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Towards Deformations of Toric Lagrangians in Weinstein Neighborhoods

Master Thesis

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Abstract

We give an introduction to symplectic geometry, focusing on Weinstein's tubular neighborhood theorem and Hamiltonian G-spaces. We then construct an example of toric Lagrangian and view deformations of it as images of one-forms in a Weinstein tubular neighborhood. Finally, we present a proof of the equivariant version of Weinstein's tubular neighborhood and use it to investigate what kind of deformations are possible for toric Lagrangians. We show that toric Lagrangians close enough to the zero section in a Weinstein neighborhood are images of invariant one-forms and we make the first steps towards the proof of a converse statement.

Contents

Introduction

Symplectic geometry originates from classical mechanics, where it was developed as a convenient tool to describe physical systems efficiently. Since then, it has become a mathematical subject of its own through the work of many important mathematicians like Arnold, Gromov, Weinstein or Floer to name a few. In very general terms, symplectic geometry is the study of symplectic manifolds, which are manifolds equipped with a closed, non-degenerate two-form, generally denoted ω. One of the first consequences of this structure is that every symplectic manifold M must be even-dimensional. One can then see that there are submanifolds of M on which ω vanishes. These submanifolds have dimension at most half the dimension of M. A submanifold of maximal dimension on which ω vanishes is called a Lagrangian submanifold. Lagrangian submanifolds are extremely useful in symplectic geometry and can often give useful information about the manifold that contains them and the dynamics on it. Weinstein proved that every Lagrangian submanifold L has a neighborhood that is diffeomorphic to a neighborhood of the zero section in T^*L in such a way that this diffeomorphism respects the symplectic structure on M and the natural symplectic structure that exists on every cotangent bundle.

In this thesis, we will look specifically at symplectic toric manifolds, which are symplectic manifolds on which there is a Lie group action by the torus $\mathbb{T}^n$ that behaves nicely with respect to the symplectic structure. Some of the Lagrangian submanifolds of symplectic toric manifolds are preserved by subgroups of the torus $\mathbb{T}^n$, which we can view as some kind of symmetry of the Lagrangian submanifolds. These submanifolds are called toric Lagrangians and will be the main subject of the last two chapters of this thesis. We will first present a proof of the equivariant version of Weinstein's theorem and then use it to view toric Lagrangians inside a cotangent bundle and investigate what kind of small deformation are possible for toric Lagrangians in this setting.

We present how this thesis is organized here. Chapter 1 is a basic introduction to symplectic geometry, where we define symplectic manifolds and Lagrangian submanifolds precisely and give examples. We also prove some foundational results of symplectic geometry, finishing with the proof of Weinstein's tubular neighborhood theorem 1.53, which we will use in later chapters to study toric Lagrangians.

Chapter 2 is a short introduction to the notions from complex geometry that are needed in the context of symplectic geometry.

Chapter 3 is about Hamiltonian G-spaces, i.e. symplectic manifolds with a Lie group action on them. We give some important properties of Hamiltonian G-spaces and define the moment map of such a space. In the second part of this chapter, we prove the Marsden-Meyer-Weinstein theorem 3.10, which gives conditions for the orbit space of a Hamiltonian G-space to still be a symplectic manifold. The first three chapters mostly follow Cannas' presentation in [5], with some additions from [13] and are an exposition of basic results from symplectic geometry. We try to make this introduction to symplectic geometry as self-contained as possible by providing most of the proofs in detail.

In Chapter 4, we construct an explicit example of toric Lagrangian and look at some of its properties.

Chapter 5 gives a precise definition of toric Lagrangians and present a very detailed proof of the equivariant version of Weinstein's tubular neighborhood theorem. We finish this work by proving that toric Lagrangians close enough to the zero section in a Weinstein neighborhood are images of invariant one-forms and by making some steps towards the proof of the converse statement.

This thesis assumes the reader to be familiar with standard differential topology and geometry, although we give a proof of some important result. Familiarity with some Lie theory is also useful. This text should be accessible to any pure mathematics student at the end of their undergraduate studies. We assume every manifold to be smooth, Hausdorff and second countable and every map to be smooth unless otherwise specified. Manifolds are usually denoted by M, their dimension by n and points in a manifold by p or x. For a smooth map φ from a manifold M_1 to a manifold M_2, we denote the derivative at $x \in M_1$ by

$$D\varphi(x) : T_x M_1 \to T_{\varphi(x)} M_2.$$

The algebra of k-forms on a manifold M is denoted by $\Omega^k(M)$, and the exterior derivative of a function $f \in C^\infty(M)$ is denoted by df. The notation conventions used are mostly the same as in [14], while trying to follow [5] as well where it is possible.

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Chapter 1

Symplectic vector spaces and manifolds

Symplectic geometry studies the properties of a special kind of manifolds, the so-called symplectic manifolds. A symplectic manifold is a manifold equipped with a non-degenerate two-form. This additional structure gives rise to an incredibly rich theory and to a variety of different topics. The aim of the first three chapters of this text is to define some basic notions and prove some important results that will allow us to investigate toric Lagrangians and some of their properties. Most of the time, we will denote a manifold by M, and the non-degenerate two-form that make it a symplectic manifold by ω. For a point p in M, we have that $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate skew-symmetric bilinear map. For this reason, studying bilinear maps with such properties is a good way to introduce the topic of symplectic geometry. The first section of this chapter introduces basic notions of symplectic linear algebra. We introduce symplectic manifolds in the second section and state and prove Weinstein's tubular neighborhood theorem in the third section. Unless otherwise specified, this chapter follows Cannas' lecture notes on symplectic geometry [5], more specifically lectures 1 to 3, 8 and 9.

1.1 Symplectic linear algebra

Definition 1.1. Let V be a real vector space of dimension n. A bilinear map $\Omega : V \times V \to \mathbb{R}$ is called *skew-symmetric* if for every u and v in V, we have $\Omega(u, v) = -\Omega(v, u)$.

Any vector space V equipped with a skew-symmetric bilinear map Ω admits a basis that has useful properties with respect to Ω, as the next theorem shows. This basis is typically not unique, although it is sometimes referred to as a canonical basis in the context of symplectic geometry.

Theorem 1.2. *Let V be an n-dimensional real vector space, and Ω a skew-symmetric bilinear map on V. Then there exists a basis $u_1, \ldots, u_k, e_1, \ldots, e_m, f_1, \ldots, f_m$ such that:*

$$\begin{aligned} &\Omega(u_i, v) = 0 \quad \forall i, \forall v \in V \\ &\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j) \quad \forall i, j \\ &\Omega(e_i, f_j) = \delta_{ij} \quad \forall i, j \end{aligned}$$

We follow the proof of theorem 1.1 in [5].

Proof. The method of this proof is similar to the Gram-Schmidt algorithm, except that we have a skew-symmetric map instead of a scalar product.
Let

$$U := \{u \in V \big| \ \Omega(u, v) = 0, \ \forall v \in V\}.$$

Choose a basis $u_1, ..., u_k$ of U and let W be a complement to U in V so that we have $U \oplus W = V$. Take $0 \neq e_1 \in W$. Then since e_1 is not in U, there exists $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. Using the bilinearity of Ω, we can assume without loss of generality that $\Omega(e_1, f_1) = 1$.
We now define W_1 to be the span of e_1 and f_1 and set

$$W_1^\Omega := \{w \in W \big| \Omega(w, v) = 0, \forall v \in W_1\}.$$

We claim that $W_1 \cap W_1^\Omega = \{0\}$. Let $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$. We have

$$\begin{aligned} 0 &= \Omega(v, e_1) = a \\ 0 &= \Omega(v, f_1) = -b \end{aligned}$$

This shows that $v = 0$.
Furthermore, we have that $W_1 + W_1^\Omega = W$. Indeed, for v in W, let us write $c := \Omega(v, e_1)$ and $d := \Omega(v, f_1)$. Then, v can be decomposed as

$$v = \underbrace{-cf_1 + de_1}_{\in W_1} + \underbrace{v + cf_1 - de_1}_{\in W_1^\Omega}.$$

We thus have that $W_1 \oplus W_1^\Omega = W$.
We can now repeat this process: take e_2 in W_1^Ω a non-zero vector, and find f_2 such that $\Omega(e_2, f_2) = 1$. Define W_2 to be the span of e_2 and f_2 and carry on in the same way. Since V has finite dimension, this process must stop eventually with the decomposition

$$V = U \oplus W_1 \oplus W_m$$

and the basis $u_1, ..., u_k, e_1, ..., e_m, f_1, ..., f_m$ which has the desired properties. ■

Remark 1.3. The dimension of the null-space U does not depend on the choice of basis and thus is an invariant of the pair (V, Ω). We then have $dim V = k + 2m$, which shows that m is also an invariant of (V, Ω). The number m corresponds to half the rank of Ω.

Apart from being skew-symmetric, a symplectic linear map is also non-degenerate, which means that the subspace U defined in the previous proof is zero. The next definition allows us to write U as the kernel of linear map.

Definition 1.4. Let V be an n-dimensional real vector space and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. We define the map $\widetilde{\Omega} : V \to V^*$ given by $\widetilde{\Omega}(u)(v) = \Omega(u, v)$. The subspace U from the above theorem then corresponds to the kernel of $\widetilde{\Omega}$.

We may now define a symplectic vector space.

Definition 1.5. Let V be an n-dimensional real vector space and let Ω be a skew-symmetric bilinear map on V. Ω is called *symplectic* or *non-degenerate* if $\widetilde{\Omega}$ is bijective, or equivalently if $U = \{0\}$. Ω is then called a *linear symplectic structure* on V, and the pair (V, Ω) is called a *symplectic vector space*.

Remark 1.6. Since $\widetilde{\Omega}$ is bijective, the subspace U has dimension zero, which implies that $dim V = n = 2m$ is even. Hence, every symplectic vector space is even-dimensional.

Linear subspaces of a symplectic linear subspace may behave in a few different ways with respect to the linear symplectic structure Ω. We use the following two definitions to distinguish between those different kind of linear subspaces. The so-called Lagrangian subspaces will be of particular interest ot us, and more generally are a fundamental concept in many areas of symplectic geometry.

Definition 1.7. Let (V, Ω) be a symplectic vector space, and let $W \subseteq V$ be a linear subspace. We define the *symplectic complement* of W to be:

$$W^{\Omega} := \{v \in V \mid \Omega(v, w) = 0, \ \forall w \in W\}.$$

Definition 1.8. Let $W \subseteq (V, \Omega)$ be a linear subspace in a symplectic vector space. We say that

- W is *symplectic* if $\Omega\big|_W$ is non-degenerate.
- W is *isotropic* if $W \subseteq W^{\Omega}$.
- W is *coisotropic* if $W^{\Omega} \subseteq W$.

- W is *Lagrangian* if $W = W^{\Omega}$.

We now take a look at how the dimension of a subspace $W \subseteq (V, \Omega)$ relate to the dimension of its symplectic complement, and use this to prove that the dimension of Lagrangian subspaces is always half the dimension of V. The next two results are summed up as lemma 2.1.1 in [13].

Lemma 1.9. *Let (V, Ω) be a symplectic vector space and let $W \subseteq V$ be a linear subspace. Then $dim W + dim W^{\Omega} = dim V$.*

Proof. Let us look at the map $\widetilde{\Omega}_W : V \to W^*$ given by $v \mapsto \Omega(v, \cdot)\big|_W$. The kernel of $\widetilde{\Omega}_W$ is clearly W^{Ω}. Moreover, since Ω is non-degenerate, $\widetilde{\Omega}_W$ is surjective, and hence the image of $\widetilde{\Omega}_W$ is the whole of W^*. This yields

$$dim(V) = dim(W^{\Omega}) + dim(W^*) = dim(W^{\Omega}) + dim(W).$$

■

Lemma 1.10. *A linear subspace W of (V, Ω) is Lagrangian if and only if W is isotropic and $dim(W) = \frac{1}{2} dim(V)$.*

Proof. Assume that W is isotropic and and that its dimension is half the dimension of V. Then we have $W \subseteq W^{\Omega}$ by definition of an isotropic subspace and by lemma 1.9

$$dim(W^{\Omega}) = dim(V) - dim(W) = \frac{1}{2} dim(V) = dim(W),$$

which implies that $W = W^{\Omega}$, and thus that W is Lagrangian.
Conversely, assume that W is Lagrangian. Then, it is immediate that W is isotropic, and using lemma 1.9 again, we have that

$$dim(W^{\Omega}) + dim(W) = 2\, dim(W) = dim(V),$$

and hence $dim(W) = \frac{1}{2}\, dim(V)$. ■

Just as we require maps between groups to respect the group structure by defining group homomorphisms, it makes sense to require a linear map between two symplectic vector spaces to respect the symplectic linear structures.

Definition 1.11. Let (V_1, Ω_1) and (V_2, Ω_2) be two symplectic vector spaces. A *symplectomorphism* $\varphi : V_1 \to V_2$ is a linear isomorphism that respects the symplectic structure, i.e. such that $\varphi^* \Omega_2 = \Omega_1$, where $\varphi^* \Omega_2(v, w) = \Omega_2(\varphi(v), \varphi(w))$. We then say that the vector spaces V_1 and V_2 are *symplectomorphic*.

We finish this section with results about Lagrangian subspaces that we will need in order to prove Weinstein's tubular neighborhood theorem 1.53, a result that will be central in constructing the objects we want to study in Chapter 5. We follow the proofs given in Cannas' lecture notes [5] in propositions 8.2, 8.3 and section 9.1.

Proposition 1.12. *Let (V, Ω) be a $2n$-dimensional symplectic vector space. Let U be a Lagrangian subspace of (V, Ω), and let W be any complement of U, not necessarily Lagrangian. Then, from W we can canonically construct a Lagrangian complement to U.*

Proof. Since Ω is non-degenerate, we have a non-degenerate pairing $U \times W \overset{\Omega'}{\to} \mathbb{R}$. This implies that the map $\widetilde{\Omega}' : U \to W^*$ given by $\widetilde{\Omega}'(u)(w) = \widetilde{\Omega}'(u, w)$ is bijective. We want to find a Lagrangian complement to U of the form

$$W' = \{w + Aw | \, w \in W\},$$

where $A : W \to U$ is a linear map. To make W' a Lagrangian subspace, one must have for every w_1, w_2 in W:

$$\begin{aligned} 0 &= \Omega(w_1 + Aw_1, w_2 + Aw_2) \\ &= \Omega(w_1, w_2) + \Omega(w_1, Aw_2) + \Omega(Aw_1, w_2) + \Omega(Aw_1, Aw_2) \\ &= \Omega(w_1, w_2) + \Omega(w_1, Aw_2) + \Omega(Aw_1, w_2). \end{aligned}$$

The term $\Omega(Aw_1, Aw_2)$ vanishes because A has image in U, and U is a Lagrangian subspace. Rewriting the equation above, we get

$$\begin{aligned} \Omega(w_1, w_2) &= \Omega(Aw_2, w_1) - \Omega(Aw_1, w_2) \\ &= \widetilde{\Omega}'(Aw_2)(w_1) - \widetilde{\Omega}'(Aw_1)(w_2) \\ &= A'(w_2)(w_1) - A'(w_1)(w_2), \end{aligned}$$

where we set $A' := \widetilde{\Omega}' \circ A : W \to W^*$. The canonical choice for A' is to put $A'(w) = -\frac{1}{2}\Omega(w, \cdot)$. We then find that $A = (\widetilde{\Omega}')^{-1} \circ A'$.
We show that W' is indeed a Lagrangian complement of U. Since W is isotropic by construction, it is enough to show that $dim(W') = \frac{1}{2}dim(V)$ by lemma 1.10. Let $w \in W$. Assume that $w + Aw \in U$. This implies that $w \in U$, but since U and W are complementary, one must have $w = 0$ and hence $w + Aw = 0$. This shows that $U \cap W' = \{0\}$. Furthermore, since we can write any v in V as a sum $u + w$ with u in U and w and W, we have:

$$v = u + w = \underbrace{u - Aw}_{\in U} + \underbrace{w + Aw}_{\in W'},$$

and hence $V = U \oplus W'$. Finally, since $dim(W') = dim(V) - dim(U) = \frac{1}{2}dim(V)$, W' is indeed a Lagrangian subset of V. This concludes the proof. ∎

The next proposition is the linear analogue of Weinstein's Lagrangian neighborhood theorem, which we will prove in this chapter as theorem 1.52.

Proposition 1.13. *Let V be a $2n$-dimensional vector space and Ω_0, Ω_1 two linear symplectic structures on it. Let U be a Lagrangian subspace of V for both Ω_0 and Ω_1, and let W be an arbitrary complement to U. Then, from W we can canonically construct a linear isomorphism $L : V \xrightarrow{\cong} V$ such that $L|_U = id_U$ and $L^*\Omega_1 = \Omega_0$.*

Proof. Using proposition 1.12 above, we canonically obtain a complement W_0 to U which is Lagrangian with respect to Ω_0 and a complement W_1 to U, which is Lagrangian with respect to Ω_1. Then, as in the proof of proposition 1.12, for $j = 0, 1$, we obtain an isomorphism $\widetilde{\Omega}'_j : W_j \to U^*$ from the non-degeneracy of the paring $\Omega_j : W_j \times U \to \mathbb{R}$.
Consider the diagram

$$\begin{array}{ccc} W_0 & \xrightarrow{\widetilde{\Omega}'_0} & U^* \\ \Big\downarrow B & & \Big\downarrow id \\ W_1 & \xrightarrow{\widetilde{\Omega}'_1} & U^* \end{array}$$

where the linear isomorphism B makes the diagram commute, i.e $\Omega_0(w_0, u) = \Omega_1(Bw_0, u)$ for every w_0 in W_0 and every u in U. We can now extend B to the whole space V by setting it to be the identity on U:

$$L := id_U \oplus B : U \oplus W_0 \to U \oplus W_1.$$

It is clear that L is an isomorphism, and that it is the identity on U. To conclude the proof, we compute:

$$\begin{aligned} (L^*\Omega_1)(u \oplus w_0, u' \oplus w_0') &= \Omega_1(u \oplus Bw_0, u' \oplus Bw_0') \\ &= \Omega_1(u, Bw_0') + \Omega_1(Bw_0, u') \\ &= \Omega_0(u, w_0') + \Omega_0(w_0, u') \\ &= \Omega_0(u \oplus w_0, u' \oplus w_0'). \end{aligned}$$

This shows that $L^*\Omega_1 = \Omega_0$ and hence finishes the proof. ■

The last result of this section shows that there is a canonical identification between the quotient V/U of a symplectic vector space by a Lagrangian subspace and the dual space U^*.

Lemma 1.14. *Let (V, Ω) be a symplectic vector space and let U be a Lagrangian subspace. Then there is a canonical non-degenerate bilinear pairing $\Omega' : V/U \times U \to \mathbb{R}$.*

Proof. Let us define $\Omega'([v], u) := \Omega(v, u)$, where $[v]$ denotes the equivalence class of v in the quotient V/U, and $u \in U$. This is well-defined, since replacing v with $v + u'$ for an arbitrary u' in U yields

$$\Omega(v + u', u) = \Omega(v, u) + \underbrace{\Omega(u', u)}_{=0} = \Omega(v, u),$$

because U is Lagrangian. We now prove that Ω' is non-degenerate. Assume for a contradiction that there is $[v] \in V/U$ such that $\Omega'([v], u) = 0$ for every u in U. This would imply that $\Omega(v, u) = 0$ for every $u \in U$, which contradicts Ω being non-degenerate. Similarly, assuming that there is a $u \in U$ such that $\Omega'(\cdot, u)$ always vanishes would imply that $\Omega(\cdot, u)$ always vanishes, which is a contradiction. This completes the proof. ■

Remark 1.15. Lemma 1.14 implies that we have an isomorphism $\widetilde{\Omega}' : V/U \to U^*$ induced by Ω' as $\widetilde{\Omega}'([v])(u) = \Omega'([v], u)$. Since the whole construction is canonical, U/V and U^* are canonically identified.

1.2 Symplectic manifolds

Now that the prerequisites from symplectic linear algebra have been covered, we have all that we need to look at symplectic manifolds. In this section, we define symplectic manifolds and prove some important results that are some of the cornerstones of symplectic geometry. Throughout this section, unless otherwise specified, we will consider smooth and connected manifolds.

Definition 1.16. Let M be a manifold. A *symplectic structure* or *symplectic form* on M is a differential 2-form $\omega \in \Omega^2(M)$ with the following properties:

- ω is closed: $d\omega = 0$, where d is the de Rham exterior derivative.
- ω is non-degenerate, i.e. (T_pM, ω_p) is a symplectic vector space for every p in M.

We call the pair (M, ω) a *symplectic manifold*.

Remark 1.17. Since every (T_pM, ω_p) is a symplectic vector space, the tangent spaces all have even dimension, and so M also has even dimension. Thus, every symplectic manifold has even dimension.

Let us go over some basic examples of symplectic manifolds.

Example 1.18. Consider the space $\mathbb{R}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. We define the 2-form $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$. It is clear from the definition that ω_0 is closed and non-degenerate, and hence is a symplectic structure on $\mathbb{R}^{2n}$, which we call the *standard symplectic structure* on $\mathbb{R}^{2n}$.

Example 1.18 is very important because it is actually a local model for every symplectic manifold. Indeed, a result of Darboux shows that every symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. We will prove this result later as theorem 1.50. Another way to describe this model is with complex coordinates.

Example 1.19. Take $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$. We define the standard symplectic form on $\mathbb{C}^n$ as $\omega_0 = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$. This form actually corresponds to the previous structure, given the identification $z_j = x_j + iy_j$. Indeed,

$$\begin{aligned}
&\frac{i}{2} dz_j \wedge d\bar{z}_j = \frac{i}{2}(dx_j + idy_j) \wedge (dx_j - idy_j) \\
=&\frac{i}{2}(-idx_j \wedge dy_j + idy_j \wedge dx_j) = dx_j \wedge dy_j.
\end{aligned}$$

It is sometimes useful to write ω_0 in polar coordinates. Given $z_j = r_j e^{i\theta_j}$, we have that $dz_j = e^{i\theta_j} dr_j + ir_j e^{i\theta_j} d\theta_j$ and $d\bar{z}_j = e^{-i\theta_j} dr_j - ir_j e^{-i\theta_j} d\theta_j$. Hence,

$$\begin{aligned}
&\frac{i}{2} dz_j \wedge d\bar{z}_j = \frac{i}{2}(-ir_j dr_j \wedge d\theta_j + ir_j d\theta_j \wedge dr_j) \\
=&r_j dr_j \wedge d\theta_j.
\end{aligned}$$

This means that in polar coordinates, we have that

$$\omega_0 = \sum_{j=1}^n r_j dr_j \wedge d\theta_j.$$

Every orientable surface equipped with its area form is also a symplectic manifold. Indeed, every area form is non-degenerate, and any two-form on a surface is closed for dimensional reasons. We illustrate this with an example taken from [6].

Example 1.20. In particular, the unit sphere in $\mathbb{R}^3$ equipped with the standard euclidean area form is a symplectic manifold. Away from the poles, this form can be expressed as

$$\omega_{\text{std}} := d\theta \wedge dh,$$

where h is the height function and θ the angle around the height axis given by h.

Just like in the case of symplectic vector spaces, we have a notion of isomorphism for symplectic manifolds.

Definition 1.21. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds and let $\varphi : M_1 \to M_2$ be a diffeomorphism. We call φ a *symplectomorphism* if $\varphi^* \omega_2 = \omega_1$. We then say that (M_1, ω_1) and (M_2, ω_2) are *symplectomorphic*. We will denote the group of symplectomorphisms between (M_1, ω_1) and (M_2, ω_2) as $\mathrm{Sympl}((M_1, \omega_1), (M_2, \omega_2))$ or sometimes just as $\mathrm{Sympl}(M_1, M_2)$ if it is clear what symplectic forms are meant. If $M_1 = M_2$, we simply write $\mathrm{Sympl}(M_1, \omega_1)$.

1.2.1 Cotangent bundles

Another important kind of symplectic manifolds are cotangent bundles of arbitrary manifolds. Cotangent bundles are naturally equipped with a symplectic form and in chapters 4 and 5, we will use this to better understand other symplectic manifolds. Let M be any n-dimensional manifold and let $(U, x_1, \dots, x_n)$ denote a coordinate chart on M. Then, at any point $p \in U$, an element $\xi \in T^*M$ can be written as $\xi|_p = \sum_{j=1}^n \xi_j(p) dx_j|_p$, where every ξ_j is a smooth function on U Thus, we obtain a chart $(U, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ on T^*M. We define a two-form on T^*U by setting:

$$\omega = \sum_{j=1}^{n} dx_j \wedge d\xi_j.$$

This two-form is clearly closed, as $d^2 = 0$ and it is non-degenerate because $(dx_1, \dots, dx_n, d\xi_1, \dots, d\xi_n)$ is a local frame for the cotangent bundle.

For this definition to be entirely satisfactory, we need to show that it is independent of the choice of coordinates. To this end, we define the one form

$$\alpha = \sum_{j=1}^{n} \xi_j dx_j.$$

We clearly have $\omega = -d\alpha$. For this reason, it is enough to show that α is independent of the choice of coordinates.

Proposition 1.22. *The* 1*-form* α *is independent of the choice of coordinates.*

We follow the proof given in [5] section 2.2.

Proof. Let $(U, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ and $(U', x'_1, \dots, x'_n, \xi'_1, \dots, \xi'_n)$ be two coordinate charts on T^*M. Let $p \in U \cap U'$ and $\xi \in T_p^*M$. We have

$$\begin{aligned} \xi = \sum_{j=1}^{n} \xi_j(p) dx_j\big|_p &= \sum_{j=1}^{n} \xi_j(p) \sum_{k=1}^{n} dx_j\big|_p \left(\frac{\partial}{\partial x'_k} \right)\bigg|_p dx'_k\big|_p \\ &= \sum_{j,k=1}^{n} \xi_j(p) \left(\frac{\partial x_j}{\partial x'_k} \right)\bigg|_p dx'_k\big|_p . \end{aligned}$$

This shows that we have $\xi'_k = \sum_{j=1}^n \xi_j \left(\frac{\partial x_j}{\partial x'_k}\right)$. Since we have that $dx'_k = \sum_{j=1}^n \frac{\partial x'_k}{\partial x_j} dx_j$, we can write

$$\alpha' = \sum_{k=1}^n \xi'_k dx'_k = \sum_{k=1}^n \sum_{j=1}^n \xi_j \left(\frac{\partial x_j}{\partial x'_k}\right) \left(\frac{\partial x'_k}{\partial x_j}\right) dx_j = \sum_{j=1}^n \xi_j dx_j = \alpha.$$

This completes the proof. ∎

The 1-form α is called the *tautological* or *Liouville* 1*-form*, and $\omega = -d\alpha$ is called the *canonical symplectic form*.

It is also possible to define α and ω in a coordinate-free way. Let

$$\begin{aligned} \pi : & T^*M \to M \\ & (p, \xi) \mapsto p \end{aligned}$$

denote the cotangent bundle projection. We define the tautological form α pointwise as $\alpha_{(p,\xi)}[v] = \xi(D\pi_{(p,\xi)}[v])$, for every v in $T_{(p,\xi)}T^*M$. The canonical symplectic form is then once again given by $\omega = -d\alpha$.

Proposition 1.23. *Let $(U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be a chart on T^*M. Then, on T^*U, $\alpha = \sum_{j=1}^n \xi_j dx_j$*

We follow the argument given on pages 105-106 in [13].

Proof. We observe that at a point $(p, \xi) \in T^*M$, the map $D\pi_{(p,\xi)} : T_{(p,\xi)}T^*M \to T^*M$ sends $\left(\frac{\partial}{\partial x_j}\right)\Big|_{(p,\xi)}$ to $\left(\frac{\partial}{\partial x_j}\right)\Big|_p$ and $\left(\frac{\partial}{\partial \xi_j}\right)\Big|_{(p,\xi)}$ to 0. This shows that $\xi \circ D\pi$ does indeed correspond to $\sum_{j=1}^n \xi_j dx_j$. ∎

The tautological 1-form and the canonical 2-form on a cotangent bundle are natural. This means that a diffeomorphism between two manifolds canonically induces a symplectomorphism between their cotangent bundles. Let M_1 and M_2 be n-dimensional manifolds and let $f : M_1 \to M_2$ be a diffeomorphism. Then there is a natural diffeomorphism

$$f_\# : T^*M_1 \to T^*M_2$$

which lifts f. More concretely, if we have (x_1, ξ_1) in T^*M_1, we define

$$f_\#(x_1, \xi_1) = (f(x_1), \xi_1 \circ (Df(x_1))^{-1}) \in T^*M_2$$

More concisely, we may write $f_\# = ((f, (Df)^{-1})^*)$. It is clear that $f_\#$ is smooth since it can be written as a pullback by a smooth map. It is also bijective since the map $(f^{-1})_\# = (f^{-1}, (Df)^*)$ clearly inverts $f_\#$. This shows that $f_\#$ is a diffeomorphism. Moreover, we observe that the following diagram commutes.

$$\begin{array}{ccc} T^*M_1 & \xrightarrow{\;f_\#\;} & T^*M_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow[\;f\;]{} & M_2 \end{array} \tag{1.1}$$

We also have the following property on the tautological one-forms and canonical symplectic forms. These results correspond to proposition 2.1 and corollary 2.3 in [5].

Proposition 1.24. *Let M_1 and M_2 be two n-dimensional manifolds and let $f : M_1 \to M_2$ be a diffeomorphism. Let α_1 and α_2 denote the tautological 1-forms on T^*M_1 and T^*M_2 respectively. Then*

$$(f_\#)^*\alpha_2 = \alpha_1.$$

Proof. Let $p_1 = (x_1, \xi_1)$ be an element of T^*M_1, and set $p_2 = f_\#(p_1)$, with $x_2 = f(x_1)$ and $\xi_2 = (Df^{-1})^*\xi_1$. What we want to show is

$$(Df_\#)^*_{p_1}(\alpha_2)_{p_2} = (\alpha_1)_{p_1}.$$

Notice that for $j = 1, 2$, we can write $(\alpha_j)_{p_j} = ((D\pi_j)^*_{p_j})\xi_j$, where π_j denote the cotangent bundle projections. Thus, we have that

$$\begin{aligned} (Df_\#)^*_{p_1}(\alpha_2)_{p_2} &= (Df_\#)^*_{p_1}((D\pi_2)^*_{p_2})\xi_2 = (D(\pi_2 \circ f_\#))^*_{p_1}\xi_2 \\ &\overset{(1.1)}{=} (D(f \circ \pi_1))^*_{p_1} = (D\pi_1)^*_{p_1}(Df)^*_{x_1}\xi_2 \\ &= (D\pi_1)^*_{p_1}\xi_1 = (\alpha_1)_{p_1} \end{aligned}$$

■

Corollary 1.25. *The lift $f_\# : T^*M_1 \to T^*M_2$ of a diffeomorphism is a symplectomorphism. In other words, if the ω_1 and ω_2 denote the canonical symplectic forms on T^*M_1 and T^*M_2 respectively, we have that*

$$(f_\#)^*\omega_2 = \omega_2$$

Proof. Use proposition 1.24 and the fact that pullbacks commute with the exterior differential. ■

It is possible to generalize the concepts of symplectic, isotropic, coisotropic and Lagrangian subspaces to submanifolds. We thus get different sorts of submanifolds depending on how their tangent spaces interact with the symplectic forms.

