$$

by checking the properties of Definition 1.25.

(i) Here we introduce polar coordinates $(r_1, \varphi_1, \ldots, r_n, \varphi_n)$ for $\mathbb{C}^n$. In these coordinates we have $\omega_0 = \sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i = \sum_{i=1}^n r_i \mathrm{d}r_i \wedge \mathrm{d}\varphi_i$. Due to linearity it is sufficient to check property (a) only basis vectors of $\mathbb{R}^n$. For a tangent vector $X_i = \frac{\partial}{\partial\theta_i} \in \mathbb{R}^n$ of the torus we have $X^\# = \frac{\partial}{\partial\varphi_i} \in \mathbb{C}^n$. We compute

$$\mathrm{d}\mu^X = \mathrm{d}\left(-\frac{1}{2}r_i^2\right) = -r_i \mathrm{d}r_i.$$

and

$$\iota_{X^{\#}}\omega = \iota_{\frac{\partial}{\partial\varphi_i}} \sum_{i=1}^{n} r_i \mathrm{d}r_i \wedge \mathrm{d}\varphi_i = -r_i \mathrm{d}r_i.$$

This shows the property (a).

(ii) Since $\mathbb{T}^n$ is a commutative Lie group we only have to check invariance of μ with respect to ψ. For this, we compute

$$\begin{aligned}
\mu(\psi\left(\left(\mathrm{e}^{\mathrm{i}\,\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\,\theta_n}\right), (z_1, \ldots, z_n)\right)) &= \mu\left(\mathrm{e}^{\mathrm{i}\,\theta_1} z_1, \ldots, \mathrm{e}^{\mathrm{i}\,\theta_n} z_n\right) \\
&= -\frac{1}{2}\left(\left|\mathrm{e}^{\mathrm{i}\,\theta_1} z_1\right|^2, \ldots, \left|\mathrm{e}^{\mathrm{i}\,\theta_n} z_n\right|^2\right) \\
&= -\frac{1}{2}\left(|z_1|^2, \ldots, |z_n|^2\right) \\
&= \mu(z_1, \ldots, z_n).
\end{aligned}$$

This indeed shows that $(\mathbb{C}^n, \omega_0, \mathbb{T}^n, \mu)$ is a symplectic toric manifold.

2.4 Action angle coordinates

Let $(M, \omega, \mathbb{T}^n, \mu)$ be symplectic toric manifold of dimension $2n$. The action of the torus allows us to define a specific choice of coordinates on M that are adapted with respect to the flow lines of the action. These are called **action angle coordinates** and will be suitable for a lot of cases in further investigations. In order to show their existence consider the following definitions and theorems. Here, we follow [AdSL00], part B, section I.1.3.

Definition 2.12 - Let (M, ω) be symplectic manifold, $f, g \in C^\infty(M)$ two functions on M and denote by $X_f, X_g \in \Gamma(TM)$ their corresponding hamiltonian vector fields. The **Poisson bracket** of f and g is the function

$$\{f, g\} := \omega(X_f, X_g).$$

Lemma/Definition 2.13 - *Let (M, ω, H) be a hamiltonian system and $f \in C^\infty(M)$ be a function. Then, $\{f, H\} = 0$ holds if and only if f is constant along integral curves of H. Such function f is called an* ***integral of motion*** *of (M, ω, H).*

Proof - We compute

$$\begin{aligned}\{f, H\} = 0 &\Leftrightarrow \omega(X_f, X_H) = 0\\ &\Leftrightarrow \mathrm{d}f(X_H) = 0\\ &\Leftrightarrow f \text{ constant along } X_H.\end{aligned}$$

This completes the proof. ∎

Remark 2.14 - Trivially, H is an integral of motion of a hamiltonian system (M, ω, H) by Remark 1.24.

Theorem 2.15 (Arnold-Louville) **-** *Let (M, ω, H) be a hamiltonian system of dimension $2n$ with n integrals of motions $f_1 = H, f_2, \dots, f_n \in C^\infty(M)$. Furthermore, let $c \in \mathbb{R}^n$ be a regular value of $f := (f_1, \dots, f_n)$. Then, the set $f^{-1}(c) \subset (M, \omega)$ is a lagrangian submanifold.*

*(i) If the flows of the corresponding vector fields $X_{f_1}, \dots, X_{f_n} \in \Gamma(TM)$ starting at $p \in f^{-1}(c)$ are complete, then on the connected component of $f^{-1}(c)$ containing p there exists an affine structure with coordinates $\varphi_1, \dots, \varphi_n$ in which the flows of the vector fields $X_{f_1}, \dots, X_{f_n}$ are linear. These coordinates are called **angle coordinates**.*

*(ii) Furthermore, there exist coordinates $\psi_1, \dots, \psi_n$ that are complementary to the angle coordinates. That is, the set of coordinates $\psi_1, \dots, \psi_n, \varphi_1, \dots, \varphi_n$ forms a Darboux chart. These coordinates are called **action coordinates**.*

Proof - A proof can be found in [Arn89]. ∎

Corollary 2.16 - *Let $(M, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold and let $\theta_1, \dots, \theta_n$ be coordinates of $\mathbb{T}^n$. Furthermore, let c be a regular level of μ. Then, for every point $p \in f^{-1}(c)$ there exists a neighbourhood U of p and a set of coordinates $\psi_1, \dots, \psi_n$ on U such that*

$$\omega = \sum_{i=1}^{n} \mathrm{d}\theta_i \wedge \mathrm{d}\psi_i.$$

Proof - Here it is sufficient to observe that every component μ_i of the moment map is an integral of motion. ∎

Corollary 2.17 - *For the complex standard toric mainfold* $(\mathbb{C}^n, \omega_0, \mathbb{T}^n \mu)$*, as in Example 1.7. Then, we have* $\psi_i \equiv \mu_i = -\frac{1}{2} r_i^2$ *for all* $i \in \{1, \ldots, n\}$.

2.5 Delzant theorem

Theorem 2.7 allows us to associate to every symplectic toric manifold a polytope in $\mathbb{R}^n$. Naturally, two questions arise. First, which kinds of polytopes occur when taking the moment image and second, we can ask the inverse question of the first statement. That is, can we associate to certain and allowed polytopes a (unique) symplectic toric manifold? The answers to these questions where given by Delzant and are presented on the following pages.

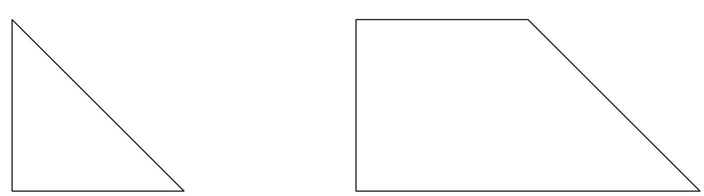
Definition 2.18 - A **Delzant polytope** $\Delta \subset \mathbb{R}^n$ is a polytope satisfying the following conditions.

(a) at every vertex there are n edges,

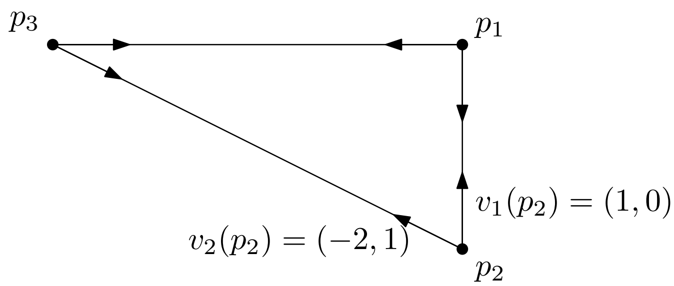
(b) edges meeting at one vertex p are rational, that is, the i-th edge is of the form $p + t \cdot v_i$, where $v \in \mathbb{Z}^n$ and

(c) for every vertex the corresponding edge vectors $v_1, \ldots, v_n \in \mathbb{Z}^n$ from above form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

The following polytopes are examples of Delzant polytopes.



The following polytope is an example of a non-Delzant polytope.



Property (c) of Definition 2.18 is not satisfied at the vertex p_2. To see this observe that the point $(1, 0) \in \mathbb{Z}^2$ can not be realized by $a_1 v_1(p_2) + a_2 v_2(p_2)$ for some $a_1, a_2 \in \mathbb{Z}$.

Theorem 2.19 (Delzant [Del]) **-** *There exists a bijection between the set of all compact sympletic toric manifolds up to equivalence and the set of Delzant polytopes, which is given by the moment map via* $(M^{2n}, \omega, \mathbb{T}^n, \mu) \mapsto \mu(M) \subset \mathbb{R}^n$.

Proof - A proof can be found in [AdSL00]. ■

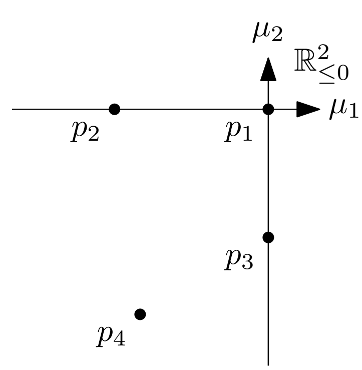
3

Toric lagrangians in the complex space

In this section we formalize the central question of this thesis. In short, we are asking for the existence of immersed and embedded lagrangians that will be mapped under the moment map to specific affine subspaces of $\mathbb{R}^n$.

