Equivariant Cohomology and the Duistermaat-Heckman Theorems

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March 26, 2018

Master Thesis

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1 Introduction

When we have a topological space M with an action of a group G on it, we would like to have a functor which reflects both the topology of the space and the action of the group. Since singular cohomology is a functor from topological spaces to rings, which turns geometric problems into easier algebraic problems, a candidate could be the singular cohomology of the orbit space M/G. It turns out that this is not a good candidate, as for example when the circle S^1 acts on the sphere S^2 in $\mathbb{R}^3$ by rotations about the z-axis, the orbit space S^2/S^1 is a closed interval and its cohomology is trivial. So we don't get any information about the group action. The main issue in working with the quotient space M/G is that when the action is not *free*, the quotient might be "not nice", for instance, not Hausdorff or not carrying a smooth structure. To overcome this, one can substitute M by a space of the same homotopy type but on which G acts freely. This is done by considering instead of M the product space $EG\times M$, where EG is a contractible space on which the group G acts freely. Then one makes G act on $EG\times M$ diagonally, that is, for $g\in G$ and $(e,x)\in EG\times M$ one sets $g\cdot(e,x)=(g\cdot e, g\cdot x)$. This action is always free, because it is in the first component, and since EG is chosen to be contractible, $EG\times M$ and M have the same homotopy type. The orbit space $M_G := (EG\times M)/G$ of this action is called the *Borel construction* (or *homotopy quotient*) on M and one defines the *equivariant cohomology* $H^*_G(M)$ of M to be the singular cohomology of the space $(EG\times M)/G$:

$$H^*_G(M) := H^*((EG\times M)/G).$$

To ensure this definition to be well defined, one has to check two things, namely,

(i) for every group G there exists a contractible space EG on which G acts freely, and

(ii) the definition of $H_G(M)$ doesn't depend on the choice of the space EG.

In algebraic topology it is known that a contractible space on which G acts freely is given by the total space of a universal principal G-bundle $\pi : EG \longrightarrow BG$. In [27] Milnor gives an explicit construction of a principal G-bundle $EG\longrightarrow BG$ for any group G and, although he proves that the spaces EG are just *weakly* contractible (that is, all their homotopy groups are trivial), it turns out (see [14]) that they are actually contractible, and thus they can be used as building spaces for the homotopy quotient of M. In the case of the circle $G=S^1$, the space EG is homeomorphic to the infinite sphere S^∞, and the quotient BG is given by the infinite projective space $\mathbb{CP}^\infty$. The advantage of considering EG as the infinite sphere is that it is a union of finite dimensional manifolds ($S^\infty = \cup_n S^n$), and one can approximate M_{S^1} by the spaces $(S^n\times M)/S^1$, in the sense that for each $i\in\mathbb{N}$ there is some n big enough such that

$$H^i(M_{S^1}) = H^i((S^n\times M)/S^1).$$

Since the action of S^1 on $S^n \times M$ is free, when M is a manifold the quotient space $(S^n \times M)/S^1$ is a manifold as well and, by de Rham's theorem, its singular cohomology coincides with its de Rham cohomology. Thus elements of the equivariant cohomology group $H^i_{S^1}(M)$ are represented by differential i-forms on $(S^n \times M)/S^1$. It is possible to use this kind of finite approximations for any compact Lie group G (Section 2.6). Therefore in the case of compact Lie groups we can work with differential forms on finite approximations, as long as we are interested in equivariant cohomology groups of fixed degree. If we consider the ring $H^*_G(M)$, this approach doesn't work anymore, and a different strategy is needed.

In 1950, H. Cartan wrote two articles on the cohomology of a manifold M acted on by a compact connected Lie group G, in which he constructed a differential complex $(\Omega^*_G(M), d_G)$ out of the differential forms on M and the Lie algebra of G. At that time, equivariant cohomology wasn't yet defined (the topological definition was first introduced by Borel in 1959), and in fact the term "equivariant cohomology" itself never appears in his papers [31]. However, it turned out that Cartan's complex computes the real singular cohomology of the Borel quotient M_G, and the key step in the proof of this *equivariant de Rham theorem* (Theorem 3.37) is given by Cartan in [12]. We call elements of the Cartan's complex $\Omega^*_G(M)$ *equivariant differential forms.* Although Cartan's model can be very convenient to work with, it is not so clear *why* it should compute the equivariant cohomology. This is better understood from another chain complex isomorphic to Cartan's, which is called the *Weil model.*

One of the features of equivariant cohomology are the "localization theorems", which enable to reduce many computations on the manifold to the fixed-point set of the group action. One example of these theorems is the *equivariant localization formula* (Theorem 6.32) for the integral of closed equivariant forms. If a torus T acts on a compact oriented manifold M with discrete fixed point set F, and ϕ is a closed equivariant form on M, then the localization theorem states that

$$\int_M \phi = \sum_{z \in F} \frac{i_z^* \phi}{e_T(\nu_z)}, \tag{1.1}$$

where $i_z^*\phi$ is the restriction to ϕ to a fixed point z and $e_T(\nu_z)$ is the equivariant Euler class of the normal bundle to z in M. Here is where symplectic geometry and Hamiltonian actions come into play. In 1982, Duistermaat and Heckman (see [15]) proved the "exact stationary phase formula", which states that if (M, ω) is a symplectic manifold with moment map $f : M \longrightarrow \mathbb{R}$ on which the circle S^1 acts with isolated fixed points, then

$$\int_M e^{-itf} \frac{\omega^n}{n!} = \sum_{z \in F} \frac{e^{-itf(z)}}{(it)^n e(z)}, \tag{1.2}$$

where $e(z)$ are certain integers (the weights) attached to the linear S^1 action on T_zM. Soon after, Berline and Vergne (see [4]) and Atiyah and Bott (see [1]) recognized this formula as a special case of the above mentioned localization theorem in equivariant cohomology. The language of equivariant cohomology actually fits very well in the context of Hamiltonian actions. For instance, when S^1 acts on a symplectic manifold (M, ω), the action admits a moment map $f : M \longrightarrow \mathbb{R}$ if and only if $\omega - fu$ is an equivariantly closed form on M (Proposition 4.14). In particular, if the action is Hamiltonian, using (1.1) one can compute the integral of the volume form ω^n.

Besides the exact phase formula, Duistermaat and Heckman proved in [15] another theorem concerning Hamiltonian actions. Namely, they showed that on a symplectic T^l-manifold (M, ω) with proper moment map $\mu : M \longrightarrow \mathfrak{t}^*$, the push-forward of the Liouville measure by μ is a piecewise polynomial multiple of the Lebesgue measure on $\mathfrak{t}^* \cong \mathbb{R}^l$ (see [11, §30]). We are going to prove this result (Proposition 5.11) using the equivariantly closed extension of the symplectic form ω in terms of the moment map μ of which we were talking above. Thus this illustrates another application of equivariant cohomology in the field of Hamiltonian actions.

1.1 Overview

The goal of this thesis is to expand and explain in detail what we have illustrated above, namely the equivariant cohomology theory and its application in the context of Hamiltonian actions.

In Section 2 we define principal G-bundles and construct a universal G-bundle for any group G using Milnor joins. Then we define the Borel construction and the equivariant cohomology of a G-space M.

In Section 3 we talk about the de Rham theoretic models of equivariant cohomology. Starting from the Weil model we define an isomorphism to the Cartan model and at the end of the section we outline the proof of the equivariant de Rham theorem.

Section 4 illustrates a first instance of the relationship between equivariant cohomology and Hamiltonian actions. Namely, we prove that a symplectic manifold admits a moment map if and only if the symplectic form extends to an equivariantly closed form. We show this both using the Cartan model and via finite approximations. Since the proof requires some knowledge about connections on principal bundles, we include an introduction about them in Appendix A.

In Section 5 we show the first Duistermaat-Heckman theorem, which states that when a torus T acts on a symplectic manifold with proper moment map μ, the reduced symplectic form ω_ξ on the quotient of a regular level $\mu^{-1}(\xi)$ depends linearly on ξ.

Section 6 is devoted to the proof of the equivariant localization formula (1.1) in the case of a circle action, from which we then deduce the Duistermaat-Heckman formula (1.2). To make this section lighter, we moved some of the used tools to Appendix B, which is dedicated to the definition of the Gysin homomorphism, passing through the equivariant tubular neighborhood theorem and the equivariant Euler class.

In the last appendix we show an explicit formula for the equivariant Euler class of the normal bundle ν_z to a fixed point z in an S^1-manifold M. This is useful for two reasons. On the one hand, it shows that the equivariant Euler class is non-zero and thus can be inverted (notice that $e_{S^1}(\nu_z)$ appears as a denominator in (1.1)). On the other hand, it allows explicit calculations (see for instance Example 6.38).

1.2 Acknowledgments

I would first like to thank my thesis advisor Dr. Ana Cannas da Silva for introducing me to this stimulating subject, and for guiding me through it. She has been of great help whenever I needed, and her door was always open. I would also like to thank Prof. Loring Tu for sharing his knowledge and insight into the topic. I cannot thank Tommaso Goldhirsch enough for his precious help and endless support. Last but not least, I would like to thank my family for all the encouragement and love.

2 Borel construction and equivariant cohomology

In this section we follow the exposition of [3]. For a topological group G we will denote its identity element by 1_G. The action of G on a topological space X is a continuous map $\phi : G \times X \longrightarrow X$, written $\phi(g, x) = g \cdot x$ or simply $\phi(g, x) = gx$.

2.1 Principal G-bundles

Definition 2.1. Let E, B and F be topological spaces and let $p : E \longrightarrow B$ be a continuous map. The triple (p, E, B) is a *fiber bundle* with base B, total space E and fiber F if

(i) the map $p : E \longrightarrow B$ is surjective,

(ii) there is an open cover $\{U_i\}_{i\in I}$ of B and homeomorphisms

$$h_i : p^{-1}(U_i) \longrightarrow U_i \times F,$$

such that the following diagram commutes

$$\begin{tikzcd}
p^{-1}(U_i) \arrow[rr, "h_i"] \arrow[dr, "p"'] & & U_i \times F \arrow[dl, "\mathrm{pr}_1"] \\
& U_i. &
\end{tikzcd}$$

The homeomorphisms h_i are called *local trivializations*.

Definition 2.2. Let G be a topological group. A *principal G-bundle* is a fiber bundle $p : E \longrightarrow B$ with fiber G together with a continuous action $G \times E \longrightarrow E$ such that

(i) G preserves the fibers of E and acts freely and transitively on them,

(ii) the local trivializations $h_i : p^{-1}(U_i) \longrightarrow U_i \times G$ satisfy $h_i(g \cdot e) = g \cdot h_i(e)$, for all $g \in G$, $e \in p^{-1}(U_i)$, where G acts on $U_i \times G$ by $g(x, g') = (x, gg')$.

We say that two principal G-bundles E_1, E_2 over B are isomorphic if there is a homeomorphism $f : E_1 \longrightarrow E_2$ such that the diagram

$$\begin{tikzcd}
E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\
& B &
\end{tikzcd}$$

commutes and f is G-equivariant, i.e.

$$f(g \cdot e_1) = g \cdot f(e_1) \quad \text{for all } g \in G, e_1 \in E_1.$$

Definition 2.3. Let X be a topological space. A family of continuous functions $\{u_i : X \longrightarrow [0,1]\}_{i\in I}$ is called *locally finite* if every point $x \in X$ has a neighborhood U such that $u_i|_U = 0$ for all but finitely many i. It is a *partition of unity* if $\sum_{i\in I} u_i(x) = 1$ for every $x \in X$.

Definition 2.4. An open covering $\{U_i\}_{i\in I}$ of a topological space X is called *numerable* if there exists a locally finite partition of unity $\{u_i\}_{i\in I}$ on X such that $\mathrm{supp}(u_i) \subset U_i$ for all $i \in I$. Such a partition of unity is said to be subordinate to the cover $\{U_i\}_{i\in I}$.

Definition 2.5. A principal G-bundle is called *numerable* if there exists a numerable covering of B which makes it locally trivial. That is, there exists a covering of B by local trivializations $\{(U_i, h_i)\}_{i\in I}$ and a locally finite partition of unity $\{u_i\}_{i\in I}$ with $\mathrm{supp}(u_i) \subset U_i$ for all $i \in I$.

In the context of manifolds[1] one has the following theorem.

Theorem 2.6 ([9, Theorem 3.15])**.** *Let M be a smooth manifold and let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of M. Then there is a countable locally finite partition of unity $\{u_n\}_{n\in\mathbb{N}}$ on M such that for every $\alpha \in I$ there is $n \in \mathbb{N}$ with* $\mathrm{supp}(u_n) \subset U_\alpha$.

In particular, we have the following corollary.

Corollary 2.7. *Let E, B be manifolds. Then every principal G-bundle $p : E \longrightarrow B$ is numerable.*

2.2 The Milnor join

Let G be a topological group. We want to show the existence of a contractible space EG on which G acts freely. In [26] Milnor gives an explicit construction of such a space, which we now illustrate. Consider the space

$$G^{n+1} \times \Delta^n := \{(x_0, t_0; \ldots; x_n, t_n) \mid x_i \in G,\ t_i \in [0,1]\}$$

and let EG_n be the quotient of $G^{n+1} \times \Delta^n$ by the equivalence relation

$$(x_0, t_0; \ldots; x_n, t_n) \sim (x_0', t_0'; \ldots; x_n', t_n') \text{ if and only if } \begin{cases} t_i = t_i' \quad \forall i, \text{ and} \\ t_i = t_i' \neq 0 \ \Rightarrow \ x_i = x_i'. \end{cases}$$

We shall write $\langle x_0, t_0; \ldots; x_n, t_n\rangle$ for the equivalence class of the element under consideration. The space EG_n is called the *Milnor join.* We equip EG_n with the initial topology with respect to the maps

$$\begin{aligned} t_i : EG_n &\longrightarrow [0,1] \\ \langle x_0, t_0; \ldots; x_n, t_n\rangle &\longmapsto t_i \end{aligned}$$

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[1] For us manifolds are always smooth, Hausdorff and second-countable.

and

$$\begin{aligned} a_i : t_i^{-1}((0,1]) &\longrightarrow G \\ \langle x_0, t_0; \ldots; x_n, t_n \rangle &\longmapsto x_i, \end{aligned}$$

that is, the smallest (finest) topology which contains

$$\{t_i^{-1}(U) \mid U \overset{\text{open}}{\subset} [0,1]\} \cup \{a_i^{-1}(V) \mid V \overset{\text{open}}{\subset} G\}.$$

Thus a subbasis for the topology is given by sets of the following two types

(1) $t_i^{-1}(U)$, $i \in \{0, \ldots, n\}$, $U \subset [0,1]$ open.

(2) $a_i^{-1}(V)$, $i \in \{0, \ldots, n\}$, $V \subset G$ open.

Lemma 2.8. *A function $f : X \longrightarrow EG_n$ is continuous if and only if $t_i \circ f : X \longrightarrow [0,1]$ and $a_i \circ f\big|_{f^{-1}(t_i^{-1}((0,1]))} : f^{-1}(t_i^{-1}((0,1])) \longrightarrow G$ are continuous for all i.*

Proof. "$\Rightarrow$": Suppose that $f : X \longrightarrow EG_n$ is continuous. Consider first an open subset $U \subset [0,1]$. Then $(t_i \circ f)^{-1}(U) = f^{-1}(t_i^{-1}(U))$ is open in X because $t_i^{-1}(U)$ is open in EG_n. This shows that $t_i \circ f$ is continuous for all i. Let $D_{a_i} := t_i^{-1}((0,1])$ denote the domain of a_i. Notice that by assumption the domain $f^{-1}(D_{a_i})$ of $a_i \circ f$ is open in X. Let now $V \subset G$ be open. Then $(a_i \circ f)^{-1}(V) = f^{-1}(a_i^{-1}(V))$ is open in X. But since $(a_i \circ f)^{-1}(V) = (a_i \circ f)^{-1}(V) \cap f^{-1}(D_{a_i})$, it is open also in $f^{-1}(D_{a_i})$. Therefore $a_i \circ f$ is continuous for all i.

"$\Leftarrow$": Since intersections and unions behave well with respect to preimages, it suffices to check that preimages of elements in the subbasis for EG_n are open in X. But for an open subset $U \subset [0,1]$ it holds $f^{-1}(t_i^{-1}(U)) = (t_i \circ f)^{-1}(U)$ which is open in X. For $V \subset G$ open, the set $f^{-1}(a_i^{-1}(V)) = (a_i \circ f)^{-1}(V)$ is open in $f^{-1}(D_{a_i}) = (t_i \circ f)^{-1}((0,1])$, and the latter is open in X by continuity of $t_i \circ f$. ☐

Example 2.9. (1) Let $G = S^1$. Then ES_n^1 can be identified with S^{2n+1} by the map

$$\begin{aligned} \phi : ES_n^1 &\longrightarrow S^{2n+1} \\ \langle z_0, t_0; \ldots; z_n, t_n \rangle &\longmapsto (\sqrt{t_0} z_0, \ldots, \sqrt{t_n} z_n). \end{aligned}$$

Here we are considering $S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}$ with the product of complex numbers as group law, and $S^{2n+1} = \{(w_0, \ldots, w_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^n |w_i|^2 = 1\} \subseteq \mathbb{C}^{n+1}$. We check the well-definedness and bijectivity of the map.

Well-definedness: Let $\langle z_0, t_0; \ldots; z_n, t_n \rangle \in ES_n^1$. Then

$$\sum_{i=0}^n |\sqrt{t_i} z_i|^2 = \sum_{i=0}^n t_i \underbrace{|z_i|^2}_{=1} = \sum_{i=0}^n t_i = 1.$$

Injectivity: Suppose that $\langle z_0, t_0; \ldots; z_n, t_n\rangle$ and $\langle z_0', t_0'; \ldots; z_n', t_n'\rangle$ are in ES^1_n with $\sqrt{t_i}z_i = \sqrt{t_i'}z_i'$ for all i. By taking modules we get that $|\sqrt{t_i}| = |\sqrt{t_i'}|$ and thus $t_i = t_i'$ for all i. If $t_i = t_i' > 0$, then $z_i = z_i'$. Else, z_i and z_i' can be any element of the group by definition of the equivalence relation on $G^{n+1} \times \Delta^n$.

Surjectivity: Let $(w_0, \ldots, w_n) \in S^{2n+1}$, so that $\sum_{i=0}^n |w_i|^2 = 1$. Then $\langle \frac{w_0}{|w_0|}, |w_0|^2; \ldots; \frac{w_n}{|w_n|}, |w_n|^2\rangle$ is an element of ES^1_n (understood that $\frac{w_i}{|w_i|}$ can be any element of S^1 when $|w_i| = 0$) which is sent to $(w_0, \ldots, w_n)$.

We show the continuity of the inverse $\psi : S^{2n+1} \longrightarrow ES^1_n$ of ϕ, given by

$$\psi(w_0, \ldots, w_n) \coloneqq \langle \frac{w_0}{|w_0|}, |w_0|^2; \ldots; \frac{w_n}{|w_n|}, |w_n|^2\rangle.$$

This suffices since S^{2n+1} is compact and ES^1_n is Hausdorff[2]. By Lemma 2.8 it suffices to check that the compositions $t_i \circ \psi$, $a_i \circ \psi$ are continuous for all i. Notice that $a_i \circ \psi$ is defined on the open set $\psi^{-1}(t_i^{-1}((0, 1]) = \{(w_0, \ldots, w_n) \in S^{2n+1} \mid w_i \neq 0\}$. Thus these maps are continuous because they are given by

$$(w_0, \ldots, w_n) \longmapsto |w_i|^2$$

and

$$(w_0, \ldots, w_n) \longmapsto \frac{w_i}{|w_i|},$$

respectively.

(2) Let $G = \mathbb{Z}/2$. Then the exact same formula as above identifies EG_n with S^n.

Injectivity: Suppose that for all i it holds $\sqrt{t_i}x_i = \sqrt{t_i'}x_i'$. If both $\sqrt{t_i}$ and $\sqrt{t_i'}$ are strictly greater than 0, then x_i and x_i' must have the same sign and thus are equal. Else, $\sqrt{t_i} = \sqrt{t_i'} = 0$ and so by definition of the equivalence relation $\langle x, t\rangle = \langle x', t'\rangle$.

Surjectivity: Given $(y_0, \ldots, y_n) \in S^n$, then $\langle \operatorname{sgn}(y_0), y_0^2; \ldots; \operatorname{sgn}(y_n), y_n^2\rangle$ is in EG_n and gets mapped to

$$(\operatorname{sgn}(y_0)|y_0|, \ldots, \operatorname{sgn}(y_n)|y_n|) = (y_0, \ldots, y_n).$$

We define $EG \coloneqq \underset{\to}{\lim} EG_n$, where the direct limit is taken over the inclusion maps

$$\begin{aligned} EG_n &\longrightarrow EG_{n+1} \\ \langle x_0, t_0; \ldots; x_n, t_n\rangle &\longmapsto \langle x_0, t_0; \ldots; x_n, t_n; 1_G, 0\rangle. \end{aligned}$$

Thus elements of EG are infinite vectors $\langle x, t\rangle = \langle x_0, t_0; \ldots; x_n, t_n; \ldots\rangle$ such that $\sum t_i = 1$ and the set $\{i \in \mathbb{N} \mid t_i \neq 0\}$ is finite; and any two such vectors

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[2] The join of Hausdorff spaces is Hausdorff, see [7, 5.7.2].

$\langle x, t\rangle$, $\langle x', t'\rangle$ are identified if and only if $t_i = t'_i$ for all i, and for those i such that $t_i = t'_i > 0$ it holds $x_i = x'_i$.

The *Milnor topology* on EG is the initial topology determined by the maps

$$\begin{aligned} t_i : EG &\longrightarrow [0,1] \\ \langle x, t\rangle &\longmapsto t_i \end{aligned}$$

and

$$\begin{aligned} a_i : t_i^{-1}((0,1]) &\longrightarrow G \\ \langle x, t\rangle &\longmapsto x_i. \end{aligned}$$

The group G acts on EG_n and EG on the left by

$$g \cdot \langle x, t\rangle \coloneqq \langle gx, t\rangle. \tag{2.1}$$

Proposition 2.10. *The action of G on both EG and EG_n defined by* (2.1) *is free and continuous.*

Proof. We prove the proposition for the action $\phi : G \times EG \longrightarrow EG$ of G on EG, and the case for EG_n can be proved in the exact same way.

To see that the action is free, consider $\langle x, t\rangle \in EG$ and let i be such that $t_i > 0$. Then $g \cdot \langle x, t\rangle = \langle x, t\rangle$ implies $gx_i = x_i$ and thus $g = 1_G$.

We show that ϕ is continuous. By definition of the topology on EG, ϕ is continuous if and only if the maps $t_i \circ \phi$ and $a_i \circ \phi$ are continuous for all i. For the first type of maps we have $t_i \circ \phi = t_i \circ \mathrm{pr}_2$, where $\mathrm{pr}_2 : G \times EG \longrightarrow EG$ denotes the (continuous) projection into the second factor, and thus they are continuous. The second kind of maps can be expressed as $a_i \circ \phi = \phi_G \circ (1_G \times a_i)$, where $\phi_G : G \times G \longrightarrow G$ is multiplication in G, and so they are continuous as well. ☐

Remark 2.11. We could also equip EG with the colimit topology, that is, the final topology with respect to the inclusions $i_n : EG_n \longrightarrow EG$. This topology is finer (i.e. larger) than the Milnor topology and in general there is no reason for the G-action to remain continuous (see [26]).

We denote by BG_n and BG the orbit spaces of the G-action on EG_n and EG, respectively. Denote by $\pi : EG \longrightarrow BG$ the projection. Elements of BG will be denoted by $[\langle x, t\rangle]$. Notice that since G doesn't act on the coordinates t_i, there are also continuous maps $t_i : BG \longrightarrow [0,1]$, such that the following diagram commutes

$$\begin{tikzcd} EG \arrow[rr, "t_i"] \arrow[dr, "\pi"'] & & {[0,1]} \\ & BG. \arrow[ur, "t_i"'] & \end{tikzcd}$$

Proposition 2.12 ([26, Theorem 3.1])**.** *$\pi : EG \longrightarrow BG$ is a principal G-bundle.*

Proof. By definition of the quotient topology π is continuous. The local trivializations are defined as follows. Let $V_i \coloneqq t_i^{-1}((0,1]) \subset BG$ and set

$$\begin{aligned} \phi_i : V_i \times G &\longrightarrow \pi^{-1}(V_i) \\ ([\langle x,t\rangle], g) &\longmapsto g x_i^{-1} \langle x,t\rangle. \end{aligned}$$

To see that it is well-defined suppose that $[\langle x,t\rangle] = [\langle y,t\rangle]$ is in V_i. So there is $h \in G$ with $\langle hx, t\rangle = \langle y,t\rangle$. Since $[\langle x,t\rangle] \in V_i$, t_i is not zero and so $hx_i = y_i$. Thus

$$g y_i^{-1}\langle y,t\rangle = g(hx_i)^{-1}\langle y,t\rangle = g x_i^{-1} h^{-1}\langle hx,t\rangle = g x_i^{-1}\langle x,t\rangle.$$

Moreover, ϕ_i is G-equivariant. For, let $g \in G$ and $([\langle x,t\rangle], h) \in V_i \times G$, then

$$\phi_i(g \cdot ([\langle x,t\rangle], h)) = \phi_i([\langle x,t\rangle], gh) = g h x_i^{-1}\langle x,t\rangle = g \cdot \phi_i([\langle x,t\rangle], h).$$

To define an inverse for ϕ_i notice first that $\pi^{-1}(V_i) \subset EG$ is the domain of $a_i : \langle x,t\rangle \longmapsto x_i$. Then the identities

- $\pi \circ \phi_i([\langle x,t\rangle], g) = \pi(g x_i^{-1}\langle x,t\rangle) = [\langle x,t\rangle]$,
- $a_i \circ \phi_i([\langle x,t\rangle], g) = a_i(g x_i^{-1}\langle x,t\rangle) = g$,
- $\phi_i\big([\langle x,t\rangle], a_i(\langle x,t\rangle)\big) = \langle x,t\rangle$,

show that $(\pi, a_i) : \pi^{-1}(V_i) \longrightarrow V_i \times G$ is an inverse to ϕ_i.

We now prove that these functions are continuous. The inverse (π, a_i) is continuous as both π and a_i are. Recall the multiplication $\phi : G \times EG \longrightarrow EG$. For $e = \langle x,t\rangle \in EG$ we have

$$\phi_i([e], 1_G) = x_i^{-1} \cdot e = \phi(x_i^{-1}, e) = \phi(a_i(e)^{-1}, e),$$

which shows that $e \longmapsto \phi_i([e], 1_G)$ is a continuous function. By the definition of quotient topology this means that $x \longmapsto \phi_i(x, 1_G)$ is continuous. To conclude, the identity

$$\phi_i(x,g) = \phi(g, \phi_i(x, 1_G))$$

shows the continuity of ϕ_i. □

Remark 2.13. It is not difficult to show that the principal G-bundle $\pi : EG \longrightarrow BG$ is actually a numerable bundle (see [14, §8]).

Example 2.14. We have seen that $ES^1_n \cong S^{2n+1}$. Under this identification the group acts by left multiplication, thus $BS^1_n = S^{2n+1}/S^1 = \mathbb{CP}^n$. The map

$$\begin{aligned} ES^1 &\longrightarrow S^\infty \\ \langle z_0, t_0; \ldots; z_n, t_n; \ldots\rangle &\longmapsto (\sqrt{t_0} z_0, \ldots, \sqrt{t_n} z_n, \ldots) \end{aligned}$$

identifies ES^1 with S^∞ and thus $BS^1 = \mathbb{CP}^\infty$.

Proposition 2.15. *The Milnor join EG is contractible for any topological group G.*

We illustrate how this comes for the case of $G = S^1$ and a proof for the general case can be found in [14, Theorem 8.1]. So consider $G = S^1$. Recall that $ES^1 \cong S^\infty$ and $BS^1 \cong \mathbb{CP}^\infty$. We show that S^∞ is contractible. For this, we may regard S^∞ as a CW complex with filtration $S^0 \subseteq \ldots \subseteq S^n \subseteq S^{n+1} \subseteq \ldots$, where S^{n+1} is obtained from S^n by attaching two $(n+1)$-dimensional cells:

$$S^{n+1} = S^n \sqcup \bigsqcup_{i=1}^2 D_i^{n+1} \Big/ \begin{array}{c} \varphi_i(x) \sim x \\ \text{for } x \in S_i^n = \partial D_i^{n+1} \end{array},$$

where $\varphi_i = id : S_i^n \longrightarrow S^n$.

Then for the homotopy groups π_n we have $\pi_n(S^\infty) = \pi_n(S^{n+1})$ for all $n \geq 0$ by cellular approximation (see [20, Theorem 4.8]), and $\pi_n(S^{n+1}) = 0$ also by cellular approximation, seeing the sphere S^{n+1} as a CW-complex with one 0-cell and a $(n+1)$-cell attached to it by collapsing the boundary. Thus $\pi_n(S^\infty) = 0$ for all $n \geq 0$. It follows from Whitehead's theorem (see [20, Theorem 4.5]) that the map $S^\infty \longrightarrow \mathrm{pt}$ is a homotopy equivalence, since it induces an isomorphism on all homotopy groups. This concludes the proof.

So for every topological group G the Milnor join EG gives a contractible space on which G acts freely, and the projection $EG \longrightarrow BG$ is a principal G-bundle.

2.3 Universal principal G-bundles

A principal G-bundle with contractible total space turns out to be a *universal* principal G-bundle. The base space of a such a bundle is called the *classifying space* of G and is unique up to homotopy type. Since it plays an important role in equivariant cohomology (its singular cohomology is the equivariant cohomology of a point) we shall briefly discuss universal principal G-bundles.

Definition 2.16. Let $p : E \longrightarrow B$ be a principal G-bundle and let $f : B' \longrightarrow B$ be a continuous map. Then the *pullback bundle* is the fiber bundle over B' with total space f^*E given by

$$f^*E = \{(b', e) \in B' \times E \mid p(e) = f(b')\}.$$

Lemma 2.17. *Let $p : E \longrightarrow B$ be a numerable principal G-bundle and let $f : B' \longrightarrow B$ be a continuous map. Then the pullback f^*E fits into the diagram*

$$\begin{tikzcd}
f^*E \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & E \arrow[d, "p"] \\
B' \arrow[r, "f"] & B.
\end{tikzcd}$$

and is a numerable principal G-bundle over B'.

Proof. Let $\{U_i\}_{i \in I}$ be a numerable cover of B, such that there are trivializations $\{h_i : p^{-1}(U_i) \longrightarrow U_i \times G\}$, and let $\{u_i\}_{i \in I}$ be a locally finite partition of unity subordinate to $\{U_i\}_{i \in I}$, so that it holds $\operatorname{supp}(u_i) \subset U_i$ for all i. Then $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of B' and the family $\{u_i \circ f\}_{i \in I}$ gives a locally finite partition of unity with

$$\operatorname{supp}(u_i \circ f) = \overline{(u_i \circ f)^{-1}((0,1])} = \overline{f^{-1}(u^{-1}((0,1])} \subset f^{-1}\big(\overline{u_i^{-1}((0,1]}\big) \subset f^{-1}(U_i).$$

Moreover, the maps

$$\begin{aligned} h'_i : \operatorname{pr}_1^{-1}(f^{-1}(U_i)) &\longrightarrow f^{-1}(U_i) \times G \\ (b', e) &\longmapsto (b', \operatorname{pr}_2 \circ h_i(e)) \end{aligned}$$

give trivializations for B'. Indeed they are continuous with continuous inverses given by

$$\begin{aligned} \varphi_i : f^{-1}(U_i) \times G &\longrightarrow \operatorname{pr}_1^{-1}(f^{-1}(U_i)) \\ (b', g) &\longmapsto (b', h_i^{-1}(f(b'), g)) \end{aligned}$$

since on one side we have

$$\varphi_i \circ h'_i(b', e) = \big(b', h_i^{-1}\big(f(b'), \operatorname{pr}_2 \circ h_i(e)\big)\big),$$

which using that $f(b') = p(e) = \operatorname{pr}_1 \circ h_i(e)$ becomes

$$\varphi_i \circ h'_i(b', e) = \big(b', h_i^{-1}\big(\operatorname{pr}_1 \circ h_i(e), \operatorname{pr}_2 \circ h_i(e)\big)\big) = (b', h_i^{-1} \circ h_i(e)) = (b', e).$$

On the other side

$$h'_i \circ \varphi_i(b', g) = h'_i\big(b', h_i^{-1}(f(b'), g)\big) = (b', \operatorname{pr}_2(f(b'), g)) = (b', g).$$

$\square$

It can be shown (see [21, §4, Theorem 9.9]) that homotopic maps give isomorphic principal bundles.

Proposition 2.18. *Let $E \longrightarrow B$ be a principal G-bundle and suppose that there is an homotopy $f_t : B' \longrightarrow B$ between $f_0 : B' \longrightarrow B$ and $f_1 : B' \longrightarrow B$. Then the principal G-bundles $f_0^* E$ and $f_1^* E$ are isomorphic.*

Definition 2.19. A numerable principal G-bundle $\mathcal{E} \longrightarrow \mathcal{B}$ is called *universal* if the following two conditions hold.

(i) For any numerable principal G-bundle $E \longrightarrow B$, there exists a map $f : B \longrightarrow \mathcal{B}$ such that E is isomorphic to $f^*\mathcal{E}$.

(ii) Two maps $f, g : B \longrightarrow \mathcal{B}$ induce isomorphic bundles $f^*\mathcal{E} \cong g^*\mathcal{E}$ if and only if they are homotopic.

We can rephrase the definition of universality as follows. Fix a principal G-bundle $\mathcal{E} \longrightarrow \mathcal{B}$. For a topological space B consider the map

$$\begin{aligned} \Phi : [B, \mathcal{B}] &\longrightarrow \{\text{numerable principal G-bundles over B}\} /_{\sim} \\ [f] &\longmapsto f^* \mathcal{E}, \end{aligned}$$

where $[B, \mathcal{B}]$ denotes the space of continuous functions $f : B \longrightarrow \mathcal{B}$ up to homotopy, and "$\sim$" indicates that we consider isomorphism classes of principal G-bundles over B. By Lemma 2.17 and Proposition 2.18 the map Φ is well-defined.

Definition 2.20. A numerable principal G-bundle $\mathcal{E} \longrightarrow \mathcal{B}$ is called *universal* if for every topological space B the map Φ is a bijection.

It turns out that any principal G-bundle with contractible total space is universal.

Theorem 2.21. *Let $E \longrightarrow B$ be any numerable principal G-bundle such that the total space E is contractible, then this is a universal principal G-bundle.*

A proof can be found in [14, Theorem 7.5].

In particular, we have the following useful corollary.

Corollary 2.22. *The numerable principal G-bundle $\pi : EG \longrightarrow BG$ is universal.*

We show that the base space BG is unique up to homotopy equivalence.

Proposition 2.23. *The base space B of a universal principal G-bundle $E \longrightarrow B$ is unique up to homotopy equivalence.*

Proof. Suppose that $E_1 \longrightarrow B_1$ and $E_2 \longrightarrow B_2$ are two universal principal G-bundles. The universality of E_1 gives a map $f : B_2 \longrightarrow B_1$ such that $f^* E_1$ is isomorphic to E_2. Similarly, since E_2 is universal there is a map $g : B_1 \longrightarrow B_2$ such that $g^* E_2$ is isomorphic to E_1. Then E_1 is isomorphic to $g^*(f^* E_1)$ and hence to $(f \circ g)^* E_1$. But since E_1 is universal and since $E_1 \cong id_{B_1}^* E_1$, it follows that $f \circ g$ is homotopic to id_{B_1}. Analogously, $g \circ f$ is homotopic to id_{B_2} and thus f and g give homotopy inverses between B_1 and B_2. ☐

Example 2.24.
(1) If H is a subgroup of G, it acts freely on EG, and as in Proposition 2.12 we see that $EG \longrightarrow EG/H$ is a principal H-bundle and thus EG/H and BH are homotopy equivalent by Proposition 2.23.

(2) If G and H are two groups, the group $G \times H$ acts componentwise on the contractible space $EG \times EH$. As the quotient map $\pi : EG \times EH \longrightarrow (EG \times EH)/(G \times H)$ is the product of the quotient maps π_G and π_H, π is a principal $G \times H$-bundle[3] and so again by Proposition 2.23 the quotient space $BG \times BH$ is homotopy equivalent to $B(G \times H)$.

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[3] The trivializations are given by the product of the trivializations on BG and BH.

Using these general examples we can describe explicitly the classifying space of some groups.

Example 2.25. (1) Consider the cyclic group $\mathbb{Z}/m$ as a subgroup of S^1:

$$\mathbb{Z}/m \cong \{\zeta \in S^1 \subseteq \mathbb{C} \mid \zeta^m = 1\} = \{e^{2\pi i \frac{k}{m}} \mid k = 0, \dots, m-1\}.$$

Thus it acts on $ES^1 = S^\infty$ by left multiplication and from Example 2.24 (1) it follows that $B(\mathbb{Z}/m)$ is homotopy equivalent to the quotient of S^∞ by the equivalence relation

$$(z_0, \dots, z_n, \dots) \sim (\zeta z_0, \dots, \zeta z_n, \dots) \quad \text{for } \zeta \in \mathbb{C} \text{ with } \zeta^m = 1.$$

This space is called the infinite-dimensional *Lens space* L_∞ (see [20, Example 1B.4]).

(2) The torus $T^m \coloneqq S^1 \times \dots \times S^1$ acts diagonally on the product $S^\infty \times \dots \times S^\infty$, therefore accordingly to Example 2.24 (2) the classifying space $B(T^m)$ is homotopy equivalent to $\mathbb{CP}^\infty \times \dots \times \mathbb{CP}^\infty$.

2.4 The Borel construction

Let G be a topological group and M be a topological space with a continuous action $G \times M \longrightarrow M$ of G on M. We call M a G-space. If M is a G-space, we know that the orbit space M/G may be not well-behaved, for example, when the action is not free it might not be Hausdorff. The Borel construction gives a reasonable substitute for this quotient. Instead of M we consider the space $EG \times M$ and we make G act on $EG \times M$ by

$$g \cdot (e, x) \coloneqq (g \cdot e, g \cdot x).$$

This action is always free, independently of how G acts on M, because it is in the first component, and since EG is contractible, $EG \times M$ has the same homotopy type as M.

Definition 2.26. The quotient space

$$M_G \coloneqq EG \times_G M \coloneqq (EG \times M)/G$$

is called the *Borel construction on M* or the *homotopy quotient of M*.

Remark 2.27. Let X and Y be G-spaces and suppose that $f : X \longrightarrow Y$ is a G-equivariant map, that is, for all $x \in X$ and $g \in G$ it holds $f(gx) = gf(x)$. Then f induces a map

$$1 \times_G f : X_G \longrightarrow Y_G$$

defined by the formula

$$(1 \times_G f)([e, x]) \coloneqq [e, f(x)] \, .$$

This map is well-defined because by the G-equivariance of f one has

$$1 \times_G f([ge, gx]) = [ge, f(gx)] = [ge, gf(x)] = [e, f(x)] = 1 \times_G f([e, x]).$$

In general, any G-equivariant map $f : X \longrightarrow Y$ between G-spaces X and Y induces a well-defined map $\bar{f} : X/G \longrightarrow Y/G$ by the formula $\bar{f}([x]) \coloneqq [f(x)]$.

Consider the two projection maps from the product $EG \times M$, they are both G-equivariant and so they induce maps p and σ, respectively, between the quotients.

(1) The projection onto EG:

$$\begin{tikzcd} EG \times M \arrow[r] \arrow[d] & EG \arrow[d, "\pi"] \\ M_G \arrow[r, "p"] & BG. \end{tikzcd}$$

(2) The projection onto M:

$$\begin{tikzcd} EG \times M \arrow[r] \arrow[d] & M \arrow[d] \\ M_G \arrow[r, "\sigma"] & M/G. \end{tikzcd}$$

Proposition 2.28 ([33, Theorem 2.1])**.** *The induced map* $M_G \xrightarrow{p} BG$, $[e, x] \longmapsto \pi(e)$ *is a fiber bundle with fiber* M.

Proof. Let $U \subset BG$ be open such that there is a local trivialization $h : \pi^{-1}(U) \longrightarrow U \times G$. Recall that $\mathrm{pr}_1 \circ h = \pi$, where $\mathrm{pr}_1 : U \times G \longrightarrow U$ is the projection onto the first component. Then define $\bar{h} : p^{-1}(U) \longrightarrow U \times M$ as

$$\bar{h}([e, x]) \coloneqq (\pi(e), \mathrm{pr}_2(h(e))^{-1} \cdot x),$$

where $\mathrm{pr}_2 : U \times G \longrightarrow G$ is the projection onto the second component. To see that $\bar{h}$ is well-defined, suppose that we have $[e, x] = [e', x'] \in M_G$, so that there is $g \in G$ with $ge = e'$ and $gx = x'$. Then $\pi(e') = \pi(ge) = \pi(e)$ and

$$\begin{aligned} \mathrm{pr}_2(h(e'))^{-1} \cdot x' &= \mathrm{pr}_2(h(ge))^{-1} \cdot gx \\ &= \mathrm{pr}_2(gh(e))^{-1} \cdot gx \\ &= (g \cdot \mathrm{pr}_2(h(e)))^{-1} \cdot gx \\ &= \mathrm{pr}_2(h(e))^{-1} \cdot x, \end{aligned}$$

where in the second equality we used that h is G-equivariant and the third equality follows from the definition of the G-action on $BG \times G$. We now define an inverse for $\bar{h}$. Consider the map $s : U \longrightarrow \pi^{-1}(U)$ defined by $s(b) = h^{-1}(b, 1_G)$. Then $f : U \times M \longrightarrow p^{-1}(U)$ given by

$$f(b, x) \coloneqq [s(b), x]$$

is the desired inverse. Indeed, on one hand we have

$$\bar{h} \circ f(b,x) = \bar{h}([s(b),x]) = (\pi(s(b)), \mathrm{pr}_2(h(s(b)))^{-1} \cdot x) = (b,x).$$

On the other hand

$$f \circ \bar{h}([e,x]) = f(\pi(e), \mathrm{pr}_2(h(e))^{-1} \cdot x) = \left[h^{-1}(\pi(e), 1_G), \mathrm{pr}_2(h(e))^{-1} \cdot x)\right].$$

Notice that since $\pi(e) = \mathrm{pr}_1 \circ h(e)$, we have

$$\mathrm{pr}_2(h(e)) \cdot h^{-1}(\pi(e), 1_G) = h^{-1}(\pi(e), \mathrm{pr}_2(h(e))) = h(h^{-1}(e)) = e,$$

and thus

$$\begin{aligned}\left[h^{-1}(\pi(e), 1_G), \mathrm{pr}_2(h(e))^{-1}x\right] &= \mathrm{pr}_2(h(e)) \cdot \left[h^{-1}(\pi(e), 1_G), \mathrm{pr}_2(h(e))^{-1}x\right] \\ &= [e,x].\end{aligned}$$

Both f and $\bar{h}$ are continuous and so they are homeomorphisms. ☐

If we look at the other projection, the induced map $\sigma : M_G \longrightarrow M/G$ is not a fiber bundle in general, however we have the following result.

Proposition 2.29. *Let G be a compact Lie group acting freely on a manifold M. Then the map*

$$\sigma : M_G \longrightarrow M/G$$

is a fiber bundle with contractible fiber EG.

Proof. Since G is a compact Lie group acting freely on M, the quotient map $M \longrightarrow M/G$ is a principal G-bundle (as shown in [33, Corollary 10.1] or in [24, §21]). The exact same proof of Proposition 2.28 shows that σ is a fiber bundle with fiber EG. ☐

Definition 2.30. A map $f : X \longrightarrow Y$ between connected topological spaces is called a *weak homotopy equivalence* if it induces an isomorphism $\pi_n(X) \longrightarrow \pi_n(Y)$ on all homotopy groups. If such a map exists, the spaces X and Y are said to be *weakly homotopy equivalent.*

Corollary 2.31. *Let G be a compact Lie group acting freely on a manifold M. Then the orbit space M/G and the homotopy quotient M_G are weakly homotopy equivalent.*

Proof. By Proposition 2.29 the map $\sigma : M_G \longrightarrow M/G$ is a fiber bundle with contractible fiber EG. Then the long exact sequence for homotopy groups ([20, Theorem 4.41]) shows that σ is a weak homotopy equivalence. ☐

Remark 2.32. Notice that in proving Proposition 2.28 and 2.29 we didn't use any particular property of the G-bundle $\pi : EG \longrightarrow BG$. In fact, they still hold true if we replace the bundle $\pi : EG \longrightarrow BG$ by any principal G-bundle $E \longrightarrow B$.

In defining the Borel construction M_G on a space M, we used the Milnor join EG to get a space homotopic to M on which G acts freely. We shall now see that, up to weak homotopy type, we can choose instead of EG the total space E of any principal G-bundle $E \longrightarrow B$, as long as it is contractible.

We remark the following fact.

Proposition 2.33 ([32, Proposition 4.11])**.** *If $E \longrightarrow B$ is a principal G-bundle and M is a G-space, then the projection $E \times M \longrightarrow E \times_G M$ is a principal G-bundle.*

Lemma 2.34. *Let E be a G-space with $\pi_i(E) = 0$ for $0 \leq i < k$ and $P \longrightarrow P/G$ be a principal G-bundle, then $\pi_i((E \times P)/G) \cong \pi_i(P/G)$ for all $0 < i < k$. In particular if E is contractible, $(E \times P)/G$ and P/G are weakly homotopy equivalent.*

Proof. Since $P \longrightarrow P/G$ is a principal G-bundle, by Proposition 2.29 the map $E \times_G P \longrightarrow P/G$ is a fiber bundle with fiber E. Then the long exact sequence of homotopy groups for fiber bundles ([20, Theorem 4.41])

$$\ldots \longrightarrow \pi_i(E) \longrightarrow \pi_i(E \times_G P) \longrightarrow \pi_i(P/G) \longrightarrow \pi_{i-1}(E) \longrightarrow \ldots$$

shows that $\pi_i(E \times_G P) \cong \pi_i(P/G)$ for all $0 < i < k$. If E is contractible all the homotopy groups $\pi_i(E)$ vanish and thus the above is an isomorphism for all $i > 0$. ☐

Theorem 2.35 ([32, Theorem 5.3])**.** *Let M be a G-space and $E \longrightarrow B$, $E' \longrightarrow B'$ be two principal G-bundles with contractible total spaces E and E'. Then there is a space X and weak homotopy equivalences*[4]

$$f : (E \times M)/G \longrightarrow X \quad \textit{and} \quad g : (E' \times M)/G \longrightarrow X.$$

Proof. Since $E \times M \longrightarrow (E \times M)/G$ is a principal G-bundle and E' is a contractible space with a G-action, we can apply the previous lemma with $P = E \times M$ and $E = E'$ to get that $(E' \times E \times M)/G$ is weakly homotopy equivalent to $(E \times M)/G$. Analogously, $(E \times E' \times M)/G$ is weakly homotopy equivalent to $(E' \times M)/G$. By setting $X \coloneqq (E \times E' \times M)/G$ this proves the theorem. ☐

Example 2.36 (Infinite Stiefel manifold)**.** We show that the infinite Stiefel manifold $V_k(\mathbb{C}^\infty)$ gives a model for the space $EU(k)$, where $U(k)$ denotes the group of unitary matrices. The Stiefel manifold $V_k(\mathbb{C}^n)$ is the set of all ordered k-tuples of orthonormal vectors in $\mathbb{C}^n$, also called *k-frames* in $\mathbb{C}^n$. Let $G_k(\mathbb{C}^n)$ be the Grassmannian manifold, i.e. the space of all k-dimensional linear subspaces of $\mathbb{C}^n$. There is a surjective map $V_k(\mathbb{C}^n) \longrightarrow G_k(\mathbb{C}^n)$, which sends an orthonormal k-frame $(v_1, \ldots, v_k)$ to its span. A fiber over a k-plane is given by all ordered orthonormal k-tuples which span the k-plane in $\mathbb{C}^n$, and so is homeomorphic to $V_k(\mathbb{C}^k)$. We can identify the k-frames in $\mathbb{C}^k$ with the unitary

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[4] This is not as strong as having a weak homotopy equivalence, but suffices for our purposes.

matrices, regarding the vectors of the frame as the columns of a $(k \times k)$-matrix. Thus the fibers are described by the unitary group $U(k)$. In fact, it turns out that the projection $V_k(\mathbb{C}^n) \longrightarrow G_k(\mathbb{C}^n)$ gives a principal $U(k)$-bundle. The infinite Stiefel manifold is defined as the limit $V_k(\mathbb{C}^\infty) = \bigcup_n V_k(\mathbb{C}^n)$ and the infinite Grasmannian manifold is $G_k(\mathbb{C}^\infty) = \bigcup_n G_k(\mathbb{C}^n)$.

Claim. The infinite Stiefel manifold is contractible.

Proof. Analogously as for the infinite sphere, we show that $V_k(\mathbb{C}^\infty)$ is weakly contractible (i.e. all homotopy groups vanish) and then apply Whitehead's theorem. So first of all we need the manifolds $V_k(\mathbb{C}^n)$ to have a CW-structure. As explained in [20, pp. 301-302] they do, and for n big enough it holds $\pi_i(V_k(\mathbb{C}^\infty)) = \pi_i(V_k(\mathbb{C}^n))$. Now, consider the fiber bundle $V_k(\mathbb{C}^n) \longrightarrow S^{2n-1}$ given by

$$(v_1, \dots, v_k) \longmapsto v_k.$$

The fibers are given by $V_{k-1}(\mathbb{C}^{n-1})$, so that the long exact sequence for homotopy groups

$$\dots \longrightarrow \pi_{i+1}(S^{2n-1}) \longrightarrow \pi_i(V_{k-1}(\mathbb{C}^{n-1})) \longrightarrow \pi_i(V_k(\mathbb{C}^n)) \longrightarrow \pi_i(S^{2n-1}) \longrightarrow \dots$$

shows that $\pi_i(V_k(\mathbb{C}^n)) \cong \pi_i(V_{k-1}(\mathbb{C}^{n-1}))$ for $i+1 < 2n-1$. Thus iterating we see that

$$\pi_i(V_k(\mathbb{C}^n)) \cong \pi_i(V_1(\mathbb{C}^{n-(k-1)})) \quad \text{for } i < 2(n-k)-2.$$

But $V_1(\mathbb{C}^{n-(k-1)}) = S^{2(n-k)+1}$, thus

$$\pi_i(V_k(\mathbb{C}^n)) \cong \pi_i(S^{2(n-k)+1}) = 0 \quad \text{for } i < 2(n-k)-2.$$

This shows that if n is big enough we have

$$\pi_i(V_k(\mathbb{C}^\infty)) = \pi_i(V_k(\mathbb{C}^n)) = 0.$$

So the infinite Stiefel manifold is weakly contractible and by Whitehead's theorem is contractible. ☐ Claim.

From Theorem 2.35 it follows that if M is a $U(k)$-manifold, then $V_k(\mathbb{C}^\infty) \times_{U(k)} M$ is weakly homotopy equivalent to $EU(k) \times_{U(k)} M$.

2.5 Equivariant cohomology

Definition 2.37. Let M be a topological space on which a topological group G acts continuously. For each n, the *equivariant cohomology of M* is the cohomology group

$$H^n_G(M) := H^n(M_G) = H^n(EG \times_G M).$$

Here $H^n(\cdot)$ denotes singular cohomology with coefficients in any ring, but we are mainly going to work with real coefficients.

The definition of equivariant cohomology doesn't depend on the space EG that we choose to define it, as long as it is contractible and G acts freely on it. To see this, recall first that weakly homotopy equivalent spaces have the same cohomology.

Theorem 2.38 ([20, Proposition 4.21])**.** *A weak homotopy equivalence $f : X \longrightarrow Y$ induces isomorphisms $f^* : H^n(Y) \longrightarrow H^n(X)$ for all n.*

Theorem 2.39. *The definition of $H_G(M)$ doesn't depend on the choice of the principal G-bundle $E \longrightarrow B$ with contractible space E used to define it.*

Proof. If $E \longrightarrow B$, $E' \longrightarrow B'$ are two principal G-bundles with E and E' contractible, then by Theorem 2.35 there is a space X and weak homotopy equivalences $f : (E \times M)/G \longrightarrow X$ and $g : (E' \times M)/G \longrightarrow X$. Thus by Theorem 2.38 it holds

$$H^n(E \times_G M) \cong H^n(X) \cong H^n(E' \times_G M)$$

for all n. Thus, it makes no difference if we take E or E' as a model for EG. □

It is often useful to work with the graded cohomology ring $H_G^*(M)$, instead of the single groups $H_G^n(M)$. So recall that $H_G^*(M)$ denotes the direct sum

$$H_G^*(M) := \bigoplus_{n=0}^{\infty} H_G^n(M)$$

which is turned into a graded ring by the cup product (see Appendix B.5).

Remark 2.40. If the action of G on M is free, then the Borel construction M_G is weakly homotopy equivalent to the quotient space M/G and thus by Theorem 2.38 we have

$$H_G^*(M) = H^*(M/G).$$

Remark 2.41. Let G be a topological group. Then the singular cohomology of the classifying space BG coincides with the equivariant cohomology of a point, since

$$\mathrm{pt}_G = (EG \times \mathrm{pt})/G = BG \times \mathrm{pt},$$

and hence

$$H_G^*(\mathrm{pt}) = H^*(\mathrm{pt}_G) = H^*(BG).$$

2.6 Finite approximations

We defined equivariant cohomology using singular cohomology, because neither the spaces EG nor M_G are manifolds and so de Rham cohomology is not at hand. However, it is much preferable to work with differential forms. In the next section we are going to construct an algebraic model which mimics the differential forms on M_G, called the Cartan complex, whose elements are called equivariant differential forms. As we shall explain next, another possibility is to consider finite approximations for the space M_G.

Throughout this subsection, M denotes a smooth manifold and G a Lie group.

Definition 2.42. Let M be a smooth manifold on which a Lie group G acts smoothly. Then we call M a G-manifold.

