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D MATH

From Symplectic Toric Manifolds to Polytopes and Back Again

Master's Thesis

Reto Kaufmann

reto.kaufmann@math.ethz.ch

Department of Mathematics
ETH Zürich

Supervisor:
Prof. Dr. Ana Cannas da Silva

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Abstract

Symplectic toric manifolds are classified by the image of their moment map which is a unimodular polytope. In this master's thesis, we present recursive aspects of this correspondence. In the first part, we show how a given unimodular polytope gives rise to new unimodular polytopes. In the second part, we present constructions for symplectic toric manifolds and show how they correspond to the constructions on unimodular polytopes. The main result is the correspondence between faces of the moment polytope and symplectic toric submanifolds.

Introduction

An action of a torus on a symplectic manifold is symplectic if it preserves the symplectic structure. It is called Hamiltonian if there is an invariant map to the dual of the Lie algebra of the torus whose components are Hamiltonian functions of the fundamental vector fields. Such a map is called a moment map. For compact and connected symplectic manifolds, the image of the moment map is the convex hull of a finite number of points, which are images of fixed points. Convex sets of this form are called polytopes. They can equivalently be characterised as bounded polyhedra, that is, a finite intersection of half-spaces which is bounded.

If the action is effective, then the dimension of the torus can be at most half of the dimension of the manifold. We speak of a symplectic toric manifold if this threshold is saturated, thus the manifold has as many circle symmetries as possible. In this case, the image of the moment map is unimodular. In fact, as shown by Delzant in 1988 [7], these unimodular polytopes completely characterise the symplectic toric manifold and one can construct a symplectic toric manifold from any unimodular polytope. This is the classification of symplectic toric manifolds by unimodular polytopes.

If a symplectic submanifold is compact, connected and invariant under the action, it is a symplectic toric manifold in its own right. Such a submanifold is then called symplectic toric submanifold and is also specified by a unimodular polytope. This unimodular polytope is a face of the moment polytope of the ambient manifold. On the other hand, all the faces of a unimodular polytope are again unimodular polytopes. One of the main results of this master's thesis is putting the folklore classification of symplectic toric submanifolds by the faces of the moment polytope on a sound footing.

Our approach to symplectic toric manifolds allows for noneffective actions whose kernel is controlled. This is the natural framework to treat symplectic toric submanifolds since one can take the same Lie group acting on the submanifold as the one acting on the ambient manifold.

A different result concerns the product of two symplectic toric manifolds. Equipped with the product action, this is again a symplectic toric manifold. The corresponding construction for the moment polytopes may be viewed as a special case of the Minkowski sum. Finally, also the constructions of symplectic cutting and symplectic reduction are addressed. In this part, the focus is on the polytope side and the possible ways that a unimodular polytope can be cut to produce a new unimodular polytope.

Overview

This master's thesis is divided in two parts. Part I is a thorough introduction to unimodular polytopes and their recursive aspects. Part II first introduces Hamiltonian spaces with a focus on recursive aspects. We then first specialise to actions by tori and eventually to symplectic toric manifolds, establishing the connection to Part I.

Chapter 1 introduces the basic formalism of polyhedra. We define faces first for general convex sets and then specialise to polyhedra. We see examples of vertices, edges and facets. The tangent space and the local cones are defined and their basic properties are shown. We also introduce the annihilator space and the support cone.

In Chapter 2, we give a brief introduction to lattices, define a lattice basis and show that every lattice admits such a basis. We also establish when a subspace and the corresponding quotient space inherit a lattice. Finally, we define rationality with respect to a given lattice for polyhedra.

Chapter 3 begins by establishing that polytopes are exactly bounded polyhedra. Then simplicity and unimodularity are first defined for polyhedral cones and this definition is then extended to polytopes. We also show that the condition on the normal vectors is equivalent to the condition on the edges.

Chapter 4 contains the main results about recursive aspects of polytopes. It starts by discussing which vector space is the natural choice to describe the faces as polyhedra in their own right. It is argued why this space inherits a lattice and that the faces are rational with respect to this new lattice. Then we show that faces of simple or unimodular polyhedral cones are again simple or unimodular polyhedral cones. This result is then extended to polytopes. Next, we introduce the direct Minkowski sum, which is the special case of the Minkowski sum mentioned above. Finally, we introduce the notion of cutting polytopes and discuss in what cases such a cut yields unimodular polytopes.

Part II begins with Chapter 5 and the definition of symplectic actions. We introduce the (symplectic) isotropy representations and discuss how the fixed point set gives rise to symplectic submanifolds. This is followed by the definition of Hamiltonian spaces and their moment maps. The chapter is concluded by an extensive presentation of recursive aspects of Hamiltonian spaces, containing in particular short overviews of symplectic reduction and symplectic cutting.