3.1 Pre-images of the moment map

The central ingredients of this thesis are the moment map and pre-images of it. In order to introduce the topic let us consider $(\mathbb{C}^2, \omega_0, \mathbb{T}^2, \mu)$ and check how certain pre-images of sets in the moment image behave. If we look at the following picture one could ask what are the pre-images of the points $p_1, p_2, p_3, p_4 \in \mathbb{R}^2_{\leqslant 0}$.



Let us start with $p_1 = (0, 0)$, where we introduce coordinates (μ_1, μ_2) for $\mathbb{R}^2_{\leqslant 0}$. From the definition of the moment map we can deduce

$$\mu^{-1}(p_1) = \mu^{-1}(0,0) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \,\middle|\, -\frac{1}{2}\left(|z_1|^2, |z_2|^2\right) = (0,0) \right\} = (0,0).$$

Thus, the pre-image in this case is just the origin of $\mathbb{R}^2_{\leqslant 0}$.

For p_2 and p_3 we have a similar situation because in both cases we have either $\mu_1 = 0$ or $\mu_2 = 0$. Therefore, it is sufficient to consider p_2. Here, we write $p_2 = (-\lambda, 0)$ for some $\lambda \in \mathbb{R}_{>0}$ and compute

$$\begin{aligned} \mu^{-1}(p_2) = \mu^{-1}(\lambda, 0) &= \left\{ (z_1, z_2) \in \mathbb{C}^2 \,\middle|\, -\frac{1}{2}\left(|z_1|^2, |z_2|^2\right) = (-\lambda, 0) \right\} \\ &= \left\{ (2\lambda\, \mathrm{e}^{\mathrm{i}\,\theta}, 0) \in \mathbb{C}^2 \,\middle|\, \theta \in \mathbb{S}^1 \right\} \simeq \mathbb{S}^1. \end{aligned}$$

For p_4 write $p_4 = (-\lambda_1, -\lambda_2)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ and compute

$$\begin{aligned} \mu^{-1}(p_4) &= \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid -\frac{1}{2}\left(|z_1|^2, |z_2|^2\right) = (-\lambda_1, -\lambda_2) \right\} \\ &= \left\{ \left(2\lambda_1\, \mathrm{e}^{\mathrm{i}\,\theta_1}, 2\lambda_2\, \mathrm{e}^{\mathrm{i}\,\theta_2}\right) \in \mathbb{C}^2 \mid \theta_1, \theta_2 \in \mathbb{S}^1 \right\} \simeq \mathbb{T}^2. \end{aligned}$$

Therefore, the pre-image is a 2-torus.

We don't give a detailed prove but these statements can be extended to $\mathbb{C}^n$ right away. Let $p = (p_1, \ldots, p_n) \in \mathbb{R}^n_{\leqslant 0}$ be a point in the moment image with $r \in \{0, \ldots, n\}$ non-vanishing coordinates. Then,

$$\pi^{-1}(p) \simeq \mathbb{T}^r.$$

We use the convention that $(\mathbb{T}^r)^0$ is just a single point.

Definition 3.1 - Let V be a vector space. A subset $W \subseteq V$ is called an **affine subspace** of V if there exists a linear subspace $W' \subseteq V$ and a vector $w \in V$ such that $W = W' + w$. An affine subspace $A \subseteq \mathbb{R}^n$ is called **rational** if W' admits a rational basis.

Remark 3.2 - The image of the moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n$ is given by $\mathbb{R}^n_{\leqslant 0}$ and is clearly not a vector space. However, we define can extend the definition of an affine subspace as follows.

Definition 3.3 - A subspace $W \subset \mathbb{R}^n_{\leqslant 0}$ is called an **affine subspace** if its linear completion W' in $\mathbb{R}^n$ is an affine space. Furthermore, it is called **rational** if W' is rational.

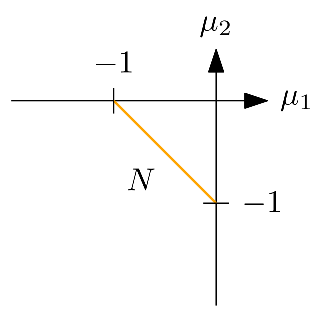
Remark 3.4 - In this thesis we are interested in rational affine subspaces of $\mathbb{R}^n_{\leqslant 0}$. If we pick coordinates $(\mu_1, \ldots, \mu_n)$ of $\mathbb{R}^n_{\leqslant 0}$ every affine subspace can be defined as a set of r equations, each of the form $a_i^1\mu_1 + \ldots + a_i^n\mu_n = k_i$ for $i \in \{1, \ldots, r\}$, where $a_i^j, k \in \mathbb{R}$. We can require $k_i \leqslant 0$ for all i and in order to have a non-empty space we can deduce that at least one of the a_i^j for a fixed i has to be greater than zero. Furthermore, by multiplication we can arrange that $a_i^j \in \mathbb{Z}$ for all $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n\}$ and such that $\sum_{j=1}^n \left|a_i^j\right|$ is as small as possible for every i. This is also known as the vector $(a_1, \ldots, a_n) \in \mathbb{R}$ is primitive. This convention is used throughout the whole thesis.

Definition 3.5 - An affine subspace $N \subset \mathbb{R}^n_{\leqslant 0}$ of codimension 1 is called **Delzant** if for every intersection point $p \in N$ with the i-th axes of $\mathbb{R}^n_{\leqslant 0}$ there exists $n-1$ tangent vectors $v_1, \ldots, v_{n-1} \in T_pN$, such that the set $\{e_i, v_1, \ldots, v_{n-1}\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ and every vector is rational, that is $v_i \in \mathbb{Z}^n$ for all $i \in \{1, \ldots, n-1\}$.

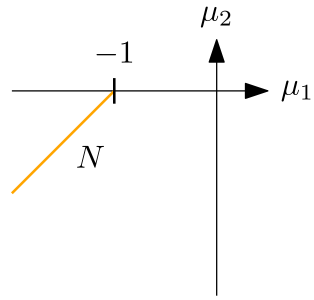
Remark 3.6 - Every Delzant affine space is rational, since the intersection being Delzant implies integer coefficients for the affine subspace.

Example 3.7 - The concept of an Delzant affine space has indeed a connection to the above Definition 2.18 for Delzant polytopes.

To make this connection visible let us consider $\mathbb{R}^2_{\leqslant 0}$. The following figure shows the Delzant affine subspace N for the equation $\mu_1 + \mu_2 = -1$. Note that the triangle with sides given by the axes and N forms a Delzant polytope.



However, our above definition allows affine spaces that not necessary enclose a Delzant polytope. The next figure shows the Delzant affine subspace for the equation $\mu_1 - \mu_2 = -1$.



Proposition 3.8 - *Let $(\mathbb{C}^n, \omega_0, \mathbb{T}^n, \mu)$ be the standard complex toric manifold and $N \subseteq \mathbb{R}^n_{\leqslant 0}$ a rational affine subspace of codimension r. Then, there exists a subgroup $H \subseteq \mathbb{R}^n$ which preserves the null foliation* $\operatorname{Ker}\left(\omega\big|_{\mu^{-1}(N)}\right)$ *and which is diffeomorphic to a torus of dimension r, i.e. $H \simeq \mathbb{T}^r$.*

Proof - Let $\mu_1, \ldots, \mu_n$ be coordinates of the moment image $\mathbb{R}^n_{\leqslant 0}$. From Linear Algebra we know that every rational affine subspace N can be charaterized by a set of r linear equations $\{a_i^1\mu_1 + \ldots + a_i^n\mu_n = k_i\}_{i\in\{1,\ldots,r\}}$ for some $a_i^j \in \mathbb{Z}$ and $k_i \in \mathbb{R}_{\leqslant 0}$, by Remark 3.4, via

$$N := \left\{(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_{\leqslant 0} \,|\, a_i^1\mu_1 + \ldots + a_i^n\mu_n = k_i \text{ for all } i \in \{1, \ldots, r\}\right\}.$$

We show that for every of these equations there exists a periodic vector field under which $\mu^{-1}(N)$ is invariant. The flow of this vector fields gives then rise to the subgroup H of $\mathbb{T}^n$. Consider the i-th defining equation of N. By differentiating we obtain on N

$$\lambda_i := a_i^1 \mathrm{d}\mu_1 + \ldots + a_i^n \mathrm{d}\mu_n = 0.$$

From this expression we see that the tangent bundle of $\mu^{-1}(N)$ is given by the union of the kernels of each of the above 1-forms λ_i, *i.e.*

$$T\left(\mu^{-1}(N)\right) = \bigcup_{i=1}^{n} \operatorname{Ker}(\lambda_i).$$

Let X_i be the vector field generated by $\frac{\partial}{\partial\theta_i}$. From the definition of the moment maps, as in Definition 1.25, we know that

$$\mathrm{d}\mu_i = \iota_{X_i}\omega$$

holds for all $i \in \{1, \ldots, n\}$.