Theorem 2.43. *Let M be a G-manifold and $E \longrightarrow B$, $E' \longrightarrow B'$ be two principal G-bundles with $\pi_i(E) = \pi_i(E') = 0$ for $0 \leq i < k$. Then $H^i(E \times_G M) \cong H^i(E' \times_G M)$ for $0 < i < k$.*

Proof. The proof is basically the same as the one of Theorem 2.35. Since $E \times M \longrightarrow (E \times M)/G$ is a principal G-bundle and $\pi_i(E') = 0$ for $0 \leq i < k$, we can apply Lemma 2.34 with $P = E \times M$ and $E = E'$ to get

$$\pi_i((E' \times E \times M)/G) \cong \pi_i((E \times M)/G) \quad \text{for all } 0 < i < k,$$

where the isomorphism is induced by a map at the level of the spaces. Symmetrically, $\pi_i((E \times E' \times M)/G) \cong \pi_i((E' \times M)/G)$ for all $0 < i < k$. Therefore by (an analogous of) Theorem 2.38 we have

$$H^i((E \times M)/G) \cong H^i((E \times E' \times M)/G) \cong H^i((E' \times M)/G)$$

for all $i < k$. □

We are ready now to describe the approximation of the singular cohomology groups $H^i_G(M)$ by de Rham cohomology groups. Consider first the case when the Lie group is $G = S^1$. We know that $ES^1 \cong S^\infty$ and $(ES^1)_n \cong S^{2n+1}$ for all n. Moreover we have

$$\pi_i(ES^1) = \pi_i(S^\infty) = \pi_i(S^{2n+1})$$

for all $i < 2n + 1$. By Theorem 2.43 it follows that

$$H^i(ES^1 \times_{S^1} M) \cong H^i(S^{2n+1} \times_{S^1} M)$$

for $i < 2n+1$. The space $S^{2n+1} \times_{S^1} M$ is a manifold as it is the quotient of the product manifold $S^{2n+1} \times M$ by a free action of S^1 ([24, §21]). Therefore, by de Rham's theorem ([24, Theorem 18.4]) it holds

$$H^i(S^{2n+1} \times_{S^1} M) \cong H^i_{\mathrm{dR}}(S^{2n+1} \times_{S^1} M).$$

Thus for all $i \in \mathbb{N}$ there is some n big enough such that

$$H^i_{S^1}(M) = H^i_{\mathrm{dR}}(S^{2n+1} \times_{S^1} M)$$

and an element of $H^i_{S^1}(M)$ is represented by a differential form of degree i on $S^{2n+1} \times_{S^1} M$.

In general, when the group G is compact we have a similar approximation. It is a theorem of representation theory that every compact Lie group H can be embedded as a closed subgroup (and hence as Lie subgroup) of a unitary group $U(k)$ for some k big enough ([6, §0, Theorem 5.1]). For a subgroup H of a Lie group G we have the following result ([32, Theorem 9.1]).

Proposition 2.44. *If $E \longrightarrow E/G$ is a principal G-bundle and H is a Lie subgroup of G, then $E \longrightarrow E/H$ is a principal H-bundle.*

Thus if G is compact we can use a principal $U(k)$-bundle to get a principal G-bundle. Let $V_k(\mathbb{C}^\infty)$ denote as before the infinite Stiefel manifold and $G_n(\mathbb{C}^\infty)$ the infinite Grassmannian manifold. Then $V_k(\mathbb{C}^\infty) \longrightarrow G_k(\mathbb{C}^\infty)$ is a principal $U(k)$-bundle and by Proposition 2.44 the bundle

$$V_k(\mathbb{C}^\infty) \longrightarrow V_k(\mathbb{C}^\infty)/G$$

is a principal G-bundle. Since both $V_k(\mathbb{C}^\infty)$ and EG are contractible, by Theorem 2.35 the spaces $V_k(\mathbb{C}^\infty) \times_G M$ and $EG \times_G M$ are both weakly homotopy equivalent to some space X, and by Theorem 2.43 this implies that they have the same cohomology groups in every degree.

Since also the finite approximations $V_k(\mathbb{C}^n)$ of $V_k(\mathbb{C}^\infty)$ are principal $U(k)$-bundles, again by Proposition 2.44 it follows that $V_k(\mathbb{C}^n) \longrightarrow V_k(\mathbb{C}^n)/G$ is a principal G-bundle for all n. Let $i \in \mathbb{N}$ be fixed. Then there is some n big enough such that we have $\pi_j(V_k(\mathbb{C}^n)) = \pi_j(V_k(\mathbb{C}^\infty))$ for all $j \leq i$. Thus by Theorem 2.43,

$$H^i(V_k(\mathbb{C}^n) \times_G M) \cong H^i(V_k(\mathbb{C}^\infty) \times_G M) \cong H^i(EG \times_G M) = H^i_G(M).$$

The space $V_k(\mathbb{C}^n) \times_G M$ is a manifold, because it is the quotient by a free action ($U(k)$ acts freely on $V_k(\mathbb{C}^n)$ and so does $G \leq U(k)$) of the product manifold $V_k(\mathbb{C}^n) \times M$. Therefore its singular cohomology coincides with its de Rham cohomology. So for each i there is n big enough such that

$$H^i_G(M) = H^i_{\mathrm{dR}}(V_k(\mathbb{C}^n) \times_G M)$$

and thus elements of $H^i(EG \times_G M)$ are represented by differential forms on $V_k(\mathbb{C}^n) \times M$.

Notation 2.45. We call the manifolds $V_k(\mathbb{C}^n)$ and $V_k(\mathbb{C}^n \times_G M)$ *finite approximations* of EG and M_G, respectively, and we denote them by EG_n and $(M_G)_n$, although this notation was already used for the construction of the Milnor join.

2.7 Properties of equivariant cohomology

Most of the properties which hold for the singular cohomology are true also in the equivariant setting.

Proposition 2.46 (Equivariant Mayer-Vietoris)**.** *Let G be a compact Lie group acting on a manifold M and let U_1, $U_2 \subseteq M$ be G-invariant open subsets. Then there exists a long exact sequence in equivariant cohomology*

$$\ldots \longrightarrow H^k_G(U_1 \cup U_2) \longrightarrow H^k_G(U_1) \oplus H^k_G(U_2) \longrightarrow H^k_G(U_1 \cap U_2) \longrightarrow \ldots.$$

Proof. Since U_1 and U_2 are G-invariant, the Borel constructions $EG \times_G U_1$ and $EG \times_G U_2$ are well-defined. The theorem follows directly from the Mayer-Vietoris theorem for singular cohomology, see [20], applied to the spaces $EG \times_G U_1$, $EG \times_G U_2$ and $EG \times_G M$. □

Let X be a G space and $U \subset X$ be a G-invariant subspace. We define the equivariant cohomology of the pair (X, U) as

$$H_G^*(X, U) \coloneqq H^*(X_G, U_G).$$

Proposition 2.47 (Long equivariant exact sequence of pairs)**.** *Let X be a G-space and $U \subset X$ be a G-invariant subspace. Then there is a long exact sequence in equivariant cohomology*

$$\ldots \longrightarrow H_G^n(X, U) \longrightarrow H_G^n(U) \longrightarrow H_G^n(X) \longrightarrow H_G^{n+1}(X, U) \longrightarrow \ldots.$$

Proof. The statement follows directly by applying the result in singular cohomology for the pair $EG \times_G U \hookrightarrow EG \times_G X$. □

Proposition 2.48 (Equivariant excision axiom)**.** *Let X be a G-space and U, V be G-invariant subspaces with $\overline{U} \subseteq V^{\mathrm{o}}$. Then the inclusion $(X \setminus U, V \setminus U) \hookrightarrow (X, V)$ induces an isomorphism*

$$H_G^*(X \setminus U, V \setminus U) \cong H_G^*(X, V).$$

Proof. For G-subsets $A, B \subset X$ the Borel construction satisfies:

$$\begin{aligned} (B^{\mathrm{o}})_G &= (B_G)^{\mathrm{o}}, \\ \overline{(A_G)} &= (\overline{A})_G, \end{aligned} \tag{2.2}$$

$$(A \setminus B)_G = A_G \setminus B_G. \tag{2.3}$$

By (2.2) we can apply the excision axiom for singular cohomology to $\overline{U_G} \subset (V_G)^{\mathrm{o}} \subset X_G$ to get

$$H_G^*(X \setminus U, V \setminus U) \overset{(2.3)}{=} H^*(X_G \setminus U_G, V_G \setminus U_G) \cong H^*(X_G, V_G) \overset{\text{def}}{=} H_G^*(X, V).$$

□

3 A de Rham model for the equivariant theory

When talking about equivariant cohomology, we would like to use the de Rham cohomology. However, neither the space EG nor M_G are manifolds in general, so that the ordinary differential forms on EG are not at hand. It turns out that a reasonable algebraic substitute for $\Omega^*(EG)$ is the Weil algebra $W\mathfrak{g}$, which is a chain complex satisfying some algebraic properties which reflect the topological properties of the space EG. For instance, since EG is contractible, its de Rham complex should be acyclic and in fact the Weil algebra is acyclic. Then, a substitute for the de Rham complex of $EG \times M$ is given by the differential algebra $W\mathfrak{g} \otimes \Omega^*(M)$. Given a principal G-bundle $\pi : P \longrightarrow B$ of manifolds, the pullback $\pi^* : \Omega^*(B) \longrightarrow \Omega^*(P)$ is injective and thus one can identify the differential forms on the base space B with the differential forms in $\pi^*\Omega^*(B)$, which are called *basic*. In particular, one has

$$H^*(B) \cong H^*(\Omega^*(P)_{\text{basic}}).$$

Thus the cohomology of the base space B can be computed out of the basic forms on P. These forms are precisely all the forms on P annihilated by the Lie derivatives $\mathcal{L}_{X^\#}$ and interior products $\iota_{X^\#}$, for all $X \in \mathfrak{g}$. Motivated by this, one defines on $W\mathfrak{g} \otimes \Omega^*(M)$ operators analogous to the Lie derivatives and interior products for manifolds, and then calls an element of $W\mathfrak{g} \otimes \Omega^*(M)$ *basic* if it is annihilated by these operators. The subcomplex of basic elements is called the *Weil model* and the equivariant de Rham theorem states that for any G-manifold M,

$$H^*((W\mathfrak{g} \otimes \Omega^*(M))_{\text{basic}}) \cong H^*_G(M).$$

The Weil model is isomorphic to another chain complex called the *Cartan model*, which can be more convenient to work with. Elements of the Cartan model are called *equivariant* differential forms.

3.1 Some differential geometry

Let M be a smooth manifold. A smooth vector field $Z \in \Gamma(TM)$ on M defines two operations on the algebra of differential forms $\Omega^*(M)$.

Definition 3.1. The *Lie derivative* $\mathcal{L}_Z : \Omega^k(M) \longrightarrow \Omega^k(M)$ is the operator of degree 0 defined by

$$\mathcal{L}_Z\omega := \frac{d}{dt}\Big|_{t=0} \rho_t^*\omega,$$

where $\rho_t : M \longrightarrow M$ is the flow of Z.

Definition 3.2. The *interior product* $\iota_Z : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ is the operator of degree -1 defined by

$$(\iota_Z\omega)_x(v_1, \dots, v_{k-1}) := \omega_x(Z(x), v_1, \dots, v_{k-1}).$$

These two operations are linked by the identity, known as *Cartan's magic formula* (see [24, Theorem 14.35]),

$$\mathcal{L}_Z = \iota_Z d + d\iota_Z,$$

where d is the exterior derivative on $\Omega^*(M)$.

When a Lie group G acts on a manifold M, it acts also on its differential forms by pullback. Moreover, every element of the Lie algebra generates a vector field on the manifold called a *fundamental vector field.* If the Lie group is connected, the forms invariant under pullback by the action are precisely the ones annihilated by the Lie derivative with respect to all fundamental vector fields.

Let now G be a Lie group and $\pi : P \longrightarrow B$ be a smooth principal G-bundle of manifolds. For $g \in G$, denote by ϕ_g the diffeomorphism given by the action of g on P,

$$\phi_g : P \longrightarrow P,\ p \longmapsto g \cdot p.$$

We denote by $\exp : \mathfrak{g} \longrightarrow G$ the exponential map of the Lie group G. Recall that for every $X \in \mathfrak{g}$ the map $t \longmapsto \exp tX$ defines a curve in G starting at the identity with initial velocity X.

Definition 3.3. Let $X \in \mathfrak{g}$. The *vector field on P generated by X* is

$$X^{\#}(p) = \frac{d}{dt}\Big|_{t=0}(\exp(tX) \cdot p). \tag{3.1}$$

It is easy to check that the flow of X is given by $\phi_{\exp tX} : P \longrightarrow P$. If the Lie group G is connected, the differential forms on P annihilated by $\mathcal{L}_{X^{\#}}$, for all $X \in \mathfrak{g}$, are called *G-invariant.* This is motivated by the following fact.

Proposition 3.4. *Suppose that the Lie group G is connected. Then for $\sigma \in \Omega^k(P)$ the following are equivalent:*

(i) $\mathcal{L}_{X^{\#}}\sigma = 0$ *for all* $X \in \mathfrak{g}$,

(ii) $\phi_g^*\sigma = \sigma$ *for all* $g \in G$.

Recall that the map $\pi : P \longrightarrow B$ induces a map $\pi^* : \Omega^*(B) \longrightarrow \Omega^*(P)$ via pullback.

Proposition 3.5. *The pullback $\pi^* : \Omega^*(B) \longrightarrow \Omega^*(P)$ is injective. In fact, for any fiber bundle $\pi : P \longrightarrow B$ with fiber F, the pullback $\pi^* : \Omega^*(B) \longrightarrow \Omega^*(P)$ is injective.*

Proof. First notice that π is a submersion, that is, for all $p \in P$ the differential $(d\pi)_p$ is surjective. Indeed, let $\pi(p) \in B$ and let U be an open neighborhood of $\pi(p)$ such that there is a diffeomorphism $h : \pi^{-1}(U) \longrightarrow U \times F$ with $\mathrm{pr}_1 \circ h = \pi$ on $\pi^{-1}(U)$. To show that $(d\pi)_p$ is surjective it suffices to show that $d(\mathrm{pr}_1 \circ h)_p = (d\mathrm{pr}_1)_{h(p)} \circ (dh)_p$ is surjective. But this is true because the projection $\mathrm{pr}_1 : U \times F \longrightarrow U$ is a submersion and h is a diffeomorphism.

Now suppose that $\omega, \mu \in \Omega^k(B)$ satisfy $\pi^*\omega = \pi^*\mu$ in $\Omega^k(P)$, that is for all $p \in P$ and $v_1, \ldots, v_k \in T_pP$ it holds $(\pi^*\omega)_p(v_1, \ldots, v_k) = (\pi^*\mu)_p(v_1, \ldots, v_k)$. By definition of pullback this means

$$\omega_{\pi(p)}(d\pi_p v_1, \ldots, d\pi_p v_k) = \mu_{\pi(p)}(d\pi_p v_1, \ldots, d\pi_p v_k).$$

But since π is a surjective submersion this readily implies that $\omega = \mu$. $\square$

Definition 3.6. We call a form in $\pi^*\Omega^*(B)$ a *basic form*, since it is the image of a form coming from the base space B under the injective map π^*.

Since pullback and exterior derivative commute, $\pi^*\Omega^*(B)$ is a subcomplex of $\Omega^*(P)$ and we denote it by $\Omega^*(P)_{\text{bas}}$.

Remark 3.7. From the injectivity of π^* it follows that $\Omega^*(P)_{\text{bas}}$ and $\Omega^*(B)$ are isomorphic. In particular,

$$H^*_{\text{dR}}(B) \cong H^*(\Omega^*(P)_{\text{bas}}),$$

where $H^*_{\text{dR}}(B)$ denotes the de Rham cohomology of the manifold B.

We can characterize basic forms in terms of the interior product and Lie derivative.

Proposition 3.8. *A differential form $\sigma \in \Omega^*(P)$ is basic if and only if $\iota_{X^\#}\sigma = 0$ and $\mathcal{L}_{X^\#}\sigma = 0$ for all $X \in \mathfrak{g}$.*

For the proof we follow the approach of [23].

Proof. "$\Rightarrow$". Let $\omega \in \Omega^k(B)$. We want to show that

$$\iota_{X^\#}\pi^*\omega = 0 \quad \text{and} \quad \mathcal{L}_{X^\#}\pi^*\omega = 0 \quad \text{for all } X \in \mathfrak{g}.$$

So let $X \in \mathfrak{g}$, $p \in P$ and $v_1, \ldots, v_k \in T_pP$, then

$$\begin{aligned}(\iota_{X^\#}\pi^*\omega)_p(v_1, \ldots, v_{k-1}) &= \omega_{\pi(p)}\Big((d\pi)_p X^\#(p), (d\pi)_p v_1, \ldots, (d\pi)_p v_{k-1}\Big) \\ &= \omega_{\pi(p)}\Big(\frac{d}{dt}\Big|_{t=0} \underbrace{\pi(\exp(tX)\cdot p)}_{=\pi(p)}, (d\pi)_p v_1, \ldots, (d\pi)_p v_{k-1}\Big) \\ &= \omega_{\pi(p)}\Big(0, (d\pi)_p v_1, \ldots, (d\pi)_p v_{k-1}\Big) \\ &= 0,\end{aligned}$$

where in the second-to-last equality we used that G preserves the fibers of π. As for the Lie derivative we have

$$\begin{aligned}(\mathcal{L}_{X^\#}\pi^*\omega)_p(v_1, \ldots, v_k) &= \frac{d}{dt}\Big|_{t=0} \big(\phi^*_{\exp(tX)}\pi^*\omega\big)_p(v_1, \ldots, v_k) \\ &= \frac{d}{dt}\Big|_{t=0} \big(\underbrace{(\pi \circ \phi_{\exp(tX)})}_{=\pi}{}^*\omega\big)_p(v_1, \ldots, v_k)\end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{dt}\Big|_{t=0} (\pi^*\omega)_p(v_1, \dots, v_k) \\
&= 0,
\end{aligned}
$$

where in the third equality we used, again, that G preserves the fibers of π. "$\Leftarrow$". Let $p \in P$ and $x = \pi(p) \in B$. We first claim that the kernel of $d\pi_p$ coincides with the space $T_p(\pi^{-1}(x))$, that is, the tangent space to the fiber $\pi^{-1}(x)$ containing p:

$$
\ker(d\pi_p) = T_p(\pi^{-1}(x)). \tag{3.2}
$$

Indeed, we have seen in Proposition 3.5 that π is a submersion, and thus $\pi^{-1}(x)$ is a submanifold of dimension $p - b$ where $p = \dim P$ and $b = \dim B$ (see [9, Theorem 1.34]). Since $d\pi_p$ is surjective, by the dimension formula we have

$$
\dim(\ker d\pi_p) = p - b = \dim T_p(\pi^{-1}(x)).
$$

Since the two subspaces of T_pP have the same dimension, it suffices to show that $T_p(\pi^{-1}(x)) \subseteq \ker d\pi_p$. Hence, suppose that $v \in T_p\pi^{-1}(x)$, so that there is a curve $t \longmapsto \gamma(t)$ in $\pi^{-1}(x)$ with $\gamma(0) = p$ and $\gamma'(0) = v$. Then $d\pi_p v = \frac{d}{dt}\big|_{t=0}\pi \circ \gamma(t) = 0$, because π is constant on γ, and thus $v \in \ker d\pi_p$. This concludes the proof of the claim.

Since the orbit map $f_p : G \longrightarrow \pi^{-1}(x)$, $g \longmapsto gp$ is a diffeomorphism, its derivative $(df_p)_e : \mathfrak{g} \longrightarrow T_p(\pi^{-1}(x))$ is an isomorphism and we have:

$$
\begin{aligned}
\ker(d\pi_p) &= T_p\big(\pi^{-1}(x)\big) \\
&= \big(df_p\big)_e(\mathfrak{g}) \\
&= \big\{(df_p)_e X \mid X \in \mathfrak{g}\big\} \\
&= \Big\{ \frac{d}{dt}\Big|_{t=0} \exp(tX)\cdot p \;\Big|\; X \in \mathfrak{g}\Big\} \\
&= \{X^{\#}(p) \;\big|\; X \in \mathfrak{g}\}.
\end{aligned}
$$

Back to our proof, let $\sigma \in \Omega^k(P)$ be such that for all $X \in \mathfrak{g}$

$$
\iota_{X^{\#}}\sigma = 0 \quad \text{and} \quad \mathcal{L}_{X^{\#}}\sigma = 0.
$$

We want to define a form $\omega \in \Omega^k(B)$ on M such that $\pi^*(\omega) = \sigma$. The condition $\iota_{X^{\#}}\sigma = 0$ implies the following useful observation.

Claim. Suppose that the vectors $v_1, \dots, v_k, w_1, \dots, w_k \in T_pP$ satisfy

$$
(d\pi_p)v_i = (d\pi_p)w_i \quad \text{for all } i = 1, \dots, k.
$$

Then $\sigma_p(v_1, \dots, v_k) = \sigma_p(w_1, \dots, w_k)$.

Proof. For all $i = 1, \dots, k$ we have $v_i - w_i \in \ker(d\pi_p)$, so that in particular for all i there exists $X_i \in \mathfrak{g}$ with $v_i - w_i = X_i^{\#}(p)$. Thus

$$
\begin{aligned}
\sigma_p(v_1, v_2, \dots, v_k) - \sigma_p(w_1, v_2, \dots, v_k) &= \sigma_p(X^{\#}(p), v_2, \dots, v_k) \\
&= \iota_{X_1^{\#}}\sigma_p\big(v_2, \dots, v_k\big) \\
&= 0.
\end{aligned}
$$

The antisymmetry of σ_p and the above give

$$\begin{aligned}\sigma_p(v_1,\ldots,v_k) &= \sigma_p(w_1,v_2,\ldots,v_k)\\ &= -\sigma_p(v_2,w_1,v_3,\ldots,v_k)\\ &= -\sigma_p(w_2,w_1,v_3,\ldots,v_k)\\ &= \sigma_p(w_1,w_2,v_3,\ldots,v_k)\\ &\ \vdots\\ &= \sigma_p(w_1,\ldots,w_k).\end{aligned}$$

This concludes the proof of the claim. ☐ Claim.

Now, suppose first that the bundle admits a section $s : B \longrightarrow P$, so that $\pi \circ s = id_B$. Set $\omega := s^*\sigma \in \Omega^k(B)$. We have to show that $\pi^*s^*\sigma = \sigma$. Let $p \in P$ and $v_1,\ldots,v_k \in T_pP$. Since π is a principal G-bundle and $\pi(s\circ\pi(p)) = \pi(p)$, there is $g \in G$ with $g\cdot(s\circ\pi)(p) = p$. Then the G-invariance of σ gives

$$\begin{aligned}(\pi^*s^*\sigma)_p(v_1,\ldots,v_k) &= (\pi^*s^*\phi_g^*\sigma)_p(v_1,\ldots,v_k)\\ &= (\phi_g\circ s\circ\pi)^*\sigma_p(v_1,\ldots,v_k)\\ &= \sigma_p(d(\phi_g\circ s\circ\pi)_pv_1,\ldots,d(\phi_g\circ s\circ\pi)_pv_k)\\ &= \sigma_p(v_1,\ldots,v_k).\end{aligned}$$

In the last equality we used the claim, since $d\pi_p(d(\phi_g\circ s\circ\pi)_pv_i) = d(\pi\circ\phi_g\circ s\circ\pi)_pv_i = d(\pi\circ s\circ\pi)_pv_i = d\pi_pv_i$ for all i. Thus if there exists a global section, we are done. If $\pi : p \longrightarrow B$ doesn't admit a section, it admits at least local sections. These can be defined as follows: let $\{U_i\}_{i\in I}$ be a cover of B with trivializations $h_i : \pi_i^{-1}(U_i) \longrightarrow U_i \times G$. Then the smooth maps $s_i : U_i \longrightarrow \pi^{-1}(U_i)$ defined as

$$s_i(x) := h_i^{-1}(x, 1_G)$$

satisfy $\pi\circ s_i(x) = \pi\circ h_i^{-1}(x,1_G) = \mathrm{pr}_1(x,1_G) = x$, and so define local sections. On each U_i we define the smooth form

$$\omega_i := s_i^*\big(\sigma\big|_{\pi^{-1}(U_i)}\big) \in \Omega^k(U_i).$$

By the previous argument it holds $\pi^*\omega_i = \sigma\big|_{\pi^{-1}(U_i)}$, and all that remains to show is that these forms glue together to a differential form ω on B. Let $b \in U_i\cap U_j$ and $v_1,\ldots,v_k \in T_bB$. Since π is a principal G-bundle and $\pi(s_i(b)) = b = \pi(s_j(b))$, there is $g \in G$ with $g\cdot s_i(b) = s_j(b)$. Thus, again using the G-invariance of σ,

$$\begin{aligned}(\omega_i)_b(v_1,\ldots,v_k) &= \sigma_{s_i(b)}\big((ds_i)_bv_1,\ldots,(ds_i)_bv_k\big)\\ &= (\phi_g^*\sigma)_{s_i(b)}\big((ds_i)_bv_1,\ldots(ds_i)_bv_k\big)\\ &= \sigma_{s_j(b)}\big(d(\phi_g\circ s_i)_bv_1,\ldots,d(\phi_g\circ s_i)_bv_k\big)\\ &= \sigma_{s_j(b)}\big((ds_j)_bv_1,\ldots(ds_j)_bv_k\big)\end{aligned}$$

$$= (\omega_j)_b(v_1, \dots, v_k),$$

where the second-to-last equality follows by the claim since

$$\begin{aligned} d\pi_{s_j(b)}\big(d(\phi_g \circ s_i)_b v_l\big) &= d(\pi \circ \phi_g \circ s_i)_b v_l \\ &= d(\pi \circ s_i)_b v_l \\ &= d(\pi \circ s_j)_b v_l \\ &= d\pi_{s_j(b)}\big((ds_j)_b v_l\big). \end{aligned}$$

This shows the well-definedness of a global preimage ω on B of σ and concludes the proof. ☐

3.2 The Weil algebra

As a first step towards an algebraic model for $(EG \times M)/G$ we define the Weil algebra, which will be our algebraic model for EG. We follow the exposition of [1].

Let $\mathfrak{g}$ denote the Lie algebra of the Lie group G.

Definition 3.9. The *Weil algebra* $W\mathfrak{g}$ is the tensor product

$$W\mathfrak{g} := \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$$

of the exterior algebra $\Lambda\mathfrak{g}^*$ and the symmetric algebra $S\mathfrak{g}^*$ on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$.

This algebra is graded by assigning dimension 1 to an element $\theta \in \mathfrak{g}^*$ in the exterior algebra, and degree 2 to the corresponding element, usually denoted by u, in the symmetric algebra. Thus we have

$$\begin{aligned} &W^1\mathfrak{g} = \Lambda^1\mathfrak{g}^* \otimes 1, \\ &W^2\mathfrak{g} = \Lambda^2\mathfrak{g}^* \otimes 1 \oplus 1 \otimes S^1\mathfrak{g}^*, \\ &W^3\mathfrak{g} = \Lambda^3\mathfrak{g}^* \otimes 1 \oplus \Lambda^1\mathfrak{g}^* \otimes S^1\mathfrak{g}^* \end{aligned}$$

and in general

$$W^p\mathfrak{g} = \bigoplus_{k+2l=p} \Lambda^k\mathfrak{g}^* \otimes S^l\mathfrak{g}^*.$$

We have that for any $\omega \in W^p\mathfrak{g}$, $\sigma \in W^q\mathfrak{g}$

$$\omega\sigma = (-1)^{pq}\sigma\omega,$$

that is to say, $W\mathfrak{g}$ is a commutative-graded algebra[5]. Suppose that G has dimension n, and let $X_1, \dots, X_n$ be a basis for $\mathfrak{g}$ and $X_1^*, \dots, X_n^*$ be the corresponding dual basis for $\mathfrak{g}^*$. Then

$$W\mathfrak{g} = \Lambda(X_1^*, \dots, X_n^*) \otimes \mathbb{R}\left[X_1^*, \dots, X_n^*\right].$$

---

[5] Recall that if A and B are two graded algebras, the product on $A \otimes B$ is defined as

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{\deg a_2 \deg b_1} a_1a_2 \otimes b_1b_2.$$

This turns $A \otimes B$ into a commutative-graded algebra.

To avoid confusions, for $\alpha = 1, \dots, n$ we define

$$\theta^\alpha := X_\alpha^* \otimes 1 \text{ and } u_\alpha := 1 \otimes X_\alpha^*.$$

Then we write

$$W\mathfrak{g} = \Lambda(\theta^1, \dots, \theta^n) \otimes \mathbb{R}\left[u_1, \dots, u_n\right].$$

Let $\{c_{\alpha\beta}^\gamma\} \subset \mathbb{R}$ be the *structure constants* of $\mathfrak{g}$, defined by the relations $[X_\alpha, X_\beta] = \sum_{\gamma=1}^n c_{\alpha\beta}^\gamma X_\gamma$. They satisfy the identity $c_{\alpha\beta}^\gamma = -c_{\beta\alpha}^\gamma$.

We endow $W\mathfrak{g}$ with a differential operator D, which on the generators is defined by

$$D\theta^\alpha = u_\alpha - \frac{1}{2}\sum_{\beta,\gamma} c_{\beta\gamma}^\alpha \theta^\beta\theta^\gamma, \tag{3.3}$$

$$Du_\alpha = \sum_{\beta,\gamma} c_{\beta\gamma}^\alpha u_\beta \theta^\gamma. \tag{3.4}$$

Then D is extended to all of $W\mathfrak{g}$ as an antiderivation, so that for $\omega \in W^p\mathfrak{g}$, $\sigma \in W^q\mathfrak{g}$ it satisfies

$$D(\omega\sigma) = D(\omega)\sigma + (-1)^p \omega D(\sigma).$$

Proposition 3.10. *$D^2 = 0$ on $W\mathfrak{g}$.*

Proof. The Jacobi identity imposes the relation

$$\sum_\beta (c_{\beta\gamma}^\alpha c_{kl}^\beta + c_{\beta k}^\alpha c_{l\gamma}^\beta + c_{\beta l}^\alpha c_{\gamma k}^\beta) = 0 \quad \text{for all } \alpha, \gamma, k, l.$$

We compute

$$\begin{aligned}
D^2\theta^\alpha &= D\big(u_\alpha - \frac{1}{2}\sum_{\beta,\gamma} c_{\beta\gamma}^\alpha \theta^\beta\theta^\gamma\big) \\
&= Du_\alpha - \frac{1}{2}\sum_{\beta,\gamma} c_{\beta\gamma}^\alpha D\theta^\beta \cdot \theta^\gamma + \frac{1}{2}\sum_{\beta,\gamma} c_{\beta\gamma}^\alpha \theta^\beta D\theta^\gamma \\
&= \sum_{\beta,\gamma} c_{\beta\gamma}^\alpha u_\beta\theta^\gamma - \frac{1}{2}\sum_{\beta,\gamma} c_{\beta\gamma}^\alpha u_\beta\theta^\gamma + \frac{1}{4}\sum_{\beta,\gamma}\sum_{k,l} c_{\beta\gamma}^\alpha c_{kl}^\beta \theta^k\theta^l\theta^\gamma \\
&\quad + \frac{1}{2}\sum_{\beta,\gamma} c_{\beta\gamma}^\alpha \theta^\beta u_\gamma - \frac{1}{4}\sum_{\beta,\gamma}\sum_{k,l} c_{\beta\gamma}^\alpha c_{kl}^\gamma \theta^\beta\theta^k\theta^l \\
&= \frac{1}{2}\sum_{\beta,\gamma} c_{\beta\gamma}^\alpha u_\beta\theta^\gamma + \frac{1}{2}\sum_{\beta,\gamma} c_{\gamma\beta}^\alpha \theta^\gamma u_\beta \\
&\quad + \frac{1}{4}\sum_{\beta,\gamma}\sum_{k,l} c_{\beta\gamma}^\alpha c_{kl}^\beta \theta^k\theta^l\theta^\gamma - \frac{1}{4}\sum_{\beta,\gamma}\sum_{k,l} c_{\gamma\beta}^\alpha c_{kl}^\beta \theta^\gamma\theta^k\theta^l \\
&= \frac{1}{2}\sum_{\gamma,k,l}\Big(\sum_\beta c_{\beta\gamma}^\alpha c_{kl}^\beta\Big)\theta^\gamma\theta^k\theta^l.
\end{aligned}$$

In the last step we used the relations

$$c^{\alpha}_{\gamma\beta} = -c^{\alpha}_{\beta\gamma},\ \ \theta^{\gamma}u_{\beta} = u_{\beta}\theta^{\gamma} \text{ and } \theta^{k}\theta^{l}\theta^{\gamma} = \theta^{\gamma}\theta^{k}\theta^{l}.$$

Notice that whenever $k = l$ or $l = \gamma$ or $k = \gamma$ we have $\theta^{k}\theta^{l}\theta^{\gamma} = 0$ and thus the sum is taken over $k \neq l \neq \gamma \neq k$. Moreover we have

$$\theta^{\gamma}\theta^{k}\theta^{l} = -\theta^{\gamma}\theta^{l}\theta^{k} = \theta^{l}\theta^{\gamma}\theta^{k} = -\theta^{l}\theta^{k}\theta^{\gamma} = \theta^{k}\theta^{l}\theta^{\gamma} = -\theta^{k}\theta^{\gamma}\theta^{l}.$$

Thus the sum becomes

$$\begin{aligned} D^2\theta^{\alpha} &= \frac{1}{2}\sum_{\gamma<k<l}\left(\sum_{\beta}\Big(c^{\alpha}_{\beta\gamma}c^{\beta}_{kl} - c^{\alpha}_{\beta\gamma}c^{\beta}_{lk} + c^{\alpha}_{\beta l}c^{\beta}_{\gamma k} - c^{\alpha}_{\beta l}c^{\beta}_{k\gamma} + c^{\alpha}_{\beta k}c^{\beta}_{l\gamma} - c^{\alpha}_{\beta k}c^{\beta}_{kl}\Big)\right)\theta^{\gamma}\theta^{k}\theta^{l} \\ &= \sum_{\gamma<k<l}\underbrace{\left(\sum_{\beta}\Big(c^{\alpha}_{\beta\gamma}c^{\beta}_{kl} + c^{\alpha}_{\beta l}c^{\beta}_{\gamma k} + c^{\alpha}_{\beta k}c^{\beta}_{l\gamma}\Big)\right)}_{=0 \text{ by the Jacobi identity}}\theta^{\gamma}\theta^{k}\theta^{l} \\ &= 0. \end{aligned}$$

It follows that $D^2u_{\alpha} = 0$ as well. Indeed,

$$\begin{aligned} D^2u_{\alpha} &= D^2\big(D\theta^{\alpha} + \frac{1}{2}\sum_{\beta,\gamma}c^{\alpha}_{\beta\gamma}\theta^{\beta}\theta^{\gamma}\big) \\ &= D\underbrace{D^2\theta^{\alpha}}_{=0} + \frac{1}{2}\sum_{\beta,\gamma}c^{\alpha}_{\beta\gamma}D\big(D\theta^{\beta}\cdot\theta^{\gamma} - \theta^{\beta}D\theta^{\gamma}\big) \\ &= \frac{1}{2}\sum_{\beta,\gamma}c^{\alpha}_{\beta\gamma}\Big(\underbrace{D^2\theta^{\beta}}_{=0}\cdot\theta^{\gamma} + \underbrace{D\theta^{\beta}D\theta^{\gamma} - D\theta^{\beta}D\theta^{\gamma}}_{=0} + \theta^{\beta}\underbrace{D^2\theta^{\gamma}}_{=0}\Big) \\ &= 0. \end{aligned}$$

Thus D^2 vanishes on the generators θ^{α}, u_{β} of $W\mathfrak{g}$. Notice that D^2 is a derivation. Indeed, for $\omega, \mu \in W\mathfrak{g}$ we have

$$\begin{aligned} D^2(\omega\cdot\mu) &= D\big(D\omega\cdot\mu + (-1)^{\deg\omega}\omega D\mu\big) \\ &= D^2\omega\cdot\mu + \underbrace{(-1)^{\deg\omega+1}D\omega D\mu + (-1)^{\deg\omega}D\omega D\mu}_{=0} + (-1)^{2\deg\omega}\omega D^2\mu \\ &= D^2\omega\cdot\mu + \omega D^2\mu. \end{aligned}$$

Since θ^{α}, u_{β} generate $W\mathfrak{g}$, it follows that $D^2 = 0$ on all of $W\mathfrak{g}$. □

Thus we have a chain complex structure on $W\mathfrak{g} = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$ with

$$W^p\mathfrak{g} = \bigoplus_{k+2l=p}\Lambda^k\mathfrak{g}^* \otimes S^l\mathfrak{g}^*$$

and $D : W^p\mathfrak{g} \longrightarrow W^{p+1}\mathfrak{g}$ satisfying $D^2 = 0$.

Example 3.11 (The torus case)**.** Consider the case when the group is a torus $G = T$. Then the Lie bracket on the Lie algebra $\mathfrak{t} \cong \mathbb{R}^n$ is trivial and thus the structure constants $c^{\alpha}_{\beta\gamma}$ vanish for all α, β, γ. In particular we have

$$D\theta^{\alpha} = u_{\alpha} \text{ and } Du_{\alpha} = 0 \text{ for all } \alpha.$$

We want $W\mathfrak{g}$ to be a model for the differential forms on EG. Since EG is contractible, its de Rham complex should give trivial cohomological groups, so we now show that the chain complex $W\mathfrak{g}$ is acyclic. First, we consider the case of the torus $G = T$ with Lie algebra $\mathrm{Lie}(T) = \mathfrak{t}$.

Proposition 3.12. *$H^*(W\mathfrak{t}, D) = H^0(W\mathfrak{t}, D) = \mathbb{R}$.*

For the proof we follow the approach of [18, pp. 33-34].

Proof. We want to find a chain homotopy $K : W\mathfrak{t} \longrightarrow W\mathfrak{t}$ between the identity on $W\mathfrak{t}$ and the 0-map, that is, a family of maps $K_p : W^p\mathfrak{t} \longrightarrow W^{p-1}\mathfrak{t}$ such that $DK_p + K_{p+1}D = id_{W^p\mathfrak{t}}$ for all $p \geq 0$.

$$\begin{tikzcd}
\dots \arrow[r] & W^{p-1}\mathfrak{t} \arrow[r, "D"] \arrow[d] & W^p\mathfrak{t} \arrow[r, "D"] \arrow[d] \arrow[dl, "K_p"'] & W^{p+1}\mathfrak{t} \arrow[r] \arrow[d] \arrow[dl, "K_{p+1}"'] & \dots \\
\dots \arrow[r] & W^{p-1}\mathfrak{t} \arrow[r, "D"] & W^p\mathfrak{t} \arrow[r, "D"] & W^{p+1}\mathfrak{t} \arrow[r] & \dots .
\end{tikzcd}$$

Consider the map Q defined on the generators of $W\mathfrak{t}$ by

$$Q\theta^{\alpha} = 0 \text{ and } Qu_{\alpha} = \theta^{\alpha}$$

and then extended to the whole of $\mathfrak{t}$ as an antiderivation. Then $DQ + QD$ is a derivation, as for $\omega \in W^p\mathfrak{t}$ and $\mu \in W^q\mathfrak{t}$ we have

$$\begin{aligned}
(DQ + QD)(\omega\mu) &= D\big(Q\omega \cdot \mu + (-1)^p \omega Q\mu\big) + Q\big(D\omega \cdot \mu + (-1)^p \omega D\mu\big) \\
&= DQ\omega \cdot \mu + (-1)^{p-1} Q\omega D\mu + (-1)^p D\omega Q\mu + (-1)^{2p} \omega DQ\mu \\
&\quad + QD\omega \cdot \mu + (-1)^{p+1} D\omega Q\mu + (-1)^p Q\omega D\mu + (-1)^{2p} \omega QD\mu \\
&= (DQ + QD)\omega \cdot \mu + \omega (DQ + QD)\mu.
\end{aligned}$$

Since on the generators of $W\mathfrak{t}$ it holds $DQ + QD = id$, it follows that

$$DQ + QD = (k + l)id \text{ on } \Lambda^k \otimes S^l.$$

Now define

$$K \coloneqq \frac{1}{k + l} Q \quad \text{on } \Lambda^k \otimes S^l, \text{ for } (k, l) \neq (0, 0).$$

Then K is chain homotopy between the identity and the zero-map defined on $W^p\mathfrak{t}$ for all $p > 0$ and thus

$$H^p(W\mathfrak{t}, D) = 0 \quad \text{for all } p > 0.$$

For $p = 0$ we have

$$H^0(W\mathfrak{t}) = \ker D\big|_{W^0\mathfrak{t}} = \mathbb{R},$$

since $W^0\mathfrak{t} = \mathbb{R}$ and $D|_{\mathbb{R}} = 0$.

□

We can extend this result to a general Lie group G.

Theorem 3.13. *$H^*(W\mathfrak{g}, D) = \mathbb{R}$ with the only non-trivial cohomology being in degree 0.*

Proof. The shifts $u_\alpha \longmapsto u_\alpha - \frac{1}{2}\sum_{\beta,\gamma} c^\alpha_{\beta\gamma}\theta^\beta\theta^\gamma =: u'_\alpha$ give a new basis $\{\theta^\alpha, u'_\alpha\}$ of $W\mathfrak{g}$ for which

$$D\theta^\alpha = u'_\alpha,$$
$$Du'_\alpha = 0.$$

Indeed, by definition we have $D\theta^\alpha = u'_\alpha$ and thus $Du'_\alpha = DD\theta^\alpha = 0$. The theorem now follows as for the example of the torus. $\square$

Similarly as in the case of a Lie group G acting on a manifold, we define for each $X \in \mathfrak{g}$ the operators ι_X and $\mathcal{L}_X$ on $W\mathfrak{g}$.

Definition 3.14. (i) On the generators θ^β, u_β of $W\mathfrak{g}$ we set

$$\iota_{X_\alpha}\theta^\beta := \delta^\beta_\alpha \quad \text{and} \quad \iota_{X_\alpha}u_\beta := 0.$$

Then we extend ι_{X_α} as an antiderivation on $W\mathfrak{g}$. For $X = \sum_\alpha c_\alpha X_\alpha \in \mathfrak{g}$ we set $\iota_X := \sum_\alpha c_\alpha \iota_{X_\alpha}$, so that ι_X is linear in $X \in \mathfrak{g}$.

(ii) For $X \in \mathfrak{g}$ we define

$$\mathcal{L}_X := \iota_X D + D\iota_X.$$

By definition ι_X is an antiderivation of degree -1, while $\mathcal{L}_X$ is a derivation of degree 0, because the composition of two antiderivations gives a derivation, as we have seen in the proof of Proposition 3.12.

Remark 3.15. These definitions are motivated by the following observation. If EG and BG were manifolds, then the principal G-bundle $EG \longrightarrow BG$ would admit a connection form $\theta \in \Omega^1(EG, \mathfrak{g})$ and a corresponding curvature form $u \in \Omega^2(EG, \mathfrak{g})$. These forms would give rise, in the obvious way, to two maps $\mathfrak{g}^* \longrightarrow \Omega^1(EG)$ and $\mathfrak{g}^* \longrightarrow \Omega^2(EG)$, respectively, which can be extended to $W\mathfrak{g}$ to generate an algebra homomorphism $f : W\mathfrak{g} \longrightarrow \Omega^*(EG)$, called the *Chern-Weil homomorphism* (see [22]). Then one defines the operators ι_X and $\mathcal{L}_X$ on $W\mathfrak{g}$ in such a way that they are compatible under the map f with the operators $\iota_{-X^\#}$ and $\mathcal{L}_{-X^\#}$ on $\Omega^*(EG)$.

We express $\mathcal{L}_X$ in terms of the generators $\{\theta^\gamma\}_\gamma$ and $\{u_\gamma\}_\gamma$ of $W\mathfrak{g}$.

$$\begin{aligned}
\mathcal{L}_{X_\beta}\theta^\alpha &= \iota_{X_\beta} D\theta^\alpha + D\iota_{X_\beta}\theta^\alpha \\
&= \iota_{X_\beta} u_\alpha - \frac{1}{2}\sum_{\gamma,\delta} c^\alpha_{\gamma\delta}\iota_{X_\beta}(\theta^\gamma\theta^\delta) + 0 \\
&= -\frac{1}{2}\sum_{\gamma,\delta} c^\alpha_{\gamma\delta}\Big(\underbrace{\iota_{X_\beta}\theta^\gamma}_{=\delta^\gamma_\beta}\cdot\theta^\delta - \theta^\gamma\underbrace{\iota_{X_\beta}\theta^\delta}_{=\delta^\delta_\beta}\Big) \\
&= -\frac{1}{2}\sum_{\delta} c^\alpha_{\beta\delta}\theta^\delta + \frac{1}{2}\sum_{\gamma} c^\alpha_{\gamma\beta}\theta^\gamma \\
&= -\sum_{\gamma} c^\alpha_{\beta\gamma}\theta^\gamma.
\end{aligned} \tag{3.5}$$

And

$$\begin{aligned}
\mathcal{L}_{X_\beta} u_\alpha &= \iota_{X_\beta} Du_\alpha + D\iota_{X_\beta} u_\alpha \\
&= \sum_{\gamma,\delta} c^\alpha_{\gamma\delta}\iota_{X_\beta}(u_\gamma\theta^\delta) \\
&= \sum_{\gamma,\delta} c^\alpha_{\gamma\delta}\Big(\underbrace{\iota_{X_\beta} u_\gamma}_{=0}\cdot\theta^\delta + u_\gamma\underbrace{\iota_{X_\beta}\theta^\delta}_{=\delta^\delta_\beta}\Big) \\
&= \sum_{\gamma} c^\alpha_{\gamma\beta} u_\gamma \\
&= -\sum_{\gamma} c^\alpha_{\beta\gamma} u_\gamma.
\end{aligned} \tag{3.6}$$

Definition 3.16. An element $\varphi \in W\mathfrak{g}$ is called *basic* if

$$\iota_X\varphi = 0 \text{ and } \mathcal{L}_X\varphi = 0 \text{ for all } X \in \mathfrak{g}.$$

We denote the subcomplex of basic elements in $W\mathfrak{g}$ by $B\mathfrak{g}$.

Remark 3.17. The basic elements $B\mathfrak{g}$ form indeed a subcomplex of $W\mathfrak{g}$, that is, the differential D maps basic elements to basic elements. In fact, suppose that $\varphi \in W\mathfrak{g}$ is basic, so that $\iota_X\varphi = 0$ and $\mathcal{L}_X\varphi = 0$ for all $X \in \mathfrak{g}$. Then the relation $\mathcal{L}_X = \iota_X D + D\iota_X$ gives

$$\iota_X D\varphi = \mathcal{L}_X\varphi - D\iota_X\varphi = 0$$

and

$$\mathcal{L}_X D\varphi = \iota_X DD\varphi + D\iota_X D\varphi = 0.$$

Thus $D\varphi$ is basic.

We now want to describe the basic elements of $W\mathfrak{g}$ more explicitly. Recall that the group G acts on its Lie algebra $\mathfrak{g}$ via the *adjoint representation* $\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$. For $g \in G$ and $X \in \mathfrak{g}$ one defines

$$\mathrm{Ad}_g(X) \coloneqq (dc_g)_e X,$$

where $c_g : G \longrightarrow G$ is conjugation by g, i.e., $c_g(h) = g \cdot h \cdot g^{-1}$. The corresponding action of G on the dual space $\mathfrak{g}^*$ is called the *coadjoint action* and we denote it by $\mathrm{Ad}^* : G \longrightarrow GL(\mathfrak{g}^*)$. For $g \in G$ and $\xi \in \mathfrak{g}^*$ it is given by

$$\mathrm{Ad}^*_g(\xi) \coloneqq \xi \circ \mathrm{Ad}_{g^{-1}}.$$

This action can be extended to $S\mathfrak{g}^* = \{\text{polynomials in } \mathfrak{g}^*\} = \mathbb{R}[u_1, \dots, u_n]$ by setting

$$\mathrm{Ad}^*_g(u_{\alpha_1} \dots u_{\alpha_k}) \coloneqq \mathrm{Ad}^*_g u_{\alpha_1} \dots \mathrm{Ad}^*_g u_{\alpha_k}.$$

We want to illustrate that $B\mathfrak{g}$ reduces to the ring of polynomials in $\mathfrak{g}^*$ which are invariant under the coadjoint action of G. First, we notice that a basic element contains no θ^α at all, i.e. it is a polynomial in the u's.

Lemma 3.18. *For $\varphi \in W\mathfrak{g}$ it holds*

$$\iota_X \varphi = 0 \textit{ for all } X \in \mathfrak{g} \quad \textit{if and only if} \quad \varphi \in 1 \otimes S\mathfrak{g}^*.$$

In particular, $B\mathfrak{g} \subseteq S\mathfrak{g}^*$.

Proof. Since ι_X is an antiderivation which vanishes on the generators u_α of $S\mathfrak{g}^*$ it is clear that if $\varphi \in 1 \otimes S\mathfrak{g}^*$, then $\iota_X \varphi = 0$ for all $X \in \mathfrak{g}$. For the other implication, following the idea of [23, Lemma 2.54], we show that on $\Lambda^n \mathfrak{g}^* \otimes S^l \mathfrak{g}^*$ the identity

$$\sum_\alpha \theta^\alpha \iota_{X_\alpha} = n \cdot id \tag{3.7}$$

holds for all $n > 1$. Then if $\iota_X \varphi = 0$ for all $X \in \mathfrak{g}$, it follows that $\varphi \notin \Lambda^n \mathfrak{g}^* \otimes S^l \mathfrak{g}^*$ for any $n > 1$. Thus $\varphi \in \Lambda^1 \mathfrak{g}^* \otimes S^l \mathfrak{g}^*$ or $\varphi \in \Lambda^0 \mathfrak{g}^* \otimes S^l \mathfrak{g}^*$ for some $l \geq 0$. In the latter case, we are done. If $\varphi \in \Lambda^1 \mathfrak{g}^* \otimes S^l \mathfrak{g}^*$, then φ is of the form

$$\varphi = \sum_{\substack{\alpha \\ \beta_1 < \dots < \beta_l}} k_{\alpha, \beta_1, \dots, \beta_l} \theta^\alpha u_{\beta_1} \dots u_{\beta_l}$$

and thus

$$\iota_{X_\gamma} \varphi = \sum_{\beta_1 < \dots < \beta_l} k_{\gamma, \beta_1, \dots, \beta_l} u_{\beta_1} \dots u_{\beta_l}$$

vanishes for all γ if and only if $k_{\gamma, \beta_1, \dots, \beta_l} = 0$ for all γ and all $\beta_1 < \dots < \beta_l$, i.e. if and only if $\varphi = 0$. Therefore if $\iota_X \varphi = 0$ for all $X \in \mathfrak{g}$, then (3.7) implies that $\varphi \in 1 \otimes S\mathfrak{g}^*$. We now prove (3.7). Since both sides of (3.7) are linear, it suffices to show this identity for elements of the form $\theta^{\alpha_1} \dots \theta^{\alpha_n} u_{\beta_1} \dots u_{\beta_l}$ in $\Lambda^n \mathfrak{g}^* \otimes S^l \mathfrak{g}^*$. Since ι_{X_α} is an antiderivation, we have

$$\begin{aligned}\sum_\alpha \theta^\alpha \iota_{X_\alpha} (\theta^{\alpha_1} \dots \theta^{\alpha_n} u_{\beta_1} \dots u_{\beta_l}) &= \sum_\alpha \theta^\alpha \iota_{X_\alpha} (\theta^{\alpha_1} \dots \theta^{\alpha_n}) u_{\beta_1} \dots u_{\beta_l} \\ &\quad + \sum_\alpha (-1)^n \theta^{\alpha_1} \dots \theta^{\alpha_n} \underbrace{\iota_{X_\alpha} (u_{\beta_1} \dots u_{\beta_l})}_{=0}\end{aligned}$$

$$= \sum_{\alpha} \theta^{\alpha} \sum_{i=1}^{n} (-1)^{i+1} \theta^{\alpha_1} \dots \underbrace{\iota_{X_\alpha} \theta^{\alpha_i}}_{=\delta_{\alpha}^{\alpha_i}} \dots \theta^{\alpha_n} u_{\beta_1} \dots u_{\beta_l}$$
$$= \sum_{i=1}^{n} (-1)^{i+1} \theta^{\alpha_i} \theta^{\alpha_1} \dots \widehat{\theta^{\alpha_i}} \dots \theta^{\alpha_n} u_{\beta_1} \dots u_{\beta_l}$$
$$= n \cdot \theta^{\alpha_1} \dots \theta^{\alpha_n} u_{\beta_1} \dots u_{\beta_l}.$$

□

Since the basic forms are contained in the symmetric algebra, we look at what it means to be $\mathcal{L}_X$-invariant for a polynomial in $\mathfrak{g}^*$.

Lemma 3.19. $\mathcal{L}_X u_\alpha = \frac{d}{dt}\big|_{t=0} \mathrm{Ad}^*_{\exp tX} u_\alpha$ *for all* $\alpha = 1, \dots, n$.

Proof. We have an action of $\mathfrak{g}$ on itself given by

$$\mathrm{ad}(X)(Y) = [X, Y] \text{ for } X, Y \in \mathfrak{g}.$$

This comes with a corresponding action of $\mathfrak{g}$ on $\mathfrak{g}^*$:

$$\mathrm{ad}^*(X)\xi = \xi \circ \mathrm{ad}(-X) \text{ for } X \in \mathfrak{g}, \xi \in \mathfrak{g}^*.$$

We have that $(d\mathrm{Ad})_e = \mathrm{ad}$ (see [10, Proposition 1.49]). Analogously we claim that

$$(d\mathrm{Ad}^*)_e = \mathrm{ad}^*.$$

We first show that Ad^* and ad^* commute with the exponential map.