In Chapter 6 we specialise to Hamiltonian actions by tori. We first introduce the basic formalism for tori. Then we use the classification of symplectic torus representations to get to the toric Darboux theorem, giving a local form around a fixed point of a Hamiltonian torus action. From this we deduce that the components of moment maps are Morse-Bott functions. We then state the convexity theorem and show how the effectiveness of the torus action gives rise to the dimension threshold.

Chapter 7 contains the main results of this master's thesis. It begins with the local picture, which is given by symplectic toric representations. The image of the corresponding moment maps are unimodular polyhedral cones and we prove that symplectic toric subrepresentations correspond to the faces of these cones. We then formally introduce symplectic toric manifolds and the Delzant correspondence. The main part of the text is then concluded by the discussion of the recursive aspects, in particular the product of symplectic toric manifolds and the classification of symplectic toric submanifolds by the faces of the moment polytope.

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Contents

Part I

Unimodular Polytopes

Foreword We will develop the theory for unimodular polytopes living in the dual V^* of a finite-dimensional vector space V. The advantage of doing this is mainly twofold:

1. We will be largely concerned with hyperplanes and affine halfspaces. These are comfortably described by their annihilating vectors (their normal vectors if we would use the identification provided by an inner product) which in turn live in the dual of V^*. Using the canonical identification $(V^*)^* \cong V$, these annihilating vectors can be seen as elements of the vector space V.

2. The goal is to characterise the polytopes which arise as images of the moment maps of symplectic toric manifolds. But the codomain of a moment map is the dual of the Lie algebra of the acting torus. Even though for tori, both the Lie algebra and its dual can be identified with $\mathbb{R}^n$, we argue that it is conceptually more coherent to work with polytopes living in the dual space V^*. However, whenever examples are presented, the identification $V \cong \mathbb{R}^n \cong V^*$ will be assumed.

We will approach the subject of polytopes in stages. Polytopes are bounded polyhedra and many useful properties for polytopes can be shown for general polyhedra. Moreover, certain unbounded polyhedra, called polyhedral cones, are central for the definition of unimodular polytopes. This is motivation enough to study the case of general polyhedra with a focus on polyhedral cones before specifying to polytopes.

One of the main goals of this work is to understand the faces of the polytopes arising from symplectic toric manifolds. This is why extra care is given to the notion of a face of a polytope. The notion of a face is defined for any convex set. This is why the latter are a natural starting point for this Master's thesis.

Chapter 1

Preliminaries on Convexity and Polyhedra

The goal of this chapter is to present the basic formalism of polyhedra and the associated objects such as the tangent space and the local cone. Many of the definitions, results and proofs presented in this chapter are taken directly from the works [3] and [4] of Alexander Barvinok. For some constructions, we took the liberty of using a different name, motivated by the comparison with symplectic toric manifolds. The main example for this is the renaming of the *cone of feasible directions* as *local cone*. For some other constructions, we are not aware of any prior formal definition. The main examples for this are the *splitting* of a convex subset or the *tangent space*, the *annihilator space* or the *support cone* to a polyhedron at a given point.

Convention. For the entire part I, we will assume V to be a finite dimensional real vector space.

1.1 Convex Sets

A convex subset of a vector space is a subset such that the straight line connecting any two points in the subset is entirely contained in the subset. Adapting from the start the convention of working in the dual of the vector space, this corresponds to the following formal definition:

Definition. Let $\varphi, \psi \in V^*$. The interval with endpoints φ and ψ is the set

$$[\varphi, \psi] = \{t\varphi + (1-t)\psi \mid t \in [0,1]\}.$$

A subset $A \subseteq V^*$ is *convex* if for every two points $\varphi, \psi \in A$, one has $[\varphi, \psi] \subseteq A$.

A particularly simple example of convex subsets are vector subspaces. Since the property of being convex is invariant under translation, also the translates of subspaces are convex.

Definition. Let $W \subseteq V^*$ be a subspace. The translation by any non-zero covector $\varphi \in V^* \setminus \{0\}$

$$A = W + \varphi = \{\psi \in V^* \mid \psi = \omega + \varphi \text{ for some } \omega \in W\}$$

is called an *affine subspace* of V^*. The dimension of A is the dimension of W and we say that A is *parallel* to W.

Examples. There are some special cases of particular importance for the present work:

1. An affine subspace A of dimension 1 is called a *line* and is of the form

   $$A = \{\varphi + t\psi \mid t \in \mathbb{R}\}$$

   where $\psi, \varphi \in V^*$ with $\psi \neq 0$.

2. Let $v \in V \setminus \{0\}$ and $c \in \mathbb{R}$. The affine subspace defined by

   $$H = \{\varphi \in V^* \mid \langle \varphi, v \rangle = c\}$$

   is called an *affine hyperplane* and has dimension $\dim V - 1$, i.e. codimension 1.