Define the vector fields $Y_i := a_i^1 X_1 + \ldots a_i^n X_n$ and we compute

$$\begin{aligned}\left(a_i^1 \mathrm{d}\mu_1 + \ldots + a_i^n \mathrm{d}\mu_n\right)(Y_i) &= a_i^1 \omega(X_1, Y_i) + \ldots + a_i^n \omega(X_n, Y_i) \\ &= a_i^1 \sum_{j=1}^n a_i^j \omega(X_1, X_j) + \ldots + a_1^n \sum_{j=1}^n a_i^j \omega(X_n, X_j) \\ &= \sum_{k=1}^n a_j^k \sum_{j=1}^n a_i^j \omega(X_k, X_j) \\ &= \sum_{j,k=1}^n a_i^k a_i^j \omega(X_k, X_j) \\ &= \sum_{j<k} a_i^j a_i^k \omega(X_k, X_j) + \sum_{k<j} a_i^j a_i^k \omega(X_k, X_j) \\ &= \sum_{j<k} a_i^j a_i^k \omega(X_k, X_j) + \sum_{j<k} a_i^k a_i^j \underbrace{\omega(X_j, X_k)}_{=-\omega(X_k, X_j)} \\ &= 0.\end{aligned}$$

Here, we only use the anti-symmetry of ω and relabeling of a sum. Hence, we get that $\mu^{-1}(N)$ is invariant with respect to Y_i. Each of the X_i vector fields is periodic and hence their linear combination is also periodic since all the a_i^j are integral numbers. Since this is true for every Y_i with $i \in \{1, \ldots, r\}$ we conclude that $\mu^{-1}(N)$ is invariant with respect to $\mathbb{T}^r \subseteq \mathbb{T}^n$. This completes the proof. ■

Proposition 3.9 - *Let $(\mathbb{C}^n, \omega_0, \mathbb{T}^n, \mu)$ be the standard complex toric manifold and $N \subseteq \mathbb{R}^n_{\leqslant 0}$ be a rational affine subspace of codimension r. Assume there exists a lagrangian submanifold $L \subseteq (\mathbb{C}^n, \omega_0)$ with $L \subset \mu^{-1}(N)$. Then, the null foliation is a subset of the tangent bundle of L, i.e.* $\operatorname{Ker}\left(\omega\big|_{\mu^{-1}(N)}\right) \subseteq TL$.

Proof - Let us write ω in action angle coordinates as $\omega = \sum_{i=1}^n \mathrm{d}\theta_i \wedge \mathrm{d}\mu_i$ on the restriction to $\mathbb{R}^n_{\leqslant 0}$ to its interior. Since N is an rational affine subspace of codimension r there exists r equations of the form $a_i^1 \mu_1 + \ldots + a_i^n \mathrm{d}\mu_n = k_i$, where $i \in \{1, \ldots, r\}$, $a_i^j \in \mathbb{Z}$ and $k_i \in \mathbb{R}_{\leqslant 0}$. Therefore, we can deduce that

$$TN = \bigcup_{i=1}^r \operatorname{Ker}\left(a_i^1 \mathrm{d}\mu_1 + \ldots + a_i^n \mathrm{d}\mu_n\right) =: \underbrace{T\Theta}_{\operatorname{span}\left\{\frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_n}\right\}} \oplus \underbrace{M}_{TN \cap \operatorname{span}\left\{\frac{\partial}{\partial\mu_1}, \ldots, \frac{\partial}{\partial\mu_n}\right\}}.$$

Define the vector fields $Y_j = a_j^1 \frac{\partial}{\partial \mu_1} + \ldots + a_j^n \frac{\partial}{\partial \mu_n}$ and from this we see that

$$\operatorname{Ker}\left(\omega\big|_{\mu^{-1}(N)}\right) = \operatorname{span}\{Y_1, \ldots, Y_r\}. \tag{3.1}$$

If $L \subseteq N$ is a submanifold of N we get

$$TL \cap \operatorname{span}\left\{\frac{\partial}{\partial \mu_1}, \ldots, \frac{\partial}{\partial \mu_n}\right\} \subseteq M.$$

Asumme that L is lagrangian and we show that $Y_1, \ldots, Y_r \in TL$ holds. We prove by contradiction. Assume that $Y_1, \ldots, Y_r \notin TL$ and that L is lagrangian. Then, we get that the space

$$TL \oplus \operatorname{span}(Y_1, \ldots, Y_r)$$

is $n + r$ dimensional and coistropic by Equation 3.1. But this contradicts the lagrangian property of L. Since every vector field Y_j is periodic, we can conclude that there exists a Lie subgroup $H \subset \mathbb{T}^n$ with $\mathbb{T}^r \subseteq H$. This completes the proof. ■

This result can be tightened even further by restricting the moment image of L as follows.

Proposition 3.10 - *Let $(\mathbb{C}^n, \omega_0, \mathbb{T}^n, \mu)$ be the standard complex toric manifold and $N \subseteq \mathbb{R}^n_{\leqslant 0}$ be a rational affine subspace of codimension r. Assume there exists a lagrangian submanifold $L \subseteq (\mathbb{C}^n, \omega_0)$ with $\mu(L) = N$ and such that $\mu\big|_L : L \to N$ is a submersion. Then, L is invariant with respect to the same torus $\mathbb{T}^r \subset \mathbb{T}^n$ as in Proposition 3.8.*

Proof - The proof is essentially the same as in the case of Proposition 3.9. However, since $\mu\big|_L$ is a submersion we get that

$$TL \cap \operatorname{span}\left\{\frac{\partial}{\partial \mu_1}, \ldots, \frac{\partial}{\partial \mu_n}\right\} = M$$

and hence we conclude $TL = \operatorname{span}\{X_1, \ldots, X_r\}$, which implies immediately that L is $\mathbb{T}^r$ invariant. This completes the proof. ■

3.2 Toric lagrangians

Lagrangian submanifolds with the property as in Proposition 3.10 are of central interest for this thesis. Therefore, we take this notion and generalize it as follows.

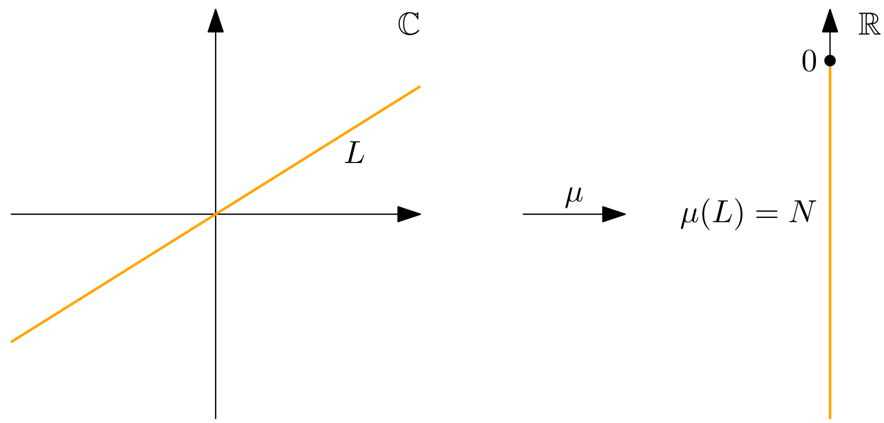
Definition 3.11 - Let $(M, \omega, \mathbb{T}^n, \mu)$ be a hamiltonian $\mathbb{T}^n$-space with an action $\psi : \mathbb{T}^n \times M \to M$. An immersed lagrangian $L \subset (M, \omega)$ is called **toric lagrangian** if there exits a proper Lie subgroup $H \subsetneq G$ such that L is invariant under ψ with respect to H, that is, if $\psi(h, L) = L$ holds for all $h \in H$.

Remark 3.12 - So far, we only considered continuous cyclic Lie subgroups of the form $\mathbb{T}^r \subseteq \mathbb{T}^n$. By Proposition 2.4 there are also subgroups that are made of cartesian products of $\mathbb{Z}_k$ for some k. However, the study of toric lagrangians with respect to these subgroups is omitted in this thesis. See Section 4.3 about further research for more details.

On the following pages we show the existence and also non-existence of toric lagrangian submanifolds for certain affine subspaces for $(\mathbb{C}^2, \omega)$. However, as a first example we start with the simpler case of $n = 1$.

3.3 Baby case $(\mathbb{C}, \omega_0)$

Since $(\mathbb{C}, \omega_0)$ is a 2-dimensional manifold, $\omega_0 \in \Omega^2(\mathbb{C})$ is automatically a volume form and hence every 1-dimensional submanifold of $\mathbb{C}$ is lagrangian. Since there is only one affine subspace in $\mathbb{R}_{\leqslant 0}$, that is, $\mathbb{R}_{\leqslant 0}$ itself, we can take any linear subspace $L \subset \mathbb{C}$, *i.e.* a ray through the origin, which is then toric by Proposition 3.10.



3.4 Existence of toric lagrangians in the complex plane

The complexity of the existence of toric lagrangians changes a lot for all the cases of $n \neq 1$. We focus here on the next simplest case of $(\mathbb{C}^2, \omega_0)$. Here, we formulate so-called ray-theorems which show the existence of toric lagrangians for rational affine subspaces of $\mathbb{R}^2_{\leqslant 0}$. This will be shown in the following.

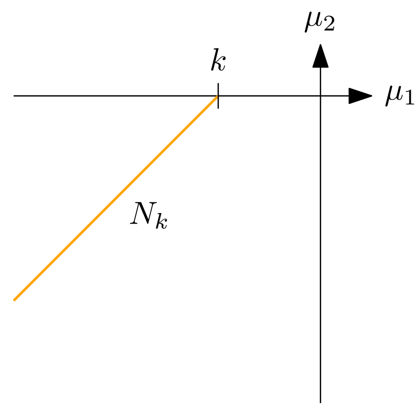
3.4.1 Proving existence by symplectic reduction

For the cases of affine spaces that enclose a 45° or 90° degree angle with one of the axes of $\mathbb{R}^2_{\leqslant 0}$ even an embedded toric lagrangian can be archived.