Claim. The diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}^*"] \arrow[d, "\exp_{\mathfrak{g}}"'] & \mathrm{End}(\mathfrak{g}^*) \arrow[d, "\exp_{\mathrm{End}\,\mathfrak{g}^*}"] \\ G \arrow[r, "\mathrm{Ad}^*"] & \mathrm{Aut}(\mathfrak{g}^*) \end{tikzcd}$$

commutes.

Proof. Recall that the exponential map on $\mathrm{End}(\mathfrak{g}^*)$ is just the matrix exponential, i.e.

$$\exp_{\mathrm{End}\,\mathfrak{g}^*}(\mathrm{ad}^*(X)) = \sum_{n=0}^{\infty} \frac{\mathrm{ad}^*(X)^n}{n!},$$

where $\mathrm{ad}^*(X)^n = \underbrace{\mathrm{ad}^*(X) \circ \dots \circ \mathrm{ad}^*(X)}_{n-\text{times}}$. We want to show that for all $X \in \mathfrak{g}, \xi \in \mathfrak{g}^*$ and $Y \in \mathfrak{g}$:

$$\mathrm{Ad}^*(\exp_{\mathfrak{g}}(X))(\xi)(Y) = \exp_{\mathrm{End}\,\mathfrak{g}^*}(\mathrm{ad}^*(X))(\xi)(Y).$$

So using the identity $\exp_{\mathfrak{g}} \circ \mathrm{ad} = \mathrm{Ad} \circ \exp_{\mathfrak{g}}$ we compute

$$\begin{aligned}
\exp_{\mathrm{End}\,\mathfrak{g}^*}(\mathrm{ad}^*(X))(\xi)(Y) &= \left(\sum_{n=0}^{\infty} \frac{\mathrm{ad}^*(X)^n}{n!}\right)(\xi)(Y) = \sum_{n=0}^{\infty} \frac{\mathrm{ad}^*(X)^n(\xi)(Y)}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\xi(\mathrm{ad}(-X)^n(Y))}{n!} = \xi\left(\sum_{n=0}^{\infty} \frac{\mathrm{ad}(-X)^n(Y)}{n!}\right) \\
&= \xi\left(\exp_{\mathfrak{g}}(\mathrm{ad}(-X))(Y)\right) = \xi\left(\mathrm{Ad}(\exp_{\mathfrak{g}}(-X))(Y)\right) \\
&= \xi\left(\mathrm{Ad}(\exp_{\mathfrak{g}}(X)^{-1})(Y)\right) = \mathrm{Ad}^*(\exp_{\mathfrak{g}}(X))(\xi)(Y).
\end{aligned}$$

☐ Claim.

Given the claim, the identity $(d\mathrm{Ad})_e^* = \mathrm{ad}^*$ follows, namely for $X \in \mathfrak{g}$ we have

$$\begin{aligned}
(d\mathrm{Ad}^*)_e X &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*(\exp tX) \\
&= \frac{d}{dt}\Big|_{t=0} \exp(\mathrm{ad}^*(tX)) \\
&= \frac{d}{dt}\Big|_{t=0} \exp(t\mathrm{ad}^*(X)) \\
&= \mathrm{ad}^*(X),
\end{aligned}$$

where the first step is the definition of derivative, the third step is the linearity of ad^* and the last one is the definition of the exponential map.

This shows in particular that for $u_\alpha \in S^1\mathfrak{g}^* = \mathfrak{g}^*$ we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp tX}(u_\alpha) &= (d\mathrm{Ad}^*)_e(X)(u_\alpha) \\
&= \mathrm{ad}^*(X)(u_\alpha) \\
&= u_\alpha \circ \mathrm{ad}(-X) \\
&= u_\alpha\left([\,\cdot\,, X]\right).
\end{aligned} \tag{3.8}$$

Claim. $\mathcal{L}_X u_\alpha = u_\alpha\left([\,\cdot\,, X]\right)$.

Proof. By linearity in X of both sides of the equality it suffices to show

$$\mathcal{L}_{X_\beta} u_\alpha = u_\alpha\left([\,\cdot\,, X_\beta]\right) \text{ for all } \beta.$$

Recall that by (3.6) we have $\mathcal{L}_{X_\beta} u_\alpha = -\sum_\delta c^\alpha_{\beta\delta} u_\delta$, so that

$$\mathcal{L}_{X_\beta} u_\alpha(X_\gamma) = -c^\alpha_{\beta\gamma}.$$

Moreover, $u_\alpha\left([X_\gamma, X_\beta]\right) = \sum_\delta c^\delta_{\gamma\beta} u_\alpha(X_\delta) = c^\alpha_{\gamma\beta} = -c^\alpha_{\beta\gamma}$. Thus for all γ we have

$$\mathcal{L}_{X_\beta} u_\alpha(X_\gamma) = u_\alpha\left([X_\gamma, X_\beta]\right)$$

and the claim follows by linearity. ☐ Claim.

The claim and (3.8) give the desired result

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp tX} u_\alpha = \mathcal{L}_X u_\alpha.$$

□

Lemma 3.20. *Let G be a connected Lie group. For every $\varphi \in 1 \otimes S\mathfrak{g}^*$ it holds*

$$\mathrm{Ad}^*_g \varphi = \varphi \textit{ for all } g \in G \textit{ if and only if } \mathcal{L}_X \varphi = 0 \textit{ for all } X \in G.$$

Proof. We first restrict ourselves to the dual of the Lie algebra $\mathfrak{g}^* = S^1\mathfrak{g}^*$. We want to prove that if G is connected, then for the generators u_α of $\mathrm{S}\mathfrak{g}^*$ the two conditions

(i) $\mathrm{Ad}^*_g u_\alpha = u_\alpha$ for all $g \in G$,

(ii) $\mathcal{L}_X u_\alpha = 0$ for all $X \in \mathfrak{g}$

are equivalent.

(i) ⇒ (ii): Suppose that $\mathrm{Ad}^*_g u_\alpha = u_\alpha$ for all $g \in G$. Let $X \in \mathfrak{g}$. Then by Lemma 3.19 we have

$$\mathcal{L}_X u_\alpha = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp tX}(u_\alpha) \overset{\text{(i)}}{=} \frac{d}{dt}\Big|_{t=0} u_\alpha = 0.$$

(ii) ⇒ (i): Since the group is connected, for any $g \in G$ there are $X_1, \dots, X_l \in \mathfrak{g}$ with $\exp(X_1) \cdot \ldots \cdot \exp(X_l) = g$. Moreover, $\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$ is a group homomorphism, so it suffices to show that

$$\mathrm{Ad}^*_{\exp X} u_\alpha = u_\alpha \text{ for all } X \in \mathfrak{g}.$$

Notice that

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_0} \mathrm{Ad}^*_{\exp tX} u_\alpha &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp((t+t_0)X)} u_\alpha \\ &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp(t_0X)\exp(tX)} u_\alpha \\ &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp t_0X} \circ \mathrm{Ad}^*_{\exp tX} u_\alpha \\ &= \mathrm{Ad}^*_{\exp t_0X} \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp tX} u_\alpha.\end{aligned}$$

In the last equality we used that $\mathrm{Ad}^*_g : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ is linear and thus commutes with the derivative. Thus Lemma 3.19 and condition (ii) imply that $\frac{d}{dt}\mathrm{Ad}^*_{\exp tX} u_\alpha$ vanishes for all times, so that $\mathrm{Ad}^*_{\exp tX} u_\alpha = \text{constant} = u_\alpha$.

Now we prove the lemma in the general case. We compute

$$\begin{aligned}\mathcal{L}_X(u_{\alpha_1} \dots u_{\alpha_k}) &= \sum_{i=1}^{k} u_{\alpha_1} \dots \mathcal{L}_X u_{\alpha_i} \dots u_{\alpha_k} \\ &\overset{\text{Lemma 3.19}}{=} \sum_{i=1}^{k} u_{\alpha_1} \dots \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}^*_{\exp tX} u_{\alpha_i}) \dots u_{\alpha_k}\end{aligned}$$

and

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}^*_{\exp tX}(u_{\alpha_1}\dots u_{\alpha_k}) \overset{\text{def}}{=} \frac{d}{dt}\Big|_{t=0}(\mathrm{Ad}^*_{\exp tX}u_{\alpha_1}\dots \mathrm{Ad}^*_{\exp tX}u_{\alpha_k})$$
$$\overset{\text{Leibniz}}{=} \sum_{i=1}^{k} u_{\alpha_1}\dots \frac{d}{dt}\Big|_{t=0}(\mathrm{Ad}^*_{\exp tX}u_{\alpha_i})\dots u_{\alpha_k}.$$

This shows that

$$\mathcal{L}_X(u_{\alpha_1}\dots u_{\alpha_k}) = \frac{d}{dt}\Big|_{t=0}\mathrm{Ad}^*_{\exp tX}\left(u_{\alpha_1}\dots u_{\alpha_k}\right).$$

By linearity it follows that for any $\varphi \in S\mathfrak{g}^* = \mathbb{R}\left[u_1,\dots,u_n\right]$ we have the identity

$$\mathcal{L}_X\varphi = \frac{d}{dt}\Big|_{t=0}\mathrm{Ad}^*_{\exp tX}\varphi. \tag{3.9}$$

So suppose that $\varphi \in S\mathfrak{g}^*$ is $\mathrm{Ad}^*(G)$-invariant. Then for any $X \in \mathfrak{g}$

$$\mathcal{L}_X\varphi = \frac{d}{dt}\Big|_{t=0}\mathrm{Ad}^*_{\exp tX}\varphi = \frac{d}{dt}\Big|_{t=0}\varphi = 0.$$

Conversely, suppose that $\mathcal{L}_X\varphi = 0$ for all $X \in \mathfrak{g}$. Then, as in the case of $\mathfrak{g}^*$, this and the identity (3.9) imply that $\frac{d}{dt}\mathrm{Ad}^*_{\exp tX}\varphi$ vanishes everywhere and so $\mathrm{Ad}^*_{\exp tX}\varphi = \varphi$ is constant. Since G is connected, this proves the lemma.

□

Denote by $(S\mathfrak{g}^*)^G$ the elements of $S\mathfrak{g}^*$ invariant with respect to the $\mathrm{Ad}^*(G)$-action. Then Lemma 3.18 and Lemma 3.20 together give the following.

Proposition 3.21. *$B\mathfrak{g} = (S\mathfrak{g}^*)^G$.*

Before passing to the actual Weil model for the equivariant cohomology, we notice that the differential D vanishes on $B\mathfrak{g}$.

Proposition 3.22. *$D|_{B\mathfrak{g}} = 0$.*

Proof. For u_α we have

$$Du_\alpha = \sum_{\beta,\gamma} c^\alpha_{\beta\gamma}u_\beta\theta^\gamma = \sum_\gamma \theta^\gamma \sum_\beta c^\alpha_{\beta\gamma}u_\beta = \sum_\gamma \theta^\gamma\big(-\sum_\beta c^\alpha_{\gamma\beta}u_b\big) \overset{(3.6)}{=} \sum_\gamma \theta^\gamma \mathcal{L}_{X_\gamma}u_\alpha.$$

Set $\delta \coloneqq \sum_\gamma \theta^\gamma\mathcal{L}_{X_\gamma}$. Then for $x, y \in S\mathfrak{g}^*$:

$$\begin{aligned}\delta(x\cdot y) &= \sum_\gamma \theta^\gamma\mathcal{L}_{X_\gamma}(x\cdot y)\\ &= \sum_\gamma \theta^\gamma\mathcal{L}_{X_\gamma}x\cdot y + \sum_\gamma \theta^\gamma x\cdot\mathcal{L}_{X_\gamma}y\\ &= \delta(x)\cdot y + x\cdot\delta(y).\end{aligned}$$

Since δ and D coincide on the generators $\{u_\alpha\}$ of $S\mathfrak{g}^*$ and they obey the same rules, we can conclude that

$$D = \delta = \sum_\gamma \theta^\gamma \mathcal{L}_{X_\gamma} \text{ on the whole space } S\mathfrak{g}^*. \tag{3.10}$$

So for $\varphi \in B\mathfrak{g}$ we have

$$D\varphi = \delta\varphi = \sum_\beta \theta^\beta \underbrace{\mathcal{L}_{X_\beta}\varphi}_{=0} = 0,$$

which finishes the proof. $\square$

Corollary 3.23. *The basic elements* $B\mathfrak{g}$ *form a subcomplex of* $W\mathfrak{g}$ *and* $H^*(B\mathfrak{g}) \cong B\mathfrak{g}$.

3.3 The Weil model

As before, let G be a Lie group and M be a G-manifold. Consider the tensor product $W\mathfrak{g} \otimes \Omega^*(M)$, equipped with the differential

$$D_W(w \otimes a) \coloneqq Dw \otimes a + (-1)^{\deg w} w \otimes da.$$

Since $d^2 = 0 = D^2$ it follows that ${D_W}^2 = 0$ as well. In each factor of the product we have the operators ι_X and $\mathcal{L}_X$ for $X \in \mathfrak{g}$. We want to extend them on the tensor product as well. The naive definition would be to set $\mathcal{L}_X \coloneqq \mathcal{L}_X \otimes 1 + 1 \otimes \mathcal{L}_{X^\#}$ on $W\mathfrak{g} \otimes \Omega^*(M)$. However, we want the infinitesimal invariance condition $\mathcal{L}_X = 0$ for all $X \in \mathfrak{g}$ to correspond to some invariance at the Lie group level. The Lie group G acts on $W\mathfrak{g} \otimes \Omega^*(M)$ by

$$g \cdot (w \otimes a) \coloneqq \mathrm{Ad}_g^* w \otimes \phi_{g^{-1}}^* a.$$

Suppose that $w \otimes a$ is G-invariant, that is, for all $g \in G$

$$w \otimes a = \mathrm{Ad}_g^* w \otimes \phi_{g^{-1}}^* a.$$

If we take $g = \exp tX$ and differentiate at $t = 0$, by the product rule we get

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp tX} w \otimes a + w \otimes \frac{d}{dt}\Big|_{t=0} \phi_{\exp(-tX)} a \\ &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp tX} w \otimes a + w \otimes \mathcal{L}_{-X^\#} a. \end{aligned}$$

So we set on $\Omega^*(M)$

$$\mathcal{L}_X \coloneqq \mathcal{L}_{-X^\#}$$

and on $W\mathfrak{g} \otimes \Omega^*(M)$

$$\mathcal{L}_X \coloneqq \mathcal{L}_X \otimes 1 + 1 \otimes \mathcal{L}_X. \tag{3.11}$$

Since we want to keep the relations between D, $\mathcal{L}_X$ and ι_X unvaried, for the interior product we also set on $\Omega^*(M)$

$$\iota_X := \iota_{-X^\#}$$

and on $W\mathfrak{g} \otimes \Omega^*(M)$

$$\iota_X(w \otimes a) := \iota_X w \otimes a + (-1)^{\deg w} w \otimes \iota_X a. \tag{3.12}$$

Remark 3.24. In most texts about equivariant cohomology (for example [18], [32]) one takes directly as fundamental vector field corresponding to $X \in \mathfrak{g}$ the vector field generated by $-X$, that is, one sets $X^\# := \frac{d}{dt}\big|_{t=0}\phi_{\exp(-tX)}$. The reason why we don't do the same here, is because in the context of Hamiltonian actions the convention is to define $X^\#$ as in (3.1). For instance, one says that f (and not $-f$) is a moment map if $\iota_{X^\#} = df$. The drawback is the risk to forget that in (3.11) and (3.12) there is an "hidden minus sign".

Definition 3.25. An element $\varphi \in W\mathfrak{g} \otimes \Omega^*(M)$ is called *horizontal* if $\iota_X\varphi = 0$ for all $X \in \mathfrak{g}$, *G-invariant* if $\mathcal{L}_X\varphi = 0$ for all $X \in \mathfrak{g}$, and is called *basic* if it is both horizontal and G-invariant.

Notice that ι_X is an antiderivation and that these operators satisfy the relation $\mathcal{L}_X = D_W\iota_X + \iota_X D_W$ and thus Remark 3.17 shows that the basic elements form a subcomplex of $W\mathfrak{g} \otimes \Omega^*(M)$.

Definition 3.26. The *Weil model* for equivariant cohomology is the chain complex

$$\Omega^*_W(M) := \big(W\mathfrak{g} \otimes \Omega^*(M)\big)_{\text{basic}}$$

of the basic elements of $W\mathfrak{g} \otimes \Omega^*(M)$, together with the differential D_W.

In the following subsections we are going to construct a chain complex isomorphic to the Weil model, called the Cartan model. Then we will outline the proof of the *equivariant de Rham theorem* which asserts that the cohomology of the Borel construction $EG \times_G M$, over $\mathbb{R}$, is isomorphic to the cohomology of the Cartan complex.

3.4 From Weil to Cartan for the circle S^1

Following [1] and [5] we analyze more explicitly the Weil model for the circle $G = S^1$. In this case $\mathfrak{g} \cong \mathbb{R}$ and $W\mathfrak{g}$ is the tensor product of the exterior algebra with a single generator θ in dimension 1 with the polynomial algebra in u with $\dim(u) = 2$:

$$W\mathfrak{g} = \Lambda(\theta) \otimes \mathbb{R}[u].$$

Since the Lie bracket on $\mathfrak{g}$ is trivial, the action of the Weil differential is described by $D\theta = u$ and $Du = 0$, and also from (3.5) and (3.6) we see that $\mathcal{L}_X\theta = 0$ and $\mathcal{L}_X u = 0$ for all $X \in \mathfrak{g}$.

Notice that an element w of $W\mathfrak{g}$ can be expressed as

$$w = \sum_k c_k u^k + \sum_l d_l u^l \theta$$

for $c_k, d_l \in \mathbb{R}$, and thus every element $\varphi \in W\mathfrak{g} \otimes \Omega^*(M)$ is of the form

$$\begin{aligned}
\varphi &= \sum_m \Big(\sum_k c_k^m u^k + \sum_l d_l^m u^l \theta \Big) \otimes \omega_m \\
&= \sum_{m,k} c_k^m u^k \otimes w_m + \sum_{l,m} d_l^m u^l \theta \otimes \omega_m \\
&= \sum_k u^k \otimes \underbrace{(\sum_m c_k^m \omega_m)}_{=:a_k} + \sum_l u^l \theta \otimes \underbrace{\sum_m d_l^m \omega_m}_{=:b_l} \\
&= \sum_k u^k \otimes a_k + \sum_l u^l \theta \otimes b_l,
\end{aligned}$$

with $a_k, b_l \in \Omega^*(M)$.

With the notation $wa = (w \otimes 1) \cdot (1 \otimes a) = w \otimes a$ the above says that every $\varphi \in W\mathfrak{g} \otimes \Omega^*(M)$ has a unique decomposition into a finite sum

$$\varphi = \sum_k u^k a_k + \sum_l u^l \theta b_l, \;\; a_k, b_l \in \Omega^*(M). \tag{3.13}$$

Let X denote the generator of $\mathfrak{g} \cong \mathbb{R}$ dual to θ, so that $\iota_X \theta = 1$, then a basic φ must satisfy the requirements

$$\begin{aligned}
0 = \iota_X \varphi &= \sum_k \iota_X (u^k a_k) + \sum_l \iota_X (u^l \theta b_l) \\
&= \sum_k \Big(\underbrace{\iota_X u^k}_{=0} \cdot a_k + (-1)^{\deg u^k} u^k \iota_X a_k \Big) \\
&\quad + \sum_l \Big(\iota_X (u^l \theta) b_l + (-1)^{\deg (u^l \theta)} u^l \theta \iota_X b_l \Big) \\
&= \sum_k u^k \iota_X a_k + \sum_l \Big(\big(\underbrace{\iota_X u^l}_{=0} \cdot \theta + u^l \underbrace{\iota_X \theta}_{=1} \big) b_l - u^l \theta \iota_X b_l \Big) \\
&= \sum_k u^k (\iota_X a_k + b_k) - \sum_l u^l \theta \iota_X b_l,
\end{aligned}$$

and

$$\begin{aligned}
0 = \mathcal{L}_X \varphi &= \sum_k \mathcal{L}_X (u^k a_k) + \sum_l \mathcal{L}_X (u^l \theta b_l) \\
&= \sum_k \Big(\underbrace{\mathcal{L}_X u^k}_{=0} \cdot a_k + u^k \mathcal{L}_X a_k \Big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_l \Big((\underbrace{\mathcal{L}_X u^l}_{=0} \cdot \theta + u^l \underbrace{\mathcal{L}_X \theta}_{=0}) b_l + u^l \theta \mathcal{L}_X b_l \Big) \\
&= \sum_k u^k \mathcal{L}_X a_k + \sum_l u^l \theta \mathcal{L}_X b_l.
\end{aligned}
$$

In view of the independence of the u^k and $u^k\theta$ over $\Omega^*(M)$ these conditions are equivalent to

$$\iota_X a_k = -b_k, \quad \mathcal{L}_X a_k = 0, \quad \iota_X b_k = 0, \quad \mathcal{L}_X b_k = 0 \quad \text{for all } k.$$

Notice that the first two conditions suffice. The first implies

$$\iota_X b_k \overset{1.}{=} -\iota_X \iota_X a_k = -\iota_X^2 a_k = 0,$$

and the first and the second together imply

$$\mathcal{L}_X b_k \overset{1.}{=} -\mathcal{L}_X \iota_X a_k = -d\underbrace{\iota_X \iota_X a_k}_{=0} - \iota_X d\iota_X a_k = -\iota_X(\mathcal{L}_X a_k - \iota_X d a_k) \overset{2.}{=} \iota_X^2 d a_k = 0.$$

Thus in order to determine whether an element $\varphi \in W\mathfrak{g} \otimes \Omega^*(M)$ is basic it suffices to check that

$$\mathcal{L}_X a_k = 0 \quad \text{and} \quad b_k = -\iota_X a_k \quad \text{for all } k. \tag{3.14}$$

Now let Ω^*_X denote the kernel of $\mathcal{L}_X$ in $\Omega^*(M)$. These are the forms on M invariant under the circle action. Let $\Omega^*_X[u]$ be the polynomial ring generated by a generator u of degree 2 over Ω^*_X, and define a graded-algebra homomorphism

$$\lambda : \Omega^*_X[u] \longrightarrow W\mathfrak{g} \otimes \Omega^*(M)$$

by the formulae:

$$
\begin{aligned}
&\lambda(a) = a - \theta \iota_X a \quad \text{for } a \in \Omega^*_X, \text{ and} \\
&\lambda(u) = u.
\end{aligned}
$$

Proposition 3.27. *λ induces a ring isomomorphism $\Omega^*_X[u] \cong \Omega^*_W(M)$.*

Proof. We first check that λ maps into the basic subcomplex $\Omega^*_W(M)$ of $W\mathfrak{g} \otimes \Omega^*(M)$. Since λ is a ring homomorphism and ι_X, $\mathcal{L}_X$ are derivations, it suffices to check that $\lambda(u)$ and $\lambda(a)$ are basic for all $a \in \Omega^*(M)$.

We have $\lambda(a) = a - \theta\iota_X a$ which in the form (3.13) is given by $a_0 = a$, $b_0 = -\iota_X a$ and $a_k = b_k = 0$ for all $k \geq 1$. Since $a \in \Omega^*_X$ the conditions of (3.14) are satisfied and thus $\lambda(a)$ is basic. In the form of (3.13) $\lambda(u) = u$ is given by $a_1 = 1$, $a_k = 0$ for all $k \neq 1$ and $b_k = 0$ for all k. Thus (3.14) is satisfied and $\lambda(u)$ is basic as well.

To see that λ is surjective, let $\varphi = \sum_k u^k a_k + \sum_l u^l \theta b_l \in \Omega^*_W(M)$, so that for all k it holds $\mathcal{L}_X a_k = 0$ and $b_k = -\iota_X a_k$. Then $\sum_k u^k a_k \in \Omega^*_X[u]$ and

$$\lambda\big(\sum_k u^k a_k\big) = \sum_k \lambda(u)^k \lambda(a_k) = \sum_k u^k a_k - \sum_k u^k \theta \iota_X a_k = \varphi.$$

It remains to show injectivity. Suppose that $\lambda(\sum_k u^k a_k) = 0$. Then

$$0 = \sum_k \lambda(u)^k \lambda(a_k) = \sum_k u^k a_k - \sum_k u^k \theta \iota_X a_k$$

and thus by the independence of u^k and $u^k\theta$ over $\Omega^*(M)$ it holds $a_k = 0$ for all k and $\sum_k u^k a_k = 0$. $\square$

By the proposition $\Omega_X^*[u]$ inherits a differential operator d_X from D_W in $\Omega_W^*(M)$, characterized by the condition

$$\lambda d_X = D_W \lambda.$$

Now for $a \in \Omega_X^*$,

$$\begin{aligned} D_W\lambda(a) &= D_W(a - \theta\iota_X a) \\ &= da - \big(D\theta \cdot \iota_X a + (-1)^{\deg\theta}\theta d\iota_X a\big) \\ &= da - \big(u \cdot \iota_X a - \theta(\mathcal{L}_X a - \iota_X da)\big) \\ &= da - u \cdot \iota_X a - \theta\iota_X da, \end{aligned}$$

and

$$\begin{aligned} \lambda(da - u\iota_X a) &= \lambda(da) - \lambda(u\iota_X a) \\ &= da - \theta\iota_X da - u\big(\iota_X a - \theta\iota_X\iota_X a\big) \\ &= da - \theta\iota_X da - u\iota_X a. \end{aligned}$$

Thus for $a \in \Omega_X^*$ we have

$$d_X a = da - u\iota_X a = da + u\iota_{X^\#} a.$$

Moreover, as u is closed in $\Omega_W^*(M)$ it follows that

$$d_X u = 0,$$

and these two conditions now uniquely determine the differential operator d_X on $\Omega_X^*[u]$ (it must be an antiderivation).

Definition 3.28. The complex $(\Omega_X^*[u], d_X)$ is the *Cartan model* for $H_{S^1}^*(M)$.

Remark 3.29. We denote the set of elements of degree k in $\Omega_X^*[u]$ by $\Omega_X^k[u]$.

For $\omega = a_0 + ua_1 + \ldots + u^n a_n$ in $\Omega_X^*[u]$, the condition $d_X\omega = 0$ is equivalent to

$$\begin{aligned} da_0 &= 0 \\ da_1 &= -\iota_{X^\#} a_0 \\ &\vdots \\ da_n &= -\iota_{X^\#} a_{n-1} \\ \iota_{X^\#} a_n &= 0. \end{aligned} \tag{3.15}$$

Indeed,

$$\begin{aligned}
d_X\big(\sum_{ik=0}^{n} u^k a_k\big) &= \sum_{k=0}^{n} d_X(u^k a_k) \\
&= \sum_{k=0}^{n} \underbrace{d_X u^k}_{=0} \cdot a_k + u^k d_X a_k \\
&= \sum_{k=0}^{n} u^k (da_k - u\iota_X a_k) \\
&= \sum_{k=0}^{n} u^k da_k - \sum_{k=0}^{n} u^{k+1}\iota_X a_k \\
&= \sum_{k=0}^{n} u^k da_k - \sum_{k=1}^{n+1} u^k \iota_X a_{k-1} \\
&= da_0 + \sum_{k=1}^{n} u^k (da_k - \iota_X a_{k-1}) - u^{n+1}\iota_X a_n,
\end{aligned}$$

and one can read the above conditions by comparing the coefficients (recall that for $a \in \Omega^*(M)$ we defined $\iota_X a := \iota_{-X^\#} a = -\iota_{X^\#} a$).

Thus a closed form in the Cartan model is represented by a polynomial in u

$$\omega = a_0 + ua_1 + \ldots + u^n a_n,$$

whose coefficients are S^1-invariant forms $a_i \in \Omega_X^*$ which satisfy (3.15). For instance, a closed equivariant 2-form in $\Omega_X^2[u]$ can be represented in the form

$$\omega = a_0 - ua_1,$$

where $a_0 \in \Omega^2(M)$ is a closed ($da_0 = 0$) and invariant ($\mathcal{L}_X a_0 = 0$) 2-form on M and $a_1 \in \Omega^0(M) = C^\infty(M)$ is a smooth function satisfying $d(-a_1) = -\iota_{X^\#} a_0$, that is,

$$da_1 = \iota_{X^\#} a_0. \tag{3.16}$$

These conditions bring us at once to the concept of moment map. Namely, suppose that M is a symplectic manifold with symplectic form ω and that S^1 acts on M by symplectomorphisms, so that in particular ω is closed and we have $\mathcal{L}_X\omega = 0$. Then the action is said to admit a moment map exactly if there is a smooth function a_1 satisfying (3.16) with $a_0 = \omega$. We are going to come back to this relation in Section 4 and we'll see also how moment maps and equivariant cohomology are related in the case of general Lie groups.

3.5 The Mathai-Quillen isomorphism

In the case of the circle $G = S^1$ we have found a chain complex isomorphic to the Weil model which makes working in $H_{S^1}^*(M)$ much easier. This can be achieved also for a general Lie group G and the resulting Cartan model of equivariant cohomology is analogous to what we found in the circle case. As in [5] and [18] we show how, in searching for the basic forms, in the Weil model $W\mathfrak{g} \otimes \Omega^*(M)$ all the θ terms can be eliminated by the condition $\iota_X \varphi = 0$, $X \in \mathfrak{g}$, so that one gets a smaller complex consisting of the $\mathcal{L}_X$-invariant forms in $S\mathfrak{g}^* \otimes \Omega^*(M)$, and there the Cartan differential is given by

$$d_G u_\alpha = 0, \quad d_G a = da - \sum_\alpha u_\alpha \iota_{X_\alpha} a.$$

We now make this more precise. First, we fix some conventions.

1. We use Einstein's convention for summations. For example,
$$c_{\alpha\beta}^{\gamma} X_\gamma$$
stands for $\sum_\gamma c_{\alpha\beta}^{\gamma} X_\gamma = [X_\alpha, X_\beta]$.
2. Suppose that A and B be two graded algebras. Whenever we have an antiderivation δ_B on B we define on the tensor product $A \otimes B$ the map $1 \otimes \delta_B$ as
$$(1 \otimes \delta_B)(a \otimes b) := (-1)^{\deg a} a \otimes \delta_B b.$$
If we have an antiderivation δ_A on A, the map $\delta_A \otimes 1$ is the usual
$$(\delta_A \otimes 1)(a \otimes b) = \delta_A a \otimes b.$$
With this notation, if δ_A and δ_B are antiderivations on A and B, respectively, then $\delta_A \otimes 1 + 1 \otimes \delta_B$ defines an antiderivation on $A \otimes B$.

For ease of notation we write $W \otimes \Omega$ for $W\mathfrak{g} \otimes \Omega^*(M)$, as well as ι_α and $\mathcal{L}_\alpha$ for ι_{X_α} and $\mathcal{L}_{X_\alpha}$, respectively. We define the degree 0 endomorphism $\eta \in \mathrm{End}(W \otimes \Omega)$ by

$$\eta := \theta^\alpha \otimes \iota_\alpha.$$

In view of the above conventions this is explicitly given by

$$\eta(x \otimes y) = (-1)^{\deg x} \sum_\alpha \theta^\alpha x \otimes \iota_\alpha y.$$

Suppose that $\dim G = n$. Then $\eta^{n+1} = 0$ since every term in its expansion involves the application of $n+1$ factors of θ and $\theta^\alpha \theta^\alpha = 0$. In particular, the sum

$$\exp(\eta) = 1 + \eta + \frac{1}{2}\eta^2 + \frac{1}{3!}\eta^3 + \dots$$

is finite, and thus $\exp(\eta)$ has an inverse given by $\exp(-\eta)$.

Definition 3.30. The automorphism $\phi : W \otimes \Omega \longrightarrow W \otimes \Omega$ defined by

$$\phi := \exp(\eta) = 1 + \eta + \frac{1}{2}\eta^2 + \frac{1}{3!}\eta^3 + \dots$$

is called the *Mathai-Quillen isomorphism.*

For any $\xi \in \operatorname{End}(W \otimes \Omega)$ we define

$$\operatorname{ad}(\eta)(\xi) \coloneqq [\eta, \xi] = \eta \circ \xi - \xi \circ \eta,$$

and

$$\operatorname{Ad}(\phi)(\xi) = \phi \circ \xi \circ \phi^{-1}.$$

Then we have the identity

$$\operatorname{Ad}(\phi)(\xi) = \phi\xi\phi^{-1} = \exp(\operatorname{ad}(\eta))(\xi). \tag{3.17}$$

Indeed, this is nothing else than the relation $\operatorname{Ad} \circ \exp = \exp \circ \operatorname{ad}$, which holds in any algebra of endomorphisms as long as the series on both sides converge. On the left hand side of (3.17) we have the converging (finite) series $\exp(\eta) = 1 + \eta + \frac{1}{2}\eta^2 + \cdots + \frac{1}{n!}\eta^n$. As for the right hand side, notice that every term of $(\operatorname{ad}(\eta))^{2n+1}\xi$ vanishes since

$$(\operatorname{ad}(\eta))^{2n+1}\xi = \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \eta^{2n+1-k}\xi\eta^k$$

and $\eta^{n+1} = 0$. Thus $\exp(\operatorname{ad}(\eta))$ is a finite sum.

Theorem 3.31 ([18, Theorem 4.1.1])**.** *The Mathai-Quillen isomorphism satisfies*

$$\phi(\iota_X \otimes 1 + 1 \otimes \iota_X)\phi^{-1} = \iota_X \otimes 1 \quad \forall X \in \mathfrak{g} \tag{3.18}$$

and

$$\phi D_W \phi^{-1} = D_W - u^\alpha \otimes \iota_\alpha + \theta^\alpha \otimes \mathcal{L}_\alpha. \tag{3.19}$$

Remark 3.32. θ^α and u_α denote as before the generators of $W\mathfrak{g}$. The superscript u^α indicates that there is a summation.

Proof. To prove (3.18) we first show the following three identities:

$$\operatorname{ad}(\eta)(\iota_\beta \otimes 1) = -1 \otimes \iota_\beta, \tag{3.20}$$
$$\operatorname{ad}(\eta)(w \otimes \iota_\beta) = 0 \quad \forall w \in W\mathfrak{g}, \tag{3.21}$$
$$\operatorname{ad}(\eta)^2(\iota_\beta \otimes 1) = 0. \tag{3.22}$$

Proof of (3.20). Let $x \in W^k\mathfrak{g}$, $y \in \Omega^l(M)$. Then

$$\eta \circ (\iota_\beta \otimes 1)(x \otimes y) = \eta(\iota_\beta x \otimes y) = (-1)^{k-1}\theta^\alpha\iota_\beta x \otimes \iota_\alpha y = -(-1)^k\theta^\alpha\iota_\beta x \otimes \iota_\alpha y,$$

and

$$\begin{aligned}(\iota_\beta \otimes 1) \circ \eta(x \otimes y) &= (\iota_\beta \otimes 1)\big((-1)^k\theta^\alpha x \otimes \iota_\alpha y\big)\\ &= (-1)^k\iota_\beta(\theta^\alpha x) \otimes \iota_\alpha y\end{aligned}$$

$$\begin{aligned} &= (-1)^k \underbrace{\iota_\beta\theta^\alpha}_{=\delta^\alpha_\beta} x \otimes \iota_\alpha y - (-1)^k \theta^\alpha \cdot \iota_\beta x \otimes \iota_\alpha y \\ &= (-1)^k x \otimes \iota_\beta y - (-1)^k \theta^\alpha \iota_\beta x \otimes \iota_\alpha y \\ &= (1 \otimes \iota_\beta)(x \otimes y) - (-1)^k \theta^\alpha \iota_\beta x \otimes \iota_\alpha y. \end{aligned}$$

Subtracting the second from the first gives the result.

Proof of (3.21). Let $x \in W^k\mathfrak{g}$, $y \in \Omega^l(M)$ and $w \in W^m\mathfrak{g}$. Then

$$\begin{aligned} \eta \circ (w \otimes \iota_\beta)(x \otimes y) &= \eta\big((-1)^k wx \otimes \iota_\beta y\big) \\ &= (-1)^k(-1)^{m+k}\theta^\alpha wx \otimes \iota_\alpha\iota_\beta y \\ &= (-1)^m \theta^\alpha wx \otimes \iota_\alpha\iota_\beta, \end{aligned}$$

and

$$\begin{aligned} (w \otimes \iota_\beta) \circ \eta(x \otimes y) &= (w \otimes \iota_\beta)\big((-1)^k\theta^\alpha x \otimes \iota_\alpha y\big) \\ &= (-1)^k(-1)^{k+1} \underbrace{w\theta^\alpha}_{=(-1)^{m\cdot 1}\theta^\alpha w} x \otimes \underbrace{\iota_\beta\iota_\alpha}_{=-\iota_\alpha\iota_\beta} y \\ &= (-1)^m\theta^\alpha wx \otimes \iota_\alpha\iota_\beta. \end{aligned}$$

Thus $[\eta, w \otimes \iota_\beta]\,(x \otimes y) = 0$.

Proof of (3.22).

$$\mathrm{ad}(\eta)^2(\iota_\beta \otimes 1) \overset{(3.20)}{=} -\mathrm{ad}(\eta)(1 \otimes \iota_\beta) \overset{(3.21)}{=} 0.$$

These identities, which we have shown for a basis element $X_\beta \in \mathfrak{g}$ extend to any $X \in \mathfrak{g}$ since both on $W\mathfrak{g}$ and on $\Omega^*(M)$ it holds $\iota_X = \iota_{\sum_\beta c_\beta X_\beta} = \sum_\beta c_\beta \iota_\beta$, and the tensor product as well as $\mathrm{ad}(\eta)$ are linear. In particular we have that

$$\mathrm{ad}(\eta)^k(\iota_X \otimes 1) = 0 \quad \forall k \geq 2 \quad \text{and} \quad \mathrm{ad}(\eta)^k(1 \otimes \iota_X) = 0 \quad \forall k \geq 1.$$

Using equation (3.17) the left-hand side of (3.18) becomes

$$\begin{aligned} \phi(\iota_X \otimes 1 + 1 \otimes \iota_X)\phi^{-1} &\overset{(3.17)}{=} \exp(\mathrm{ad}\eta)(\iota_X \otimes 1 + 1 \otimes \iota_X) \\ &= \iota_X \otimes 1 + 1 \otimes \iota_X + \mathrm{ad}(\eta)(\iota_X \otimes 1 + 1 \otimes \iota_X) \\ &\quad \underbrace{+ \sum_{k \geq 2} \frac{1}{k!}\mathrm{ad}(\eta)^k(\iota_X \otimes 1 + 1 \otimes \iota_X)}_{=0} \\ &\overset{(3.21),(3.22)}{=} \iota_X \otimes 1 + 1 \otimes \iota_X + \mathrm{ad}(\eta)(\iota_X \otimes 1) \\ &\overset{(3.20)}{=} \iota_X \otimes 1 + 1 \otimes \iota_X - 1 \otimes \iota_X \\ &= \iota_X \otimes 1. \end{aligned}$$

The proof of (3.19) also requires three further identities:

$$\begin{aligned}
\mathrm{ad}(\eta)(D_W) &= -D\theta^\alpha \otimes \iota_\alpha + \theta^\alpha \otimes \mathcal{L}_\alpha, && (3.23)\\
\mathrm{ad}(\eta)^2(D_W) &= -c^\gamma_{\alpha\beta}\theta^\alpha\theta^\beta \otimes \iota_\gamma, && (3.24)\\
\mathrm{ad}(\eta)^3(D_W) &= 0. && (3.25)
\end{aligned}$$

Proof of (3.23). Recall that using the above convention we write $D_W = D \otimes 1 + 1 \otimes d$. Let $x \in W^k\mathfrak{g}$, $y \in \Omega^l(M)$.

$$\begin{aligned}
\eta \circ (D \otimes 1 + 1 \otimes d)(x \otimes y) &= \eta(Dx \otimes y) + (-1)^k \eta(x \otimes dy)\\
&= (-1)^{k+1}\theta^\alpha Dx \otimes \iota_\alpha y + (-1)^{2k}\theta^\alpha x \otimes \iota_\alpha dy\\
&= -(-1)^k \theta^\alpha Dx \otimes \iota_\alpha y + \theta^\alpha x \otimes \iota_\alpha dy,
\end{aligned}$$

and

$$\begin{aligned}
(D \otimes 1 + 1 \otimes d) \circ \eta(x \otimes y) &= (-1)^k D(\theta^\alpha x) \otimes \iota_\alpha y + (-1)^k(-1)^{k+1}\theta^\alpha x \otimes d\iota_\alpha y\\
&= (-1)^k (D\theta^\alpha) x \otimes \iota_\alpha y - (-1)^k \theta^\alpha Dx \otimes \iota_\alpha y - \theta^\alpha x \otimes d\iota_\alpha y\\
&= D\theta^\alpha \otimes \iota_\alpha (x \otimes y) - (-1)^k \theta^\alpha Dx \otimes \iota_\alpha y - \theta^\alpha x \otimes d\iota_\alpha y.
\end{aligned}$$

Subtracting the second from the first we get

$$\begin{aligned}
\mathrm{ad}(\eta)(D_W)(x \otimes y) &= \theta^\alpha x \otimes (\iota_\alpha d + d\iota_\alpha) y - (D\theta^\alpha \otimes \iota_\alpha)(x \otimes y)\\
&= (\theta^\alpha \otimes \mathcal{L}_\alpha)(x \otimes y) - (D\theta^\alpha \otimes \iota_\alpha)(x \otimes y).
\end{aligned}$$

Proof of (3.24). By (3.21) it holds $\mathrm{ad}(\eta)(D\theta^\alpha \otimes \iota_\alpha) = 0$ and thus with (3.23):

$$\mathrm{ad}(\eta)^2(D_W) = \mathrm{ad}(\eta)(\theta^\alpha \otimes \mathcal{L}_\alpha).$$

We have

$$\begin{aligned}
\mathrm{ad}(\eta)(\theta^\alpha \otimes \mathcal{L}_\alpha) &= (\theta^\beta \otimes \iota_\beta) \circ (\theta^\alpha \otimes \mathcal{L}_\alpha) - (\theta^\alpha \otimes \mathcal{L}_\alpha) \circ (\theta^\beta \otimes \iota_\beta)\\
&= -\theta^\beta\theta^\alpha \otimes \iota_\beta \mathcal{L}_\alpha - \theta^\alpha\theta^\beta \otimes \mathcal{L}_\alpha \iota_\beta\\
&= \theta^\alpha\theta^\beta \otimes [\iota_\beta, \mathcal{L}_\alpha]\\
&= -\theta^\alpha\theta^\beta \otimes [\mathcal{L}_\alpha, \iota_\beta]\\
&= -c^\gamma_{\alpha\beta}\theta^\alpha\theta^\beta \otimes \iota_\gamma,
\end{aligned}$$

where in the last step we used the identity $\left[\mathcal{L}_{-X^\#_\alpha}, \iota_{-X^\#_\beta}\right] = \iota_{-[X_\alpha, X_\beta]^\#}$ on $\Omega^*(M)$(see [30, Problem 20.8]), so that

$$[\mathcal{L}_\alpha, \iota_\beta] = \iota_{-c^\gamma_{\alpha\beta}X^\#_\gamma} = c^\gamma_{\alpha\beta}\iota_\gamma.$$

Proof of (3.25).

$$\begin{aligned}
\mathrm{ad}(\eta)^3 D_W &\overset{(3.24)}{=} -\mathrm{ad}(\eta)(c^\gamma_{\alpha\beta}\theta^\alpha\theta^\beta \otimes \iota_\gamma) \\
&= -c^\gamma_{\alpha\beta}(\theta^\delta \otimes \iota_\delta)(\theta^\alpha\theta^\beta \otimes \iota_\gamma) + c^\gamma_{\alpha\beta}(\theta^\alpha\theta^\beta \otimes \iota_\gamma)(\theta^\delta \otimes \iota_\delta) \\
&= -c^\gamma_{\alpha\beta}\theta^\delta\theta^\alpha\theta^\beta \otimes \iota_\delta\iota_\gamma - c^\gamma_{\alpha\beta}\theta^\alpha\theta^\beta\theta^\delta \otimes \iota_\gamma\iota_\delta \\
&= +c^\gamma_{\alpha\beta}\theta^\alpha\theta^\beta\theta^\delta \otimes \iota_\gamma\iota_\delta - c^\gamma_{\alpha\beta}\theta^\alpha\theta^\beta\theta^\delta \otimes \iota_\gamma\iota_\delta \\
&= 0,
\end{aligned}$$

where in the second to last step we used that $\theta^\alpha\theta^\beta\theta^\delta = \theta^\delta\theta^\alpha\theta^\beta$ and $\iota_\gamma\iota_\delta = -\iota_\delta\iota_\gamma$.

By (3.25) it holds $\sum_{k\geq 3}\frac{1}{k!}\mathrm{ad}(\eta)^k(D_W) = 0$. Now, again using (3.17) the left-hand side of (3.19) is

$$\begin{aligned}
\phi D_W \phi^{-1} &\overset{(3.17)}{=} \exp\big(\mathrm{ad}(\eta)\big) D_W \\
&\overset{(3.25)}{=} D_W + \mathrm{ad}(\eta)(D_W) + \frac{1}{2}\mathrm{ad}(\eta)^2(D_W) \\
&\overset{(3.23),(3.24)}{=} D_W - D\theta^\gamma \otimes \iota_\gamma + \theta^\gamma \otimes \mathcal{L}_\gamma - \frac{1}{2}c^\gamma_{\alpha\beta}\theta^\alpha\theta^\beta \otimes \iota_\gamma \\
&= D_W + \theta^\gamma \otimes \mathcal{L}_\gamma - \underbrace{(D\theta^\gamma + \frac{1}{2}c^\gamma_{\alpha\beta}\theta^\alpha\theta^\beta)}_{=u_\gamma} \otimes \iota_\gamma \\
&= D_W + \theta^\gamma \otimes \mathcal{L}_\gamma - u^\gamma \otimes \iota_\gamma,
\end{aligned}$$

which concludes the proof. □

3.6 The Cartan model

We want to describe the image under the Mathai-Quillen isomorphism ϕ of the basic elements of $W\mathfrak{g} \otimes \Omega^*(M)$. Recall that an element φ of $W\mathfrak{g} \otimes \Omega^*(M)$ is basic if and only if it is horizontal and G-invariant, that is, $(\iota_X \otimes 1 + 1 \otimes \iota_X)\varphi = 0$ and $(\mathcal{L}_X \otimes 1 + 1 \otimes \mathcal{L}_X)\varphi = 0$ for all $X \in \mathfrak{g}$. Equation (3.18) implies that ϕ carries the horizontal subspace $(W \otimes \Omega)_{\mathrm{hor}}$ into $W_{\mathrm{hor}} \otimes \Omega$ and viceversa; and since $W_{\mathrm{hor}} = S\mathfrak{g}^*$ this means:

$$\phi : (W \otimes \Omega)_{\mathrm{hor}} \xrightarrow{\cong} S\mathfrak{g}^* \otimes \Omega.$$

Moreover, notice that for all $X \in \mathfrak{g}$ it holds

$$(\mathcal{L}_X \otimes 1 + 1 \otimes \mathcal{L}_X) \circ \eta = \eta \circ (\mathcal{L}_X \otimes 1 + 1 \otimes \mathcal{L}_X).$$

Indeed, for $x \in W^k\mathfrak{g}$ and $y \in \Omega^l(M)$ we have

$$\begin{aligned}
&(\mathcal{L}_\beta \otimes 1 + 1 \otimes \mathcal{L}_\beta) \circ \eta(x \otimes y) \\
&\quad = (-1)^k(\mathcal{L}_\beta \otimes 1 + 1 \otimes \mathcal{L}_\beta)(\theta^\alpha x \otimes \iota_\alpha y) \\
&\quad = (-1)^k\big(\mathcal{L}_\beta(\theta^\alpha x) \otimes \iota_\alpha y + \theta^\alpha x \otimes \mathcal{L}_\beta\iota_\alpha y\big)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^k \mathcal{L}_\beta \theta^\alpha \cdot x \otimes \iota_\alpha y + \underbrace{(-1)^k \big(\theta^\alpha \mathcal{L}_\beta x \otimes \iota_\alpha y + \theta^\alpha x \otimes \iota_\alpha \mathcal{L}_\beta y\big)}_{\substack{=(\theta^\alpha \otimes \iota_\alpha)(\mathcal{L}_\beta x \otimes y + x \otimes \mathcal{L}_\beta y) \\ =(\theta^\alpha \otimes \iota_\alpha)(\mathcal{L}_\beta \otimes 1 + 1 \otimes \mathcal{L}_\beta)(x \otimes y)}} \\
&\quad + (-1)^k c^\gamma_{\beta\alpha} \theta^\alpha x \otimes \iota_\gamma y \\
&= (-c^\alpha_{\beta\gamma} \theta^\gamma \otimes \iota_\alpha)(x \otimes y) + (c^\gamma_{\beta\alpha} \theta^\alpha \otimes \iota_\gamma)(x \otimes y) \\
&\quad + (\theta^\alpha \otimes \iota_\alpha)(\mathcal{L}_\beta \otimes 1 + 1 \otimes \mathcal{L}_\beta)(x \otimes y) \\
&= \eta \circ (\mathcal{L}_\beta \otimes 1 - 1 \otimes \mathcal{L}_\beta)(x \otimes y).
\end{aligned}
$$

In the third equality we used that on $\Omega^*(M)$ it holds

$$
\mathcal{L}_\beta \iota_\alpha = \iota_\alpha \mathcal{L}_\beta + \iota_{\left[X^\#_\beta, X^\#_\alpha\right]} = \iota_\alpha \mathcal{L}_\beta + \iota_{-[X_\beta, X_\alpha]^\#} = \iota_\alpha \mathcal{L}_\beta + c^\gamma_{\beta\alpha} \iota_\gamma,
$$

and in the fourth we used equation (3.5), that is $\mathcal{L}_\beta \theta^\alpha = -c^\alpha_{\beta\gamma} \theta^\gamma$.

So the same identity holds for ϕ as well:

$$
(\mathcal{L}_X \otimes 1 + 1 \otimes \mathcal{L}_X) \circ \phi = \phi \circ (\mathcal{L}_X \otimes 1 + 1 \otimes \mathcal{L}_X).
$$

This shows that ϕ carries G-invariant elements to G-invariant elements and hence restricts to an isomorphism

$$
\phi : (W \otimes \Omega)_{\text{bas}} \xrightarrow{\cong} (S\mathfrak{g}^* \otimes \Omega)^G.
$$

Recall that by equation (3.10) we have $D\big|_{S\mathfrak{g}^*} = \theta^\alpha \mathcal{L}_\alpha$. Thus, according to (3.19), ϕ conjugates $D_W = D \otimes 1 + 1 \otimes d$ into

$$
\begin{aligned}
\theta^\alpha \mathcal{L}_\alpha \otimes 1 + 1 \otimes d - u^\alpha \otimes \iota_\alpha + \theta^\alpha \otimes \mathcal{L}_\alpha \\
&= (\theta^\alpha \otimes 1)(\mathcal{L}_\alpha \otimes 1 + 1 \otimes \mathcal{L}_\alpha) + 1 \otimes d - u^\alpha \otimes \iota_\alpha \\
&= 1 \otimes d - u^\alpha \otimes \iota_\alpha
\end{aligned}
$$

on $(S\mathfrak{g}^* \otimes \Omega)^G$.

Definition 3.33. The *Cartan model* for the equivariant cohomology of M is

$$
C_G(M) \coloneqq (S\mathfrak{g}^* \otimes \Omega^*(M))^G
$$

together with the differential

$$
\begin{aligned}
d_G : C_G(M) \longrightarrow C_G(M),\ d_G &\coloneqq 1 \otimes d - u^\alpha \otimes \iota_\alpha \\
&= 1 \otimes d + u^\alpha \otimes \iota_{X^\#_\alpha}.
\end{aligned}
$$

Since $\phi : \big((W \otimes \Omega)_{\text{bas}}, D_W\big) \longrightarrow \big(C_G(M), d_G\big)$ is an isomorphism of chain complexes we recover the following theorem.

Theorem 3.34 ([18, Theorem 4.2.1])**.** $H^*\big((W \otimes \Omega)_{\text{bas}}, D_W\big) = H^*\big(C_G(M), d_G\big)$.

That is to say that the Cartan model gives the same cohomology as the Weil model. Notice that when $G = S^1$ this coincides with the results of the previous subsection.

Remark 3.35 (G-invariance). Assume that the Lie group G is connected. On $S\mathfrak{g}^* \otimes \Omega^*(M)$ we have an action of the group G given by the coadjoint action on the symmetric part and by pullback on the differential forms. Explicitly for $w \otimes a \in S\mathfrak{g}^* \otimes \Omega^*(M)$ we have

$$g \cdot (w \otimes a) \coloneqq \mathrm{Ad}_g^* w \otimes \phi_{g^{-1}}^* a.$$

Being G-invariant with respect to this action means that for all $g \in G$ it holds

$$\mathrm{Ad}_g^* w \otimes \phi_{g^{-1}}^* a = w \otimes a. \tag{3.26}$$

As one would expect this invariance is equivalent to the one with respect to the operator $\mathcal{L}_X \otimes 1 + 1 \otimes \mathcal{L}_X$. Indeed, if $w \otimes a$ satisfies (3.26), then taking $g = \exp tX$, $X \in \mathfrak{g}$ and differentiating at $t = 0$ gives

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \big(\mathrm{Ad}_{\exp tX}^* w \otimes \phi_{\exp(-tX)}^* a\big) \\ &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp tX}^* w \otimes a + w \otimes \frac{d}{dt}\Big|_{t=0} \phi_{\exp(-tX)}^* a \\ &= \mathcal{L}_X w \otimes a + w \otimes \mathcal{L}_{-X^\#} a \\ &= (\mathcal{L}_X \otimes 1 + 1 \otimes \mathcal{L}_X)(w \otimes a). \end{aligned}$$

In the second equality we applied the Leibniz rule and in the third equality we used the identity (3.9). Since the Lie group is connected, the infinitesimal invariance implies the invariance at the Lie group level, and thus the two are equivalent, justifying the notation $(S\mathfrak{g}^* \otimes \Omega^*(M))^G$.