Definition 1.26. Let (M, ω) be a symplectic manifold and let $X \subseteq M$ be a submanifold of M. We say that:

- X is *symplectic* if for every $p \in X$, T_pX is a symplectic subspace of T_pM.
- X is *isotropic* if for every $p \in X$, T_pX is a isotropic subspace of T_pM.
- X is *coisotropic* if for every $p \in X$, T_pX is a coisotropic subspace of T_pM.
- X is *Lagrangian* if for every $p \in X$, T_pX is a Lagrangian subspace of T_pM.

Remark 1.27. By lemma 1.10, a submanifold $X \subseteq (M, \omega)$ is Lagrangian if and only if $i^*\omega = 0$ and $dim(X) = \frac{1}{2}dim(M)$, where $i : X \hookrightarrow M$ denotes the inclusion.

We now investigate the Lagrangian submanifolds of T^*M for an arbitrary manifold M. Let us denote by $M_0 = \{(p, 0) | p\ inM\} \subseteq T^*M$ the zero section of the cotangent bundle.

Lemma 1.28. *The zero section M_0 is a Lagrangian submanifold of T^*M.*

We follow the ideas from [5] section 3.2.

Proof. We directly have that $dim(M_0) = dim(M) = \frac{1}{2}dim(T^*M)$. Let $i : M_0 \hookrightarrow T^*M$ denote the inclusion. We need to show that $i^*\omega = 0$, as explained in remark 1.27. Let $(U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be a local cotangent chart. Then $M_0 \cap T^*U$ is given by the equation $\xi_1 = \cdots = \xi_n = 0$. This implies that $\alpha = \sum_{j=1}^{n} \xi_j dx_j = 0$ on $M_0 \cap T^*U$. Since the charts U cover T^*M, α vanishes on M_0, which is we can write as $i^*\alpha = 0$. Thus, $i^*\omega = -di^*\alpha = 0$. This concludes the proof. ∎

After proving that the zero section is Lagrangian, a natural question is to ask whether the image of other sections are also Lagrangian submanifolds. Let $\mu \in \Omega_1(M)$ be a smooth 1-form on M. We can see μ as a smooth map from M to T^*M such that $\pi \circ \mu = id$, with $\pi : T^*M \to M$ the projection map. Let us denote by $M_\mu := \{(p, \mu_p) | p \in M\}$ the image of μ. To avoid confusing notation later, we denote by $s_\mu : M \to T^*M$ the map sending p to (p, μ_p) considered as a map only and not as a 1-form. We can then show the following fact, which correspond to proposition 3.4 in [5]:

Lemma 1.29. *Let α be the tautological* 1*-form on T^*M. Then $s_\mu^*\alpha = \mu$.*

Proof. Let $(p, \xi) \in T^*M$. The coordinate-free definition of α at (p, ξ) is $\alpha_{(p,\xi)} = \xi \circ D\pi(p, \xi)$, and thus

$$\alpha_{(p,\mu_p)} = \mu_p \circ D\pi(p, \mu_p).$$

We can then compute

$$(s_\mu^*\alpha)_p = \alpha_{(p,\mu_p)} \circ Ds_\mu(p) = \mu_p \circ D\pi(p,\mu_p) \circ Ds_\mu(p)$$
$$= \mu_p \circ D \underbrace{(\pi \circ s_\mu)}_{=id}(p) = \mu_p.$$

■

Since s_μ is an embedding, its image M_μ is diffeomorphic to M. Let $\tau : M \to M_\mu$, $\tau(p) = (p, \mu)$ denote this diffeomorphism. We have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\quad s_\mu \quad} & T^*M \\ & {\scriptstyle \tau}\searrow^{\simeq} \quad \nearrow{\scriptstyle i} & \\ & M_\mu & \end{array}$$

We look for a condition on μ that would make M_μ a Lagrangian submanifold of T^*M.

$$i^*d\alpha = 0 \iff \tau^*i^*d\alpha = 0 \iff (i \circ \tau)^*d\alpha = 0$$
$$\iff s_\mu^*d\alpha = 0 \iff ds_\mu^*\alpha = 0 \overset{L.1.29}{\iff} d\mu = 0$$

This shows that the image of a 1-form is a Lagrangian submanifold if and only this 1-form is closed. One of our main goals at the end of chapter 5 will be to few certain Lagrangian submanifolds as images of closed 1-forms inside a cotangent bundle. To do this, we will need to find a symplectomorphism between an arbitrary symplectic manifold M and the cotangent bundle of some other manifold. In that way, we will be able to transfer the Lagrangian submanifolds of M to this cotangent bundle to understand them better. The next section ends with the proof of Weinstein's tubular neighborhood theorem, which gives a way to construct a symplectomorphism between a symplectic manifold and a cotangent bundle.

1.3 The Moser and Weinstein theorems

The main goal of this section is proving Weinstein's tubular neighborhood theorem 1.53, as it will be one of the central tools used to investigate toric Lagrangians in chapters 4 and 5. However, this section also presents other important results and concepts of symplectic geometry, both because they are needed in the proof of theorem 1.53, and because they are foundational in and of themselves. We begin by recalling of few concepts from differential geometry. Again, unless otherwise specified, all our manifolds are smooth and connected. This section follows the presentation given in [5] lectures 6 to 9.

1.3.1 Isotopies

Definition 1.30. Let M be a manifold, and $\rho : M \times \mathbb{R} \to M$ a smooth map. Let us write $\rho_t(p) = \rho(p,t)$. We call ρ an *isotopy* if every map $\rho_t : M \to M$ is a diffeomorphism, and $\rho_0 = id$.

To every such isotopy corresponds a family $\{X_t\}$ of vector fields. Indeed, let $p \in M$ and set $q := \rho_t^{-1}(p)$, then we define

$$X_t(p) = \frac{d}{ds}\bigg|_{s=t} \rho_s(q).$$

This is equivalent to

$$X_t(\rho_t(p)) = \frac{d}{ds}\bigg|_{s=t} \rho_s(\rho_t^{-1}\rho_t(p)) = \frac{d}{ds}\bigg|_{s=t} \rho_s(p) = \frac{d}{dt}\rho_t(p).$$

In short, we have

$$X_t \circ \rho_t = \frac{d}{dt}\rho_t. \tag{1.2}$$

The family $\{X_t\}$ is called a *time-dependent vector field*.

We can also do this the other way round. If $\{X_t\}$ is a time-dependent vector field, provided that M is compact (or that the X_t are compactly supported), there exists an isotopy satisfying the ODE 1.2. Thus, for M compact there is a bijective correspondence between isotopies of M and time-dependent vector fields on M.

Remark 1.31. When $X_t = X$ is independent of time, the isotopy is simply the flow of the vector field X. One can think of an isotopy as a kind of "time-dependent flow".

We now recall the definition of Lie derivative and extend it to time-dependent vector fields. These definitions are valid for any smooth manifold M, since the flow or the isotopy exist at least locally by Picard's theorem.

Definition 1.32. Let M be manifold and X a vector field with (local) flow θ_t. The *Lie derivative* by X is the operator $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ given by

$$\mathcal{L}_X \eta = \frac{d}{dt}\bigg|_{t=0} \theta_t^* \eta.$$

Similarly, given a time-dependent vector field X_t with corresponding (local) isotopy denoted by ρ_t , we define the *Lie derivative* along X_t by

$$\mathcal{L}_{X_t} \eta = \frac{d}{dt}\bigg|_{t=0} \rho_t^* \eta.$$

Following the same logic, we recall Cartan's magic formula and present the analogue for time-dependent vector fields.

Theorem 1.33 (Cartan's magic formula)**.** $\mathcal{L}_X\eta = \iota_X d\eta + d\iota_X\eta$*, where* d *is the exterior differential and* ι *denotes the interior product.*

Proposition 1.34. *Let* X_t *be a time-dependent vector field on a manifold* M*, and denote by* ρ_t *the corresponding (local) isotopy. Take* $\eta \in \Omega^k(M)$*. The following formula holds:*

$$\frac{d}{dt}\rho_t^*\eta = \rho_t^*\mathcal{L}_{X_t}\eta.$$

Since these theorems are standard results in differential geometry, we will not give a full proof here. We only sketch the strategy of the proofs.
Both theorem are proved in the exact same way. We notice that both sides are graded derivation of degree zero of the algebra of differential forms, which is generated by functions and exact one-forms. It is thus enough to check the equality on those. A proof of Cartan's magic formula can be found in Merry's differential geometry lecture notes [14] as theorem 20.6.

The formula above can be generalized again to the following. This is proposition 6.4 in [5].

Proposition 1.35. *Let* η_t *be a smooth family of k-forms on* M*, and* X_t *a time-dependent vector field with isotopy* ρ_t*. Then*

$$\frac{d}{dt}\rho_t^*\eta_t = \rho_t^*\big(\mathcal{L}_{X_t}\eta_t + \frac{d}{dt}\eta_t\big).$$

Proof. By the chain rule, we have

$$\begin{aligned}\frac{d}{dt}\rho_t^*\eta_t &= \frac{d}{dx}\bigg|_{x=t}\rho_x^*\eta_t + \frac{d}{dy}\bigg|_{y=t}\rho_t^*\eta_y \\ &\overset{Prop.1.34}{=} \rho_x^*\mathcal{L}_{X_x}\eta_t\bigg|_{x=t} + \rho_t^*\frac{d}{dy}\bigg|_{y=t}\eta_y \\ &= \rho_t^*\left(\mathcal{L}_{X_t}\eta_t + \frac{d}{dt}\eta_t\right)\end{aligned}$$

∎

The concepts of isotopy and time-dependent vector filed can easily be adapted to the symplectic setting. The rest of the material in this subsection follows section 3.1 in McDuff and Salamon's book [13] between pages 94 and 103.

Definition 1.36. Let (M, ω) by a symplectic manifold. A *symplectic isotopy* $\rho : M \times \mathbb{R} \to M$ is an isotopy such that every diffeomorphism ρ_t is a symplectomorphism.

With this stronger definition of isotopy, it is logical to expect that the vector fields that generate symplectic isotopies also have special properties.

Definition 1.37. Let (M, ω) be a symplectic manifold, and let X be a vector field on M. X is called *symplectic* if $\iota_X\omega$ is a closed 1-form, i.e. $d\iota_X\omega = 0$.

The next result proves the link between symplectic isotopies and symplectic vector fields by showing that a symplectic isotopy generates a symplectic vector field and vice versa. This result can be found in [13] as proposition 3.1.5 and we follow the proof given there.

Proposition 1.38. *Let (M, ω) be a closed symplectic manifold. If $t \mapsto \rho_t \in$ Diff(M) is an isotopy generated by the family of vector fields X_t, then the isotopy ρ is symplectic if and only if every X_t is a symplectic vector field. Moreover, if both X and Y are symplectic vector fields then $[X, Y]$ is also symplectic and $\iota_{[X,Y]}\omega = dH$ for $H = \omega(Y, X)$.*

Proof. Using Cartan's magic formula 1.33 and proposition 1.34, we have the following identity:

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*(\mathcal{L}_{X_t}\omega) = \rho_t^*(d\iota_{X_t}\omega + \iota_{X_t}d\omega) = \rho_t^*(d\iota_{X_t}\omega).$$

This shows that every ρ_t is symplectic if and only if every X_t is symplectic. Next, we note that a vector field X is symplectic if and only if $\mathcal{L}_X\omega = 0$, which is again a consequence of Cartan's magic formula 1.33. Now let X and Y be symplectic vector fields with flows φ_t and ψ_t. We have that

$$[X, Y] = \mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* Y$$

which yields

$$\begin{aligned}\iota_{[X,Y]}\omega &= \left.\frac{d}{dt}\right|_{t=0} \iota_{\varphi_t^* Y}\omega \overset{\varphi_t^*\omega=\omega}{=} \left.\frac{d}{dt}\right|_{t=0} \varphi_t^*\iota_Y\omega = \mathcal{L}_X\iota_Y\omega \\ &= d\iota_X\iota_Y\omega + \iota_X \underbrace{d\iota_Y\omega}_{=0} = d(\omega(Y, X)).\end{aligned}$$

This concludes the proof. ∎

We now introduce the notions of Hamiltonian functions, Hamiltonian vector fields and Hamiltonian isotopies. The non-degeneracy of symplectic forms implies that every function on a symplectic manifold gives rise to a vector field whose flow preserves the symplectic structure.

Definition 1.39. Let (M, ω) be a symplectic manifold, and let $H : M \to \mathbb{R}$ be a smooth function. Since ω is non-degenerate, there exists exactly one vector field X_H such that

$$-\iota_{X_H}\omega = dH.$$

X_H is called the *Hamiltonian vector field* associated to the *Hamiltonian function* H. If M is a closed manifold, X_H generates a smooth one-parameter group of diffeomorphisms φ_H^t satisfying

$$\frac{d}{dt}\varphi_H^t = X_H \circ \varphi_H^t, \quad \varphi_H^0 = id.$$

φ_H^t is called the *Hamiltonian flow* associated to H.

Remark 1.40.
- The following identity holds:

  $$dH(X_H) = -\iota_{X_H}\omega(X_H) = -\omega(X_H, X_H) = 0.$$

  The geometric interpretation of this equation is that the vector field X_H is tangent to the level sets of the Hamiltonian function H.
- φ_H^t preserves the symplectic structure for all t, i.e. $(\varphi_H^t)^*\omega = \omega$. Indeed, we have that

  $$\begin{aligned} \frac{d}{dt}(\varphi_H^t)^*\omega &\overset{L.1.34}{=} (\varphi_H^t)^*\mathcal{L}_{X_H}\omega = (\varphi_H^t)^*(d\iota_{X_H}\omega + \iota_{X_H}d\omega) \\ &= (\varphi_H^t)^*(d(-dH) + 0) = 0. \end{aligned}$$

- The minus sign in the definition of Hamiltonian function is a choice, and some authors define it with the equation

  $$dH = \iota_{X_H}\omega.$$

  It is a fact of symplectic geometry that there is no sign convention that makes every formula appear without somewhat unintuitive minus signs somewhere. In our case, the motivation behind this choice of a minus sign here comes from the fact that it will make our first example of toric Lagrangian in chapter 4 easier to visualize.

If we make the Hamiltonian function H time-dependent, we get the more general concept of a Hamiltonian isotopy.

Definition 1.41. Let (M, ω) be a symplectic manifold, and let $\{\rho_t\}_{0 \le t \le 1}$ be a symplectic isotopy on M with time-dependent vector field X_t. We say that ρ_t is a *Hamiltonian isotopy* if the 1-form $\iota_{X_t}\omega$ is exact for every $t \in [0, 1]$. This means that every X_t is a Hamiltonian vector field and so there exists a smooth function $H : [0,1] \times M \to \mathbb{R}$ such that for every t in $[0,1]$, we have that

$$-\iota_{X_t}\omega = dH_t,$$

where $H_t = H(t,\cdot)$. The function H is called a *time-dependent Hamiltonian*. It is determined by a Hamiltonian isotopy only up to an additive function $c : [0,1] \to \mathbb{R}$. Note that if M is simply connected, every closed 1-form is exact, and hence every symplectic isotopy is actually a Hamiltonian isotopy.

1.3.2 Weinstein's tubular neighborhood

This subsection is devoted to the detailed proof of Weinstein's tubular neighborhood 1.53 and of every intermediary result needed. The reason for going into so much detail is that we will need to adapt Weinstein's tubular neighborhood theorem to the setting of Hamiltonian G-spaces in section 5.3. We will do this by improving every proof in this section to them make compatible with a Lie group action. This is why it is preferable that every proof is written out in full detail.

The first step is a standard result of differential geometry, namely the tubular neighborhood theorem. We begin by defining the normal bundle of a submanifold. The strategy of the proof comes from [5], lectures 6 to 9.

Definition 1.42. Let M be an n-dimensional manifold, and$X \subseteq M$ a k-dimensional submanifold such that $k < n$. Denote by $i : X \hookrightarrow M$ the inclusion. For every point p in X, we can view the tangent space T_pX as subspace of the vector space T_pM via the linear inclusion $Di(p) : T_pX \to T_pM$, where we write p for $i(p)$. We define the *normal space* to X at p to be the quotient space $N_pX := T_pM/T_pX$. This is an $(n-k)$-dimensional space. Putting every fiber together, we define the *normal bundle* of X as

$$NX = \{(p,v) | p \in M, v \in N_pX\}.$$

This is a rank $(n-k)$ vector bundle over X, and thus an n-dimensional manifold.

If X is a Lagrangian submanifold, we have the following fact, given as theorem 9.1 in [5].

Proposition 1.43. *Let (M,ω) be a symplectic manifold and X a Lagrangian submanifold of M. Then the vector bundles NX and T^*X are canonically identified.*

Proof. For any $x \in X$, the space T_xX is a Lagrangian subspace of T_xM. By remark 1.15, we have a canonical identification between T_x^*X and $T_xM/T_xX = N_xX$. Thus, as a bundle we indeed have a canonical identification $T^*X \simeq NX$. ∎

We can imbed X into NX with the zero section map $i_0 : X \to NX, i_0(p) = (p, 0)$. We denote by X_0 the zero section (i.e. the copy of X inside NX).

Definition 1.44. A neighborhood U_0 of the zero section X_0 in NX is called *convex* if the intersection $U_0 \cap N_pX$ is convex for all p in X_0.

Here is the statement of the tubular neighborhood theorem.

Theorem 1.45 (Tubular Neighborhood Theorem)**.** *There exists a convex neighborhood U_0 of X in NX, a neighborhood U of X in M and a diffeomorphism $\varphi : U_0 \to U$ such that the following diagram commutes:*

$$\begin{array}{ccc} NX \supseteq U_0 & \xrightarrow[\simeq]{\varphi} & U \subseteq M \\ & {\scriptstyle i_0} \nwarrow \quad \nearrow {\scriptstyle i} & \\ & X & \end{array} .$$

The first part of this proof follows what is done in [14] for theorem 6.10. In the second part we extend these ideas to arbitrary Riemannian manifolds.

Proof. 1. To prove the tubular neighborhood theorem, we first look at the special case of the Euclidean space, so let us assume that $M = \mathbb{R}^n$. In this case, notice that for every $x \in X$, N_xX is the orthogonal complement of T_xX. Let us define a map

$$\begin{aligned} T : & NX \to \mathbb{R}^n \\ & (x, v) \mapsto x + v. \end{aligned}$$

It is clear that the map T restricted to the zero section X_0 is a diffeomorphism onto X, and thus the map $DT_{(x,0)}$ maps $T_{(x,0)}X_0$ isomorphically onto T_xX. Next, if we restrict T to the fiber N_xX, T just becomes the affine map $v \mapsto x + v$ and thus $DT_{(x,0)}$ maps $T_{(x,0)}N_xX$ isomorphically onto N_xX. Thus, by the inverse function theorem, for each $x \in X$ there exists a positive number $\varepsilon_x > 0$ such that with the set

$$U(x, \varepsilon_x) := \{(y, v) \in NX \,|\, |y - x| < \varepsilon_x, |v| < \varepsilon_x\},$$

we have that $T|_{U(x,\varepsilon_x)}$ is a diffeomorphism.
Now, let $\varepsilon : X \to \mathbb{R}$ denote the function that maps every point $x \in X$ to the supremum of all $\varepsilon \leq 1$ such that T is a diffeomorphism on $U(x, \varepsilon)$. By construction, ε is a strictly positive function. We claim that ε is continuous. Take $x, y \in X$ such that $|x - y| < \varepsilon(x)$ and define $\delta := \varepsilon(x) - |x - y|$. It follows from the triangle inequality and the definition of δ that $U(y, \delta) \subseteq U(x, \varepsilon(x))$ and hence $\varepsilon(y) \geq \varepsilon(x) - |x - y|$. Thus, if $|x - y| < \varepsilon(x)$, we have that

$$\varepsilon(x) - \varepsilon(y) \leq |x - y|.$$

Conversely, let us assume that $|x-y| \geq \varepsilon(x)$, then since ε is a positive function, we clearly have, once again that

$$\varepsilon(x) - \varepsilon(y) \leq |x-y|.$$

Reversing the role of x and y gives us

$$|\varepsilon(x) - \varepsilon(y)| \leq |x-y|,$$

which shows the continuity of ε. We now define the set

$$U_0 := \Big\{(x,v) \in NX \big| |v| < \frac{1}{2}\varepsilon(x)\Big\}.$$

and show that T is injective on this set. Let $(x,v),(y,w)$ be such that $x+v = T(x,v) = T(y,w) = y+w$. Without loss of generality, we may assume that $\varepsilon(y) \leq \varepsilon(x)$. Since $x-y = w-v$, we can write

$$|x-y| = |v-w| \leq |v| + |w| \leq \frac{1}{2}\varepsilon(x) + \frac{1}{2}\varepsilon(x) = \varepsilon(x).$$

This implies that both (x,v) and (y,w) belong to $U(x,\varepsilon(x))$. But T is injective on $U(x,\varepsilon(x))$ by construction. This show that $(x,v) = (y,w)$ and hence T is injective on U_0. Now set $T(U_0) = U$. Then U is open because T is a local diffeomorphism. Moreover, since $T|_{U_0}$ is injective, $T : U_0 \to U$ is a smooth bijection that is also a local diffeomorphism. Hence, it is a diffeomorphism, as we wanted. Note that the intersection of U_0 with each fiber is just a ball, and hence it is convex. This proves the theorem in $\mathbb{R}^n$.

2. We now prove the general case. Let us put a Riemannian metric m on M. We will denote by $||\cdot||$, the norm on TM coming from m, and by $d : M \times M \to \mathbb{R}$ the induced metric on the manifold. Just as in the Euclidean case, we can view the normal bundle of X as the orthogonal complement of the tangent space to X with respect to the given Riemannian metric, namely for $x \in X$:

$$N_xX \simeq \{v \in T_xM | m_x(v,w) = 0, \quad \forall w \in T_xX\}.$$

We now look at the exponential map $exp : TM \to M$ coming from the Riemannian metric. It is clear that restricting exp to the zero section $X_0 \subseteq NX \subseteq TX$ gives a diffeomorphism onto X, and hence $Dexp(x,0)$ maps $T_{(x,0)}X_0$ isomorphically onto T_xX. Next, it is a property of the exponential map that a small neighborhood of $0_x \in T_xM$ is mapped diffeomorphically onto a small ball around x with respect to the metric d and that its derivative at zero is the identity. Hence, if we reduce to the subspace N_xX, we see that $Dexp(x,0_x)$ maps $T_{(x,0_x)}N_xX$ isomorphically onto N_xX. Thus, as in the Euclidean case, we may use the inverse function theorem such that for every $x \in X$ there exists a positive number ε_x such that exp is a diffeomorphism on the set

$$U(x,\varepsilon_x) := \{(y,v) \in NX \big| d(x,y) < \varepsilon_x, ||v|| < \varepsilon_x\},$$

We can then define $\varepsilon : X \to \mathbb{R}$ as in the first case. The proof that ε is continuous is analogous; it just uses the metric d instead of the Euclidean distance. We can now define the set

$$U_0 := \{(x,v) \in NX \big| ||v|| < \frac{1}{2}\varepsilon(x)\}.$$

Let us prove the injectivity of exp on U_0. Let us assume that $exp(x,v) = exp(y,w)$ for some $(x,v),(y,w) \in U_0$, and without loss of generality, take $\varepsilon(y) \leq \varepsilon(x)$. This means that if we denote by $\gamma_{(x,v)}$ the geodesic starting at x with initial velocity v and by $\gamma_{(y,w)}$ the geodesic starting at y with initial velocity w, we have that $\gamma_{(x,v)}(1) = \gamma_{(y,w)}(1) =: z$. The distance between the points x and y cannot be greater than the distance of the path given by following $\gamma_{(x,v)}$ and then following $\gamma_{(y,w)}$ backwards. Thus

$$d(x,y) \leq ||v|| + ||w|| \leq \frac{1}{2}\varepsilon(x) + \frac{1}{2}\varepsilon(x) = \varepsilon(x).$$

Thus, both (x,v) and (y,w) belong to $U(x,\varepsilon(x))$, on which exp is injective. Hence, $(x,v) = (y,w)$, which proves the injectivity of exp on U_0. We conclude the proof as in step 1: $exp|_{U_0}$ is a diffeomorphism onto its image because it is an injective local diffeomorphism. This concludes the proof.

■

In the next remark, we discuss another way to approach the tubular neighborhood theorem that is closer to what is done in [5] on pages 37 to 39.

Remark 1.46. We give a more explicit picture of what the tubular neighborhood constructed above looks like. Let us denote by $\pi_{NX} : NX \to X$ the normal bundle projection. If we restrict π_{NX} to the neighborhood U_0 constructed in the previous proof, we obtain a submersion $\pi_{NX} : U_0 \to X$ where every fiber $\pi_{NX}^{-1}(x)$ is convex. We can carry this construction over to U by setting $\pi := \pi_{NX} \circ exp^{-1}$. Let us look at this construction in more detail.
Let q be in U. Then, by definition of U, we have that $q = \gamma_{(x,v)}(1)$, where $\gamma_{(x,v)}$ is the geodesic starting at $x \in X$ with initial velocity $v \in N_xX$ such that $||v|| < \frac{1}{2}\varepsilon(x)$. We then have that

$$\pi(q) = \pi_{NX}(exp^{-1}(q)) = \pi_{NX}(x,v) = x.$$

We claim that x is the closest point to q that lies in X. Indeed, let us set $d(q,\pi(q)) = d(q,x) =: \delta$ and let $y \in X$ be a closest point to q. Hence, we have that $d(q, y) \leq \delta$. Notice that this claim is trivial if q lies in X since the we would have $\pi(q) = q$ so we may assume that q is not in X. Let γ_q be a geodesic from q to y. We must have that the vector $\tilde{w}$ tangent to γ_q at y belongs to N_yX, since otherwise the geodesic would have hit X at a point before y, which contradicts the definition of y. Let us denote by $\gamma_{(y,w)}$ the inverted geodesic from y to q, where $w \in N_yX$ denotes the initial velocity vector. In other words, $w = -\tilde{w}$. We then have that $exp(x, v) = exp(y, w) = q$. Moreover, since we assumed $d(y, q)$ to be at most δ, we have that $||w|| \leq ||v|| < \frac{1}{2}\varepsilon(x)$. Finally, $d(x, y) \leq ||v|| + ||w|| < \varepsilon(x)$. This shows that $(y, w) \in U(x, \varepsilon(x))$ and hence since the exponential map is injective on this set, we have that $(x, v) = (y, w)$. This shows that π maps every point of U to the point in X that is the closest to it.

Secondly, we notice that the neighborhood U is actually the set $U_\varepsilon := \{p \in M | d(p, x) < \frac{1}{2}\varepsilon(x) \quad \text{for some } x \in X\}$. Indeed, it is clear that $U = exp(U_0)$ is included in this set, since following geodesics starting in X and having initial velocity with norm strictly smaller than $\frac{1}{2}\varepsilon(x)$ for some $x \in X$ will give points that cannot be further away from X than $\frac{1}{2}\varepsilon(x)$. Conversely, let q be in U_ε. If q is in X, then it is clear that q is in U so let us assume that q is not in X. By definition of U_ε, there is an x in X such that $d(x, q) < \frac{1}{2}\varepsilon(x)$. Assume that the geodesic in from x to q has the initial velocity vector $v \in T_xX$, so the geodesic stays in X for a while. Then, we may take the last point y in X along the geodesic and the velocity vector w at this point, which by the same reasoning as above belongs to N_yX. The exact same computation as in the previous paragraph then shows that $(x, v) = (y, w)$ which is a contradiction. This implies that v is in N_xX, and hence that q lies in U.

The next important step towards the proof of Weinstein's tubular neighborhood 1.53 is Moser's theorem. This result is not only useful for what we need to show however. It is also needed for a lot of other symplectic geometry arguments, where the technique today known as "Moser's trick" is often part of the proof. We begin by discussing the so-called homotopy formula. We follow the presentation given in [5] for proposition 6.8.

Proposition 1.47 (The homotopy formula)**.** *Let U be a tubular neighborhood of a submanifold $X \subseteq M$. Let us denote by $i : X \to U$ the inclusion. Let $\omega \in \Omega^k(U)$ be a closed k-form on U such that the restriction $i^*\omega = 0$. Then ω is exact, i.e. there is a $\mu \in \Omega^{k-1}(U)$ such that $d\mu = \omega$. Moreover, we can choose μ such that $\mu_x = 0$ for every x in X.*

Proof. Via the isomorphism $\varphi : U_0 \to U$ constructed in theorem 1.45, we can work on U_0 instead of U. For every $0 \leq t \leq 1$, we define

$$\begin{aligned} \rho_t : & U_0 \to U_0 \\ & (x, v) \mapsto (x, tv). \end{aligned}$$

This map is well-defined by the convexity of U_0. We have, using the same notation as in theorem 1.45 that $\rho_1 = id, \rho_0 = i_0 \circ \pi_0$, and that for every t, $\rho_t \circ i_0 = i_0$, which means that X is fixed under ρ_t. This shows that ρ_t is a homotopy relative to X and hence X is a deformation retract of U_0 with retraction $\pi_0 : U_0 \to X$.
We look for a homotopy operator between ρ_0 and ρ_1, i.e. a map $Q : \Omega^k(U_0) \to \Omega^{k-1}(U_0)$ such that $\rho_1 - \rho_0 = id - (i_0 \circ \pi_0)^* = dQ + Qd$. We define Q by

$$Q\sigma = \int_0^1 \rho_t^*(\iota_{v_t}\sigma)dt,$$

where v_t, at the point $q = \rho_t(p)$ is the vector tangent to the curve $\rho_s(p)$ at $s = t$. Let us prove that Q is indeed a homotopy operator.

$$\begin{aligned} Qd\sigma + dQ\sigma &= \int_0^1 \rho_t^*(\iota_{v_t}d\sigma)dt + d\int_0^1 \rho_t^*(\iota_{v_t}\sigma)dt = \int_0^1 \rho_t^*(\iota_{v_t}d\sigma + d\iota_{v_t}\sigma)dt \\ &\overset{thm1.33}{=} \int_0^1 \rho_t^*(\mathcal{L}_{v_t}\sigma)dt \overset{prop1.34}{=} \int_0^1 \frac{d}{dt}\rho_t^*\sigma = \rho_1^*\sigma - \rho_0^*\sigma, \end{aligned}$$

where the last equality comes from the fundamental theorem of calculus.
In our case, we have that $d\omega = 0$ since ω is closed, and that $i_0^*\omega = 0$ by assumption. Thus, using Q on ω yields

$$\omega = dQ\omega.$$

This means we can take $\mu = Q\omega$. Note that for $x \in X$, $\rho_t(x)$ is constant, and thus v_t vanishes at all $x \in X$, which proves that $\mu_x = 0$ for every x in X. This concludes the proof. ∎

We can now prove Moser's theorem, where we construct an isotopy between two symplectic forms fulfilling certain conditions. As before, we follow [5], where this result is theorem 7.2.