3. Let $H \subseteq V^*$ be an affine hyperplane given by $v \in V \setminus \{0\}$ and $c \in \mathbb{R}$. The complement of H in V^* is the union of two convex sets, called *open affine halfspaces*

   $$V^* \setminus H = H_> \cup H_<$$

   with

   $$H_> = \{\varphi \in V^* \mid \langle \varphi, v \rangle > c\} \quad \text{and} \quad H_< = \{\varphi \in V^* \mid \langle \varphi, v \rangle < c\}.$$

   The sets $H_\geq = H \cup H_>$ and $H_\leq = H \cup H_<$ are again convex subsets and are called *closed affine halfspaces*

   $$H_\geq = \{\varphi \in V^* \mid \langle \varphi, v \rangle \geq c\} \quad \text{and} \quad H_\leq = \{\varphi \in V^* \mid \langle \varphi, v \rangle \leq c\}.$$

Notation. We will extensively work with affine hyperplanes and closed affine halfspaces, so we adapt a shorthand notation for it. For $v \in V \setminus \{0\}$ and $c \in \mathbb{R}$ we denote

$$H_{(v,c)} = \{\varphi \in V^* \mid \langle \varphi, v \rangle \leq c\}$$
$$H^=_{(v,c)} = \partial H_{(v,c)} = \{\varphi \in V^* \mid \langle \varphi, v \rangle = c\}.$$

Convention. We can extend the definition to also allow for the zero normal $0 \in V$ if we interpret the halfspaces as defined by the inequality. However, in this case we do not get a halfspace but either all of V^* or the empty set depending on the sign of $c \in \mathbb{R}$:

$$H_{(0,c)} = \{\varphi \in V^* \mid \langle \varphi, 0 \rangle \leq c\} = \begin{cases} V^* & \text{if } c \geq 0 \\ \emptyset & \text{if } c < 0 \end{cases}$$

Faces of Convex Sets

Informally, a face of a convex set is obtained by laying a hyperplane from the outside onto the boundary of the convex set and then taking the intersection. The first step towards the rigorous definition is to formalise the notion of "from the outside":

Definition. Let $A \subseteq V$ be a convex subset and $H_{(v;c)} \subseteq V$ be an affine hyperplane. $H_{(v;c)}$ is said to (strictly) isolate A if A is contained in one of the closed (open) affine halfspaces defined by $H_{(v;c)}$ i.e. if

$$\langle \varphi, v \rangle \leq c \quad \text{for all} \quad \varphi \in A$$

with a strict inequality for strict isolation.

Theorem 1.1.1 (Isolation Theorem). *Let $A \subseteq V$ be an open convex set and let $\varphi \notin A$ be a point. Then there exists an affine hyperplane H which contains φ and strictly isolates A.*

Proof. ([3], Theorem 1.6) We proceed by induction on $n = \dim(V)$ and assume without loss of generality, that $\varphi = 0$. The case $n = 1$ is trivial since in that case a hyperplane is just a point and we may choose 0 (i.e. φ itself) to be H.

Assume now that the statement is true for all dimensions smaller than n. Let $H \subseteq V$ be the maximal (meaning of largest possible dimension) subspace such that $0 \in H$ and $H \cap A = \emptyset$.

Claim. H is a hyperplane.

Proof of Claim. Consider the quotient V/H with the associated projection $\pi : V \to V/H$. If H is not a hyperplane, then $\dim(V/H) \geq 2$. By linearity of the projection, $\pi(A)$ is an open convex subset in V/H. By the induction hypothesis, there is a hyperplane $H' \subseteq V/H$ such that $0 \in H'$ and $H' \cap \pi(A) = \emptyset$. But then the preimage $\pi^{-1}(H')$ is a subset of V such that $0 \in \pi^{-1}(H')$ and $A \cap \pi^{-1}(H') = \emptyset$. But $\pi^{-1}(H')$ is strictly larger than H because $\dim(H') = \dim(V/H) - 1 \geq 1$ while $\dim(\pi(H)) = 0$. This contradicts maximality of H and proves the claim.

Corollary 1.1.2. *Let $A \subseteq V$ be a convex set with a non-empty interior and let $\varphi \in \partial A$ be a boundary point. Then there exists an affine hyperplane H such that $\varphi \in H$ and H isolates A.*

Proof. ([3], Theorem 2.7) $\operatorname{int}(A)$ is a non-empty convex open set such that $\varphi \notin \operatorname{int}(A)$. Then by the isolation theorem, there exists an affine hyperplane H containing φ and isolating $\operatorname{int}(A)$. Finally, it follows by continuity of linear maps that H also isolates A.

Figure 1.1: Tangent lines to the circle are support hyperplanes.

This last result is the formalisation of "laying a hyperplane onto the boundary":

Definition. Let $A \subseteq V$ be a convex set and $\varphi \in \partial A$ be a boundary point. A *support hyperplane* to A at φ is an affine hyperplane $H \subseteq V$ such that $\varphi \in H$ and H isolates A.