As a first example consider for a fixed $k \in \mathbb{R} \setminus \{0\}$ the affine space

$$N_k := \left\{ (\mu_1, \mu_2) \in \mathbb{R}^2_{\leqslant 0} \,\middle|\, \mu_1 - \mu_2 = k \right\} \tag{3.2}$$

which is drawn in the following picture.



The pre-image with respect to the moment map of N is given via the equation

$$
\begin{aligned}
N_k' &:= \mu^{-1}(N_k) \\
&= \left\{ (r_1\,\mathrm{e}^{\mathrm{i}\,\theta_1}, r_2\,\mathrm{e}^{\mathrm{i}\,\theta_2}) \in \mathbb{C}^2 \,\middle|\, r_1, r_2 \in \mathbb{R}_{\geqslant 0},\ r_1^2 - r_2^2 = -2k,\ \theta_1, \theta_2 \in [0, 2\pi) \right\}.
\end{aligned}
$$

We use symplectic reduction in order to show that a lagrangian submanifold in $N_k' \subset (\mathbb{C}^2, \omega_0)$ exists, which covers N_k.

Lemma 3.13 - *Consider $(\mathbb{C}^2, \omega_0)$ and $N_k' \subset \mathbb{C}^2$ as above. Then, the map μ' : $N_k' \to \mathbb{R}_{\leqslant 0}$ defined as $\mu_k'(z_1, z_2) = -\frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2$ is a moment map on N_k' for the anti-diagonal action given by $\psi'(\mathrm{e}^{\mathrm{i}\,\theta}, (z_1, z_2)) = \left(\mathrm{e}^{\mathrm{i}\,\theta} z_1, \mathrm{e}^{-\mathrm{i}\,\theta} z_2\right)$. Furthermore, this action is free on N'.*

Proof - Since $\mathbb{S}^1 \subset \mathbb{T}^n$ is a Lie subgroup with inclusion map

$$
\begin{array}{cccc}
i: & \mathbb{S}^1 & \longrightarrow & \mathbb{T}^2 \\
 & \mathrm{e}^{\mathrm{i}\,\theta} & \longmapsto & (\mathrm{e}^{\mathrm{i}\,\theta}, \mathrm{e}^{-\mathrm{i}\,\theta})
\end{array}
$$

we can conclude by Proposition 1.26 that ψ' defines indeed a hamiltonian action with moment map μ'. What is left to show is that ψ_g' is free. We compute in action angle coordinates

$$
\psi_\theta'(p) = \left(\mu_1 \mathrm{e}^{\mathrm{i}\,(\theta+\theta_1)}, \mu_2 \mathrm{e}^{\mathrm{i}\,(-\theta+\theta_2)}\right) \overset{!}{=} \left(\mu_1 \mathrm{e}^{\mathrm{i}\,\theta_1}, \mu_2 \mathrm{e}^{\mathrm{i}\,\theta_2}\right),
$$

which implies immediately $\theta = 0$ for all $\mu_1, \mu_2 \in \mathbb{R}_{\leqslant 0}$ and $\theta_1, \theta_2 \in \mathbb{S}^1$. Hence the action ψ' is free. This completes the proof. ■

Proposition 3.14 - *There exists a lagrangian submanifold $L_k \subset (\mathbb{C}^2, \omega_0)$ such that $\mu(L_k) = N_k$ for all $k \in \mathbb{R} \setminus \{0\}$.*

Proof - With Lemma 3.13 we have all the assumptions in order to apply Theorem 2.7 to perform symplectic reduction. We get that $N'_k/_{\mathbb{S}^1} \simeq \mathbb{C}$. We are now looking for a lagrangian $L_{\mathrm{red}} \subset \mathbb{C}$ whose image by the residual moment map μ_{red} is everything for the circle action. Take $L_{\mathrm{red}} = \mathbb{R} \subset \mathbb{C}$ and define $L_k := \pi^{-1}(L_{\mathrm{red}})$. By Proposition 1.28 we conclude that L_k is an embedded lagrangian with $\mu(L_k) = N_k$. This completes the proof. ∎

The case of $k = 0$ in the above setting, that is, N_0 is a ray starting at the origin in a $45°$ degree slope, can be treated separately. This example goes back to [ALP94]. But here we construct a corresponding lagrangian submanifold in $\mathbb{C}^2$ explicitly as follows.

Lemma 3.15 - *Let $(\mathbb{C}^2, \omega_0, \mathbb{T}^n, \mu)$ be the standard complex hamiltonian $\mathbb{T}^n$-space. The set*

$$L_0 := \left\{ (z, \overline{z}) \in \mathbb{C}^2 \,\middle|\, z \in \mathbb{C} \right\}$$

is a lagrangian submanifold of $(\mathbb{C}^2, \omega_0)$ with moment image $\mu(L_0) = N_0$.

Proof - We use Lemma 1.12 in order to show that L_0 is lagrangian in $(\mathbb{C}^2, \omega_0)$. For that purpose write L_0 in real coordinates as

$$L_0 = \left\{ (x, y, x, -y) \in \mathbb{R}^4 \,\middle|\, x, y \in \mathbb{R} \right\}$$

and hence we get

$$\mathrm{i}\, L_0 = \left\{ (-y, x, y, x) \in \mathbb{R}^4 \,\middle|\, x, y \in \mathbb{R} \right\}.$$

So if we take $(x_1, y_1, x_1, -y_1) \in L_0$ and $\mathrm{i}\,(x_2, y_2, x_2, -y_2) = (-y_2, x_2, y_2, x_2) \in \mathrm{i}\, L_0$ we compute

$$\langle (x_1, y_1, x_1, -y_1), (-y_2, x_2, y_2, x_2) \rangle = -x_1 y_2 + x_2 y_1 + x_1 y_2 - y_1 x_2 = 0$$

and conclude that $L_0^{\perp} \supseteq \mathrm{i}\, L_0$. For the other direction let $(x_1, y_1, x_2, y_2) \in L_0^{\perp}$ and for every $(x, y, x, -y) \in L_0$ we need to have

$$\begin{aligned} \langle (x_1, y_1, x_2, y_2), (x, y, x, -y) \rangle &= x_1 x + y_1 y + x x_2 - y_2 y \\ &= x \cdot (x_1 + x_2) + y(y1 - y_2) \\ &\overset{!}{=} 0. \end{aligned}$$

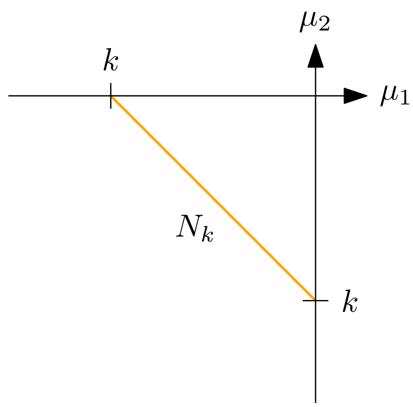
Since this equation has to hold for all $x, y \in \mathbb{R}$ simultaneously we get that $x_1 = -x_2$ and $y_1 = y_2$. This implies that $(-x_2, y_2, x_2, y_2) \in \mathrm{i}\, L_0$ and yields $L_0^{\perp} \subseteq \mathrm{i}\, L_0$. So in total we get $L_0^{\perp} = \mathrm{i}\, L_0$, which implies that L_0 is a lagrangian of $(\mathbb{C}^2, \omega_0)$.

To show that it is also embedded note that L_0 can be given as the graph of the map $z \mapsto \overline{z}$ and this map is injective. Therefore, L_0 is a lagrangian submanifold of $(\mathbb{C}^2, \omega_0)$. This completes the proof. ■

As another example we want to consider the following case. For a fixed $k \in \mathbb{R}_{<0}$ define the set

$$N_k = \left\{ (\mu_1, \mu_2) \in \mathbb{R}^2_{\leqslant 0} \,\middle|\, \mu_1 + \mu_2 = k \right\} \tag{3.3}$$

which has the following picture. The story for this case is indeed very similar to the first case.