Theorem 1.48 (Moser's theorem)**.** *Let M be a compact $2n$-dimensional manifold and ω_0, ω_1 two symplectic forms on M such that $[\omega_0] = [\omega_1]$. Assume that the form $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic for every t in $[0,1]$. Then there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^*\omega_t = \omega_0$ for all t in $[0,1]$.*

Proof. The following argument is often useful in symplectic geometry and is called Moser's trick.
Suppose that there exists an isotopy ρ such that $\rho_t^*\omega_t = \omega_0$ for all t in $[0,1]$,

and let $X_t = \frac{d}{dt}\rho_t \circ \rho_t^{-1}$ be the corresponding time-dependent vector field. Then

$$0 = \frac{d}{dt}\rho_t^* \omega_t \overset{Prop.1.35}{=} \rho_t^* \left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t \right) \iff \mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t = 0. \tag{1.3}$$

Conversely, suppose that there exists a time-dependent vector field X_t such that equation 1.3 holds for every $0 \leq t \leq 1$. Since M is compact, we can integrate X_t to get an isotopy ρ with the property that

$$\frac{d}{dt}\rho_t^* \omega_t = 0.$$

But this tells us that $\rho_t^* \omega_t = \rho_0^* \omega_0 = \omega_0$, which is what we want.
This show that the whole proof boils down to finding a time-dependent vector field that fulfills equation 1.3. First, since $\omega_t = (1-t)\omega_0 + t\omega_1$, we have

$$\frac{d}{dt}\omega_t = \omega_1 - \omega_0.$$

Then, by our assumption that $[\omega_0] = [\omega_1]$, there must exist a 1-form μ such that $\omega_1 - \omega_0 = d\mu$. These observations take care of the second term of equation 1.3. For the first term, we use Cartan's magic formula (theorem 1.33) to write

$$\mathcal{L}_{X_t}\omega_t = d\iota_{X_t}\omega_t + \iota_{X_t}\underbrace{d\omega_t}_{=0} = d\iota_{X_t}\omega_t.$$

Hence, equation 1.3 becomes

$$d\iota_{X_t}\omega_t + d\mu = 0.$$

It is enough to solve

$$\iota_{X_t}\omega_t + \mu = 0. \tag{1.4}$$

Equation (1.4) is called the *Moser equation*. Since ω_t is non-degenerate, there is exactly one vector field X_t that fulfills this equation for a fixed t. Since ω_t is a smooth family of differential forms, we get a time-dependent vector field X_t. This concludes the proof. ■

Another version of Moser's theorem is the following result that shows what we can say about two symplectic forms that agree on a submanifold. We follow what is done in [5] theorem 7.4.

Theorem 1.49 (Moser's theorem relative version). *Let M be a compact manifold with symplectic forms ω_0 and ω_1. Let X be a compact submanifold of M with inclusion map $i : X \hookrightarrow M$. Suppose that for every $q \in X$, we have $\omega_0|_q = \omega_1|_q$.*

Then there exists neighborhoods U_0 and U_1 of X and a diffeomorphism $\varphi : U_0 \to U_1$ such that the following diagram commutes:

$$\begin{array}{ccc} U_0 & \xrightarrow{\quad\varphi\quad} & U_1 \\ & {}^{i}\nwarrow \qquad \nearrow^{i} & \\ & X & \end{array}$$

and such that $\varphi^*\omega_1 = \omega_0$.

Proof. Take a tubular neighborhood U_0 of X. The form $\omega_1 - \omega_0$ is closed on U_0 because it is the difference of two symplectic forms, and by assumption, $(\omega_1 - \omega_0)_x = 0$ for every $x \in X$. Thus we can use the homotopy formula from proposition 1.47 to get a 1-form μ on U_0 such that $\omega_1 - \omega_0 = d\mu$ and $\mu_x = 0$ for every x in X. Shrinking U_0 if necessary, we may assume that the family $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic for $0 \leq t \leq 1$. We can now use Moser's trick (or Moser's theorem 1.48) to get an isotopy $\rho : U_0 \times [0,1] \to M$ such that $\rho_t^*\omega_t = \omega_0, \forall t \in [0,1]$. Moreover, we notice that the vector field X_t from Moser's trick vanishes on X. Indeed, the Moser equation was given by $\iota_{X_t}\omega = -\mu$ and μ vanishes on X. This implies that $\frac{d}{dt}\rho_t = 0$ on X and so $\rho_t|_X = id_X$. Setting $\rho_1 = \varphi$ and $U_1 = \varphi(U_0)$ concludes the proof. ■

Here, we make a slight detour on our way to the proof of Weinstein's tubular neighborhood theorem to prove the Darboux theorem. As mentioned before, this is a very important result in symplectic geometry that states that every symplectic manifold of dimension $2n$ locally looks like $\mathbb{R}^{2n}$ equipped with the standard symplectic form defined in example 1.18. We follow the proof of theorem 8.1 in [5].

Theorem 1.50 (Darboux theorem)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold and let p be a point in M. Then there exists a coordinate chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on U, the symplectic form ω is given by*

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

Proof. The idea of this proof is to use the relative version of Moser theorem with $X = \{p\}$. Let us pick any symplectic basis of the vector space T_pM and use this basis to construct coordinates $x_1', \ldots, x_n', \ldots, y_1', \ldots, y_n'$ on a neighborhood U' centered at p and such that

$$\omega_p = \sum_{j=1}^{n} dx_j' \wedge dy_j'\big|_p.$$

Thus, by the relative Moser theorem, there exists neighborhoods U_0 and U_1 of p and a diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi(p) = p$ and

$$\omega = \varphi^* \sum_{j=1}^{n} dx'_j \wedge dy'_j = \sum_{j=1}^{n} \varphi^*(dx'_j) \wedge \varphi^*(dy'_j) = \sum_{j=1}^{n} d(x'_j \circ \varphi) \wedge d(y'_j \circ \varphi).$$

Hence, our coordinates are given by $x_j = x'_j \circ \varphi$ and $y_j = y'_j \circ \varphi$. This is well-defined because we can always shrink U_0 and U_1 enough so that $U_1 \subseteq U'$. This concludes the proof. ■

Let us now go back to the main goal of this section. We need one last result to be able to prove Weinstein's tubular neighborhood theorem. This is another neighborhood theorem known as Weinstein's Lagrangian neighborhood theorem. It states that if a submanifold X is Lagrangian with respect to two different symplectic structures ω_0 and ω_1, then there is a neighborhood of (X, ω_0) that is symplectomorphic to a neighborhood of (X, ω_1). To prove this result, we need another standard piece of differential geometry called the Whitney extension theorem. We give a proof detailing the sketch that is proposed in [5] for the proof of theorem 8.5.

Theorem 1.51 (Whitney Extension Theorem)**.** *Let M be a n-dimensional manifold and let $X \subset M$ be a k-dimensional submanifold such that $k < n$. Suppose that at each $p \in X$, we are given a linear isomorphism $L_p : T_pM \xrightarrow{\cong} T_pM$ such that $L_p|_{T_pX} = id_{T_pX}$ and such that L_p depends smoothly on p. Then there exists a neighborhood U of X in M and an embedding $h : U \to M$ such that $h|_X = id_X$ and $Dh(p) = L_p$ for all p in X.*

Proof. Let us choose a Riemannian metric m on M, and let U be a tubular neighborhood of X in M. As discussed in remark 1.46, U is the set of all the points q such that there is an x in X with $d(q, x) < \frac{1}{2}\varepsilon(x)$, where d is the distance induced by m and ε the positive function constructed in the proof of the tubular neighborhood theorem 1.45. Moreover, there is a submersion $\pi : U \to X$ that maps every point of U to the point in X that is closest to it. Every point q in U can be written as $exp(x, v) = \gamma_{(x,v)}(1)$, where $\gamma_{(x,v)}$ is the geodesic starting at $x \in X$ with initial velocity $v \in N_xX$ and such that $||v|| < \frac{1}{2}\varepsilon(x)$. We define, for $q = exp(x, v)$

$$\begin{aligned} h : &U \to M \\ &q \mapsto exp(x, L_x[v]). \end{aligned}$$

It is clear that if q belongs to X, the corresponding v will be 0, and hence $h(q) = exp(q, 0) = q$, which shows that h is the identity on X. Moreover, for q in X still,

$$Dh(q)[w] = Dexp(q, \overbrace{L_q 0}^{=0}) \circ \overbrace{DL_q(0)}^{=L_q}[w] = L_q[w].$$

Finally, let us denote by U_0 the tubular neighborhood of X as the zero section in NX, i.e. $exp(U_0) = U$. Since the maps L_p are isomorphisms and are the identity on T_pX for every p in X, we have that $L_p|_{N_pX}$ has image in N_pX for every p in X. By shrinking U_0 (and incidentally U as well) if necessary, we can ensure that $V := L_p(U_0|_p)$ is still small enough so that the exponential map is still a diffeomorphism on V. This proves that h is an embedding and hence concludes the proof. ■

Here is now the proof of Weinstein's Lagrangian neighborhood theorem. Both Whitney's extension theorem 1.51 and Moser's theorem 1.49 are needed in this argument. We follow the argument proposed in [5] for theorem 8.4.

Theorem 1.52 (Weinstein's Lagrangian neighborhood theorem)**.** *Let M be a $2n$-dimensional manifold, X a compact n-dimensional submanifold with inclusion map $i : X \hookrightarrow M$, and ω_0 and ω_1 two symplectic forms on M such that $i^*\omega_0 = i^*\omega_1 = 0$, i.e. X is Lagrangian for both symplectic structures. Then there exist neighborhoods U_0 and U_1 of X, as well as a diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi^*\omega_1 = \omega_0$ and such that the following diagram commutes:*

$$\begin{array}{ccc} U_0 & \xrightarrow{\ \varphi\ } & U_1 \\ & {}_{i}\nwarrow \quad \nearrow_{i} & \\ & X & \end{array} .$$

Proof. Choose a Riemannian metric m on M, and fix a point p in X. Let us denote by V the space T_pM, and by U the space T_pX. Let $U^\perp$ be the orthogonal complement of U with respect to the scalar product m_p. By assumption, U is a Lagrangian subspace of both $(V, \omega_0|_p)$, and $(V, \omega_1|_p)$. By proposition 1.13, we canonically get a linear isomorphism $L_p : T_pM \to T_pM$ from $U^\perp$, such that $L_p|_{T_pX} = id_{T_pX}$ and $L_p^*\omega_1|_p = \omega_0|_p$. Since this construction is canonical, L_p varies smoothly with respect to p.
Then, by the Whitney extension theorem 1.51, there exist a neighborhood $\mathcal{N}$ of X, and an embedding $h : \mathcal{N} \hookrightarrow M$ with $h|_X = id_X$ and $Dh(p) = L_p$ for $p \in X$. Hence, at any point p in X, we have

$$(h^*\omega_1)_p = (Dh(p))^*\omega_1|_p = L_p^*\omega_1|_p = \omega_0|_p.$$

Hence, we may use Moser's relative theorem 1.49 on ω_0 and $h^*\omega_1$ to get a neighborhood U_0 of X and an embedding $f : U_0 \to \mathcal{N}$ such that $f|_X = id_X$ and $f^*(h^*\omega_1) = \omega_0$ on U_0. Set $\varphi = h \circ f$ and $U_1 = \varphi(U_0)$. ■

We now have all the pieces for Weinstein's tubular neighborhood theorem. The proof is now just a matter of putting everything together. This follows the proof of theorem 9.3 in [5].

Theorem 1.53 (Weinstein's tubular neighborhood)**.** *Let (M, ω) be a symplectic manifold and $X \subseteq M$ a Lagrangian submanifold of M with inclusion $i : X \hookrightarrow M$. Let us denote by $i_0 : X \hookrightarrow T^*X$ the embedding of X into its cotangent bundle via the zero section. Then there are neighborhoods U_0 of X in T^*X and U of X in M, as well as a diffeomorphism $\varphi : U_0 \to U$ such that*

$$\begin{array}{ccc} U_0 & \xrightarrow{\varphi} & U \\ & {\scriptstyle i_0}\nwarrow \quad \nearrow{\scriptstyle i} & \\ & X & \end{array}$$

commutes and such that $\varphi^*\omega = \omega_0$, where ω_0 is the canonical symplectic form on T^*X.

Proof. Since $NX \simeq T^*X$ by prop 1.43, we may use the tubular neighborhood theorem 1.45 to get a neighborhood $\mathcal{N}_0$ of X in T^*X and a neighborhood $\mathcal{N}$ of X in M, as well as a diffeomorphism $\psi : \mathcal{N}_0 \to \mathcal{N}$ such that the following commutes

$$\begin{array}{ccc} \mathcal{N}_0 & \xrightarrow{\psi} & \mathcal{N} \\ & {\scriptstyle i_0}\nwarrow \quad \nearrow{\scriptstyle i} & \\ & X & \end{array} .$$

Let us denote by ω_0 the canonical form on T^*X and by ω_1 the form $\psi^*\omega$. Since the zero section is a Lagrangian submanifold of any cotangent bundle, we have that X is Lagrangian with respect to ω_0. Furthermore using the diagram, $i_0^*\omega_1 = i_0^*(\psi^*\omega) = i^*\omega = 0$, since X is a Lagrangian submanifold of M. This shows that X is also a Lagrangian submanifold in T^*X with respect to ω_1.
Then, by Weinstein's Lagrangian neighborhood theorem 1.52, there are neighborhoods U_0 and U_1 of X in $\mathcal{N}_0$ and a diffeomorphism $\theta : U_0 \to U_1$ such that $\theta^*\omega_1 = \omega_0$ and such that the following commutes

$$\begin{array}{ccc} U_0 & \xrightarrow{\theta} & U_1 \\ & {\scriptstyle i_0}\nwarrow \quad \nearrow{\scriptstyle i_0} & \\ & X & \end{array} .$$

Now set $\varphi = \psi \circ \theta$ and $U = \varphi(U_0)$. We have that $\varphi^*\omega = \theta^*\psi^*\omega = \theta^*\omega_1 = \omega_0$, which is what we wanted. ∎

The main advantage of this theorem is that we can look at a neighborhood U of the Lagrangian submanifold X inside T^*X, where the symplectic structure is the canonical one, which gives us a clearer picture of what this structure look like. We will use this extensively in chapters 4 and 5.

Chapter 2

Complex geometry

Symplectic geometry has a strong connection with complex geometry. Indeed, every symplectic manifold possesses a so-called almost complex structure, which is "compatible" with the symplectic structure. This connection is actually the starting point of some important work in symplectic geometry, namely the theory of pseudo-holomorphic curves. In this chapter, we develop the concepts and results from complex geometry that relate to the symplectic setting. As it was the case in the previous chapter, we prove most of the results in detail, both to make this text as self-contained as possible and to make sure that everything that we need in subsequent chapters has been covered. The presentation we make in this chapter follows what is done in lectures 12 to 16 of Cannas' lecture notes [5].

2.1 Almost complex structures

We begin by defining almost complex structures on manifolds. As it was the case with symplectic structures, it makes sense to start at the level of linear algebra.

Definition 2.1. Let V be a real vector space. A *complex structure* on V is a map $J : V \to V$ such that $J^2 = -id$. The pair (V, J) is called a complex vector space. Notice that a complex structure J is actually equivalent to a structure of a vector space over $\mathbb{C}$, if we identify J with multiplication by i.

Remark 2.2. Assume that (V, J) is a complex vector space, and has dimension n. Then, since $J^2 = -Id$, $det(J^2) = (det(J))^2 = (-1)^n$. Now, since J is a linear map from a real vector space to itself, it must have only real eigenvalues, and thus $det(J)$ must be real. This implies that n is even, and hence any complex vector space must have even dimension.

We will see that every symplectic vector space has a complex structure. Moreover, this complex structure is compatible in the following sense.

Definition 2.3. Let (V, Ω) be a symplectic vector space. A complex structure J on V is said to *compatible* with Ω (or Ω-*compatible*), if

$$G_J(u, v) := \Omega(u, Jv), \quad \forall u, v \in V$$

is a positive inner product on V.

Here is another way to characterize compatible complex structures.

Lemma 2.4. *J is compatible with Ω if and only if*

1. *$\Omega(Ju, Jv) = \Omega(u, v)$ for every u, v in V*
2. *$\Omega(u, Ju) > 0$ for every $u \neq 0$ in V.*

Proof. G_J is bilinear by definition, since Ω is bilinear and J is linear. Next, G_J is positive definite if and only if condition 2. holds. This again follows directly from the definition of G_J. Finally, the following chain of equalities show that symmetry of G_J is equivalent to the condition 1. above:

$$\Omega(Ju, Jv) = G_J(Ju, v) = G_J(v, Ju) = \Omega(v, JJu) = -\Omega(v, u) = \Omega(u, v).$$

■

We can now establish our first connection between complex and symplectic geometry by proving that every symplectic vector space admits a compatible complex structure. We follow the proof of proposition 12.3 in [5].

Proposition 2.5. *Let (V, Ω) be a symplectic vector space. Then there exists a compatible complex structure on V.*

Proof. Choose a positive inner product on G on V. Since both those pairing are non-degenerate, we get two isomorphisms

$$u \mapsto \Omega(u, \cdot)$$
$$u \mapsto G(u, \cdot)$$

between V and V^*. This implies that $\Omega(u, v) = G(Au, v)$ for some linear isomorphism $A : V \to V$. The maps A is skew-symmetric:

$$G(A^*u, v) = G(u, Av) = G(Av, u) = \Omega(v, u) = -\Omega(u, v) = G(-Au, v),$$

where A^* denotes the transpose of A. Hence, we have that AA^* is symmetric since $(AA^*)^* = AA^*$. Moreover, $G(AA^*u, u) = G(A^*u, A^*u) > 0$ for $u \neq 0$ and thus AA^* is positive. These properties imply that AA^* is diagonalizable with positive eigenvalues λ_j.

$$AA^* = B\text{diag}(\lambda_1, \ldots, \lambda_n)B^{-1}$$

We may then define

$$\sqrt{AA^*} = B\mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})B^{-1}.$$

Then $\sqrt{AA^*}$ is symmetric and positive definite. Let us define

$$J := (\sqrt{AA^*})^{-1}A.$$

The decomposition $A = \sqrt{AA^*}J$ is called the *polar decomposition* of A. It is clear from the diagonalization above that A commutes with $\sqrt{AA^*}$, and hence J commutes with $\sqrt{AA^*}$. We compute

$$JJ^* = (\sqrt{AA^*})^{-1}AA^*((\sqrt{AA^*})^{-1})^* = AA^*(AA^*)^{-1} = id.$$

Moreover, $J^* = -J$ and hence $J^2 = -JJ^* = -id$. This shows that J is a complex structure. It remains to see whether this structure is actually compatible.

$$\begin{aligned}\Omega(Ju, Jv) &= G(AJu, Jv) = G(JAu, Jv) = G(Au, J^*Jv)\\ &= G(Au, v) = \Omega(u, v)\end{aligned}$$

$$\begin{aligned}\Omega(u, Ju) &= G(Au, Ju) = G(-JAu, u) = G(-J\sqrt{AA^*}Ju, u)\\ &= G(\sqrt{AA^*}u, u) > 0 \quad \forall u \neq 0\end{aligned}$$

■

Remark 2.6. The inner product G_J induced by the complex structure J is in general not the same as the inner product G chosen at the beginning of the proof.

We now extend the concepts above to the category of manifolds. We have seen in the previous chapter that a symplectic structure on a manifold is a smooth family of linear symplectic structures on the tangent spaces. Similarly, an almost complex structure is smooth family of complex structures on the tangent spaces.

Definition 2.7. An *almost complex structure* on a manifold M is a smooth field a complex structure on the tangent spaces. This means that we have a smooth map $x \mapsto J_x$ such that J_x is a complex structure on T_xM for every x in M. The pair (M, J) is called an *almost complex manifold*. By remark 2.2, an almost complex manifold always has even dimension.

The definition of compatibility for an almost complex structure is the obvious one.

Definition 2.8. Let (M, ω) be a symplectic manifold. An almost complex structure J is called *compatible* with ω (or ω*-compatible*) if the mapping

$$\begin{aligned} x \mapsto & g_x : T_xM \times T_xM \to \mathbb{R} \\ & g_x(u, v) = \omega_x(u, J_x v) \end{aligned}$$

is a Riemannian metric on M. The triple (ω, g, J) is called a *compatible triple*.

We can now show that every symplectic manifold admits a compatible almost complex structure. In fact, most of the work has already been done in the proof of proposition 2.5.

Proposition 2.9. *Let (M, ω) be a symplectic manifold and m a Riemannian metric on M. Then there exists a canonical compatible almost complex structure J on M.*

We follow the proof of proposition 12.6 in [5].

Proof. Notice that the construction in the proof of proposition 2.5 is canonical once we fix the inner product G, hence the fiberwise construction on each tangent space will be smooth by smoothness of m. ∎

Since we can always equip a smooth manifold with a Riemannian metric, we get the following result. This is labeled as corollary 12.7 in [5].

Corollary 2.10. *Any symplectic manifold admits compatible almost complex structures.*

2.2 Complex-valued differential forms

In this section, we define complex-valued differential forms for an almost complex manifold. By investigating the structure of the space of these differential form, we will be able to understand the symplectic form of some almost complex manifolds better. We start by looking at the complexified tangent bundle of an almost complex manifold. The source for this material is [5], lecture 14.

Definition 2.11. Let (M, J) be an almost complex manifold. The *complexified tangent bundle* of M is the tangent bundle $TM \otimes \mathbb{C} \to M$, where the fiber at a point $p \in M$ is the space $(TM \otimes \mathbb{C})_p = T_pM \otimes \mathbb{C}$.

Since T_pM is a real vector space of dimension $2n$, then $T_pM \otimes \mathbb{C}$ is a complex vector space of complex dimension $2n$. Notice that every vector v in $TM \otimes \mathbb{C}$ can be written uniquely as $v = v_1 \otimes 1 + v_2 \otimes i$ for some v_1 and v_2 in TM.

The almost complex structure J can be extended to the complexified tangent bundle as follows:

$$J(v \otimes c) = Jv \otimes c \quad \forall v \in TM, c \in \mathbb{C}.$$

Notice that defining J on decomposable elements is enough since J is additive. By a slight abuse of notation, we will also denote by J the map on the complexified bundle. For any p in M, the map J_p on the complex vector space $T_pM \otimes \mathbb{C}$ is a linear map that squares to $-Id$. This implies that J_p has eigenvalues i and $-i$. The eigenspaces get their own names:

Definition 2.12. Let (M, J) be an almost complex manifold, and let J also denote the extension of the almost complex structure on $TM \otimes \mathbb{C}$. The i-eigenspace of J is called the space of *(J-)holomorphic tangent vectors* and is denoted by

$$T_{1,0} = \{v \in TM \otimes \mathbb{C} | Jv = iv\}.$$

Similarly, the $(-i)$-eigenspace of J is called the space of *(J-)anti-holomorphic tangent vectors* and is denoted by

$$T_{0,1} = \{v \in TM \otimes \mathbb{C} | Jv = -iv\}.$$

Let us give a more explicit description of these two eigenspaces.

Lemma 2.13. *We have the following characterization for holomorphic and anti-holomorphic tangent vectors:*

$$T_{1,0} = \{v \otimes 1 - Jv \otimes i |\ v \in TM\}$$
$$T_{0,1} = \{v \otimes 1 + Jv \otimes i |\ v \in TM\}$$

Proof. We show the result only for holomorphic tangent vectors as the proof follows the exact same idea for anti-holomorphic tangent vectors.
Let w be in TM. Then

$$J(w \otimes 1 - Jw \otimes i) = Jw \otimes 1 + w \otimes i = i(w \otimes 1 - Jw \otimes i),$$

so any vector of the form $v \otimes 1 - Jv \otimes i$ indeed belongs to $T_{1,0}$. For the converse inclusion, take $v = w \otimes c$ in $T_{1,0}$ with $w \in TM$ and $c \in \mathbb{C}$. Let us also set $c = a + bi$ for $a, b \in \mathbb{R}$. With this notation, we have that

$$J(w \otimes c) = Jw \otimes (a + bi) = aJw \otimes 1 + bJw \otimes i$$

and that

$$i(w \otimes c) = -bw \otimes 1 + aw \otimes i.$$

Then, since $v = w \otimes c$ belongs to $T_{1,0}$ we have that $Jv = iv$ and hence

$$aw = bJw \quad \text{and} \quad aJw = -bw.$$

This implies that

$$v = w \otimes c = aw \otimes 1 + bw \otimes i = aw \otimes 1 - Jaw \otimes i.$$

We may thus write w in the form we wanted. Since an arbitrary element of $T^{1,0}$ is a finite sum of decomposable elements, and every decomposable element can be written in the form we want, the reverse inclusion is proved. ∎

We may now define the maps

$$\begin{aligned} \pi_{1,0} :& TM \to T_{1,0} \\ & v \mapsto \frac{1}{2}(v \otimes 1 - Jv \otimes i) \end{aligned}$$

and

$$\begin{aligned} \pi_{0,1} :& TM \to T_{0,1} \\ & v \mapsto \frac{1}{2}(v \otimes 1 + Jv \otimes i). \end{aligned}$$

It is clear that those maps are invertible, and thus both $\pi_{1,0}$ and $\pi_{0,1}$ are (real) vector bundle isomorphisms. Next, we have that $\pi_{1,0} \circ J(v) = \frac{1}{2}(Jv \otimes 1 + v \otimes i) = i(\pi_{1,0}(v))$ for every v in TM, and similarly $\pi_{0,1} \circ J = -i\pi_{0,1}$. Since $T_{1,0}$ and $T_{0,1}$ can be seen as complex bundles where the complex structure is just multiplication by i, this implies that we have isomorphisms of complex vector bundles

$$(TM, J) \simeq T_{1,0} \simeq \overline{T}_{0,1}$$

where $\overline{T}_{0,1}$ is the conjugate bundle to $T_{0,1}$. Let us now extend the maps $\pi_{1,0}$ and $\pi_{0,1}$ to the projections onto $T_{1,0}$ and $T_{0,1}$ by

$$\begin{aligned} \pi_{1,0} :& TM \otimes \mathbb{C} \to T_{1,0} \\ & v \mapsto \frac{1}{2}(v - iJv) \\ \pi_{0,1} :& TM \otimes \mathbb{C} \to T_{0,1} \\ & v \mapsto \frac{1}{2}(v + iJv). \end{aligned}$$

A straightforward computation shows that the extensions of $\pi_{1,0}$ and $\pi_{0,1}$ are indeed projections onto the i, respectively $(-i)$-eigenspaces. We only show the computation for $\pi_{1,0}$ as the other is identical. Let us write $v =$

$v_1 \otimes 1 + v_2 \otimes i$ with $v_1, v_2 \in TM$.

$$\begin{aligned} J(\frac{1}{2}(v - iJv)) &= J(\frac{1}{2}(v_1 \otimes 1 + v_2 \otimes i - Jv_1 \otimes i + Jv_2 \otimes 1)) \\ &= \frac{1}{2}(Jv_1 \otimes 1 + Jv_2 \otimes i + v_1 \otimes i - v_2 \otimes 1) \\ &= i(\frac{1}{2}(v_1 \otimes 1 + v_2 \otimes i - Jv_1 \otimes i + Jv_2 \otimes 1)) \\ &= i(\frac{1}{2}(v - iJv)). \end{aligned}$$

Hence, we get an isomorphism

$$(\pi_{1,0}, \pi_{0,1}) : TM \otimes \mathbb{C} \xrightarrow{\sim} T_{1,0} \oplus T_{0,1}.$$

The exact same construction can applied to the cotangent bundle.

Definition 2.14. We define the space of *complex-linear cotangent vectors* as the subspace of $T^*M \otimes \mathbb{C}$ given by

$$T^{1,0} = (T_{1,0})^* = \{\eta \in T^*M \otimes \mathbb{C} | \eta(Jv) = i\eta(v), \forall v \in TM \otimes \mathbb{C}\}.$$

Similarly, the space of *complex- antilinear cotangent vectors* is given by

$$T^{0,1} = (T_{0,1})^* = \{\eta \in T^*M \otimes \mathbb{C} | \eta(Jv) = -i\eta(v), \forall v \in TM \otimes \mathbb{C}\}.$$

As with the complexified tangent bundle, we have a useful way of writing elements of $T^{1,0}$ and $T^{0,1}$.

Lemma 2.15. *We have that*

$$\begin{aligned} T^{1,0} &= \{\xi \otimes 1 - (\xi \circ J) \otimes i | \xi \in T^*M\}, \\ T^{0,1} &= \{\xi \otimes 1 + (\xi \circ J) \otimes i | \xi \in T^*M\}. \end{aligned}$$

The proof is identical to the one for the tangent bundle, i. e. the proof of lemma 2.13, so we will not rewrite it here. Lemma 2.15 also shows that $T^{1,0}$ and $T^{0,1}$ are complex conjugate to each other, as we have seen with $T_{1,0}$ and $T_{0,1}$. Still in analogy with what we did on $TM \otimes \mathbb{C}$, we define the two natural projection maps

$$\begin{aligned} \pi^{1,0} : & T^*M \otimes \mathbb{C} \to T^{1,0} \\ & \eta \mapsto \frac{1}{2}(\eta - i\eta \circ J), \\ \pi^{0,1} : & T^*M \otimes \mathbb{C} \to T^{0,1} \\ & \eta \mapsto \frac{1}{2}(\eta + i\eta \circ J). \end{aligned}$$

This gives us the following splitting of the complexified cotangent bundle:

$$(\pi^{1,0}, \pi^{0,1}) : T^*M \otimes \mathbb{C} \xrightarrow{\sim} T^{1,0} \oplus T^{0,1}.$$

The splitting of the complexified cotangent bundle yields a splitting of the k-th exterior algebra $\Lambda^k(T^*M \otimes \mathbb{C})$. Let us denote by $\Lambda^{l,m}$ the subalgebra of $\Lambda^{l+m}(T^*M \otimes \mathbb{C})$ given by

$$\Lambda^{l,m} := \{v_1 \wedge \cdots \wedge v_l \wedge w_1 \wedge \cdots \wedge w_m | v_1, \dots, v_l \in T^{1,0}, w_1, \dots, w_m \in T^{0,1}\}.$$

The splitting of $\Lambda^k(T^*M \otimes \mathbb{C})$ is then given by

$$\Lambda^k(T^*M \otimes \mathbb{C}) = \Lambda^k(T^{1,0} \oplus T^{0,1}) = \bigoplus_{l+m=k} \Lambda^{l,m}.$$

We can now define complex-valued differential forms, and the splitting of $\Lambda^k(T^*M \otimes \mathbb{C})$ will give us a splitting of the space of differential forms.

Definition 2.16. Let (M, J) be an almost complex manifold. Then the *complex-valued k-forms* on M are defined to be the sections of $\Lambda^k(T^*M \otimes \mathbb{C})$ and are denoted by $\Omega^k(M; \mathbb{C})$. We define $\Omega^{l,m}$ to be the space of sections of $\Lambda^{l,m}$. These sections are called *forms of type* (l, m) or just (l, m)*-forms*. By construction, we have the splitting

$$\Omega^k(M; \mathbb{C}) = \bigoplus_{l+m=k} \Omega^{l,m}.$$

Let $l + m = k$ and let $\pi^{l,m} : \Lambda^k(T^*M \otimes \mathbb{C}) \to \Lambda^{l,m}$ denote the projection map. Composing the exterior derivative d with the projections maps allows us to define two differential operators on the forms of type (l, m):

$$\begin{aligned} \partial :&\Omega^{l,m} \to \Omega^{l+1,m} \\ &\sigma \mapsto \pi^{l+1,m} d\sigma \\ \bar{\partial} :&\Omega^{l,m} \to \Omega^{l,m+1} \\ &\sigma \mapsto \pi^{l,m+1} d\sigma. \end{aligned}$$

Thus, if σ is a form of type (l, m) with $l + m = k$, then $d\sigma$ lies in $\Omega^{k+1}(M; \mathbb{C})$ and can be decomposed as

$$d\sigma = \sum_{r+s=k+1} \pi^{r,s} d\sigma = \pi^{k+1,0} d\sigma + \cdots + \partial\sigma + \bar{\partial}\sigma + \cdots + \pi^{0,k+1} d\sigma.$$

Notice that for a complex-valued function $f : M \to \mathbb{C}$, we have that $df = \partial f + \bar{\partial} f$, and hence the differential of any function can be decomposed into its linear and antilinear part. We have the following definitions.