Proposition 3.36 (Pullback)**.** *A G-equivariant map $f : M \longrightarrow N$ induces pullback maps on differential forms and cohomology, $f^* : C_G(N) \longrightarrow C_G(M)$ and $f^* : H_G^*(N) \longrightarrow H_G^*(N)$.*

Proof. For $\varphi \in S\mathfrak{g}^*$ and $\omega \in \Omega^*(N)$ we set

$$f^*(\varphi \otimes \omega) \coloneqq \varphi \otimes f^*\omega$$

and then we extend this linearly on $S\mathfrak{g}^* \otimes \Omega^*(N)$. We prove that f^* maps G-invariant elements to G-invariant elements. So let $\sum_i \varphi_i \otimes \omega_i \in (S\mathfrak{g}^* \otimes \Omega^*(N))^G$. Then using the G-equivariance of f we have

$$\begin{aligned} g \cdot f^*\big(\sum_i \varphi_i \otimes \omega_i\big) &= \sum_i \mathrm{Ad}_g^* \varphi_i \otimes \phi_{g^{-1}} f^* \omega_i \\ &= \sum_i \mathrm{Ad}_g^* \varphi_i \otimes f^* \phi_{g^{-1}} \omega_i \\ &= f^*(g \cdot \sum_i \varphi_i \otimes \omega_i) \\ &= f^*\big(\sum_i \varphi_i \otimes \omega_i\big). \end{aligned}$$

Therefore f^* is a map from $C_G(N)$ to $G(M)$. To check that f^* induces a map in cohomology we need to check that it commutes with the Cartan differential. First, notice that since f is G-equivariant, for a differential form ω on N it holds $\iota_{X^\#} f^*\omega = f^*\iota_{X^\#}\omega$ for all $X \in \mathfrak{g}$. Indeed,

$$\begin{aligned}(\iota_{X^\#} f^*\omega)_x(v_1, \ldots, v_k) &= \omega_{f(x)}(df_x X^\#(x), df_x v_1, \ldots, df_x v_k) \\ &= \omega_{f(x)}(X^\#(f(x)), df_x v_1, \ldots, df_x v_k) \\ &= (\iota_{X^\#}\omega)_{f(x)}(df_x v_1, \ldots, df_x v_k) \\ &= (f^*\iota_{X^\#}\omega)_x(v_1, \ldots, v_k),\end{aligned} \tag{3.27}$$

where in the second equality we used that

$$df_x X^\#(x) = \frac{d}{dt}\Big|_{t=0} f \circ \phi^M_{\exp tX}(x) = \frac{d}{dt}\Big|_{t=0} \phi^N_{\exp tX} \circ f(x) = X^\#(f(x)).$$

Now, let $\varphi \otimes \omega \in C_G(N)$, then

$$\begin{aligned} d_G f^*(\varphi \otimes \omega) &= \varphi \otimes df^*\omega - \sum_\alpha \varphi u_\alpha \otimes \iota_{X^\#_\alpha} f^*\omega \\ &\overset{(3.27)}{=} \varphi \otimes f^* d\omega - \sum_\alpha \varphi u_\alpha \otimes f^*\iota_{X^\#_\alpha}\omega \\ &= f^*(\varphi \otimes d\omega - \sum_\alpha \varphi u_\alpha \otimes \iota_{X^\#_\alpha}\omega) \\ &= f^* d_G(\varphi \otimes \omega).\end{aligned}$$

□

3.7 The equivariant de Rham Theorem

Theorem 3.37 (Equivariant de Rham Theorem)**.** *Let G be a compact connected Lie group and let M be a G-manifold. Then*

$$H^*_G(M; \mathbb{R}) = H^*(C_G(M), d_G).$$

Following [32] we outline the proof of the theorem.

Outline of Proof. Suppose that we are given the following theorem.

Theorem (Cartan's Theorem, [12, §5])**.** *If $E \longrightarrow B$ is a principal G-bundle of manifolds, then for all i we have*

$$H^i(B) \cong H^i((W\mathfrak{g} \otimes \Omega^*(E))_{\mathrm{bas}}).$$

Suppose first that the Lie group G acts freely on M. We know that if a compact Lie group G acts freely on a manifold M, then the quotient M/G is a manifold and $M \longrightarrow M/G$ is a principal G-bundle. By Cartan's theorem,

$$H^*(M/G) = H^*((W\mathfrak{g} \otimes \Omega^*(M))_{\mathrm{bas}}) = H^*((S\mathfrak{g}^* \otimes \Omega^*(M))^G, d_G).$$

On the other hand, when G acts freely, M_G and M/G are weakly homotopy equivalent, and thus have the same cohomology. Hence,

$$H_G^*(M) = H^*(M_G) = H^*(M/G) = H^*(C_G(M), d_G).$$

Suppose now that we have an arbitrary action (not necessarily free) of a compact connected Lie group G. Then the homotopy quotient M_G need not be homotopy equivalent to the actual quotient M/G, and the idea is to apply Cartan's theorem to the principal G-bundle $EG \times M \longrightarrow M_G$. However, these spaces are not manifolds. Since the Lie group G is compact, it embeds as a Lie subgroup of some unitary $U(k)$, for k big enough. Then for each i there is n big enough such that

$$H_G^i(M) = H^i(V_k(\mathbb{C}^n) \times_G M)$$

(see Section 2.6) and one can apply Cartan's theorem to the principal G-bundle $V_k(\mathbb{C}^n) \times M \longrightarrow V_k(\mathbb{C}^n) \times_G M$ to recover the equivariant de Rham theorem in the general case.

Finally, to prove Cartan's theorem we look at the short exact sequence of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & \Omega(E)_{\mathrm{bas}} \arrow[r] \arrow[d, phantom, sloped, "=:"] & (W\mathfrak{g} \otimes \Omega(E))_{\mathrm{bas}} \arrow[r] \arrow[d, phantom, sloped, "=:"] & (W\mathfrak{g} \otimes \Omega(E))_{\mathrm{bas}}/\Omega(E)_{\mathrm{bas}} \arrow[r] \arrow[d, equal] & 0, \\
& P & \overline{P} & P/\overline{P} &
\end{tikzcd}$$

which induces a long exact sequence in cohomology

$$\longrightarrow H^{i-1}(P/\overline{P}) \longrightarrow H^i(\Omega(E)_{\mathrm{bas}}) \longrightarrow H^i((W\mathfrak{g} \otimes \Omega(E))_{\mathrm{bas}}) \longrightarrow H^i(P/\overline{P}) \longrightarrow .$$

By Remark 3.7, we have $H^i(\Omega(E)_{\mathrm{bas}}) = H^i(B)$ for all i. Thus to prove the theorem, it suffices to show that the cohomology groups $H^i(P/\overline{P})$ are zero in each degree i. For this, one constructs a chain homotopy between the identity map and the zero map (see [32]). $\square$

4 Relationship between equivariant cohomology and Hamiltonian actions

In this section we shall illustrate why equivariant cohomology can be a very convenient language in which to speak of Hamiltonian group actions. We start by recalling some symplectic geometry notions, as Hamiltonian functions, Hamiltonian vector fields, moment maps and comoment maps. Equivariant cohomology comes into play as soon as we have a Lie group acting on a symplectic manifold (M, ω). Indeed, we'll see that the action admits a moment map if and only if the symplectic form ω extends to an equivariantly closed differential form on M. We give two proofs of this fact, which have rather different flavours. The first one is an application of the Cartan model for $H^*(M_G)$ that we discussed in the previous section. An analogous proof can be given also using the Borel model, see [1]. For the second proof, instead of working with the Cartan or Borel model for the equivariant cohomology, we represent an element of $H^2_G(M) = H^2(M_G)$ by a differential 2-form β on a finite approximation $(M_G)_n$, for some n big enough.

4.1 Some symplectic geometry

For all the material covered in this subsection we refer to [25, §3, §5.2], [11, §18, §21, §22] and [1, §6]. Let (M, ω) be a symplectic manifold and let $f : M \longrightarrow \mathbb{R}$ be a smooth function. The differential df is a 1-form on M. By non-degeneracy of ω, for each $p \in M$ there is a unique vector $v_p \in T_pM$ with $\omega_p(v_p, \cdot) = (df)_p(\cdot)$.

Definition 4.1. For any smooth function $f : M \longrightarrow \mathbb{R}$ the unique vector field $X_f : M \longrightarrow TM$ determined by the identity

$$\iota_{X_f}\omega = df$$

is called the *Hamiltonian vector field* associated to the *Hamiltonian function* f.

Hamiltonian vector fields allow us to define an operation on real-valued functions on M similar to the Lie bracket of vector fields.

Definition 4.2. Given f, $g \in C^\infty(M)$ we define their *Poisson bracket* $\{f, g\} \in C^\infty(M)$ by any of the following equivalent formulae:

$$\{f, g\} \coloneqq \omega(X_f, X_g) = df(X_g) = X_g(f).$$

Remark 4.3. Our definition of the Poisson bracket agrees with the one of [25, Remark 3.3]. However, our Lie bracket differs from theirs by a minus sign. As a result we get the minus sign in the formula $X_{\{f,g\}} = -[X_f, X_g]$, which means that the assignment $f \longrightarrow X_f$ is an anti-homomorphism. This has no remarkable consequence for what concerns our work here. To keep things clear, we recall the definition of the Lie bracket. If X, Y are two vector fields, we set

$$[X, Y] \coloneqq \frac{d}{dt}\Big|_{t=0} (\theta_t)_* X, \tag{4.1}$$

where $\theta_* X(\theta(x)) \coloneqq d\theta_x X(x)$ denotes the push-forward of the vector field X under the diffeomorphism θ and θ_t denotes the flow of Y.

Definition 4.4. A vector field $X \in \Gamma(TM)$ on M is called *symplectic* if $\iota_X\omega$ is closed. We denote the space of symplectic vector fields on M by $\Gamma_\omega(TM)$.

For a vector field $X \in \Gamma(TM)$ the following conditions are equivalent:

- X is symplectic,
- $\mathcal{L}_X\omega = 0$,
- The flow ρ_t of X is a symplectomorphism for all t, that is, $\rho_t^*\omega = \omega \ \forall t$.

The first two conditions are equivalent by Cartan's magic formula $\mathcal{L}_X = \iota_X d + d\iota_X$, while the equivalence of the last two conditions follows from

$$\frac{d}{dt}\Big|_{t=s}\rho_t^*\omega = \frac{d}{dt}\Big|_{t=0}\rho_{t+s}^*\omega = \frac{d}{dt}\Big|_{t=0}\rho_s^*\rho_t^*\omega = \rho_s^*\mathcal{L}_X\omega.$$

Notice that Hamiltonian vector fields are symplectic since

$$d\iota_{X_f}\omega = ddf = 0.$$

Proposition 4.5 ([25, Proposition 3.6])**.** *Let (M,ω) be a symplectic manifold.*

(i) *For every Hamiltonian function $f : M \longrightarrow \mathbb{R}$ and every symplectomorphism $\psi \in \mathrm{Sympl}(M,\omega)$ we have $X_{f\circ\psi^{-1}} = \psi_* X_f$.*

(ii) $X_{\{f,g\}} = -[X_f, X_g]$ *for all $f, g \in C^\infty(M)$.*

Proof. To prove (i), we show that $\iota_{\psi_* X_f}\omega = d(f \circ \psi^{-1})$. Let $x = \psi(y) \in M$, then

$$\begin{aligned}
\iota_{\psi_* X_f}\omega_x &= \omega_x(\psi_* X_f(x), \cdot) \\
&= \omega_{\psi(y)}(d\psi_y X_f(y), d\psi_y(d\psi_y)^{-1}\cdot) \\
&= \psi^*\omega_y(X_f(y), d\psi_x^{-1}\cdot\,) \\
&= \omega_y(X_f(y), d\psi_x^{-1}\cdot\,) \\
&= (\iota_{X_f}\omega)_{\psi^{-1}(x)}(d\psi_x^{-1}\cdot\,) \\
&= ((\psi^{-1})^*\iota_{X_f}\omega)_x \\
&= ((\psi^{-1})^* df)_x \\
&= d(f\circ\psi^{-1})_x.
\end{aligned}$$

To prove statement (ii) let $\rho_t^g \in \mathrm{Sympl}(M,\omega)$ be the flow of X_g. Then using (i) we have

$$[X_f, X_g] = \frac{d}{dt}\Big|_{t=0}(\rho_t^g)_* X_f = \frac{d}{dt}\Big|_{t=0} X_{f\circ(\rho_t^g)^{-1}} = \frac{d}{dt}\Big|_{t=0} X_{f\circ(\rho_{-t}^g)}.$$

Hence

$$\begin{aligned}\iota_{[X_f,X_g]}\omega &= \omega([X_f,X_g]\,,\cdot)\\ &= \frac{d}{dt}\Big|_{t=0}\omega(X_{f\circ\rho^g_{-t}},\cdot)\\ &= \frac{d}{dt}\Big|_{t=0}d(f\circ\rho^g_{-t})\\ &= d\frac{d}{dt}\Big|_{t=0}(f\circ\rho^g_{-t})\\ &= d(df(-X_g))\\ &= -d\{f,g\},\end{aligned}$$

where the second to last equality follows from the fact that ρ^g_{-t} is the flow of the vector field $-X_g$. $\square$

Now let G be a compact connected Lie group acting on a symplectic manifold (M,ω) by symplectomorphisms. Denote by $\phi : G \longrightarrow \mathrm{Sympl}(M,\omega)$ the action, so that for all $g\in G$ the map $\phi_g : M \longrightarrow M$ is a symplectomorphism.

As before, we let $X^\#$ be the vector field on M generated by $X\in\mathfrak{g}$, i.e.,

$$X^\# = \frac{d}{dt}\Big|_{t=0}\phi_{\exp tX}.$$

Notice that $X^\# = (d\phi)_e X$ and one can prove (see [11, §26]) that the map $X \longmapsto X^\#$ is an anti-homomorphism, that is, for all $X,Y\in\mathfrak{g}$ it holds

$$[X,Y]^\# = -\left[X^\#,Y^\#\right]. \tag{4.2}$$

Remark 4.6. The vector field $X^\#$ satisfies

$$X^\#(\phi_{\exp tX}(\cdot)) = \frac{d}{ds}\Big|_{s=t}\phi_{\exp sX}(\cdot),$$

that is to say, its flow is given by the symplectomorphisms $\phi_{\exp tX}$ and thus $X^\#$ is symplectic.

Definition 4.7. A *moment map* for the action of G on (M,ω) is a map $\mu : M \longrightarrow \mathfrak{g}^*$ such that it holds

(i) For $X\in\mathfrak{g}$ let μ^X denote the map $\mu^X : M \longrightarrow \mathbb{R}$, $\mu^X(x) := \mu(x)(X)$. Then
$$d\mu^X = \iota_{X^\#}\mu.$$

(ii) μ is equivariant with respect to the action ϕ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$, that is,
$$\mu\circ\phi_g = \mathrm{Ad}^*_g\circ\mu.$$

Definition 4.8. Suppose that a Lie group G acts on a symplectic manifold (M, ω) by symplectomorphism. We call the action *Hamiltonian* if it admits a moment map.

The conditions of Definition 4.7 can be rephrased in terms of comoment maps.

Definition 4.9. A *comoment map* for the action of G on (M, ω) is a map $\Phi : \mathfrak{g} \longrightarrow C^\infty(M)$ with the two conditions rephrased as

(i) For every $X \in \mathfrak{g}$ the Hamiltonian vector field corresponding to $\Phi(X)$ is $X^\#$:
$$X_{\Phi(X)} = X^\#,$$
or equivalently,
$$d\Phi(X) = \iota_{X^\#}\omega.$$

(ii) Φ is a Lie algebra homomorphism:
$$\Phi([X, Y]) = \{\Phi(X), \Phi(Y)\}.$$

There is a one-to-one correspondence between moment and comoment maps given by
$$\mu \longmapsto \Phi := \mu^{(\cdot)}.$$
It follows directly from the definitions that the condition $d\mu^X = \iota_{X^\#}\omega$ is equivalent to $X_{\mu^X} = X^\#$. It is however slightly more technical to prove that, given these equivalent conditions, μ is G-equivariant if and only if Φ is a Lie algebra homomorphism. For this, we need the following lemma.

Lemma 4.10 ([25, Lemma 5.16])**.** *Let G be connected. Suppose that $\Phi : \mathfrak{g} \longrightarrow C^\infty(M)$ is a Lie algebra homomorphism satisfying $\xi^\# = X_{\Phi(\xi)}$ for all $\xi \in \mathfrak{g}$. Then for all $g \in G$*
$$\Phi(\mathrm{Ad}_{g^{-1}}\xi) = \Phi(\xi) \circ \phi_g.$$

Proof. First notice that using Proposition 4.5 we have
$$X_{\Phi(\mathrm{Ad}_{g^{-1}}\xi)} = (\mathrm{Ad}_{g^{-1}}\xi)^\# \overset{\text{Prop. A.3}}{=} (\phi_g^{-1})_* X_{\Phi(\xi)} = X_{\Phi(\xi)\circ\phi_g}.$$
Thus
$$d\big(\Phi(\xi) \circ \phi_g - \Phi(\mathrm{Ad}_{g^{-1}}\xi)\big) = \iota_{X_{\Phi(\xi)\circ\phi_g}}\omega - \iota_{X_{\Phi(\mathrm{Ad}_{g^{-1}}\xi)}}\omega = 0,$$
and so $\Phi(\xi) \circ \phi_g - \Phi(\mathrm{Ad}_{g^{-1}}\xi)$ is constant. This implies
$$\begin{aligned}\Phi(\mathrm{Ad}_{g^{-1}}[\xi, \eta]) &= \Phi\big(\,[\mathrm{Ad}_{g^{-1}}\xi, \mathrm{Ad}_{g^{-1}}\eta]\,\big)\\ &= \{\Phi(\mathrm{Ad}_{g^{-1}}\xi), \Phi(\mathrm{Ad}_{g^{-1}}\eta)\}\\ &= \{\Phi(\xi) \circ \phi_g, \Phi(\eta) \circ \phi_g\}\\ &= \{(\Phi(\xi), \Phi(\eta)\} \circ \phi_g\\ &= \Phi\big(\,[\xi, \eta]\,\big) \circ \phi_g.\end{aligned} \tag{4.3}$$

Now let $g : [0,1] \longrightarrow G$ be a path in G joining the identity to any point $g(1)$ in G. We want to show that $\Phi(\xi) \circ \phi_{g(t)} - \Phi(\mathrm{Ad}_{g(t)^{-1}}\xi) = \text{constant} = \Phi(\xi) - \Phi(\xi) = 0$ for all t. Denote by $R_g : G \longrightarrow G$ the right-multiplication by g in G and set

$$\eta(t) := (dR_{g(t)^{-1}})_{g(t)} g'(t) \in \mathfrak{g}.$$

Notice that $\eta(t) = \frac{d}{ds}\big|_{s=0} R_{g(t)^{-1}}(g(t+s))$, since $s \longmapsto g(t+s)$ is a path starting at $g(t)$ with initial velocity $g'(t)$. Denote by $\gamma_t(s) := g(t+s)g(t)^{-1}$, so that $\eta(t) = \gamma_t'(0)$.

Claim (1). $\eta(t)^{\#} \circ \phi_{g(t)} = \frac{d}{dt}\big|_t \phi_{g(t)}$.

Proof. Recall that $\eta(t)^{\#} = (d\phi)_e(\eta(t))$, where $\phi : G \longrightarrow \mathrm{Sympl}(M, \omega)$ is the action of G on M. Thus

$$\begin{aligned}
\eta(t)^{\#} \circ \phi_{g(t)} &= (d\phi)_e(\eta(t)) \circ \phi_{g(t)} \\
&= \frac{d}{ds}\bigg|_{s=0} (\phi \circ \gamma_t(s)) \circ \phi_{g(t)} \\
&= \frac{d}{ds}\bigg|_{s=0} \phi_{\gamma_t(s)} \circ \phi_{g(t)} \\
&= \frac{d}{ds}\bigg|_{s=0} \phi_{\gamma_t(s)g(t)} \\
&= \frac{d}{ds}\bigg|_{s=0} \phi_{g(t+s)} \\
&= \frac{d}{dt}\bigg|_{t} \phi_{g(t)}.
\end{aligned}$$

☐ Claim (1).

Claim (2). $\frac{d}{dt}\big|_t \mathrm{Ad}_{g(t)^{-1}}\xi = \mathrm{Ad}_{g(t)^{-1}}[\xi, \eta(t)]$.

Proof.

$$\begin{aligned}
\frac{d}{dt}\bigg|_t \mathrm{Ad}_{g(t)^{-1}}\xi &= \frac{d}{ds}\bigg|_{s=0} \mathrm{Ad}_{g(t+s)^{-1}}\xi \\
&= \frac{d}{ds}\bigg|_{s=0} \mathrm{Ad}_{g(t)^{-1}} \circ \mathrm{Ad}_{\gamma_t(s)^{-1}}\xi \\
&= \mathrm{Ad}_{g(t)^{-1}} \circ \frac{d}{ds}\bigg|_{s=0} \mathrm{Ad}_{\gamma_t(s)^{-1}}\xi \\
&= \mathrm{Ad}_{g(t)^{-1}}\big((d\mathrm{Ad})_e(-\eta(t))(\xi)\big) \\
&= \mathrm{Ad}_{g(t)^{-1}}[-\eta(t), \xi] \\
&= \mathrm{Ad}_{g(t)^{-1}}[\xi, \eta(t)].
\end{aligned}$$

In the fourth equality we used that $\frac{d}{ds}\big|_{s=0} \gamma_t(s)^{-1} = -\eta(t)$. ☐ Claim (2).

In particular, for $x = g(t)^{-1}y \in M$ Claim (1) implies that

$$\begin{aligned}
d\big(\Phi(\xi)\circ\phi_{g(t)}\big)_x(\phi_{g(t)^{-1}})_*\eta(t)^{\#}(x) &= d\big(\Phi(\xi)\circ\phi_{g(t)}\big)_{g(t)^{-1}y}(d\phi_{g(t)^{-1}})_y(\eta(t)^{\#}(y)) \\
&= d\Phi(\xi)_y\eta(t)^{\#}(y) \\
&= d\Phi(\xi)_y(\eta(t)^{\#}\circ\phi_{g(t)}(x)) \\
&\overset{\text{Claim (1)}}{=} d\Phi(\xi)_y\big(\frac{d}{dt}\Big|_t\phi_{g(t)}(x)\big) \\
&= d\Phi(\xi)_y(d\phi_{g(\cdot)}(x))_t(1) \\
&= d\big(\Phi(\xi)\circ\phi_{g(\cdot)}(x)\big)_t(1) \\
&= \frac{d}{dt}\Big|_t\Phi(\xi)\circ\phi_{g(t)}(x).
\end{aligned}$$

Using Claim (2) and (4.3) we have

$$\begin{aligned}
\frac{d}{dt}\Big|_t\Phi(\mathrm{Ad}_{g(t)^{-1}}\xi) &= \Phi(\frac{d}{dt}\Big|_t\mathrm{Ad}_{g(t)^{-1}}\xi) \\
&\overset{\text{Claim (2)}}{=} \Phi(\mathrm{Ad}_{g(t)^{-1}}[\xi,\eta(t)]) \\
&\overset{(4.3)}{=} \Phi([\xi,\eta(t)])\circ\phi_{g(t)}.
\end{aligned}$$

Putting all together we get

$$\begin{aligned}
&\frac{d}{dt}\Big|_t\big(\Phi(\xi)\circ\phi_{g(t)} - \Phi(\mathrm{Ad}_{g(t)^{-1}}\xi)\big) = \\
&\qquad = d\big(\Phi(\xi)\circ\phi_{g(t)}\big)_x\big(\phi_{g(t)^{-1}}\big)_*\eta(t)^{\#} - \Phi([\xi,\eta(t)])\circ\phi_{g(t)} \\
&\qquad = \omega(X_{\Phi(\xi)\circ\phi_{g(t)}}, (\phi_{g(t)^{-1}})_*X_{\Phi(\eta(t))}) - \{\Phi(\xi),\Phi(\eta(t))\}\circ\phi_{g(t)} \\
&\qquad \overset{\text{Prop 4.5}}{=} \omega(X_{\Phi(\xi)\circ\phi_{g(t)}}, X_{\Phi(\eta(t))\circ\phi_{g(t)}}) - \{\Phi(\xi),\Phi(\eta(t))\}\circ\phi_{g(t)} \\
&\qquad = \{\Phi(\xi)\circ\phi_{g(t)}, \Phi(\eta(t))\circ\phi_{g(t)}\} - \{\Phi(\xi),\Phi(\eta(t))\}\circ\phi_{g(t)} \\
&\qquad = 0.
\end{aligned}$$

Therefore $\Phi(\xi)\circ\phi_g - \Phi(\mathrm{Ad}_{g^{-1}}\xi)$ is constant on any path in G connecting the identity to any other point in G. Since G is (path) connected and $\Phi(\xi)\circ\phi_e - \Phi(\mathrm{Ad}_{e^{-1}}\xi) = 0$, this concludes the proof of the lemma. ☐

Now we are in the position to prove the equivalence of moment and comoment maps.

Proposition 4.11. *$\mu : M \longrightarrow \mathfrak{g}^*$ is a moment map for the action of G on M if and only if $\Phi : \mathfrak{g} \longrightarrow C^\infty(M)$, $\Phi(X) := \mu^X$ is a comoment map for the action.*

Proof. As noticed above, the condition $d\mu^X = \iota_{X^{\#}}\omega$ is equivalent to $X_{\mu^X} = X^{\#}$ since X_{μ^X} is characterized by the identity $d\mu^X = \iota_{X_{(\mu^X)}}\omega$. Thus we have to show that assuming the above equivalent properties it holds

$$\mu\circ\phi_g = \mathrm{Ad}_g^*\circ\mu \text{ if and only if } \mu^{[X,Y]} = \{\mu^X,\mu^Y\}.$$

Assume first that μ is G-equivariant. For $X, Y \in \mathfrak{g}$ and $x \in M$ we have:

$$\begin{aligned}
\{\mu^X, \mu^Y\}(x) &= X_{\mu^Y}(\mu^X)(x) \\
&= Y^{\#}(\mu^X)(x) \\
&= (d\mu^X)_x Y^{\#}(x) \\
&= \frac{d}{dt}\Big|_{t=0} \mu^X \circ \phi_{\exp tY}(x) \\
&= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp tY} \circ \mu(x)(X) \\
&= \frac{d}{dt}\Big|_{t=0} \mu(x)\big(\mathrm{Ad}_{\exp(-tY)}(X)\big) \\
&= \mu(x)\big(\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(-tY)}(X)\big) \\
&= \mu(x)\big(\,[-Y, X]\,\big) \\
&= \mu^{[X,Y]}(x).
\end{aligned}$$

For the other implication, assume that $\mu^{(\cdot)}$ is a Lie algebra homomorphism. Thus $\Phi = \mu^{(\cdot)}$ satisfies the conditions of Lemma 4.10 and we have

$$\begin{aligned}
\mu(\phi_g(x))(X) &= \Phi(X) \circ \phi_g(x) \\
&\overset{\text{Lemma 4.10}}{=} \Phi(\mathrm{Ad}_{g^{-1}} X)(x) \\
&= \mu(x)(\mathrm{Ad}_{g^{-1}} X) \\
&= \mathrm{Ad}^*_g \circ \mu(x)(X)
\end{aligned}$$

for all $x \in M$, $g \in G$ and $X \in \mathfrak{g}$. ☐

Example 4.12 (Moment map for the S^1-action)**.** Suppose that $G = S^1$ acts on (M, ω) by symplectomorphisms. We have $\mathfrak{g} \cong \mathbb{R}$ and $\mathfrak{g}^* \cong \mathbb{R}$. Let $\mu : M \longrightarrow \mathbb{R}$ be a moment map and $X = 1$ be a generator of $\mathfrak{g}$. Then $\mu^X(x) = \mu(x) \cdot 1$, that is $\mu^X = \mu$. Therefore the first condition of Definition 4.7 becomes simply

$$d\mu = \iota_{X^{\#}}\omega. \tag{4.4}$$

Moreover, since S^1 is abelian we have that $\mathrm{Ad}_g X = d(g \cdot g^{-1})_e X = X$ and thus $\mathrm{Ad}^*_g \circ \mu = \mu$. So the G-equivariance condition for μ is actually invariance: $\mu \circ \phi_g = \mu$, which is equivalent to $\mathcal{L}_{X^{\#}}\mu = 0$ by Proposition 3.4. If (4.4) holds, this condition is automatically satisfied. Indeed,

$$\mathcal{L}_{X^{\#}}\mu = \iota_{X^{\#}} d\mu + d\iota_{X^{\#}}\mu \overset{(4.4)}{=} \iota_{X^{\#}}\iota_{X^{\#}}\omega = 0.$$

4.2 Equivariant extension in the Cartan model

We want to illustrate how equivariant cohomology fits in the context of moment maps. We start by considering the case of the circle $G = S^1$, as in [1].

Example 4.13. Let $G = S^1$ act on a symplectic manifold (M, ω) by symplectomorphisms. Retain the notation of Section 3.4. Let $X \in \mathfrak{g}$. Since the action is symplectic we have $\mathcal{L}_{X^\#}\omega = 0$, thus $\omega \in \Omega^2_X[u] = \{\text{degree 2 elements of } \Omega^*_X[u]\}$. However, in general ω doesn't represent an element in the equivariant cohomology group $H^2_{S^1}(M)$ of M since need not be closed with respect to the Cartan differential d_X, as $d_X\omega = -u\iota_X\omega$. It is natural to ask if ω can be "extended" to an equivariantly closed 2-form. Since u has degree 2, the only way to extend it in $\Omega^2_X[u]$ is to add a multiple of u to ω. So any extension takes the form

$$\omega^\# = \omega - u \cdot f,$$

where $f : M \longrightarrow \mathbb{R}$ is some S^1-invariant smooth function on M. It holds:

$$\begin{aligned} d_X\omega^\# &= d_X\omega - d_X(u \cdot f) \\ &= d\omega + u\iota_{X^\#}\omega - d_X u \cdot f - u d_X f \\ &= u\iota_{X^\#}\omega - udf - u^2\iota_{X^\#} f \\ &= u(\iota_{X^\#}\omega - df). \end{aligned}$$

Thus ω has an equivariantly closed extension if and only if the action admits a moment map.

This phenomenon is actually quite general. Let (M, ω) be a symplectic manifold and G be a connected Lie group acting by symplectomorphisms on M. As in Section 3, let $\{X_\alpha\}$ be a basis of $\mathfrak{g}$ with dual basis $\{\theta^\alpha\} \in \mathfrak{g}^*$ and $\{u_\alpha\} \in S^1\mathfrak{g}^*$ be the corresponding elements in the symmetric part of $W\mathfrak{g}$. The Cartan model for the differential forms on M_G is the chain complex

$$C_G(M) = (S\mathfrak{g}^* \otimes \Omega^*(M))^G,$$

with differential $d_G = 1 \otimes d - \sum_\alpha u_\alpha \otimes \iota_\alpha$, where $\iota_\alpha \overset{\text{def}}{=} \iota_{-X^\#_\alpha}$ on $\Omega^*(M)$. In what follows instead of the symbol $\otimes$ we mainly write a plain dot $\cdot$ and most times we even omit the dot. In particular for a differential form $a \in \Omega^*(M)$ the Cartan differential $d_G a$ is given by

$$d_G a = da - \sum_\alpha u_\alpha \iota_\alpha a$$

and

$$d_G(u_\alpha a) = u_\alpha da - \sum_\beta u_\beta u_\alpha \iota_\beta a.$$

The symplectic form $\omega \in \Omega^2(M)$ can be seen as an element of

$$C^2_G(M) = \{\text{degree 2 elements of } (S\mathfrak{g}^* \otimes \Omega^*(M))^G\},$$

since $\mathcal{L}_{X^\#}\omega = 0$ for all $X \in \mathfrak{g}$. However, as in the case of S^1 in general it is not closed with respect to the Cartan differential:

$$d_G\omega = d\omega - \sum_\alpha u_\alpha \iota_\alpha \omega = -\sum_\alpha u_\alpha \iota_\alpha \omega.$$

By dimensional reasons, any extension of ω to an equivariant form in $C^2_G(M)$ is of the form

$$\omega^{\#} = \omega - \sum_\alpha u_\alpha \cdot f_\alpha,$$

where $f_\alpha \in \Omega^0(M)$ are smooth functions on M.

Proposition 4.14. *The formula $\omega^{\#} = \omega - \sum_\alpha u_\alpha \cdot f_\alpha$ defines an equivariantly closed extension of ω if and only if the linear assignment $\Phi : \mathfrak{g} \longrightarrow C^\infty(M)$, $X_\alpha \longmapsto f_\alpha$ gives a comoment map for the action.*

Proof. We compute $d_G\omega^{\#}$ and $\mathcal{L}_{X_\alpha}\omega^{\#}$ and then show that they vanish for all α if and only if $X_\alpha \longmapsto f_\alpha$ is a comoment map. We start by computing the differential.

$$\begin{aligned} d_G\omega^{\#} &= d_G\omega - \sum_\alpha d_G(u_\alpha \cdot f_\alpha) \\ &= d\omega - \sum_\alpha u_\alpha \iota_\alpha \omega - \sum_\alpha u_\alpha df_\alpha + \sum_\alpha \sum_\beta u_\beta u_\alpha \cdot \underbrace{i_\beta f_\alpha}_{=0} \\ &= -\sum_\alpha u_\alpha (df_\alpha + \iota_\alpha \omega). \end{aligned}$$

As for the Lie derivative we have

$$\begin{aligned} \mathcal{L}_\beta \omega^{\#} &= \mathcal{L}_\beta \omega - \sum_\alpha \mathcal{L}_\beta (u_\alpha \cdot f_\alpha) \\ &= -\sum_\alpha \mathcal{L}_\beta u_\alpha \cdot f_\alpha - \sum_\alpha u_\alpha \mathcal{L}_\beta f_\alpha \\ &\overset{(3.6)}{=} \sum_\alpha \sum_\gamma c^\alpha_{\beta\gamma} u_\gamma \cdot f_\alpha - \sum_\alpha u_\alpha \iota_\beta df_\alpha - \sum_\alpha u_\alpha d\underbrace{\iota_\beta f_\alpha}_{=0} \\ &= \sum_\alpha u_\alpha (\sum_\gamma c^\gamma_{\beta\alpha} f_\gamma - \iota_\beta df_\alpha). \end{aligned}$$

The independence over the u_α's gives

$$d_G\omega^{\#} = 0 \quad \text{if and only if} \quad df_\alpha = -\iota_\alpha \omega = \iota_{X^{\#}_\alpha}\omega \quad \text{for all } \alpha \tag{4.5}$$

and

$$\mathcal{L}_\beta \omega^{\#} = 0 \quad \forall \beta \quad \text{if and only if} \quad \sum_\gamma c^\gamma_{\beta\alpha} f_\gamma = \iota_\beta df_\alpha = -\iota_{X^{\#}_\beta} df_\alpha \quad \forall \alpha, \beta.$$

Notice that $df_\alpha = \iota_{X^{\#}_\alpha}\omega$ if and only if $X^{\#}_\alpha$ is the Hamiltonian vectorfield corresponding to f_α, that is, $X_{f_\alpha} = X^{\#}_\alpha$. Thus assuming that the two equivalent conditions of (4.5) hold we have

$$\iota_{X^{\#}_\beta} df_\alpha \overset{(4.5)}{=} \iota_{X^{\#}_\beta} \iota_{X^{\#}_\alpha} \omega = \omega(X^{\#}_\alpha, X^{\#}_\beta) = \omega(X_{f_\alpha}, X_{f_\beta}) = \{f_\alpha, f_\beta\} = \{\Phi(X_\alpha), \Phi(X_\beta)\}.$$

Moreover, by the linearity of the assignment $\Phi : X_\alpha \longmapsto f_\alpha$, it holds

$$\sum_\gamma c^\gamma_{\beta\alpha} f_\gamma = \sum_\gamma c^\gamma_{\beta\alpha} \Phi(X_\gamma) = \Phi\big(\sum_\gamma c^\gamma_{\beta\alpha} X_\gamma\big) = \Phi([X_\beta, X_\alpha]) = -\Phi([X_\alpha, X_\beta]).$$

Thus $\mathcal{L}_\beta \omega^\# = 0$ for all β if and only if $\Phi([X_\alpha, X_\beta]) = \{\Phi(X_\alpha), \Phi(X_\beta)\}$ for all α, β. Putting everything together, we have the following sequence of equivalences

- $\omega^\# = \omega - \sum_\alpha u_\alpha \cdot f_\alpha$ is a d_G-closed G-invariant element of $S\mathfrak{g}^* \otimes \Omega^*(M)$.
- $d_G \omega^\# = 0$ and $\mathcal{L}_\beta \omega^\# = 0$ for all β.
- $df_\alpha = \iota_{X^\#_\alpha} \omega$ for all α and $\sum_\gamma c^\gamma_{\beta\alpha} f_\gamma = -\iota_{X^\#_\beta} df_\alpha$ for all α, β.
- $X_{f_\alpha} = X^\#_\alpha$ for all α and $\Phi([X_\alpha, X_\beta]) = \{\Phi(X_\alpha), \Phi(X_\beta)\}$ for all α, β.
- Φ is a comoment map for the action.

□

4.3 Equivariant extension via finite approximations

We now want to give another version of Proposition 4.14, without working in the Cartan or Weil model, but using finite approximations for M_G. To define an equivariant extension for the symplectic form on the manifold M, we are going to use a connection form θ on (a finite approximation of) the principal G-bundle $EG \longrightarrow BG$. Recall that θ is a $\mathfrak{g}$-valued 1-form on EG which satisfies

(i) $\iota_{X^\#} \theta = X$ for all $X \in \mathfrak{g}$.

(ii) $\phi^*_g \theta = \mathrm{Ad}_g \circ \theta$ for all $g \in G$.

For the main properties of connections forms on principal G-bundles and their existence we refer to Appendix A.

Let (M, ω) be a symplectic manifold on which a compact Lie group G acts by symplectomorphisms. Let n be big enough such that $H^2(M_G) = H^2((M_G)_n)$ (see Section 2.6). Thus elements of $H^2_G(M)$ are represented by 2-forms on the manifold $(M_G)_n$. Let θ be a connection form on the principal G-bundle $EG_n \longrightarrow BG_n$[6]. In the following for simplicity we'll write M_G, EG and BG for $(M_G)_n$, EG_n and BG_n, respectively. Given a map $\mu : M \longrightarrow \mathfrak{g}^*$ we can define a $\mathfrak{g} \otimes \mathfrak{g}^*$-valued 1-form $\theta \otimes \mu$ on $EG \times M$ by

$$(\theta \otimes \mu)_{(e,x)}((v, w)) \coloneqq \theta_e(v) \otimes \mu(x),$$

where we identified $T_{(e,x)}(EG \times M) \cong T_e EG \times T_x M$. Furthermore, the pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$ allows making $\theta \otimes \mu$ into an $\mathbb{R}$-valued 1-form which we denote by $\theta \cdot \mu$:

$$(\theta \cdot \mu)_{(e,x)}((v, w)) \coloneqq \langle \mu(x), \theta_e(v) \rangle.$$

---

[6] It exists since $EG_n \longrightarrow BG_n$ is a principal G-bundle of manifolds, see Proposition A.8.

Proposition 4.15 ([3, Proposition VI.2.1])**.** *Suppose that a connected compact Lie group G acts by symplectomorphisms on the symplectic manifold (M,ω), and let θ be a connection on the bundle $EG \longrightarrow BG$. Denote by $\mathrm{pr}_2 : EG \times M \longrightarrow M$ the projection onto the second component and let $\mu : M \longrightarrow \mathfrak{g}^*$ be any differentiable map. Then the 2-form $\widetilde{\beta}$ on $EG \times M$ given by*

$$\widetilde{\beta} = \mathrm{pr}_2^*\omega + d(\theta \cdot \mu)$$

descends to a closed 2-form on M_G if and only if the action is Hamiltonian with moment map μ.

Proof. Assume that $\mu : M \longrightarrow \mathfrak{g}^*$ is a moment map for the action and let θ be a connection form on $EG \longrightarrow BG$. Let $\theta \cdot \mu$ be as above and set

$$\widetilde{\beta} \coloneqq \mathrm{pr}_2^*\omega + d(\theta \cdot \mu),$$

where $\mathrm{pr}_2 : EG \times M \longrightarrow M$ is the projection onto the second factor. This is a closed 2-form on $EG \times M$. We show that it descends to M_G, that is, there is a 2-form β on $EG \times_G M$ whose lift to $EG \times M$ is $\widetilde{\beta}$:

$$\pi^*\beta = \widetilde{\beta}.$$

That is to say, we show that $\widetilde{\beta}$ is basic. By Proposition 3.8 we need to verify the two conditions

(i) $\mathcal{L}_{X^\#}\widetilde{\beta} = 0$ for all $X \in \mathfrak{g}$,

(ii) $\iota_{X^\#}\widetilde{\beta} = 0$ for all $X \in \mathfrak{g}$.

We prove (i). Let $\phi_g : M \longrightarrow M$ denote the action of $g \in G$ on M, $\phi_g^{EG} : EG \longrightarrow EG$ the action of $g \in G$ on EG, and $\widetilde{\phi}_g : EG \times M \longrightarrow EG \times M$ denote the diagonal action of $g \in G$ on $EG \times M$. Notice that $\mathrm{pr}_2 \circ \widetilde{\phi}_g = \phi_g \circ \mathrm{pr}_2$. We show that for all $g \in G$ it holds $\widetilde{\phi}_g^*\widetilde{\beta} = \widetilde{\beta}$. Indeed,

$$\begin{aligned}
\widetilde{\phi}_g^*\widetilde{\beta} &= \widetilde{\phi}_g^*\mathrm{pr}_2^*\omega + \widetilde{\phi}_g^* d(\theta \cdot \mu) \\
&= \mathrm{pr}_2^*(\phi_g)^*\omega + d\widetilde{\phi}_g^*(\theta \cdot \mu) \\
&= \mathrm{pr}_2^*\omega + d\widetilde{\phi}_g^*(\theta \cdot \mu),
\end{aligned}$$

as G acts on M by symplectomorphisms. Notice that

$$\widetilde{\phi}_g^*(\theta \cdot \mu) = (\phi_g^{EG})^*\theta \cdot (\phi_g^M)^*\mu = (\mathrm{Ad}_g \circ \theta) \cdot (\mathrm{Ad}_g^* \circ \mu) = \theta \cdot \mu$$

Thus $\widetilde{\phi}_g^*\widetilde{\beta} = \mathrm{pr}_2^*\omega + d(\theta \cdot \mu) = \widetilde{\beta}$ and by Proposition 3.4 infinitesimally

$$\mathcal{L}_{X^\#}\widetilde{\beta} = 0.$$

We prove (ii). We compute $\iota_{X^\#}\mathrm{pr}_2^*\omega$ and $\iota_{X^\#}d(\theta\cdot\mu)$ separately. Let $(e,x)\in EG\times M$ and $(v,w)\in T_eEG\times T_xM$. Then

$$\begin{aligned}(\iota_{X^\#}\mathrm{pr}_2^*\omega)_{(e,x)}((v,w)) &= (\mathrm{pr}_2^*\omega)_{(e,x)}\big(X^\#(e,x),(v,w)\big)\\ &= \omega_x\big((d\mathrm{pr}_2)_{(e,x)}X^\#(e,x),(d\mathrm{pr}_2)_{(e,x)}(v,w)\big).\end{aligned}$$

Recall that $X^\#(e,x)=\frac{d}{dt}\big|_{t=0}\big(\exp(tX)\cdot e,\exp(tX)\cdot x\big)\in T_{(e,x)}EG\times M$ and so $(d\mathrm{pr}_2)_{(e,x)}X^\#(e,x)=\frac{d}{dt}\big|_{t=0}\mathrm{pr}_2\big(\exp(tX)\cdot e,\exp(tX)\cdot x\big)=X^\#(x)$. Thus

$$\begin{aligned}(\iota_{X^\#}\mathrm{pr}_2^*\omega)_{(e,x)}((v,w)) &= \omega_x(X^\#(x),(d\mathrm{pr}_2)_{(e,x)}(v,w))\\ &= (\iota_{X^\#}\omega)_x((d\mathrm{pr}_2)_{(e,x)}(v,w))\\ &= (\mathrm{pr}_2^*\iota_{X^\#}\omega)_{(e,x)}((v,w)).\end{aligned}$$

As for the second term we have

$$\iota_{X^\#}d(\theta\cdot\mu)=\mathcal{L}_{X^\#}(\theta\cdot\mu)-d\iota_{X^\#}(\theta\cdot\mu)=-d\iota_{X^\#}(\theta\cdot\mu),$$

since in the proof of (i) we have seen that $\widetilde{\phi}_g^*(\theta\cdot\mu)=\theta\cdot\mu$. We compute $\iota_{X^\#}(\theta\cdot\mu)$ explicitly, so for $(e,x)\in EG\times M$ we have

$$\begin{aligned}\iota_{X^\#}(\theta\cdot\mu)(e,x) &= (\theta\cdot\mu)_{(e,x)}(X^\#(e,x))\\ &= (\theta\cdot\mu)_{(e,x)}(X^\#(e),X^\#(x))\\ &= \langle\mu(x),\theta_e(X^\#(e))\rangle\\ &= \langle\mu(x),\iota_{X^\#}\theta(e)\rangle\\ &= \langle\mu(x),X\rangle\\ &= \mu^X(x)\\ &= (\mathrm{pr}_2^*\mu^X)(e,x).\end{aligned}$$

Finally, since μ is a moment map for the action we get

$$\iota_{X^\#}\widetilde{\beta}=\iota_{X^\#}\mathrm{pr}_2^*\omega+\iota_{X^\#}d(\theta\cdot\mu)=\mathrm{pr}_2^*\iota_{X^\#}\omega-d\mathrm{pr}_2^*\mu^X=\mathrm{pr}_2^*d\mu^X-\mathrm{pr}_2^*d\mu^X=0.$$

We now prove the other direction. So suppose that there is $\beta\in\Omega^2(EG\times_G M)$ with $\pi^*\beta=\widetilde{\beta}$. Then for all $X\in\mathfrak{g}$ we have $\mathcal{L}_{X^\#}\widetilde{\beta}=0$ and $\iota_{X^\#}\widetilde{\beta}=0$. Since G is connected the first condition is equivalent as saying that for all $g\in G$ it holds $\widetilde{\phi}_g^*\widetilde{\beta}=\widetilde{\beta}$. This implies that $\widetilde{\phi}_g^*(\theta\cdot\mu)=\theta\cdot\mu$ for all $g\in G$. Therefore,

$$(\phi_g^{EG})^*\theta\cdot(\phi_g)^*\mu=\widetilde{\phi}_g^*(\theta\cdot\mu)=\theta\cdot\mu. \tag{4.6}$$

Let $X\in\mathfrak{g}$. Then $\theta_e(X^\#(e))=X$, by definition of connection. Thus for all $g\in G$, $x\in M$ and $X\in\mathfrak{g}$ we have

$$\begin{aligned}\langle\mu(gx),\mathrm{Ad}_gX\rangle &= \langle\mu(gx),\mathrm{Ad}_g\theta_eX^\#(e)\rangle\\ &= \langle\phi_g^*\mu(x),((\phi_g^{EG})^*\theta)_e(X^\#(e))\rangle\\ &\overset{(4.6)}{=}\langle\mu(x),\theta_e(X^\#(e))\rangle\\ &= \langle\mu(x),X\rangle.\end{aligned}$$

This shows that $\mu \circ \phi_g = \mathrm{Ad}_g^* \circ \mu$. The same computations as in the proof of (ii) give

$$\iota_{X^\#}\widetilde{\beta} = \mathrm{pr}_2^* \iota_{X^\#}\omega - \mathrm{pr}_2^* d\mu^X.$$

Since $\iota_{X^\#}\widetilde{\beta} = 0$ and pr_2^* is injective, it follows that $\iota_{X^\#}\omega = d\mu^X$, which concludes the proof. $\square$

We conclude this section by computing the derivative $d(\theta \cdot \mu)$ of the 1-form $\theta \cdot \mu$ on $EG \times M$.

Lemma 4.16. *Let $\theta \in \Omega^1(EG, \mathfrak{g})$ and $\mu : M \longrightarrow \mathfrak{g}^*$. Then*

$$d(\theta \cdot \mu) = d\theta \cdot \mu - \theta \cdot d\mu,$$

where $\theta \cdot d\mu$ is the 2-form on $EG \times M$ defined by

$$(\theta \cdot d\mu)_{(e,x)}((v_1, w_1), (v_2, w_2)) = \langle d\mu_x(w_2), \theta_e(v_1)\rangle - \langle d\mu_x(w_1), \theta_e(v_2)\rangle.$$

Proof. Choose local coordinates $\{dx_i\}_i$ on EG and $\{dy_j\}_j$ on M, so that

$$\theta = \sum_i f_i dx_i \text{ for some } f_i : EG \longrightarrow \mathfrak{g} \quad \text{and} \quad d\mu = \sum_j \frac{\partial \mu}{\partial y_j} dy_j.$$

We want to express $\theta \cdot \mu$ in local coordinates. For $(e, x) \in EG \times M$ and $(v, w) \in T_e EG \times T_x M$ we have

$$\begin{aligned}(\theta \cdot \mu)_{(e,x)}(v, w) &= \langle \mu(x), \theta_e(v)\rangle \\ &= \langle \mu(x), \sum_i f_i(e)(dx_i)_e v\rangle \\ &= \sum_i \langle \mu(x), f_i(e)\rangle (dx_i)_e v + \sum_j 0 \cdot (dy_j)_x w.\end{aligned}$$

Set $h_i(e, x) := \langle \mu(x), f_i(e)\rangle$. Recall that for a 1-form $\sigma = \sum_i \sigma_i dx_i$ on a manifold X in local coordinates one has

$$d\sigma = \sum_i d\sigma_i \wedge dx_i = \sum_{i,j} \frac{\partial \sigma_i}{\partial x_j} dx_j \wedge dx_i.$$

Then for $(v_1, w_1), (v_2, w_2) \in T_e EG \times T_x M$ we have

$$\begin{aligned}&d(\theta \cdot \mu)_{(e,x)}((v_1, w_1), (v_2, w_2)) = \\ &\qquad = \sum_{i,k} \frac{\partial h_i}{\partial e_k}(e, x)(dx_k \wedge dx_i)_{(e,x)}((v_1, w_1), (v_2, w_2)) \\ &\qquad\quad + \sum_{i,l} \frac{\partial h_i}{\partial x_l}(e, x)(dy_l \wedge dx_i)_{(e,x)}((v_1, w_1), (v_2, w_2)) \\ &\qquad = \sum_{i,k} \frac{\partial h_i}{\partial e_k}(e, x)(dx_k \wedge dx_i)_e(v_1, v_2)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,l}\frac{\partial h_i}{\partial x_l}(e,x)\Big((dy_l)_x w_1 (dx_i)_e v_2 - (dy_l)_x w_2 (dx_i)_e v_1\Big)\\
&= \langle \mu(x), \sum_{i,k}\frac{\partial f_i}{\partial e_k}(e)(dx_k \wedge dx_i)(v_1, v_2)\rangle\\
&\quad + \sum_{i,l}\langle \frac{\partial \mu}{\partial x_l}(x), f_i(e)\rangle\Big((dy_l)_x w_i (dx_i)_e - (dy_l)_x w_2 (dx_i)_e v_1\Big)\\
&= \langle \mu(x), d\theta_e(v_1, v_2)\rangle + \big\langle \sum_l \frac{\partial \mu}{\partial x_l}(x)(dy_l)_x w_1, \sum_i f_i(e)(dx_i)_e v_2\big\rangle\\
&\quad - \big\langle \sum_l \frac{\partial \mu}{\partial x_l}(x)(dy_l)_x w_2, \sum_i f_i(e)(dx_i)_e v_1\big\rangle\\
&= (d\theta\cdot\mu)_{(e,x)}((v_1,w_1),(v_2,w_2)) + \langle d\mu_x w_1, \theta_e v_2\rangle - \langle d\mu_x w_2, \theta_e v_1\rangle.
\end{aligned}
$$

This concludes the proof. □

5 The first Duistermaat-Heckman formula

5.1 Ehresmann connections and parallel translation

We define the notion of (Ehresmann) connection for a submersion $f : M \longrightarrow N$ between manifolds. A connection gives a way to locally lift paths in N to paths in M, in a unique way if one fixes a starting point of the lift. If the lift is not only local but global, then one can define parallel translation, which allows to identify the level sets of the submersion.

Definition 5.1. Let M, N be smooth manifolds, $f : M \longrightarrow N$ a submersion and $m \in M$. The *vertical space* V_m at m is the subspace $\ker df_m$ of T_mM. An *(Ehresmann) connection* for f is the choice of a *horizontal subspace* H_m of T_mM for each $m \in M$, such that $T_mM = V_m \oplus H_m$.

Definition 5.2. Let $f : M \longrightarrow N$ be a submersion and let $\gamma : [0,1] \longrightarrow N$ be a path in N. A *lift of* γ is a path $\widetilde{\gamma}$ in M satisfying $f \circ \widetilde{\gamma} = \gamma$. If we have a connection $H = \{H_m\}_{m \in M}$ on M, then we call a lift $\widetilde{\gamma}$ in M *horizontal* if for all times t it holds $\widetilde{\gamma}'(t) \in H_{\widetilde{\gamma}(t)}$.

Theorem 5.3. *Let $f : M \longrightarrow N$ be a proper*[7] *submersion. Then f admits a connection and for every path $\gamma : [0,1] \longrightarrow N$ in N with $\gamma' \neq 0$ everywhere and any fixed $m_0 \in f^{-1}(\gamma(0))$, there is a unique horizontal lift $\widetilde{\gamma}$ of γ starting at m_0.*

For the proof of the theorem we need the following lemma.

Lemma 5.4. *Let $f : M \longrightarrow N$ be a submersion with a choice of a connection $H = \{H_m\}_{m \in M}$. Let X be a vector field on N. Then there is a unique vector field $\widetilde{X}$ on M such that $\widetilde{X}(m) \in H_m$ and $df_m\widetilde{X}(m) = X(f(m))$ for all $m \in M$.*

We call the vector field $\widetilde{X}$ the *horizontal lift* of X.