Example. The tangent space to the closed ball $A \subseteq \mathbb{R}^n$ is a support hyperplane. See figure 1.1 for the example of $n = 2$.

A face of a convex set is now the intersection of the convex set with a support hyperplane. However, it is convenient to also consider the empty set and the whole convex set to be faces. To include the first it is sufficient to allow for hyperplanes which do not intersect the convex set i.e. strictly isolate it, but the second has to be added manually:

Definition. Let $A \subseteq V$ be convex. A *face* F of A is either all of A or a (possibly empty) set of the form

$$F = A \cap H$$

where H is an affine hyperplane which isolates A. The face F is called a *proper* face if it is neither empty nor all of A.

Example. Consider the triangle T with corners A, B and C shown in figure 1.2. The corners e.g. $C = T \cap H_1$ and the edges e.g. $[A, B] = T \cap H_2$ are faces of the triangle.

Notation. Building upon the convention for the notation of hyperplanes, we write

$$F_{(v,c)}(A) = A \cap H_{(v,c)} = \{\varphi \in A \mid \langle \varphi, v \rangle = c\}.$$

Faces of convex sets are a special case of a more general class of subsets with interesting properties:

Figure 1.2: The point C and the interval $[A; B]$ are faces of the triangle with cornerpoints $A; B$ and C.

Definition. Let $A \subseteq V$ be a convex subset. A subset $E \subseteq A$ is called an *extreme set* of A if for any $\varphi \in E$ such that $\varphi = (\psi_1 + \psi_2)/2$ for some $\psi_1; \psi_2 \in A$, we must have $\psi_1; \psi_2 \in E$.

Proposition 1.1.3. *A face of a convex set is an extreme set.*

Proof. Let $F \subseteq A \subseteq V$ be a face, that is, there exist $v \in V \setminus \{0\}$ and $c \in \mathbb{R}$ such that

$$A \subseteq H_{(v;c)} \quad \text{and} \quad F = A \cap \partial H_{(v;c)}:$$

Take then $\varphi \in F$ such that $\varphi = (\psi_1 + \psi_2)/2$ for some $\psi_1; \psi_2 \in A$. Since $A \subseteq H_{(v;c)}$, we have $\langle \psi_1; v\rangle ; \langle \psi_2; v\rangle \leq c$. But then we observe that

$$\begin{aligned} c &= \langle \varphi; v\rangle \\ &= \left\langle \frac{\psi_1 + \psi_2}{2}; v \right\rangle \\ &= \frac{\langle \psi_1; v\rangle + \langle \psi_2; v\rangle}{2} \\ &\leq c \end{aligned}$$

with equality if and only if $\langle \psi_1; v\rangle = c = \langle \psi_2; v\rangle$. But this means that $\psi_1; \psi_2 \in \partial H_{(v;c)} \cap A$ and hence F is an extreme set of A.

1.1.1 Convex Cones

We will now turn our attention to a special class of convex sets which will play a crucial role in the upcoming chapters: cones. The guiding principle to keep in mind is that cones will be used to characterise the local properties of more general convex sets.

Definition. A point $\varphi \in V$ is called the *conic combination* of points $\varphi_1, \ldots, \varphi_n \in V$ if

$$\varphi = \sum_{i=1}^{n} \lambda_i \varphi_i \quad \text{for some } \lambda_i \in \mathbb{R}_{\geq 0}.$$

The set of all conic combinations of points from a given set $A \subseteq V$ is called the *conic hull* of A and denoted by $\mathrm{co}(A)$.

Example. The conic hull $\mathrm{co}(\varphi)$ of a single, non-zero point $\varphi \in V$ is called the *ray* spanned by φ.

Definition. A convex set $C \subseteq V$ is called a *convex cone* if $0 \in C$ and the ray spanned by any non-zero element $\varphi \in C \setminus \{0\}$ is entirely contained in C:

$$\lambda \varphi \in C \quad \text{for every } \lambda \in \mathbb{R}_{\geq 0}, \varphi \in C.$$

The study of convex cones is considerably easier than the study of arbitrary convex sets. This is due to the fact that any support hyperplane is a hyperplane in the strict sense, that is, it contains the origin and is a vector subspace.

Lemma 1.1.4. *Let $C \subseteq V$ be a cone and let $H \subseteq V$ be an affine hyperplane isolating and intersecting C i.e. $C \cap H \neq \emptyset$. Then $0 \in H$.*

Proof. ([3], Lemma 8.2) Write $H = H_{(v,c)} = \partial H_{(v,c)}$ for some $v \in V \setminus \{0\}$ and $c \in \mathbb{R}$ such that $C \subseteq H_{(v,c)}$. Since $0 \in C$, we must have $\langle 0, v \rangle = 0 \leq c$. Assume that $c > 0$. Since $H_{(v,c)}$ is a support hyperplane, there is a $\varphi \in C$ such that $\langle \varphi, v \rangle = c > 0$. But then for $\lambda > 1$, we have $\langle \lambda\varphi, v \rangle = \lambda \langle \varphi, v \rangle = \lambda c > c$, so that $\lambda\varphi \notin H_{(v,c)}$. But as C is a cone, we have $\lambda\varphi \in C$ which then contradicts $C \subseteq H_{(v,c)}$.