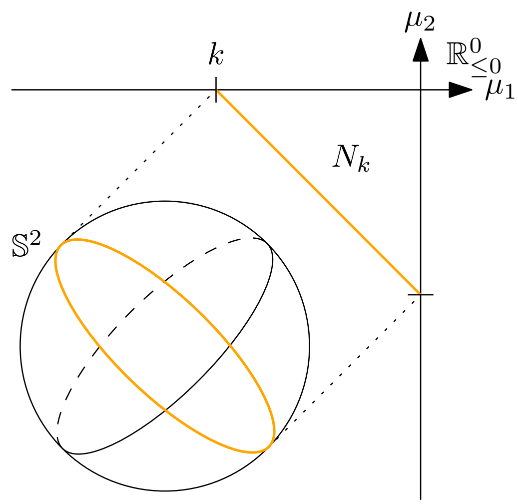


Proposition 3.16 - *There exists a lagrangian aubmanifold $L_k \subset (\mathbb{C}^2, \omega_0)$ such that $\mu(L_k) = N_k$ for all $k \in \mathbb{R}\setminus\{0\}$ and N_k, as in Equation 3.3, holds.*

Proof - First of all note that $\mu^{-1}(N_k)$ is diffeomorphic to a 3-sphere as the following expression shows:

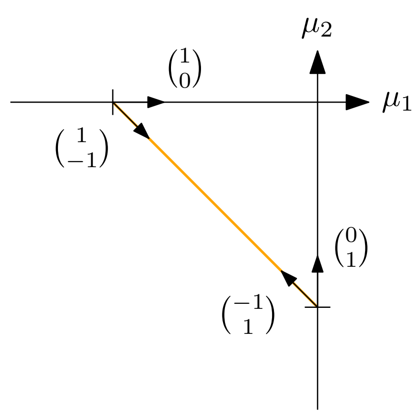
$$\mu^{-1}(N_k) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 2\,|k| \right\}.$$

Here, we have a free diagonal action $\psi'(\mathrm{e}^{\mathrm{i}\,\theta}, (z_1, z_2)) = \left(\mathrm{e}^{\mathrm{i}\,\theta} z_1, \mathrm{e}^{\mathrm{i}\,\theta} z_2\right)$ and for this action the moment map is given by $\mu' = \mu_1 + \mu_2$, by Proposition 1.26. This map is nothing else than the **Hopf fibration** and we therefore obtain $\mathbb{S}^3 / \mathbb{S}^1 = \mathbb{S}^2$.



Since $\mathbb{S}^2$ is of dimension two any 1-dimensional subspace is lagrangian. Therefore, if we pick any great circle L_{red} through the north and south-pole we get that $L_k := \mu^{-1}(L_{\text{red}})$ is a lagrangian submanifold in $(\mathbb{C}^2, \omega_0)$ with $\mu(L_k) = N_k$. This completes the proof. ■

Remark 3.17 - Note that all affine subspaces of the above cases are Delzant, *e.g.* in the second case we have the edge vectors as in the following picture.



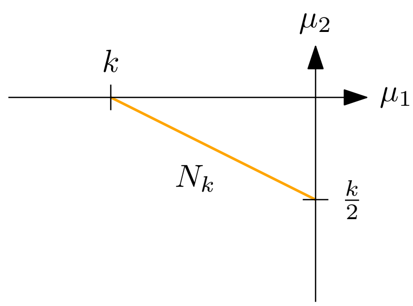
Remark 3.18 - The above cases only considered 45° degree angles to the axes. For a 90° angle to one axis the situation is of a simple nature. This case can be seen as

neglecting one of the radius values of $\mathbb{C}^2$ and is therefore equivalent to the $\mathbb{C}$ case, as in Section 3.3.

Remark 3.19 - These considerations show that for Dezlant affine subspaces we can use symplectic reduction in order to produce a lagrangian submanifold. This observation is one of two ingredients for the later central conjecture of this thesis.

3.4.2 Limitations of symplectic reduction

In the previous section we only considered angles of $0°, 45°$ and $90°$ degree between an affine space N_k and one the of the axes of $\mathbb{R}^2_{\leqslant 0}$. Therefore, the natural question arises of how the story changes if we consider different angles. To show the existence of a lagrangian with moment image N_k we used symplectic reduction via Theorem 2.7. Let us consider the following case, which is not Delzant as shown in following picture.



For a fixed $k \in \mathbb{R}_{<0}$ we can write N_k as

$$N_k = \left\{ (\mu_1, \mu_2) \in \mathbb{R}^2_{\leqslant 0} \,\middle|\, \mu_1 + 2\,\mu_2 = k \right\}.$$

Also, by Proposition 1.26 we can argue that $\mu' = \mu_1 + 2\mu_2$ is a moment map for the pseudo-diagonal action $\psi'(\theta, (z_1, z_2)) = \left(\mathrm{e}^{\mathrm{i}\,\theta} z_1, \mathrm{e}^{\mathrm{i}\,2\theta} z_2\right)$.

As a last assumption we have to verify if the given action is free at every point. However, this is wrong as we show in the following computation. For all $(z_1, z_2) \in N_k$ we need to check

$$\psi'_\theta(z_1, z_2) = \left(\mathrm{e}^{\mathrm{i}\,\theta} z_1, \mathrm{e}^{\mathrm{i}\,2\theta} z_2\right) \stackrel{!}{=} (z_1, z_2)\,. \tag{3.4}$$

only holds for $\theta = 0$. But for the element $(0, z_1)$ we have $\theta = 0$ and $\theta = \pi$ fulfill Equation (3.4) and hence the given action is not free. Therefore, we need other methods in order to show existence for affine space that has a slope different from $45°$ degrees.

3.5 Lagrangians in $(\mathbb{C}^2, \omega_0)$

3.5.1 Properties

Before moving on to the existence of lagrangians with certain moment images we derive and sum up some obstructions to lagrangian submanifolds in $(\mathbb{C}^2, \omega_0)$.

Theorem 3.20 (Neighbourhood theorem) **-** *Let M be a compact manifold, $N \subseteq M$ a compact manifold and $\omega_0, \omega_1 \in \Omega^2(M)$ two 2-forms which are equal and non-degenerate on $TM\big|_N$. Then, there exist neighborhoods $N_0, N_1 \subseteq M$ of N and a diffeomorphism $\psi : N_0 \to N_1$, which is the identity on N, such that $\psi^*\omega_1 = \omega_0$.*

Proof - A proof can be found in [MS17]. ■

Corollary 3.21 - *A compact lagrangian submanifold $L \subseteq (M, \omega)$ has a neighbourhood, which is symplectomorphic to a neighbourhood of the zero-section in T^*L.*

Proof - A proof can be found in [MS17]. ■

Lemma 3.22 - *Let $L \subset (M, \omega)$ be a compact orientable lagrangian submanifold and $i : L \to M$ be the inclusion map. The self intersection $i_*[L] \cdot i_*[L]$ is the negative of the Euler characteristic of L.*

Proof - By Corollary 3.21 this is precisely the number of zeros with sign of a generic 1-form, *i.e.* the Euler characteristic of T^*L. Therefore we get

$$i_*[L] \cdot i_*[L] = -\chi(TL) = -\chi(L).$$

This completes the proof. ■

Corollary 3.23 - *The only compact orientable lagrangian submanifolds of $\mathbb{C}^2$ are tori.*

Proof - Since $H_2(\mathbb{C}^2;\mathbb{Z}) = 0$ we have that $i_*[L] = 0$ and hence $\chi(L) = 0$. By standard topology, every orientable compact connected surface with Euler characteristic zero is diffeomorphic to a 2-torus. This completes the proof. ■

This corollary shows us that if we are capable of finding a compact connect orientable lagrangian submanifold of $(\mathbb{C}^2, \omega_0)$ that admits a certain affine space N as its moment image, it must be a torus.

3.5.2 Ray theorems in $(\mathbb{C}^2, \omega_0)$

As already explained in Section 3.4.2 symplectic reduction can not be applied. However, we prove the existence of toric lagrangians also for other affine space and these results will be called the Ray Theorems in $(\mathbb{C}^2, \omega_0)$.

Theorem 3.24 (First Ray Theorem) **-** *Let $k \in \mathbb{R}_{>0}$, $a, b \in \mathbb{N}$ and consider the rational affine space*

$$N_k := \left\{ (\mu_1, \mu_2) \in \mathbb{R}^2_{\leqslant 0} \,\middle|\, a\,\mu_1 + b\,\mu_2 = -2k \right\}.$$

Then, there exists an toric lagrangian $L_k \subset (\mathbb{C}^2, \omega_0, \mathbb{T}^n, \mu)$ such that $\mu(L_k) = N_k$.

In order to prove this theorem we give an explicit construction of a lagrangian L_k with moment image N_k.

Proposition 3.25 - *Let $k \in \mathbb{R}_{>0}$ and $a, b \in \mathbb{N}$ be two natural numbers, the set*

$$L_k := \left\{ (r_1 \mathrm{e}^{\mathrm{i}\,a\,\theta}, r_2 \mathrm{e}^{\mathrm{i}\,b\,\theta}) \in \mathbb{C}^2 \,\middle|\, r_1, r_2 \in \mathbb{R},\ a\,r_1^2 + b\,r_2^2 = k, \theta \in [0, 2\pi) \right\} \tag{3.5}$$

is an immersed lagrangian of $(\mathbb{C}^2, \omega_0)$ with moment image $\mu(L_k) = N_k$ as defined in the First Ray Theorem 3.24.

This proof can be divided into the following steps.