Proof. Since f is a submersion, for every $m \in M$ the differential $df_m : T_mM \longrightarrow T_{f(m)}N$ is surjective and so restricted to H_m gives an isomorphism $df_m\big|_{H_m} : H_m \longrightarrow T_{f(m)}N$. Thus the assignment

$$\widetilde{X} : M \longrightarrow TM,\ m \longmapsto (df_m\big|_{H_m})^{-1}(X(f(m)))$$

defines a smooth vector field on M with $\widetilde{X}(m) \in H_m$ for all $m \in M$ and $df_m\widetilde{X}(m) = X(f(m))$. $\square$

We can now prove Theorem 5.3. The proof is based on [13, Appendix C] and [29, §VII.I].

Proof of Theorem 5.3. Since any manifold M admits a Riemannian metric we define the horizontal subspaces H_m, $m \in M$, as the orthogonal complements to $\ker df_m$ with respect to the metric. To prove the theorem we show that every

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[7] Recall that proper means that the preimage of every compact set is compact.

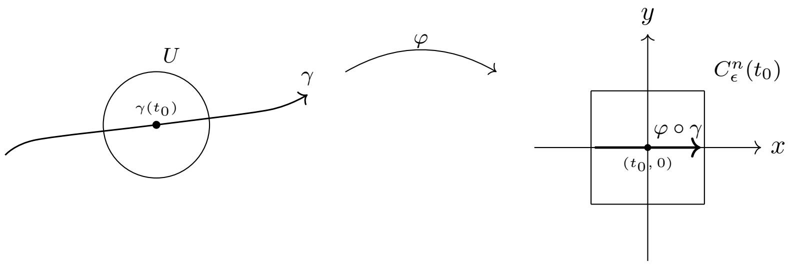

Figure 1: Proof of Claim in Theorem 5.3.

point $\gamma(t_0)$ on the curve γ has a neighborhood where, given a fixed starting point, the lift exists and is unique. Then since $\gamma(I)$ is compact it can be covered with finitely many such neighborhoods, where the lifts exist. By uniqueness, these lifts can be glued together to give a unique lift of γ. We start by showing that locally γ is the integral curve of some vector field.

Claim. Let $t_0 \in I$. Then there is an open neighborhood $U \subset M$ of $\gamma(t_0)$ and a vector field X on U with $X(\gamma(t)) = \gamma'(t)$.

Proof. Since $\gamma' \neq 0$, by dimensional reasons γ has constant rank 1 and thus by the constant rank theorem (see [9, Corollary 1.32]) there are neighborhoods J of t_0 and U of $\gamma(t_0)$ and a diffeomorphism

$$\varphi : U \longrightarrow C^n_\epsilon(t_0) := (t_0 - \epsilon, t_0 + \epsilon) \times (-\epsilon, \epsilon)^{n-1} \subset \mathbb{R}^n,$$

such that $\gamma(J) \subseteq U$ and $\varphi \circ \gamma(t) = (t, 0, \ldots, 0)$ for all $t \in J$. Let V be the vector field on $\gamma(J)$ defined as $V(\gamma(t)) = \gamma'(t)$. Then $\varphi_* V$ is the constant vector field $(1, 0, \ldots 0)$ on $\varphi(\gamma(J)) = J \times \{0\} \subseteq C^n_\epsilon(t_0)$, which can be extended smoothly by translations to all of $C^n_\epsilon(t_0)$. The pullback of this extension to U gives the required extension X of V. Notice that for any t with $\gamma(t) \in U$, γ is the integral curve of X passing through $\gamma(t)$. ☐ Claim.

By the reasoning above it suffices to show that γ has a unique lift when restricted to an open set as the one given by the claim. So we may assume for simplicity that γ lies entirely in an open set U and that for $0 \leq t \leq 1$ it is the integral curve of a vector field X on U. Let $\widetilde{X}$ be the horizontal vector field on $f^{-1}(U)$ with $df_m \widetilde{X} = X \circ f$ given by Lemma 5.4. Let $m_0 \in f^{-1}(\gamma(0))$. Then a curve $\widetilde{\gamma}$ in M is a horizontal lift of γ starting at m_0 if and only if it is the integral curve of $\widetilde{X}$ starting at m_0. Indeed, if it is a horizontal lift of γ, then

$$df_{\widetilde{\gamma}(t)} \widetilde{X}(\widetilde{\gamma}(t)) = X(f(\widetilde{\gamma}(t))) = X(\gamma(t)) = \gamma'(t) = (f \circ \widetilde{\gamma})'(t) = df_{\widetilde{\gamma}(t)} \widetilde{\gamma}'(t)$$

and thus $\widetilde{X}(\widetilde{\gamma}(t)) = \widetilde{\gamma}'(t)$ since $df_{\widetilde{\gamma}(t)}|_{H_{\widetilde{\gamma}(t)}}$ is injective. Conversely, if $\widetilde{\gamma}$ is the integral curve of $\widetilde{X}$, then $\widetilde{X}(\widetilde{\gamma}(t)) = \widetilde{\gamma}'(t)$ and as above one gets $X(f \circ \widetilde{\gamma}(t)) = (f \circ \widetilde{\gamma})'(t)$ which by uniqueness of the integral curve shows that $f \circ \widetilde{\gamma} = \gamma$. Thus horizontal lifts of γ correspond to integral curves of $\widetilde{X}$. Recall that the latter exist for small times and whenever defined they are unique (see [10, Corollary 1.21]). Thus all that remains to show is that the lift $\widetilde{\gamma}$ of γ starting at $m_0 \in f^{-1}(\gamma(0))$ is defined for all times $0 \leq t \leq 1$. To this end, consider the set

$$\mathcal{I} \coloneqq \left\{ t \in [0,1] \,\middle|\, \begin{array}{l} \gamma|_{[0,t]} \text{ has a horizontal lift } \widetilde{\gamma} \text{ defined on } [0,t] \\ \text{for every starting point } x \in f^{-1}(\gamma(0)) \end{array} \right\}.$$

Our aim is to show that $\mathcal{I} = [0,1]$.

Step 1. $\mathcal{I} \neq \emptyset$.

Proof. Let $m \in f^{-1}(\gamma(0))$. Then there is an open neighborhood U_m of m in M and $T_m > 0$ such that for all $x \in U_m$ the integral curve of $\widetilde{X}$ starting at x is defined on $[0, T_m]$. In particular, for all $x \in U_m \cap f^{-1}(\gamma(0))$ the local lift of γ starting at x is defined on $[0, T_m]$. Then $\{U_m\}_{m \in f^{-1}(\gamma(0))}$ is an open cover of the compact set $f^{-1}(\gamma(0))$ and thus $f^{-1}(\gamma(0)) \subseteq \bigcup_{i=1}^{k} U_{m_i}$ for finitely many $m_i \in f^{-1}(\gamma(0))$. Let $T \coloneqq \min\limits_{i=1,\ldots,k} T_{m_i} > 0$. Then for all $x \in f^{-1}(\gamma(0))$, it holds $x \in U_{m_i} \cap f^{-1}(\gamma(0))$ for some i and the lift of γ starting at x is defined on $[0,T]$, which shows that $T \in \mathcal{I}$. ☐ Step 1.

Step 2. $\mathcal{I}$ is closed.

Proof. Let $\{t_n\}$ be a sequence in $\mathcal{I}$ converging to t. Without loss of generality we may assume that $t_n \nearrow t$. For each n there is a lift $\widetilde{\gamma}_n$ of γ defined on $[0, t_n]$ and by uniqueness all these lifts agree whenever defined, so there is a lift $\widetilde{\gamma}$ of γ defined on $[0, t_n]$ for all n. Since $f^{-1}(\gamma([0,t]))$ is compact, the sequence $\{\widetilde{\gamma}(t_n)\}_n$ has a subsequence $\{\widetilde{\gamma}(t_{n_k})\}_k$ converging to some $x \in f^{-1}(\gamma([0,t]))$. Set $\widetilde{\gamma}(t) \coloneqq x$. This extends $\widetilde{\gamma}$ continuously to a lift of γ defined on $[0,t]$ since

$$f(\widetilde{\gamma}(t)) = f(x) = f(\lim_k \widetilde{\gamma}(t_{n_k})) = \lim_k f(\widetilde{\gamma}(t_{n_k})) = \lim_k \gamma(t_{n_k}) = \gamma(t).$$

☐ Step 2.

Step 3. $\mathcal{I} = [0,1]$.

Proof. Let $\tau > 0$ be the upper bound of $\mathcal{I}$. Since $\mathcal{I}$ is closed, $\tau \in \mathcal{I}$. Suppose that $\tau < 1$. As in Step 1 by replacing the fiber $f^{-1}(\gamma(0))$ by $f^{-1}(\gamma(\tau))$ we find $T' > 0$ such that every local lift $\widetilde{\gamma}_\tau$ of γ starting at any $m_\tau \in f^{-1}(\gamma(\tau))$ is defined on $[0, T']$. For every $m_0 \in f^{-1}(\gamma(0))$, the lift $\widetilde{\gamma}_0$ of γ starting at m_0 has as endpoint some $m_\tau \in f^{-1}(\gamma(\tau))$. Notice that both $\widetilde{\gamma}_\tau$ and $\widetilde{\gamma}_0$ give a lift of $\gamma|_{(\tau-\epsilon,\tau)}$ and so by uniqueness they glue continuously to a lift $\widetilde{\gamma}$ defined on $[0, \tau + T']$, contradicting the maximality of τ. ☐ Step 3.

This completes the proof of the theorem. ☐

Let $H = \{H_m\}_{m \in M}$ be a connection associated to the submersion $f : M \longrightarrow N$. Let $\gamma : [0,1] \longrightarrow B$ be a path with nowhere vanishing derivative and let $\widetilde{\gamma}_m$ be the horizontal lift of γ starting at $m \in f^{-1}(\gamma(0))$ with respect to the connection H.

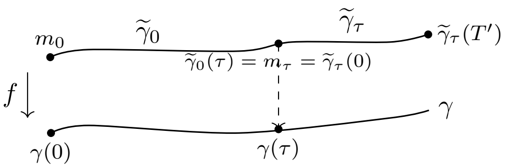

Figure 2: Construction of the lift in Step 3 of Theorem 5.3.

Definition 5.5. The map

$$\begin{aligned} P_\gamma : f^{-1}(\gamma(0)) &\longrightarrow f^{-1}(\gamma(1)) \\ m &\longmapsto \widetilde{\gamma}_m(1) \end{aligned}$$

is called *parallel translation along* γ with respect to H.

Proposition 5.6. *Let* $f : M \longrightarrow N$ *be a proper submersion, then the parallel translation* $P_\gamma : f^{-1}(\gamma(0)) \longrightarrow f^{-1}(\gamma(1))$ *along any path* γ *with* $\gamma' \neq 0$ *everywhere, is a diffeomorphism.*

Proof. The map P_γ is smooth because the lift $\widetilde{\gamma}$ of γ is the integral curve of some vector field $\widetilde{X}$ and the map

$$M \times [0,1] \longrightarrow M,\ (m,t) \longmapsto \begin{array}{l}\text{position at time } t \text{ of the integral curve of } \widetilde{X} \\ \text{starting at } m\end{array}$$

is smooth, as solutions of differential equations depend smoothly on the initial conditions (see [10, Corollary 1.21]). The inverse of P_γ is given by parallel translation along the path $\gamma^{-1} : t \longmapsto \gamma(1-t)$. Indeed, let $m \in f^{-1}(\gamma(0))$. Then $P_{\gamma^{-1}}(P_\gamma(m))$ is the endpoint of the unique lift of γ^{-1} starting in $P_\gamma(m) = \widetilde{\gamma}_m(1)$. But $(\widetilde{\gamma}_m)^{-1}$ starts in $(\widetilde{\gamma}_m)^{-1}(0) = \widetilde{\gamma}_m(1)$ and is a lift of γ^{-1} since

$$f \circ (\widetilde{\gamma}_m)^{-1}(t) = f \circ \widetilde{\gamma}_m(1-t) = \gamma(1-t) = \gamma^{-1}(t).$$

Thus $P_\gamma^{-1}(P_\gamma(m)) = (\widetilde{\gamma}_m)^{-1}(1) = \widetilde{\gamma}_m(0) = m$. Analogously, $P_\gamma \circ P_{\gamma^{-1}} = id_{f^{-1}(\gamma(1))}$. ☐

Proposition 5.7. *Suppose that* M *and* N *are* G*-manifolds and let* $f : M \longrightarrow N$ *be a* G*-invariant proper submersion, so that* $f(g \cdot m) = f(m)$ *for all* $m \in M$*. Then parallel translation* $P_\gamma : f^{-1}(\gamma(0)) \longrightarrow f^{-1}(\gamma(1))$ *along any path* γ *is* G*-equivariant.*

Proof. We want to show that

$$P_\gamma(g \cdot m) = g \cdot P_\gamma(m)$$

for all $m \in M$, $g \in G$. Notice that since f is G-invariant the spaces $f^{-1}(n)$, $n \in N$, are G-invariant, so that the above expression is well-defined. Let $\widetilde{\gamma}_m$ be the lift of γ starting at m. We claim that $g \cdot \widetilde{\gamma}_m$ is also a lift of γ starting at $g \cdot m$. In fact, $g \cdot \widetilde{\gamma}_m(0) = g \cdot m$ and since f is G-invariant it holds $f(g \cdot \widetilde{\gamma}_m(t)) = f(\widetilde{\gamma}_m(t)) = \gamma(t)$. Thus

$$P_\gamma(g \cdot m) = (g \cdot \widetilde{\gamma}_m)(1) = g \cdot \widetilde{\gamma}_m(1) = g \cdot P_\gamma(x).$$

□

5.2 The first Duistermaat-Heckman formula

In this subsection we consider an Hamiltonian action of a torus T on a symplectic manifold (M, ω). We want to prove a theorem of Duistermaat and Heckman [15], which asserts that if the moment map μ is proper, then the reduced symplectic form ω_ξ on the quotient of a regular level $\mu^{-1}(\xi)$ depends linearly on ξ. The proof we are going to present follows the strategy of [3, §VI.2] and is based on the equivariant cohomology tools that we developed in the previous section, in particular on the equivariant extension of the symplectic form ω that we defined in Proposition 4.15.

Consider a torus T acting on a symplectic manifold (M, ω) by symplectomorphisms with a proper moment map $\mu : M \longrightarrow \mathfrak{t}^*$. Let $\xi \in \mathfrak{t}^*$ be a regular value of μ. Recall that by Sard's Theorem there are (plenty of) regular values. Let $V_\xi := \mu^{-1}(\xi)$ denote the level set. Since the torus T is abelian, μ is T-invariant and hence T acts on $\mu^{-1}(\xi)$. Assume that the action of T on V_ξ is free, so that by the Marsden-Weinstein-Meyer theorem (see [11, Theorem 23.1]) the quotient space

$$B_\xi := V_\xi / T$$

is a $(\dim M - 2\dim T)$-dimensional manifold with a symplectic structure given by the reduced symplectic form ω_ξ. We consider the two maps

$$\begin{tikzcd} V_\xi \arrow[r, hook, "j_\xi"] \arrow[d, "q"'] & M \\ B_\xi & \end{tikzcd}$$

namely the inclusion $V_\xi \subset M$ and the quotient map $V_\xi \longrightarrow V_\xi/T = B_\xi$. Recall that the reduced form ω_ξ on B_ξ satisfies

$$j_\xi^* \omega = q^* \omega_\xi. \tag{5.1}$$

As the inclusion j_ξ is T-equivariant, the product map $1 \times j_\xi : ET \times V_\xi \longrightarrow ET \times M$ induces a map $1 \times_T j_\xi : ET \times_T V_\xi \longrightarrow ET \times_T M$. This means that we have the commutative diagram

$$\begin{tikzcd} ET \times V_\xi \arrow[r, "1 \times j_\xi"] \arrow[d, "p"] & ET \times M \arrow[d, "\pi"] \\ ET \times_T V_\xi \arrow[r, "1 \times_T j_\xi"] & ET \times_T M. \end{tikzcd}$$

Let $\omega^{\#} := \beta$ be the closed 2-form on $ET \times_T M$ given by Proposition 4.15 with $\pi^*\omega^{\#} = \mathrm{pr}_2^*\omega + d(\theta \cdot \mu)$, where $\mathrm{pr}_2 : ET \times M \longrightarrow M$ is the projection onto the second factor and $\theta \in \Omega^1(ET, \mathfrak{t})$ is a connection 1-form on ET. Denote by $\omega_\xi^{\#}$ the closed 2-form on $ET \times_T V_\xi$ given by

$$\omega_\xi^{\#} := (1 \times_T j_\xi)^* \omega^{\#}.$$

Let $\mathrm{pr}_1 : ET \times M \longrightarrow ET$ be the projection onto the first factor, and for a $\mathfrak{t}$-valued k-form $\nu \in \Omega^k(ET, \mathfrak{t})$ let $\nu \cdot \xi$ be the k-form on ET defined by

$$\nu \cdot \xi_e(\cdot) := \langle \xi, \nu_e(\cdot) \rangle \quad \text{for } e \in ET.$$

Lemma 5.8. *$p^*\omega_\xi^{\#} = \mathrm{pr}_2^* j_\xi^* \omega + \mathrm{pr}_1^* d\theta \cdot \xi$ on $ET \times V_\xi$.*

Proof. From the above diagram we see that

$$p^*\omega_\xi^{\#} = (1 \times j_\xi)^*(\mathrm{pr}_2^*\omega + d(\theta \cdot \mu)).$$

We compute the right hand side explicitly. Let $(e, x) \in ET \times V_\xi$ and (v_1, w_1), $(v_2, w_2) \in T_eET \times T_xV_\xi$. Then

$$\begin{aligned}
&(1 \times j_\xi)^*\big(\mathrm{pr}_2^*\omega + d(\theta \cdot \mu)\big)_{(e,x)}\big((v_1, w_1), (v_2, w_2)\big) = \\
&\qquad = \big(\mathrm{pr}_2^*\omega + d\theta \cdot \mu - \theta \cdot d\mu\big)_{(e, j_\xi(x))}\big((v_1, (dj_\xi)_x w_1), (v_2, (dj_\xi)_x w_2)\big) \\
&\qquad = \omega_{j_\xi(x)}\big((dj_\xi)_x w_1, (dj_\xi)_x w_2\big) + \langle \underbrace{\mu(j_\xi(x))}_{=\xi}, (d\theta)_e(v_1, v_2) \rangle \\
&\qquad\quad - \langle (d\mu)_{j_\xi(x)}((dj_\xi)_x w_2), \theta_e(v_1) \rangle + \langle (d\mu)_{j_\xi(x)}((dj_\xi)_x w_1), \theta_e(v_2) \rangle \\
&\qquad = (j_\xi^*\omega)_x(w_1, w_2) + \langle \xi, (\mathrm{pr}_1^* d\theta)_{(e,x)}\big((v_1, w_1), (v_2, w_2)\big) \rangle \\
&\qquad = (\mathrm{pr}_2^* j_\xi^* \omega)_{(e,x)}\big((v_1, w_1), (v_2, w_2)\big) + (p_1^* d\theta \cdot \xi)_{(e,x)}\big((v_1, w_1), (v_2, w_2)\big),
\end{aligned}$$

where in the second to last equality the last two summands vanish because $w_1, w_2 \in T_xV_\xi = \ker(d\mu_x)$. ☐

Since T acts freely on V_ξ, the quotient map $q : V_\xi \longrightarrow B_\xi$ is a principal G-bundle and by Corollary 2.31 we have a weak homotopy equivalence

$$\sigma_\xi : ET \times_T V_\xi \longrightarrow B_\xi, \ [(e, x)] \longmapsto [x].$$

Let $\tau \in \Omega^1(B_\xi, \mathfrak{t})$ be a connection form on the principal T-bundle $q : V_\xi \longrightarrow B_\xi$, with connection form $\alpha \in \Omega^2(B_\xi, \mathfrak{t})$[8]. Then $\alpha \cdot \xi \in \Omega^2(B_\xi)$. Let ω_ξ denote the reduced symplectic form on B_ξ.

Lemma 5.9. *$\sigma_\xi^*\big([\omega_\xi] + [\alpha \cdot \xi]\big) = [\omega_\xi^{\#}]$ in $H^2(ET \times_T V_\xi) = H_T^2(V_\xi)$.*

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[8] See Appendix A.3.

Proof. We have the commutative diagram

$$\begin{tikzcd}
V_\xi \arrow[d, "q"'] & ET \times V_\xi \arrow[l, "\mathrm{pr}_2"'] \arrow[r, "\mathrm{pr}_1"] \arrow[d, "p"] & ET \arrow[d, "\pi"] \\
B_\xi & ET \times_T V_\xi \arrow[l, "\sigma_\xi"'] \arrow[r, "\sigma"] & BT.
\end{tikzcd}$$

The identity that we want to show lies in the middle lower entry. Since $p^* : \Omega^*(ET \times_T V_\xi) \longrightarrow \Omega^*(ET \times V_\xi)$ is injective, it suffices to show that

$$p^*(\sigma_\xi^*(\omega_\xi + \alpha \cdot \xi)) = p^*(\omega_\xi^\# + dF)$$

for some $F \in \Omega^1(ET \times_T V_\xi)$. By the commutativity of the above diagram we have

$$\begin{aligned}
p^*\sigma_\xi^*(\omega_\xi + \alpha \cdot \xi) &= p^*\sigma_\xi^*\omega_\xi + p^*\sigma_\xi^*(\alpha \cdot \xi) \\
&= \mathrm{pr}_2^* q^* \omega_\xi + p^*\sigma_\xi^*\alpha \cdot \xi \\
&\overset{(5.1)}{=} \mathrm{pr}_2^* j_\xi^* \omega + p^*\sigma_\xi^*\alpha \cdot \xi.
\end{aligned} \tag{5.2}$$

Since α is a curvature form of τ, we have $q^*\alpha = d\tau$. Denote by η the curvature 2-form on BT with $\pi^*\eta = d\theta$. Then by Proposition A.6 both $\mathrm{pr}_2^*\tau$ and $\mathrm{pr}_1^*\theta$ are connection forms on the bundle $ET \times V_\xi \xrightarrow{p} ET \times_T V_\xi$. Moreover, we have

$$p^*\sigma_\xi^*\alpha = p_2^* q^* \alpha = dp_2^*\tau \quad \text{ and } \quad p^*\sigma^*\eta = p_1^*\pi^*\eta = dp_1^*\theta.$$

Thus $\sigma_\xi^*\alpha$ and $\sigma^*\eta$ are both curvature forms on $ET \times_T V_\xi$ and according to Proposition A.12 there is $F \in \Omega^1(ET \times_T V_\xi, \mathfrak{t})$ with $\sigma_\xi^*\alpha = \sigma^*\eta + dF$. Therefore,

$$\begin{aligned}
p^*\sigma_\xi^*\alpha \cdot \xi &= p^*\sigma^*\eta \cdot \xi + p^*dF \cdot \xi \\
&= \mathrm{pr}_1^*\pi^*\eta \cdot \xi + p^*dF \cdot \xi \\
&= \mathrm{pr}_1^* d\theta \cdot \xi + p^*dF \cdot \xi.
\end{aligned}$$

Inserting in (5.2) we get

$$p^*\sigma_\xi^*(\omega_\xi + \alpha \cdot \xi) = \mathrm{pr}_2^* j_\xi^* \omega + \mathrm{pr}_1^* d\theta \cdot \xi + p^*dF \cdot \xi \overset{\text{Lemma 5.8}}{=} p^*\omega_\xi^\# + p^*(dF \cdot \xi).$$

Thus by injectivity of p^* we have

$$\sigma_\xi^*(\omega_\xi + \alpha \cdot \xi) = \omega_\xi^\# + dF \cdot \xi.$$

By Lemma 4.16 we have $d(F \cdot \xi) = dF \cdot \xi - F d\xi = dF \cdot \xi$, since ξ is constant as function $M \longrightarrow \mathfrak{g}^*$. Thus taking cohomology classes

$$\sigma_\xi^*([\omega_\xi] + [\alpha \cdot \xi]) = [\omega_\xi^\#].$$

This concludes the proof. Notice at last that by Proposition A.12 and Lemma 4.16 the class $[\alpha \cdot \xi]$ doesn't depend on the choice of the connection form τ on the principal T-bundle $V_\xi \longrightarrow B_\xi$. $\square$

Now let U be a convex open subset of $\mathfrak{t}^*$ containing only regular values of μ. By Sard's theorem the set of regular values is dense in $\mathfrak{t}^*$ and, since μ is proper, it is also open, so a convex open set of regular values exists. It follows that $\mu|_{\mu^{-1}(U)} : \mu^{-1}(U) \longrightarrow U$ is a proper submersion and in particular by Theorem 5.3 admits an Ehresmann connection and parallel translation is defined. Moreover, since T is abelian, it acts on $\mu^{-1}(U)$ and we assume the action to be free. Let $\xi \in U$ and denote by $j_\xi : V_\xi \hookrightarrow \mu^{-1}(U)$ the inclusion, which descends to the quotients giving the map $i_\xi : B_\xi \longrightarrow \mu^{-1}(U)/T$. These maps fit into the commutative diagram

$$\begin{tikzcd}
ET \times_T V_\xi \arrow[r, "1\times_T j_\xi"] \arrow[d, "\sigma_\xi"'] & ET \times_T \mu^{-1}(U) \arrow[d, "\sigma_U"] \\
B_\xi \arrow[r, "i_\xi"] & \mu^{-1}(U)/T.
\end{tikzcd} \tag{5.3}$$

Lemma 5.10. *j_ξ is a homotopy equivalence.*

Proof. Fix a connection for $\mu : \mu^{-1}(U) \longrightarrow U$. Given $\xi, \chi \in U$ let $P_{\xi,\chi} : \mu^{-1}(\xi) \longrightarrow \mu^{-1}(\chi)$ denote parallel translation along the curve $\gamma(t) = (1-t)\xi + t\chi$. Then an homotopy inverse of j_ξ is given by

$$k : \mu^{-1}(U) \longrightarrow \mu^{-1}(\xi),\ x \longmapsto P_{\mu(x),\xi}(x).$$

- For $x \in V_\xi = \mu^{-1}(\xi)$ it holds $k \circ j_\xi(x) = P_{\mu(x),\xi}(x) = P_{\xi,\xi}(x) = x$ by uniqueness of the lift.
- An homotopy h from $j_\xi \circ k$ to $id_{\mu^{-1}(U)}$ is given by parallel translation along the curves $\gamma_s(t) = (1-st)\mu(x) + st\xi$, that is,

  $$\begin{aligned} h :\ & \mu^{-1}(U) \times I \longrightarrow \mu^{-1}(U) \\ & (x,s) \longmapsto j_{\gamma_s(1)} \circ P_{\gamma_s(t)}(x), \end{aligned}$$

  where $j_{\gamma_s(1)} : \mu^{-1}(\gamma_s(1)) \hookrightarrow \mu^{-1}(U)$ denotes the inclusion. We have,
  - $h(x,0) = x$ since $\gamma_0(t)$ is the constant path to $\mu(x)$, and
  - $h(x,1) = j_\xi \circ P_{\mu(x),\xi}(x) = j_\xi \circ k(x)$.

□

Since the moment map μ is T-invariant, by Proposition 5.7 parallel translation along a curve is T-equivariant and therefore both k and the homotopy h are T-equivariant. Thus j_ξ and k descend to homotopy inverses to each other between the quotients $\mu^{-1}(\xi)/T$ and $\mu^{-1}(U)/T$. In particular, also $1 \times_T j_\xi$ is a homotopy equivalence. Thus taking cohomologies in (5.3) we get the commutative diagram

$$\begin{tikzcd}
H_T^*(V_\xi) & H_T^*(\mu^{-1}(U)) \arrow[l, "(1\times_T j_\xi)^*"'] \\
H^*(B_\xi) \arrow[u, "\sigma_\xi^*"] & H^*(\mu^{-1}(U)/T), \arrow[l, "i_\xi^*"] \arrow[u, "\sigma_U^*"']
\end{tikzcd}$$

where all the maps are isomorphisms. By restricting to $\mu^{-1}(U)$ consider $\left[\omega^{\#}\right]$ as an element of $H^2(ET \times_T \mu^{-1}(U)) = H^2_T(\mu^{-1}(U))$. There is a unique class

$$\nu = (\sigma_U^*)^{-1}\left[\omega^{\#}\right] \in H^2(\mu^{-1}(U)/T).$$

Notice that $\sigma_\xi^* i_\xi^* \nu = [\omega_\xi^{\#}]$. Hence using Lemma 5.9 we deduce the following proposition.

Proposition 5.11. *Let U be a convex open subset of $\mathfrak{t}^*$ consisting of regular values of a proper moment map $\mu : M \longrightarrow \mathfrak{t}^*$ for the action. Assume that T acts freely on $\mu^{-1}(U)$. There exists a class $\nu \in H^2(\mu^{-1}(U)/T)$ such that*

$$i_\xi^*\nu = [\omega_\xi] + [\alpha \cdot \xi]\,, \textit{ for all } \xi \in U,$$

where ω_ξ is the reduced symplectic form at the level ξ and α is a curvature form on the bundle $\mu^{-1}(\xi) \longrightarrow \mu^{-1}(\xi)/T$.

Parallel translation along the lifts gives equivariant diffeomorphisms $V_\xi \xrightarrow{\cong} V_{\xi_0}$ for all $\xi, \xi_0 \in U$, which descend to diffeomorphisms $B_\xi \cong B_{\xi_0}$. Thus we can identify each $H^2(B_\xi)$ with $H^2(B_{\xi_0})$ for a fixed $\xi_0 \in U$ and compare the various classes $[\omega_\xi]$.

Theorem 5.12 (Duistermaat and Heckman, [15])**.** *Let U be a convex open subset of $\mathfrak{t}^*$ consisting of regular values of the proper moment map $\mu : M \longrightarrow \mathfrak{t}^*$, and assume that T acts freely on $\mu^{-1}(U)$. Then for every $\xi, \xi_0 \in U$*

$$[\omega_{\xi_0}] - [\omega_\xi] = [-\alpha \cdot (\xi_0 - \xi)]\,,$$

where $\alpha \in \Omega^2(B_{\xi_0}, \mathfrak{t})$ is a curvature form of the bundle $V_{\xi_0} \longrightarrow B_{\xi_0}$ and we have identified the $H^2(B_\xi)$ via parallel translation along the lines in U.

Proof. Let $\xi, \xi_0 \in U$. Denote by $P_{\xi_0,\xi} : \mu^{-1}(\xi_0) \longrightarrow \mu^{-1}(\xi)$ parallel translation along the curve $t \longmapsto (1-t)\xi_0 + t\xi$. For $s \in I$ consider the family of curves

$$\gamma_s : I \longrightarrow U,\ \gamma_s(t) := (1-ts)\xi_0 + ts\xi,$$

and let $j_{\gamma_s(1)} : \mu^{-1}(\gamma_s(1)) \hookrightarrow \mu^{-1}(U)$ denote the inclusion. Then the map

$$h : \mu^{-1}(\xi_0) \times I \longrightarrow \mu^{-1}(U),\ h(x,s) := j_{\gamma_s(1)} \circ P_{\gamma_s}(x)$$

defines a T-equivariant homotopy between j_{ξ_0} and $j_\xi \circ P_{\xi_0,\xi}$. Thus $1 \times_T (j_\xi \circ P_{\xi,\xi})$ and $1 \times_T j_{\xi_0}$ are homotopic maps and therefore induce the same map in cohomology. Denote by $\widetilde{P}_{\xi_0,\xi}$ the mapping $B_{\xi_0} \longrightarrow B_\xi$ induced by $P_{\xi_0,\xi}$. Then also $i_\xi \circ \widetilde{P}_{\xi_0,\xi}$ and i_{ξ_0} are homotopic. So we have the diagram

$$\begin{tikzcd}
ET \times_T V_{\xi_0} \arrow[r, "1\times_T P_{\xi_0,\xi}"] \arrow[d, "\sigma_{\xi_0}"] \arrow[rr, bend left, "1\times_T j_{\xi_0}"] & ET \times_T V_\xi \arrow[r, "1\times_T j_\xi"] \arrow[d, "\sigma_\xi"] & ET \times_T \mu^{-1}(U) \arrow[d, "\sigma_U"] \\
B_{\xi_0} \arrow[r, "\widetilde{P}_{\xi_0,\xi}"] \arrow[rr, bend right, "i_{\xi_0}"'] & B_\xi \arrow[r, "i_\xi"] & \mu^{-1}(U)/T,
\end{tikzcd}$$

which commutes up to homotopies. Denote $\varphi := P_{\xi_0,\xi}$. Let $\alpha \in \Omega^2(B_\xi, \mathfrak{t})$ and $\alpha_{\xi_0} \in \Omega^2(B_\xi, \mathfrak{t})$ be curvature forms and let $\nu \in H^2(\mu^{-1}(U)/T)$ be given by Proposition 5.11. Then we have

$$\begin{aligned}
[\omega_{\xi_0}] + [\alpha_{\xi_0} \cdot \xi_0] &= \iota_{\xi_0}^* \nu \\
&= (\iota_\xi \circ \widetilde{\varphi})^* \nu \\
&= \widetilde{\varphi}^* i_\xi^* \nu \\
&= \widetilde{\varphi}^* [\omega_\xi] + \widetilde{\varphi}^* [\alpha_\xi \cdot \xi] \\
&= \widetilde{\varphi}^* [\omega_\xi] + [\widetilde{\varphi}^* \alpha_\xi \cdot \xi] \\
&= \widetilde{\varphi}^* [\omega_\xi] + [\alpha_{\xi_0} \cdot \xi] ,
\end{aligned}$$

since $\widetilde{\varphi}^* \alpha_\xi$ is a curvature form on $V_{\xi_0} \longrightarrow B_{\xi_0}$. So

$$[\omega_{\xi_0}] - \widetilde{\varphi}^* [\omega_\xi] = [-\alpha_{\xi_0} \cdot (\xi_0 - \xi)] .$$

□

Remark 5.13. Let (M, ω) be a symplectic manifold of dimension $2n$. Then $\frac{\omega^n}{n!}$ is the symplectic volume form and the *Liouville measure* of a Borel subset U of M is defined as

$$m_\omega(U) := \int_U \frac{\omega^n}{n!}.$$

Now, suppose that we have an l-dimensional torus T acting symplectically on (M, ω) with proper moment map $\mu : M \longrightarrow \mathfrak{t}^*$. Then we can consider the push-forward measure $\mu_* m_\omega$ of m_ω by μ. This is a measure on $\mathfrak{t}^* \cong \mathbb{R}^l$, called the *Duistermaat-Heckman measure*, and we denote it by m_{DH}. From Theorem 5.12 one can deduce that the push-forward of the Liouville measure is a piecewise polynomial multiple of the Lebesgue measure on $\mathfrak{t}^* \cong \mathbb{R}^l$. More precisely, for any Borel subset $V \subset \mathfrak{t}^*$,

$$m_{DU}(V) = \int_V f(x) dx,$$

where we integrate with respect to the Lebesgue measure on V and $f : \mathfrak{t}^* \longrightarrow \mathbb{R}$ is piecewise polynomial on any region consisting of regular values of μ. For the details we refer to [11, §30].

6 Localization at fixed points and the Duistermaat-Heckman theorem

In this section we want to prove another theorem of Duistermaat and Heckman (namely, Theorem 4.1 in [15]) to illustrate once more how well the equivariant cohomology language fits in the context of Hamiltonian actions. When a torus T acts on a compact symplectic manifold (M, ω) with moment map H, and the fixed point set F of the action is discrete, the Duistermaat-Heckman theorem states

$$\int_M \frac{\omega^n}{n!} e^{-uH} = \sum_{z \in F} \frac{e^{-uH(z)}}{u^n \prod \alpha_i(z)},$$

where $\alpha_i(z)$ are the weights of the action of T on the tangent space $T_z M$. This formula was first realized as a special case of a "localization theorem" in equivariant cohomology by Berline and Vergne in [4] and Atiyah and Bott in [1], independently.

Since we are going to make use of some notions, in particular the ones of equivariant Euler class and Gysin homomorphism, which require a bit of work to be properly exposed, we have illustrated them in Appendix B, where the reader can also find an outline of the proof of the equivariant tubular neighborhood theorem (Theorem B.10), which will be extensively used in this section.

6.1 H^*BG-algebra structure on $H_G^*(M)$

Let G be a Lie group. Recall that the singular cohomology of the classifying space BG coincides with the equivariant cohomology of a point, since

$$\mathrm{pt}_G = (EG \times_G \mathrm{pt})/G = BG \times \mathrm{pt},$$

and hence

$$H_G^*(\mathrm{pt}) = H^*(\mathrm{pt}_G) = H^*BG.$$

If M is a G-manifold, the G-equivariant map $M \longrightarrow \mathrm{pt}$ induces a map $M_G \longrightarrow \mathrm{pt}_G$ on the Borel constructions and hence in cohomology

$$H_G^*(\mathrm{pt}) = H^*BG \xrightarrow{\alpha} H_G^*(M).$$

Therefore we can equip $H_G^*(M)$ with the structure of an H^*BG-algebra by setting for $u \in H^*BG$ and $x \in H_G^*(M)$

$$u \cdot x \coloneqq \alpha(u)x.$$

If $f : M \longrightarrow N$ is a G-equivariant map of G-manifolds, then it induces a map on equivariant cohomology $f^* : H_G^*(N) \longrightarrow H_G^*(M)$. This map is actually an H^*BG-algebra homomorphism. Indeed, the commutative diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & N \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

induces the commutative diagram in equivariant cohomology

$$\begin{tikzcd}
H^*_G(M) & & H^*_G(N) \arrow[ll, "f^*"'] \\
& H^*BG, \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

and for $u \in H^*BG$, $x \in H^*_G(N)$ we have

$$f^*(u \cdot x) = f^*(\beta(u)x) = f^*(\beta(u))f^*(x) = \alpha(u)f^*(x) = u \cdot f^*(x).$$

6.2 The support of an H^*BT-module

We want to investigate the $H^*(BG)$-structure of $H^*_G(M)$, in the case when the Lie group is a torus. For this, we follow the approach of [3, §VI.3.a].

Let $T = S^1 \times \ldots \times S^1$ be an m-dimensional torus. Then we have

$$H^*BT \cong S\mathfrak{t}^* \cong \mathbb{R}\left[u_1, \ldots, u_m\right].$$

Indeed, $H^*BT \cong H^*_T(\mathrm{pt})$ and, in the Cartan model, $H^*_T(\mathrm{pt})$ is the cohomology of the chain complex $C_T(\mathrm{pt}) = (S\mathfrak{t}^* \otimes \Omega^*(\mathrm{pt}))^T = (S\mathfrak{t}^*)^T$, which are the $\mathrm{Ad}^*(T)$-invariant polynomials in $\mathfrak{t}^*$. Since the torus is abelian, these are all the polynomials. Moreover, since the manifold is just a point, the Cartan differential d_T is 0 and thus

$$H^*_T(\mathrm{pt}) = H^*(C_T(\mathrm{pt}), d_T) = S\mathfrak{t}^*.$$

By choosing a basis for the Lie algebra $\mathfrak{t}$ and a corresponding dual basis $u_1, \ldots, u_m$ for $\mathfrak{t}^*$ we can identify $S\mathfrak{t}^* \cong R\left[u_1, \ldots, u_m\right]$.

First, we consider H^*BT-modules in general.

Any polynomial $f \in S\mathfrak{t}^*$ can be regarded as a function on $\mathfrak{t}$. So for $f \in H^*BT = S\mathfrak{t}^*$, let V_f be the set of zeros of the polynomial f

$$V_f \coloneqq \{X \in \mathfrak{t} \mid f(X) = 0\} \subseteq \mathfrak{t}.$$

Definition 6.1. Let M be an H^*BT-module. The *support of* M is the intersection

$$\mathrm{Supp}M \coloneqq \bigcap_{\{f \mid f \cdot M = 0\}} V_f \subseteq \mathfrak{t},$$

over all polynomials $f \in H^*BT$ which annihilate M.

Example 6.2. (1) The module $\{0\}$ has empty support.

(2) Let M be a free module, that is, a module with a generating set of linearly independent elements. If f annihilates M, then f must be zero so that V_f is $\mathfrak{t}$, and therefore $\mathrm{Supp}M = \mathfrak{t}$.

(3) Suppose that Supp$M \subsetneq \mathfrak{t}$. Since the zero polynomial $0 \in H^*BT$ annihilates M and $V_0 = \mathfrak{t}$, there must be $f \neq 0$ with $f \cdot M = 0$. Thus M is a torsion module[9].

(4) If $T = S^1$, the ring H^*BT is a ring of polynomials in one variable, hence a principal ideal domain. Since the set of all polynomials which annihilate a module M is an ideal, suppose that $f \neq 0$ is a generator for it. Then Supp$M = V_f \subsetneq \mathfrak{t}$.

We now establish some properties of the support.

Lemma 6.3. *If $M' \xrightarrow{a} M \xrightarrow{b} M''$ is an exact sequence of H^*BT-modules, then* Supp$M \subseteq$ Supp$M' \cup$ SuppM''.

Proof. Write $S =$ Supp$M$, $S' =$ SuppM' and $S'' =$ SuppM''. Let $x \notin S' \cup S''$. Then as $x \notin S'$, there exists a polynomial f with $f \cdot M' = 0$ such that $f(x) \neq 0$. For the same reason, there exists a polynomial g annihilating M'' but not x. But

$$b(g \cdot M) = g \cdot b(M) \subseteq g \cdot M'' = 0.$$

Thus by exactness

$$g \cdot M \subseteq \ker(b) = a(M')$$

and

$$f \cdot (g \cdot M) \subseteq f \cdot a(M') = a(f \cdot M') = 0,$$

hence

$$(fg) \cdot M = 0 \quad \text{and} \quad f(x)g(x) \neq 0,$$

so that $x \notin S$. ☐

Lemma 6.4. *Let M and M' be H^*BT-algebras with units. Let $a : M' \longrightarrow M$ be a morphism of algebras. Then* Supp$M \subseteq$ SuppM'.

Proof. If $x \notin$ SuppM', there is a polynomial f annihilating M' but not x. So

$$0 = a(f \cdot 1_{M'}) = f \cdot a(1_{M'}) = f \cdot 1_M.$$

Thus f kills M and $x \notin$ Supp$M$. ☐

Proposition 6.5. *If $0 \longrightarrow M' \xrightarrow{a} M \xrightarrow{b} M'' \longrightarrow 0$ is an exact sequence of H^*BT-modules, then* Supp$M =$ Supp$M' \cup$ SuppM''.

Proof. By Lemma 6.3 we have Supp$M \subseteq$ Supp$M' \cup$ SuppM''. For the other inclusion suppose that $x \notin$ Supp$M$, so that there is a polynomial f with $f \cdot M = 0$ and $f(x) \neq 0$. We have $a(f \cdot M') = f \cdot a(M') \subseteq f \cdot M = 0$. Hence by injectivity of a it holds $f \cdot M' = 0$ and so $x \notin$ SuppM'. Moreover, by surjectivity of b we have $f \cdot M'' = f \cdot b(M) = b(f \cdot M) = 0$, so $x \notin$ SuppM'' as well. ☐

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[9] *torsion element* if there exists $0 \neq g \in H^*BT$ with $g \cdot m = 0$. A module M is called a *torsion module* if all its elements are torsion elements. It is called *torsion-free* if zero is the only torsion element.

Lemma 6.6. *Let H be a closed subgroup (hence a Lie subgroup) of the torus T with Lie algebra $\mathfrak{h}$. Then*

$$\mathrm{Supp} H_T^*(T/H) = \mathfrak{h} \subseteq \mathfrak{t}.$$

Proof. First notice that $ET \times_T T/H \cong ET/H$. By example 2.24(1) the spaces BH and ET/H are homotopy equivalent and thus $H_T^*(T/H) = H^*BH$, with the H^*BT-module structure induced by the inclusion $BH \overset{i}{\hookrightarrow} BT$. Moreover, since all the closed subgroups of a torus are a product of a torus H_0 with some finite group, we can write $H = H_0 \times K$, where H_0 is a torus and K is finite.

Claim. $H^*BH = H^*BH_0$.

Proof. For any Lie group G we have $H^*BG = H_G^*(\mathrm{pt})$. So we show the claim by looking at the Cartan model for $H_H^*(\mathrm{pt})$ and $H_{H_0}^*(\mathrm{pt})$. Recall that $H_H^*(\mathrm{pt}) = S\mathfrak{h}^*$, since H is abelian, and also $H_{H_0}^*(\mathrm{pt}) = S\mathfrak{h}_0^*$. Since $H = H_0 \times K$ and K is finite, at the Lie algebras level it holds

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{k} = \mathfrak{h}_0.$$

☐ Claim.

Therefore we may assume without loss of generality that H is a torus. The structure of $H^*BH \cong S\mathfrak{h}^*$ as an $H^*BT \cong S\mathfrak{t}^*$-module is rather easy to understand. Let $\{X_i\}_{i=1}^m$ be a basis for $\mathfrak{t}$ such that $\{X_i\}_{i=0}^k$ is a basis for $\mathfrak{h}$. Denote by u_i the dual elements in $\mathfrak{t}^*$. Then the restriction $i^* : H^*BT \longrightarrow H^*BH$ translates into the restriction of polynomials

$$i^* : S\mathfrak{t}^* \longrightarrow S\mathfrak{h}^*, \; f = \sum_{j=1}^m a_j u_j \longmapsto \sum_{j=1}^k a_j u_j.$$

Suppose that $f \in S\mathfrak{t}^*$ annihilates $H_T^*(T/H) = S\mathfrak{h}^*$, that is $i^*(f) \cdot S\mathfrak{h}^* = 0$ in $S\mathfrak{h}^*$. Then since $S\mathfrak{h}^*$ is free, $i^*(f) = 0$ and thus f has the form $f = \sum_{j=k+1}^m a_j u_j$. We claim that

$$\mathfrak{h} = \bigcap_{\substack{f \in S\mathfrak{t}^* \\ f \cdot H_T^*(T/H) = 0}} V_f.$$

To see this, let $X = \sum_{i=1}^m b_i X_i \in \mathfrak{t}$. Then for $f = \sum_{j=k+1}^m a_j u_j$ in the annihilator of $H_T^*(T/H)$ we have

$$f(X) = \sum_{i=1}^m \sum_{j=k+1}^m b_i a_j u_j(X_i) = \sum_{i=k+1}^m b_i a_i.$$

Thus $X \in \bigcap_f V_f$ if and only if $\sum_{i=k+1}^m b_i a_i = 0$ for all $(a_{k+1}, \ldots, a_m) \in \mathbb{R}^{m-k}$, that is, if and only if $b_{k+1} = \ldots = b_m = 0$, i.e. $X \in \mathfrak{h}$. This proves the claim and the proposition follows by the definition of the support. ☐

Corollary 6.7. *Let V be a T-manifold. Assume there exists an equivariant map $f : V \longrightarrow T/H$ for some closed subgroup H of T. Then* $\mathrm{Supp} H^*_T(V) \subseteq \mathfrak{h}$.

Proof. The induced map $f^* : H^*_T(T/H) \longrightarrow H^*_T(V)$ is a ring homomorphism. By Lemma 6.4 and Lemma 6.6 we have

$$\mathrm{Supp} H^*_T(V) \subseteq \mathrm{Supp} H^*_T(T/H) = \mathfrak{h}.$$

□

Consider now a T-manifold M and for $x \in M$ let $T \cdot x$ be an orbit of the T-action on M. Since T is compact, the space $T \cdot x$ is a T-invariant compact submanifold of M (see [3, Corollary I.1.2]) and thus by the equivariant tubular neighborhood theorem (Theorem B.10), $T{\cdot}x$ has an open invariant neighborhood V in M. Let T_x denote the stabilizer of x in T,

$$T_x \coloneqq \mathrm{stab}_T(x) = \{g \in T \mid gx = x\}.$$

This is a closed subgroup of T, hence a Lie subgroup.

Corollary 6.8. *Let V be an equivariant tubular neighborhood of an orbit $T \cdot x$ in the T-manifold M. Then* $\mathrm{Supp} H^*_T(V) \subseteq \mathrm{Lie}(T_x)$.

Proof. The neighborhood V is equivariantly homotopy equivalent to $T \cdot x$, thus $H^*_T(V) \cong H^*_T(T{\cdot}x)$ and the two modules they have the same support. Moreover, the map

$$T/T_x \longrightarrow T \cdot x,\ gT_x \longmapsto gx$$

is continuous by definition of the quotient topology, since it comes from the continuous map $T \longrightarrow T{\cdot}x$, $g \longmapsto gx$. It is easy to see that it is a bijection. Since T is compact, so is T/T_x and since any continuous bijection from a compact space onto a Hausdorff space is a homeomorphism, it follows that T/T_x and $T \cdot x$ are homeomorphic. Moreover, the above map is equivariant with respect to left multiplication of T on T/T_x and the restricted action of T on $T \cdot x \subseteq M$. Thus by Corollary 6.7 we have

$$\mathrm{Supp} H^*_T(T \cdot x) \subseteq \mathrm{Lie}(T_x).$$

□

Proposition 6.9. *Let M be a T-manifold. The set F of all fixed points is a topologically closed submanifold of M. In particular, if M is compact, the set F is a compact submanifold of M.*

Proof. To see that F is closed in M, we consider for every $g \in T$ the map $\lambda_g : M \longrightarrow M \times M$, $x \longmapsto (x, g \cdot x)$. Then the set of elements of M fixed by the action of an element $g \in T$ is given by

$$\lambda_g^{-1}(\Delta_{M \times M}),$$

where $\Delta_{M\times M} = \{(x,x) \mid x \in M\}$ is the diagonal of $M \times M$, which is closed because M is Hausdorff. Thus the fixed point set

$$F = \bigcap_{g\in T} \lambda_g^{-1}(\Delta)$$

is closed in M.

We now prove that F is a submanifold of M. Since the torus T is compact, we can put a T-invariant Riemannian metric on M. Let m be a point in F and let $V \subseteq T_mM$ be an open neighborhood of $0 \in T_mM$ such that the exponential map

$$\exp : V \longrightarrow U \subset M$$

is a diffeomorphism. Up to restricting V to a ball of some radius $\epsilon > 0$ small enough, we can assume that V is T-invariant, i.e. for each $g \in T$ we have $\phi_g(V) = V$. Then we have the commutative diagram

$$\begin{tikzcd}
V \arrow[r, "(d\phi_g)_m"] \arrow[d, "\exp"'] & V \arrow[d, "\exp"] \\
U \arrow[r, "\phi_g"] & U.
\end{tikzcd}$$

In particular, $\exp^{-1}$ maps $U \cap F$ to the fixed point set of T in V. Thus

$$\exp^{-1}(U \cap F) = V \cap (T_mM)^T = V \cap \text{linear subspace},$$

because the action of T on T_mM is linear. Hence the pair

$$(U, \varphi := \exp^{-1} : U \longrightarrow V)$$

gives a chart of M at m, such that $\varphi(U \cap F)$ is the intersection of a linear plane with $\varphi(U) = V$. This shows that F is a submanifold of M and concludes the proof. $\square$

Proposition 6.10. *Let F be the fixed-point set of a T-action on a compact manifold M. Then*

$$\mathrm{Supp} H_T^*(M \setminus F) \subseteq \bigcup_{x\in M\setminus F} \mathrm{Lie}(T_x).$$

Remark 6.11. Suppose we have a Lie group G acting on a manifold M. To each orbit $G \cdot x$ we associate a conjugation class of subgroups of G, called the *orbit type*, by the assignment $G \cdot x \longmapsto (G_x)$, where the parentheses $(\,\cdot\,)$ indicate that we are considering the conjugation class of G_x. It is not difficult to see that if $G \cdot x = G \cdot y$, then G_x and G_y are conjugate, so that the above assignment is well-defined. Moreover, if the manifold M is compact, there are only finitely many orbit types (see [3, Proposition I.2.4]). In the case of a torus T, each stabilizer sits alone in its conjugacy class and thus if a torus acts on a compact manifold there are only finitely many stabilizers. In particular, the union in Proposition 6.10 is a union of finitely many proper subspaces of $\mathfrak{t}$ and so is itself a proper subspace of the Lie algebra $\mathfrak{t}$.

Proof of Proposition 6.10. Let V be an invariant tubular neighborhood of F. Then $M \setminus F$ has the same (equivariant) homotopy type of $M \setminus V$. The subset $M \setminus V$ can be covered by the union of the orbits $T \cdot x$, $x \in M \setminus F$. Each orbit has an invariant tubular neighborhood U. Since $M \setminus V$ is closed, it is compact, and thus we can cover it by finitely many invariant tubular neighborhoods $U_1, \ldots, U_\ell$ of orbits. Notice that each of these covers an orbit $T \cdot x$ with $x \in M \setminus F$ and thus with $T_x \subsetneq T$. Applying Corollary 6.8 to each U_i we get

$$\mathrm{Supp} H_T^*(U_i) \subseteq \mathrm{Lie}(T_x) =: \mathfrak{h}_\mathfrak{i}$$

for some $x \in M \setminus F$. We prove by induction on ℓ that

$$\mathrm{Supp} H_T^*(U_1 \cup \ldots \cup U_\ell) \subseteq \mathfrak{h}_1 \cup \ldots \cup \mathfrak{h}_\ell.$$

For $\ell = 1$ this is Corollary 6.8. Suppose that the result holds for $i = 1, \ldots, \ell$. Let $V_\ell := U_1 \cup \ldots \cup U_\ell$, so that $V_{\ell+1} = V_\ell \cup U_{\ell+1}$. By the induction hypothesis we have

$$\mathrm{Supp} H_T^*(V_\ell) \subseteq \bigcup_{i=1}^{\ell} \mathfrak{h}_i.$$

By the equivariant version of Mayer-Vietoris (Theorem 2.46) we have the exact sequence

$$H_T^*(V_\ell \cap U_{\ell+1}) \longrightarrow H_T^*(V_{\ell+1}) \longrightarrow H_T^*(V_\ell) \oplus H_T^*(U_{\ell+1}).$$

We want to apply Lemma 6.3 to it. The (equivariant) inclusion $V_\ell \cap U_{\ell+1} \hookrightarrow U_{\ell+1}$ induces a H^*BT-ring homomorphism $H_T^*(U_{\ell+1}) \longrightarrow H_T^*(V_\ell \cap U_{\ell+1})$ and so by Lemma 6.4, $\mathrm{Supp} H_T^*(V_\ell \cap U_{\ell+1}) \subseteq \mathfrak{h}_{\ell+1}$. Moreover, the exact sequence

$$0 \longrightarrow H_T^*(V_\ell) \hookrightarrow H_T^*(V_\ell) \oplus H_T^*(U_{\ell+1}) \twoheadrightarrow H_T^*(U_{\ell+1}) \longrightarrow 0$$

together with Proposition 6.5 give

$$\mathrm{Supp}(H_T^*(V_\ell) \oplus H_T^*(U_{\ell+1})) = \mathrm{Supp} H_T^*(V_\ell) \cup \mathrm{Supp} H_T^*(U_{\ell+1}) \subseteq \bigcup_{i=1}^{\ell+1} \mathfrak{h}_i.$$

Now Lemma 6.3 concludes the induction. □

Corollary 6.12. *Assume that T acts freely on M. Then* $\mathrm{Supp} H_T^*(M) = 0$.