1.2 Polyhedra

The first goal of part I is to properly define what unimodular polytopes are. Polytopes are special cases of convex sets which belong to a larger class of convex sets, namely polyhedra: In section 3.1 we will see that convex polytopes can be seen as exactly those polyhedrons which are bounded. This will allow us to treat polytopes with the same formalism as polyhedra. This is the motivation for this extensive presentation of the formalism of polyhedra which we begin with the formal definition.

Definition. A *polyhedron* $P \subseteq V$ is the intersection of finitely many closed affine halfspaces:

$$P = \bigcap_{i \in I} H_{(v_i, c_i)} = \{\varphi \in V \mid \langle \varphi, v_i \rangle \leq c_i \text{ for all } i \in I\}$$

where I is some finite set. By convention, if I is the empty set, P corresponds to all of V.

Figure 1.3: Polyhedron with outward pointing normal vectors $v_1; v_2$ and v_3.

Remark. 1. Since the intersection of convex sets is again convex, polyhedra are convex.

2. The intersection of two polyhedra is again a polyhedron. This is obvious as the intersection of two finite intersection is again a finite intersection.

Examples. 1. Taking $|I| = 1$, we see that affine halfspaces are polyhedra.

2. For $|I| = 2$ and choosing the two distinct closed halfspaces defined by a single hyperplane, we get that hyperplanes are polyhedra as well.

3. For $|I| = 3$, the polyhedron

$$P = \bigcap_{i=1}^{3} H_{(v_i;c_i)}$$

with

$$v_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}; \quad v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as well as $c_1 = c_2 = 0$ and $c_3 = 1$ is shown in figure 1.3. This polyhedron is clearly not bounded.

As can be seen already from the first two examples, one must be careful when talking about the dimension of a polyhedron. The key to a sensible notion is the observation that for any polyhedron which does not contain an interior point, there is a proper affine subspace containing the whole polyhedron. More formally, we have the following result:

Lemma 1.2.1. Any non-empty polyhedron either contains an interior point or lies in a proper affine subspace.

Proof. ([4], Theorem 4.15). Consider an arbitrary polyhedron $P = \bigcap_{i \in I} H_{(v_i;c_i)}$ and assume first that for every $i \in I$ there is a point $\varphi_i \in P$ such that $\langle \varphi_i; v_i \rangle < c_i$. Then for the average

$$\varphi = \frac{1}{|I|} \sum_{i \in I} \varphi_i$$

we have

$$\langle \varphi; v_i \rangle < c_i \quad \text{for all} \quad i \in I$$

and φ lies in the interior of P. If on the other hand, for some $i \in I$ we have $\langle \varphi; v_i \rangle = c_i$ for every $\varphi \in P$, then P lies in the affine hyperplane $\partial H_{(v_i;c_i)}$.

Definition. Let $P \subseteq V$ be a polyhedron.

1. The dimension of P is the dimension of the smallest affine subspace that contains P. By convention, the dimension of the empty set is -1.

2. The *relative interior* $\mathrm{int}(P)$ is the interior of P with respect to the smallest affine subspace containing P. For brevity, we will often just refer to the relative interior as the interior.

Since polyhedra are convex sets, they inherit the notion of a face established in section 1.1: A face F of a polyhedron P is either all of P or a (possibly empty) set of the form

$$F = P \cap H_{(v;c)} = \{ \varphi \in P \mid \langle \varphi; v \rangle = c \}$$

where $H_{(v;c)}$ is an affine hyperplane which isolates P. Moreover, since hyperplanes are examples of polyhedra and intersections of polyhedra are again polyhedra, the faces of a polyhedron are again polyhedra. In particular, we can thus speak of the dimension of a face and classify faces according to their dimension. Of particular importance are faces of small dimension or codimension which therefore get their own name:

Definition. Let $P \subseteq V$ be a polyhedron and $F \subseteq P$ a face of P.

1. If $\dim(F) = 0$, F is called a *vertex*. We denote the set of vertices of P by $\mathrm{Vert}(P)$.

2. If $\dim(F) = 1$, F is called an *edge*.

3. If $\mathrm{codim}(F) = 1$, F is called a *facet*.

Example. For the standard cube $[0;1]^3 \subseteq \mathbb{R}^3$ the notions of a vertex, and edge and a facet are exactly what one would expect. See figure 1.4 for a particularly colourful example.

Remark. Note that being a vertex, that is a face of dimension zero, is equivalent to being an extreme point.