Lemma 3.26 - *The set $L_k \subset \mathbb{C}^2$ as defined in Proposition 3.25 is a differentiable manifold for every $k \in \mathbb{R}_{>0}$.*

Proof - Here we take a direct approach and give a set of parameterizations that fulfill the requirements of a parameterization of a differentiable manifold. Since there are various definitions of differentiable manifolds we attached the ones we chose in Appendix A and we refer to Definition A.1. One can easily verify that the following four maps cover L_k and all of them are injective. This shows property (a).

$$\begin{array}{rccc} g_1 : & \left(0, \sqrt{\frac{k}{b}}\right) \times (0, 2\pi) & \longrightarrow & L \\ & (r,\theta) & \longmapsto & \left(\sqrt{\frac{k}{a} - \frac{b}{a} r^2}\mathrm{e}^{\mathrm{i}\,a\,\theta}, r\mathrm{e}^{-\mathrm{i}\,b\,\theta}\right) \end{array}$$

$$\begin{array}{rccc} g_2 : & \left(0, \sqrt{\frac{k}{b}}\right) \times (-\pi, \pi) & \longrightarrow & L \\ & (r,\theta) & \longmapsto & \left(\sqrt{\frac{k}{a} - \frac{b}{a} r^2}\mathrm{e}^{\mathrm{i}\,a\,\theta}, r\mathrm{e}^{-\mathrm{i}\,b\,\theta}\right) \end{array}$$

$$\begin{array}{rccc} h_1 : & \left(0, \sqrt{\frac{k}{a}}\right) \times (0, 2\pi) & \longrightarrow & L \\ & (r,\theta) & \longmapsto & \left(r\mathrm{e}^{-\mathrm{i}\,a\,\theta}, \sqrt{\frac{k}{b} - \frac{a}{b} r^2}\mathrm{e}^{\mathrm{i}\,b\,\theta}\right) \end{array}$$

$$\begin{array}{rccc} h_2 : & \left(0, \sqrt{\frac{k}{a}}\right) \times (-\pi, \pi) & \longrightarrow & L \\ & (r,\theta) & \longmapsto & \left(r\mathrm{e}^{-\mathrm{i}\,a\,\theta}, \sqrt{\frac{k}{b} - \frac{a}{b} r^2}\mathrm{e}^{\mathrm{i}\,b\,\theta}\right) \end{array}$$

For property (b) note that the intersections of the images of all possible pairs are open since every image is given open in L. So it is left to show that a change of parametrization is differentiable. For that purpose we need to compute the inverse maps. For $i \in \{1, 2\}$ we get

$$g_i^{-1}(z_1, z_2) = \left(|z_2|, -\frac{\mathrm{Arg}(z_2)}{b}\right) \quad \text{and} \quad h_i^{-1}(z_1, z_2) = \left(|z_1|, -\frac{\mathrm{Arg}(z_1)}{a}\right).$$

Changes of parametrizations between any of the g's or the h's are the identity on their intersection of domains. The only non-trivial reparameterizations are therefore between g's and h's. For all $i, j \in \{1,2\}$ we compute

$$\begin{aligned}\left(g_i^{-1} \circ h_j\right)(r,\theta) &= g_i^{-1}\left(r\mathrm{e}^{\mathrm{i}\,a\,\theta}, \sqrt{\frac{k}{b} - \frac{a}{b}r^2}\,\mathrm{e}^{\mathrm{i}\,b\,\theta}\right) \\ &= \left(\sqrt{\frac{k}{b} - \frac{a}{b}r^2}, -\theta\right) \quad \text{and} \\ \left(h_j^{-1} \circ g_i\right)(r,\theta) &= h_j^{-1}\left(\sqrt{\frac{k}{a} - \frac{b}{a}r^2}\,\mathrm{e}^{\mathrm{i}\,a\,\theta}, r\,\mathrm{e}^{-\mathrm{i}\,b\,\theta}\right) \\ &= \left(\sqrt{\frac{k}{a} - \frac{b}{a}r^2}, -\theta\right),\end{aligned}$$

for $(r,\theta) \in \mathrm{Dom}(g_i) \cap \mathrm{Dom}(h_j)$. The square root function is differentiable for every value that is non-zero and positive. Since $r \in \left(0, \min\left\{\sqrt{\frac{k}{b}}, \sqrt{\frac{k}{a}}\right\}\right)$ we get that these map is indeed differentiable, *i.e.*

$$g_i^{-1} \circ h_j, h_j^{-1} \circ g_i \in C^\infty$$

for all $i, j \in \{1,2\}$. This completes the proof. ■

Lemma 3.27 - *The manifold $L_k \subset \mathbb{C}^2$ as defined in Proposition 3.25 is immersed in $\mathbb{C}^2$.*

Proof - In order to be immersed we have to show that every parametrization has an injective differential. For simplicity, we consider g_1 and all to other maps can be treated again in a similar fashion. The differential computes to

$$\mathrm{d}g_1(r,\theta) = \begin{pmatrix} -\frac{b\,r}{a}\frac{1}{\sqrt{\frac{k}{b} - \frac{a}{b}r^2}}\mathrm{e}^{\mathrm{i}\,a\,\theta} & \mathrm{i}\,a\sqrt{\frac{k}{b} - \frac{a}{b}r^2}\,\mathrm{e}^{\mathrm{i}\,a\,\theta} \\ \mathrm{e}^{-\mathrm{i}\,b\,\theta} & \mathrm{i}\,b\,r\mathrm{e}^{-\mathrm{i}\,b\,\theta} \end{pmatrix}$$

and for the determinant we get

$$\begin{aligned}\det\left(\mathrm{d}g_1(r,\theta)\right) &= -\mathrm{i}\left(\frac{b^2}{a}r\frac{1}{\sqrt{\frac{k}{b} - \frac{a}{b}r^2}} + a\sqrt{\frac{k}{b} - \frac{a}{b}r^2}\right)\mathrm{e}^{\mathrm{i}\,(a-b)\theta} \\ &= -\frac{\mathrm{i}}{a}\underbrace{\frac{1}{\sqrt{\frac{k}{a} - \frac{b}{a}r^2}}}_{>0}\left(\underbrace{b^2r^2}_{>0} + a^2\underbrace{\left(\frac{k}{a} - \frac{b}{a}r^2\right)}_{>0}\right)\mathrm{e}^{\mathrm{i}\,(a-b)\theta} \neq 0.\end{aligned}$$

This shows that the differential of $\mathrm{d}g_1$ is injective. This completes the proof. ■

Lemma 3.28 - *The manifold $L \subset \mathbb{C}^2$ as defined in Proposition 3.25 is coisotropic in $(\mathbb{C}^2, \omega_0)$.*

Proof - Write ω_0 in polar coordinates as $\omega_0 = r_1 \mathrm{d}r_1 \wedge \mathrm{d}\theta_1 + r_2 \mathrm{d}r_2 \wedge \mathrm{d}\theta_2$ and denote by $i : L \hookrightarrow \mathbb{C}^2$ the inclusion map. Observe, that by definition, on L_k we have that the arguments of z_1 and z_2 are not independent. They satisfy the relation

$$a\,\theta_2 = b\,\theta_1 + k \cdot 2\pi \quad \Leftrightarrow \quad b\,\theta_1 - a\,\theta_2 \equiv 0 \bmod 2\pi.$$

Also by differentiation we get

$$a\,r_1 \mathrm{d}r_1 + b\,r_2 \mathrm{d}r_2 = 0 \quad \text{and} \quad b\,\mathrm{d}\theta_1 = a\,\mathrm{d}\theta_2.$$

So if we compute the pullback of ω_0 on L we get

$$\begin{aligned} i^*\omega_0 &= r_1 \mathrm{d}r_1 \wedge \mathrm{d}\theta_1 + r_2 \mathrm{d}r_2 \wedge \frac{b}{a}\mathrm{d}\theta_1 \\ &= \underbrace{(a\,r_1 \mathrm{d}r_1 + b\,r_2 \mathrm{d}r_2)}_{=0} \wedge \frac{1}{a}\mathrm{d}\theta_1 \\ &= 0. \end{aligned}$$

Therefore, ω_0 vanishes on L_k, which proves the Lemma. Alternatively one could also use Lemma 1.12 by showing L_k is lagrangian and show that $L_k^{\perp} = \mathrm{i}\,L_k$. This completes the proof. ■

Remark 3.29 - The above parameterizations were chosen such that the Jacobians of each reparametrization have a positive determinant. Therefore, we have an immersion of an orientable smooth manifold into $\mathbb{C}^2$.

Having these three lemmata we are now able to prove the First Ray Theorem.

Proof (Proof of First Ray Theorem 3.24) **-** By Lemma 3.26 and 3.27 we see that L_k is an immersed manifold and since it is also coisotropic by Lemma 3.28 we only need to check that the moment image of L_k is N_k. This follows directly from the definition of L_k and the fact that $-\frac{1}{2}r_i^2 = \mu_i$ for $i \in \{1, 2\}$. This completes the proof. ■

Theorem 3.30 - *The lagrangian submanifold L_k defined as Equation 3.5 is an immersed torus.*

Proof - For this we give an immersion as follows. Consider the map

$$\begin{array}{rccc} \psi: & \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2 & \longrightarrow & L \\ & (r_1, r_2, \theta) & \longmapsto & (r_1 \mathrm{e}^{\mathrm{i}\, a\, \theta}, r_2 \mathrm{e}^{\mathrm{i}\, b\, \theta}), \end{array}$$

where we use for the first $\mathbb{S}^1$ the coordinates $(r_1, r_2) \in \mathbb{R}^2$ with $a\, r_1^2 + b\, r_2^2 = k$ and for the second $\mathbb{S}^1$ that $\theta \in [0, 2\pi)$. This map is indeed an immersion as we show now. For this, we need to compute its Jacobian and check if this map is injective.