Proof. If T acts freely, the fix point set F is empty and all stabilizers T_x, $x \in M$, are trivial. □

6.3 Algebraic localization

Let T be a torus and M be a compact T-manifold. Let $i : F \hookrightarrow M$ denote the inclusion of fixed points of the action into the manifold M.

Theorem 6.13. *Retain the above notation. Then the supports of both the kernel and cokernel of*

$$i^* : H_T^*(M) \longrightarrow H_T^*(F)$$

are contained in the proper subspace $\bigcup\limits_{x \in M\setminus F} \mathrm{Lie}(T_x) \subsetneq \mathfrak{t}$.

Proof. Let U be an invariant tubular neighborhood of the fixed point set F. Then

$$\mathrm{Supp} H_T^*(M \setminus U) \subseteq \bigcup_{x \in M\setminus F} \mathrm{Lie}(T_x)$$

by Proposition 6.10 since $M \setminus U$ and $M \setminus T$ are (equivariantly) homotopy equivalent. The same holds for $\mathrm{Supp}H_T^*(\partial(M \setminus U))$, applying Lemma 6.4 to the inclusion $\partial(M \setminus U) \hookrightarrow M \setminus U$. Using the long equivariant exact sequence of the pair $(M \setminus U, \partial M \setminus U)$ (Proposition 2.47) and Lemma 6.5, the same is true for the support of $H_T^*(M \setminus U, \partial(M \setminus U))$. Let V be another equivariant tubular neighborhood a little larger than U, so that $V \setminus U \simeq \partial(M \setminus U) = \partial U$. We have isomorphisms

$$H_T^*(M, F) \cong H_T^*(M, V) \underset{\cong}{\overset{\substack{\text{Prop.2.48} \\ \text{excision}}}{\longrightarrow}} H_T^*(M \setminus U, \partial(M \setminus U)).$$

In particular,

$$\mathrm{Supp} H_T^*(M, F) \subseteq \bigcup_{x \in M\setminus F} \mathrm{Lie}(T_x).$$

The long exact sequence of the pair (M, F)

$$H_T^*(M, F) \xrightarrow{\alpha} H_T^*(M) \xrightarrow{i^*} H_T^*(F) \xrightarrow{\beta} H_T^*(M, F)$$

allows us to conclude. In fact, the kernel of i^* coincides with the image of α and so by Lemma 6.4 its support is contained in the one of $H_T^*(M, F)$. As for the cokernel, consider the map

$$\mathrm{coker}\, i^* = H_T^*(F)/\mathrm{im}\, i^* = H_T^*(F)/\ker\beta \longrightarrow H_T^*(M, F).$$

Then again by Lemma 6.4, $\mathrm{Supp}(\mathrm{coker}\, i^*) \subseteq \mathrm{Supp}H_T^*(M, F)$. □

To give a more precise statement about the relation between $H_T^*(M)$ and $H_T^*(F)$ we need the notion of localization. If $f \in S\mathfrak{t}^* = \mathbb{R}[u_1, \ldots, u_m]$ is any non-zero polynomial, localizing to the open set $\mathfrak{t} \setminus V_f$ means that we allow ourselves to divide by powers of f. Formally, we define the localization of an H^*BT-module as follows.

Definition 6.14. Let $f \in H^*BT$ be a polynomial in $u_1, \ldots, u_m$.

(i) We denote by $(H^*BT)_f$ the ring of fractions of H^*BT, which consists of all polynomial fractions which have a power of f as denominator.

(ii) For any H^*BT-module M we define the *localization of* M to be the $(H^*BT)_f$-module

$$M_f := M \otimes_{H^*BT} (H^*BT)_f.$$

A morphism of modules $a : M \longrightarrow N$ induces a map $a_f : M_f \longrightarrow N_f$ by the formula

$$a_f := a \otimes id_{(H^*BT)_f}.$$

Remark 6.15. Elements of M_f are of the form $\sum_{i=1}^k m_i \otimes \frac{1}{f^{k_i}}$. Moreover, $M_f = \{0\}$ if and only if for all $m \in M$ there exists $\ell \in \mathbb{N}$ with $f^\ell \cdot m = 0$ in M.

Lemma 6.16. *If* Supp$M \subsetneq \mathfrak{t}$*, then there is* $0 \neq f \in H^*BT$ *such that* $M_f = \{0\}$.

Proof. Suppose that Supp$M \subsetneq \mathfrak{t}$. By Example 6.2 (3), there is $f \neq 0$ with $f \cdot M = 0$. Let $\sum_{i=1}^k m_i \otimes \frac{1}{f^{k_i}} \in M_f$ be any element. Then

$$\sum_{i=1}^k m_i \otimes \frac{1}{f^{k_i}} = \sum_{i=1}^k m_i \otimes \frac{f}{f^{k_i+1}} = \sum_{i=1}^k \underbrace{fm_i}_{=0} \otimes \frac{1}{f^{k_i+1}} = 0.$$

□

Lemma 6.17. *Assume*

$$M' \xrightarrow{a} M \xrightarrow{b} M''$$

is an exact sequence of modules. Then the sequence

$$M'_f \xrightarrow{a_f} M_f \xrightarrow{b_f} M''_f$$

is exact.

Proof. Since $b \circ a = 0$, also $b_f \circ a_f = 0$. This shows

$$\operatorname{Im} a_f \subseteq \operatorname{Ker} b_f.$$

For the other containment, let $x \in \ker b_f \subseteq M_f$. Write $x = \sum_{i=1}^k m_i \otimes \frac{1}{f^i}$. Then

$$f^k \cdot x = \sum_{i=1}^k m_i \otimes f^{m-i} = \sum_{i=1}^k f^{k-i} m_i \otimes 1 = (\sum_{i=1}^k f^{k-i} m_i) \otimes 1.$$

Thus

$$0 = f^k \cdot b_f(x) = b_f(f^k \cdot x) = b\big(\sum_{i=1}^k f^{k-i} m_i\big) \otimes 1 \quad \text{in } M''_f.$$

This means that there exists $\ell \in \mathbb{N}$ with

$$0 = f^\ell \cdot b\big(\sum_{i=1}^k f^{k-i} m_i\big) = b(f^\ell \sum_{i=1}^k f^{k-i} m_i).$$

By exactness there is $y \in M'$ with $a(y) = f^\ell \sum_{i=1}^k f^{k-i} m_i$. Thus

$$a_f(y \otimes 1) = a(y) \otimes 1 = f^\ell f^k \sum_{i=1}^k (m_i \otimes \frac{1}{f^i}) = f^{\ell+k} x.$$

Therefore $x = a_f\big(\frac{1}{f^{k+\ell}}(y \otimes 1)\big) \in \operatorname{Im} a_f$. $\square$

Let $f, g \in H^*BT$ be two polynomials. Denote by $(H^*BT)_{f,g}$ the ring of all polynomial fractions with denominator in the subring $\langle f, g \rangle$ generated by f and g. For any H^*BT-module M, let $M_{f,g}$ be the corresponding localization.

Lemma 6.18. *Let $f, g \in H^*BT$ be two polynomials and M be an H^*BT-module. Then*

(i) $M_{f,g} = M_{fg}$.

(ii) *If $M_f = \{0\}$, then also $M_{fg} = \{0\}$.*

Proof. We prove (i). Let $x = m \otimes \frac{1}{f^k g^l} \in M_{f,g}$, then $x = f^l g^k m \otimes \frac{1}{(fg)^{k+l}} \in M_{fg}$. For the other inclusion, let $y = n \otimes \frac{1}{(fg)^s} \in M_{fg}$, then $y = n \otimes \frac{1}{f^s g^s} \in M_{f,g}$.

We prove (ii). Let $m \otimes \frac{1}{(fg)^k} \in M_{fg}$. Then, since $M_f = \{0\}$, there is $\ell \in \mathbb{N}$ with $f^\ell \cdot m = 0$. Thus $m \otimes \frac{1}{(fg)^k} = m \otimes \frac{(fg)^\ell}{(fg)^{k+\ell}} = g^\ell \cdot (f^\ell \cdot m) \otimes \frac{1}{(fg)^{k+\ell}} = 0$. $\square$

Proposition 6.19. *There exists a non-zero polynomial $f \in H^*BT$ such that*

$$i_f^* : (H_T^*(M))_f \longrightarrow (H_T^*(F))_f$$

is an isomorphism, where $i : F \hookrightarrow M$ denotes the inclusion of the fixed point set F in M.

Proof. We claim that for any polynomial f the kernel and the cokernel of the localized morphism i_f^* are the localizations of the kernel and the cokernel of i^*. Indeed, consider the exact sequence

$$0 \longrightarrow \ker i^* \overset{j}{\hookrightarrow} H_T^*(M) \xrightarrow{i^*} H_T^*(F).$$

By Lemma 6.17 applied both to the first three slots and to the last three ones it follows that the induced sequence

$$0 \longrightarrow (\ker i^*)_f \overset{j_f}{\hookrightarrow} H_T^*(M)_f \xrightarrow{i_f^*} H_T^*(F)_f$$

is exact. Thus $\ker(i_f^*) = \operatorname{im} j_f = (\ker i^*)_f$.

Applying the same lemma to the exact sequence

$$H_T^*(M) \xrightarrow{i^*} H_T^*(F) \xrightarrow{q} \operatorname{coker} i^* \longrightarrow 0$$

gives the exact sequence

$$H_T^*(M)_f \xrightarrow{i_f^*} H_T^*(F)_f \xrightarrow{q_f} (\operatorname{coker} i^*)_f \longrightarrow 0.$$

Thus

$$\text{coker}\,(i_f^*) \overset{\text{def}}{=} H_T^*(F)_f/\text{im}\, i_f^* = H_T^*(F)_f/\ker q_f \cong \text{im}\, q_f = (\text{coker}\, i^*)_f,$$

where we used the exactness in the second and last equality. This proves the claim.

By Theorem 6.13 we have $\text{Supp}\ker i^* \subsetneq \mathfrak{t}$ and $\text{Suppcoker}\; i^* \subsetneq \mathfrak{t}$. Thus by Lemma 6.16 there are two non-zero polynomials $f_1, f_2 \in H^*BT$, such that $(\ker i^*)_{f_1} = \{0\}$ and $(\text{coker}\, i^*)_{f_2} = \{0\}$. Let $f \coloneqq f_1 f_2 \in H^*BT$. Then, using Lemma 6.18 and the above claim, we see that $\ker(i_f^*)$ and coker (i_f^*) are zero and thus i_f^* is an isomorphism. ☐

6.4 Inverting the equivariant Euler class

We now restrict to connected components Z of the fixed point set F and invert the equivariant Euler class, that is, we find a polynomial $f \in H^*BT$ such that $e_T(\nu_Z)$ has an inverse in $H_T^*(F)_f$.

Let M be a compact orientable manifold on which an m-dimensional torus T acts smoothly. As before, let F be the fixed point set of the action and let Z be a connected component of F. Then Z is a submanifold of M (Proposition 6.9) and is orientable (see [6, Theorem 2.1]). Moreover, we assume that it has codimension $2n$ in M[10]. Since the torus T is compact, we can fix a T-invariant Riemannian metric on M. Let

$$\nu_Z \coloneqq \bigcup_{z\in Z} \{z\} \times N_z Z \longrightarrow Z$$

denote the normal bundle to Z in M, where $N_z Z$ is the orthogonal complement to $T_z Z$ in $T_z M$ with respect to the metric, that is,

$$N_z Z = (T_z Z)^\perp \subset T_z M.$$

Since Z is orientable, the normal bundle $\nu_Z \longrightarrow Z$ is an orientable T-vector bundle and we write $e_T(\nu_Z) \in H_T^{2n}(Z)$ for its equivariant Euler class (see Appendix B). Since the T-action on Z is trivial we have

$$ET \times_T Z \cong ET/T \times Z$$

and thus

$$H_T^*(Z) \cong H^*(Z) \otimes H^*BT.$$

(see [20, §3.2]). The equivariant Euler class is in $H_T^{2n}(Z) \cong \bigoplus_{i=0}^{2n} H^i(Z) \otimes H^{2n-i}BT$ and therefore is of the form

$$e_T(\nu_Z) = \sum_{i=0}^{2n} \sum_j a_i^j \otimes f_{2n-i}^j,$$

---

[10] If M is a symplectic manifold, then Z is a symplectic submanifold (see [25, Lemma 5.53]) and thus has even codimension.

where the lower index denotes the degree of the cohomology class under consideration. Since for k big enough $H^k(Z) = 0$ (de Rham cohomology!), all elements of positive degree in $H^*(Z)$ are nilpotent. So we have

$$a_i^j \in H^i(Z) \quad \text{is nilpotent for all } i, j > 0.$$

Since $(a_i^j \otimes f_{2n-i}^j)^n = (a_i^j)^n \otimes (f_{2n-i}^j)^n$, also $a_i^j \otimes f_{2n-i}^j$ is nilpotent for all $i, j > 0$. The product on $H^*(Z) \otimes H^*BT$ is graded-commutative, thus a sum of nilpotent elements is nilpotent as well, in particular

$$\sum_{\substack{i,j \\ i>0}} a_i^j \otimes f_{2n-i}^j$$

is nilpotent. By linear algebra if A is nilpotent, then $x + A$ is invertible if and only if x is invertible. Thus

$$(\sum_j a_0^j \otimes f_{2n}^j) + \sum_{\substack{i,j \\ i>0}} a_i^j \otimes f_{2n-i}^j \text{ is invertible if and only if } \sum_j a_0^j \otimes f_{2n}^j \text{ is.}$$

Embed both sums in $(H^*(Z) \otimes H^*BT)_f$. By properties of the tensor product (see [2, Exercise 2.15]) it holds

$$\begin{aligned}(H^*(Z) \otimes H^*BT)_f &= (H^*(Z) \otimes_{\mathbb{R}} H^*BT) \otimes_{H^*BT} (H^*BT)_f \\ &\cong H^*(Z) \otimes_{\mathbb{R}} (H^*BT \otimes H^*BT_f) \\ &\cong H^*(Z) \otimes_{\mathbb{R}} (H^*BT)_f.\end{aligned}$$

So write $\sum_j a_0^j \otimes f_{2n}^j = 1 \otimes \sum_j a_0^j f_{2n}^j$ (using that $a_0^j \in H^0(Z) \cong \mathbb{R}$). By setting $f := \sum_j a_0^j f_{2n}^j \in H^*BT$ we see that if $f \neq 0$, then $\sum_j a_0^j \otimes f_{2n}^j$ is invertible in $(H^*(Z) \otimes H^*BT)_f$. Therefore we have proved the following lemma.

Lemma 6.20. *There exists $0 \neq f \in H^*BT$ such that $e_T(\nu_Z)$ is invertible in $H_T^*(Z)_f$ if and only if its component $\sum_j a_0^j \otimes f_{2n}^j$ in $H^0(Z) \otimes H^{2n}BT$ is non-zero.*

Our aim is now to show that $\sum_j a_0^j \otimes f_{2n}^j$ is non-zero. To this end, let $z \in Z$ be a fixed point and let $j : \{z\} \hookrightarrow Z$ denote the inclusion.

Lemma 6.21. $j^*(e_T(\nu_Z)) = \sum_j a_0^j \otimes f_{2n}^j \in H^0(\{z\}) \otimes H^{2n}BT$.

Proof. We have

$$j^*(\sum_{i,j} a_i^j \otimes f_{2n-i}^j) = \sum_{i,j} j^* a_i^j \otimes f_{2n-i}^j = \sum_j a_0^j \otimes f_{2n}^j,$$

since for $i > 0$ it holds $j^* a_i^j \in H^i(\{z\}) = 0$, and $H^0(Z) \cong H^0(\{z\}) \cong \mathbb{R}$ via the restriction. $\square$

By naturality of the Euler class[11] we have

$$j^*(e_T(\nu_Z)) = e_T(j^*\nu_Z),$$

where $j^*\nu_Z$ is the pullback bundle. By definition $j^*\nu_Z = \{z\} \times N_zZ$ is a vector bundle over $\{z\}$ of rank $2n =$codim(Z). Thus it gives a representation of T on the vector space N_zZ:

$$\rho : T \longrightarrow GL(N_zZ), g \longmapsto (d\phi_g)_z.$$

Every torus representation decomposes as a sum of irreducible representations. The real irreducible representations of the torus are ([8, Proposition 15.5])

(1) the trivial representation $1 : T \longrightarrow \{1\} \subset GL(\mathbb{R})$, and

(2) the two-dimensional representations

$$\begin{aligned} L^k :\, T &\longrightarrow GL(\mathbb{R}^2) \\ (e^{ix_1}, \ldots, e^{ix_m}) &\longmapsto \begin{pmatrix} \cos(\sum_i k_i x_i) & \sin(\sum_i k_i x_i) \\ -\sin(\sum_i k_i x_i) & \cos(\sum_i k_i x_i) \end{pmatrix}, \end{aligned}$$

for $k = (k_1, \ldots, k_m) \neq (0, \ldots, 0) \in \mathbb{Z}^m$.

Notice that the representations $L^{k_1,\ldots,k_m}$ and $L^{-k_1,\ldots,-k_m}$ are equivalent, that is, there is an isomorphism $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$ such that the following diagram commutes.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "L^k"] \arrow[d, "f"] & \mathbb{R}^2 \arrow[d, "f"] \\ \mathbb{R}^2 \arrow[r, "L^{-k}"] & \mathbb{R}^2. \end{tikzcd}$$

Indeed, $f(x, y) := (y, x)$ is such an isomorphism. Notice that it is orientation reversing. Thus we have

$$\rho = \rho_1 \oplus \ldots \oplus \rho_k \oplus 1 \oplus \ldots \oplus 1,$$

for some $\rho_j = L^{j_1,\ldots,j_m}$, determined up to sign.

We show that actually there cannot be any trivial representation in the decomposition.

Lemma 6.22. *No non-zero vector in N_zZ is fixed by the action of T.*

Proof. By definition of the normal bundle we have

$$N_zZ \cap T_zZ = \{0\}.$$

Moreover, we claim that the tangent space to Z coincides with the subspace of T-invariant vectors tangent to M at z, that is,

$$T_zZ = (T_zM)^T.$$

---

[11] It follows directly from the naturality of the Thom class, see [27, Property 9.2].

Indeed, suppose first that $v \in T_zZ$, so that $v = \gamma'(0)$ for a curve γ in Z which satisfies $g \cdot \gamma = \gamma$ for all $g \in T$. Then for all $g \in G$,

$$(d\phi_g)_z v = \frac{d}{dt}\Big|_{t=0} g \cdot \gamma(t) = \frac{d}{dt}\Big|_{t=0} \gamma(t) = v,$$

that is, v is a fixed point and so is a vector in $(T_zM)^{T\text{-inv}}$. For the other inclusion, suppose that $v \in (T_zM)^T$. Since by assumption the Riemannian metric on M is T-invariant, T acts by isometries on TM. Let $\gamma(t) = \exp_z(tv)$ be the geodesic through z with initial velocity v. Isometries send geodesics to geodesics and any geodesic is characterized by its starting point and initial velocity. For all $g \in T$ we have

$$g \cdot \exp_z(0) = g \cdot z = z \quad \text{and} \quad \frac{d}{dt}\Big|_{t=0} g \cdot \exp_z(tv) = (d\phi_g)_z v = v,$$

since $v \in (T_zM)^T$. It follows that

$$g \cdot \exp_z(tv) = \exp_z(tv) \quad \text{for all } g \in T.$$

Thus γ is a curve in the fixed point set F and $v \in T_zF = T_zZ$. This proves the claim and concludes the proof of the lemma. ☐

From the lemma it follows that T doesn't act trivially on any subspace of N_zZ and thus we have the decomposition

$$N_zZ \cong L_1 \oplus \ldots \oplus L_n,$$

where each $L_j \cong \mathbb{R}^2$ is T-invariant and T acts on L_j by ρ_j. As noticed above, the spaces L_j are determined up to sign. Since we are working with the *oriented* vector bundle T_zN we fix the signs by requiring the isomorphism $N_zZ \cong L_1 \oplus \ldots \oplus L_n$ to be orientation preserving. Consider for a moment the case $n = 1$. We have an orientation on N_zZ determined by the oriented submanifold Z, and we can fix an orientation on the vector space $L_j = \mathbb{R}^2$. The isomorphism $\Phi : N_zZ \longrightarrow L_j = \mathbb{R}^2$ is chosen such that the diagram

$$\begin{tikzcd} N_zZ \arrow[r, "(d\phi_g)_z"] \arrow[d, "\Phi"] & N_zZ \arrow[d, "\Phi"] \\ \mathbb{R}^2 \arrow[r, "L^{j_1,\ldots,j_m}"] & \mathbb{R}^2 \end{tikzcd}$$

commutes, and can be orientation preserving or reversing. In the former case, we are happy with the chosen orientation on $\mathbb{R}^2$. In the latter case, we make Φ orientation preserving by composing Φ with the orientation reversing isomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x, y) \longmapsto (y, x)$. This gives an orientation preserving isomorphism $\widetilde{\Phi} : N_zZ \longrightarrow \mathbb{R}^2$ and the commutative diagram

$$\begin{tikzcd} N_zZ \arrow[r, "(d\phi_g)_z"] \arrow[d, "\widetilde{\Phi}"] & N_zZ \arrow[d, "\widetilde{\Phi}"] \\ \mathbb{R}^2 \arrow[r, "L^{-j_1,\ldots,-j_m}"] & \mathbb{R}^2. \end{tikzcd}$$

Thus we have $N_z Z \cong L_{-j}$. This shows that by changing the sign of $j \in \mathbb{Z}^m$ we can preserve the orientation in the identification $N_z Z \cong L_j$. For the general case $n \geq 1$, by choosing an orientation on each space in the decomposition we get an orientation on the direct sum, and up to switching one of the signs in the decomposition, the identification $N_z Z \cong L_1 \oplus \ldots \oplus L_n$ is orientation preserving.

Since each representation ρ_j is a group homomorphism $\rho_j : T \longrightarrow SO(2) = S^1$, it is determined by its derivative at the identity

$$\alpha_j \coloneqq (d\rho_j)_{1_T} : \mathfrak{t} \longrightarrow \mathrm{Lie}(S^1) \cong \mathbb{R},$$

so that we have

$$\rho_j(\exp_T \xi) = e^{i\alpha_j(\xi)}$$

for all $\xi \in \mathfrak{t}$. We call $\alpha_j \in \mathfrak{t}^*$ the *weight* of the action of T on L_j. It can be shown that the equivariant Euler class of the oriented T-vector bundle

$$L_j \longrightarrow \{z\}$$

(which is an element of $H^2_T(\{z\}) \cong (S\mathfrak{t}^*)^2 = \mathfrak{t}^*$) is given by α_j (see Appendix C). By Lemma B.18 it holds

$$e_T(\bigoplus_{i=1}^{m} L_j) = e_T(L_1) \cdot \ldots \cdot e_T(L_n),$$

hence the Euler class $e_T(j^*\nu_Z)$ is given by the product of the weights $\alpha_1 \cdot \ldots \cdot \alpha_n$.

Notice that $\exp_T(\ker \alpha_j) \subseteq \mathrm{stab}_T(L_j)$. Indeed, if $g = \exp_T(\xi)$ for some $\xi \in \ker \alpha_j$, then for all $v \in L_j$ it holds

$$g \cdot v \overset{\text{def}}{=} (d\phi_g)_z v = \rho_j(g)v = e^{i\alpha_j(\xi)}v = e^0 v = v.$$

Thus g fixes L_j. By Lemma 6.22 the stabilizer $\mathrm{stab}_T(L_j)$ cannot be the whole torus, thus $\ker \alpha_j$ is a proper subspace of $\mathfrak{t}$. As the union of finitely many proper subspaces is a proper subspace, the set $\mathfrak{t} \setminus \cup_j \ker \alpha_j$ is non-empty. So there is $\xi \in \mathfrak{t}$ with $\alpha_j(\xi) \neq 0$ for all j and $(\alpha_1 \cdot \ldots \cdot \alpha_n)(\xi) \neq 0$.

Therefore,

$$\sum_j a_0^j \otimes f_{2n}^j = j^*(e_T(\nu_Z)) = e_T(j^*\nu_Z) = \alpha_1 \cdot \ldots \cdot \alpha_n \neq 0.$$

This shows that the equivariant Euler class $e_T(\nu_Z)$ is invertible in $H^*_T(Z)_{f_Z}$ with $f_Z = \alpha_1 \cdot \ldots \cdot \alpha_n \in H^*BT = S\mathfrak{t}^*$. Let $f \coloneqq \prod_Z f_Z \in H^*BT$. Then for all $Z \subseteq F$ each Euler class $e_T(\nu_Z)$ is invertible also in $H^*_T(Z)_f$. Since F is the disjoint union of its connected components, $F = Z_1 \sqcup \ldots \sqcup Z_l$, each of which is T-invariant, we have

$$H^*_T(F) = \bigoplus_{j=1}^{l} H^*_T(Z_j)$$

(this can be seen for example using Mayer-Vietoris) and also

$$H_T^*(F)_f = \bigoplus_{j=1}^{l} H_T^*(Z_j)_f$$

so that via the inclusion $H_T^*(Z_j)_f \hookrightarrow H_T^*(F)_f$ all the Euler classes $e_T(\nu_{Z_j})$ are invertible in $H_T^*(F)_f$.

Remark 6.23. If the fixed points of the action are isolated, each connected component Z of F consists of a single point z. Thus the Euler class $e_T(\nu_Z)$ is an element of $H_T^{2n}(\{z\}) \subseteq \mathbb{R}[u_1, \ldots, u_m]$, that is, it is a polynomial in $u_1, \ldots, u_m$. Then with $f \coloneqq e_T(\nu_z)$ we can localize to $H_T^*(M)_f$ and the inverse of the Euler class is given precisely by $\frac{1}{f}$.

6.5 The localization theorem

As in section 6.4, let M be a compact orientable manifold with a torus action on it. Denote by F the fixed point set and let $Z_1, \ldots, Z_\ell$ be the connected components of F, which are finitely many since F, being closed in M, is compact. We assume that they have even codimension in M. Denote by $i_Z : Z \hookrightarrow M$ the inclusion of a connected component Z in M, and let $(i_Z)_* : H_T^*(Z) \longrightarrow H_T^*(M)$ be the Gysin homomorphism, defined in Appendix B.9.

We have seen that there are polynomials f and g in H^*BT such that each equivariant Euler class $e_T(\nu_Z)$ has an inverse $\frac{1}{e_T(\nu_Z)}$ in $H_T^*(Z)_f$ and the restriction $i_g^* : H_T^*(M)_g \longrightarrow H_T^*(F)_g$ is an isomorphism. So, up to considering instead of f the product fg, we can assume that in a suitable localization $\mathfrak{t} \setminus V_f$, each equivariant Euler class $e_T(\nu_Z)$ has an inverse $\frac{1}{e_T(\nu_Z)}$ in $H_T^*(Z)_f$ and the restriction $i_f^* : H_T^*(M)_f \longrightarrow H_T^*(F)_f$ is an isomorphism.

Theorem 6.24 (Localization at fixed points)**.** *For $\beta \in H_T^*(M)$, in a suitable localization (or also over the full field of rational functions) we have*

$$\beta = \sum_{Z \subset F} (i_Z)_* \frac{i_Z^* \beta}{e_T(\nu_Z)},$$

where $i_Z : Z \hookrightarrow M$ denotes the inclusion of the connected component Z of F in M, and $(i_Z)_* : H_T^*(Z) \longrightarrow H_T^*(M)$ is the Gysin homomorphism.

Proof. Let $i_F : F \hookrightarrow M$ denote the inclusion of the fixed point set F in M. By Proposition 6.19 and Section 6.4, there is a polynomial $f \in H^*BT$ such that each equivariant class $e_T(\nu_Z)$ has an inverse $\frac{1}{e_T(\nu_Z)}$ in $H_T(Z)_f$ and the restriction $i_F^* : H_T^*(M)_f \longrightarrow H_T^*(F)_f$ is an isomorphism. Thus it suffices to show

$$i_F^* \sum_{Z \subset F} (i_Z)_* \left(\frac{i_Z^* \beta}{e_T(\nu_Z)} \right) = i_F^* \beta.$$

Notice that the map induced by the inclusion $i_{Z_j,F} : Z_j \hookrightarrow F$ in (equivariant) cohomology is the projection into the j-th component

$$i^*_{Z_j,F} : H^*_T(F) \cong \bigoplus_{j=1}^{\ell} H^*_T(Z_j) \longrightarrow H^*_T(Z_j)$$

and we have the commutative diagram

$$\begin{tikzcd}
H^*_T(M) \arrow[r, "i^*_F"] \arrow[dr, "i^*_{Z_j}"'] & H^*_T(F) \arrow[d, "i^*_{Z_j,F}"] \\
& H^*_T(Z_j).
\end{tikzcd}$$

Therefore for $\beta \in H^*_T(M)$ it holds

$$i^*_F\beta = \sum_{j=1}^{\ell} i^*_{Z_j}\beta. \tag{6.1}$$

Recall also that in each connected component Z we have $i^*_Z(i_Z)_*(\sigma) = \sigma e_T(\nu_Z)$ for all $\sigma \in H^*_T(Z)$ by Proposition B.23. Thus

$$\begin{aligned}
i^*_F \sum_{Z\subset F} (i_Z)_* \left(\frac{i^*_Z\beta}{e_T(\nu_Z)}\right) &= \sum_{Z\subset F} i^*_Z(i_Z)_* \left(\frac{i^*_Z\beta}{e_T(\nu_Z)}\right) \\
&\overset{\text{Prop } B.23}{=} \sum_{Z\subset F} \frac{i^*_Z\beta}{e_T(\nu_Z)} e_T(\nu_Z) \\
&= \sum_{Z\subset F} i^*_Z\beta \\
&\overset{(6.1)}{=} i^*_F\beta.
\end{aligned}$$

□

6.6 Integration along the fibers

For all the details we refer to [28, §9.2]. Let M, N be manifolds of dimensions m and n, respectively. Let $\pi : N \longrightarrow M$ be a fiber bundle with compact oriented fibers of dimension $k = n - m$.

Definition 6.25. For $0 \leq l \leq m$ we define a linear map

$$\pi_* : \Omega^{k+l}(N) \longrightarrow \Omega^l(M)$$

as follows. Let $\omega \in \Omega^{k+l}(N)$ be given. Then $\pi_*\omega \in \Omega^l(M)$ is defined by

$$(\pi_*\omega)_x(X_1, \ldots, X_l) := \int_{\pi^{-1}(x)} \alpha_{x,X_1,\ldots,X_l}$$

where $\alpha_{x,X_1,\dots,X_l} \in \Omega^k(\pi^{-1}(x))$ is the differential form defined as follows. Given $y \in \pi^{-1}(x)$ choose lifts $\widetilde{X}_1, \dots, \widetilde{X}_l \in T_yN$ such that

$$d\pi_y \widetilde{X}_i = X_i.$$

Then for $v_1, \dots, v_k \in T_y(\pi^{-1}(x))$ set

$$(\alpha_{x,X_1,\dots,X_l})_y(v_1, \dots, v_k) \coloneqq \omega_y(\widetilde{X}_1, \dots, \widetilde{X}_l, v_1, \dots, v_k)$$

for $v_1, \dots, v_k \in T_y(\pi^{-1}(x))$. The expression on the right is independent of the choice of the lifts $\widetilde{X}_i$. Namely, suppose that Y_i, $Z_i \in T_yN$ are such that $d\pi_y Y_i = X_i = d\pi_y Z_i$. Then $Y_i - Z_i \in \ker d\pi_y = T_y(\pi^{-1}(x))$ (notice that it is k-dimensional). If $v_1, \dots, v_k$ are linearly dependent, then

$$\omega_y(Y_1, \dots, Y_l, v_1, \dots, v_k) = 0 = \omega_y(Z_1, \dots, Z_l, v_1, \dots, v_k).$$

Else, $Y_i - Z_i$ is a linear combination of $v_1, \dots, v_k$ and thus

$$\begin{aligned} \omega_y(Y_1, \dots, Y_i, \dots, Y_l, v_1, \dots, v_k) - \omega_y(Y_1, \dots, Z_i, \dots, Y_l, v_1, \dots, v_k) \\ = \omega_y(Y_1, \dots, Y_i - Z_i, \dots, Y_l, v_1, \dots, v_k) \\ = 0. \end{aligned}$$

So iterating for all i we get $\omega_y(Y_1, \dots, Y_l, v_1, \dots, v_k) = \omega_y(Z_1, \dots, Z_l, v_1, \dots, v_k)$.

Remark 6.26 ([28, Exercise 9.24]). $\pi_*\omega$ so defined is indeed a smooth differential k-form.

Example 6.27. When $M = \text{pt}$, for all $\omega \in \Omega^n(N)$ it holds

$$\pi_*\omega = \int_N \omega.$$

Proposition 6.28 (Properties of integration along the fibers, [28, Lemma 9.27])**.**

(i) $\pi_* d = d\pi_*$.

(ii) *For every* $\omega \in \Omega^{k+l}(N)$ *and every* $x \in \Omega^{m-l}(N)$

$$\int_M x \wedge \pi_*\omega = \int_N \pi^* x \wedge \omega.$$

Remark 6.29. (1) From Proposition 6.28 it follows that integration along the fibers induces a map $\pi_* : H^{k+l}(N) \longrightarrow H^l(M)$.

(2) The identity $\int_M x \wedge \pi_*\omega = \int_N \pi^* x \wedge \omega$ characterizes $\pi_*\omega$, as an l-differential form σ on M with the property

$$\int_M x \wedge \sigma = 0 \quad \text{for all } x \in \Omega^{m-l}(M)$$

must vanish.

Remark 6.30 (The push-forward). Integration along the fibers is a special case of the push-forward homomorphism which is defined more generally for any continuous map $f : N \longrightarrow M$ of compact oriented manifolds of dimensions n and m, respectively (see [19, Example C.58]). The push-forward

$$f_* : H^*(N) \longrightarrow H^{*-(n-m)}(M)$$

is characterized by the property

$$\int_M f_* v \wedge x = \int_N v \wedge f^* x, \quad \text{for all } v \in H^*(N),\ x \in H^*(M).$$

The push-forward is functorial (i.e. $(fg)_* = f_* g_*$) and if $i : N \hookrightarrow M$ is the inclusion of a submanifold it coincides with the Gysin homomorphism ([19, Proposition C.59]). In particular, if $\pi : E \longrightarrow B$ is a fiber bundle and $s : B \hookrightarrow E$ is the inclusion of a section it holds

$$\underbrace{\pi_*}_{\substack{\text{int}\\ \text{along fiber}}} \underbrace{s_*}_{\substack{\text{Gysin}\\ \text{homo}}} = \underbrace{\pi_*}_{\substack{\text{push}\\ \text{forw}}} \underbrace{s_*}_{\substack{\text{push}\\ \text{forw}}} = (\pi \circ s)_* = id.$$

6.7 Equivariant integration along the fibers

Let G be a compact Lie group and let $\pi : N \longrightarrow M$ be a G-fiber bundle between manifolds. Then up to finite approximations $EG \times_G N \longrightarrow EG \times_G M$ is a fiber bundle of manifolds with compact orientable fiber G as well. We define equivariant integration along the fibers as the map

$$(\pi_G)_* : H_G^{k+l}(N) \longrightarrow H_G^l(M)$$

given by integration along the fibers of $EG \times_G N \longrightarrow EG \times_G M$.

In the Cartan model, $(\pi_G)_*$ is given by the map which sends an element

$$\omega = \sum_\alpha \theta_\alpha \otimes \omega_\alpha \ \ \in (S\mathfrak{g}^* \otimes \Omega^*(N))^G$$

to

$$(\pi_G)_* \omega = \sum_\alpha \theta_\alpha \otimes \pi_* \omega_\alpha.$$

Claim. If ω is G-invariant, so is $(\pi_G)_*\omega$.

Proof. First notice that π_* commutes with the action of G on the differential forms on N. Indeed, for $v \in \Omega^{k+l}(N)$, $g \in G$, in view of Remark 6.29, $\pi_* \phi_g^* v$ is characterized by the identity

$$\int_M x \wedge \pi_* \phi_g^* v = \int_N \pi^* x \wedge \phi_g^* v \quad \text{for all } x \in \Omega^{m-l}(M).$$

So to show that $\pi_* \phi_g^* v = \phi_g^* \pi_* v$ we compute

$$\int_M x \wedge \phi_g^* \pi_* v = \int_M \phi_{g^{-1}}^* (x \wedge \phi_g^* \pi_* v)$$

$$\begin{aligned} &= \int_M \phi_{g^{-1}}^* x \wedge \pi_* v \\ &= \int_N \pi^* \phi_{g^{-1}}^* x \wedge v \\ &= \int_N \phi_{g^{-1}}^* \pi^* x \wedge v \\ &= \int_N \pi^* x \wedge \phi_g^* v. \end{aligned}$$

Therefore $\pi_* \phi_g^* = \phi_g^* \pi_*$. Now let $\omega = \sum_\alpha \theta_\alpha \otimes \omega_\alpha \in S\mathfrak{g}^* \otimes \Omega^*(E)$ be G-invariant.

$$\begin{aligned} g \cdot (\pi_G)_* \omega &= g \cdot \sum_\alpha \theta_\alpha \otimes \pi_* \omega_\alpha \\ &= \sum_\alpha \mathrm{Ad}_g^* \theta_\alpha \otimes \phi_{g^{-1}}^* \pi_* \omega_\alpha \\ &= \sum_\alpha \mathrm{Ad}_g^* \theta_\alpha \otimes \pi_* \phi_{g^{-1}} \omega_\alpha \\ &= (\pi_G)_* (g \cdot \omega) \\ &= (\pi_G)_* \omega \end{aligned}$$

since ω is G-invariant. ☐

This shows that $(\pi_G)_*$ is well-defined on the Cartan complex. Since we want it to induce a map in cohomology, we have to check that it commutes with the Cartan differential.

Claim. $d_G (\pi_G)_* = (\pi_G)_* d_G$.

Proof. We first show that integration along the fibers commutes with the interior product. Let $Z \in \mathfrak{g}$, $y \in N$, $\pi(y) = x$ and let $X_1, \dots, X_l \in T_x M$. Choose lifts $\widetilde{X}_1, \dots \widetilde{X}_l \in T_y N$ such that $d\pi_y \widetilde{X}_i = X_i$. Notice that since π is equivariant it holds $d\pi_y Z^\#(y) = Z^\#(x)$. Let $\omega \in \Omega^{l+k+1}(N)$ and let $\alpha, \gamma \in \Omega^k(N)$ be such that

$$(i_{Z^\#} \pi_* \omega)_x (X_1, \dots, X_l) = (\pi_* \omega)_x (Z^\#(x), X_1, \dots X_l) = \int_{\pi^{-1}(x)} \alpha$$

and

$$(\pi_* \iota_{Z^\#} \omega)_x (X_1, \dots, X_l) = \int_{\pi^{-1}(x)} \gamma.$$

It suffices to show that $\alpha = \gamma$. To this end let $v_1 \dots, v_k \in T_y N$. Then

$$\begin{aligned} \gamma_y (v_1, \dots v_k) &= (\iota_{Z^\#} \omega)_y (\widetilde{X}_1, \dots, \widetilde{X}_l, v_1, \dots, v_k) \\ &= \omega_y (Z^\#(y), \widetilde{X}_1, \dots, \widetilde{X}_l, v_1, \dots, v_k) \\ &= \alpha_y (v_1, \dots, v_k) \end{aligned}$$

since $Z^{\#}(y)$ is a lift of $Z^{\#}(x)$.

So now let $\omega = \sum_\alpha \theta_\alpha \otimes \omega_\alpha \in C_G(N)$. Then

$$\begin{aligned}
\pi_* d_G \omega &= \pi_* \sum_\alpha \theta_\alpha \otimes d\omega_\alpha - \pi_* \sum_\alpha \sum_\beta u_\beta \theta_\alpha \otimes \iota_\beta \omega_\alpha \\
&= \sum_\alpha \theta_\alpha \otimes \pi_* d\omega_\alpha - \sum_{\alpha,\beta} u_\beta \theta_\alpha \otimes \pi_* \iota_\beta \omega_\alpha \\
&= \sum_\alpha \theta_\alpha \otimes d\pi_* \omega_\alpha - \sum_{\alpha,\beta} u_\beta \theta_\alpha \otimes \iota_\beta \pi_* \omega_\alpha \\
&= d_G \Big(\sum_\alpha \theta_\alpha \otimes \pi_* \omega_\alpha - \sum_\alpha \theta_\alpha \otimes \pi_* \omega_\alpha \Big) \\
&= d_G \pi_* \omega.
\end{aligned}$$

□

It is also clear that integration along the fibers extends to a linear homomorphism

$$H^*_G(N) \otimes H^* BT_f \longrightarrow H^*_G(M) \otimes H^* BT_f.$$

6.8 Equivariant localization and the Duistermaat-Heckman formula

Following [3], we restrict ourselves to a circle action and use the tools developed in the previous sections to prove the equivariant localization theorem, which gives a formula to express the integral of an equivariantly closed differential form as a sum of its restrictions over the fixed points, with some corrections coming from the action of S^1 on the tangent spaces at the fixed points. Explicitly,

$$\int_M \phi = \sum_{z \in F} \frac{i_z^* \phi}{e_{S^1}(\nu_z)},$$

where ϕ is an equivariantly closed form on M, F is the fixed point set of the action and $e_{S^1}(\nu_z)$ denotes the equivariant Euler class of the normal bundle to z in M.

In the particular case of a symplectic manifold with an Hamiltonian action, we know that the symplectic form ω extends to an equivariantly closed equivariant form $\omega^{\#} = \omega - Hu$, where H is the moment map. By applying the equivariant localization theorem to all the powers $e^{\omega - Hu}$ of $\omega - Hu$, we are going to recover the Duistermaat-Heckman theorem, which states that

$$\int_M \frac{\omega^n}{n!} e^{-Hu} = \sum_{z \in F} \frac{e^{-H(z)u}}{e_{S^1}(\nu_z)}.$$

To conclude the section, we are going to give an application of the above mentioned theorems.

Consider a compact oriented manifold M on which the circle S^1 acts smoothly. Recall that in this case the Cartan model for the equivariant cohomology is given by

$$C_{S^1}(M) = \Omega^*_X\,[u]\,,$$

where Ω^*_X is the complex of S^1-invariant differential forms on M. Thus an equivariant form $\beta \in C_{S^1}(M)$ is given by a polynomial in u with coefficients in the S^1-invariant differential forms:

$$\beta = \beta_0 + \beta_1 u + \ldots + \beta_k u^k,$$

with $\beta_i \in \Omega^*_X$.

Definition 6.31. Given an equivariant form $\beta = \beta_0 + \beta_1 u + \ldots + \beta_k u^k$ with $\beta_i \in \Omega^*_X$ we define its integral by the formula

$$\int_M \beta = \int_M \beta_0 + \Big(\int_M \beta_1\Big)u + \ldots + \Big(\int_M \beta_k\Big)u^k.$$

Theorem 6.32 (Equivariant localization)**.** *Let S^1 act on a compact oriented $2n$-dimensional manifold M with discrete fixed point set F. Let $\phi \in C^*_{S^1}(M)$ be an equivariantly closed form on M. Then*

$$\int_M \phi = \sum_{z\in F} \frac{i_z^*\phi}{e_{S^1}(\nu_z)},$$

where $e_{S^1}(\nu_z)$ denotes the equivariant Euler class of the normal bundle ν_z of z in M.

Proof. We apply Theorem 6.24 to $[\phi] \in H^*_{S^1}(M)$,

$$[\phi] = \sum_{z\in F} (i_z)_* \frac{i_z^*\,[\phi]}{e_{S^1}(\nu_z)}.$$

Then we equivariantly integrate along the fibers of the projection

$$\pi : M \longrightarrow \mathrm{pt}$$

that is, we apply $(\pi_{S^1})_*$ to both sides of the equation:

$$(\pi_{S^1})_*\,[\phi] = (\pi_{S^1})_* \sum_{z\in F} (i_z)_* \frac{i_z^*\,[\phi]}{e_{S^1}(\nu_z)}. \tag{6.2}$$

By Remark 6.30 we have

$$\pi_*(i_z)_* \overset{\text{Remark 6.30}}{=} (\pi \circ i_z)_* \overset{\text{Example 6.27}}{=} \text{integral over } \{z\} = id.$$

Thus the right-hand side of (6.2) becomes

$$\sum_{z\in F} \frac{i_z^*\,[\phi]}{e_{S^1}(\nu_z)} = \sum_{z\in F} \frac{[i_z^*\phi]}{e_{S^1}(\nu_z)}.$$

On the other hand, since $H^*_{S^1}(\{z\}) = S\mathfrak{g}^* = \mathbb{R}[u]$ we have

$$(\pi_{S^1})_*[\phi] = [(\pi_{S^1})_*\phi] = (\pi_{S^1})_*\phi.$$

Suppose that $\phi = \phi_0 + \phi_1 u + \ldots + \phi_k u^k$ for some $\phi_i \in \Omega^*_X$. Then by Example 6.27, the left-hand side of (6.2) reads

$$\begin{aligned}(\pi_{S^1})_*\phi &= \pi_*\phi_0 + (\pi_*\phi_1)u + \ldots (\pi_*\phi_k)u^k \\ &= \int_M \phi_0 + (\int_M \phi_1)u + \ldots + (\int_M \phi_k)u^k \\ &\overset{\text{def}}{=} \int_M \phi.\end{aligned}$$

Therefore

$$\int_M \phi = \sum_{z \in F} \frac{i_z^*\phi}{e_{S^1}(\nu_z)}.$$

□

Remark 6.33. There is a more general statement, which doesn't require the fixed points to be isolated. Namely, suppose that S^1 acts on a compact oriented manifold M with fixed point set F, and let $\phi \in C^*_{S^1}(M)$ be an equivariantly closed form on M. Then

$$\int_M \phi = \sum_{Z \subset F} \int_Z \frac{i_Z^*\phi}{e_{S^1}\nu_Z}, \tag{6.3}$$

where $i_Z : Z \hookrightarrow M$ is the inclusion of a connected component Z of F and $e_{S^1}(\nu_Z)$ is the equivariant Euler class of the normal bundle to Z in M. Let $2k$ be the codimension of a connected component Z in M. Then each equivariant Euler class $e_{S^1}(\nu_Z)$ is represented in the Cartan model by a closed equivariant form β of degree $2k$ on Z, i.e. by a polynomial in u with coefficients in the S^1-invariant differential forms:

$$\beta = \beta_{2n} + \beta_{2n-2}u + \ldots + \beta_0 u^k \in \Omega^*_X[u],$$

where each β_i is an S^1-invariant differential form on Z of degree i. Since $\beta_{2k} + \ldots + \beta_2 u^{k-1}$ is nilpotent, and $\beta_0 u^k$ is invertible in $\mathbb{R}[u, u^{-1}]$, the equivariant form β as an inverse in $\Omega^*_X[u, u^{-1}]$. Thus the right hand side of (6.3) is a polynomial in u and u^{-1}, as in the case of isolated fixed points.

To prove (6.3) one proceeds as in the case of isolated fixed points. First, we apply the localization Theorem 6.24 to the closed equivariant form ϕ and then we integrate along the fibers of the projection $\pi : M \longrightarrow \text{pt}$. Then the composition of integration along the fibers and the Gysin homomorphism $(i_Z)_*$ gives integration over Z, and thus we recover the right-hand side of (6.3). For the left-hand side, the proof is exactly the same as the one in Theorem 6.32.

Remark 6.34. It is also not necessary to consider only circle actions, analogous results hold also for higher dimensional tori. However, the following corollaries and examples already illustrate the utility of the formula for the one-dimensional case.

Corollary 6.35. *Let S^1 act on a compact oriented $2n$-dimensional manifold M with discrete fixed point set F. Let*

$$\omega = \phi_{2n} + \phi_{2n-2}u + \ldots + \phi_0 u^n \in C^{2n}_{S^1}(M)$$

be an equivariantly closed differential form of degree $2n$ with $\phi_i \in \Omega^i(M)$. Then

$$\int_M \phi_{2n} = \sum_{z\in F} \frac{\phi_0(z)u^n}{e_{S^1}(\nu_z)}, \tag{6.4}$$

where $e_{S^1}(\nu_z)$ denotes the equivariant Euler class of the normal bundle ν_z of z in M.

Proof. By dimension reasons we have

$$\int_M \phi = \int_M \phi_{2n} + \Big(\int_M \phi_{2n-2}\Big)u + \ldots + \Big(\int_M \phi_0\Big)u^n = \int_M \phi_{2n}.$$

Moreover, $\Omega^k(z) = 0$ for all $k > 0$ and thus

$$i_z^*\phi = i_z^*\phi_{2n} + i_z^*\phi_{2n-2}u + \ldots + i_z^*\phi_0 u^n = i_z^*\phi_0 u^n = \phi_0(z)u^n.$$

Therefore by Theorem 6.32 we have

$$\int_M \phi_{2n} = \int_M \phi = \sum_{z\in F} \frac{i_z^*\phi}{e_{S^1}(\nu_z)} = \sum_{z\in F} \frac{\phi_0(z)u^n}{e_{S^1}(\nu_z)}.$$

□

Theorem 6.36 (Duistermaat-Heckman formula)**.** *Let (M,ω) be a compact symplectic manifold of dimension $2n$ on which S^1 acts by symplectomorphisms with only isolated fixed points and suppose that $H : M \longrightarrow \mathbb{R}$ is a moment map for the action. Then in a suitable localization*

$$\int_M e^{-Hu}\frac{\omega^n}{n!} = \sum_{z\in F} \frac{e^{-uH(z)}}{e_{S^1}(\nu_z)}.$$

Remark 6.37. The Euler class $e_{S^1}(\nu_z)$ is an element of $H^*_{S^1}(\{z\}) = \mathbb{R}[u]$. Thus it is invertible in the localized ring $H^*_{S^1}(\{z\})_u = \mathbb{R}[u, u^{-1}]$ and its inverse $\frac{1}{e_{S^1}(\nu_z)}$ is a polynomial in u and u^{-1}. In particular, the above formula must be read as an equality of formal series in the variables u and u^{-1}. The notation will become clearer during the proof.

Proof. Let $\ell \geq 0$. We want to apply Theorem 6.32 with

$$\phi = \sum_{k=0}^{n+\ell} \frac{(\omega - Hu)^k}{k!}.$$

First of all, notice that ϕ is equivariantly closed. Indeed, since H is a moment map for the action, the form $\omega - Hu$ is equivariantly closed (see Examoke 4.13) and since the Cartan differential d_X is an antiderivation, this implies that $d_X\phi = 0$. Hence ϕ represents an element of $H^*_{S^1}(M)$ and we can apply Theorem 6.32 to ϕ to get

$$\int_M \phi = \sum_{z\in F} \frac{i_z^*\phi}{e_{S^1}(\nu_z)}. \tag{6.5}$$

We compute $\int_M \phi$ more explicitly. For an equivariant form $\beta = \beta_0 + \beta_1 f_1 + \ldots + \beta_k f_k$ with $\beta_i \in \Omega^i(M)$, $f_i \in \mathbb{R}[u]$ by definition we have

$$\int_M \beta = \int_M \beta_0 + (\int_M \beta_1) f_1 + \ldots + (\int_M \beta_k) f_k.$$

Since $\dim M = 2n$, the only terms which don't vanish during the integration process are the $2n$-dimensional forms β_{2n}, which in our case are the terms which contain ω^n. Therefore we have

$$\int_M \phi = \sum_{k=0}^{n+\ell} \int_M \frac{(\omega - Hu)^k}{k!} = \sum_{s=0}^{\ell} \int_M \frac{(\omega - Hu)^{n+s}}{(s+k)!}.$$

By the binomial formula we have

$$\begin{aligned}\int_M \frac{(\omega - Hu)^{n+s}}{(n+s)!} &= \sum_{i=0}^{n+s} \binom{n+s}{i} \int_M \frac{\omega^i(-Hu)^{n+s-i}}{(n+s)!} \\ &= \binom{n+s}{n} \int_M \frac{\omega^n(-Hu)^s}{(n+s)!} \\ &= \int_M \frac{\omega^n}{n!}\frac{(-Hu)^s}{s!}.\end{aligned}$$

Therefore,

$$\int_M \phi = \sum_{s=0}^{\ell} \int_M \frac{\omega^n}{n!}\frac{(-Hu)^s}{s!}.$$

On the other hand, the right-hand side of (6.5) reads

$$\sum_{z\in F} \frac{i_z^*\phi}{e_{S^1}(\nu_z)} = \sum_{z\in F} \left(\sum_{k=0}^{n+\ell} \frac{(-H(z)u)^k}{k!} \right) \frac{1}{e_{S^1}(\nu_z)},$$

since

$$i_z^*(\omega - Hu)^k = \sum_{j=0}^{k} \binom{k}{j} i_z^*(\omega^j(-Hu)^{k-j}) = i_z^*(-H)^k u^k = (-H(z))^k u^k.$$

Putting everything together, for all $\ell \geq 0$ we have

$$\sum_{s=0}^{\ell} \int_M \frac{\omega^n}{n!} \frac{(-Hu)^s}{s!} = \sum_{z \in F} \left(\sum_{k=0}^{n+\ell} \frac{(-H(z)u)^k}{k!} \right) \frac{1}{e_{S^1}(\nu_z)}.$$

Letting $\ell \to \infty$ we get

$$\int_M \frac{\omega^n}{n!} e^{-Hu} := \sum_{k=0}^{\infty} \int_M \frac{\omega^n}{n!} \frac{(-Hu)^k}{k!} = \sum_{z \in F} \frac{e^{-H(z)u}}{e_{S^1}(\nu_z)}.$$

□

Example 6.38. Let (M, ω) be a $2n$-dimensional compact symplectic manifold on which S^1 acts by symplectomorphisms with only isolated fixed points. Suppose that the action admits a moment map H and denote by F the fixed point set. Using Theorem 6.36 we compute the integral of ω^n. So, consider $z \in F$. The normal bundle to $\{z\}$ in M is the whole tangent space T_zM and it decomposes as a direct sum of 2-dimensional S^1-invariant subspaces

$$T_zM \cong L_{m_1} \oplus \ldots \oplus L_{m_n},$$

so that S^1 acts on L_{m_j} with weight $m_j \in \mathbb{Z}$, that is, for $g = e^{it} \in S^1$ and $v \in L_{m_j}$ we have

$$e^{it} \cdot v = \rho_{m_j}(e^{it})v = \begin{pmatrix} \cos(m_jt) & -\sin(m_jt) \\ \sin(m_jt) & \cos(m_jt) \end{pmatrix} v.$$

The symplectic form ω determines an orientation on the manifold M and thus on each tangent space T_zM (an ordered basis $(v_1, \ldots, v_{2n})$ of T_zM is positively oriented if $\omega_z^n(v_1, \ldots, v_{2n}) > 0$), and we choose the weights m_j such that the isomorphism $T_zM \cong L_{m_1} \oplus \ldots \oplus L_{m_n}$ is orientation preserving (see Section 6.4).