Figure 1.4: Colourful example of a cube in $\mathbb{R}^3$ with a vertex V, an edge E and a facet F.

This equivalence can for instance be used to show that any vertex of a face is also a vertex of the polyhedron.

Lemma 1.2.2. *Let $F \subseteq P$ be a face of a polyhedron P and let φ be a vertex of F. Then φ is also a vertex of P.*

Proof. ([4], Lemma 4.6) Suppose that φ is a vertex of F, or equivalently that it is an extreme point. Write then $\varphi = (\varphi_1 + \varphi_2)/2$ for $\varphi_1, \varphi_2 \in P$. Since F is an extreme set by Proposition 1.1.3, it follows that $\varphi_1, \varphi_2 \in F$. But then $\varphi_1 = \varphi = \varphi_2$ since φ is an extreme point in F. Thus φ is an extreme point of P, hence a vertex of P.

It is clear from the definition of a polyhedron that the description in terms of halfspaces is not unique. For instance, one can use a certain halfspace twice and still get the same subset of V. More generally, there might be halfspaces involved in the intersection which are not needed to describe the subset:

Definition. Let $P = \bigcap_{i \in I} H_{(v_i;c_i)}$ be a polyhedron. An index $j \in I$ such that

$$P = \bigcap_{i \in I} H_{(v_i;c_i)} = \bigcap_{i \in I \setminus \{j\}} H_{(v_i;c_i)}$$

is called *redundant*. The index set I of a polyhedron is called *irredundant* or *minimal* if no index $i \in I$ is redundant, that is, if for every $j \in I$ we have

$$P = \bigcap_{i \in I} H_{(v_i;c_i)} \neq \bigcap_{i \in I \setminus \{j\}} H_{(v_i;c_i)}.$$

Remark. 1. From every index set of a polyhedron we can obtain a minimal one by just omitting any redundant index. In the following, we will hence assume without loss of generality the index sets to be minimal.

2. Any index $j \in I$ such that $H_{(v_j;c_j)}$ is a hyperplane isolating $\bigcap_{i \in I \setminus \{j\}} H_{(v_i;c_i)}$ is redundant. Hence we might assume that if I is minimal, all the indices $i \in I$ actually give rise to support hyperplanes $H_{(v_i;c_i)}$ such that

$$H_{(v_j;c_j)} \cap \operatorname{int}\left(\bigcap_{i \in I \setminus \{j\}} H_{(v_i;c_i)} \right) \neq \emptyset \, .$$

But this implies that for $P = \bigcap_{i \in I} H_{(v_i;c_i)}$ (with I minimal) the face

$$F_{(v_i;c_i)}(P) = P \cap H_{(v_i;c_i)}$$

has codimension 1, and is hence a facet. Hence for any index $i \in I$ of a minimal index set, there is a well-determined facet of the polyhedron.

Recall that a polyhedron is not required to be bounded and may therefore contain directions which extend to infinity. The existence of a direction which extends to infinity in both directions can be linked to the polytope containing a vertex.

Lemma 1.2.3. 1. A polyhedron is bounded if and only if it does not contain a ray.

2. A non-empty polyhedron contains a vertex if and only if it does not contain a line.

Proof. ([4], Theorem 4.8 and Lemma 4.2.)

1. If a polyhedron contains a ray, it is clearly unbounded.

   Reversely, let $P = \bigcap_{i \in I} H_{(v_i;c_i)}$ be an unbounded polyhedron. Choose an inner product on V and note that since P is unbounded, there exists a sequence $u_n \in P$ with $u_1 = 0$ (if $0 \neq u_1$, just translate the whole polyhedron and sequence by u_1) and $\|u_n\| \to \infty$. Consider then the normalised sequence $w_n = u_n / \|u_n\|$. Note that each point w_n lies on the interval with endpoints 0 and u_n i.e. $w_n \in [u_1; u_n] = [0; u_n]$ and hence by convexity $w_n \in P$. Since P is closed it follows that any limit point w of w_n is contained in P as well. It follows that the ray spanned by $w = \lim_{n \to \infty} w_n$ will lie in P.

2. Let $P = \bigcap_{i \in I} H_{(v_i;c_i)}$ be a polyhedron. Suppose first that P contains a line $\{\varphi + t\lambda \mid t \in \mathbb{R}\}$ for some $\lambda \neq 0$ through $\varphi \in P$. By linearity, we must have

$$\langle \lambda ; v_i \rangle = 0 \quad \text{for all} \quad i \in I \, :$$

   since otherwise there would be a $t \in \mathbb{R}$ large enough such that

$$\langle \varphi + t\lambda ; v_i \rangle = \langle \varphi ; v_i \rangle + t \langle \lambda ; v_i \rangle > c_i$$

meaning that the line is not contained in P. But then we can write

$$\varphi = \frac{\varphi_1 + \varphi_2}{2} \quad \text{with} \quad \varphi_1 = \varphi + \varepsilon; \ \varphi_2 = \varphi - \varepsilon \in P.$$

This shows that no point can be an extreme point and hence also that no point can be a vertex.