Definition 2.17. Let (M, J) be an almost complex manifold. A function $f \in C^\infty(M; \mathbb{C})$ is called $(J-)$*holomorphic* at $p \in M$ if df_p is complex linear, i.e. $df_p \circ J = i df_p$. The function f is called $(J-)$*holomorphic* if it is holomorphic at every point. Similarly, we have the notion of a $(J-)$*anti-holomorphic* function if df is complex anti-linear.

Any holomorphic function has that $df = \partial f$, because its differential is complex linear and $d = \partial + \bar{\partial}$ for functions. Thus, f is holomorphic if and only if $\bar{\partial} f = 0$. Similarly, f is anti-holomorphic if and only if $\partial f = 0$.

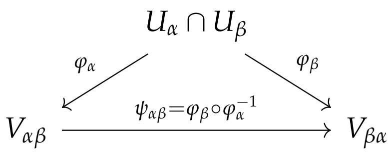
2.3 Kähler manifolds

In this section, we look at Kähler manifolds, which are symplectic manifolds that are also complex manifolds. We start by defining complex manifolds and showing that, as the name suggests, every complex manifold is also an almost complex manifold. We will then see that the space of complex-valued forms of a complex manifold has some useful properties that we will use to understand the symplectic structure of Kähler manifolds. We follow lectures 15 and 16 in [5].

Definition 2.18. A *complex manifold* of complex dimension n is a topological manifold equipped with a complex atlas

$$\mathcal{A} = \{(U_\alpha, V_\alpha, \varphi_\alpha)\}_{\alpha \in I},$$

where I is an index set, $M = \bigcup_\alpha U_\alpha$, the V_α's are open subsets of $\mathbb{C}^n$, and the φ_α's are homeomorphisms making the transition functions $\psi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ biholomorphic. This means that we have the following diagram:



where $V_{\alpha\beta} = \varphi_\alpha(U_\alpha \cap U_\beta)$, and $V_{\beta\alpha} = \varphi_\beta(U_\alpha \cap U_\beta)$.

If a manifold is complex, it automatically has an almost complex structure.

Proposition 2.19. *Any complex manifold M admits a canonical almost complex structure*

The proof comes from [5], theorem 15.2.

Proof. We first define the almost complex structure J locally, and then show that it is globally well-defined.
Let $(U, V, \varphi : U \to V)$ be a complex chart on M. Let us denote by $z_1, \ldots, z_n$, $z_j = x_j + iy_j$ the complex coordinates on U. For any point p in U, the tangent space is the $\mathbb{R}$-span of the vectors

$$\left\{ \frac{\partial}{\partial x_j}\bigg|_p, \frac{\partial}{\partial y_j}\bigg|_p \right\}_{j=1,\ldots,n}.$$

We define J on U by setting

$$J_p\left(\frac{\partial}{\partial x_j}\bigg|_p\right) = \frac{\partial}{\partial y_j}\bigg|_p$$
$$J_p\left(\frac{\partial}{\partial y_j}\bigg|_p\right) = -\frac{\partial}{\partial x_j}\bigg|_p$$

for $j = 1, \ldots, n$. We now need to check that this is well-defined on the whole of M. Let (U, V, φ) and (U', V', φ') be two charts on M such that $U \cap U' \neq \emptyset$. We want to show that $J = J'$ on $U \cap U'$. Let us denote by $z_j = x_j + iy_j$ the coordinates on U and by $w_j = u_j + iv_j$ the coordinates on U', so that we can write $\varphi = (z_1, \ldots, z_n)$ and $\varphi' = (w_1, \ldots, w_n)$. If we denote by ψ the transition map between φ and φ', we have $\psi(z_1, \ldots, z_n) = (w_1, \ldots, w_n)$. Using the chain rule, we have that

$$\frac{\partial}{\partial x_k} = \sum_{j=1}^{n} \left(\frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k}\frac{\partial}{\partial v_j} \right)$$
$$\frac{\partial}{\partial y_k} = \sum_{j=1}^{n} \left(\frac{\partial u_j}{\partial y_k}\frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial y_k}\frac{\partial}{\partial v_j} \right).$$

Since ψ is a transition map, it is biholomorphic and thus it must satisfy the Cauchy-Riemann equations for each of its components. With our notation, this yields

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k}$$
$$\frac{\partial u_j}{\partial y_k} = -\frac{\partial v_j}{\partial x_k}$$

for $j,k = 1,\dots,n$. Now for any point p in $U \cap U'$ we have that

$$\begin{aligned}
J'_p\left(\frac{\partial}{\partial x_k}\bigg|_p\right) &= J'_p\left(\sum_{j=1}^{n}\left(\frac{\partial u_j}{\partial x_k}(p)\frac{\partial}{\partial u_j}\bigg|_p + \frac{\partial v_j}{\partial x_k}(p)\frac{\partial}{\partial v_j}\bigg|_p\right)\right)\\
&= \sum_{j=1}^{n}\left(\frac{\partial u_j}{\partial x_k}(p)J'_p\left(\frac{\partial}{\partial u_j}\bigg|_p\right) + \frac{\partial v_j}{\partial x_k}(p)J'_p\left(\frac{\partial}{\partial v_j}\bigg|_p\right)\right)\\
&= \sum_{j=1}^{n}\left(\frac{\partial u_j}{\partial x_k}(p)\frac{\partial}{\partial v_j}\bigg|_p + \frac{\partial v_j}{\partial x_k}(p)\left(-\frac{\partial}{\partial u_j}\bigg|_p\right)\right)\\
&= \sum_{j=1}^{n}\left(\frac{\partial v_j}{\partial y_k}(p)\frac{\partial}{\partial v_j}\bigg|_p + \frac{\partial u_j}{\partial y_k}(p)\frac{\partial}{\partial u_j}\bigg|_p\right) = \frac{\partial}{\partial y_k}\bigg|_p,
\end{aligned}$$

and a similar computation shows that

$$J'_p\left(\frac{\partial}{\partial y_k}\bigg|_p\right) = -\frac{\partial}{\partial x_k}$$

for any $k \in \{1,\dots,n\}$. This concludes the proof. ■

If an almost complex structure is induced from a complex manifold structure, it gets a special name.

Definition 2.20. An almost complex structure J on M is called *integrable* if and only if it is induced by a structure of complex manifold on M.

For a complex manifold M, the splitting $\Omega^k(M;\mathbb{C}) = \bigoplus_{l+m=k}\Omega^{l,m}$ is easier to describe than for an arbitrary almost complex manifold. To see this, we investigate what complex-valued differential forms look like locally on a complex manifold. Let M be a complex manifold and denote by J the induced almost complex structure on M. Let U be a coordinate chart on M with complex coordinate functions $z_1,\dots,z_n$ and real coordinate functions $x_1,y_1,\dots,x_n,y_n$ such that $z_j = x_j + iy_j$. Let us fix a point p in U. We have that

$$T_pM = \mathbb{R} - \operatorname{span}\left\{\frac{\partial}{\partial x_j}\bigg|_p, \frac{\partial}{\partial y_j}\bigg|_p\right\}$$

and similarly

$$T_pM \otimes \mathbb{C} = \mathbb{C} - \operatorname{span}\left\{\frac{\partial}{\partial x_j}\bigg|_p, \frac{\partial}{\partial y_j}\bigg|_p\right\}.$$

Let us now see what $(T_{1,0})_p$ and $(T_{0,1})_p$ look like in this coordinate chart. By what we have seen previously, $(T_{1,0})_p$ is the subspace containing vectors of the form $\frac{1}{2}(v - iJv)$ for $v \in TM \otimes \mathbb{C}$. This implies that $(T_{1,0})_p$ can be described as the $\mathbb{C}$-span of

$$\frac{1}{2}\left(\frac{\partial}{\partial x_j}\bigg|_p - iJ\left(\frac{\partial}{\partial x_j}\bigg|_p\right)\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\bigg|_p - i\frac{\partial}{\partial y_j}\bigg|_p\right) =: \frac{\partial}{\partial z_j}\bigg|_p$$

and

$$\frac{1}{2}\left(\frac{\partial}{\partial y_j}\bigg|_p - iJ\left(\frac{\partial}{\partial y_j}\bigg|_p\right)\right) = \frac{1}{2}\left(\frac{\partial}{\partial y_j}\bigg|_p + i\frac{\partial}{\partial x_j}\bigg|_p\right) =: i\frac{\partial}{\partial z_j}\bigg|_p$$

for $j = 1, \ldots, n$. Since the two expressions are $\mathbb{C}$-colinear, we may write

$$(T_{1,0})_p = \mathbb{C} - \operatorname{span}\left\{\frac{\partial}{\partial z_j}\bigg|_p\right\}_{j=1,\ldots,n}.$$

The same computations on $(T_{0,1})_p$ yield

$$(T_{0,1})_p = \mathbb{C} - \operatorname{span}\left\{\frac{\partial}{\partial \bar{z}_j}\bigg|_p\right\}_{j=1,\ldots,n},$$

where

$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Once again, the exact same idea applies to the complexified cotangent bundle. We have that

$$T_p^* M \otimes \mathbb{C} = \mathbb{C} - \operatorname{span}\{(dx_j)_p, (dy_j)_p\}_{j=1,\ldots n}.$$

Notice that

$$\begin{aligned}
(dx_j)_p \circ J\left(\frac{\partial}{\partial x_k}\bigg|_p\right) &= (dx_j)_p\left(\frac{\partial}{\partial y_k}\bigg|_p\right) = 0, \\
(dx_j)_p \circ J\left(\frac{\partial}{\partial y_k}\bigg|_p\right) &= -(dx_j)_p\left(\frac{\partial}{\partial x_k}\bigg|_p\right) = -\delta_{jk},
\end{aligned}$$

and so we have that $dx_j \circ J = -dy_j$. A similar computation shows that $dy_j \circ J = dx_j$. We can now do the same kind of computations we did for the tangent bundle the get bases for $(T^{1,0})_p$ and $(T^{0,1})_p$. If we define

$$\begin{aligned}
dz_j &:= dx_j + idy_j, \\
d\bar{z}_j &:= dx_j - idy_j,
\end{aligned}$$

we get

$$\begin{aligned}
(T^{1,0})_p &= \mathbb{C} - \operatorname{span}\{dx_j + idy_j\}_{j=1,\ldots,n} = \mathbb{C} - \operatorname{span}\{dz_j\}_{j=1,\ldots,n}, \\
(T^{0,1})_p &= \mathbb{C} - \operatorname{span}\{dx_j - idy_j\}_{j=1,\ldots,n} = \mathbb{C} - \operatorname{span}\{d\bar{z}_j\}_{j=1,\ldots,n}.
\end{aligned}$$

With this local description of $T^*M \otimes \mathbb{C}$, we can write complex-valued forms on U in terms of the local frame given by the dz_j's and the $d\bar{z}_j$'s. Let us denote by $\Omega^{l,m}(U;\mathbb{C})$ the (l,m)-forms on U. We have that:

$$\Omega^{1,0}(U;\mathbb{C}) = \left\{ \sum_j b_j dz_j \,\middle|\, b_j \in C^\infty(U;\mathbb{C}) \right\},$$

$$\Omega^{0,1}(U;\mathbb{C}) = \left\{ \sum_j b_j d\bar{z}_j \,\middle|\, b_j \in C^\infty(U;\mathbb{C}) \right\},$$

$$\Omega^{1,1}(U;\mathbb{C}) = \left\{ \sum_{j_1,j_2} b_{j_1,j_2} dz_{j_1} \wedge d\bar{z}_{j_2} \,\middle|\, b_{j_1,j_2} \in C^\infty(U;\mathbb{C}) \right\},$$

$$\Omega^{2,0}(U;\mathbb{C}) = \left\{ \sum_{j_1<j_2} b_{j_1,j_2} dz_{j_1} \wedge dz_{j_2} \,\middle|\, b_{j_1,j_2} \in C^\infty(U;\mathbb{C}) \right\},$$

$$\Omega^{0,2}(U;\mathbb{C}) = \left\{ \sum_{j_1<j_2} b_{j_1,j_2} d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \,\middle|\, b_{j_1,j_2} \in C^\infty(U;\mathbb{C}) \right\}.$$

More generally, if we use multi-index notation with $J = (j_1, \dots, j_m)$, $1 \leq j_1 < \cdots < j_m \leq n$, $|J| = m$ and $dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_m}$, we have that

$$\Omega^{l,m}(U;\mathbb{C}) = \left\{ \sum_{|J|=l,|K|=m} b_{J,K} dz_J \wedge d\bar{z}_K \,\middle|\, b_{J,K} \in C^\infty(U;\mathbb{C}) \right\}.$$

Hence, any form $\sigma \in \Omega^k(U;\mathbb{C})$ can be written as

$$\sum_{l+m=k} \sum_{|J|=l,|K|=m} b_{J,K} dz_J \wedge d\bar{z}_K,$$

with $b_{J,K} \in C^\infty(U;\mathbb{C})$. Taking the exterior derivative of σ yields

$$d\sigma = \sum_{l+m=k} \sum_{|J|=l,|K|=m} db_{J,K} \wedge dz_J \wedge d\bar{z}_K.$$

Recall that on functions $f \in C^\infty(U;\mathbb{C})$, the exterior derivative decomposes as

$$df = \partial f + \bar{\partial} f$$

and thus $d\sigma$ can be written as

$$\begin{aligned} d\sigma &= \sum_{l+m=k} \sum_{|J|=l,|K|=m} (\partial b_{J,K} + \overline{\partial} b_{J,K}) \wedge dz_J \wedge d\overline{z}_K \\ &= \sum_{l+m=k} \Bigg(\underbrace{\sum_{|J|=l,|K|=m} \partial b_{J,K} \wedge dz_J \wedge d\overline{z}_K}_{\in \Omega^{l+1,m}} + \underbrace{\sum_{|J|=l,|K|=m} \overline{\partial} b_{J,K} \wedge dz_J \wedge d\overline{z}_K}_{\in \Omega^{l,m+1}} \Bigg) \\ &= \partial\sigma + \overline{\partial}\sigma, \end{aligned}$$

where we slightly abuse the notation by writing ∂, respectively $\overline{\partial}$ for $\sum_{l,m} \partial_{l,m}$, respectively $\overline{\partial}_{l,m}$. Here $\partial_{l,m}$ denotes the map $\Omega^{l,m} \to \Omega^{l+1,m}$ and similarly for $\overline{\partial}_{l,m}$.

The conclusion of all of this is that for complex manifold, the exterior differential is given by

$$d\sigma = \partial\sigma + \overline{\partial}\sigma.$$

A first consequence of this is that

$$d^2\sigma = \partial^2\sigma + (\partial\overline{\partial} + \overline{\partial}\partial)\sigma + \overline{\partial}^2\sigma = 0.$$

Since each of these three terms if of a different type, they must all individually be zero. Hence, on a complex manifold, we have that

$$\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{2.1}$$

Remark 2.21. The fact that $d = \partial + \overline{\partial}$ does not hold for an arbitrary almost complex manifold (M, J) because one does not have a nice local frame for $T^*M|_U \otimes \mathbb{C}$, as we had in the complex case with the dz_j's and the $d\overline{z}_j$'s.

Remark 2.22. For a function $f \in C^\infty(U; \mathbb{C})$, we have nice expressions for ∂f and $\overline{\partial} f$ in terms of the coordinates.

$$\begin{aligned} df &= \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \right) \\ &= \sum_{j=1}^n \left[\frac{1}{2} \left(\frac{\partial f}{\partial x_j} - i \frac{\partial f}{\partial y_j} \right) (dx_j + i dy_j) + \frac{1}{2} \left(\frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j} \right) (dx_j - i dy_j) \right] \\ &= \sum_{j=1}^n \left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \right). \end{aligned}$$

This implies that we have the expressions

$$\begin{aligned} \partial f &= \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j, \\ \overline{\partial} f &= \sum_{j=1}^n \frac{\partial}{\partial \overline{z}_j} d\overline{z}_j. \end{aligned} \tag{2.2}$$

We now use (2.1) to investigate the symplectic structure of manifolds that are also complex, namely Kähler manifolds.

Definition 2.23. Let (M, ω) be a symplectic manifold. M is called a *Kähler manifold* if it is equipped with an integrable and compatible almost complex structure. The symplectic form ω is then called a *Kähler form*.

It is immediate from the definition that every Kähler manifold is a complex manifold. We can now use what we now about differential forms on a complex manifold to look at what Kähler forms look like locally. Let (M, ω) be a Kähler manifold. First, since ω is a two-form, on a complex chart $(U, z_1, \dots, z_n)$, we have that

$$\omega = \sum_{j<k} a_{jk} dz_j \wedge dz_k + \sum_{j,k} b_{jk} dz_j \wedge d\bar{z}_k + \sum_{j<k} c_{jk} d\bar{z}_j \wedge d\bar{z}_k$$

for functions $a_{jk}, b_{jk}, c_{jk} \in C^\infty(U, \mathbb{C})$.
Next, the induced almost complex structure J of M must be compatible with ω. In particular, J must be a symplectomorphism, i.e. we must have $J^*\omega = \omega$ where $(J^*\omega)(v, w) = \omega(Jv, Jw)$ for any tangent vectors v and w. We already know that $dz_j \circ J = idz_j$ and $d\bar{z}_j \circ J = -id\bar{z}_j$ so

$$J^*\omega = \sum_{j<k} -a_{jk} dz_j \wedge dz_k + \sum_{j,k} b_{jk} dz_j \wedge d\bar{z}_k + \sum_{j<k} -c_{jk} d\bar{z}_j \wedge d\bar{z}_k.$$

This implies that $J^*\omega = \omega$ if and only if the functions a_{jk} and c_{jk} are zero for all j and k. Hence, ω belongs to $\Omega^{1,1}$.

Let us now use the fact that ω is closed. Since M is a complex manifold, $d = \partial + \bar{\partial}$ and thus

$$0 = d\omega = \partial\omega + \bar{\partial}\omega.$$

Since $\partial\omega$ is a $(2, 1)$-form and $\bar{\partial}\omega$ a $(1, 2)$-form, both must be zero. This shows that ω is both ∂ and $\bar{\partial}$-closed.

Let us slightly rewrite the expression we have for ω in order to ease the notation. Setting $b_{jk} = \frac{i}{2} h_{jk}$ yields

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk} dz_j \wedge d\bar{z}_k.$$

with $h_{jk} \in C^\infty(U; \mathbb{C})$. Since ω is a symplectic form, it is by definition real-valued, which is equivalent to asking that $\omega = \bar{\omega}$.

$$\bar{\omega} = -\frac{i}{2} \sum_{j,k=1}^{n} \bar{h}_{jk} d\bar{z}_j \wedge dz_k = \frac{i}{2} \sum_{j,k=1}^{n} \bar{h}_{jk} dz_k \wedge d\bar{z}_j = \frac{i}{2} \sum_{j,k=1}^{n} \bar{h}_{kj} dz_j \wedge d\bar{z}_k.$$

Hence, ω is real-valued if and only if $h_{jk} = \overline{h}_{kj}$. Hence, at every point p in U, the matrix $(h_{jk}(p))$ is hermitian.

To deduce the next property of the Kähler form ω, we will need to prove that the following formula holds:

$$\omega^n = n!\left(\frac{i}{2}\right)^n det(h_{jk}) dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n. \tag{2.3}$$

First, recall that $dz_j \wedge dz_j = d\overline{z}_k \wedge d\overline{z}_k = 0$. This implies that every non-vanishing term in the expression for ω^n will be of the form

$$\frac{i}{2} h_{j_1k_1} h_{j_2k_2} \dots h_{j_nk_n} dz_{j_1} \wedge d\overline{z}_{k_1} \wedge dz_{j_2} \wedge d\overline{z}_{k_2} \wedge \cdots \wedge dz_{j_n} \wedge d\overline{z}_{k_n}, \tag{2.4}$$

where $\overline{J} = (j_1, \dots, j_n)$ and $\overline{K} = (k_1, \dots, k_n)$ are permutations of the numbers $\{1, \dots, n\}$. Let $\rho \in S_n$ denote an arbitrary permutation of $\{1, \dots, n\}$. Notice that

$$\begin{aligned} &\frac{i}{2} h_{j_1k_1} \dots h_{j_nk_n} dz_{j_1} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge dz_{j_n} \wedge d\overline{z}_{k_n} \\ =&\frac{i}{2} h_{j_{\rho(1)}k_{\rho(1)}} \dots h_{j_{\rho(n)}k_{\rho(n)}} dz_{j_{\rho(1)}} \wedge d\overline{z}_{k_{\rho(1)}} \wedge \cdots \wedge dz_{j_{\rho(n)}} \wedge d\overline{z}_{k_{\rho(n)}}. \end{aligned} \tag{2.5}$$

This is true because for any $1 \leq l \leq n-1$, permuting $(dz_{j_{\rho(l)}} \wedge d\overline{z}_{k_{\rho(l)}})$ with $(dz_{j_{\rho(l+1)}} \wedge d\overline{z}_{k_{\rho(l+1)}})$ does not change the sign of the expression, as we are permuting two-forms. The expression for ω^n will then be given by a sum of terms as in (2.4), where we sum over pairs of arrangements $\overline{J} = (j_1, \dots, j_n)$ and $\overline{K} = (k_1, \dots, k_n)$ of the numbers $\{1, \dots, n\}$ as follows:

$$\omega^n = \left(\frac{i}{2}\right)^n \sum_{(\overline{J}, \overline{K})} h_{j_1k_1} \dots h_{j_nk_n} dz_{j_1} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge dz_{j_n} \wedge d\overline{z}_{k_n}.$$

Now, let us fix an arrangement $\overline{J} = (j_1, \dots, j_n)$. Then there exists a unique permutation $\rho \in S_n$ such that $(j_{\rho(1)}, \dots, j_{\rho(n)}) = (1, \dots, n)$. Thus, using equation (2.5), we have that

$$\begin{aligned} &\sum_{\overline{K}} h_{j_1k_1} \dots h_{j_nk_n} dz_{j_1} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge dz_{j_n} \wedge d\overline{z}_{k_n} \\ =&\sum_{\overline{K}} h_{1k_{\rho(1)}} \dots h_{nk_{\rho(n)}} dz_1 \wedge d\overline{z}_{k_{\rho(1)}} \wedge \cdots \wedge dz_n \wedge d\overline{z}_{k_{\rho(n)}}. \end{aligned}$$

Since there is exactly one such ρ for every arrangement $\overline{J}$, summing over all the $\overline{J}$'s is the same thing as summing over every $\rho \in S_n$. Moreover for an arbitrary $\overline{K} = (k_1, \dots, k_n)$, taking $(k_{\rho(1)}, \dots, k_{\rho_n})$ for every $\rho \in S_n$ just gives back every permutation of $\{1, \dots, n\}$, and so the term

$$\sum_{\rho \in S_n} h_{1k_{\rho(1)}} \dots h_{nk_{\rho(n)}} dz_1 \wedge d\overline{z}_{k_{\rho(1)}} \wedge \dots \wedge dz_n \wedge d\overline{z}_{k_{\rho(n)}}$$

is the same for every arrangement $\overline{K}$. Thus, since the cardinality of S_n is $n!$, our formula for ω^n becomes

$$\begin{aligned}
&\left(\frac{i}{2}\right)^n \sum_{\overline{J}} \sum_{\overline{K}} h_{j_1k_1} \dots h_{j_nk_n} dz_{j_1} \wedge d\overline{z}_{k_1} \wedge \dots \wedge dz_{j_n} \wedge d\overline{z}_{k_n} \\
=&\left(\frac{i}{2}\right)^n \sum_{\rho \in S_n} \sum_{\overline{K}} h_{1k_{\rho(1)}} \dots h_{nk_{\rho(n)}} dz_1 \wedge d\overline{z}_{k_{\rho(1)}} \wedge \dots \wedge dz_n \wedge d\overline{z}_{k_{\rho(n)}} \\
=&n!\left(\frac{i}{2}\right)^n \sum_{\overline{K}} h_{1k_{\rho(1)}} \dots h_{nk_{\rho(n)}} dz_1 \wedge d\overline{z}_{k_{\rho(1)}} \wedge \dots \wedge dz_n \wedge d\overline{z}_{k_{\rho(n)}}.
\end{aligned}$$

Now, notice that for every arrangement K, there exists a unique permutation η such that $(k_{\rho(1)}, \dots, k_{\rho(n)}) = (\eta(1), \dots, \eta(n))$. Hence, we can rewrite the sum as

$$n!\left(\frac{i}{2}\right)^n \sum_{\eta \in S_n} h_{1\eta(1)} \dots h_{n\eta(n)} dz_1 \wedge d\overline{z}_{\eta(1)} \wedge \dots \wedge dz_n \wedge d\overline{z}_{\eta(n)}.$$

For the last step, we need to rearrange $dz_1 \wedge d\overline{z}_{\eta(1)} \wedge \dots \wedge dz_n \wedge d\overline{z}_{\eta(n)}$ to $dz_1 \wedge d\overline{z}_1 \wedge \dots \wedge dz_n \wedge d\overline{z}_n$. The dz_j's are already in the right order, so we only need to permute the $d\overline{z}_j$'s. We have that

$$\begin{aligned}
&dz_1 \wedge d\overline{z}_{\eta(1)} \wedge \dots \wedge d\overline{z}_{\eta(l)} \wedge dz_{l+1} \wedge d\overline{z}_{\eta(l+1)} \wedge \dots \wedge dz_n \wedge d\overline{z}_{\eta(n)} \\
=& - dz_1 \wedge d\overline{z}_{\eta(1)} \wedge \dots \wedge d\overline{z}_{\eta(l+1)} \wedge dz_{l+1} \wedge d\overline{z}_{\eta(l)} \wedge \dots \wedge dz_n \wedge d\overline{z}_{\eta(n)}
\end{aligned}$$

for any $1 \leq l \leq n-1$, and so permuting two $d\overline{z}_j$'s without changing the rest of the wedge product gives a minus sign. This implies that

$$dz_1 \wedge d\overline{z}_{\eta(1)} \wedge \dots \wedge dz_n \wedge d\overline{z}_{\eta(n)} = \operatorname{sgn}(\eta) dz_1 \wedge d\overline{z}_1 \wedge \dots \wedge dz_n \wedge d\overline{z}_n,$$

where $\operatorname{sgn}(\eta)$ denotes the sign of the permutation η. We can now conclude using the Leibniz formula for the determinant of a matrix:

$$\begin{aligned}
\omega^n =& n!\left(\frac{i}{2}\right)^n \sum_{\eta \in S_n} h_{1\eta(1)} \dots h_{n\eta(n)} dz_1 \wedge d\overline{z}_{\eta(1)} \wedge \dots \wedge dz_n \wedge d\overline{z}_{\eta(n)} \\
=&n!\left(\frac{i}{2}\right)^n \sum_{\eta \in S_n} \operatorname{sgn}(\eta) h_{1\eta(1)} \dots h_{n\eta(n)} dz_1 \wedge d\overline{z}_1 \wedge \dots \wedge dz_n \wedge d\overline{z}_n \\
=&n!\left(\frac{i}{2}\right)^n det(h_{jk}) dz_1 \wedge d\overline{z}_{\eta(1)} \wedge \dots \wedge dz_n \wedge d\overline{z}_{\eta(n)}.
\end{aligned}$$

This proves that (2.3) holds. We can then use it to see that ω is non-degenerate if and only if $\det_{\mathbb{C}} \neq 0$, since non-degeneracy is equivalent to $\omega^n \neq 0$. This shows that at every point p in U, $(h_{jk}(p))$ is non-singular.

Finally, let us notice that for any tangent vector v, we have that

$$\begin{aligned}\omega(v, Jv) &= \frac{i}{2}\sum_{j,k=1}^{n} h_{jk} dz_j \wedge d\overline{z}_k(v, Jv) \\ &= \frac{i}{2}\sum_{j,k=1}^{n} h_{jk}(-2i) v_j \overline{v}_k = \sum_{j,k=1}^{n} h_{jk} v_j \overline{v}_k = v^*(h_{jk})v.\end{aligned} \tag{2.6}$$

Hence, $\omega(\cdot, J\cdot)$ is strictly positive for every v if and only if (h_{jk}) is a positive definite hermitian matrix. Since any compatible almost complex structure must fulfill $\omega(\cdot, J\cdot) > 0$, we have that the matrix (h_{jk}) is positive definite.

This whole development shows that Kähler forms have a nice local expression that allows us to understand the symplectic structure well. We finish this section by showing that it is possible to construct a Kähler form from certain kind of functions on a complex manifold. This will be useful in chapter 4, where we construct an example of toric Lagrangian using this method.

Definition 2.24. Let M be a complex manifold. A function $\rho \in C^\infty(M, \mathbb{R})$ is called *strictly plurisubharmonic* (abbreviated s.p.s.h.) if on every complex chart $(U, z_1, \ldots, z_n)$, where $n = dim_{\mathbb{C}}(M)$, the matrix $\left(\frac{\partial \rho}{\partial z_j \partial \overline{z_k}}(p)\right)$ is positive definite for every p in U.

Being strictly plurisubharmonic is exactly the condition we need to construct a Kähler form from a function. This is proposition 16.4 in [5].

Proposition 2.25. *Let M be a complex manifold and let $\rho \in C^\infty(M, \mathbb{R})$ be strictly plurisubharmonic. Then the form*

$$\omega := \frac{i}{2}\partial\overline{\partial}\rho$$

is a Kähler form.