According to Appendix C.4, the equivariant Euler class $e_{S^1}(L_{m_j}) \in H^2_{S^1}(\{z\}) = \{\text{polynomials of degree 2 in } \mathbb{R}[u]\}$ is given by

$$e_{S^1}(L_{m_j}) = \frac{-1}{2\pi} m_j u.$$

Therefore,

$$e_{S^1}(T_zM) = \frac{(-1)^n}{(2\pi)^n} m_1 \cdot \ldots \cdot m_n u^n \in H^{2n}_{S^1}(\{z\}) \subseteq \mathbb{R}[u].$$

Thus by Theorem 6.36 we have

$$\int_M \frac{\omega^n}{n!} e^{-Hu} = (-2\pi)^n \sum_{z \in F} \frac{e^{-uH(z)}}{m_1 \cdot m_n(z)} u^{-n}.$$

By equating the coefficient of u^0 on both sides we obtain

$$\int_M \frac{\omega^n}{n!} = (-2\pi)^n \sum_{z \in F} \frac{(-H(z))^n}{m_1 \cdot \ldots \cdot m_n(z)}. \tag{6.6}$$

Example 6.39. We apply the example above to compute the volume of the 2-sphere. Consider the 2-sphere $S^2 \subset \mathbb{R}^3$. It is a compact symplectic manifold with symplectic form

$$\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$$

The circle S^1 acts on S^2 by rotations about the z-axis. Explicitly, for $e^{it} \in S^1$, $(x, y, z) \in S^2$ we have

$$e^{it} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The fixed points of the action are the north pole $p = (0, 0, 1)$ and the south pole $q = (0, 0, -1)$. Notice that they are isolated. We check that S^1 acts by symplectomorphisms and that the action is Hamiltonian. Let $X := 1 \in \mathrm{Lie}(S^1) = \mathbb{R}$. The vector field generated by X is given by

$$\begin{aligned} X^{\#}(x, y, z) &= \frac{d}{dt}\Big|_{t=0} e^{it} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \end{aligned}$$

We compute $\iota_{X^{\#}}\omega$. We have

- $\iota_{X^{\#}} dx = dx(X^{\#}) = -y.$
- $\iota_{X^{\#}} dy = dy(X^{\#}) = x.$
- $\iota_{X^{\#}} dz = 0.$

Thus

$$\begin{aligned} \iota_{X^{\#}}\omega &= x^2 dz - y(-y)dz + z(-ydy - xdx) \\ &= (x^2 + y^2 + z^2)dz - z(ydy + xdx + zdz) \\ &= dz, \end{aligned}$$

since applying the operator d to the identity $x^2 + y^2 + z^2 = 1$ shows that $ydy + xdx + zdz = 0$. In particular,

$$\mathcal{L}_{X^{\#}}\omega = \iota_{X^{\#}} d\omega + d\iota_{X^{\#}}\omega = ddz = 0.$$

Since S^1 is connected, this implies that ω is invariant with respect to the action. Moreover, the identity $\iota_{X^{\#}}\omega = dz$ shows that $H(x, y, z) := z$ is a moment map.

We now want to compute the weights of the S^1-action on T_pS^2 and T_qS^2, which is also given by rotations. The orientation on T_pS^2 is given by the basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and the orientation preserving isomorphism

$$T_pS^2 \longrightarrow \mathbb{R}^2, \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

makes the following diagram commute

$$\begin{tikzcd}
T_pS^2 \arrow[r, "e^{it}"] \arrow[d] & T_pS^2 \arrow[d] \\
\mathbb{R}^2 \arrow[r, "\rho_1(e^{it})"] & \mathbb{R}^2.
\end{tikzcd}$$

Thus S^1 acts on T_pS^1 with weight $m(p) = 1$. At the south pole q, we have $\omega_q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = -1 < 0$ and thus a positively oriented basis is given by $(\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$. Therefore, the weight of the action is $m(q) = -1$. We can finally apply formula (6.6) to get

$$\int_{S^2} \omega = -2\pi\Big(\frac{-H(p)}{m(p)} + \frac{-H(q)}{m(q)}\Big) = -2\pi\big(\frac{-1}{1} + \frac{1}{-1}\big) = 4\pi.$$

A Connections on principal G-bundles

A.1 Definitions and existence of connection forms

The main reference for the material covered in this section is [16].

Let G be a Lie group. Let P, B be manifolds and $\pi : P \longrightarrow B$ be a principal G-bundle. Let $e \in G$ be the identity element and denote by $\phi_g : P \longrightarrow P$ the action of $g \in G$ on P.

Definition A.1. A connection form on a principal G-bundle $\pi : P \longrightarrow B$ is a $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(P, \mathfrak{g})$ on P satisfying:

(i) $\iota_{X^\#}\theta = X$ for all $X \in \mathfrak{g}$.

(ii) $\phi_g^*\theta = \mathrm{Ad}_g \circ \theta$ for all $g \in G$.

Remark A.2 ($\mathfrak{g}$-valued 1-forms). Let $X_1, \dots, X_n$ be a basis of the Lie algebra $\mathfrak{g}$. A $\mathfrak{g}$-valued 1-form on P is an element of $\Omega^1(P) \otimes \mathfrak{g}$ and it can be written as $\theta = \sum_{i=1}^n \theta_i \otimes X_i$, where $\theta_i \in \Omega^i(M)$ for $i = 1, \dots, n$. The various operations for $\mathbb{R}$-valued differential forms as pullback, exterior derivative, interior product and Lie derivative, carry over to $\mathfrak{g}$-valued forms by applying them to the first components of the tensor product. For example, if $f : E \longrightarrow P$ is a smooth map between manifolds and θ as above is a $\mathfrak{g}$-valued form on P, then the pullback $f^*\theta$ is a $\mathfrak{g}$-valued form on E given by

$$f^*\theta = \sum_{i=1}^n f^*\theta_i \otimes X_i.$$

Notice also that if $\varphi \in \mathrm{Hom}(\mathfrak{g})$ and θ is as above, then $\varphi \circ \theta = \sum_{i=1}^n \theta_i \otimes \varphi(X_i)$. Properties (i) and (ii) of Definition A.1 in terms of the θ_i are equivalent to

- $\iota_{X_j^\#}\theta_i = \delta_{ij}$ for all $i, j = 1, \dots n$.
- $\phi_g^*\theta_i = \sum_j a_j^i(g)\theta_j$ for all $g \in G$, $i = 1, \dots, n$,

where $a_i^j(g)$ are coefficients such that $\mathrm{Ad}_g X_i = \sum_j a_i^j(g) X_j$.

It is easy to check that a connection θ determines a complement H_x of $\ker d\pi_x$ in T_xP by the formula

$$H_x := \ker \theta_x,$$

so that for all $x \in P$ it holds

$$T_xP = H_x \oplus \ker d\pi_x.$$

The vectors in $V_x := \ker d\pi_x$ are called *vertical* and H_x is called an *horizontal* subspace of T_xP. Conversely, if we are given a family of horizontal subspaces H_x of T_xP, $x \in P$, such that $H_x \oplus V_x = T_xP$ for all $x \in P$, then the formula

$$\theta_x(v) := \mathrm{proj}_{V_x}(v)$$

defines a connection form $\theta \in \Omega^1(P, \mathfrak{g})$, with $\ker \theta_x = H_x$ for all $x \in P$ (see [11, §25.2]). Such a family $\{H_x\}_{x \in P}$ of subspaces is called an *Ehresmann connection* for the map π and is defined more generally when π is just a submersion (see Section 5.1).

The space of vertical vectors corresponds to the Lie algebra of G, since (see the proof of Proposition 3.8)

$$\ker d\pi_x = \{X^\#(p) \mid X \in \mathfrak{g}\}.$$

Proposition A.3. *For all $g \in G$ and $X \in \mathfrak{g}$ it holds*

$$(d\phi_g)_x X^\#(x) = (\mathrm{Ad}_g)^\#(gx).$$

Proof.

$$\begin{aligned}
(\mathrm{Ad}_g X)^\#(gx) &= \frac{d}{dt}\Big|_{t=0} \phi_{\exp(t\mathrm{Ad}_g X)} \circ \phi_g(x) \\
&= \frac{d}{dt}\Big|_{t=0} \phi_g \circ \phi_{\exp tX}(x) \\
&= (d\phi_g)_x X^\#(x).
\end{aligned}$$

In the second equality we used that $\exp(t\mathrm{Ad}_g X) = \exp \mathrm{Ad}_g tX = g \cdot \exp tX \cdot g^{-1}$, since $\mathrm{Ad}_g = dc_g$, where $c_g : G \longrightarrow G$ is the Lie group homomorphisms $c_g(h) = ghg^{-1}$, and $\exp \circ (dc_g) = c_g \circ \exp$ (the Lie group exponential commutes with Lie group homomorphisms). ☐

Proposition A.4. *Condition (ii) of Definition A.1 is equivalent to the invariance of H_x under the action of G, that is*

(ii)′ $(\phi_g)_* H_x = H_{gx}$ *for all $x \in P$, $g \in G$.*

Proof. (ii)⇒(ii)′: Let $v \in H_x = \ker \theta_x$. Then $(\phi_g)_* v = (d\phi_g)_x v \in H_{gx}$ satisfies

$$\theta_{gx}((d\phi_g)_x v) = (\phi_g^* \theta)_x(v) \overset{(ii)}{=} \mathrm{Ad}_g(\theta_x v) = 0.$$

So $(\phi_g)_* H_x \subseteq H_{gx}$. Analogously, $(\phi_{g^{-1}})_* H_{gx} \subseteq H_x$. Since $(\phi_g)_*(\phi_{g^{-1}})_* = id$ it follows that $H_{gx} \subseteq (\phi_g)_* H_x$ and so we have equality.

(ii)′ ⇒(ii): Notice that if $v \in H_x$ we have

$$(\phi_g^* \theta)_x(v) = \theta_{gx}((d\phi_g)_x v) = 0,$$

since by (ii)′ $(d\phi_g)_x v \in H_{gx}$. So both sides of (ii) vanish on horizontal vectors. By linearity it suffices to check (ii) on vertical vectors. Let $v \in T_x P$ be a vertical vector. Then there is $X \in \mathfrak{g}$ with $v = X^{\#}(x)$. Thus

$$\begin{aligned}(\phi_g^* \theta)_x(v) &= \theta_{gx}((d\phi_g)_x X^{\#}(x)) \\ &\overset{\text{Prop. } A.3}{=} \theta_{gx}((\mathrm{Ad}_g)^{\#}(gx)) \\ &\overset{(i)}{=} \mathrm{Ad}_g X \\ &\overset{(i)}{=} \mathrm{Ad}_g(\theta_x X^{\#}(x)) \\ &= \mathrm{Ad}_g \circ \theta_x(v).\end{aligned}$$

□

We want to show that every principal G-bundle admits a connection form. We start with the trivial case.

Proposition A.5 ([16, §3, Remark 2]). *The trivial bundle $B \times G \longrightarrow B$ has a connection form given by*

$$\theta_{(b,g)} = d(R_{g^{-1}} \circ \mathrm{pr}_2)_{(b,g)},$$

where $\mathrm{pr}_2 : B \times G \longrightarrow G$ is the projection onto the second factor and $R_{g^{-1}}$ is right multiplication in G by g^{-1}.

Proof. We have to check that θ satisfies properties (i) and (ii) of Definition A.1. Let $(b, g) \in B \times G$ and $X \in \mathfrak{g}$. Notice that for $X \in \mathfrak{g}$ it holds

$$X^{\#}(b, g) = \frac{d}{dt}\Big|_{t=0} (b, \exp tX \cdot g).$$

So we have

$$\begin{aligned}\theta_{(b,g)}(X^{\#}(b, g)) &= d(R_{g^{-1}} \circ \mathrm{pr}_2)_{(b,g)}(X^{\#}(b, g)) \\ &= \frac{d}{dt}\Big|_{t=0} R_{g^{-1}} \circ \mathrm{pr}_2(b, \exp tX \cdot g) = \frac{d}{dt}\Big|_{t=0} \exp tX = X.\end{aligned}$$

Denote by $\widetilde{\phi}$ the action of G on $B \times G$, given by the identity on B and left multiplication on G. To check the second property let $(b, g) \in B \times G$ and $h \in G$. Then

$$\begin{aligned}(\widetilde{\phi}_h^* \theta)_{(b,g)} &= \theta_{(b,hg)} \circ (d\widetilde{\phi}_h)_{(b,g)} \\ &= d(R_{(hg)^{-1}} \circ \mathrm{pr}_2)_{(b,hg)} \circ (d\widetilde{\phi}_h)_{(b,g)} \\ &= d(R_{(hg)^{-1}} \circ \mathrm{pr}_2 \circ \widetilde{\phi}_h)_{(b,g)},\end{aligned}$$

and

$$\begin{aligned}\mathrm{Ad}_h \circ \theta_{(b,g)} &= \mathrm{Ad}_h \circ d(R_{g^{-1}} \circ \mathrm{pr}_2)_{(b,g)} \\ &= dc_h)_e d(R_{g^{-1}} \circ \mathrm{pr}_2)_{(b,g)} \\ &= d(L_h \circ R_{h^{-1}})_e d(R_{g^{-1}} \circ \mathrm{pr}_2)_{(b,g)} \\ &= d(L_h \circ R_{h^{-1}} \circ R_{g^{-1}} \circ \mathrm{pr}_2)_{(b,g)}.\end{aligned}$$

Since $R_{(hg)^{-1}} \circ \mathrm{pr}_2 \circ \widetilde{\phi}_h = L_h \circ R_{h^{-1}} \circ R_{g^{-1}} \circ \mathrm{pr}_2$, this shows that $\widetilde{\phi}_h^* \theta = \mathrm{Ad}_h \circ \theta$. □

The connection constructed in Proposition A.5 is called the *flat connection.*

Proposition A.6. *Let $P \longrightarrow B$ and $P' \longrightarrow B'$ be two principal G-bundles and $\varphi : P' \longrightarrow P$ be a morphism of $G-$ bundles. If θ is a connection on P, then $\varphi^*\theta$ is a connection on P'.*

Proof. We check properties (i) and (ii) of Definition A.1.

(i) For $X \in \mathfrak{g}$ by equation (3.27) we have

$$\iota_{X^\#}\varphi^*\theta = \varphi^*\iota_{X^\#}\theta = \iota_{X^\#}\theta \circ \varphi = X.$$

(ii) Since φ commutes with the G-actions on P and P', it holds $\varphi \circ \phi_g = \phi_g \circ \varphi$ for every $g \in G$. Thus

$$\phi_g^*\varphi^*\theta = \varphi^*\phi_g^*\theta = \varphi^*(\mathrm{Ad}_g \circ \theta) = \mathrm{Ad}_g \circ \varphi^*\theta.$$

□

Proposition A.7 ([16, Proposition 3.10])**.** *Any convex combination of connections is again a connection. More precisely, let $\theta_1, \dots, \theta_k$ be connections on the G-bundle $\pi : P \longrightarrow B$ and let $\lambda_1, \dots, \lambda_k$ be real-valued functions on B with $\sum_{i=1}^k \lambda_i \equiv 1$. Then $\theta = \sum_{i=1}^k (\lambda_i \circ \pi)\theta_i$ is again a connection on P.*

Proof. Let $x \in P$ and $X \in \mathfrak{g}$. Then

(i) $\iota_{X^\#}\theta = \sum_{i=1}^k (\lambda_i \circ \pi)\iota_{X^\#}\theta_i = \sum_{i=1}^k (\lambda_i \circ \pi)X = X.$

(ii) For every $g \in G$ we have

$$\begin{aligned}(\phi_g^*\theta)_x &= \sum_i (\lambda_i \circ \pi \circ \phi_g)(x)(\phi_g^*\theta_i)_x \\ &= \sum_i \lambda_i(\pi(gx))\mathrm{Ad}_g(\theta_i)_x \\ &= \mathrm{Ad}_g\big(\sum_i \lambda_i(\pi(x))(\theta_i)_x\big) \\ &= \mathrm{Ad}_g \circ \theta_x.\end{aligned}$$

□

Finally we are ready to prove that any principal G-bundle admits a connection.

Proposition A.8 ([16, Corollary 3.11])**.** *Any principal G-bundle $\pi : P \longrightarrow B$ admits a connection.*

Proof. Let $\{U_i\}$ be a cover of B with trivializations $h_i : \pi^{-1}(U_i) \longrightarrow U_i \times G$. Let θ_0 be the flat connection on $U_i \times G$ constructed as in Proposition A.5. Then $\theta_i := h_i^* \theta_0$ is a connection on $\pi\big|_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \longrightarrow U_i$. Recall that since $\pi : P \longrightarrow B$ is a principal G-bundle of manifolds, by Corollary 2.7 it is numerable and so there is a locally finite partition of unity $\{\lambda_i\}$ on B with $\operatorname{supp}(\lambda_i) \subseteq U_i$. Set

$$\theta := \sum_i (\lambda_i \circ \pi)\theta_i,$$

Then for every $x \in P$ there is an open neighborhood U of $\pi(x)$ which intersects only finitely many U_i's, so that $\theta\big|_U$ is a finite convex combination of connections and as in Proposition A.7 one can show that θ is a connection. $\square$

Example A.9 (Connections on S^1-bundles, [3, Example V.4.4])**.** Assume that the Lie group G is the circle S^1. Since its Lie algebra is $\mathfrak{g} \cong \mathbb{R}$ with trivial Lie bracket, if $X = 1$ is a generator for $\mathbb{R}$, the two conditions for being a connection of Definition A.1 become:

(i) $\iota_{X^\#}\theta \equiv 1$,

(ii) $\phi_g^*\theta = \theta$ for all $g \in G$, that is, $\mathcal{L}_{X^\#}\theta = 0$.

Consider now the S^1-bundle $S^{2n-1} \longrightarrow \mathbb{CP}^{n-1}$, where S^1 acts on $S^{2n-1} \subseteq \mathbb{C}^n$ by

$$w \cdot (z_1, \dots, z_n) = (wz_1 \dots, wz_n).$$

The exponential map $\exp : \mathfrak{g} \cong \mathbb{R} \longrightarrow S^1$ is given by $\exp(Y) = e^{iY}$. Thus the fundamental vector field associated to $X = 1 \in \mathfrak{g}$ is

$$X^\#(z_1, \dots, z_n) = \frac{d}{dt}\Big|_{t=0} (e^{itX} z_1, \dots, e^{itX} z_n) = (iz_1, \dots, iz_n).$$

Let $\{x_j, y_j\}$ be the coordinates on $\mathbb{C}^n$, so that $z_j = x_j + iy_j$ and $\big\{\big(\frac{\partial}{\partial x_j}\big), \big(\frac{\partial}{\partial y_j}\big)\big\}_{j=1,\dots,n}$ is the corresponding basis of $T_z\mathbb{C}^n = \mathbb{C}^n$. In these coordinates we have

$$X^\# = \sum_{j=1}^n x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}.$$

Let θ be the 1-form on S^{2n-1} defined by

$$\theta = \sum_{j=1}^n (-y_j dx_j + x_j dy_j).$$

Then θ is a connection on S^{2n-1}. Indeed, we check the two defining properties.

(i) For $z \in S^{2n-1}$ we have

$$\iota_{X^\#}\theta(z) = \theta_z(X^\#(z)) = \sum_{j=1}^n x_j^2 + y_j^2 = |z|^2 = 1.$$

(ii) First notice that

$$d\theta = \sum_{j=1}^{n} \big(- dy_j \wedge dx_j + dx_j \wedge dy_j \big) = 2 \sum_{j=1}^{n} dx_j \wedge dy_j = 2\omega,$$

where $\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$ is the canonical symplectic form on $\mathbb{C}^n$. The function $H(z) = -\frac{1}{2}\sum_{j=1}^{n}|z_j|^2 = -\frac{1}{2}\sum_{j=1}^{n} x_j^2 + y_j^2$ is a moment map for the S^1-action on $(\mathbb{C}^n, \omega)$ as

$$\begin{aligned} \iota_{X^\#}\omega = \omega(X^\#, \cdot) &= \sum_{j=1}^{n} (dx_j \wedge dy_j)(X^\#, \,\cdot\,) \\ &= \sum_{j=1}^{n} -dx_j(X^\#)dy_j(\cdot) - dx_j(\cdot)dy_j(X^\#) \\ &= \sum_{j=1}^{n} -y_j dy_j - x_j dx_j = dH. \end{aligned}$$

Thus finally we compute

$$\mathcal{L}_{X^\#}\theta = d\iota_{X^\#}\theta + \iota_{X^\#}d\theta = \iota_{X^\#}d\theta = 2\iota_{X^\#}\omega = 2dH = 0,$$

because H is constant on the sphere. Hence θ satisfies also condition (ii) and defines a connection form on S^{2n+1}.

A.2 The curvature form

Associated to a connection form on a principal G-bundle there is a $\mathfrak{g}$-valued 2-form, called the curvature form. To define it, we need to define a wedge product for $\mathfrak{g}$-valued forms. For two 1-forms η and ξ on P with values in $\mathfrak{g}$ their wedge is defined as follows. We denote by $\eta \wedge \xi$ the $\mathfrak{g} \otimes \mathfrak{g}$-valued 2-form on P given by

$$(\eta \wedge \xi)_x(v, w) = \eta_x(v) \otimes \xi_x(w) - \eta_x(w) \otimes \xi_x(v).$$

Then the image of $\eta \wedge \xi$ under the bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ gives a $\mathfrak{g}$-valued 2-form on P which we denote by $[\eta \wedge \xi]$. Explicitly for $x \in P$ and $v, w \in T_xP$ we have

$$\left[\eta \wedge \xi\right]_x (v, w) = \left[\eta_x(v), \xi_x(w)\right] - \left[\eta_x(w), \xi_x(v)\right].$$

Definition A.10. Let $\pi : P \longrightarrow B$ be a principal G-bundle with connection form $\theta \in \Omega^1(P, \mathfrak{g})$. The *curvature form* (associated to θ) is the $\mathfrak{g}$-valued 2-form u on P defined as

$$u \coloneqq d\theta - \frac{1}{2}\left[\theta \wedge \theta\right].$$

We want to show that the curvature form is horizontal and G-equivariant. First, we recall briefly some facts about derivations and vector fields on a manifold M. A *derivation* of $C^\infty(M)$ is a linear map $\delta : C^\infty(M) \longrightarrow C^\infty(M)$ such that $\delta(fg) = \delta(f)g + f\delta(g)$ for all $f, g \in C^\infty(M)$.

There is a one-to-one correspondence between derivations and vector fields on M, given by the formula (see [10, Proposition 1.15]).

$$\begin{aligned} \Gamma(TM) &\longrightarrow \mathrm{Der}(C^\infty(M)) \\ X &\longmapsto \left\{ \begin{aligned} C^\infty(M) &\longrightarrow C^\infty(M) \\ f &\longmapsto X(f)(p) := df_p X(p) \end{aligned} \right\}. \end{aligned}$$

Moreover, for two vector fields X, Y on M (considered as derivations on $C^\infty(M)$) and $\mu \in \Omega^1(M)$ it holds the identity

$$d\mu(X,Y) = X\mu(Y) - Y\mu(X) - \mu([X,Y]). \tag{A.1}$$

(see [24, Proposition 14.29]). The same holds for a $\mathfrak{g}$-valued form μ, as both the exterior derivative d and the derivations X, Y act just on the first component of μ, according to Remark A.2.

Proposition A.11 ([16, Proposition 3.12])**.** *The curvature form $u \in \Omega^2(P, \mathfrak{g})$ is horizontal and G-equivariant, i.e. $\iota_{X^\#} u = 0$ for all $X \in \mathfrak{g}$ and $\phi_g^* u = \mathrm{Ad}_g \circ u$ for all $g \in G$.*

Proof. Since θ is G-equivariant we have

$$\phi_g^* d\theta = d\phi_g^* \theta = d\mathrm{Ad}_g \circ \theta = \mathrm{Ad}_g \circ d\theta,$$

and for $x \in P$, $v, w \in T_x P$ we have

$$\begin{aligned} \phi_g^* \left[\theta \wedge \theta\right]_x (v,w) &= \left[\theta \wedge \theta\right]_{gx} ((d\phi_g)_x v, (d\phi_g)_x w) \\ &= 2\left[\theta_{gx}((d\phi_g)_x v), \theta_{gx}((d\phi_g)_x w)\right] \\ &= 2\left[\phi_g^* \theta_x v, \phi_g^* \theta_x w\right] \\ &= 2\left[\mathrm{Ad}_g \circ \theta_x(v), \mathrm{Ad}_g \theta_x(w)\right] \\ &= 2\mathrm{Ad}_g \circ \left[\theta_x v, \theta_x w\right] \\ &= \mathrm{Ad}_g(2\left[\theta_x v, \theta_x w\right]) \\ &= \mathrm{Ad}_g \circ \left[\theta \wedge \theta\right]_x (v,w). \end{aligned}$$

Thus $\phi_g^* u = \mathrm{Ad}_g \circ u$, that is, u is G-equivariant. To prove that u is horizontal we must show that for all $x \in P$, $v \in T_x P$ and $X \in \mathfrak{g}$ it holds

$$\iota_{X^\#} u_x(v) = 0$$

that is

$$u_x(X^\#(x), v) = (d\theta)_x(X^\#(x), v) - \frac{1}{2}\left[\theta \wedge \theta\right]_x (X^\#(x), v) = 0. \tag{A.2}$$

By linearity it suffices to consider the cases of v vertical and of v horizontal. If v is vertical, then $v = Y^\#(x)$ for some $Y \in \mathfrak{g}$ and so we have to show that

$$\frac{1}{2}\left[\theta \wedge \theta\right]_x (X^\#(x), Y^\#(x)) = (d\theta)_x(X^\#(x), Y^\#(x)).$$

Now, the identity (A.1) applied to the vector fields $X^{\#}, Y^{\#}$ and to the 1-form θ gives

$$d\theta(X^{\#}, Y^{\#}) = -\theta(\left[X^{\#}, Y^{\#}\right]),$$

since $x \longmapsto \theta_x(X^{\#}(x)) = X$ and $x \longmapsto \theta_x(Y^{\#}(x)) = Y$ are constant. Moreover, by the identity (4.2) we have $[X, Y]^{\#} = -\left[X^{\#}, Y^{\#}\right]$. So we compute

$$\begin{aligned} \frac{1}{2}\left[\theta \wedge \theta\right](X^{\#}, Y^{\#}) &= \left[\theta(X^{\#}), \theta(Y^{\#})\right] \\ &= [X, Y] \\ &= \theta([X, Y]^{\#}) \\ &= -\theta(\left[X^{\#}, Y^{\#}\right]) \\ &= d\theta(X^{\#}, Y^{\#}). \end{aligned}$$

If instead v is horizontal, i.e. $v \in \ker \theta_x$, then $\left[\theta \wedge \theta\right]_x (X^{\#}(x), v) = 0$. To see that also $(d\theta)_x(X^{\#}(x), v)$ vanishes, extend v to an horizontal vector field V on P. This can be done by first extending v to any vector field W on P (using local coordinates at x and a bump function) and then set $V(y) := W(y) - (\theta_y W(y))^{\#}$, so that $\theta_y(V(y)) = \theta_y(W(y)) - \theta_y((\theta_y W(y))^{\#}) = 0$.

Notice that $X^{\#}\theta(V) = 0$ since V is horizontal, and also $V\theta(X^{\#}) = 0$ because $\theta(X^{\#}) = X$ is constant. Thus

$$d\theta(X^{\#}, V) \overset{(A.1)}{=} -\theta(\left[X^{\#}, V\right]) = \theta(\left[V, X^{\#}\right]) \overset{(4.1)}{=} \frac{d}{dt}\Big|_{t=0} \theta((\phi_{\exp tX})_* V),$$

as the flow associated to $X^{\#}$ is $t \longmapsto \phi_{\exp tX}$. But for any $g \in G$ we have

$$\begin{aligned} \theta_x((\phi_g)_* V(x)) &= \theta_x((d\phi_g)_x V(g^{-1}x)) \\ &= (\phi_g^* \theta)_{g^{-1}x}(V(g^{-1}x)) \\ &= \mathrm{Ad}_g \circ \underbrace{\theta_{g^{-1}x}(V(g^{-1}x))}_{=0}. \end{aligned}$$

So $d\theta(X^{\#}, V) = 0$ as well. ☐

A.3 Case of G abelian

Suppose that the group G is abelian. Then $\mathrm{Ad}_g = id_{\mathfrak{g}}$ and in particular $\phi_g^* u = u$ for all $g \in G$, so that $\mathcal{L}_{X^{\#}} u = 0$ for all $X \in \mathfrak{g}$. Since u is also horizontal, it follows that it is basic and thus there is $\eta \in \Omega^2(B, \mathfrak{g})$ with $\pi^*\eta = u$. Since the Lie algebra of an abelian Lie group has trivial bracket, it follows that $[\theta \wedge \theta] = 0$ and thus $u = d\theta$. Therefore we have $\pi^*\eta = d\theta$ for some $\eta \in \Omega^2(B, \mathfrak{g})$. We call η the *curvature form* of θ on B.

Proposition A.12. *Suppose that* G *is abelian and let* $\pi : P \longrightarrow B$ *be a principal* G*-bundle of manifolds. If* θ, α *are two connections on* P *with curvature forms on* B *given by* η *and* ξ *respectively, then there is* $\mu \in \Omega^1(B, \mathfrak{g})$ *with* $\eta - \xi = d\mu$.

Notice that in the case of $G = S^1$, since $\mathfrak{g}$-valued forms are the ordinary real-valued differential forms, the above proposition says that $[\eta] = [\xi]$ in $H^2(B)$, that is, the equivalence class of a curvature form on the bundle is independent of the connection form that we choose.

Proof. The 1-form $\theta - \alpha$ is basic since $\iota_{X^\#}(\theta - \alpha) = \iota_{X^\#}\theta - \iota_{X^\#}\alpha = X - X = 0$ and $\mathcal{L}_{X^\#}(\theta - \alpha) = \mathcal{L}_{X^\#}\theta - \mathcal{L}_{X^\#}\alpha = 0$. So there is $\mu \in \Omega^1(B, \mathfrak{g})$ with $\pi^*\mu = \theta - \alpha$. It follows that

$$\pi^* d\mu = d\pi^*\mu = d(\theta - \alpha) = \pi^*(\eta - \xi),$$

and the injectivity of π^* gives

$$d\mu = \eta - \xi.$$

□

B Orientability, Thom class, Euler class and the Gysin homomorphism

B.1 Orientable G-vector bundles

Definition B.1. A *vector bundle* of rank n is a triple (p, E, B) consisting of a surjective continuous map $p : E \longrightarrow B$ between topological spaces such that the following two conditions hold.

(i) There is an open cover $\{U_i\}_{i\in I}$ of B and a collection of homeomorphisms $h_i : p^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$ such that $h_i(p^{-1}(x)) = \{x\} \times \mathbb{R}^n$.

(ii) For all $i, j \in I$ the map $h_i(h_j|_{\{x\}\times\mathbb{R}^n})^{-1} : \{x\} \times \mathbb{R}^n \longrightarrow \{x\} \times \mathbb{R}^n$ is given by $h_i \circ h_j^{-1}(x, v) = (x, g_{ij}(x)(v))$ for $g_{ij}(x) \in GL_n(\mathbb{R})$.

Definition B.2. Let E and B be two topological spaces both endowed with a (continuous) action of a Lie group G. A vector bundle $p : E \longrightarrow B$ is called a *G-vector bundle* if p is G-equivariant and the action is linear in the fibers. Precisely, for any $g \in G$ and for any trivializations (U_i, h_i) at $x \in B$ and (U_j, h_j) at gx the composition

$$h_j \circ \phi_g^E \circ h_i^{-1}|_{\{x\}\times\mathbb{R}^n} : \{x\} \times \mathbb{R}^n \longmapsto \{gx\} \times \mathbb{R}^n$$

is linear.

In what follows we'll use several times the next easily provable lemma.

Lemma B.3. *Let $p : E \longrightarrow B$ be a vector bundle, and suppose that we have the commutative diagram*

$$\begin{tikzcd} M \arrow[r, "\bar{\varphi}"] \arrow[d, "q"] & E \arrow[d, "p"] \\ N \arrow[r, "\varphi"] & B, \end{tikzcd}$$

where $\bar{\varphi}$ and φ are homeomorphisms. Then $q : M \longrightarrow N$ is a vector bundle.

We want to show that a G-vector bundle induces a vector bundle on the Borel spaces. The proof of this fact is rather technical.

Proposition B.4. *Let $p : E \longrightarrow B$ be a G-vector bundle. Then the induced map*

$$1 \times_G p : EG \times_G E \longrightarrow EG \times_G B$$

is a vector bundle.

Proof. Since p is G-equivariant, $1 \times_G p$ is well-defined. The strategy of the proof is as follows: first, we show that $1 \times_G p$ is locally isomorphic to the G-vector bundle $1 \times p : BG \times E \longrightarrow BG \times B$ and thus it is locally a vector bundle. Then analyzing the local isomorphisms we show that the trivializations are compatible in intersecting neighborhoods, so that $1 \times_G p$ is actually a global vector bundle.

The second part of the proof is quite technical and explicit on the constructions we made, as for example we will use the explicit definition of local trivializations of the principal G-bundle $EG \longrightarrow BG$.

Let $U \subseteq BG$ be open such that $\pi : EG \longrightarrow BG$ is trivial over U and let $h : \pi^{-1}(U) \xrightarrow{\cong} U \times G$ be the trivialization. Recall that h is G-equivariant.

Claim. $\pi^{-1}(U) \times_G B$ is homeomorphic to $U \times B$.

Proof. Let $\varphi : \pi^{-1}(U) \times_G B \longrightarrow U \times B$ be given by the composition

$$\begin{tikzcd}
\varphi : \pi^{-1}(U) \times_G B \arrow[r, "h\times_G 1"] & (U \times G \times B)/G \arrow[r, "r"] & U \times B \\
{[(y,b)]} \arrow[r, mapsto] & {[(h(y),b)]} & \\
& {[(x,1_G,b)]} \arrow[r, mapsto] & (x,b).
\end{tikzcd}$$

where 1_G denotes the identity of G and on $U \times G \times B$ we have the G-action; $g \cdot (x, h, b) = (x, gh, gb)$. Notice that $h \times_G 1$ is well-defined as h is equivariant and it is a homeomorphism with inverse $h^{-1} \times_G 1$. Also, r is well-defined as every element of $(U \times G \times B)/G$ has a unique representative of the form $(x, 1_G, b)$. Since r is a homeomorphism as well, φ is, and its inverse is given by

$$\begin{aligned}
\varphi^{-1} : U \times B &\longrightarrow \pi^{-1}(U) \times_G B \\
(x, b) &\longmapsto \left[(h^{-1}(x, 1_G), b)\right].
\end{aligned}$$

☐ Claim.

Analogously, one shows that $\pi^{-1}(U) \times_G E \cong U \times E$, where the homeomorphism $\overline{\varphi} : \pi^{-1}(U) \times_G E \longrightarrow U \times E$ is given by the exact same formula as above. In particular, the following diagram commutes

$$\begin{tikzcd}
\pi^{-1}(U) \times_G E \arrow[r, "\overline{\varphi}"] \arrow[d, "1\times_G p"] & U \times E \arrow[d, "1\times p"] \\
\pi^{-1}(U) \times_G B \arrow[r, "\varphi"] & U \times B.
\end{tikzcd}$$

Since $1 \times p : U \times E \longrightarrow U \times B$ is a vector bundle, by Lemma B.3 so is

$$1 \times_G p : \pi^{-1}(U) \times_G E \longrightarrow \pi^{-1}(U) \times_G B.$$

So $1 \times_G p$ is locally a vector bundle. The problem now is the compatibility of the trivializations, that is, we have to show that condition (ii) of Definition B.1 holds. For this, we need to work more explicitly.

Let $\{U_i\}_i \subseteq BG$ be a trivializing cover of BG with trivializations $h_i : \pi^{-1}(U_i) \longrightarrow U_i \times G$. The restriction

$$1 \times p : U_i \times E \longrightarrow U_i \times B$$

is a G-vector bundle and we denote by $(V_{ij}, k_{ij})_j$ the trivializing cover of $U_i \times B$. Set $\widetilde{V}_{ij} := \varphi_i^{-1}(V_{ij})$ and let $\widetilde{k}_{ij} : (1 \times_G p)^{-1}(\widetilde{V}_{ij}) \longrightarrow \widetilde{V}_{ij} \times \mathbb{R}^n$ be given by

$$\widetilde{k}_{ij} := (\varphi_i^{-1} \times id_{\mathbb{R}^n}) \circ k_{ij} \circ \overline{\varphi}_i|_{(1 \times_G p)^{-1}(\widetilde{V}_{ij})}.$$

This are trivializations on $\bigcup_j \widetilde{V}_{ij} = \pi^{-1}(U_i) \times_G B$ by Lemma B.3. We have to check that if $\pi^{-1}(U_i) \times_G B$ intersects $\pi^{-1}(U_l) \times_G B$ they are compatible, that is, if $\widetilde{V}_{ij} \cap \widetilde{V}_{lm} \neq \emptyset$, then

$$\widetilde{k}_{ij} \circ (\widetilde{k}_{lm})^{-1}(z, v) = (z, \eta_{ij,lm}(z)(v))$$

with $\eta_{ij,lm}(z) \in GL_n\mathbb{R}$ for all $z \in \widetilde{V}_{ij} \cap \widetilde{V}_{lm}$. So, suppose $\widetilde{V}_{ij} \cap \widetilde{V}_{lm} \neq \emptyset$. The commutative diagram

$$\begin{tikzcd}
\widetilde{V}_{ij} \cap \widetilde{V}_{lm} \times \mathbb{R}^n \arrow[dr, "\mathrm{pr}_1"'] & (1 \times_G p)^{-1}(\widetilde{V}_{ij} \cap \widetilde{V}_{lm}) \arrow[l, "\widetilde{k}_{ij}"'] \arrow[r, "\widetilde{k}_{lm}"] \arrow[d, "1 \times_G p"] & \widetilde{V}_{ij} \cap \widetilde{V}_{lm} \times \mathbb{R}^n \arrow[dl, "\mathrm{pr}_1"] \\
& \widetilde{V}_{ij} \cap \widetilde{V}_{lm} &
\end{tikzcd}$$

implies that $\widetilde{k}_{ij} \circ \widetilde{k}_{lm}^{-1}$ is the identity on the first coordinate. As for the second coordinate, let $z \in \widetilde{V}_{ij} \cap \widetilde{V}_{lm}$. Then

$$\begin{aligned}
\widetilde{k}_{ij} \circ \widetilde{k}_{lm}(z, \cdot) &= (\varphi_i^{-1} \times id_{\mathbb{R}^n}) \circ k_{ij} \circ \overline{\varphi}_i|_{(1 \times_G p)^{-1}(\widetilde{V}_{ij} \cap \widetilde{V}_{lm})} \circ \overline{\varphi}_l^{-1} \circ k_{lm}^{-1} \circ (\varphi_l(z), \cdot\,) \\
&= (\varphi_i^{-1} \times id_{\mathbb{R}^n}) \circ k_{ij} \circ \overline{\varphi}_i \circ \overline{\varphi}_l^{-1} \circ k_{lm}^{-1}|_{\{\varphi_l(z)\} \times \mathbb{R}^n}(\varphi_l(z), \cdot\,).
\end{aligned}$$

We compute $\overline{\varphi}_i \circ \overline{\varphi}_l^{-1}$ on $U_i \cap U_l \times E$. Let $(u, e) \in U_i \cap U_j \times E$. Then

$$\overline{\varphi}_i \circ \overline{\varphi}_l^{-1}(u, e) = \overline{\varphi}_i\big(\big[(h_l^{-1}(u, 1_G), e)\big]_G \big) = r_i\big(\big[h_i \circ h_l^{-1}(u, 1_G), e\big]_G \big).$$

We have to express $\big[h_i \circ h_l^{-1}(u, 1_G), e\big]_G$ in the form $(u, 1_G, e)$. Recall from Proposition 2.12 that if $u = [\langle x, t\rangle] \in U_i \cap U_l$, then

$$h_i \circ h_l^{-1}(u, 1_G) = (u, x_l^{-1}x_i).$$

Thus

$$\overline{\varphi}_i \circ \overline{\varphi}_l^{-1}(u, e) = (u, x_i^{-1}x_l \cdot e)$$

for $u = [\langle x, t\rangle] \in U_i \cap U_l$. Let $f_{il} : U_i \cap U_l \times E \longrightarrow G$ be given by $f_{il}(u, e) := x_i^{-1}x_l$. Notice that it depends only on the first coordinate. Then

$$\overline{\varphi}_i \circ \overline{\varphi}_l^{-1} = \phi^{BG \times E}_{f_{il}(\cdot)}(\cdot).$$

Now go back to the composition

$$\overline{\varphi}_i \circ \overline{\varphi}_l^{-1} \circ k_{lm}^{-1}|_{\{\varphi_l(z)\} \times \mathbb{R}^n}(\varphi_l(z), v) = \phi^{BG \times E}_{f_{il}\left(k_{lm}^{-1}(\varphi_l(z), v)\right)} \circ k_{lm}^{-1}(\varphi_l(z), v).$$

If we write $k_{lm}^{-1}(\varphi_l(z), v) = (u, e) \in (1 \times p)^{-1}(V_{ij} \cap V_{lm})$, the commutative diagram

$$\begin{tikzcd}
(1 \times p)^{-1}(V_{ij} \cap V_{lm}) \arrow[rr, "k_{lm}"] \arrow[dr, "1\times p"'] & & (V_{ij} \cap V_{lm}) \times \mathbb{R}^n \arrow[dl, "\mathrm{pr}_1"] \\
& V_{ij} \cap V_{lm} &
\end{tikzcd}$$

shows that u is the first coordinate of $\varphi_l(z)$ and thus doesn't depend on v. Therefore, $f_{il} \circ k_{lm}^{-1}$ is constant on $\{\varphi_l(z)\} \times \mathbb{R}^n$ and we have

$$k_{ij} \circ \overline{\varphi}_i \circ \overline{\varphi}_l^{-1} \circ k_{lm}|_{\{\varphi_l(z)\}\times\mathbb{R}^n}(\varphi_l(z), \cdot\,) = k_{ij} \circ \phi_g^{BG\times E} \circ k_{lm}^{-1}|_{\{\varphi_l(z)\}\times\mathbb{R}^n}(\varphi_l(z), \cdot\,)$$

which is linear since $1 \times p : BG \times E \longrightarrow BG \times B$ is a G-vector bundle. This shows that the second coordinate of $\widetilde{k}_{ij} \circ \widetilde{k}_{lm}(z, \cdot\,)$ is linear, hence in $GL_n\mathbb{R}$, and concludes the proof. □

B.2 Orientability

The main reference for this subsection is [27].

Let V be an n-dimensional vector space. An orientation on V is a choice of an equivalence class of bases, where $(v_1, \dots, v_n) \sim (v'_1, \dots, v'_n)$ if the matrix $(a_{ij})_{i,j}$ defined by $v'_i = \sum_j a_{ij} v_j$ has positive determinant. Every vector space has exactly two possible orientations.

Let $V_0 := V \setminus \{0\text{-vector}\}$. Notice that $H_n(V, V_0; \mathbb{Z}) \cong H_{n-1}(V_0; \mathbb{Z}) \cong H_{n-1}(S^{n-1}; \mathbb{Z}) \cong \mathbb{Z}$ and thus has only two possible generators. A choice of orientation for V corresponds to a choice of one of the two possible generators for the singular homology group $H_n(V, V_0; \mathbb{Z})$, and by the universal coefficient theorem to a choice of a generator of $H^n(V, V_0; \mathbb{Z})$[12].

Now consider a vector bundle $p : E \longrightarrow B$ of rank $n > 0$.

Definition B.5. A vector bundle $p : E \longrightarrow B$ is called *orientable* if its local trivializations $\{h_i : p^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n\}_i$ can be chosen such that the transition maps

$$h_j \circ h_i^{-1} : \{x\} \times \mathbb{R}^n \longrightarrow \{x\} \times \mathbb{R}^n$$

are in $GL_n^+\mathbb{R}$, that is, have determinant strictly greater than 0, for all $x \in U_i \cap U_j$.

Proposition B.6. *If $p : E \longrightarrow B$ is an orientable G-vector bundle, the induced vector bundle $1 \times_G p : EG \times_G E \longrightarrow EG \times_G B$ is orientable as well.*

Proof. We use the same notation as in Proposition B.4. We have to show that the transition maps $\widetilde{k}_{ij} \circ \widetilde{k}_{lm}^{-1} : \{z\} \times \mathbb{R}^n \longrightarrow \{z\} \times \mathbb{R}^n$ have determinant greater than 0 for all $z \in \widetilde{V}_{ij} \cap \widetilde{V}_{lm}$. These maps are given by

$$(\varphi_i^{-1} \times id_{\mathbb{R}^n}) \circ k_{ij} \circ \phi_g \circ k_{lm}^{-1} \circ (\varphi_l \times id_{\mathbb{R}^n}) : \{z\} \times \mathbb{R}^n \longrightarrow \{z\} \times \mathbb{R}^n,$$

---

[12] See [27] for more details.

where $g \in G$ is some fixed element depending on z, l and i. Thus it suffices to show that

$$k_{ij} \circ \phi_g \circ k_{lm}^{-1} : \{\varphi_l(z)\} \times \mathbb{R}^n \longrightarrow \{g \cdot \varphi_l(z)\} \times \mathbb{R}^n$$

has positive determinant, where $\{(V_{ij}, k_{ij})\}_{i,j}$ are the trivializations of the G-vector bundle $1 \times P : BG \times E \longrightarrow EG \times B$. Hence the proposition follows from the following claim.

Claim. Let $q : P \longrightarrow X$ be a G-vector bundle. Let $x \in X$, $g \in G$ and suppose that (V_i, k_i) and (V_j, k_j) are trivializations on X with $x \in V_i$ and $gx \in V_j$. Then

$$k_j \circ \phi_g \circ k_i^{-1} : \{x\} \times \mathbb{R}^n \longrightarrow \{gx\} \times \mathbb{R}^n$$

has positive determinant.

Proof. Suppose first that $V_i \cap V_j \neq \emptyset$ and that there is a path $t \longmapsto g(t)$ in G starting at $g(0) = 1_G$ and ending at $g(1) = g$, such that $g(t) \cdot x \subset V_i \cup V_j$ for all t and there is $t_0 \in I$ such that $g(t_0) \cdot x \in V_i \cap V_j$, $g(t) \cdot x \subset V_i$ for all $t \in [0, t_0]$ and $g(t) \cdot x \subset V_j$ for all $t \in [t_0, 1]$.

Step 1. $k_i \circ \phi_{g(t_0)} \circ k_i^{-1} : \{x\} \times \mathbb{R}^n \longrightarrow \{g(t_0)x\} \times \mathbb{R}^n$ has positive determinant.

Proof. For $t \in [0, t_0]$ consider the composition

$$\{x\} \times \mathbb{R}^n \xrightarrow{k_i^{-1}} q^{-1}(x) \xrightarrow{\phi_{g(t)}} q^{-1}(g(t)x) \xrightarrow{k_i} \{g(t)x\} \times \mathbb{R}^n.$$

It is well-defined and in $GL_n\mathbb{R}$. In particular, $\det(k_i \circ \phi_{g(t)} \circ k_i^{-1}) \neq 0$ for all $t \in [0, t_0]$. As for $t = 0$ it is the identity, which has positive determinant, it has positive determinant for all $t \in [0, t_0]$ and thus

$$\det(k_i \circ \phi_{g(t_0)} \circ k_i^{-1}) > 0.$$

$\square$ Step 1.

Step 2. $k_j \circ \phi_{gg(t_0)^{-1}x} \circ k_j^{-1} : \{g(t_0)\} \times \mathbb{R}^n \longrightarrow \{gx\} \times \mathbb{R}^n$ has positive determinant.

Proof. For $t \in [t_0, 1]$ let $\gamma(t) := g(t)g(t_0)^{-1}$. Then $\gamma(t_0) = 1_G$, $\gamma(1) = gg(t_0)^{-1}$ and $\gamma(t)g(t_0)x \subset V_j$ for all $t \in [t_0, 1]$. Thus Step 2 follows by the exact same proof as above. $\square$ Step 2.

Since we can write

$$\begin{aligned} k_j \circ \phi_g \circ k_i^{-1}|_{\{x\}\times\mathbb{R}^n} &= k_j \circ \phi_{g\cdot g(t_0)^{-1}} \circ \phi_{g(t_0)} \circ k_i^{-1}|_{\{x\}\times\mathbb{R}^n} \\ &= \underbrace{k_j \circ \phi_{g\cdot g(t_0)^{-1}} \circ k_j^{-1}}_{\substack{\det>0 \\ \text{by Step 2.}}} \circ \underbrace{k_j \circ k_i^{-1}}_{\substack{\det>0 \text{ since} \\ q \text{ orientable.}}} \circ \underbrace{k_i \circ \phi_{g(t_0)} \circ k_i^{-1}}_{\substack{\det>0 \\ \text{by Step 1.}}} \end{aligned}$$

the claim in the special case follows by Step 1, Step 2 and the assumption of q orientable.

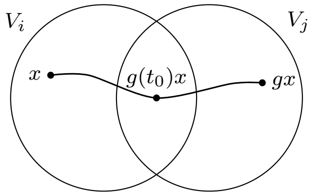

Figure 3: Special case in Proposition B.6.

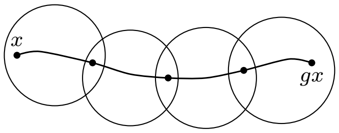

Figure 4: General case in Proposition B.6.

In the general case, since any connected Lie group is path connected we can choose a path $t \longrightarrow g(t)$ in G from the identity element to g. Since $g(I) \cdot x \subset BG \times B$ is compact, we can cover it with finitely many V_i's such that the special case above always applies piecewise. Then a decomposition analogous to the one above gives the desired result. ☐ Claim.

☐

Remark B.7. The claim in the proof of Proposition B.6 shows that if $q : P \longrightarrow X$ is an orientable G-vector bundle, then the action of G on the fibers is orientation preserving.

B.3 The normal bundle

Let M be an m-dimensional manifold with a Riemannian metric $\langle \cdot, \cdot \rangle$. Let $N \subseteq M$ be an n-dimensional submanifold of M, with $i : N \hookrightarrow M$ the inclusion map, so that for all $x \in N$ we have $di_x(T_xN) \subseteq T_xM$.

Definition B.8. For $x \in N$ the set

$$N_x^\perp \coloneqq \{v \in T_xM \mid \langle v, di_x u \rangle = 0 \text{ for all } u \in T_xN\}$$

is called the *normal space to N at x*. Let $\nu \coloneqq \bigcup_{x \in N} \{x\} \times N_x^\perp$. Then

$$p : \nu \longrightarrow N,\ (x, v) \longmapsto x$$

is called the *normal bundle to N in M*.

The normal bundle is in fact a vector bundle over N of rank $m-n$. If we have an action of a compact Lie group G on the manifold M, averaging a Riemannian metric over the group gives a G-invariant Riemannian metric, that is, a metric such that the group acts by isometries on M. Then if the submanifold N is invariant, the normal bundle ν has a natural structure of G-vector bundle, given by

$$g \cdot (x, v) = (gx, (d\phi_g)_x v).$$

Since every vector bundle over an orientable manifold is orientable (see [28, Exercise 9.20]), from Proposition B.6 we get the following corollary.

Corollary B.9. *Let M be an m-dimensional manifold acted on by a Lie group G. Suppose that $N \subseteq M$ is an orientable G-invariant submanifold of M of codimension n. Then the normal bundle $\nu \longrightarrow N$ to N in M induces an orientable vector bundle*

$$EG \times_G \nu \longrightarrow EG \times_G N$$

of rank n.

B.4 The equivariant tubular neighborhood theorem

Let G be a compact Lie group, M be a G-manifold and $N \subseteq M$ be a G-invariant compact submanifold. Equip M with a G-invariant Riemannian metric.