$$(J\psi)_{(r_1, r_2, \theta)} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\, a\, \theta} & 0 & \mathrm{i}\, a\, r_1 \mathrm{e}^{\mathrm{i}\, a\, \theta} \\ 0 & \mathrm{e}^{\mathrm{i}\, b\, \theta} & \mathrm{i}\, b\, r_2 \mathrm{e}^{\mathrm{i}\, b\, \theta} \end{pmatrix}$$

Let $v = (v_1, v_2, v_3) \in T_{(r_1, r_2, \theta)}\mathbb{T}^2$ be a tangent vector. For injectivity it is sufficient to compute the kernel of $(J\psi)$ via

$$(J\psi)_{(r_1, r_2, \theta)} v = \begin{pmatrix} (v_1 + \mathrm{i}\, a\, r_1\, v_3) \mathrm{e}^{\mathrm{i}\, a\, \theta} \\ (v_2 + \mathrm{i}\, b\, r_2\, v_3) \mathrm{e}^{\mathrm{i}\, b\, \theta} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.6}$$

This yields

$$\mathrm{i}\, v_1 = a\, r_1 v_3 \quad \text{and} \quad \mathrm{i}\, v_2 = b\, r_2 v_3, \quad \Leftrightarrow \quad -\frac{v_1^2}{a\, v_3^2} = a\, r_1^2 \quad \text{and} \quad -\frac{v_2^2}{b\, v_3^2} = b\, r_2^2.$$

Adding these two equations we get

$$\frac{v_1^2}{a} + \frac{v_2^2}{b} = -k v_3^2.$$

Since $a, b, k > 0$ this equation only holds if $v_1 = v_2 = v_3 = 0$ and hence ψ is a surjective immersion. This completes the proof. ■

The First Ray Theorem 3.24 only considered rational affine subspaces that intersect both axes of $\mathbb{R}^2_{\leqslant 0}$. However, the same construction can be extended easily.

Theorem 3.31 (Second Ray Theorem) **-** *Let $k \in \mathbb{R}$, $a, b \in \mathbb{N}$ two natural numbers and consider the rational affine space*

$$N_k := \left\{ (\mu_1, \mu_2) \in \mathbb{R}^2_{\leqslant 0} \,\middle|\, a\, \mu_1 - b\, \mu_2 = -2k \right\}.$$

Then, there exists a toric lagrangian $L_k \subset (\mathbb{C}^2, \omega_0, \mathbb{T}^n, \mu)$ such that $\mu(L_k) = N_k$.

Proof - The computation is a straight forward generalization of the one in the proof of the First Ray Theorem 3.24. Define

$$L_k = \left\{(r_1 e^{i\,a\,\theta}, r_2 e^{i\,b\,\theta}) \in \mathbb{C}^2 \,|\, a\,r_1^2 - b r_2^2 = k\right\}.$$

Then, one can check again that this is indeed a immersed lagrangian in $(\mathbb{C}^2, \omega_0)$. This completes the proof. ■

The above results can be summarized as follows.

Corollary 3.32 (Ray Theorem) **-** *Let $N \subset \mathbb{R}^2_{\leqslant 0}$ be an affine subspace with rational slope. Then, there exists an immersed lagrangian L of $(\mathbb{C}^2, \omega, \mathbb{T}^n, \mu)$ with $\mu(L) = N$.*

From the results about the Delzant affine spaces and the Ray Theorem we are able to formulate the central conjecture of this thesis. Further details of why we think this might be true are given in the concluding chapter.

Conjecture 3.33 - *Let $N \subset \mathbb{R}^2_{\leqslant 0}$ an affine subspace. Then, there exists a lagrangian submanifold L of $(\mathbb{C}^2, \omega, \mathbb{T}^2, \mu)$ with $\mu(L) = N$ if and only if N is Delzant.*

Remark 3.34 - In the above considerations we were only interested in affine spaces with rational slopes. However, one could ask if there are also toric lagrangians in the pre-image of non-rational affine spaces. This question can be answered with a clear no. The central reason is, that if such a subgroup $H \subseteq \mathbb{T}^2$ exists, it is no closed. For example $\mathbb{R} \subset \mathbb{S}^1$ is a dense subgroup, but in our definition we neglected these cases. We are only interested in closed subgroups.

4 Summary

In order to close this thesis we give some further remarks on the presented material. Also, we give a conclusion of our findings and point out the future research that could be done to extend our findings.

4.1 Remarks

In Section 3.5.2 we have proven that the set L_k as in Proposition 3.25 is an immersed lagrangian of $(\mathbb{C}^2, \omega_0, \mathbb{T}^n, \mu)$ with $\mu(L_k) = N_k$. Here, we give some remarks why it is not easy, or even not possible at all to find an embedded lagrangian with the same moment image property.

As we have seen in Section 3.4.2 the obtained action on $\mu^{-1}(N_k)$ is not free if $a \neq b$, which is a crucial. Namely, in this case, the quotient $\mu^{-1}(N_k) /_{\mathbb{S}^1}$ is in general not a manifold. It is rather a so-called orbifold and therefore the procedure of quotiening manifolds is not closed. See Appendix B for the exact definition and some properties of orbifolds. However, Eugene Lermann and Susan Tolmann showed in 1977 [LT97] that the whole concept of symplectic quotients and the Delzant Theorem 2.19 can be generalized to symplectic orifolds and is in that category closed. Therefore, the notion of a symplectic orbifolds seems to be the more natural definition. However, a symplectic orbifold can have a very complicated shape due to the occurrences of singularities. Also in physical fields like String Theory a lot of attention is drawn towards symplectic orbifolds.

The study of singularies of quotient spaces is very old in its own. In particular, singularities of $\mathbb{C}^2$ over some subgroup $G \subseteq \mathrm{SL}(n, \mathbb{C})$ were studied and classified in

1884 by Klein. In this case it is possible to resolve every singularity by changing coordinates or "blowing" them up. For details consider Theorem B.14. If we apply this to our setting one could achieve that $\mu^{-1}(N_k)$ can be deformed to a smooth manifold. However, it is far from obvious how to prove the existence of a smooth lagrangian in this manifold with corresponding moment image.

4.2 Conclusion

The aim of this thesis was to study the relationships between Lie subgroups of $\mathbb{T}^n$, affine subspaces of the moment image and lagrangian subspaces with corresponding moment image of $(\mathbb{C}^n, \omega_0)$. Due to its complexity we studied the most simple cases of $\mathbb{C}$ and $\mathbb{C}^2$. There, we gave explicit constructions of immersed and embedded lagrangians with respect to given rational affine subspaces. We discovered that finding an embedded lagrangian on a non-Delzant affine space is quite challenging due to the nature of the occurring orbifold singularities. At most, we could only archive immersed lagrangians, which are of interest in their own. This relationship we formulated at the above Conjecture 3.33. By combining the Ray Theorem 3.32 with the invariance Theorem 3.10 even more can be archived. With these it is possible to draw a connection between the non-discrete Lie subgroups of $\mathbb{T}^2$ and rational affine subspaces of $\mathbb{C}^n$, *i.e.* for every non-discrete subgroup we have a principle of how to pick an affine rational subspace which has a toric lagrangian in its pre-image with invariance group is the given Lie subgroup. However, these statements are here only explained and explored in $\mathbb{C}^2$, but we think that they can be extended generally to $\mathbb{C}^n$.

4.3 Future research

Further work on this could be done in four directions.

1. One could try to proceed further and try to extend the construction of immersed lagrangians with given rational affine subspace moment image to $\mathbb{C}^3$ or generally $\mathbb{C}^n$. However, the difficulty may lie in proving the immersion property for this general construction. Furthermore, a generlization of Definition 3.5 has to be found. If this would work out one could adapt the framework such that Conjecture 3.33 could be formulated for the case of general $(\mathbb{C}^n, \omega_0, \mathbb{T}^n, \mu)$ instead of only for $n = 2$.

2. The most interesting and challenging task would be to try to prove or disprove the central Conjecture 3.33 for the case of $\mathbb{C}^2$ or even for $\mathbb{C}^n$.

3. Conjecture 3.33 might also give a connection to the concepts of symplectic cutting and symplectic blow-up which are not presented in this thesis. For further details consider [AdSL00], part B, sections I.3.5 and I.3.6. There, a relation between "cutting out" certain areas of a manifold are also related to affine subspaces in the moment image that are Delzant. Maybe this would yield further insights into the nature of pre-images of affine subspaces.

4. In the conclusion we mentioned a connection from non-discrete subgroups to rational affine subspaces and their lagrangians. By Theorem 1.26 there also exists subgroups that include $\mathbb{Z}_k$ for some $k \in \mathbb{N}$. Therefore, a further task would be to check if we can find immersed lagrangians or even embedded lagrangians that are invariant with respect to these groups.

$\mathcal{A}$ Appendix - Differentiable Manifold

Here, we recall some of the basics definitions of embedded submanifolds from [dC92] in order to fix the notation we are using the proof of the Ray Theorems in Chapter 3.

Definition A.1 - A set M is called a **differentiable manifold** of dimension n if there exists a family, labeled by some index set I, of injective maps $\varphi_\alpha : U_\alpha \subset \mathbb{R}^n \to M$, where U_α are open subsets of $\mathbb{R}^n$, such that

(a) M is covered by $\varphi_\alpha(U_\alpha)$, that is $\bigcup_{\alpha\in I} \varphi_\alpha(U_\alpha) = M$,

(b) for any pair $\alpha, \beta \in I$ with $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) = W \neq \varnothing$ the sets $\varphi_\alpha^{-1}(W)$ and $\varphi_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ and the map $\varphi_\beta^{-1} \circ \varphi_\alpha$ is differentiable and

(c) the family $\{(\varphi_\alpha, U_\alpha)\}_{\alpha\in I}$ is maximal relative to the conditions (a) and (b).