Proof. First, let us show that ω is closed. Since M is complex manifold, we know that $d = \partial + \overline{\partial}$. Moreover, by equation (2.1), we have

$$\partial\omega = \frac{i}{2}\underbrace{\partial\partial}_{=0}\overline{\partial}\rho = 0$$

and

$$\overline{\partial}\omega = \frac{i}{2}\underbrace{\overline{\partial}\partial}_{=-\partial\overline{\partial}}\overline{\partial}\rho = -\frac{i}{2}\partial\underbrace{\overline{\partial}\overline{\partial}}_{=0}\rho = 0.$$

This implies that $d\omega = 0$ and hence ω is closed.
We now want to show that ω is real-valued. Since we know that $T^{1,0}$ is complex-conjugate to $T^{0,1}$, the complex conjugate of $\partial\sigma$ will be $\overline{\partial}\sigma$,and the complex-conjugate of $\overline{\partial}\sigma$ will be $\partial\sigma$ for any form $\sigma \in \Omega^k(M;\mathbb{C})$, $k = 0,1,2,\dots$. Using this and (2.1) yields

$$\overline{\omega} = -\frac{i}{2}\overline{\partial}\partial\rho = \frac{i}{2}\partial\overline{\partial}\rho = \omega,$$

which shows that ω is indeed real-valued.
By remark 2.22, we can compute ω explicitly in local coordinates $z_1,\dots,z_n$, where n is the dimension of M:

$$\begin{aligned}\omega &= \frac{i}{2}\partial\sum_{k=1}^{n}\left(\frac{\partial\rho}{\partial\overline{z}_k}d\overline{z}_k\right) = \frac{i}{2}\sum_{j,k=1}^{n}\frac{\partial}{\partial z_j}\left(\frac{\partial\rho}{\partial\overline{z}_k}\right)dz_j\wedge d\overline{z}_k\\ &= \frac{i}{2}\sum_{j,k=1}^{n}\left(\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\right)dz_j\wedge d\overline{z}_k =: \frac{i}{2}\sum_{j,k=1}^{n}h_{jk}dz_j\wedge d\overline{z}_k.\end{aligned}$$

Notice that since ρ is real-valued, the matrix (h_{jk}) is hermitian. It is also positive definite since by assumption ρ is s.p.s.h. In particular, we have that $\det(h_{jk}) > 0$ and so ω is non-degenerate.
We have shown that ω is a symplectic form. It now remains to prove that it is compatible with the almost complex structure J coming from the complex structure of M. We have that

$$\begin{aligned}J^*\omega &= J^*\left(\frac{i}{2}\sum_{j,k=1}^{n}\left(\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\right)dz_j\wedge d\overline{z}_k\right) = \frac{i}{2}\sum_{j,k=1}^{n}h_{jk}dz_j\circ J\wedge d\overline{z}_k\circ J\\ &= \frac{i}{2}\underbrace{(i)(-i)}_{=1}\sum_{j,k=1}^{n}h_{jk}dz_j\wedge d\overline{z}_k = \omega.\end{aligned}$$

Moreover, using the fact that (h_{jk}) is positive definite again, the same computation as in (2.6) shows that $\omega(Jv,v) > 0$ for every v. Thus, ω and J are compatible and ω is a Kähler form. ■

Chapter 3

Moment maps and symplectic quotients

This chapter presents the concepts of Hamiltonian G-spaces and symplectic quotients. Both of these notions are classical constructions in symplectic geometry that are interesting in their own right, but we introduce them here because toric Lagrangians are Lagrangians inside some Hamiltonian $\mathbb{T}^n$-space. Furthermore, a lot of nice examples of toric Lagrangians can be constructed via symplectic reduction, as we will see in chapter 4. The material for this chapter follows what is presented in [5] lectures 21-22 and [13] section 5.2 for the section about Hamiltonian G-spaces and moment maps and in [12] for the section about symplectic reduction.

3.1 Hamiltonian G-spaces and moment maps

Broadly speaking, Hamiltonian G-spaces are symplectic manifolds on which a Lie group G is acting. Moreover, the action of G behaves nicely with respect to the symplectic structure. One important feature of Hamiltonian G-spaces is the moment map. This map gives a lot of information about the space, especially when G is the torus $\mathbb{T}^n$.

We first recall the definition of a smooth action.

Definition 3.1. Let M be a manifold and let G be a Lie group. A *(left) action* of G on M is a group homomorphism

$$\begin{aligned} \psi :&G \to \mathrm{Diff}(M) \\ &g \mapsto \psi_g. \end{aligned}$$

The action is said to be smooth if the map $(p, g) \mapsto \psi_g(p)$ is smooth as a map from $M \times G$ to M.

In the rest of this text, any action of a Lie group will implicitly be assumed to be smooth. Lie group actions are a common concept of differential geometry. Let us adapt this concept to the symplectic setting.

Definition 3.2. Let (M, ω) be a symplectic manifold and G a Lie group. Let $\psi : G \to \text{Diff}(M)$ denote a smooth action. We say that ψ is a *symplectic action* if $im(\psi) \subseteq \text{Sympl}(M, \omega)$, which means that every ψ_g is a symplectomorphism.

We now recall the definitions of the adjoint and coadjoint representations of a Lie group.

Definition 3.3. Let G be a Lie group with Lie algebra $\mathfrak{g}$. G acts on itself by conjugation via: $g \mapsto \psi_g$, where $\psi_g(a) = gag^{-1}$. Differentiating ψ_g at the identity $e \in G$ yields an invertible linear map $\text{Ad}_g : \mathfrak{g} \to \mathfrak{g}$. We define the *adjoint representation of* G as the map

$$\begin{aligned} \text{Ad} : G &\to \text{GL}(\mathfrak{g}) \\ g &\mapsto \text{Ad}_g. \end{aligned}$$

This immediately induces the following dual notion.

Definition 3.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\langle \cdot, \cdot \rangle$ denote the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, i.e. given $\lambda \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$, we have that $\langle \lambda, \xi \rangle = \lambda(\xi) \in \mathbb{R}$. We define $\text{Ad}_g^* \lambda$ by

$$\langle \text{Ad}_g^* \lambda, \xi \rangle = \langle \lambda, \text{Ad}_{g^{-1}} \xi \rangle \quad \forall \xi \in \mathfrak{g}.$$

This allows us to define the *coadjoint representation of* G as the map

$$\begin{aligned} \text{Ad}^* : G &\to \text{GL}(\mathfrak{g}^*) \\ g &\mapsto \text{Ad}_g^*. \end{aligned}$$

Remark 3.5. The reason for Ad_g^* corresponding to $\text{Ad}_{g^{-1}}$ is that this choice of definition makes the coadjoint representation into a group homomorphism (and not an anti-homomorphism).

We can now give the most important definition of this section. The moment map μ is defined in such a way that the vector fields given by the infinitesimal action of G are Hamiltonian.

Definition 3.6. Let (M, ω) be a symplectic manifold, G a Lie group with Lie algebra $\mathfrak{g}$, and $\psi : G \to \text{Sympl}(M, \omega)$ a symplectic action of G on M. We say that ψ is a *Hamiltonian action* if there exists a map

$$\mu : M \to \mathfrak{g}^*$$

with the following properties.
For each $\xi \in \mathfrak{g}$, let $\mu^{\xi} : M \to \mathbb{R}$ denote the function defined as $\mu^{\xi}(p) = \langle \mu(p), \xi \rangle$. Furthermore, let X_{ξ} denote the vector field on M defined by

$$X_{\xi}(p) = \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(t\xi)}(p).$$

Then the map μ must satisfy

$$d\mu^{\xi} = -\iota_{X_{\xi}}\omega. \tag{3.1}$$

In other words, μ^{ξ} should be a hamiltonian function for the vector field X_{ξ}. Moreover, μ must be equivariant with respect to the action ψ and the coadjoint action Ad^*:

$$\mu \circ \psi_g = \mathrm{Ad}^*_g \circ \mu, \quad \forall g \in G. \tag{3.2}$$

The map μ is called a *moment map*, or sometimes a *momentum map* and the space (M, ω, G, μ) is called a *Hamiltonian G-space*.

Let us introduce some notation that will make the following results easier to formulate. Let (M, ω, G, μ) be a Hamiltonian G-space. For every point p in M, let us denote by

$$I_p : T_pM \to T^*_pM$$

the isomorphism induced by ω. In other words, $I_p(v) = \omega_p(v, \cdot)$. Moreover, let us denote by

$$L_p : \mathfrak{g} \to T_pM$$

the infinitesimal action, i. e. $L_p(\xi) = X_{\xi}(p)$.

We now present some useful properties regarding Hamiltonian G-spaces and moment maps. This result and its proof follow what is done in [13], lemma 5.2.5.

Lemma 3.7. *Let (M, ω, G, μ) be a Hamiltonian G-space, and let p be in M. Then the following statements hold:*

1. *The dual map of $D\mu(p) : T_pM \to \mathfrak{g}^*$ is given by*

$$D\mu(p)^* = -I_p \circ L_p : \mathfrak{g} \to T^*_pM. \tag{3.3}$$

2. *The symplectic complement of the kernel of $D\mu(p)$ is the tangent space of the group action orbit $\mathcal{O}_p := \{\psi_g(p) | g \in G\}$, where ψ denotes the action map. In other words*

$$(ker(D\mu(p)))^{\omega} = im(L_p). \tag{3.4}$$

3. *Let $\xi \in \mathfrak{g}$ and denote by $ad(\xi)^* : \mathfrak{g}^* \to \mathfrak{g}^*$ the dual of the map $ad(\xi)$. Then*

$$D\mu(p)[L_p\xi] = -ad(\xi)^*\mu(p). \tag{3.5}$$

Proof. 1. Let us take $v \in T_pM$ and $\xi \in \mathfrak{g}$. Then by definition of the moment map, more specifically by equation 3.1, we have

$$\langle D\mu(p)[v], \xi \rangle = \omega(X_\xi(p), v).$$

Then, we have

$$\begin{aligned} I_p \circ L_p(\xi)[v] &= I_p(X_\xi(p))[v] = \omega(X_\xi(p), v) \\ &= -\langle D\mu(p)[v], \xi \rangle = -\langle D\mu(p)^*(\xi), v \rangle, \end{aligned}$$

which proves equation 3.3.

2. For the second equation, notice that (3.1) gives us the inclusion $im(L_p) \subseteq (kerD\mu(p))^\omega$. Then using equation (3.3) and the fact that I_p is an isomorphism, we have

$$\begin{aligned} dim(imL_p) &= dim(im(D\mu(p)^*)) = dim(im(D\mu(p))) \\ &= codim(ker(D\mu(p))) = dim((kerD\mu(p))^\omega), \end{aligned}$$

which shows that $im(L_p) = (kerD\mu(p))^\omega$.

3. For the last equality we first observe that

$$\mu^{Ad_{g^{-1}}\xi} = \mu^\xi \circ \psi_g. \tag{3.6}$$

Indeed, if we use the equivariance of the moment map, we notice that

$$\begin{aligned} \mu^{Ad_{g^{-1}}\xi}(p) &= \langle \mu(p), Ad_{g^{-1}}\xi \rangle = \langle Ad_g^*\mu(p), \xi \rangle \\ &= \langle \mu(\psi_g(p)), \xi \rangle = \mu^\xi(\psi_g(p)). \end{aligned}$$

In particular, we can write

$$\mu^\xi \circ \psi_{exp(t\eta)} = \mu^{Ad_{exp(-t\eta)}\xi}.$$

Differentiating this equality at $t = 0$ yields

$$\begin{aligned} &\left.\frac{d}{dt}\right|_{t=0} \mu^\xi \circ \psi_{exp(t\eta}(p) = D\mu^\xi(p)\left[\left.\frac{d}{dt}\right|_{t=0} \psi_{exp(t\eta}(p)\right] = -\iota_{X_\xi}\omega(X_\eta(p)) \\ &= -\omega(X_\xi, X_\eta)(p) = \left.\frac{d}{dt}\right|_{t=0} \mu^{Ad_{exp(-t\eta)}\xi}(p) = \left.\frac{d}{dt}\right|_{t=0} \langle \mu(p), Ad_{exp(-t\eta)}\xi \rangle \\ &= \langle \mu(p), \left.\frac{d}{dt}\right|_{t=0} Ad_{exp(-t\eta)}\xi \rangle = \langle \mu(p), ad(-\eta)(\xi) \rangle = \langle \mu(p), [\xi, \eta] \rangle \\ &= \mu^{[\xi,\eta]}(p). \end{aligned}$$

This shows that

$$-\omega(X_\xi, X_\eta) = \mu^{[\xi,\eta]}(p). \tag{3.7}$$

We can now finish the proof. We have that

$$\begin{aligned}\langle D\mu(p)L_p\xi, \eta\rangle &= \langle D\mu(p)[X_\xi(p)], \eta\rangle \overset{(3.1)}{=} -\omega(X_\eta, X_\xi)(p) \\ &\overset{(3.7)}{=} -\langle \mu(p), [\xi, \eta]\rangle = -\langle \mu(p), ad(\xi)\eta\rangle = -\langle ad^*(\xi)\eta\rangle,\end{aligned}$$

which is what we wanted.

■

Since we will focus on toric Lagrangians in the last two chapters of this text, we end this section by giving one important result about moment maps in the case where the group G is a torus. We do not give the proof here, as this is not the main focus of this text.

Theorem 3.8 (Atiyah-Guillemin-Sternberg). *Let (M, ω) be a compact and connected symplectic manifold. Assume that we have Hamiltonian action of the torus $\mathbb{T}^m$ on M with moment map $\mu \to \mathbb{R}^m$. Then:*

1. *the level sets of μ are connected;*
2. *the image of μ is the convex hull of the images of the fixed points of the action, and hence it is a convex polytope that we call the* moment polytope*.*

A proof of this theorem can be found in [5], theorem 27.1, or [13] theorem 5.5.1.

3.2 Symplectic quotients

We now cover the Marsden-Weinstein-Meyer theorem 3.10, a result that asserts that the orbit space of a Hamiltonian G-space is, under certain conditions, a symplectic manifold as well. This is a common construction in symplectic geometry and we will use it to construct our first example of toric Lagrangian in Chapter 4.

One important step in the proof of the Marsden-Weinstein-Meyer theorem is the following technical proposition, which is a standard result of differential geometry

Proposition 3.9. *Let P be a manifold and let G be a Lie group acting freely and properly on P. Then P/G admits a unique smooth structure such that $\pi : P \to P/G$ is a surjective submersion, where P/G is endowed with the quotient topology.*

A proof of this result can be found in [1], where it is labeled as proposition 4.1.23 and we will not repeat it here.

We now have all the results we need to state and prove the Marsden-Weinstein-Meyer theorem. This theorem gives the assumptions necessary for the quotient of a Hamiltonian G-space by a group to be a symplectic manifold.

Theorem 3.10 (Marsden-Weinstein-Meyer theorem)**.** *Let (M, ω, G, μ) be a Hamiltonian G-space and let $j \in \mathfrak{g}^*$ be a regular value of μ. Let us denote by $M_j := \mu^{-1}(j)$ the submanifold of M given by the preimage by μ of j, and by $i_j : M_j \hookrightarrow M$ the inclusion. Assume that the isotropy group $G_j := \{g \in G | Ad^*_g(j) = j\}$ acts freely and properly on M_j. Then $\pi_j : M_j \to P_j := M_j/G_j$ is a submersion onto a manifold P_j. Moreover, P_j has a unique symplectic structure ω_j such that*

$$i_j^*\omega = \pi_j^*\omega_j. \tag{3.8}$$

This construction is called symplectic reduction at the level j.

We follow the proof given in [12], theorem 13.23.

Proof. Since j is a regular value of μ, M_j is an embedded submanifold of M. Let us take $x \in M_j$, and $g \in G_j$. We have that

$$\mu(\psi_g(x)) = Ad^*_g(\mu(x)) = Ad^*_g(j) = j.$$

This shows that M_j is G_j-invariant. Since G_j acts freely and properly on M_j, we can use proposition 3.9 to conclude that P_j is a manifold and $\pi_j : M_j \to P_j$ is a surjective submersion.
Let ω_1 and ω_2 be two-forms on P_j, and assume that $\pi_j^*\omega_1 = \pi_j^*\omega_2$. Then, for every p in M_j and every v, w in T_pM_j we have

$$\begin{aligned} \pi_j^*\omega_1|_p(v_1, v_2) &= \omega_1|_{\pi_j(p)}(D\pi_j(p)[v], D\pi_j(p)[w]) \\ =\pi_j^*\omega_2|_p(v_1, v_2) &= \omega_2|_{\pi_j(p)}(D\pi_j(p)[v], D\pi_j(p)[w]). \end{aligned}$$

Since π_j is a surjective submersion, both π_j and $D\pi_j$ are surjective, and hence we must have $\omega_1 = \omega_2$. It is easy to see that the computation above can be extended to any r-forms on P_j. Thus, we have showed that the pullback of differential forms on P_j by π_j is injective. This means that there can be at most one symplectic form ω_j on P_j that satisfies equation 3.8. We now construct this form. Set

$$\omega_j|_{\pi_j(p)}(D\pi_j(p)[v], D\pi_j(p)[w]) := \omega|_p(v, w),$$

for $p \in M_j$ and $v, w \in T_pM_j$. It is clear that equation 3.8 holds by construction, but we need to know that ω_j is well-defined. Take $p' = \psi_g(p)$ and assume that we have v' and w' in $T_{p'}M_j$ such that

$$D\pi_j(p)[v] = D\pi_j(p')[v']$$

and

$$D\pi_j(p)[w] = D\pi_j(p')[w'].$$

We need to show that $\omega_p(v,w) = \omega_{p'}(v',w')$. First, since G acts by symplectomorphisms, notice that

$$\omega_{p'}(v',w') = \omega_{\psi_g(p)}(D\psi_g(p)\circ D\psi_{g^{-1}}(p')[v'], D\psi_g(p)\circ D\psi_{g^{-1}}(p')[w'])$$
$$=(\psi_g^*\omega)_p(D\psi_{g^{-1}}(p')[v'], D\psi_{g^{-1}}(p')[w']) = \omega_p(D\psi_{g^{-1}}(p')[v'], D\psi_{g^{-1}}(p')[w']).$$

Moreover, since $\pi_j \circ \psi_g = \pi_j$ we have that,

$$D\pi_j(p)\circ D\psi_{g^{-1}}(p')[v'] = D\pi_j(p')[v'] = D\pi_j(p)[v]$$

and so we can assume that $p = p'$ without loss of generality. Now, to show that ω_j is well-defined, it is enough to prove that for every $v \in T_pM_j$ and every $w \in ker(D\pi_j(p))$ we have

$$\omega_p(v,w) = 0.$$

If w belongs to $ker(D\pi_j(p))$, then w is tangent to the G_j-orbit of p. In other words, we have that

$$ker(D\pi_j(p)) = \{X_\xi(p)|\ \xi \in \text{Lie}(G_j)\}. \tag{3.9}$$

This means that if w belongs to $ker(D\pi_j(p))$ then in particular there exists a ξ in $\mathfrak{g}$ such that $X_\xi(p) = w$. Next, using the definition of a moment map, we have that $-\iota_{X_\xi}\omega = d\mu^\xi$. But for any x in M_j, we have

$$\mu^\xi(x) = \langle\mu(x),\xi\rangle = j(\xi),$$

and thus μ^ξ is constant on M_j. This implies that $-\iota_{X_\xi}\omega|_{M_j} = 0$ and thus shows that ω_j is well-defined.

What now remains to show is that ω_j is actually a symplectic form. Observe that

$$\pi_j^* d\omega_j = d\pi_j^*\omega_j = di_j^*\omega = i_j^* d\omega = 0.$$

Now, as we have observed before, the pullback of differential forms by π_j is injective, which implies that $d\omega_j = 0$ and hence that ω_j is closed.

Showing that ω_j is non-degenerate is a little trickier. Since j is a regular value of μ, we can identify T_pM_j with $ker(D\mu(p))$. Then, lemma 3.7 tells us that

$$(T_pM_j)^\omega = (ker(D\mu(p)))^\omega = im(L_p).$$

We will show that $im(L_p)\cap T_pM_j = \{X_\xi(p)|\xi \in \text{Lie}(G_j)\} =: W$, because, as we have already seen with equation 3.9, W is the kernel of $D\pi_j(p)$ and hence this will prove that ω_j is non-degenerate. We have that

$$X_\xi(p) \in T_pM_j \iff D\mu(p)[X_\xi(p)] = 0$$
$$\iff D\mu(p)[L_p(\xi)] \overset{\text{L. 3.7}}{=} -ad(\xi)^*\mu(p) = -ad(\xi)^*j = 0.$$

Furthermore[1], the Lie algebra $\mathrm{Lie}(G_j)$ of G_j is the set given by

$$\mathrm{Lie}(G_j) = \{\xi \in \mathfrak{g} |\ exp(t\xi) \in G_j,\ \forall t \in \mathbb{R}\}$$

and hence for ξ in $\mathrm{Lie}(G_j)$ we have that for every t in $\mathbb{R}$, $Ad^*_{exp(t\xi)} j = j$. Differentiating this equality at $t = 0$ yields

$$\begin{aligned} &\left.\frac{d}{dt}\right|_{t=0} \langle Ad^*_{exp(t\xi)} j, \eta\rangle = \left.\frac{d}{dt}\right|_{t=0} \langle j, Ad_{exp(-t\xi)} \eta\rangle = j\left(\left.\frac{d}{dt}\right|_{t=0} Ad_{exp(-t\xi)}\eta\right) \\ &= -\, j(ad(\xi)\eta) = -\langle j, ad(\xi)\eta\rangle = -\langle ad(\xi)^* j, \eta\rangle = 0 \end{aligned}$$

for every η in $\mathfrak{g}$. This shows that $\mathrm{Lie}(G_j) \subseteq \{\xi \in \mathfrak{g} |\ ad^*(\xi) j = 0\}$.
For the converse inclusion, take ξ in $\mathfrak{g}$ such that $ad^*(\xi)j = 0$. The map $-ad^*(\xi)$ can be seen as the infinitesimal generator of the one-parameter group of diffeomorphisms on $\mathfrak{g}^*$ given by $Ad^*_{exp(t\xi)}$, and thus we see that the constant curve $\gamma(t) = j$ in $\mathfrak{g}^*$ is an integral curve of $ad^*(\xi)$ passing through j at $t = 0$. By uniqueness of integral curves, this implies that $Ad^*_{exp(t\xi)} j = \gamma(t) = j$. This shows that $exp(t\xi)$ is in G_j for all t and hence ξ belongs to $\mathrm{Lie}(G_j)$. We have thus proved that

$$ad^*(\xi)j = 0 \iff \xi \in \mathrm{Lie}(G_j).$$

This shows that ω_j is well-defined and hence concludes the proof. ■

The Marsden-Weinstein-Meyer theorem is often used with $j = 0 \in \mathfrak{g}^*$. In this case, we have that $G_j = G$, and we claim that 0 is automatically a regular value. To see this, we need the following result, which is taken from [13], proposition 5.4.13, step 2.

Lemma 3.11. *Zero is a regular value of μ if and only if the stabilizer subgroup*

$$G_p := \{g \in G |\ \psi_g(p) = p\}$$

is discrete for every p in $\mu^{-1}(0)$.

Proof. Let p be in $\mu^{-1}(0)$. We want the show that the condition above is equivalent to $D\mu(p)$ being surjective. By equation (3.3) in lemma 3.7, $D\mu(p) : T_pM \to \mathfrak{g}^*$ is surjective if and only if the map $L_p : \mathfrak{g} \to T_pM$ is injective. Observe that the kernel of L_p is exactly the Lie algebra $\mathrm{Lie}(G_p)$ associated to G_p. Hence, L_p is injective if and only if G_p is discrete, which concludes the proof. ■

[1] The idea for the last part of this proof was taken from [17]

Now, by assumption $G_0 = G$ acts freely and properly on $M_0 = \mu^{-1}(0)$, which means that for every $p \in \mu^{-1}(0)$, we have that G_p is trivial and thus in particular discrete. Hence, by lemma 3.11, 0 is automatically a regular value of μ. The quotient $\mu^{-1}(0)/G$ is called the *reduction* of (M, ω) (at 0), *symplectic quotient* of (M, ω) or *Marsden-Weinstein-Meyer quotient* of M.

If G is the torus $\mathbb{T}^n$, the coadjoint action is trivial, and so $G_j = G$ for any $j \in \mathfrak{g}*$. Moreover, reducing at j is the same thing as reducing at 0 when considering the shifted moment map $\mu - j$ instead of μ.

A proof for the special case $j = 0$ of the Marsden-Weinstein-Meyer can also be found in [5] or in [13].

Chapter 4

An example of toric Lagrangian

In this chapter, we construct a first example of toric Lagrangian in the setting of Weinstein's tubular neighborhood theorem. We first fix the dimension by looking at an example in $\mathbb{C}^2$. This will allow us to visualize the situation clearly and do some very concrete computations. We will then generalize the example to an arbitrary dimension and extend the results that we had in $\mathbb{C}^2$.

4.1 First construction

We start with the symplectic manifold $M = (\mathbb{C}^2, \omega_0)$, where ω_0 is the standard symplectic form given by $\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ as in example 1.19. We have a Hamiltonian action of the torus $\mathbb{T}^2 = S^1 \times S^1$ on $\mathbb{C}^2$ by multiplication, namely:

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2) = \psi_{(e^{i\theta_1}, e^{i\theta_2})}(z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2).$$

We claim that the moment map of this action is given by

$$\begin{aligned} \mu : \mathbb{C}^2 &\to \mathbb{R}^2 \\ (z_1, z_2) &\mapsto \left(\frac{|z_1|^2}{2}, \frac{|z_2|^2}{2} \right). \end{aligned}$$

Let $\xi = (\xi_1, \xi_2) \in \mathfrak{g} = \mathbb{R}^2$ and $z = (z_1, z_2) \in M$. The infinitesimal action vector field is given by

$$X_\xi(z) = \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(t\xi)}(z) = \frac{d}{dt}\bigg|_{t=0} (e^{it\xi_1} z_1, e^{it\xi_2} z_2) = (i\xi_1 z_1, i\xi_2 z_2).$$

Recall that $\mu^\xi(z) = \langle \mu(z), \xi \rangle$, where the brackets denote the evaluation of $\mu(z)$ at ξ.

The first condition for μ to be a moment map is that the equation

$$d\mu^{\xi} = -\iota_{X_\xi}\omega_0$$

must hold. Fix a point $z \in \mathbb{C}^2$, and a tangent vector $v = (v_1, v_2) \in T_z\mathbb{C}^2 \simeq \mathbb{C}^2$. We have:

$$\begin{aligned}
\omega_0(X_\xi(z), v) &= \left[\frac{i}{2}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2)[(i\xi_1 z_1, i\xi_2 z_2), (v_1, v_2)]\right] \\
&= \frac{i}{2}(i\xi_1 z_1 \overline{v}_1 - \overline{i\xi_1 z_1} v_1) + \frac{i}{2}(i\xi_2 z_2 \overline{v}_2 - \overline{i\xi_2 z_2} v_2) \\
&= -\frac{1}{2}(\xi_1 z_1 \overline{v}_1 + \xi_1 \overline{z}_1 v_1 + \xi_2 z_2 \overline{v}_2 + \xi_2 \overline{z}_2 v_2).
\end{aligned}$$

On the other hand, since $\mathfrak{g} \simeq \mathfrak{g}^* = \mathbb{R}^2$, the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is simply given by the standard scalar product. Hence, the map μ^{ξ} can be written as $\mu^{\xi} = \frac{1}{2}(\xi_1 z_1 \wedge \overline{z}_1 + \xi_2 z_2 \wedge \overline{z}_2)$ and the exterior derivative is

$$\frac{1}{2}(\xi_1 \overline{z}_1 dz_1 + \xi_1 z_1 d\overline{z}_1 + \xi_2 \overline{z}_2 dz_2 + \xi_2 z_2 d\overline{z}_2).$$

Feeding v to $d\mu^{\xi}(z)$, we obtain

$$d\mu^{\xi}(z)[v] = \frac{1}{2}(\xi_1 \overline{z}_1 v_1 + \xi_1 z_1 \overline{v}_1 + \xi_2 \overline{z}_2 v_2 + \xi_2 z_2 \overline{v}_2) = -\iota_{X_\xi(z)}\omega_0[v],$$

which is what we wanted.

It remains to check whether the map μ is equivariant. Since $\mathbb{T}^2$ is an abelian Lie group, the coadjoint action is trivial and so this boils down to checking that $\mu \circ \psi_{(e^{i\theta_1}, e^{i\theta_2})} = \mu$, which is clear since the action rotates the elements of M without changing their norm. This shows that μ is indeed a moment map for the torus-action on $\mathbb{C}^2$.

The map μ gives us new coordinates on $\mathbb{C}^2$. Indeed, if we write $\mu = (\mu_1, \mu_2)$, the functions $(\mu_1, \theta_1, \mu_2, \theta_2)$ form a set of coordinates, which are called *action-angle coordinates*. Notice that in polar coordinates, we have that

$$\mu_j = \frac{r_j^2}{2} \quad \text{and hence} \quad d\mu_j = r_j dr_j,$$

which implies that $\omega_0 = d\mu_1 \wedge d\theta_1 + d\mu_2 \wedge d\theta_2$ in action-angle coordinates by the computation we did in example 1.19.

Now, let us look at the map $R : \mathbb{C}^2 \to \mathbb{R}$ given by

$$R(z_1, z_2) = \mu_1(z_1, z_2) + \mu_2(z_1, z_2) = \frac{1}{2}(z_1\overline{z}_1 + z_2\overline{z}_2).$$

Using the computations above, one can see that R is Hamiltonian for the diagonal circle action $e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$, $\theta \in [0, 2\pi]$. If we take a number $r > 0$, we see that the level set $R^{-1}(r)$ is a 3-sphere of radius $\sqrt{2r}$ in $\mathbb{C}^2$. This sphere is preserved by the diagonal circle action. Since S^1 is compact, the circle action is proper and it is clear that it is also free. We can thus use the Marsden-Weinstein-Meyer theorem 3.10. The symplectic quotient is given by:

$$\begin{aligned} M_{\text{red}} &= R^{-1}(r)/S^1 \simeq S^2 \simeq \mathbb{CP}^1 \\ &= \{(z_1, z_2) \in \mathbb{C}^2 \big| \ |z_1|^2 + |z_2|^2 = 2r\}/ \sim \end{aligned}$$

where $(z_1, z_2) \sim (e^{i\theta} z_1, e^{i\theta} z_2)$. We denote the equivalence class of (z_1, z_2) by $[z_1 : z_2]$.

We take on M_{red} the real projective manifold

$$\mathbb{RP}^1 = \{[x_1 : x_2] \big| \ x_1, x_2 \in \mathbb{R}\} \simeq M_{\text{red}}^{\mathbb{R}} \subseteq M_{\text{red}}.$$

Since $M_{\text{red}}^{\mathbb{R}}$ has both dimension and codimension 1 in M_{red}, it is both isotropic and coisotropic and hence Lagrangian.

Consider now the preimage L_r of this Lagrangian submanifold in $R^{-1}(r)$:

$$S^1 \curvearrowright R^{-1}(r) \supseteq L_r = \pi^{-1}(M_{\text{red}}^{\mathbb{R}}),$$

where $\pi : R^{-1}(r) \to M_{\text{red}}$ denotes the projection. By theorem 3.10, we know that ω_0 restricted to $R^{-1}(r)$ is equal to the pullback by π of the symplectic form of the reduced space ω_{red}. Since ω_{red} vanishes on $M_{\text{red}}^{\mathbb{R}}$, L_r is an isotropic submanifold of $R^{-1}(r)$ and thus of $\mathbb{C}^2$. L_r is actually a two-torus, as it is given by

$$L_r = \{(e^{i\theta} x_1, e^{i\theta} x_2) \big| \ \theta \in [0, 2\pi], \ x_1, x_2 \in \mathbb{R}, \ x_1^2 + x_2^2 = 2r\}.$$

In particular, L_r has dimension 2 and so it is a Lagrangian submanifold of $M = (\mathbb{C}^2, \omega_0)$. It is also a circle bundle over $M_{\text{red}}^{\mathbb{R}}$, and its image by μ is the line segment in the first quadrant, given by the equation $\mu_1 + \mu_2 = r$. If we now let $r > 0$ vary, we get a family of such Lagrangian submanifolds.