Theorem B.10. *In the above setting there exists a G-invariant open neighborhood U of N in M which is equivariantly diffeomorphic to the total space of the normal bundle to N. Moreover, the diffeomorphism maps every point x of N to the zero normal vector at x, that is, it restricts to the zero section on N.*

$$\begin{tikzcd}
U \arrow[r, "\varphi", "\cong"'] & \nu \\
N \arrow[u, phantom, sloped, "\subseteq", "\textit{open}"'] \arrow[r, "\cong"] & \bigcup_{x \in N} \{x\} \times \{0\}. \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

We sketch the proof Theorem B.10. For more details, we refer to [6, §VI, Theorem 2.2].

Outline of proof. Given a G-invariant metric on M, one lets $E(\epsilon) \subseteq \nu$ be the open subset of ν given by

$$E(\epsilon) = \{(x, v) \in \nu \mid |v| < \epsilon\}.$$

Then one considers the exponential map $\exp : E(\epsilon) \longrightarrow M$ which assigns to each $(x, v) \in \nu$ with $|v|$ sufficiently small the endpoint $\gamma(1)$ of the unique geodesic $\gamma : [0, 1] \longrightarrow M$ with $\gamma(0) = x$ and $\gamma'(0) = v$. If $\epsilon > 0$ is small enough, exp is defined on $E(\epsilon)$[13]. Then one shows that up to make ϵ smaller, $E(\epsilon)$ is mapped

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[13] See [10, Corollary 2.50].

diffeomorphically onto an open set $U \subseteq M$, containing $N = \exp(\bigcup_{x \in N}\{(x,0)\})$. Since the exponential map is defined canonically in terms of the Riemannian metric, the map exp and its domain are G-equivariant. $\square$

B.5 The cup product

Let X be any topological space, R be any ring. In this section we consider singular cohomology. We define a product for cohomology classes which turns $H^*(X;R)$ into a graded ring. As this product is well-known, for all the details we refer to [20, §3.2]. Denote by $C_p(X)$ the free group of p-chains on X and let $C^p(X) \coloneqq \mathrm{Hom}(C_p(X), R)$.

Definition B.11. Let $\alpha \in C^p(X)$ and $\beta \in C^q(X)$. For $\sigma \in C_{p+q}(X)$, $\sigma : \Delta^{p+q} \longrightarrow X$ we define

$$(\alpha \smile \beta)(\sigma) \coloneqq \alpha(\sigma \circ \iota_{0,1,\dots,p}) \cdot \beta(\sigma \circ \iota_{p,p+1,\dots,p+q}),$$

where for $S \subseteq \{0,1,\dots,p+q\}$ the map ι_S is the canonical embedding of the simplex spanned by S into the $(p+q)$-simplex whose vertices are indexed by $\{0,\dots,p+q\}$. So $\sigma \circ \iota_{0,\dots,p} = \sigma\big|_{\text{face } \{0,\dots,p\}}$ and $\sigma \circ \iota_{p,\dots,p+q} = \sigma\big|_{\text{face } \{p,\dots,p+q\}}$.

The cup product induces a product

$$\smile : H^p(X;R) \times H^q(X;R) \longrightarrow H^{p+q}(X;R).$$

Proposition B.12 ([20, Proposition 3.10])**.** *For a map $f : X \longrightarrow Y$, the induced maps $f^* : H^n(Y;R) \longrightarrow H^n(X;R)$ satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.*

If $A \subseteq X$ is a subspace, elements of $H^n(X,A;R)$ are represented by cochains in $C^n(X,A;R)$, which are the same as homomorphisms $C_n(X) \longrightarrow R$ which vanish on $C_n(A)$. In particular, the cup product

$$\smile : H^p(X) \otimes H^q(X,A) \longrightarrow H^{p+q}(X,A)$$

is well-defined, because if $\alpha \in C^p(X;R)$ and $\beta : C_q(X) \longrightarrow R$ is a homomorphism which vanishes on $C_q(A)$, then $\alpha \smile \beta : C_{p+q}(X) \longrightarrow R$ is a homomorphism vanishing on $C_{p+q}(A)$ as well.

At last, notice also that if $\alpha \in H^p(X)$, $\beta \in H^q(X,A)$ and $j : (X,\emptyset) \hookrightarrow (X,A)$ is the inclusion of pairs, then

$$j^*(\alpha \smile \beta) = \alpha \smile j^*\beta \in H^{p+q}(X). \tag{B.1}$$

Remark B.13. For a manifold M there is an isomorphism

$$I : H^*_{\mathrm{dR}}(M) \xrightarrow{\cong} H^*_{\mathrm{sing}}(M)$$

between de Rham and singular cohomology (see [24, Theorem 18.4]). It is an isomorphism of graded rings, that is, for ω and η in $\Omega^*(M)$ it holds

$$I(\omega \wedge \eta) = I(\omega) \smile I(\eta).$$

B.6 The Thom isomorphism

Let E, B be topological spaces and consider and orientable vector bundle $p : E \longrightarrow B$. Let $E_0 \subseteq E$ be the complement of the zero section in E[14]. Then the standard orientation $(e_1, \dots, e_n)$ on $\mathbb{R}^n$ induces an orientation on each fiber $F_x = p^{-1}(x)$. Namely, if (U, h) is a trivialization at x, one chooses on $p^{-1}(x)$ the orientation given by $(h^{-1}(e_1), \dots, h^{-1}(e_n))$. This doesn't depend on the chosen chart at x. Indeed, suppose that $x \in U_i \cap U_j$ and set $v_l \coloneqq h_i^{-1}(e_l)$ and $v_l' \coloneqq h_j^{-1}(e_l)$. Then

$$v_l' = h_j^{-1} \circ h_i(v_l) = h_i^{-1} \circ (h_i \circ h_j^{-1}) \circ h_i(v_l),$$

and $h_i^{-1} \circ (h_i \circ h_j^{-1}) \circ h_i \in GL_n^+(p^{-1}(x))$ with the vector space structure induced by $h_i : p^{-1}(x) \cong \{x\} \times \mathbb{R}^n$.

In terms of cohomology, this means that to each fiber F_x there is assigned a preferred generator $u_{F_x} \in H^n(F, F_0; \mathbb{Z})$ (see Section B.2). The Thom isomorphism theorem asserts the existence of a cohomology class on E , which on each fiber F_x restricts to U_{F_x}.

Theorem B.14 ([27, Theorem 9.1])**.** *Let $p : E \longrightarrow B$ be an oriented n-vector bundle, Then there exists a unique cohomology class $\tau \in H^n(E, E_0; \mathbb{Z})$ whose restriction to $H^n(F, F_0; \mathbb{Z})$ is u_F for every fiber F of p. Furthermore, the map*

$$\begin{aligned} H^k(E; \mathbb{Z}) &\longrightarrow H^{k+n}(E, E_0; \mathbb{Z}) \\ y &\longmapsto y \smile \tau \end{aligned}$$

is an isomorphism for all k.

Definition B.15. The cohomology class $\tau \in H^n(E, E_0; \mathbb{Z})$ of Theorem B.14 is called the *Thom class* or *fundamental class* of the bundle.

Since the map $p : E \longrightarrow B$ is a homotopy equivalence with inverse given by the zero section $s : B \longrightarrow E$, it follows that $H^{k+n}(E; \mathbb{Z})$ is isomorphic to the cohomology group $H^k(B; \mathbb{Z})$ of the base space. In fact the Thom isomorphism

$$\Phi : H^k(B; \mathbb{Z}) \xrightarrow{\cong} H^{k+n}(E, E_0; \mathbb{Z})$$

is defined by the formula

$$\Phi(x) = p^*(x) \smile \tau. \tag{B.2}$$

B.7 The Euler class

Given an oriented n-vector bundle, the inclusion $(E, \emptyset) \overset{j}{\hookrightarrow} (E, E_0)$ gives rise to a restriction homomorphism

$$H^*(E, E_0; \mathbb{Z}) \xrightarrow{j^*} H^*(E; \mathbb{Z}).$$

---

[14] The zero section is $s : B \longrightarrow E$, $x \longmapsto h^{-1}(x, 0)$, where (U, h) is any trivialization at x. Then $E_0 = E \setminus s(B)$.

Recall that the Thom class τ is an element of $H^n(E, E_0; \mathbb{Z})$ and that the latter is canonically isomorphic to $H^n(B; \mathbb{Z})$.

Definition B.16. The *Euler class* of an oriented n-vector bundle $p : E \longrightarrow B$ is the unique cohomology class $e(E) \in H^n(B; \mathbb{Z})$ with $p^*(e(E)) = j^*\tau$.

Remark B.17. Since the map s^* induced by the zero section $s : B \longrightarrow E$ is the inverse of p^* we have the identity

$$e(E) = s^* j^*(\tau).$$

Since the Euler class of a Whitney sum is given by the cup product of the Euler classes (see [27, Property 9.6]), and the Whitney sum of vector bundles over a point coincides with the direct sum, we have the following result.

Lemma B.18. *Let $L_1, \ldots, L_k$ be vector bundles over a point $\{x\}$. Then*

$$e(\bigoplus_{i=1}^{k} L_i) = e(L_1) \smile \ldots \smile e(L_k).$$

B.8 The equivariant Euler class

Suppose that we have an oriented G-vector bundle $p : E \longrightarrow B$. Then

$$1 \times_G p : EG \times_G E \longrightarrow EG \times_G B$$

is an oriented vector bundle and the Thom isomorphism theorem gives an equivariant class

$$\tau_G \in H^n_G(E, E_0; \mathbb{Z})$$

and an isomorphism

$$\begin{aligned} H^k_G(B; \mathbb{Z}) &\longrightarrow H^{k+n}_G(E, E_0; \mathbb{Z}) \\ x &\longmapsto (1 \times_G p)^*(x) \smile \tau_G \end{aligned}$$

for all k.

Definition B.19. The *equivariant Euler class* $e_G(E)$ is the Euler class of the oriented vector bundle $EG \times_G E \longrightarrow EG \times_G B$.

Remark B.20. $e_G(E)$ is determined by the identity

$$(1 \times_G p)^*(e_G(E)) = j^* \tau_G$$

where $j : (EG \times_G E, \emptyset) \hookrightarrow (EG \times_G E, EG \times_G E_0)$ is the inclusion.

Remark B.21 (Coefficients). In the previous subsections we considered cohomology with integral coefficients. However, if $p : E \longrightarrow B$ is an oriented n-vector bundle and we are given a preferred generator u_F of $H^n(F, F_0; \mathbb{Z})$, then the ring homomorphism $\mathbb{Z} \longrightarrow \mathbb{R}$, $k \longmapsto k$ gives rise to a corresponding generator for $H^n(F, F_0; \mathbb{R})$ and we get an analogous of the Thom isomorphism theorem with real coefficients (see [27, Theorem 10.4]). Therefore everything in this section can be carried over real coefficients and from now on they are to be understood.

B.9 The Gysin homomorphism

We finally have all the ingredients to define the Gysin homomorphism, which we'll do directly in the equivariant case, although of course one could define everything for the finite approximations of M_G and then take the limit to get to the equivariant cohomology (as done for example in [3, §VI.4.c]).

We are considering a compact manifold M on which a compact Lie group G acts by diffeomorphisms. Suppose that $N \subseteq M$ is a topologically closed submanifold of M of codimension n and let $i : N \hookrightarrow M$ denote the inclusion. Let $\nu \xrightarrow{p} N$ be the normal bundle to N in M, which we assume to be oriented (as it is for example when N is oriented). Let $U \subseteq M$ be a G-invariant tubular neighborhood of N in M, so that there is a G-equivariant diffeomorphism $\varphi : U \longrightarrow \nu$ which restricted to N gives the zero section.

Apply equivariant excision (Proposition 2.48) to the G-invariant spaces

$$M \setminus U \subseteq M \setminus N \subseteq M$$

For this we need $\overline{M \setminus U} \subseteq (M \setminus N)^\circ$, which holds since N is closed. Thus we get

$$H^*_G(M \setminus (M \setminus U), (M \setminus N) \setminus (M \setminus U)) \cong H^*_G(M, M \setminus N).$$

Since $M \setminus (M \setminus U) = U \cong \nu$ and $(M \setminus N) \setminus (M \setminus U) = U \setminus N \cong \nu_0$ via equivariant diffeomorphisms, we get

$$H^*_G(\nu, \nu_0) \cong H^*_G(M, M \setminus N).$$

Definition B.22. The *Gysin homomorphism*

$$i_* : H^k_G(N) \longrightarrow H^{k+n}_G(M)$$

is defined as the composition

$$H^k_G(N) \xrightarrow[\cong]{\Phi_G} H^{k+n}_G(\nu, \nu_0) \xrightarrow[\cong]{\text{excision}} H^{k+n}_G(M, M \setminus N) \longrightarrow H^{k+n}_G(M) \tag{B.3}$$

for all k, where Φ_G is the Thom isomorphism and the last map is the restriction induced by the inclusion $(M, \emptyset) \hookrightarrow (M, M \setminus N)$.

Proposition B.23 ([3, Proposition VI. 4.6])**.** *$i^* i_*(x) = x \smile e_G(\nu)$ for all $x \in H^k_G(N)$.*

Proof. We have the commutative diagram

$$\begin{tikzcd}
H^k_G(N) \arrow[r, "\Phi_G", "\cong"'] & H^{k+n}_G(\nu, \nu_0) \arrow[d, "j^*"] & H^{k+n}_G(M, M \setminus N) \arrow[l, "\text{excision}"', "\cong"] \arrow[r, "\text{restr.}"] & H^{k+n}_G(M) \arrow[d, "i^*"] \\
 & H^k_G(\nu) \arrow[rr, shift left, "(1 \times_G s)^*"] & & H^{k+n}_G(N) \arrow[ll, shift left, "(1 \times_G p)^*"]
\end{tikzcd}$$

Indeed, if we consider the maps that induce the above diagram, starting from the bottom right going left we have

- $N \xrightarrow{s} (\nu, \emptyset) \overset{j}{\hookrightarrow} (\nu, \nu_0) \overset{\varphi^{-1}}{\cong} (U, M \setminus U) \hookrightarrow (M, M \setminus N)$.

Notice that since $\varphi|_N = s$, the composition $\varphi^{-1} \circ s$ is the inclusion of N in T, and thus the above is just a diagram of inclusions. So the diagram commutes. If from the bottom right we go up we have

- $N \overset{i}{\hookrightarrow} (M, \emptyset) \hookrightarrow (M, M \setminus N)$,

which are also just inclusions.

Therefore for $x \in H_G^k(N)$ we have:

$$\begin{aligned} (1 \times_G p)^*(i^* i_* x) &\overset{\circlearrowleft}{=} j^*((1 \times_G p)^* x \smile \tau_G) \\ &\overset{(B.1)}{=} (1 \times_G p)^* x \smile j^* \tau_G \\ &= (1 \times_G p)^* x \smile (1 \times_G p)^* e_G(\nu) \\ &= (1 \times_G p)^*(x \smile e_G(\nu)). \end{aligned}$$

Thus $i^* i_* x = x \smile e_G(\nu)$. $\square$

C Equivariant Euler class and weights

In this section we exploit the relation between the equivariant Euler class of a G-vector bundle and the weights of the G-action on the total space of the bundle[15]. Although this relation was used in Section 6 (to invert the equivariant Euler class) in the case of a torus action, for simplicity we will restrict ourselves to the one-dimensional case $G = S^1$, as anyway the main application is Theorem 6.32. It is convenient to work with a definition of Euler class in terms of characteristic classes, rather than using the algebraic topological one of Section B.7. Of course the two notions coincide, see [16, §7]. Recall that the Euler class is a cohomology class associated to a vector bundle. Given any oriented n-vector bundle $E \longrightarrow B$, we can consider its oriented orthonormal frame bundle $F(E)$, which is the bundle of all the oriented orthonormal bases of the fibers of E. This is a principal $SO(n)$-bundle. Since there is a correspondence between vector bundles and their frame bundles, we start by considering principal K-bundles with $K = SO(n)$.

Remark C.1. In this section we make use of the Einstein convention for summations.

C.1 Chern-Weil theory for principal bundles

Let $K = SO(n)$ and consider a principal K-bundle $P \xrightarrow{\pi} X$. Denote by ψ_k the action of $k \in K$ on P. Let $v_1, \ldots, v_n$ be a basis for the Lie algebra $\mathfrak{k}$. Then for all $k \in K$ we have

$$\mathrm{Ad}_k v_j = a^i_j(k) v_i, \tag{C.1}$$

for some coefficients $a^i_j(k)$. We denote by $v^\#$ the vector field on P generated by $v \in \mathfrak{k}$ (see Definition 3.3). First we prove one identity that we'll need later on.

Proposition C.2. *For $k \in K$, $v \in \mathfrak{k}$ and $\sigma \in \Omega^*(P)$ it holds*

$$\psi_k^* \iota_{v^\#} \sigma = \iota_{(\psi_{k^{-1}})_* v^\#} \psi_k^* \sigma.$$

Proof.

$$\begin{aligned}
\iota_{(\psi_{k^{-1}})_* v^\#} \psi_k^* \sigma_p(\cdot) &= (\psi_k^* \sigma)_p \big((\psi_{k^{-1}})_* v^\#(p), \cdot \big) \\
&= \sigma_{kp} \big((d\psi_k)_p (d\psi_{k^{-1}})_{kp} v^\#(p), (d\psi_k)_p \,\cdot \big) \\
&= \sigma_{kp} \big(v^\#(kp), (d\psi_k)_p \,\cdot \big) \\
&= \iota_{v^\#} \sigma_{kp} \big(d\psi_k)_p \,\cdot \big) \\
&= \psi_k^* \iota_{v^\#} \sigma_p(\cdot).
\end{aligned}$$

□

Now, recall from Definition A.1 that a *connection form* on P is a $\mathfrak{k}$-valued 1-form on P satisfying

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[15] I'm very grateful to Prof. Ana Cannas da Silva for explaining this relation to me.

(i) $\iota_{v^\#}\theta = v \quad$ for all $v \in \mathfrak{g}$.

(ii) $\psi_k^*\theta = \mathrm{Ad}_k \circ \theta$ for all $k \in K$.

If we write $\theta = \sum_{i=1}^n \theta^i \otimes v_i$, then the 1-forms $\theta^j \in \Omega^1(P)$ must satisfy

(i) $\iota_{v_i^\#}\theta^j = \delta_i^j$.

(ii) $\psi_k^*\theta^j = a_i^j(k)\theta^i$.

The *curvature form* (associated to the connection θ) is the $\mathfrak{k}$-valued 2-form u on P given by

$$u \coloneqq d\theta - \frac{1}{2}\left[\theta \wedge \theta\right],$$

where $\left[\theta \wedge \theta\right]_p(v, w) = 2\left[\theta_p(v), \theta_p(w)\right]$ for $v, w \in T_pP$.

Let $\{c_{ij}^k\}$ denote the structure constants of $\mathfrak{k}$. If we write $u = \sum_{i=1}^n u^i \otimes v_i$, then the 2-forms $u^j \in \Omega^2(P)$ satisfy two main relations.

Lemma C.3 (First structure equation)**.** *The connection 2-forms u^j satisfy*

$$u^j = d\theta^j - \frac{1}{2}c_{i\ell}^j\theta^i \wedge \theta^\ell. \tag{C.2}$$

Proof. We first express $[\theta \wedge \theta]$ with respect to the basis v_i in $\Omega(P) \otimes \mathfrak{k}$. Let $p \in P$ and $v, w \in T_pP$.

$$\begin{aligned}
\left[\theta \wedge \theta\right]_p(v, w) &= 2\left[\theta_p(v), \theta_p(w)\right] \\
&= 2\,\theta_p^i(v)\theta_p^\ell(w)\left[v_i, v_\ell\right] \\
&= 2\,\theta_p^i(v)\theta_p^\ell(w)c_{i\ell}^j v_j \\
&= \theta_p^i(v)\theta_p^\ell(w)c_{i\ell}^k v_j - \theta_p^i(v)\theta_p^\ell(w)c_{\ell i}^j v_j \\
&= \big(\theta_p^i(v)\theta_p^\ell(w) - \theta_p^\ell(v)\theta_p^i(w)\big)c_{i\ell}^j v_j \\
&= c_{i\ell}^j(\theta^i \wedge \theta^\ell)_p(v, w)v_j.
\end{aligned}$$

Thus $u = d\theta - \frac{1}{2}\left[\theta \wedge \theta\right] = \sum_j (d\theta^j - \frac{1}{2}c_{i\ell}^j\theta^i \wedge \theta^\ell) \otimes v_j$, which concludes the proof. ☐

Lemma C.4 (Second structure equation)**.** *The connection forms u^j satisfy*

$$du^j = -c_{i\ell}^j u^i \wedge \theta^\ell. \tag{C.3}$$

Proof.

$$\begin{aligned}
0 = dd\theta^j &\overset{(C.2)}{=} d(u^j + \frac{1}{2}c_{i\ell}^i\theta^i \wedge \theta^\ell) \\
&= du^j + \frac{1}{2}c_{i\ell}^j d\theta^i \wedge \theta^\ell - \frac{1}{2}c_{i\ell}^j\theta^i \wedge d\theta^\ell \\
&\overset{(C.2)}{=} du^j + \frac{1}{2}c_{i\ell}^j(u^i + \frac{1}{2}c_{pm}^i\theta^p \wedge \theta^m) \wedge \theta^\ell - \frac{1}{2}c_{i\ell}^j(\theta^i \wedge (u^\ell + \frac{1}{2}c_{pm}^\ell\theta^p \wedge \theta^m))
\end{aligned}$$

$$
\begin{aligned}
&= du^j + \frac{1}{2}c^j_{i\ell}u^i \wedge \theta^\ell + \frac{1}{2}c^j_{i\ell}c^i_{pm}\theta^p \wedge \theta^m \wedge \theta^\ell - \frac{1}{2}c^j_{i\ell}\theta^i \wedge u^\ell \\
&= du^j + c^j_{i\ell}u^i \wedge \theta^\ell,
\end{aligned}
$$

since $\frac{1}{2}c^j_{i\ell}c^i_{pm}\theta^p \wedge \theta^m \wedge \theta^\ell = 0$ by the Jacobi identity, as in the proof of Proposition 3.10. ☐

As explained in Appendix A, a connection form θ corresponds to a choice of an horizontal subspace $H_p := \ker \theta_p \subset T_pP$ for each $p \in P$, so that it holds

$$T_pP = H_p \oplus \ker d\pi_p.$$

For a k-form $\sigma \in \Omega^k(P)$ we call the restriction $\sigma|_{H_p \times \ldots \times H_p}$ the horizontal component of σ. Recall that $\ker d\pi_p = \{X^{\#}(p) \mid X \in \mathfrak{g}\}$. Thus for $\sigma \in \Omega^k(P)$ we have

$$\iota_{X^{\#}}\sigma = 0 \text{ for all } X \in \mathfrak{g} \text{ if and only if } \sigma = \sigma|_{H_p \times \ldots \times H_p}.$$

Since $\theta^\ell|_{H_p} \equiv 0$ for all ℓ, from Lemma C.4 it follows that $du^j|_{H_p \times H_p} \equiv 0$, that is, du^j has no horizontal component. Moreover, recall that $\iota_{v^{\#}}u = 0$ for all $v \in \mathfrak{k}$ (by Proposition A.2), and thus $u = u|_{H_p \times H_p}$ is purely horizontal.

The curvature forms are K-equivariant in the following sense.

Lemma C.5. *For all $k \in K$ and $i, j = 1, \ldots, n$ it holds $\psi_k^* u^j = a^j_i(k)u^i$.*

Proof. The identities $\mathrm{Ad}_k\,[v_m, v_p] = [\mathrm{Ad}_k v_m, \mathrm{Ad}_k v_p]$ and (C.1) together give $c^i_{mp}a^j_i(k) = a^i_m(k)a^l_p(k)c^j_{il}$ for all j, m, p. Thus we have

$$
\begin{aligned}
\psi_k^* u^j &\overset{(C.2)}{=} \psi_k^* d\theta^j - \frac{1}{2}c^j_{i\ell}(\psi_k^*\theta^i \wedge \psi_k^*\theta^\ell) \\
&= d\psi_k^*\theta^j - \frac{1}{2}c^j_{i\ell}a^i_m(k)a^\ell_p(k)(\theta^m \wedge \theta^p) \\
&= a^j_i(k)d\theta^i - \frac{1}{2}c^i_{mp}a^j_i(k)(\theta^m \wedge \theta^p) \\
&\overset{(C.2)}{=} a^j_i(k)u^i.
\end{aligned}
$$

☐

C.2 Characteristic classes

As in the previous section let $v_1, \ldots, v_n$ be a basis for $\mathfrak{k}$ and let $\xi^1, \ldots, \xi^n$ be a basis for $\mathfrak{k}^*$ dual to $v_1, \ldots, v_n$. Let $\pi : P \longrightarrow X$ be a principal K-bundle with connection θ and curvature u. Denote by $S^l(\mathfrak{k}^*)^K$ the polynomials in $\mathfrak{k}^*$ of homogeneous degree l which are invariant under the adjoint action of K. Let $p = p(\xi^1, \ldots, \xi^n)$ be a polynomial in $S^l(\mathfrak{k}^*)^K$. We consider $p(u) := p(u^1, \ldots, u^n) \in \Omega^{2l}(P)$.

Lemma C.6. *$p(u)$ is K-invariant, that is, $\psi_k^* p(u) = p(u)$ for all $k \in K$.*

Proof. Since $p(\xi^1, \dots, \xi^n)$ is invariant under the coadjoint action of K we have

$$\begin{aligned} p(\xi^1, \dots, \xi^n) &= \mathrm{Ad}^*_{k^{-1}} p(\xi^1, \dots, \xi^n) \\ &= p(\mathrm{Ad}^*_{k^{-1}} \xi^1, \dots, \mathrm{Ad}^*_{k^{-1}} \xi^n) \\ &= p(\xi^1 \circ \mathrm{Ad}_k, \dots, \xi^n \circ \mathrm{Ad}_k) \\ &\overset{(C.1)}{=} p(a^1_i(k)\xi^i, \dots, a^n_i(k)\xi^i). \end{aligned} \tag{C.4}$$

Therefore for $k \in K$ we have

$$\begin{aligned} \psi^*_k p(u^1, \dots, u^n) &= p(\psi^*_k u^1, \dots, \psi^*_k u^n) \\ &\overset{\text{Lemma } C.5}{=} p(a^1_i(k)u^i, \dots, a^n_i(k)u^i) \\ &\overset{(C.4)}{=} p(u^1, \dots, u^n). \end{aligned}$$

□

Lemma C.7. *$p(u)$ is horizontal, i.e. $\iota_{v^\#} p(u) = 0$ for all $v \in \mathfrak{k}$.*

Proof. This follows from the fact that each u^j is horizontal and the wedge of horizontal forms is horizontal. □

From Lemma C.6 and Lemma C.7 it follows that $p(u)$ is a basic form in $\Omega^{2l}(P)$. Thus it exists $\gamma_p \in \Omega^{2l}(X)$ with

$$\pi^* \gamma_p = p(u).$$

Lemma C.8. *γ_p is closed.*

Proof. It suffices to show $\pi^* d\gamma_p = 0$. On one hand $\pi^* d\gamma_p$ is horizontal because it is a basic element. On the other hand $\pi^* d\gamma_p = d\pi^*\gamma_p = dp(u)$ and since $du^i|_{H_p \times H_p} = 0$, also $dp(u)|_{H_p \times \ldots \times H_p} = 0$. Thus $dp(u)$ has no horizontal component and therefore $\pi^* d\gamma_p = 0$. □

Lemma C.9 ([27, Appendix C])**.** *If we change the connection θ, then γ_p changes only by an exact form.*

Proof. Let θ_0, $\theta_1 \in \Omega^1(P, \mathfrak{k})$ be two connection forms on P with curvature forms u_0 and u_1, respectively. Let $\gamma_0, \gamma_1 \in \Omega^{2l}(X)$ be such that $p(u_0) = \pi^*\gamma_0$ and $p(u_1) = \pi^*\gamma_1$. Consider the principal K-bundle $\pi \times 1 : P \times [0,1] \longrightarrow X \times [0,1]$, and let $\widetilde{\theta} \in \Omega^1(P \times [0,1]\,, \mathfrak{k})$ be the form given by

$$\widetilde{\theta}_{(p,s)} = (1-s)(\theta_0)_p + s(\theta_1)_p.$$

This is a connection form on $P \times [0,1]$. Let $\widetilde{u}$ be its curvature form. For $\epsilon \in \{0,1\}$, denote by $\iota^P_\epsilon : P \longrightarrow P \times [0,1]$ the inclusion $x \longmapsto (x, \epsilon)$. Moreover, let $\gamma \in \Omega^{2l}(X \times [0,1])$ be such that $p(\widetilde{u}) = (\pi \times 1)^*\widetilde{\gamma}$.

From $(\iota^P_\epsilon)^*\widetilde{\theta} = \theta_\epsilon$, it follows that $(\iota^P_\epsilon)^*\widetilde{u} = u_\epsilon$. Therefore for $\epsilon = 0, 1$ it holds

$$(\iota^P_\epsilon)^* p(\widetilde{u}) = p((\iota^P_\epsilon)^*\widetilde{u}) = p(u_\epsilon) = \pi^*\gamma_\epsilon.$$

We claim that $(\iota_\epsilon^X)^*\widetilde{\gamma} = \gamma_\epsilon$. It suffices to show $\pi^*(\iota_\epsilon^X)^*\widetilde{\gamma} = \pi^*\gamma_\epsilon$. In fact, we have

$$\pi^*(\iota_\epsilon^X)^*\widetilde{\gamma} = (\iota_\epsilon^X \circ \pi)^*\widetilde{\gamma} = (\iota_\epsilon^P)^*(\pi \times 1)^*\widetilde{\gamma} = (\iota_\epsilon^P)^* p(\widetilde{u}) = \pi^*\gamma_\epsilon.$$

This proves the claim. Now, to see that $[\gamma_0] = [\gamma_1]$, notice that ι_0^X and $\iota_1^X : X \longrightarrow X \times [0,1]$ are homotopic maps and thus induce the same map in cohomology. Therefore

$$[\gamma_0] \overset{\text{Claim}}{=} \left[(\iota_0^X)^*\widetilde{\gamma}\right] = (\iota_0^X)^*\left[\widetilde{\gamma}\right] = (\iota_1^X)^*\left[\widetilde{\gamma}\right] = \left[(\iota_1^X)^*\widetilde{\gamma}\right] \overset{\text{Claim}}{=} [\gamma_1].$$

□

By Lemma C.9 we can associate to each polynomial p in $(S^l\mathfrak{k}^*)^K$ a cohomology class $[\gamma_p] \in H^{2l}(X)$ and this assignment doesn't depend on the choice of the connection θ on P.

Definition C.10. The *characteristic class* of a principal K-bundle $P \longrightarrow X$ corresponding to $p \in (S^l\mathfrak{k}^*)^K$ is $[\gamma_p] \in H^{2l}(X)$.

Now consider $K = SO(2n)$. The Euler class of a principal $SO(2n)$-bundle is defined as the characteristic class corresponding to a particular polynomial called the Pfaffian. Recall that the Lie algebra $\mathfrak{k} = \mathfrak{so}(2\mathfrak{n})$ consists of all $2n \times 2n$ skew-symmetric matrices, i.e. matrices $A = (a_{ij})_{ij}$ with $a_{ij} = -a_{ji}$.

Definition C.11. The *Pfaffian* Pf is the homogeneous polynomial of degree n in $\mathfrak{so}(2\mathfrak{n})^*$ which for $A = (a_{ij}) \in \mathfrak{so}(2\mathfrak{n})$ is given by

$$\mathrm{Pf}(A) := \frac{1}{2^n n!}\frac{1}{(2\pi)^n}\sum_\sigma \mathrm{sgn}(\sigma) a_{\sigma_1\sigma_2}\dots a_{\sigma_{2n-1}\sigma_{2n}},$$

where the sum is taken over all permutations $\sigma \in S_{2n}$ of the set $\{1,\dots,2n\}$.

Lemma C.12 ([16, §4, Example 3])**.** *Pf is K-invariant.*

Proof. We want to show that for all $k \in SO(2n)$ it holds $\mathrm{Ad}_k^*\mathrm{Pf} = \mathrm{Pf}$. Since K is a matrix Lie group, the adjoint representation is given by conjugation: $\mathrm{Ad}_k(A) = kAk^{-1}$. Therefore we claim that

$$\mathrm{Pf}(kAk^{-1}) = \mathrm{Pf}(A) \quad \text{for all } A \in \mathfrak{so}(2\mathfrak{n}),\ k \in SO(2n).$$

Notice first that if $k = (x_{ij}) \in SO(2n)$ then $kAk^{-1} = kAk^\intercal = A'$, where $A' = (a'_{ij})$ is given by

$$a'_{ij} = \sum_{\ell_1,\ell_2} x_{i\ell_1}a_{\ell_1\ell_2}x_{j\ell_2}.$$

Thus,

$$\mathrm{Pf}(A') = \frac{1}{2^n n!}\sum_{\ell_1,\dots,\ell_{2n}} a_{\ell_1\ell_2}\dots a_{\ell_{2n-1}\ell_{2n}} \underbrace{\sum_\sigma \mathrm{sgn}(\sigma)x_{\sigma_1\ell_1}x_{\sigma_2\ell_2}\dots x_{\sigma_{2n-1}\ell_{2n-1}}x_{\sigma_{2n}\ell_{2n}}}_{=\det(x_{i\ell_j})_{ij}}.$$

If $(\ell_1, \ldots, \ell_{2n})$ is not a permutation of $(1, \ldots, 2n)$, then at least two columns of the matrix $(x_{i\ell_j})_{ij}$ are the same, and so the determinant of the matrix is zero. If $(\ell_1, \ldots, \ell_{2n})$ is a permutation of $(1, \ldots, 2n)$, then the matrix $(x_{i\ell_j})_{ij}$ is obtained from (x_{ij}) by permuting the columns and since $\det(x_{ij}) = 1$, its determinant is the sign of the permutation. Thus

$$\mathrm{Pf}(A') = \frac{1}{2^n n!(2\pi)^n} \sum_{\eta \in S_{2n}} a_{\eta_1\eta_2} \ldots a_{\eta_{2n-1}\eta_{2n}} \mathrm{sgn}(\eta) = \mathrm{Pf}(A).$$

□

Definition C.13. The *Euler class* of a $SO(2n)$-bundle $\pi : P \longrightarrow X$ is the characteristic class $[\gamma_p]$ corresponding to $p = \mathrm{Pf} \in (S^n \mathfrak{so}(\mathfrak{n})^*)^{SO(2n)}$.

Oriented vector bundles of dimension $2n$ (with a Riemannian metric on the total space) correspond bijectively to principal $SO(2n)$-bundles, the bijection is given by assigning to a vector bundle its oriented orthonormal frame bundle (see [17, §2 and §3]). Thus, given an oriented $2n$-dimensional vector bundle its Euler class is defined as the Euler class of its associated frame bundle. To see that this definition of Euler class and the one we gave in Appendix B.7 agree, one shows that both satisfy a list of properties and that these properties determine their cohomology classes, see [16, §7 and §8].

C.3 Equivariant characteristic classes

A before, let $\pi : P \longrightarrow X$ be a principal K-bundle, with $K = SO(n)$. Suppose now that we have a compact connected Lie group G acting as bundle morphism of π, that is, G acts on P and X and π is G-equivariant. Furthermore suppose that the G- and K-actions on P commute (both are left actions). Denote by ϕ_g the G-action and by ψ_k the K-action.

Let $X_1, \ldots, X_m$ be a basis of the Lie algebra $\mathfrak{g}$ and let $z^1, \ldots, z^m$ be the dual basis in $S^1(\mathfrak{g}^*)$. As before, $v_1, \ldots, v_n$ is a basis for $\mathfrak{k}$ and $\xi^1, \ldots, \xi^n$ is the dual basis for $\mathfrak{k}^*$.

We consider the chain complex

$$\widetilde{\Omega}_H(P) := (\mathbb{R}\left[z^1, \ldots, z^m\right] \otimes \Omega_H(P))^G$$

with the differential

$$\widetilde{d}_H := 1 \otimes d_H + z^j \otimes \iota_{X_j^\#},$$

where $\Omega_H(P)$ are the horizontal forms on P and $d_H :=$ (proj. to horiz.) $\circ\, d$ is a differential on $\Omega_H(P)$. Recall that the superscript $(\cdot)^G$ denotes the G-invariance with respect to the action given by $\mathrm{Ad}_g^* \otimes \phi_{g^{-1}}^*$.

Choose a G-invariant connection form θ on the K-bundle $P \longrightarrow X$. This can be done by first choosing any connection for $P \longrightarrow X$ (which exists by Proposition A.8) and then averaging it over the compact group G. We denote the connection forms as before by $\theta^1, \dots, \theta^n \in \Omega^1(P)$ with corresponding curvature forms $u^1, \dots, u^n \in \Omega^2(P)$. Since we chosed θ to be G-invariant it holds

$$\phi_g^* \theta^j = \theta^j \quad \text{and} \quad \phi_g^* u^j = u^j \quad \text{for all } g \in G, j = 1, \dots n.$$

Notice that for each $v_j \in \mathfrak{k}$ it holds $\iota_{v_j^\#}\theta^i = \delta_i^j$, but we don't have any condition on the interior product by the vector fields generated by the G-action. So we can define the *moment functions* $\phi_j^i \in C^\infty(P)$ as

$$\phi_j^i := \iota_{X_j^\#}\theta^i.$$

Lemma C.14. *The moment functions are G-equivariant, in the sense that for all $g \in G$,*

$$(\mathrm{Ad}_g^* \otimes \phi_{g^{-1}}^*)(z^j \otimes \phi_j^i) = z^j \otimes \phi_j^i.$$

Proof. For $g \in G$, let $c_i^k(g) \in \mathbb{R}$ be such that

$$\mathrm{Ad}_g X_i = c_i^k(g) X_k.$$

Then $\mathrm{Ad}_g^* z^j = c_k^j(g^{-1}) z^k$.

Claim. $c_k^j(g^{-1}) c_i^k(g) = \delta_i^j$ and $c_k^j(g) c_i^k(g^{-1}) = \delta_i^j$.

Proof. It holds $(\mathrm{Ad}_g^* z^j)(\mathrm{Ad}_g X_i) = \delta_i^j$. On the other hand

$$(\mathrm{Ad}_g^* z^j)(\mathrm{Ad}_g X_i) = (c_k^j(g^{-1}) z^k)(c_i^l(g) X_l) = c_k^j(g^{-1}) c_i^l(g) z^k(X_l) = c_k^j(g^{-1}) c_i^k(g).$$

Analogously, $\delta_i^j = (\mathrm{Ad}_{g^{-1}}^* z^j)(\mathrm{Ad}_{g^{-1}} X_i) = c_k^j(g) c_i^k(g^{-1})$. ☐ Claim.

Notice that

$$\begin{aligned}
\phi_{g^{-1}}^* \phi_j^i &= \phi_{g^{-1}}^*(\iota_{X_j^\#}\theta^i) \\
&\overset{\text{Prop. } C.2}{=} \iota_{(\phi_g)_* X_j^\#} \phi_{g^{-1}}^* \theta^i \\
&\overset{G\text{-inv.}}{=} \iota_{(\phi_g)_* X_j^\#} \theta^i \\
&\overset{\text{Prop. } A.3}{=} \iota_{(\mathrm{Ad}_g X_j)^\#} \theta^i \\
&= c_j^l(g) \iota_{X_l^\#} \theta^i \\
&= c_j^l(g) \phi_l^i.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\mathrm{Ad}_g^* z^j \otimes \phi_{g^{-1}}^* \phi_j^i &= c_m^j(g^{-1}) z^m \otimes c_j^l(g) \phi_l^i \\
&= z^m \otimes c_j^l(g) c_m^j(g^{-1}) \phi_l^i \\
&\overset{\text{Claim}}{=} z^m \otimes \phi_m^i.
\end{aligned}$$

☐

In the following we mostly omit the symbol $\otimes$.

Definition C.15. The *equivariant curvature forms* are

$$\widetilde{u}^i := u^i + z^j \phi^i_j \ \in \mathbb{R}\left[z^1, \dots, z^m\right] \otimes \Omega^*(P),$$

for $i = 1, \dots, n$.

Proposition C.16. *The equivariant curvature forms $\widetilde{u}^i$ satisfy*

(i) $\widetilde{u}^i \in \widetilde{\Omega}^2_H(P)$.

(ii) $\widetilde{d}_H \widetilde{u}^i = 0$.

(iii) *For all $k \in K$: $\psi_k^* \widetilde{u}^i = a^i_j(k) \widetilde{u}^j$.*

Proof. We prove (i). The curvature forms u^i are horizontal and ϕ^i_j are functions, thus horizontal as well. Therefore $\widetilde{u}^i \in \mathbb{R}\left[z^1, \dots, z^m\right] \otimes \Omega_H(P)$. Moreover, all $\widetilde{u}^i$ are G-invariant, because the u^i's are and by Lemma C.14 the $z^j\phi^i_j$'s are as well.

We prove (ii). We compute

$$\begin{aligned}
\widetilde{d}_H \widetilde{u}^i &= \widetilde{d}_H u^i + \widetilde{d}_H (z^j \phi^i_j) \\
&= \underbrace{d_H u^i}_{=0} + z^j \iota_{X_j^\#} u^i + z^j d_H \phi^i_j + z^l z^i \underbrace{\iota_{X_l^\#} \phi^i_j}_{=0} \\
&= z^j \iota_{X_j^\#} u^i + z^j \cdot (\text{proj. to hor.}) \circ d\phi^i_j,
\end{aligned}$$

where $d_H u^i = 0$ because du^i has no horizontal component. Now we compute

$$\begin{aligned}
d\phi^i_j = d\iota_{X_j^\#} \theta^i = \ &\underset{\substack{=0 \text{ since } \theta^i \\ \text{is } G\text{-invariant}}}{\mathcal{L}_{X_j^\#} \theta^i} - \iota_{X_j^\#} d\theta^i \\
&\overset{(C.2)}{=} -\iota_{X_j^\#} \left(\frac{1}{2} c^i_{kl} \theta^k \wedge \theta^l + u^i\right) \\
&= -\frac{1}{2} c^i_{kl} \iota_{X_j^\#} (\theta^k \wedge \theta^l) - \iota_{X_j^\#} u^i \\
&= -\frac{1}{2} c^i_{kl} \phi^k_j \cdot \theta^l + \frac{1}{2} c^i_{kl} \theta^k \cdot \phi^l_j - \iota_{X_j^\#} u^i \\
&= -\frac{1}{2} c^i_{kl} \phi^k_j \theta^l - \frac{1}{2} c^i_{kl} \phi^k_j \theta^l - \iota_{X_j^\#} u^i \\
&= -c^i_{kl} \phi^k_j \theta^l - \iota_{X_j^\#} u^i.
\end{aligned}$$

Since $c^i_{kl} \phi^k_j \theta^l$ vanishes on horizontal vectors and u^i is horizontal, it follows $d_H \phi^i_j = -\iota_{X_j^\#} u^i$ and thus $\widetilde{d}_H \widetilde{u}^i = 0$.

We prove (iii). The K-action on $\mathbb{R}\left[z^1, \dots, z^n\right] \otimes \Omega_H(P)$ is given by the identity in the first component and by pullback in the second component. Notice that for $k \in K$ by equation (3.27) we have $\psi_k^* \iota_{X_j^\#} \theta^i = \iota_{X_j^\#} \psi_k^* \theta^i$, because since the K and G actions commute, ψ_k is G-equivariant. Thus

$$\psi_k^* \phi_j^i = \psi_k^* \iota_{X_j^\#} \theta^i = \iota_{X_j^\#} \psi_k^* \theta^i = \iota_{X_j^\#} (a_\ell^i(k) \theta^\ell) = a_\ell^i(k) \phi_j^\ell .$$

So for $k \in K$ we compute

$$\begin{aligned} \psi_k^* \widetilde{u}^i \overset{\text{def}}{=} \psi_k^* u^i + z^j \cdot \psi_k^* \phi_j^i \overset{\substack{\text{Lemma}\\ C.5}}{=} & \, a_j^i(k) u^j + z^j \cdot a_\ell^i(k) \phi_j^\ell \\ = & \, a_j^i(k)(u^j + z^l \phi_l^j) \\ = & \, a_j^i(k) \widetilde{u}^j . \end{aligned}$$

□

Now let $p \in (S^l \mathfrak{k}^*)^K$, $p = p(\xi^1, \ldots, \xi^n)$ as before. Then $p(\widetilde{u}) \coloneqq p(\widetilde{u}^1, \ldots, \widetilde{u}^n)$ is in $\widetilde{\Omega}_H^{2l}(P)$ since all $\widetilde{u}^i$ are.

Lemma C.17. *$p(\widetilde{u})$ is K-invariant.*

Proof. By Proposition C.16 (iii) the equivariant curvature forms $\widetilde{u}^i$ satisfy the same K-equivariant law as the u^i. Thus the lemma follows as in the non-equivariant case (Lemma C.6). □

It follows that $p(\widetilde{u})$ is basic (horizontal and K-invariant) and thus there is $\widetilde{\gamma}_p \in \widetilde{\Omega}(X) \coloneqq \mathbb{R}\left[z^1, \ldots, z^m\right] \otimes \Omega^*(X)$ with $\pi^* \widetilde{\gamma}_p = p(\widetilde{u})$.

Remark C.18. Here the reasoning is as follows. Let $\{f_\alpha\}_\alpha$ be a basis for $\mathbb{R}\left[z^1, \ldots, z^m\right]$, so that we can write $p(\widetilde{u}) = \sum_\alpha f_\alpha \otimes \omega_\alpha$ for some $\omega_\alpha \in \Omega_H(P)$. Since $p(\widetilde{u})$ is in $\widetilde{\Omega}_H(P)$, all the ω_α are indeed horizontal. By K-invariance of $p(\widetilde{u})$ it holds

$$\sum_\alpha f_\alpha \otimes \omega_\alpha = p(\widetilde{u}) = \psi_k^* p(\widetilde{u}) = \sum_\alpha f_\alpha \otimes \psi_k^* \omega_\alpha$$

and thus $\psi_k^* \omega_\alpha = \omega_\alpha$ for all α. Thus ω_α is basic for all α and there are $\sigma_\alpha \in \Omega^*(X)$ with $\pi^* \sigma_\alpha = \omega_\alpha$ for all α. It follows that $\widetilde{\gamma}_p \coloneqq \sum_\alpha f_\alpha \otimes \sigma_\alpha$ is an element of $\widetilde{\Omega}(X)$ with $\pi^* \widetilde{\gamma}_p = p(\widetilde{u})$.

Lemma C.19. *$\widetilde{\gamma}_p$ is G-invariant.*

Proof. By injectivity of π^* it suffices to show that $\pi^*(\phi_g^* \widetilde{\gamma}_p) = \pi^* \widetilde{\gamma}_p$. Since G acts as a bundle morphism and since $p(\widetilde{u})$ is G-invariant we have

$$\pi^* \phi_g^* \widetilde{\gamma}_p = \phi_g^* \pi^* \widetilde{\gamma}_p = \phi_g^* p(\widetilde{u}) = p(\widetilde{u}) = \pi^* \widetilde{\gamma}_p .$$

□

From the lemma it follows that $\widetilde{\gamma}_p$ is an element of the Cartan complex $C_G(X) = (\mathbb{R}\left[z^1, \ldots, z^m\right] \otimes \Omega^*(X))^G$. In order for it to represent an element of the equivariant cohomology $H_G^*(X)$ it must be closed with respect to the Cartan differential.

Lemma C.20. *$\widetilde{\gamma}_p$ is closed with respect to the Cartan differential d_G on X.*

Proof. Notice first that $\widetilde{d}_H p(\widetilde{u}) = 0$, because $\widetilde{d}_H p(\widetilde{u})$ is a sum in which every term contains at least one $\widetilde{d}_H \widetilde{u}^i$ and $\widetilde{d}_H \widetilde{u}^i = 0$ by Proposition C.16 (ii). It suffices to show that $\pi^* d_G \widetilde{\gamma}_p = 0$. Let d_G^P denote the Cartan differential on P and notice that it differs by $\widetilde{d}_H$ just by the projection to the horizontal differential forms in the first term. Since $d\pi^* \widetilde{\gamma}_p$ is horizontal (because it is basic!), it holds $\widetilde{d}_H \pi^* \widetilde{\gamma}_p = d_G^P \pi^* \widetilde{\gamma}_p$. Thus we have

$$\pi^* d_G \widetilde{\gamma}_p = d_G^P \pi^* \widetilde{\gamma}_p = \widetilde{d}_H \pi^* \widetilde{\gamma}_p = \widetilde{d}_H p(\widetilde{u}) = 0.$$

□

Therefore $\widetilde{\gamma}_p$ represents an element in equivariant cohomology and as in the non-equivariant case its class doesn't depend on the choice of the connection.

Definition C.21. The *equivariant characteristic class* of a principal K-bundle $P \longrightarrow X$ associated to the polynomial $p \in (S^l \mathfrak{k}^*)^K$ is $[\widetilde{\gamma}_p] \in H_G^{2l}(X)$.

Now suppose that we have an oriented G-vector bundle $E \longrightarrow X$ of rank $2n$, with a family of G-invariant metrics on each fiber E_x. Then we can consider its oriented orthonormal frame bundle $F_{SO}(E) \longrightarrow X$. Recall that

$$F(E) := \bigsqcup_{x \in X} F(E_x),$$

where for a vector space V the space $F(V)$ is defined as the set of all bases for V. The subscript F_{SO} indicates that we consider the set of all orthonormal positively oriented bases. By Remark B.7, G acts by orientation preserving vector space isomorphisms on the fibers, thus we have an action of G on $F_{SO}(E)$. With this action $\pi : F_{SO}(E) \longrightarrow X$ is G-equivariant. Moreover, $F_{SO}(E) \longrightarrow X$ is a principal $SO(2n)$-bundle (see [17]).

We define the equivariant Euler class of the oriented G-vector bundle $E \longrightarrow X$ as the equivariant Euler class of its oriented orthogonal frame bundle.

Definition C.22. The *equivariant Euler class* of an oriented G-vector bundle $E \longrightarrow X$ is the equivariant characteristic class of the frame bundle $F_{SO}(E) =: P \longrightarrow X$ associated to the polynomial $p = \mathrm{Pf} \in (S^n \mathfrak{so}(\mathfrak{n})^*)^{SO(2n)}$.

C.4 The equivariant Euler class of a S^1-vector bundle over a point

The aim of this section is to show that the equivariant Euler class of an S^1-vector bundle $L \longrightarrow \{x\}$ of rank 2 is the weight of the S^1-action on L (see Section 6.4). Suppose that we have a 2-dimensional oriented S^1-vector bundle $L \longrightarrow \{x\}$, so that $L \cong \mathbb{R}^2$. By definition of S^1-vector bundle, S^1 acts linearly on $L = \mathbb{R}^2$ and thus the action is described by a representation $S^1 \longrightarrow GL_1(L)$. This is given by

$$\begin{aligned} \rho :\ S^1 &\longrightarrow GL(\mathbb{R}^2) \\ e^{it} &\longmapsto \begin{pmatrix} \cos(mt) & \sin(mt) \\ -\sin(mt) & \cos(mt) \end{pmatrix}, \end{aligned}$$

for some integer $m \in \mathbb{Z}$, which we call the weight of the action. In order to talk about the equivariant Euler class of this vector bundle we consider its orthonormal frame bundle $P \longrightarrow \{x\}$, which is a principal $SO(2)$-bundle on which S^1 acts by the same weights.

Let φ be an angle variable on

$$P \cong SO(2) = \Big\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \Big\}.$$

Moreover, let $v = 1 \in \mathfrak{so}(2)$ be the generator of the Lie algebra of $SO(2)$. Then we have $v^{\#} = \frac{\partial}{\partial\varphi}$. Let $X = i \in \mathfrak{g} \cong i\mathbb{R}$ generator of the Lie algebra. Then the exponential map is given by $\exp_{S^1}(\lambda X) = e^{i\lambda}$ and thus

$$X^{\#}(p) = \frac{d}{dt}\Big|_{t=0} \phi_{\exp tX}(p) = \frac{d}{dt}\Big|_{t=0} e^{imt}p = imp = m\frac{\partial}{\partial\varphi}(p).$$

Let $\theta := d\varphi \in \Omega^1(P)$. This is a connection form on the $SO(2)$-bundle $P \longrightarrow \{x\}$. Indeed, on one hand we have

$$\iota_{v^{\#}}\theta = d\varphi(\frac{\partial}{\partial\varphi}) = 1.$$

On the other hand, the $SO(2)$-equivariance condition for a connection is actually $SO(2)$-invariance, since the Lie algebra $\mathfrak{so}(2)$ has trivial Lie bracket. Since $SO(2)$ is connected, the infinitesimal invariance $\mathcal{L}_{v^{\#}} = 0$ suffices. This condition is satisfied, since

$$\mathcal{L}_{v^{\#}}\theta = d\underbrace{\iota_{v^{\#}}\theta}_{=1} + \iota_{v^{\#}}\underbrace{d\theta}_{=d^2\varphi} = 0.$$

Since $SO(2)$ is connected, the infinitesimal invariance $\mathcal{L}_{v^{\#}}$ suffices. The moment function is given by $\phi = \iota_{X^{\#}}\theta = m$. Thus if $u \in \Omega^2(P)$ is the curvature form and $z \in \mathfrak{g}^*$ is dual to X, the equivariant curvature form is given by

$$\widetilde{u} = u + mz.$$

For $SO(2)$ the Pfaffian is given by

$$\mathrm{Pf}(\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}) = \frac{1}{2}\frac{1}{2\pi}(a_{12} - a_{21}) = \frac{-1}{2\pi}\lambda.$$

Thus $\mathrm{Pf}(\widetilde{u}) = \frac{-1}{2\pi}(u + mz)$. Since $SO(2)$ is abelian, the curvature u is a basic form (see Remark A.3), so it comes from a differential form in $\Omega^2(\{x\}) = 0$ and therefore $u = 0$. It follows that $\widetilde{\gamma}_{\mathrm{Pf}} = \frac{-1}{2\pi}mz \in \Omega^2_C(\{x\})$ and thus

$$e_{S^1}(P) = \frac{-1}{2\pi}mz \in H^2_{S^1}(\{x\}) \cong S^1\mathfrak{g}^* = \mathbb{R}\left[z\right].$$

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