Reversely, assume that P does not contain a line and proceed by induction over $n = \dim(V)$. The case $n = 0$ is clear. Assume now the that the statement holds for $n - 1$ and choose an arbitrary $\varphi \in P$. We distinguish two cases:

(a) Suppose first that φ is part of some proper face. This face lies in a proper affine subspace and hence contains a vertex by the induction hypothesis.

(b) Suppose now that φ is not contained in any face F. Consider then a line in V through φ. Since P does not contain any lines, the intersection of this line with P is either a ray emanating from a certain point ψ in a proper face or an interval with endpoints $\psi_1; \psi_2$ in proper faces. In both cases, we apply the induction hypothesis as above to the appropriate face of P to see that this face contains a vertex.

In both cases we conclude since any vertex of a face is also a vertex of P by Lemma 1.2.2.

1.2.1 Polyhedral Cones

Polyhedral cones are a special class of polyhedra. They are a useful tool to understand the local behaviour of a point in a more general polyhedron as we will see in section 1.3. Just as for convex cones, polyhedral cones are considerably simpler to study than general polyhedra because the halfspaces characterising them arise from proper hyperplanes, not affine ones.

Definition. A polyhedron $C \subseteq V$ is called a *polyhedral cone* if $0 \in C$ and for every $\varphi \in C$ and every $\lambda \in \mathbb{R}_{\geq 0}$ we have $\lambda\varphi \in C$.

Remark. 1. It follows directly from Lemma 1.1.4, that a polyhedral cone is the finite intersection of halfspaces that is affine halfspaces with 0 on their boundary. This condition is equivalent to all the coefficients $c_i \in \mathbb{R}$ being zero. For brevity, when talking about cones we will thus omit them from the notation and write

$$C = \bigcap_{i \in I} H_{v_i} \quad \text{with} \quad H_{v_i} = \{\varphi \in V \mid \langle \varphi; v_i \rangle \leq 0\}$$

2. For polyhedral cones, only the origin can possibly be a vertex. A cone where the origin is indeed a vertex, is called *pointed*. By Lemma 1.2.3, a polyhedral cone is pointed if and only if it does not contain a line.

3. The conic hull of a finite set of points is necessarily a polyhedral cone. Reversely, a pointed polyhedral cone can be written as the conic hull of a finite set of points.

1.3 Tangent Spaces and Local Cones

In the preceding sections we have exhibited several global properties of polyhedra and introduced polyhedral cones. This was motivated by saying that these objects can be used to characterise local properties of points in the polyhedron. In this section we will introduce the constructions which establish this link.

Consider first a boundary point of a polyhedron. As such it is surely contained in all the closed affine halfspaces which define the polyhedron. However, since it is a boundary point, for some of those halfspaces, the point lies on the boundary, that is, in the defining hyperplane. These are the affine halfspaces that determine the local properties of the point. Since we will use them extensively, we introduce some useful terminology and notation for them.

Definition. Take a polyhedron $P = \bigcap_{i \in I} H_{(v_i;c_i)}$ and let $\varphi \in P$ be a point. The indices

$$I_\varphi = \{ i \in I \mid \varphi \in \partial H_{(v_i;c_i)} \} = \{ i \in I \mid \langle \varphi ; v_i \rangle = c_i \}$$

are called *active* on φ.

Notation. We will denote by $\widehat{I_\varphi}$ the complement of I_φ in I, that is

$$\widehat{I_\varphi} = \{ i \in I \mid \varphi \notin \partial H_{(v_i;c_i)} \} = \{ i \in I \mid \langle \varphi ; v_i \rangle < c_i \}.$$

Tangent Spaces and Annihilator Spaces

We start with the definition of a tangent space to a polyhedron at a given point. This definition is inspired by the definition of the tangent space in differential geometry: Instead of taking smooth curves and considering its velocity vectors, for polyhedra we take straight lines.

Definition. Let P be a polyhedron and $\varphi \in P$ a point.

1. The *tangent space* $L_\varphi(P)$ to P at φ is the set

$$L_\varphi(P) = \{ \xi \in V \mid \varphi + t\xi \in P \quad \text{for} \quad t \in [-\varepsilon ; \varepsilon] \quad \text{for some} \quad \varepsilon > 0 \}.$$

2. The annihilator space $W_\varphi(P)$ at φ is the annihilator in $(V^*)^*$ of the tangent space to P at φ:
$$W_\varphi(P) = (L_\varphi(P))^0.$$

Remark. Since V is assumed to be finite-dimensional, there is a canonical identification $V = (V^*)^*$. Under this identification, for any subspace $W \subseteq V$ it holds that $(W^0)^0 = W$. Hence the annihilator space $W_\varphi(P)$ can be interpreted as the subspace of V such that
$$(W_\varphi(P))^0 = L_\varphi(P).$$