Now, by Weinstein's tubular neighborhood theorem 1.53, a neighborhood of a fixed L_{r_0} is modeled on a neighborhood of the zero section in $T^*L_{r_0}$. We want to look at the various L_r's as sections of this cotangent bundle given by closed one-forms α_r on L_{r_0}. We will prove that the L_r's close enough to L_{r_0} are indeed images of closed 1-forms in $T^*L_{r_0}$ in the general n-dimensional setting later in this chapter with proposition 4.4 and the discussion right after this proposition. The fact that these 1-forms must be closed follows from our discussion on the Lagrangian submanifolds of the cotangent bundle at the end of section 1.2.

The first thing to notice in this case is that the cotangent bundle of a 2-torus is trivial. We begin by giving an explicit trivialization of the tangent bundle $T\mathbb{T}^2$. Since the Lie algebra $\text{Lie}(\mathbb{T}^2)$ of $\mathbb{T}^2$ is just $\mathbb{R}^2$, we have:

$$\begin{aligned} \varphi : \mathbb{T}^2 \times \mathbb{R}^2 &\to T\mathbb{T}^2 \\ (g, (v, w)) &\mapsto (g, (Dl_g(e)[v], Dl_g(e)[w]), \end{aligned}$$

where the left translation l_g is just multiplication in this case. As the tangent bundle is trivial, it admits a global frame. This implies that the dual frame on the cotangent bundle is also global, and so $T^*\mathbb{T}^2$ is trivial.

Since the cotangent bundle is trivial, we can identify the one-forms α_r representing the L_r's with functions from L_{r_0} to $\mathbb{R}^2$. Then, the fact that the α_r's are closed implies that those functions must be locally constant, and thus globally constant by smoothness.

Another fact worth noticing is that, except for the zero section, the α_r's cannot vanish since the L_r's do not intersect. This implies that the α_r's cannot be exact, as a function on a compact manifold must admit critical points. Thus, for $r \neq r_0$, the α_r represent a non-zero cohomology class.

We see that in this simple example, the Weinstein's neighborhood theorem gives a very easy way to describe the Lagrangians L_r's as constant functions. This is where the motivation to use this theorem comes from.

Another thing worth noticing is that L_{r_0} has a very nice description in terms of the action-angles coordinates defined above:

$$\begin{aligned} \mu_1 + \mu_2 &= r_0 \\ \theta_1 - \theta_2 &= 0. \end{aligned}$$

Let us choose the coordinates $\mu_1 - \mu_2$ and $\theta_1 + \theta_2$ on L_{r_0}. We define additional coordinates on the cotangent fibers by $\frac{1}{2}(\theta_1 - \theta_2)$ and $-\frac{1}{2}(\mu_1 + \mu_2)$. We claim that this is a Darboux chart on $T^*L_{r_0}$. Indeed, we compute

$$\begin{aligned} &d(\mu_1 - \mu_2) \wedge d(\frac{1}{2}(\theta_1 - \theta_2)) + d(\theta_1 + \theta_2) \wedge d(-\frac{1}{2}(\mu_1 + \mu_2)) \\ &= \frac{1}{2}(d\mu_1 \wedge d\theta_1 - d\mu_1 \wedge d\theta_2 - d\mu_2 \wedge d\theta_1 + d\mu_2 \wedge d\theta_2) \\ &- \frac{1}{2}(d\theta_1 \wedge d\mu_1 + d\theta_1 \wedge d\mu_2 + d\theta_2 \wedge d\mu_1 + d\theta_2 \wedge d\mu_2) \\ &= d\mu_1 \wedge d\theta_1 + d\mu_2 \wedge d\theta_2 = \omega_0. \end{aligned}$$

Hence, we see that the action-angle coordinates on $\mathbb{C}^2$ can be used to construct useful coordinates on $T^*L_{r_0}$ as well.

Now that we have a very clear description of the L_r's, we want to see what other kind of Lagrangians there are close to L_{r_0}. Let $\tilde{L}$ be another Lagrangian manifold, that is close to L_{r_0}. We will specify what topology we are using to talk about two manifolds being close to each other in definitions 4.1 and 4.2. Proposition 4.4 below will show that in the Weinstein's model, being close means that $\tilde{L}$ corresponds to a closed one-form on $T^*L_{r_0}$. Then, $\tilde{L}$ is also an S^1-bundle over $\mathbb{RP}^1$(i.e. it has the same topological type as the zero section). Furthermore, if we assume that $\tilde{L}$ is invariant under the diagonal S^1-action, the image $\mu(\tilde{L})$ of this Lagrangian by the moment map is contained in the line segment $\mu_1 + \mu_2 = a$ for some a. We will prove these two facts in the general n-dimensional setting as propositions 4.4 and 4.5.

To give a clear illustration of what we mean by deforming toric Lagrangians, let us get ahead and assume that the image by the moment map of $\tilde{L}$ is not just contained in the line segment $\mu_1 + \mu_2 = a$, but equal to it. The question we want to answer is now the following: Is $\tilde{L}$ Hamiltonian-isotopic to one of the L_r's? We can simplify this question directly by observing that since $\tilde{L}$ and the L_r's are S^1-bundles, we can solve this problem fiberwise. This then boils down to finding out whether a Lagrangian real projective space X in $\mathbb{CP}^1$ with $\mu(X) = \mu(\mathbb{RP}^1) = \mu(\mathbb{CP}^1)$ is always Hamiltonian isotopic to the standard $\mathbb{RP}^1$ in $\mathbb{CP}^1$. The answer to this question is actually no. Indeed, a submanifold X that is topologically an $\mathbb{RP}^1$ and has moment map image $\mu(X) = \mu(\mathbb{RP}^1) = \mu(\mathbb{CP}^1)$ must be a closed curve through the poles of the sphere given by the points $[1 : 0]$ and $[0 : 1]$. The extension of this example to higher dimensions discussed below will establish that the reduced form that we have on $\mathbb{CP}^1$ is the Fubini-Study form ω_{FS}, and, as it is discussed at the end of appendix A, on $\mathbb{CP}^1$, we have that

$$\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{std}},$$

where ω_{std} denotes the standard area form on the sphere defined in example 1.20. Hence, an isotopy preserving ω_{FS} must also preserve the area. If we assume that X is a closed curve through the poles that separates the sphere into two regions of different areas, then there is no way to find a Hamiltonian isotopy moving X to the standard $\mathbb{RP}^1$, which splits the sphere into two regions of equal area.

Since the answer to our first question is no, the next natural step is to ask what further conditions we need to put on the manifold X to ensure the existence of a Hamiltonian isotopy between X and the standard $\mathbb{RP}^1$. Let us assume that $X \subset \mathbb{CP}^1$ is invariant under the action of the finite subgroup $\{\pm 1\}$ where $(-1) \cdot [z_0 : z_1] = [z_0 : -z_1]$.[1] Then, using the diffeomorphism between $\mathbb{CP}^1$ and S^2 that is described at the end of appendix A, (which is

[1]We will see in chapter 5 that is kind of symmetry condition is a property of toric Lagrangians.

the standard diffeomorphism given by the stereographic projection), we see that if the curve X goes through the point (x, y, z), it must also go through $(-x, -y, z)$. This ensures that the curve X splits the sphere into two surfaces of equal area. It is actually possible to show that this symmetry condition is enough to ensure the existence of a Hamiltonian isotopy. A proof of this can be found in appendix A of [2] by Akveld and Salamon.

4.2 Generalization to higher dimensions

This example is a good start, since $\mathbb{CP}^1$ and $\mathbb{RP}^1$ are objects that are easy to visualize. However, one can generalize the above construction to higher dimensions. Let us take the manifold $\mathbb{C}^n$ with the standard symplectic form $\omega_0 = \frac{i}{2}\sum_{k=1}^n dz_k \wedge d\bar{z}_k$. We have the action of the n-dimensional torus $\mathbb{T}^n$ by component-wise multiplication. The moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n$ is then given by

$$\mu(z_1, \dots, z_n) = (\frac{|z_1|^2}{2}, \dots, \frac{|z_n|^2}{2}).$$

We thus see that μ maps $\mathbb{C}^n$ to the portion of $\mathbb{R}^n$ with only positive coordinates. Again, $\mu = (\mu_1, \dots, \mu_n)$ gives action-angle coordinates on $\mathbb{C}^n$, which allow us to express the symplectic form ω_0 as $\sum_{k=1}^n d\mu_k \wedge d\theta_k$. We then define the function $R : \mathbb{C}^n \to \mathbb{R}$ as we did above, namely as the sum of the components of the moment map:

$$R = \sum_{k=1}^n \mu_k.$$

For a positive number r, the level set $R^{-1}(r)$ is a $(2n-1)$-sphere of radius $\sqrt{2r}$ around the origin in $\mathbb{C}^n$. The function R is again Hamiltonian with respect to the diagonal circle action, and the level spheres are invariant under this action. Moreover, the action is once again free and proper on the spheres. Hence, we may once again take the quotient to obtain a reduced manifold

$$M_{\text{red}} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \big| |z|^2 = 2r\}/\sim \, \simeq \mathbb{CP}^{n-1},$$

where $(z_1, \dots, z_n) \sim (e^{i\theta}z_1, \dots, e^{i\theta}z_n)$, as before. This M_{red} is a $\mathbb{CP}^{n-1}$ and the real part $M_{\text{red}}^{\mathbb{R}}$ is a copy of $\mathbb{RP}^{n-1}$. Here, it becomes necessary to examine the symplectic structure of complex projective spaces in more details. To avoid interrupting our construction, we do so in appendix A, where we construct a symplectic structure on $\mathbb{CP}^n$ and show that it matches the symplectic form that we get in our construction by symplectic reduction. By what is discussed in appendix A, we now know that the reduced symplectic form is the Fubini-Study form, and that $M_{\text{red}}^{\mathbb{R}}$ is a Lagrangian submanifold of $(M_{\text{red}}, \omega_{\text{red}})$.

As we did with $n = 2$, we take the preimage

$$L_r := \pi^{-1}(M^{\mathbb{R}}_{\text{red}})$$

of $M^{\mathbb{R}}_{\text{red}}$ in $R^{-1}(r)$. By the same argument as in dimension 2, we can see that the L_r's are Lagrangian submanifolds of $(\mathbb{C}^n, \omega_0)$. They are also S^1-bundles over $\mathbb{RP}^{n-1}$, and their image under the moment map is the $(n-1)$-dimensional simplex given by $\mu_1 + \cdots + \mu_n = r$.

We now develop the construction further in an arbitrary dimension n. Let us fix a Lagrangian L_{r_0}. Its image by the moment map is the simplex given by $\mu_1 + \cdots + \mu_n = r_0$. The image by the moment map of any member of the family $\{L_r\}$ is a simplex parallel to the simplex corresponding to L_{r_0}. Let us consider, using Weinstein's tubular neighborhood 1.53 theorem once again, a neighborhood of the zero section in $T^*L_{r_0}$. Let $\tilde{L}$ denote a Lagrangian submanifold close to L_{r_0}. We are interested in what $\tilde{L}$ can look like. First, let us formalize what we mean by $\tilde{L}$ being close to L_{r_0}.

Definition 4.1. Let X and Y be smooth manifolds. A sequence of maps $f_j : X \to Y$ converges to $f : X \to Y$ in the C^1*-topology* if and only if the derivatives $Df_j : TX \to TY$ converge uniformly to Df on compact set.

The C^1-topology is the notion that we will use to talk about two functions being close to one another. We say that a map $f : X \to Y$ is *sufficiently* C^1*-close* to $g : X \to Y$ if it is in a sufficiently small neighborhood of g in the C^1-topology. We can now talk about two submanifolds being close to one another in the following way:

Definition 4.2. Let M be a manifold and X and Y two submanifolds of M, and let us denote by $i_X : X \hookrightarrow M$ and $i_Y : Y \hookrightarrow M$ the inclusion maps. We say that Y is C^1*-close* to X if there exists a diffeomorphism $f : X \to Y$ such that $i_Y \circ f$ is C^1-close to i_X.

Being C^1-close to the identity is a strong condition on maps, as the next result illustrates. The idea for this proof comes from the discussion in [16], in particular from Eric Wofsey's comment.

Proposition 4.3. *Let M be a compact, connected manifold and $f : M \to M$ a smooth map that is C^1-close to the identity. Then f is a diffeomorphism.*

Proof. First of all, being C^1-close to the identity implies that the derivative of f cannot vanish anywhere, and thus f is a local diffeomorphism.
Next, we show that f is surjective. Indeed, f is an open map since it is a

local diffeomorphism, and hence $f(M)$ is open. Since M is compact, $f(M)$ is a compact subspace of a Hausdorff space and is thus closed. Since we assume M to be connected, this implies that f is surjective.
It remains to show that f is injective. Indeed, a local diffeomorphism that is injective is always a diffeomorphism onto its image, and since we have just shown surjectivity, the proof will be complete. Let us put a Riemannian metric on M. Since M is compact, we can cover it with finitely many coordinate charts. By the Lebesgue number lemma, there exists a number $\varepsilon > 0$ such that every subset U with diameter $\mathrm{diam}(U) \leq \varepsilon$ is contained in one of the coordinate charts that we chose. For a contradiction, let us assume that we have two points $p \neq q$ in M such that $f(p) = f(q)$. Since f is sufficiently C^1-close to the identity, we must have that the distance between p and q is small. Indeed, the distance between p and $f(p)$, respectively q and $f(q)$ cannot be too big without violating the C^1-close condition. Since $f(p) = f(q)$, we then have p and q cannot be too far apart. By taking f closer to the identity if we need to, we can without loss of generality assume that p and q are less than ε away from each other, and hence the geodesic between p and q lies in one coordinate chart. We now look at f through this coordinate chart and hence in Euclidean space. Since $f(p) = f(q)$, the image by f of the geodesic between p and q must be a loop, and hence the directional derivative of f along this segment must be perpendicular to it at some point x. Since the derivative of the identity is again the identity, the distance between the derivative of f and the identity at x cannot be as small as we want, which contradicts the C^1-close hypothesis. This implies that $p = q$ and hence that f is injective. This concludes the proof. ■

We can now formally show that it makes sense to view Lagrangians close to L_{r_0} as images of one-forms. The main result in that direction is the following proposition. As above, the idea for this proof comes from [16].

Proposition 4.4. *Let M be a compact and connected manifold such that $g : M \to T^*M$ is an embedding that is sufficiently C^1-close to the canonical embedding of the zero section. Then $g(M)$ is the image of a one-form.*

Proof. Let $\pi : T^*M \to M$ denote the canonical projection, and $i_0 : M \hookrightarrow T^*M$ the canonical embedding of the zero section. Let us look at the composition

$$\pi \circ g : M \to M.$$

Since $\pi \circ i_0 = id$, we have that $\pi \circ g$ is C^1-close to the identity, and hence is a diffeomorphism by proposition 4.3. We may then take

$$\mu := g \circ (\pi \circ g)^{-1}$$

as the desired 1-form. ■

By Weinstein's tubular neighborhood theorem 1.53, we can see that $\tilde{L}$ being C^1-close to L_{r_0} in $\mathbb{C}^n$ is equivalent to $\tilde{L}$ being C^1-close to the zero section in $T^*L_{r_0}$. Hence, proposition 4.4 tells us that we can view $\tilde{L}$ and all the L_r's close enough to L_{r_0} as images of one-forms inside $T^*L_{r_0}$. Moreover, all these one-forms will be closed since we are dealing with Lagrangian submanifolds. In particular, this implies that $\tilde{L}$ (as well as the L_r's but that was already clear from the construction) is diffeomorphic to L_{r_0}, and thus is also an S^1-bundle over $\mathbb{RP}^{n-1}$. We now impose an additional symmetry condition on $\tilde{L}$ and see what new restrictions it imposes on its topology.

Proposition 4.5. *Assume that $\tilde{L}$ is invariant under the diagonal S^1-action (multiplication by $(e^{i\theta}, \dots, e^{i\theta})$). Then the image of $\tilde{L}$ by the moment map is contained in a simplex $\mu_1 + \cdots + \mu_n = b$ for some positive number b.*

Proof. We claim that $\mu_1 + \cdots + \mu_n\big|_{\tilde{L}}$ is constant. Let $\xi \in \mathrm{Lie}(S^1) \simeq \mathbb{R}$, and denote by $\overline{\xi}$ the vector $(\xi, \dots, \xi) \in \mathrm{Lie}(\mathbb{T}^n) \simeq \mathbb{R}^n$. By definition of the moment map

$$d\mu_1^{\xi} + \cdots + d\mu_n^{\xi} = d\mu^{\overline{\xi}} = -\iota_{X_{\overline{\xi}}}\omega.$$

If we take p in $\tilde{L}$, $\psi_{\exp(t\overline{\xi})}(p)$ will be a curve in $\tilde{L}$ by invariance, and thus $X_{\overline{\xi}}(p)$ will belong to $T_p\tilde{L}$. Since $\tilde{L}$ is Lagrangian, $-\iota_{X_{\overline{\xi}}}\omega\big|_{\tilde{L}} = 0$ and hence $d\mu_1 + \cdots + d\mu_n\big|_{\tilde{L}} = 0$ because ξ was arbitrary. This implies that $\mu_1 + \cdots + \mu_n\big|_{\tilde{L}}$ is constant. We have shown that $\mu(\tilde{L})$ is included in the simplex Δ_b for some b. This completes the proof. ■

To finish this example, we give the formula to find a Darboux chart on $T^*L_{r_0}$ as we did in dimension 2. Since L_{r_0} is not a torus anymore for $n > 2$, it does not have a trivial cotangent bundle, and we can only do this locally. We start by choosing appropriate local coordinates on the base manifold L_{r_0} inside our Weinstein tubular neighborhood model. We use the action-angle coordinate functions and take coordinates $\mu_1 - \mu_2, \mu_2 - \mu_3, \dots, \mu_{n-1} - \mu_n, \theta_1 + \cdots + \theta_n$ on L_{r_0}, which is a natural generalization of what we did in dimension 2.

We then define the following coordinates on the cotangent fibers:

$$\frac{1}{n}\left[\sum_{k=1}^{j}(n-j)\theta_k - \sum_{k=j+1}^{n} j\theta_k\right] \quad j = 1, \dots, n-1$$

for the first $n-1$ coordinates and

$$-\frac{1}{n}\sum_{j=1}^{n}\mu_j$$

for the last one. Let us verify that this choice of coordinates is correct. We have that

$$\sum_{j=1}^{n-1}\Big[d(\mu_j-\mu_{j+1})\wedge\frac{1}{n}d\big(\sum_{k=1}^{j}(n-j)\theta_k-\sum_{k=j+1}^{n}j\theta_k\big)\Big]+d\big(\sum_{j=1}^{n}\theta_j\big)\wedge-\frac{1}{n}d\big(\sum_{j=1}^{n}\mu_j\big)$$

$$=\sum_{j=1}^{n-1}\Big[\sum_{k=1}^{j}\frac{n-j}{n}d\mu_j\wedge d\theta_k-\sum_{k=j+1}^{n}\frac{j}{n}d\mu_j\wedge d\theta_k-\sum_{k=1}^{j}\frac{n-j}{n}d\mu_{j+1}\wedge d\theta_k+\sum_{k=j+1}^{n}\frac{j}{n}d\mu_{j+1}\wedge d\theta_k\Big]$$

$$+\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}d\mu_j\wedge d\theta_k$$

$$=\frac{n-1}{n}d\mu_1\wedge d\theta_1-\frac{1}{n}\sum_{k=2}^{n}d\mu_1\wedge d\theta_k-\frac{1}{n}\sum_{k=1}^{n-1}d\mu_n\wedge d\theta_k+\frac{n-1}{n}d\mu_n\wedge d\theta_n$$

$$+\sum_{j=2}^{n-1}\Big[\sum_{k=1}^{j}\frac{n-j}{n}d\mu_j\wedge d\theta_k-\sum_{k=j+1}^{n}\frac{j}{n}d\mu_j\wedge d\theta_k\Big]$$

$$+\sum_{j=1}^{n-2}\Big[\sum_{k=1}^{j}-\frac{n-j}{n}d\mu_{j+1}\wedge d\theta_k+\sum_{k=j+1}^{n}\frac{j}{n}d\mu_{j+1}\wedge d\theta_k\Big]$$

$$+\frac{1}{n}d\mu_1\wedge d\theta_1+\frac{1}{n}\sum_{k=2}^{n}d\mu_1\wedge d\theta_k+\frac{1}{n}\sum_{k=1}^{n-1}d\mu_n\wedge d\theta_k+\frac{1}{n}d\mu_n\wedge d\theta_n+\frac{1}{n}\sum_{j=2}^{n-1}\sum_{k=1}^{n}d\mu_j\wedge d\theta_k$$

$$=d\mu_1\wedge d\theta_1+d\mu_n\wedge d\theta_n+\sum_{j=2}^{n-1}\Big[\sum_{k=1}^{j}\frac{n-j}{n}d\mu_j\wedge d\theta_k-\sum_{k=j+1}^{n}\frac{j}{n}d\mu_j\wedge d\theta_k\Big]$$

$$+\sum_{j=2}^{n-1}\Big[\sum_{k=1}^{j-1}-\frac{n-j+1}{n}d\mu_j\wedge d\theta_k+\sum_{k=j}^{n}\frac{j-1}{n}d\mu_j\wedge d\theta_k\Big]+\frac{1}{n}\sum_{j=2}^{n-1}\sum_{k=1}^{n}d\mu_j\wedge d\theta_k$$

$$=d\mu_1\wedge d\theta_1+d\mu_n\wedge d\theta_n$$

$$+\sum_{j=2}^{n-1}\Big[\sum_{k=1}^{j-1}\Big(-\frac{1}{n}d\mu_j\wedge d\theta_k\Big)+\frac{n-j}{n}d\mu_j\wedge d\theta_j+\sum_{k=j+1}^{n}\Big(-\frac{1}{n}d\mu_j\wedge d\theta_k\Big)+\frac{j-1}{n}d\mu_j\wedge d\theta_j\Big]$$

$$+\frac{1}{n}\sum_{j=2}^{n-1}\sum_{k=1}^{n}d\mu_j\wedge d\theta_k$$

$$=d\mu_1\wedge d\theta_1+d\mu_n\wedge d\theta_n$$

$$+\sum_{j=2}^{n-1}\Big[\sum_{\substack{k=1\\k\neq j}}^{n}\Big(-d\mu_j\wedge d\theta_k+d\mu_j\wedge d\theta_k\Big)+\frac{n-1}{n}d\mu_j\wedge d\theta_j+\frac{1}{n}d\mu_j\wedge d\theta_j\Big]$$

$$=\sum_{j=1}^{n}d\mu_j\wedge d\theta_j.$$

This shows that these coordinates are the right ones.

With the construction of this example, we have seen that the Weinstein's tubular neighborhood theorem allows us to view the Lagrangian submanifolds that are close to a fixed Lagrangian L_{r_0} as images of one-forms on L_{r_0}. We have also seen that imposing symmetry conditions on the Lagrangians such as invariance under a group action gives us additional information about what these Lagrangian submanifolds look like. The next chapter will be devoted to generalizing these ideas. We will see that the so-called symplectic toric manifolds and toric Lagrangians have the right kind of symmetry properties to push these ideas further.

Chapter 5

Toric Lagrangians

In this chapter, we want to extend the ideas discussed in chapter 4 to other Lagrangian submanifolds, namely toric Lagrangians. We use the first to sections to define toric Lagrangians and symplectic toric manifolds and give some of their properties. Section 5.3 is a proof of Weinstein's tubular neighborhood theorem adapted to the setting of toric Lagrangians. Finally, we present results extending the observations from the last chapter in section 5.4. The structure of this chapter follows what has been discussed with my advisor in [4] and has been inspired by the work done in [7].

5.1 Elementary subgroups

As the name suggests, symplectic toric manifolds are Hamiltonian G-spaces where the Lie group G is a torus $\mathbb{T}^n$. Toric Lagrangians are Lagrangians submanifolds of symplectic toric manifolds that are invariant under the action of a subgroup of $\mathbb{T}^n$. We begin by defining the kind of subgroups we will be focusing on.

Definition 5.1. Let $\mathbb{T}^n$ denote the n-dimensional torus. The *k-dimensional basic subgroup* of $\mathbb{T}^n$ is the subgroup

$$T^k := \underbrace{S^1 \times \cdots \times S^1}_{\text{first } k \text{ factors}} \times \underbrace{\{\pm 1\} \times \cdots \times \{\pm 1\}}_{\text{last } n-k \text{ factors}} \subseteq \mathbb{T}^n \quad k = 0, 1, \dots, n.$$

In terms of the exponential map, we can write $T^k = exp(\mathfrak{t}^k)$, where we define $\mathfrak{t}^k$ as

$$\mathfrak{t}^k := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{k} \times \underbrace{\pi\mathbb{Z} \times \cdots \times \pi\mathbb{Z}}_{n-k} \subseteq \mathbb{R}^n.$$

We are interested in subgroups of $\mathbb{T}^n$ that are just a lattice isomorphism away from T^k. By lattice isomorphism, we mean an element of $\mathrm{GL}(n, \mathbb{Z})$, which is an $n \times n$-matrix with integer entries and determinant ± 1.

Definition 5.2. Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be a lattice isomorphism. We call *k-dimensional subgroup given by A* the subgroup

$$T^{k,A} = exp(A\mathfrak{t}^k).$$

Remark 5.3. We have that $T^0 = T^{0,A} = \{\pm 1\}^n$ for every $A \in GL(n, \mathbb{Z})$.

The connected component containing the identity in $T^{k,A}$ is denoted by $T_0^{k,A}$. It is a k-dimensional torus with Lie algebra

$$\mathfrak{t}_0^{k,A} := A\big(\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{k} \times \underbrace{\{0\} \times \cdots \times \{0\}}_{n-k}\big).$$

This corresponds the the span of the first k columns of A, hence $\mathfrak{t}_0^{k,A} = im(AI_{n\times k})$, where $I_{n\times k}$ is the $n \times k$ matrix corresponding to the first k columns of $Id_{n\times n}$. The fact that the Lie algebra $\mathfrak{t}_0^{k,A}$ has such an easy description in terms of the matrix A motivates the next definition.

Definition 5.4. We denote by $\mathcal{S}_b^{k,A}$ the set of solutions $u \in \mathbb{R}^n$ to the equation

$$(AI_{n\times k})^\top u = b,$$

where $(AI_{n\times k})^\top$ denotes the transpose of $AI_{n\times k}$ and b is an element of $\mathbb{R}^k$. $\mathcal{S}_b^{k,A}$ is called an *$n - k$-dimensional rational affine subspace of* $\mathbb{R}^n$.

Remark 5.5. Notice that if $k = 0$, then there is no equation, and hence $\mathcal{S}^{0,A} = \mathbb{R}^n$.

Let us now give a concrete example for these definitions to make them more tangible.

Example 5.6. Let us take $n = 2$ and $k = 1$. For $\mathbb{T}^2$, we have

$$\mathfrak{t}^1 = \left\{ \begin{pmatrix} \theta \\ j\pi \end{pmatrix} \Big| \; \theta \in \mathbb{R}, j \in \mathbb{Z} \right\}.$$

Let us now choose the matrix $A = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$, where $m \in \mathbb{Z}$ is fixed. The 1-dimensional subgroup given by A is then

$$T^{1,A} = exp\left\{ \begin{pmatrix} \theta \\ m\theta + j\pi \end{pmatrix} \Big| \; \theta \in \mathbb{R}, j \in \mathbb{Z} \right\} = \{(e^{i\theta}, \pm e^{im\theta}) | \; \theta \in \mathbb{R}\}.$$

The connected component of the identity and its Lie algebra are given by

$$T_0^{1,A} = \{(e^{i\theta}, e^{im\theta}) | \; \theta \in \mathbb{R})\} \quad \mathfrak{t}_0^{1,A} = \mathrm{Lie}(T_0^{1,A}) = \left\{ \begin{pmatrix} \theta \\ m\theta \end{pmatrix} \Big| \theta \in \mathbb{R} \right\},$$

and $\mathcal{S}_b^{1,A} \subseteq \mathbb{R}^2$ is the subspace given by the equation

$$\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = b \iff u_1 + mu_2 = b$$

If a toric subgroup $T_0^{k,A}$ preserves a Lagrangian submanifold, then the image of this Lagrangian by the moment map will be contained in a rational affine subspace. The result below comes from [7].

Lemma 5.7. *Let L be a connected Lagrangian in $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ and let $A \in GL(n, \mathbb{Z})$ be a lattice isomorphism. Then there exists $b \in \mathbb{R}^k$ such that $\mu(L) \subseteq \mathcal{S}_b^{k,A}$ if and only if L is preserved by the $T_0^{k,A}$.*

Proof. First, let us assume that $\mu(L) \subseteq \mathcal{S}_b^{k,A}$. Then, $(AI_{n\times k})^\top \mu|_L \equiv b$. Let $\varphi_1, \ldots, \varphi_k$ denote the coordinate functions of $(AI_{n\times k})^\top \mu$ and let $Y_1, \ldots, Y_k$ denote the first k columns of the matrix A. For p in M and $j = 1, \ldots k$, we have that $d\varphi_j(p)$ is the jth row of the matrix

$$d(AI_{n\times k})^\top \mu(p) = (AI_{n\times k})^\top d\mu(p).$$

Notice that the jth row of $(AI_{n\times k})^\top$ corresponds to the jth column Y_j of A. Hence,

$$d\varphi_j(p) = \langle d\mu(p), Y_j \rangle = -\iota_{Y_j^\#}\omega,$$

where $Y_j^\#$ denote the vector field generated by Y_j. This shows that the functions φ_j are hamiltonian for the vector fields $Y_j^\#$. But since every φ_j is constant on L, we have that $d\varphi_j|_{TL} \equiv 0$, and hence by the computation above $-\iota_{Y_j^\#}\omega|_{TL} \equiv 0$. Since L is maximally isotropic, we must have that $Y_j^\#$ is tangent to L at every point of L and for every $j = 1, \ldots, k$. Since none of the $Y_j's$ are zero, the vector fields $Y_j^\#$ never vanish on L and hence each of them defines a one-dimensional distribution on L. Since any one-dimensional distribution is integrable, the Frobenius theorem tells us that an integral curve of any $Y_j^\#$ starting in L must stay in L. Since the Y_j's are linearly independent, the same argument works with any linear combination of the $Y_j^\#$. Now, since $T_0^{k,A}$ is a connected Lie group, it is generated by a neighborhood of the identity. In particular, any element $g \in T_0^{k,A}$ can be written as $g = exp(a_1Y_1 + \cdots + a_kY_k)$, where the a_j's are real numbers and exp denotes the exponential map of the Lie group $T_0^{k,A}$. The path $\psi_{exp(ta_1Y_1+\cdots+ta_kY_k)}(p)$ for $t \in [0,1]$ and $p \in L$ is an integral curve of $a_1Y_1^\# + \cdots + a_kY_k^\#$ starting in L, and thus by what we explained above, the whole path must lie in L. In particular, gp is in L, and since g and p were arbitrary, this shows that $T_0^{k,A}$ preserves L.

Conversely, suppose that L is preserved by $T_0^{k,A}$. Let us once again denote by $Y_1, \dots, Y_k$ the first k columns of A, and by $Y_1^\#, \dots, Y_k^\#$ the induced vector fields. Those vector fields must by tangent to L, and thus $-\iota_{Y_j^\#}\omega|_{TL} \equiv 0$ since L is isotropic. Then, using the same notation as in the previous part $d\varphi_j|_{TL} \equiv 0$ for every $j = 1, \dots k$. Since L is connected this implies that every φ_j is constant on L, which we can rewrite as $(AI_{n\times k})^\top \mu|_L \equiv b$ for some $b \in \mathbb{R}^k$. This shows that $\mu(L)$ lies in the rational affine subspace $\mathcal{S}_b^{k,A}$. ■

5.2 Toric Lagrangians

The previous proposition gives us motivation to investigate Lagrangian submanifolds that are preserved by some elementary subgroup. In this section, we define symplectic toric manifolds and toric Lagrangians.