The family $\{(\varphi_\alpha, U_\alpha)\}_{\alpha\in I}$ is called a **parametrization** of M.

Definition A.2 - Let M and N be two differentiable manifold. A smooth map $\varphi : M \to N$ is said to be an **immersion** if the differential map $\mathrm{d}\varphi_p : T_pM \to T_{\varphi(p)}N$ is injective for all $p \in M$. If furthermore φ is a homeomorphism onto its image $\varphi(M) \subset N$ then it is called an **embedding** and M is called a submanifold.

B

Appendix - Orbifolds

In Chapter 3 we constructed lagrangian submanifolds L_k that are immersed into $\mathbb{C}^2$ but not embedded. The central reason for this is that the corresponding action is not free and yields to a singularity after quotening. Instead of producing a manifold the corresponding space is an orbifold, which is a generalization of the concept of a manifold. Since the are only a few texts about orbifolds we sum up the most important definitions and theorems about them. We follow the textbooks [ALR07] and the excellent work [Thu02].

Definition B.1 - Let X be a topological space and $n \in \mathbb{N}$. A n-dimensional **orbifold chart** on X is given by a connected open subset $\tilde{U} \subseteq \mathbb{R}^n$, a finite group $G \subseteq \operatorname{Aut}(\tilde{U})$ and a map $\psi : \tilde{U} \to X$ such that it is G-invariant and induces a homeomorphism $\psi : {}^{\tilde{U}}/_{G} \to U$ onto an open subset $U \subseteq X$.

Definition B.2 - Let X be a topological space and $\psi : \tilde{U} \to X$, $\varphi : \tilde{V} \to X$ be two n-dimensional orbifold charts. Then, a map $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ is called an **embedding** if it is smooth and $\psi \circ \lambda = \varphi$.

Definition B.3 - An **orbifold atlas** on X is a family $\mathcal{U} = \big\{(\tilde{U}_\alpha, G_\alpha, \psi_\alpha)\big\}\big)_{\alpha \in I}$ of orbifold charts that cover X and are locally compatible. That is, given two charts $(\tilde{U}_\alpha, G_\alpha, \psi_\alpha)$ and $(\tilde{U}_\beta, G_\beta, \psi_\beta)$ for $\alpha, \beta \in I$, a point $x \in U_\alpha \cap U_\beta$, there exists an open neighbourhood $W \subseteq U \cap V$ and a chart $(\tilde{V}, H, \varphi)$ for W such that there are embeddings $(\tilde{V}, H, \varphi) \hookrightarrow (\tilde{U}_i, G_i, \psi_i)$ for $i \in \{\alpha, \beta\}$.

Definition B.4 - An atlas $\mathcal{U}$ is said to **refine** on another atlas $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of $\mathcal{V}$. Two orbifold atlases are said to be **equivalent** if they have a common refinement.

Definition B.5 - An **effective orbifold** $\mathcal{X}$ of dimension n is a paracompact Hausdorff space X equipped with an equivalence class $[\mathcal{U}]$ of n-dimensional orbifold atlases.

Remark B.6 - The concept of an orbifold is closely related to the one of a manifold as the following remarks will state.

- We assume that for each orbifold chart $(\tilde{U}, G, \psi)$ the group G is acting smoothly and effectively.
- Since every smooth action is locally smooth, any orbifold has an atlas consiting of linear charts, which are of the form $(\mathbb{R}^n, G, \psi)$, where G acts on $\mathbb{R}^n$ via an orthogonal representation, that is, $G \subseteq \mathrm{O}(n, \mathbb{R})$.
- If every finite group action on a orbifold chart is free the above definition is the one of a manifold.

Definition B.7 - Let $x \in X$ and $\mathcal{X} = (X, \mathcal{U})$ be an orbifold. If $(\tilde{U}, G, \psi)$ is a local chart around $x = \psi(y)$ for some $y \in \tilde{U}$ we define the **local group** at x as

$$G_x := \{g \in G \,|\, gy = y\}\,.$$

Note that this group is unique up to conjugacy in G.

Definition B.8 - For an orbifold $\mathcal{X} = (X, \mathcal{U})$ we define its **singular set** as

$$\Sigma(\mathcal{X}) := \{x \in X \,|\, G_x \neq \{e\}\}\,,$$

i.e. the set of all points in X on which its local group is non-trivial.

The following part considers the most common case of orbifolds, that is, orbifolds which are obtained by quotiening.

Definition B.9 - An **effective quotient orbifold** $\mathcal{X} = (X,\mathcal{U})$ is an orbifold given as the quotient of an effective, smooth and almost free action of a compact Lie group G on a smooth manifold M.

Remark B.10 - If G is a compact Lie group that acts smoothly, effectively and almost freely on a manifold M, then for every $x \in M$ its local group is simply the isotropy group on a local chart around x in M.

Definition B.11 - An orbifold $\mathcal{X} = (X,\mathcal{U})$ is called a **complex orbifold** if X is a complex manifold.

Definition B.12 - Let $\mathcal{X}$ be a complex orbifold and $f : Y \to \mathcal{X}$ a holomorphic map from a smooth complex manifold Y to $\mathcal{X}$. Then, f is called a **resolution** if $f|_{\mathcal{X}\setminus\Sigma(\mathcal{X})}$ is biholomorphic and $f^{-1}(\Sigma(\mathcal{X}))$ is an analytic subset of Y. A resolution F is called **crepant** if $f^*K_{\mathcal{X}} = K_Y$, where $K_X = \bigwedge_{\mathbb{C}}^n T^*\mathcal{X}$ denotes the canonical bundle over a orbifold $\mathcal{X}$ and $n = \dim \mathcal{X}$.

Definition B.13 - A n-dimensional complex orbifold $\mathcal{X}$ is called **Gorenstein** if all the local group G_x are subgroups of $\mathrm{SL}(n, \mathbb{C})$.

Theorem B.14 - *For the complex case of* $\mathbb{C}^2$ *and* $G \subseteq \mathrm{SL}(2, \mathbb{C})$ *every singularity of* $\mathbb{C}^2/G$ *admits a unique crepant resolution* (Y, f).

Proof - A proof of this can be found in [ALR07], Example 1.59. Furthermore this result goes back to the first classification by Klein in 1884. ∎

Remark B.15 - Theorem B.14 is also true for the case of $n = 3$. However, for $n \geqslant 4$ the possible resolutions of singularities in $\mathbb{C}^n$ are not well understood. See for example [Ade02] for further details. Also these constructions are part of the so-called McKay correspondence, see [Ade02] and [McK80].

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Notation index

$\delta_{ij} = \begin{cases} 1 & \text{, if } i = j \\ 0 & \text{, if } i \neq j \end{cases}$	Kronecker delta
$\mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$	Set of positive real numbers
$\mathbb{R}_{<0} := \{x \in \mathbb{R} \mid x < 0\}$	Set of negative real numbers
$\mathbb{R}_{\geqslant 0} := \{x \in \mathbb{R} \mid x \geqslant 0\}$	Set of non-negative real numbers
$\mathbb{R}_{\leqslant 0} := \{x \in \mathbb{R} \mid x \leqslant 0\}$	Set of non-positive real numbers
$\mathrm{Dom}(f) \subseteq X$	Domain of the map $f : X \to Y$
$\mathrm{Imag}(f) \subseteq Y$	Image of the map $f : X \to Y$
$\mathrm{Ker}(f) \subseteq X$	Kernel of the linear map $f : X \to Y$
V	$\mathbb{K}$-vector space
V^*	Dual vector space to V
$\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$	Natural pairing of a vector space with its dual
$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$	Euclidean inner product on $\mathbb{R}^n$
M, N	Standard symbols for differentiable manifolds
$\Gamma(TM)$	Section of the tangent bundle TM, *i.e.* vector field on M
$\exp : TM \to M$	Exponential map from tangent bundle to manifold
$C^\infty(M, N)$	Group of differentiable maps from M to N
$C^\infty(M) := C^\infty(M, \mathbb{R})$	Group of differentiable real-valued functions on M
$i : N \hookrightarrow M$	Inclusion map of $N \subseteq M$ into M
$\chi(M)$	Euler characteristic of M
$\Omega^k(M)$	Space of differentiable k-forms on M
$\iota_X \omega$	($k-1$-form obtained by inserting the vector field X in

	the first entry of the k-form ω
$\mathrm{Diff}(M,N)$	Group of diffeomorphism of M to N
$\mathrm{Diff}(M) := \mathrm{Diff}(M,M)$	Group of diffeomorphisms of M onto itself
$\mathrm{Symp}(M,\omega)$	Group of symplectomorphisms of (M,ω)
G, H	Standard symbols for Lie (sub)groups
$\mathfrak{g}, \mathfrak{h}$	Standard symbols for Lie algebras of Lie groups G, H
$L_g, R_g : G \to G$	Left and right multiplication in G by $g \in G$
$\mathrm{Aut}(G)$	Group of Automorphisms of G

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