Definition 5.8. A *symplectic toric manifold* is a compact and connected symplectic manifold (M, ω) with a maximal hamiltonian torus symmetry, i.e. an effective action of a half-dimensional torus.

The idea behind requiring the action to be effective is that we want to consider "maximal" torus symmetries. In the following, let $(M, \omega, \mathbb{T}^n, \mu)$ denote a (compact) symplectic toric manifold or $(\mathbb{C}^n, \omega_0, \mathbb{T}^n, \mu_0)$.

To define toric Lagrangians properly, we will need the notion of a clean intersection. Indeed, a toric Lagrangian is a Lagrangian submanifold that intersects the torus-orbits in a specific way.

Definition 5.9. Let M be a manifold and K and L two embedded submanifolds. We say K and L have a *clean intersection* if $K \cap L$ is an embedded submanifold (or empty) and $T_p(K \cap L) = T_pK \cap T_pL$ for every p in $K \cap L$.

We have the following equivalent characterization of a clean intersection.

Proposition 5.10. *Let M be a manifold and let K and L be two embedded submanifolds. Then K and L intersect cleanly if and only if for every x in $K \cap L$ there is a chart (U, φ) around x such that $\varphi(U \cap K)$ and $\varphi(U \cap L)$ are open subsets of affine subspaces of $\mathbb{R}^n$, where n is the dimension of M.*

A proof of this can be found in appendix C.3 of [11] (proposition C.3.1).

Definition 5.11. A *toric Lagrangian* in $(M, \omega, \mathbb{T}^n, \mu)$ is a pair $(L, T^{k,A})$ such that L is a proper (meaning that the inclusion map $i : L \hookrightarrow M$ is proper) connected Lagrangian submanifold of (M, ω) and $T^{k,A}$ is an elementary subgroup of of $\mathbb{T}^n$ such that the intersection of L with each $\mathbb{T}^n$-orbit is clean and exactly one $T^{k,A}$-orbit (or empty). Then $T^{k,A}$ is called the *symmetry group* of the toric Lagrangian. A *real toric Lagrangian* in $(M, \omega, \mathbb{T}^n, \mu)$ is a toric Lagrangian where the symmetry group is $T^0 = \{\pm 1\}^n$.

Remark 5.12. The definition of toric Lagrangian means in particular that:

- L is preserved by the action of $T^{k,A}$.
- $T^{k,A}$ is maximal in the sense that no other Lie subgroup of $\mathbb{T}^n$ containing $T^{k,A}$ preserves L.

By lemma 5.7, the image of a toric Lagrangian by the moment map is contained in a rational affine subspace.

5.3 Toric Weinstein Neighborhoods

In this section, we prove an equivariant version of Weinstein's tubular neighborhood theorem, i.e. we make the construction from proof 1.53 compatible with a Lie group action. As we have seen in subsection 1.3.2, the proof of Weinstein's tubular neighborhood theorem requires some preliminary results. We will follow this sequence of results once again and adapt each of them to the context of Lie group actions. The adaptation of most of these statements and proofs where discussed with my advisor in [4].

We start by noticing that a Lie group action on a manifold also induces actions on the tangent and cotangent bundles.

Let $\psi : G \to Diff(M)$ denote the action of a Lie group on a manifold. This action also induces actions of G on the tangent and cotangent bundles of M, which we denote as follows for $g \in G$:

$$\begin{aligned} T\psi_g :& TM \to TM \\ & (x,v) \mapsto (\psi_g(x), D\psi_g(x)[v]), \\ T^*\psi_g :& T^*M \to T^*M \\ & (x,\lambda) \mapsto (\psi_g(x), \lambda \circ (D\psi_g(y))^{-1}), \quad y = \psi_g(x). \end{aligned}$$

Remark 5.13.

- Notice that the action $T^*\psi_g$ is exactly the lift of the diffeomorphism ψ_g as we defined just before proposition 1.24. Hence, by this same proposition, $T^*\psi_g$ preserves the canonical form on the cotangent bundle.
- For some of the calculations in this section, it will be beneficial to ease the notation a bit, and we will sometimes do so by writing the action of a Lie group G simply by concatenation: gx for the action on the manifold itself, $g(x,v)$ for the action on the tangent bundle, and $g(x,\lambda)$ for the action on the cotangent bundle. This is a slight abuse of notation, but the action we are using should be clear from the context.

In the example we want to consider, we often see Lagrangians as zero section of their cotangent bundle via theorem 1.53. In the case of toric Lagrangians, this model only makes sense if the Weinstein construction also respects the toric structure. This is why we need to strengthen theorem 1.53 to get the following.

Theorem 5.14 (Equivariant Weinstein Tubular Neighborhood)**.** *Let G be a compact Lie group acting symplectically on a symplectic manifold (M, ω) and preserving a compact Lagrangian submanifold X. Let ω_0 be the canonical form on T^*X, $i_0 : X \hookrightarrow T^*X$ the Lagrangian embedding as the zero section and $i : X \hookrightarrow M$ the Lagrangian embedding given by the inclusion.*
Then there are G-invariant neighborhoods U_0 of X in T^*X, U of X in M, and a G-equivariant diffeomorphism $\varphi : U_0 \to U$ such that $\varphi^*\omega = \omega_0$ and such that the following diagram commutes

$$\begin{array}{ccc} U_0 & \xrightarrow{\ \varphi\ } & U \\ {\scriptstyle i_0}\nwarrow & & \nearrow{\scriptstyle i} \\ & X & \end{array} .$$

The proof of theorem 5.14, will basically be identical to the non-equivariant version, except that we need to make sure that every intermediary result can be extended to an invariant/equivariant statement. We begin with the tubular neighborhood theorem. We formulate this result in the more general setting of a compact Lie group acting on a manifold. Ideas for this proof were taken from Bredon's book [3], chapter VI section 2, as well as from [8], theorem 7.1.

Theorem 5.15 (Equivariant tubular neighborhood theorem)**.** *Let G be a compact Lie group acting on a manifold M, with the action map denoted by ψ, and let X be a submanifold that is invariant under the G-action. Then, there exists invariant neighborhoods U_0 of X in NX, and U of X in M, as well as an equivariant diffeomorphism $\varphi : U_0 \to U$ such that the following commutes*

$$\begin{array}{ccc} NX \supseteq U_0 & \xrightarrow{\ \varphi\ } & U \subseteq M \\ {\scriptstyle i_0}\nwarrow & & \nearrow{\scriptstyle i} \\ & X & \end{array} .$$

Proof. The proof of the standard tubular neighborhood 1.45 starts with the choice of a Riemannian metric and then uses the exponential map as the desired diffeomorphism. To improve this result, we first need to construct an invariant Riemannian metric. This can be done as follows. Let m be any Riemannian metric on M. Let $p \in M$ and $u, v \in T_pM$. To ease the notation, we will write gp for the G-action on the point $p \in M$, and similarly we will simply use gu for the G-action on the tangent bundle given by $(d\psi_g)_p[u]$. Let us define the new metric $\tilde{m}$ as

$$\tilde{m}_p(u,v) = \int_G m_{gp}(gu, gv) d\nu,$$

where ν is a Haar measure on G. We have

$$\begin{aligned}\tilde{m}_{hp}(hu, hv) &= \int_G m_{g(hp)}(g(hu), g(hv)) d\nu \\ &= \int_G m_{ap}(au, av) d\nu = \tilde{m}_p(u,v),\end{aligned}$$

and hence the metric $\tilde{m}$ is G-invariant. We now want to take the exponential map corresponding to $\tilde{m}$ as our diffeomorphism. Let $\gamma_{(x,v)}$ be the geodesic starting at x with $\gamma'_{(x,v)}(0) = v$. We know that $\gamma_{(x,v)}(t)$ is locally length minimizing, and hence there is a t_0 such that $\gamma_{(x,v)}(t)$ is the shortest path from x to $\gamma_{(x,v)}(t_0) =: y$. We now look at the curve $g\gamma_{(x,v)}(t)$ for some $g \in G$. For a contradiction, let us assume that $g\gamma_{(x,v)}(t)$ is not the shortest path from gx to gy and let $\gamma(t)$ be such a shortest path. Then, by G-invariance of the metric, $g^{-1}\gamma(t)$ would be a path from x to y that is shorter than $\gamma_{(x,v)}$, which is impossible. This show that $g\gamma_{(x,v)}(t)$ is locally length minimizing, and hence a geodesic. In other words, the group G maps geodesics to geodesics. We have that $exp(gx, gv)$ is the position of the geodesic with initial conditions (gx, gv) at time $t = 1$. By what we just saw, $g(exp(x,v))$ is also the position of a geodesic at time $t = 1$, and it has initial conditions

$$\left.\frac{d}{dt}\right|_{t=0} g(exp(x,v))(t) = D\psi_g(x)[v] = (gx, gv).$$

Thus, by uniqueness we must have

$$g(exp(x,v)) = exp(gx, gv).$$

This shows that the exponential map is G-equivariant, and so the construction that we had in the proof of theorem 1.45 can be made equivariant. The last thing we need to show is that the neighborhoods that we called V in U in the standard proof are invariant. V was defined as

$$V = \{(x,v) \in NX \,|\, ||v|| < \frac{1}{2}\varepsilon(x)\}.$$

Let (x,v) be in V. Then since X is invariant, gx belongs to X. Then, by equivariance of exp, we have that $\varepsilon(gx) = \varepsilon(x)$. Finally, since the metric is invariant $||v|| = ||gv||$. This shows that V is invariant, and since the neighborhood U was defined as $exp(V)$, and since exp is equivariant, U is also clearly invariant. This concludes the proof. ■

The next big result to generalize is the Moser theorem, for which we first need to take a look at the homotopy formula from proposition 1.47 in the equivariant setting. The idea for the discussion below comes from [10], pages 155-160. However, we give explicit calculations here to make everything as transparent as possible.

In the proof of proposition 1.47, we worked on a tubular neighborhood U of a submanifold $X \subseteq M$, and we had defined a homotopy $\rho_t : U_0 \to U_0$ by $(x, v) \mapsto (x, tv)$ for $0 \leq t \leq 1$. If we want to describe that homotopy in terms of the neighborhood U instead of U_0, we have

$$exp(x, v) \mapsto exp(x, tv) \quad 0 \leq t \leq 1.$$

To avoid cluttering the notation we will denote the homotopy on U by ρ_t as well and we will keep working with the neighborhood U for the rest of this discussion.
If we now assume that a compact Lie group G acts on M in such a way that X is invariant under the action of G, as in the proof of the equivariant tubular neighborhood theorem 5.15, we may choose a G-invariant Riemannian metric, which makes the exponential map G-equivariant. Hence, the homotopy ρ_t will also be G-equivariant. Let

$$v_t(\rho(x)) = \frac{d}{ds}\bigg|_{s=t} \rho_t(x),$$

exactly as in proposition 1.47 and its proof. We can write the following claim for any $g \in G$:

$$\psi_g^* v_t = v_t.$$

Indeed, the G-equivariance of ρ_t gives us that for any x in U and $0 \leq t \leq 1$,

$$\begin{aligned}
&\psi_g^* v_t(\rho_t(x)) = D\psi_{g^{-1}}(\psi_g(\rho_t(x)))[v_t(\psi_g(\rho_t(x)))] \\
&= D\psi_{g^{-1}}(\rho_t(\psi_g(x)))[v_t(\rho_t(\psi_g(x)))] = \frac{d}{dt}\psi_{g^{-1}} \circ \rho_t(\psi_g(x)) \\
&= \frac{d}{dt}\rho_t(\psi_{g^{-1}}(\psi_g(x))) = \frac{d}{dt}\rho_t = v_t(\rho_t(x)).
\end{aligned}$$

This then allows us to show that the homotopy equivalence Q constructed in 1.47 commutes with pullbacks by the action. Indeed, for any $g \in G$ and for an arbitrary differential form σ defined on U, we have that

$$\begin{aligned}
\psi_g^* Q\sigma &= \int_0^1 \psi_g^* \rho_t^* (\iota_{v_t}\sigma) dt = \int_0^1 \rho_t^* \psi_g^* (\iota_{v_t}\sigma) dt \\
&= \int_0^1 \rho_t^* \iota_{\psi_g^* v_t} \psi_g^* \sigma dt = \int_0^1 \rho_t^* \iota_{v_t} \psi_g^* \sigma dt = Q\psi_g^* \sigma.
\end{aligned}$$

We can thus adapt the homotopy formula to make it work nicely with respect to the group action.
We can use this to see how the Moser trick can be adapted to this setting. Again, the ideas for this can be found in [10], pages 155-160. Let ρ be the isotopy described in the Moser theorem 1.48 and let $X_t = \frac{d}{dt}\rho_t \circ \rho_t^{-1}$ be the corresponding time-dependent vector field. Observe that if we have ρ_t equivariant, then, simplifying the notation for the G-action again, we have

$$\rho_t^{-1}(gx) = g\rho_t^{-1}(x) \iff gx = \rho_t(\rho_t^{-1}(gx)) = \rho_t(g\rho_t^{-1}(x)) \overset{equiv.}{=} gx,$$

and hence ρ_t^{-1} is also equivariant. Moreover, if we differentiate $\rho_t(gx) = g\rho_t(x)$ on both sides, we get

$$\begin{aligned} &\frac{d}{dt}\rho_t(gx) = X_t(\rho_t(gx)) = X_t(g\rho_t(x)) \\ &= \frac{d}{dt}(g\rho_t(x)) = D\psi_g(\rho_t(x))[X_t(\rho_t(x))] = g \cdot X_t(\rho_t(x)). \end{aligned}$$

Setting $x = \rho_t^{-1}(y)$ for some $y \in M$ shows that X_t is G-invariant.

Conversely, if X_t is a G-equivariant vector field that fulfills equation (1.3), then its flow will also be equivariant[1]. To see this, let us define $\tilde{\rho}_t = \psi_g \circ \rho_t \circ \psi_{g^{-1}}$. We compute

$$\begin{aligned} &\frac{d}{dt}\tilde{\rho}_t(\psi_g(x)) = D\psi_g\left[X_t(\rho_t(\psi_{g^{-1}}(\psi_g(x))))\right] \\ &D\psi_g\left[X_t(\psi_{g^{-1}} \circ \tilde{\rho}_t(\psi_g(x)))\right] \overset{equiv.}{=} X_t(\tilde{\rho}_t(\psi_g(x))). \end{aligned}$$

By uniqueness of flows, we must have that $\rho_t = \tilde{\rho}_t$, which shows that ρ_t is equivariant.

Thus, using the notation of theorem 1.48, the equivariant version of Moser's theorem will hold, as long as we can find an equivariant vector field X_t solving

$$\iota_{X_t}\omega_t = -\mu,$$

where $\omega_1 - \omega_0 = d\mu$. Recall that in theorem 1.48, we know that there exists such a μ since we assume that ω_0 and ω_1 belong to the same cohomology class.

We can now prove the equivariant version of the relative Moser theorem. This result is named the Darboux-Weinstein theorem 22.1 in [10]. We follow the ideas for the proof given there, but we adapt them to the structure we had when proving theorem 1.49.

[1]The idea to prove this comes from [15], prop. 3.3.2

Theorem 5.16 (equivariant relative Moser theorem)**.** *Let M be a compact manifold, and ω_0, ω_1 two symplectic forms on M. Let X be a compact submanifold of M with inclusion map $i : X \hookrightarrow M$. Suppose that for every $q \in X$, we have $\omega_0|_q = \omega_1|_q$. Moreover, suppose there is an action of a compact Lie group G on M preserving both ω_0 and ω_1 and such that X is G-invariant. Then, there exist G-invariant neighborhoods U_0 and U_1 of X and a G-equivariant diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi^*\omega_1 = \omega_0$ and such that the following diagram commutes:*

$$\begin{array}{ccc} U_0 & \xrightarrow{\quad\varphi\quad} & U_1 \\ & {}_{i}\nwarrow \quad \nearrow_{i} & \\ & X & \end{array} .$$

Proof. Take a G-invariant tubular neighborhood U_0 of$X$ as constructed in theorem 5.15. As in the standard proof, $\omega_1 - \omega_0$ is closed on U_0 and it vanishes on X by assumption. Thus, by the homotopy formula (proposition 1.47) there is a 1-form μ on U_0 such that $\omega_1 - \omega_0 = d\mu$ and $\mu_x = 0, \forall x \in X$. Moreover, as explained in the proof of the homotopy formula 1.47, the 1-form μ is given by $Q(\omega_1 - \omega_0)$, where Q denotes the homotopy equivalence from the homotopy formula. By the discussion above about adapting the homotopy formula to the equivariant case, we know that we can choose Q to commute with pullbacks by the group action, which means that for any g in G, we have

$$\begin{aligned} \psi_g^*\mu &= \psi_g^* Q(\omega_1 - \omega_0) \\ &= Q\psi_g^*(\omega_1 - \omega_0) = Q(\omega_1 - \omega_0) = \mu. \end{aligned}$$

This shows that μ is invariant under pullbacks via the group action. Shrinking U_0 if necessary ensures that $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic on U_0 for $t \in [0,1]$. (Notice that shrinking U_0 does not break the invariance assumption since any ε-neighborhood is invariant if we take an invariant metric). We may now use the equivariant Moser trick. If we solve the Moser equation (1.4) pointwise with the invariant 1-form μ, we get a time-dependent vector field X_t. We claim that this vector field is G-equivariant. Indeed, let p be a point in M, g an element of G and let $u \in T_{gp}M$ and define the tangent vector v as $v = g^{-1}u$. We then have for any $t \in [0,1]$

$$\omega_t(X_t(gp), u) \overset{(1.4)}{=} -\mu|_{gp}[u] = -\mu|_{gp}[gv] = -\mu|_p[v]$$

and on the other hand, using the fact that G acts via symplectomorphisms also on every ω_t by linearity, we have

$$\omega_t(gX_t(p), u) = \omega_t(gX_t(p), gv) = \omega_t(X_t(p), v) = -\mu|_p[v].$$

This shows that

$$\iota_{X_t(g\cdot)}\omega_t = \iota_{gX_t}\omega_t.$$

But since every ω_t is non-degenerate by assumption this implies that X_t is G-equivariant for every t in $[0,1]$.
Since X_t is equivariant, our above discussion of the equivariant Moser trick shows that we will get an isotopy $\rho : U_0 \times [0,1] \to M$ such that every ρ_t is G-equivariant and $\rho_t^* \omega_t = \omega_0$ for all t in $[0,1]$. For the same reason as in the standard proof, we must have $\rho_t|_X = id_X$ for every $0 \leq t \leq 1$. Let us now define $\varphi := \rho_1$ and $U_1 = \varphi(U_0)$. The diffeomorphism φ is G-equivariant by construction, and since we chose U_0 to be invariant, U_1 will be invariant as well. This concludes the proof. ■

One other result that we need to adapt to prove theorem 5.14 is Weinstein's Lagrangian neighborhood theorem 1.52. Following what was done in subsection 1.3.2, we start by proving an equivariant version of Whitney's extension theorem.

Theorem 5.17 (Equivariant Whitney extension theorem)**.** *Let M be an n-dimensional manifold and X a k-dimensional submanifold of M such that $k < n$. Suppose that we have a smooth family of linear isomorphisms $L_p : T_pM \xrightarrow{\cong} T_pM$ for $p \in X$ with the property that $L_p|_{T_pX} = id_{T_pX}$. Moreover, let G be a compact Lie group acting on M such that X is G-invariant. Furthermore, assume that the L_p's are compatible with the action of G in the following sense: for every p in X, every g in G and every v in T_pM, we have*

$$D\psi_g(p)[L_p v] = L_{\psi_g(p)} D\psi_g(p)[v].$$

Then, there exists a G-invariant neighborhood U of X and a G-equivariant embedding $h : U \to M$ such that $h|_X = id_X$ and $Dh(p) = L_p$ for every p in X.

Proof. The proof of this theorem follows the same construction as the standard result 1.51. We just need to check that we can make every step of the construction G-equivariant. Let us choose a G-invariant Riemannian metric on M. Then, the tubular neighborhood that we denote by U as in the proof of the standard theorem 1.51 will be invariant, and the exponential map will be G-equivariant. The embedding h is defined as in the standard proof by

$$h(exp(x,v)) = exp(x, L_x v).$$

But by the assumption on the L_p's and the equivariance of the exponential map, we have that for any g in G

$$\begin{aligned} &\psi_g(h(exp(x,v))) = \psi_g(exp(x, L_x v)) = exp(\psi_g(x), D\psi_g(x)[L_x v]) \\ &= exp(\psi_g(x), L_{\psi_g(x)} D\psi_g(x)[v]) = h(exp(\psi_g(x), D\psi_g(x)[v])) \\ &= h(\psi_g(exp(x,v))). \end{aligned}$$

This shows that h is G-equivariant. Moreover, U in invariant by construction. This concludes the proof. ■

Here is now the equivariant version of Weinstein's Lagrangian neighborhood theorem. The main adaptation we have to make is at the level of proposition 1.13, which was an essential tool in proving Weinstein's Lagrangian neighborhood theorem.

Theorem 5.18 (equivariant Weinstein's Lagrangian neighborhood theorem)**.** *Let M be a $2n$-dimensional manifold M with two symplectic structures ω_0 and ω_1, and let X be a compact submanifold of M with inclusion map $i : X \hookrightarrow M$ and such that it is Lagrangian with respect to both ω_0 and ω_1. Let G be a compact Lie group acting via symplectomorphism on both (M, ω_0) and (M, ω_1) and such that X is G-invariant. Then there exists G-invariant neighborhoods U_0 and U_1 of X, as well as a G-equivariant diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi^*\omega_1 = \omega_0$ and such that the following diagram commutes:*

$$\begin{array}{ccc} U_0 & \xrightarrow{\varphi} & U_1 \\ & {\scriptstyle i}\nwarrow \quad \nearrow{\scriptstyle i} & \\ & X & \end{array} .$$

Proof. As in the proof of theorem 5.15, we may choose a G-invariant Riemannian metric m on M. Let us fix $p \in X$ and denote by U the space T_pX, by V the space T_pM and by $U^\top$ the orthogonal complement of U in V with respect to m_p. We now follow the same construction as in the standard version of the theorem to canonically get linear isomorphisms $L_p : T_pM \to T_pM$ such that $L_p|_{T_pX} = id|_{T_pX}$ and $L_p^*\omega_1|_p = \omega_0|_p$. As in the standard version, L_p varies smoothly in p. We now need to check that the L_p's fulfill the compatibility condition needed to make the equivariant Whitney extension theorem 5.17 work. Looking back to proposition 1.13, we see that for every p, the map L_p is given by

$$L_p = Id_{U_p} \oplus B_p : U_p \oplus W_0|_p \to W_1|_p$$

where U_p denotes T_pX, $W_0|_p$ is the canonical Lagrangian complement to U_p with respect to $\omega_0|_p$, $W_1|_p$ is the Lagrangian complement to U_p with respect to $\omega_1|_p$ and $B_p : W_0|_p \to W_1|_p$ is the map such that for every $u \in U_p$ and every $w_0 \in W_0|_p$, we have

$$\omega_0|_p(w_0, u) = \omega_1|_p(Bw_0, u).$$

We claim that B is G-equivariant. We write gp for $\psi_g(p)$ and gv for $D\psi_g(p)[v]$ to ease the notation. We have that

$$\begin{aligned} \omega_1|_{gp}(gB_pw_0, gu) &\overset{\text{inv. of } \omega_1}{=} \omega_1|_p(B_pw_0, u) = \omega_0|_p(w_0, u) \\ &\overset{\text{inv. of } \omega_0}{=} \omega_0|_{gp}(gw_0, gu) = \omega_1|_{gp}(B_{gp}gw_0, gu). \end{aligned}$$

By the non-degeneracy of ω_1, this implies that $gB_pw_0 = B_{gp}gw_0$. This in turn implies that for every p in M, g in G and v in T_pM, we have (switching back to the more exact notation)

$$D\psi_g(p)[L_pv] = L_{\psi_g(p)}D\psi_g(p)[v],$$

which is exactly the compatibility condition we need to use the equivariant Whitney extension theorem 5.17. Thus, there exists a G-invariant neighborhood $\mathcal{N}$ of X and a G-equivariant embedding $h : \mathcal{N} \to M$ such that $h|_X = id_X$ and $Dh(p) = L_p$ for every p in X. As in the standard proof, the forms ω_0 and $h^*\omega_1$ agree on X. Moreover, let p be a point in $\mathcal{N}$ and let g be an element of G. Using the G-equivariance of h and the fact that the G-action preserves ω_1, we have that

$$\begin{aligned}&(D\psi_g)_p^*(Dh)_{\psi_g(p)}^*(\omega_1)_{h(\psi_g(p))} = (D(h\circ\psi_g))_p^*(\omega_1)_{h(\psi_g(p))}\\ &= (D(\psi_g\circ h))_p^*(\omega_1)_{h(\psi_g(p))} = (Dh)_p^*(D\psi_g)_{h(p)}^*(\omega_1)_{\psi_g(h(p))}\\ &= (Dh)_p^*(\omega_1)_{h(p)}.\end{aligned}$$

This shows that $(\psi_g)^*(h^*\omega_1) = (h^*\omega_1)$. Thus, we may use the equivariant Moser relative theorem on the forms ω_0 and $h^*\omega_1$ to get a G-invariant neighborhood U_0 of X and a G-equivariant embedding $f : U_0 \to \mathcal{N}$ such that $f|_X = id_X$ and $f^*(h^*\omega_1) = \omega_0$. Setting $\varphi = h\circ f$ and $U_1 = \varphi(U_0)$ completes the proof, since φ is equivariant as the composition of two equivariant maps and U_1 is invariant by invariance of U_0 and equivariance of φ. ■

The last important element of the proof of the Weinstein tubular neighborhood theorem is the identification between the normal bundle and the cotangent bundle of a Lagrangian submanifold, as we proved in proposition 1.43.

Proposition 5.19. *Let (M,ω) be a symplectic manifold and X a Lagrangian submanifold of M. Let G be a compact Lie group acting on M via symplectomorphisms such that X is G-invariant. Then NX and T^*X are canonically identified, and the bundle isomorphism is G-equivariant.*

Proof. The bundle isomorphism between NX and T^*X constructed in proposition 1.43 is a smooth collection of linear isomorphisms φ_x making the following diagram commute for every x in X:

$$\begin{array}{ccc} N_xX & \xrightarrow{\varphi_x} & T_x^*X \\ {\scriptstyle\pi_{NX}}\downarrow & & \downarrow{\scriptstyle\pi_{T^*X}} \\ X & \xrightarrow{id} & X \end{array},$$

where π_{NX} and π_{T^*X} denote bundle projections. For $v \in N_xX$, $\varphi_x(v)$ is given by

$$\varphi_x(v) = \omega_x(v,\cdot).$$

Thus, taking $g \in G$, $(x,v) \in NX$, $u \in T_{\psi_g(x)}X$ and denoting the bundle isomorphism by φ, we can write

$$(\varphi \circ T\psi_g(x,v))[u] = \omega_{\psi_g(x)}(D\psi_g(x)[v], u) = \omega_x(v, D\psi_{g^{-1}}(\psi_g(x))[u])$$
$$= T^*\psi_g \varphi(x,v)[u].$$

This shows that this identification is equivariant as we wanted. ■

We are now finally ready to put all the pieces together and prove theorem 5.14.

Proof. By proposition 5.19, we know that NX and T^*X are identified in a G-equivariant way. Thus, using the equivariant tubular neighborhood theorem 5.15, we get a G-invariant neighborhood $\mathcal{N}_0$ of X in T^*X, a G-invariant neighborhood $\mathcal{N}$ of X in M, and a G-equivariant diffeomorphism $\eta : \mathcal{N}_0 \to \mathcal{N}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{N}_0 & \xrightarrow{\eta} & \mathcal{N} \\ {\scriptstyle i_0}\nwarrow & & \nearrow{\scriptstyle i} \\ & X & \end{array}.$$

Let us denote by ω_0 the canonical symplectic form on T^*X and by ω_1 the form $\eta^*\omega$. As in the standard proof, X is Lagrangian with respect to both ω_0 and ω_1. Moreover, by what we said in remark 5.13, G acts by symplectomorphisms on (T^*X, ω_0). To be able to use the equivariant Weinstein's Lagrangian neighborhood theorem, we also need to show that G acts by symplectomorphisms on (M, ω_1), which means that we must show that for every g in G

$$(T^*\psi_g)^*\omega_1 = \omega_1.$$

If we fix $p = (x, \xi)$ in T^*X, this amounts to show the equality

$$(D(T^*\psi_g))_p^*((\omega_1)_{T^*\psi_g(p)}) = (\omega_1)_p.$$

Let us set $y := \eta(p)$. Using the definition of ω_1, we have that $(\omega_1)_{T^*\psi_g(p)} = (D\eta)^*_{T^*\psi_g(p)}(\omega_{\psi_g y})$. This yields

$$(D(T^*\psi_g))_p^*(\omega_1)_{T^*\psi_g(p)} = (D(T^*\psi_g))_p^*((D\eta)^*_{T^*\psi_g(p)}\omega_{\psi_g y}) = (D(\eta \circ T^*\psi_g))_p^*(\omega_{\psi_g y})$$
$$= (D(\psi_g \circ \eta))_p^*(\omega_{\psi_g y}) = (D\eta)_p^*(D\psi_g)_y^*(\omega_{\psi_g y})$$
$$= (D\eta)_p^*\omega_y = (\omega_1)_p,$$

where the third equality follows from the equivariance of η, and the fifth from the assumption that G acts symplectically on (M, ω). This computation shows that G acts via symplectomorphisms on (M, ω_1) as well and thus by the equivariant version of Weinstein Lagrangian neighborhood theorem 5.18, there exists G-invariant neighborhoods U_0 and U_1 of X in $\mathcal{N}_0$ and a G-equivariant diffeomorphism $\theta : U_0 \to U_1$ such that $\theta^*\omega_1 = \omega_0$, and such that the following diagram commutes

$$\begin{array}{ccc} U_0 & \xrightarrow{\quad\theta\quad} & U_1 \\ & {\scriptstyle i_0}\nwarrow \qquad \nearrow{\scriptstyle i_0} & \\ & X & \end{array} .$$

As in the standard proof, we may now set $\varphi := \eta \circ \theta$ and $U := \varphi(U_0)$. Since U_0 in invariant and both η and θ are equivariant diffeomorphisms, U will be invariant as well. Finally, $\varphi^*\omega = \theta^*\eta^*\omega = \theta^*\omega_1 = \omega_0$. This concludes the proof. ∎

Remark 5.20. A more general statement known as the equivariant isotropic embedding theorem can is proved in [10] as theorem 39.1.

Although everything we have done in this section holds for any compact Lie group G, we will mostly use Weinstein's tubular neighborhood theorem for toric Lagrangians. We formulate this special case here.

Corollary 5.21 (Toric Weinstein Tubular Neighborhood)**.** *Let $(L, T^{k,A})$ be a toric Lagrangian in a (compact) symplectic toric manifold $(M, \omega, \mathbb{T}^n, \mu)$. Let ω_0 be the canonical symplectic form on T^*L, $i_0 : L \hookrightarrow T^*L$ the Lagrangian embedding as the zero section, and $i : L \hookrightarrow M$ the Lagrangian embedding given by the inclusion. Then there are $T^{k,A}$-invariant neighborhoods U_0 of L in T^*L, U of L in M and a $T^{k,A}$-equivariant diffeomorphism $\varphi : U \to U_0$ such that $\varphi^*\omega = \omega_0$ and such that the following commutes*

$$\begin{array}{ccc} U_0 & \xrightarrow{\quad\varphi\quad} & U \\ & {\scriptstyle i_0}\nwarrow \qquad \nearrow{\scriptstyle i} & \\ & X & \end{array} .$$

Proof. Use theorem 5.14 with $G = T^{k,A}$. ∎

5.4 Towards deformation of toric Lagrangians

We will now work in the setting that we constructed with corollary 5.21. A 1-form ν on L induces an embedding $i_\nu : L \to T^*L$ given by $i_\nu(p) = \nu_p$.

Definition 5.22. If a group G acts on L via the map ψ, we say that the 1-form ν on L is *G-invariant* if the submanifold $i_\nu(L) \subseteq T^*L$ is G-invariant with respect to the lifted action, i.e.

$$\nu_p = \nu_q \circ (d\psi_g)_p \quad \forall p \in L, \forall g \in G,$$

where we have $q = \psi_g(p)$.

Let now$(L, T^{k,A})$ be a toric Lagrangian in a (compact) symplectic toric manifold $(M, \omega, \mathbb{T}^n, \mu)$. Consider a $T^{k,A}$-equivariant tubular neighborhood model as in corollary 5.21:

$$\begin{array}{ccc} U_0 & \xrightarrow{\ \varphi\ } & U \\ {\scriptstyle i_0}\nwarrow & & \nearrow{\scriptstyle i} \\ & L & \end{array}$$

such that $\varphi^*\omega = \omega_0$. We want to extend the result from our example in chapter 4 to this more general setting with the two following claims.

Proposition 5.23. *Let $i_\nu : L \to U_0$ be the embedding of L given by a $T^{k,A}$-invariant closed 1-form ν. Then, $L_\nu := (\varphi \circ i_\nu)(L)$ is a proper and connected Lagrangian submanifold of M lying in U and is $T^{k,A}$-invariant.*

Proof. We start by showing that L_ν is a Lagrangian submanifold. By definition, we have the following commutative diagram:

$$\begin{array}{ccc} U_0 & \xrightarrow{\ \varphi\ } & U \\ {\scriptstyle i_\nu}\nwarrow & & \nearrow{\scriptstyle i} \\ & L_\nu & \end{array} .$$

We compute

$$i^*\omega = (\varphi \circ i_\nu)^* = i_\nu^* \circ (\varphi^*\omega) = i_\nu^*\omega_0 = 0,$$

where the last equality comes from ν being a closed 1-form. Since L_ν has the same dimension as L, we conclude that L_ν is a Lagrangian submanifold.
We now prove $T^{k,A}$-invariance. Let p be a point in L_ν and let g be in $T^{k,A}$. Then we have an x in L such that $p = \varphi(i_\nu(x))$. Using the equivariance of φ and the invariance of ν, this yields

$$\begin{aligned} \psi_g(p) &= \psi_g(\varphi(i_\nu(x))) = \varphi(T^*\psi_g(x, \nu_x)) = \varphi(\psi_g(x), \nu_x \circ D\psi_{g^{-1}}(\psi_g(x))) \\ &= \varphi((\psi_g(x), \nu_{\psi_g(x)})) = \varphi(i_\nu(\psi_g(x))). \end{aligned}$$

Since L is $T^{k,A}$-invariant, $\psi_g(x)$ also belongs to L, and thus by the computation above, $\psi_g(p)$ belongs to L_ν, which shows that L_ν is $T^{k,A}$-invariant.
Since i_ν is an embedding and has closed image, it is a proper map, as is φ because it is a diffeomorphism. Hence, L_ν is a proper submanifold. Connectedness of L_ν also follows directly from the connectedness of L. ■

We see that requiring the 1-form ν to be invariant already gives L_ν some nice properties. We now show that any toric Lagrangian close enough to the zero section is of the form L_ν for some $T^{k,A}$-invariant form ν.

Proposition 5.24. *Any toric Lagrangian $(\tilde{L}, T^{k,A})$ in U that is sufficiently C^1-close to L is of the form L_ν as in proposition 5.23*

Proof. Let $i_L : L \hookrightarrow M$ and $i_{\tilde{L}} : \tilde{L} \hookrightarrow M$ denote the inclusion maps. By definition of $\tilde{L}$ being sufficiently C^1-close to L, we have a diffeomorphism $f : L \to \tilde{L}$ such that $i_{\tilde{L}} \circ f$ if sufficiently C^1-close to i_L. This implies that

$$\varphi^{-1} \circ i_{\tilde{L}} \circ f : L \to \varphi^{-1}(\tilde{L})$$

is sufficiently C^1-close to $i_0 = \varphi^{-1} \circ i_L$, which is the canonical embedding of the zero section. Thus, we have that $\varphi^{-1}(\tilde{L})$ is sufficiently C^1-close to the zero section, and by proposition 4.4, $\varphi^{-1}(\tilde{L})$ is the image of a 1-form ν. It remains to show that ν is $T^{k,A}$-invariant. This is equivalent to the image of the embedding i_ν being invariant under the lifted action of $T^{k,A}$. But the image of i_ν is precisely $\varphi^{-1}(\tilde{L})$. Let λ be an element of $\varphi^{-1}(\tilde{L})$. We have that

$$\begin{aligned} & T^*\psi_g\lambda \in \varphi^{-1}(\tilde{L}) \\ \iff & \varphi(T^*\psi_g\lambda) = \psi_g\varphi(\lambda) \in \tilde{L}. \end{aligned}$$

Since $\tilde{L}$ is $T^{k,A}$-invariant by definition, we directly have that $\psi_g(\lambda) \in \tilde{L}$ for any $g \in T^{k,A}$ and any $\lambda \in \varphi^{-1}(\tilde{L})$. This shows that ν is $T^{k,A}$-invariant, and hence concludes the proof. ■

We conclude this work with a question and some remarks pointing at ways to answer that question.

Question 5.25. Are Lagrangians L_ν defined as in proposition 5.23 toric Lagrangians with symmetry group $T^{k,A}$ if the invariant one-form ν is close enough to the zero section?

We already know from proposition 5.23 that L_ν is a proper and connected $T^{k,A}$-invariant Lagrangian submanifold. What remains to be shown is that the intersection between L_ν and every $\mathbb{T}^n$-orbit is clean and either empty or exactly one $T^{k,A}$-orbit. Here are some remarks that begin to tackle this question.

Remark 5.26.
- If $T^{k,A} = \mathbb{T}^n$, then L is one $\mathbb{T}^n$-principal orbit, and the only closed $\mathbb{T}^n$-invariant one-form is the zero section. Thus, the answer is trivial in that case. There is no deformation of the toric Lagrangian.
- In the other extreme case where $T^{k,A} = \{\pm 1\}^n$, there are a lot of deformations. We have seen this in our example from chapter 4 when we considered all $\{\pm 1\}$-symmetric curves through the poles of the sphere $S^2 \simeq \mathbb{CP}^1$.
- We expect the general case to be a combination of the two extreme cases. We can start by restricting our attention to the set

$$M^o := \mu^{-1}(\Delta^o),$$

where Δ denotes the moment polytope $\mu(M)$ and Δ^o its interior. Take $x \in M^o$ and $g \in \mathbb{T}^n$. Since μ is invariant under the action of $\mathbb{T}^n$, we have that $\mu(x) = \mu(gx)$, and hence $gx \in M^o$, which shows that M^o is $\mathbb{T}^n$-invariant. By the stratified structure of the moment polytope, we have that $\mathbb{T}^n$ acts freely on M^o, and for any x in Δ^o, $\mu^{-1}(x)$ is exactly the $\mathbb{T}^n$-orbit through x. A proof of these facts can be found in [9] on pages 12 and 49. These results about the structure of the moment map of toric symplectic manifolds imply that M^o has the structure of a $\mathbb{T}^n$-principal bundle over the orbit space Δ^o.

We show that $L \cap M^o$ has a similar structure over $\mu(L \cap M^o)$. Since L is a toric Lagrangian, it is $T^{k,A}$-invariant, and since M^o is $\mathbb{T}^n$-invariant, it is in particular $T^{k,A}$-invariant. This implies that $L \cap M^o$ is $T^{k,A}$-invariant. Next, if $\mathbb{T}^n$ acts freely on M^o, in particular $T^{k,A}$ acts freely on $L \cap M^o$. By definition of a toric Lagrangian, the intersection of an orbit passing through L with L is exactly one $T^{k,A}$-orbit and hence $\mu(L \cap M^o)$ is exactly the $T^{k,A}$-orbit space. Thus, in the same way as above, we can see that $L \cap V$ has the structure of a $T^{k,A}$-principal bundle over $\mu(L \cap M^o)$. We take a local trivialization of this principal bundle. Let x be an arbitrary point in $\mu(L \cap M^o)$. Then there exists an open neighborhood W of x in $\mu(L \cap M^o)$ such that

$$\mu^{-1}(W) \cap (L \cap M^o) \simeq W \times T^{k,A}.$$

Since $\mu(L \cap M^o)$ is the orbit space, W is diffeomorphic to a subset S of $\mu^{-1}(W) \cap (L \cap M^o)$ containing exactly one representative for each $T^{k,A}$-orbit through $\mu^{-1}(W)$. We call this set S a slice. We can thus write

$$\mu^{-1}(W) \cap (L \cap M^o) \simeq S \times T^{k,A}.$$

In other words, every point p in $L \cap M^o$ admits an open neighborhood $W' \subseteq (L \cap M^o)$ such that

$$W' \simeq S \times T^{k,A} \tag{5.1}$$

with $S \subseteq W'$.

The local product structure of $L \cap M^o$ induces a local product structure on $\varphi(i_\nu(L \cap M^o))$. Indeed, let q be an arbitrary point in $\varphi(i_\nu(L \cap M^o))$ and take $p \in L \cap M^o$ such that $\varphi(i_\nu(p)) = q$. By the local product structure of $L \cap M^o$, $p \in L \cap M^o$ can be written as $\psi_g(p_0)$, with $g \in T^{k,A}$ and such that p_0 belongs to some slice S. Using the equivariance of φ and the invariance ν, this implies that

$$q = \varphi(i_\nu(p)) = \varphi(i_\nu(\psi_g(p_0))) = \varphi(T^*\psi_g i_\nu(p_0)) = \psi_g(\varphi(i_\nu(p_0))).$$

This computation shows that any point in $\varphi(i_\nu(L \cap M^o))$ can be written as $\psi_g(q_0)$ with $q_0 \in S_\nu := \varphi(i_\nu(S))$. Hence, $\varphi(i_\nu(L \cap M^o))$ has a local product structure of the form

$$S_\nu \times T^{k,A}.$$

Studying the intersection with $\mathbb{T}^n$-orbits locally could be a first step, since we are given this product structure. Of course, we have only shown here that the points in $\varphi(i_\nu(L \cap M^o))$ admit such a product structure in a neighborhood and we haven't even talked about points who do not belong to M^o, so there remains work to be done to show that L_ν is a toric Lagrangian if ν is close enough to the zero section.

Appendix A

The Fubini-Study form

In this appendix, we investigate the Fubini-Study symplectic form and show it is the form that we have on the projective spaces $\mathbb{CP}^n$ after symplectic reduction. The presentation of this material follows homework 12 in [5] and we provide the missing proofs.

Let us look at the function $\rho : \mathbb{C}^n \setminus \{0\} \to \mathbb{R}, z \mapsto log(|z|^2 + 1)$. Our goal is to show that this function is strictly plurisubharmonic so that it defines a Kähler form on $\mathbb{C}^n \setminus \{0\}$ by theorem 2.25. We compute:

$$\frac{\partial}{\partial z_j}\frac{\partial \rho}{\partial \bar{z}_k} = \frac{\partial}{\partial z_j}\frac{z_k}{|z|^2+1} = \frac{\delta_{jk}(|z|^2+1) - z_k\bar{z}_j}{(|z|^2+1)^2}.$$

We need to show that $H := \left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right)$ is positive definite, i.e. that

$$v^* H v > 0, \quad \forall v \in \mathbb{C}^n \setminus \{0\}.$$

By linearity, it is sufficient to check this on unit vectors, so for $v \in S^{2n-1} \subseteq \mathbb{C}^n \setminus \{0\}$. Another observation is that ρ is $U(n)$-invariant, where $U(n)$ denotes the Lie group of unitary matrices. This means that H is also $U(n)$-invariant. Since $U(n)$ acts transitively on S^{2n-1}, it is enough to check positive-definiteness in just one direction, for example on the vector $v_1 = (1, 0, \ldots, 0)$:

$$v_1^* H v_1 = \frac{|z|^2 + 1 - z_1\bar{z}_1}{(|z|^2+1)^2} > 0.$$

This show that H is positive definite, and hence that ρ is strictly plurisubharmonic. Using proposition 2.25, we can thus conclude that the 2-form

$$\rho_{\mathrm{FS}} := \frac{i}{2}\partial\bar{\partial}\rho$$

is a Kähler form. This form is called the *Fubini-Study form* on $\mathbb{C}^n \setminus \{0\}$. We now use this form to construct one on the complex projective spaces.
Let V_1 be the open subset of $\mathbb{C}^n$ defined by $z_1 \neq 0$, and let $\varphi : V_1 \to V_1$ be the map $\varphi(z_1, \dots, z_n) = \frac{1}{z_1}(1, z_2, \dots, z_n)$. We have that $\varphi \circ \varphi = id$, and since $z_1 \mapsto z_1^{-1}$ is holomorphic on V_1, φ is holomorphic and hence biholomorphic. We have that:

$$\begin{aligned}\varphi^*\rho(z) = \rho(\varphi(z)) &= log(\frac{1}{|z_1|^2}(1 + |z_2|^2 + \dots + |z_n|^2) + 1) \\ &= log(\frac{1}{|z_1|^2}(1 + |z_2|^2 + \dots + |z_n|^2 + |z_1|^2)) = \rho(z) + log\frac{1}{|z_1|^2}.\end{aligned}$$

Notice that the last term of this equation is the sum of a holomorphic and an anti-holomorphic function:

$$log\frac{1}{|z_1|^2} = log\frac{1}{z_1}\frac{1}{\overline{z}_1} = -log z_1 - log\overline{z}_1.$$

This yields:

$$\begin{aligned}\partial\overline{\partial}\varphi^*\rho(z) &= \partial\overline{\partial}(\rho(z) - log z_1 - log\overline{z_1}) \\ &= \partial\overline{\partial}\rho(z) - \partial\overline{\partial}log z_1 + \overline{\partial}\partial log\overline{z_1} = \partial\overline{\partial}\rho(z).\end{aligned}$$

Next, we have that:

$$\begin{aligned}\varphi^* dz_j\big|_p[v] &= dz_j\big|_{\varphi(p)}[d\varphi_p[v]] = d(z_j \circ \varphi)\big|_p[v] \\ &= d\varphi_j\big|_p[v] = \partial\varphi_j\big|_p[v] + \overline{\partial}\varphi_j\big|_p[v] = \partial\varphi_j\big|_p[v] \quad \in \Omega^{1,0}.\end{aligned}$$

where $\overline{\partial}\varphi_j\big|_p[v]$ vanishes because φ is holomorphic.

Similarly,

$$\begin{aligned}\varphi^* d\overline{z}_j\big|_p[v] &= d\overline{z}_j\big|_{\varphi(p)}[d\varphi\big|_p[v]] \\ &= d\overline{z}_j\big|_p[v] = \overline{\partial}\overline{\varphi}_j\big|_p[v] \quad \in \Omega^{0,1}.\end{aligned}$$

These last two computations show pulling back by φ preserves type, i.e. $\varphi^*(\Omega^{p,q}) \subseteq \Omega^{p,q}$, which implies that φ^* commutes with ∂ and $\overline{\partial}$. We have shown that ρ_{FS} is invariant under pullback by φ:

$$\varphi^*\rho_{\mathrm{FS}} = \rho_{\mathrm{FS}}.$$

We now put everything together to define a Kähler form on the complex projective spaces $\mathbb{CP}^n$. Let us denote by $U_i = \{[z_0 : \dots : z_n] \in \mathbb{CP}^n \big| z_i \neq 0\}$ the standard charts on $\mathbb{CP}^n$ with maps $\varphi_i([z_0 : \cdots : z_n]) = (\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i})$. The maps φ_0 and φ_1 send $U_0 \cap U_1$ to the set V_1 defined above, and the transition map

$$\varphi_{0,1} := \varphi_0^{-1} \circ \varphi_1 : \varphi_0(U_0 \cap U_1) \to \varphi_1(U_0 \cap U_1)$$

is exactly the map φ that we defined along with V_1. We compute:

$$\varphi_1^* \rho_{\mathrm{FS}} = (\varphi_{0,1} \circ \varphi_0)^* \rho_{\mathrm{FS}} = \varphi_0^* \circ \varphi_{0,1}^* \rho_{\mathrm{FS}} = \varphi_0^* \rho_{\mathrm{FS}}.$$

Doing the same computations for all the U_i's shows that all the Kähler forms $\varphi_i^* \rho_{\mathrm{FS}}$ glue together to define a symplectic form ω_{FS} on $\mathbb{CP}^n$, that we also call the *Fubini-Study form*.

Using the local coordinates given by the charts U_i, we write the following local expression for ρ_{FS}:

$$\begin{aligned}
\frac{i}{2}\partial\bar{\partial}\rho &= \frac{i}{2}\partial \sum_k \frac{\partial \rho}{\partial \overline{z_k}} d\overline{z_k} = \frac{i}{2}\partial \sum_k \frac{z_k}{|z|^2+1} d\overline{z_k} \\
&= \frac{i}{2} \sum_{k,j} \frac{\partial}{\partial z_j}\Big(\frac{z_k}{|z|^2+1}\Big) dz_j \wedge d\overline{z_k} \\
&= \frac{i}{2} \sum_{k,j} \frac{\delta_{jk}(|z|^2+1) - z_k \overline{z_j}}{(|z|^2+1)^2} dz_j \wedge d\overline{z_k} \\
&= \frac{i}{2} \left(\frac{dz \wedge d\bar{z}}{|z|^2+1} - \frac{\bar{z}dz \wedge z d\bar{z}}{(|z|^2+1)^2} \right),
\end{aligned}$$

where $dz \wedge d\bar{z} = \sum_j dz_j \wedge \overline{z_j}$, and $\bar{z}dz = \sum_j \overline{z_j} dz_j$.

Remark A.1. The last expression for ρ_{FS} in the computation above shows that it vanishes when restricted to $\mathbb{R}^n \setminus \{0\}$, which implies that $\mathbb{RP}^n$ is a Lagrangian submanifold of $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$. This will allow us to take $\mathbb{RP}^n$ as a Lagrangian submanifold of $\mathbb{CP}^n$ just like we did for $n = 1$ at the beginning of this chapter.

To make use of this construction in our example, we need to show that ω_{FS} is actually (up the a scaling factor) the form that we obtain when doing symplectic reduction on the level spheres. Let us first fix some notation. Let S_r^{2n+1} denote the sphere of radius r centered around the origin in $\mathbb{C}^{n+1}$, and let $\iota_r : S_r^{2n+1} \hookrightarrow (\mathbb{C}^{n+1}, \omega_0)$ be the inclusion. We write $\pi_r : S_r^{2n+1} \to \mathbb{CP}^n$ for the projection. We claim that $\omega_{\mathrm{FS}} = k\omega_{\mathrm{red},r}$, where $\omega_{\mathrm{red},r}$ is the form obtained by symplectic reduction of the sphere of radius r, and k is a constant factor depending on the radius r.

By theorem 3.10, we need to prove that $\pi^*\omega_{\text{FS}} = k\iota_r^*\omega_0$. Taking without loss of generality a chart φ_k, we compute:

$$\begin{aligned}
\pi_r^*\omega_{\text{FS}}\big|_z &= \pi_r^*\varphi_k^*\frac{i}{2}\partial\overline{\partial}\rho(z) = \frac{i}{2}\partial\overline{\partial}\rho \circ \varphi_k \circ \pi_r(z_0, \dots, z_n) \\
&= \frac{i}{2}\partial\overline{\partial}\rho(\frac{z_0}{z_k}, \dots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \dots, \frac{z_n}{z_k}) \\
&= \frac{i}{2}\partial\overline{\partial}log(\frac{1}{|z_k|^2}(|z_0|^2 + \cdots + |z_{k-1}|^2 + |z_{k+1}|^2 + \cdots + |z_n|^2) + 1) \\
&= \frac{i}{2}\partial\overline{\partial}log(\frac{|z|^2}{|z_k|^2}) \\
&= \frac{i}{2}\partial\overline{\partial}log(|z|^2),
\end{aligned}$$

since $\partial\overline{\partial}log(\frac{1}{|z_k|^2}) = 0$ by what we computed above.

We now differentiate in the same way we did when defining the Fubini-Study form and restrict to S_r^{2n+1}. Using coordinates x_k, y_k such that $z_k = x_k + iy_k$, we obtain:

$$\begin{aligned}
&\frac{i}{2}\partial\overline{\partial}log(|z|^2) = \frac{i}{2}\sum_{j,k=0}^{n}\frac{\delta_{jk}|z|^2 - \overline{z_j}z_k}{r^4}dz_j \wedge d\overline{z_k} \\
&= \frac{i}{2r^4}\sum_{j,k=0}^{n}\delta_{jk}r^2 - (x_j - iy_j)(x_k + iy_k)(dx_j + idy_j) \wedge (dx_k - idy_k) \\
&= \frac{i}{2r^4}\sum_{j=0}^{n}r^2 - x_j^2 - y_j^2(dx_j \wedge dx_j - 2idx_j \wedge dy_j + dy_j \wedge dy_j) \\
&+ \frac{i}{2r^4}\sum_{j=0}^{n}\sum_{k\neq j} -x_jx_k - ix_jy_k + iy_jx_k - y_jy_k(dx_j \wedge dx_k - idx_j \wedge dy_k + idy_j \wedge dx_k + dy_j \wedge dy_k)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{r^4} \sum_{j=0}^{n} r^2 - x_j^2 - y_j^2 dx_j \wedge dy_j \\
&+ \frac{1}{2r^4} \sum_{j=0}^{n} \sum_{j \neq k} x_j y_k - y_j x_k (dx_j \wedge dx_k + dy_j \wedge dy_k) \\
&+ \frac{1}{2r^4} \sum_{j=0}^{n} \sum_{j \neq k} -x_j x_k - y_j y_k (dx_j \wedge dy_k + dx_k \wedge dy_j) \\
&= \frac{1}{r^4} \sum_{j=0}^{n} r^2 - x_j^2 - y_j^2 dx_j \wedge dy_j \\
&+ \frac{1}{2r^4} \sum_{j=0}^{n} \sum_{j \neq k} [x_j dx_j \wedge y_k dx_k + x_j dy_j \wedge y_k dy_k - y_j dx_j \wedge x_k dx_k - y_j dy_j \wedge x_k dy_k \\
&- x_j dx_j \wedge x_k dy_k + x_k dx_k \wedge (-x_j) dy_j - y_j dx_j \wedge y_k dy_k - y_k dx_k \wedge y_j dy_j].
\end{aligned}
$$

Observe that since we are restricting to the sphere, the one-form $\sum_{j=0}^{n} x_j dx_j + \sum_{j=0}^{n} y_j dy_j$ vanishes because the tangent space at a given point is the orthogonal complement of that point. If we look at the double sum above, it is composed of eight two-forms. Adding the first and the last of those forms yields

$$
\begin{aligned}
\sum_{j=0}^{n} \sum_{j \neq k} (x_j dx_j + y_j dy_j) \wedge y_k dx_k &= \sum_{k=0}^{n} (-x_k dx_k - y_k dy_k) \wedge y_k dx_k \\
&= \sum_{k=0}^{n} -y_k^2 dy_k \wedge dx_k = \sum_{k=0}^{n} y_k^2 dx_k \wedge dy_k.
\end{aligned}
$$

Similarly, the fourth and fifth terms add up to

$$
\sum_{j=0}^{n} \sum_{k \neq k} (-x_j dx_j - y_j dy_j) \wedge x_k dy_k = \sum_{k=0}^{n} (x_k dx_k + y_k dy_k) \wedge x_k dy_k = \sum_{k=0}^{n} x_k^2 dx_k \wedge dy_k.
$$

Putting the second and sixth term together gives

$$
\sum_{j=0}^{n} \sum_{j \neq k} x_j dy_j \wedge (x_k dx_k + y_k dy_k) = \sum_{j=0}^{n} x_j dy_j \wedge (-x_j dx_j + y_j dy_j) = \sum_{j=0}^{n} x_j^2 dx_j \wedge dy_j.
$$

An identical computation shows that the sum of the third and seventh form is $\sum_{j=0}^{n} y_j^2 dx_j \wedge dy_j$.

Adding everything up together yields:

$$\frac{i}{2}\partial\bar{\partial}log(|z|^2) = \frac{1}{r^4}\sum_{j=0}^{n} r^2 - x_j^2 - y_j^2 dx_j \wedge dy_j$$
$$+ \frac{1}{2r^4}\left[\sum_{j=0}^{n} y_j^2 dx_j \wedge dy_j + x_j^2 dx_j \wedge dy_j + x_j^2 dx_j \wedge dy_j + y_j^2 dx_j \wedge dy_j\right]$$
$$= \frac{1}{r^4}\sum_{j=0}^{n} r^2 - x_j^2 - y_j^2 + x_j^2 + y_j^2 dx_j \wedge dy_j = \frac{1}{r^4}\sum_{j=0}^{n} r^2 dx_j \wedge dy_j = \frac{1}{r^2}\omega_0.$$

Hence, we have what we wanted, namely that $\pi^*\omega_{\text{FS}} = k\iota_r^*\omega_0$, which shows that the reduced form in our example is the Fubini-Study form.

We conclude this discussion of the Fubini-Study form with an explicit computation for $\mathbb{CP}^1$. As a real manifold, $\mathbb{CP}^1$ is diffeomorphic to the unit sphere S^2 via the stereographic projection. We will show that the Fubini-Study symplectic form ω_{FS} on $\mathbb{CP}^1$ and the standard symplectic form ω_{std} on S^2 from example 1.20 are proportional to each other. Let us start by describing the diffeomorphism between $\mathbb{CP}^1$ and S^2 a bit more explicitly. Let $U_0 = \{[z_0 : z_1] \in \mathbb{CP}^1 | \, z_0 \neq 0\}$ denote the standard complex chart for $\mathbb{CP}^1$. Let $\frac{z_1}{z_0} =: z = x + iy$. The chart map

$$\begin{aligned}\varphi_0 : & U_0 \to \mathbb{C} \\ & [z_0 : z_1] \mapsto z = \frac{z_1}{z_0}\end{aligned}$$

sends U_0 diffeomorphically onto $\mathbb{C} \simeq \mathbb{R}^2$. The last map needed to construct our diffeomorphism is the stereographic projection through the south pole given by

$$\begin{aligned}\varphi_S : & S^2 \setminus \{(-1,0,0)\} \to \mathbb{R}^2 \\ & (x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}\right).\end{aligned}$$

We then have a diffeomorphism $\Phi : S^2 \to \mathbb{CP}^1$ given by $\varphi_0^{-1} \circ \varphi_S$ on $S^2 \setminus \{(-1,0,0)\}$ and by sending $(-1,0,0)$ to $[0 : 1]$. Showing that is map is indeed smooth is a standard computation from differential geometry, which we will not do here as it would take us a bit too far afield.

By what we have seen above, the Fubini-Study form on U_0 is given by

$$\frac{i}{2}\left(\frac{dz \wedge d\bar{z}}{|z|^2+1} - \frac{\bar{z}dz \wedge zd\bar{z}}{(|z|^2+1)^2}\right) = \frac{i}{2}\left(\frac{-2idx \wedge dy}{x^2+y^2+1} - \frac{(x^2+y^2)(-2i)dx \wedge dy}{(x^2+y^2+1)^2}\right)$$
$$= \frac{(x^2+y^2+1)dx \wedge dy - (x^2+y^2)dx \wedge dy}{(x^2+y^2+1)^2} = \frac{dx \wedge dy}{(x^2+y^2+1)^2}.$$

We now compute the pullback of this form to the sphere via the stereographic projection φ_S. We have that

$$\varphi_S^*(dx) = d(x \circ \varphi_S) = d\left(\frac{x_1}{1+x_3}\right) = \frac{1}{1+x_3}dx_1 - \frac{x_1}{(1+x_3)^2}dx_3,$$

and similarly

$$\varphi_S^*(dy) = \frac{1}{1+x_3}dx_2 - \frac{x_2}{(1+x_3)^2}dx_3,$$

where (x_1, x_2, x_3) are the standard Euclidean coordinate on S^2 as the unit sphere in $\mathbb{R}^3$. Thus, pulling back the Fubini-Study form yields

$$\varphi_S^*\omega_{FS} = \frac{(1+x_3)^2 dx_1 \wedge dx_3 - (1+x_3)x_2 dx_1 \wedge dx_3 - (1+x_3)x_1 dx_3 \wedge dx_2}{(x_1^2 + x_2^2 + (1+x_3)^2)^2}.$$

We change to cylindrical coordinates (θ, h) by setting

$$\begin{aligned} x_1 &= \sqrt{1-h^2}\cos\theta, \\ x_2 &= \sqrt{1-h^2}\sin\theta, \\ x_3 &= h. \end{aligned}$$

The exterior derivatives are then given by

$$\begin{aligned} dx_1 &= -\sqrt{1-h^2}\sin\theta d\theta - \frac{h}{\sqrt{1-h^2}}\cos\theta dh, \\ dx_2 &= \sqrt{1-h^2}\cos\theta d\theta - \frac{h}{\sqrt{1-h^2}}\sin\theta dh, \\ dx_3 &= dh. \end{aligned}$$

Plugging this back into our expression for $\varphi_S^*\omega_{FS}$ yields

$$\begin{aligned} &\frac{1}{4(1+h)^2}\Big[(1+h^2)[h\sin^2\theta d\theta \wedge dh - h\cos^2\theta dh \wedge d\theta] \\ &\qquad + (1+h)(1-h^2)\sin^2\theta d\theta \wedge dh - (1+h)(1-h^2)\cos^2\theta dh \wedge d\theta\Big] \\ &= \frac{1}{4(1+h^2)}\Big[(1+h^2)h d\theta \wedge dh + (1+h)(1-h^2)d\theta \wedge dh\Big] \\ &= \frac{1}{4}h d\theta \wedge dh + \frac{1}{4}(1-h)d\theta \wedge dh = \frac{1}{4}d\theta \wedge dh = \frac{1}{4}\omega_{\text{std}}. \end{aligned}$$

This shows that the Fubini-Study form on $\mathbb{CP}^1$ and the area form on S^2 are proportional. In particular, any symplectomorphism on $(\mathbb{CP}^1, \omega_{FS})$ must be area-preserving if we identify $\mathbb{CP}^1$ with the two-sphere.

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