While the definition of the tangent space above exhibits very well its local character, it is not very useful for explicit computations. The concept of active indices provides a rather natural expression for the tangent space in terms of the defining hyperplanes:

Lemma 1.3.1. Let $P = \bigcap_{i\in I} H_{(v_i,c_i)}$ be a polyhedron and $\varphi \in P$ a point. Write d_φ for the dimension of the smallest face containing φ. Then

1. the tangent space is given explicitly by
$$L_\varphi(P) = \bigcap_{i\in I_\varphi} \partial H_{v_i}$$
and has dimension d_φ and

2. the annihilator space is given explicitly by
$$W_\varphi(P) = \mathrm{Span}_{\mathbb{R}} \{v_i\}_{i\in I_\varphi}$$
and has codimension d_φ.

Proof. 1. Suppose $\psi \in L_\varphi(P)$. Then there is an $\varepsilon > 0$ such that for any $t \in [-\varepsilon, \varepsilon]$, $\varphi + t\psi \in P$. It follows that for $i \in I_\varphi$
$$\begin{aligned}\langle \varphi + t\psi, v_i\rangle &= \langle \varphi, v_i\rangle + t\langle \psi, v_i\rangle \\ &= c_i + t\langle \psi, v_i\rangle \\ &\geq c_i.\end{aligned}$$
Therefore $t\langle \psi, v_i\rangle \geq 0$ for all $t \in [-\varepsilon, \varepsilon]$ which implies that $\langle \psi, v_i\rangle = 0$, and hence $\psi \in \partial H_{v_i}$. Since $i \in I_\varphi$ and $\psi \in L_\varphi(P)$ were arbitrary, this shows that $L_\varphi(P) \subseteq \bigcap_{i\in I_\varphi} \partial H_{v_i}$.

Conversely, take $\psi \in \bigcap_{i\in I_\varphi} \partial H_{v_i}$. Then for $i \in I_\varphi$ we have
$$\langle \varphi + t\psi, v_i\rangle = \langle \varphi, v_i\rangle + t\langle \psi, v_i\rangle = c_i + 0$$

while for $i \in I \setminus I_\varphi$ we have $c_i - \langle \varphi, v_i \rangle > 0$. Choose $\epsilon > 0$ sufficiently small such that

$$|\langle \psi, v_i \rangle| < c_i - \langle \varphi, v_i \rangle .$$

Then for all $t \in [-\epsilon, \epsilon]$ we have

$$\begin{aligned}
\langle \varphi + t\psi, v_i \rangle &= \langle \varphi, v_i \rangle + t\langle \psi, v_i \rangle \\
&\le \langle \varphi, v_i \rangle + |t||\langle \psi, v_i \rangle| \\
&\le \langle \varphi, v_i \rangle + \epsilon|\langle \psi, v_i \rangle| \\
&< \langle \varphi, v_i \rangle + c_i - \langle \varphi, v_i \rangle \\
&= c_i
\end{aligned}$$

Repeating the same argument for the other indices $i \in I \setminus I_\varphi$ and choosing the minimal ϵ implies that $\psi \in L_\varphi(P)$.

$\dim(L_\varphi(P)) = d_\varphi$ follows immediately from the definition of $L_\varphi(P)$.

2. $\operatorname{codim}(W_\varphi(P)) = d_\varphi$ is a consequence of $\dim(L_\varphi(P)) = d_\varphi$. The explicit expression for $L_\varphi(P)$ implies that the codimension of $L_\varphi(P)$ corresponds to the dimension of $\operatorname{Span}_{\mathbb{R}}\{v_i\}_{i \in I_\varphi}$. Hence we have

$$\dim(W_\varphi(P)) = \dim \operatorname{Span}_{\mathbb{R}}\{v_i\}_{i \in I_\varphi}$$

and it suffices to show one inclusion. Take $v \in \operatorname{Span}_{\mathbb{R}}\{v_i\}_{i \in I_\varphi}$ and write it as $v = \sum_{i \in I_\varphi} \lambda_i v_i$ for some real coefficients $\lambda_i \in \mathbb{R}$. Then for all $\psi \in L_\varphi(P)$

$$\langle \psi, v \rangle = \sum_{i \in I_\varphi} \lambda_i \langle \psi, v_i \rangle = 0$$

where we used that $\psi \in \partial H_{v_i}$ for all $i \in I_\varphi$. Hence $\operatorname{Span}_{\mathbb{R}}\{v_i\}_{i \in I_\varphi} \subseteq W_\varphi(P)$ and we conclude by equality of the dimensions.

For interior points, no index is active and hence the tangent space is all of V^*. For a boundary point φ on the other hand, it follows by Corollary 1.1.2 that it is contained in some face F and by minimality of the index set, I_φ is non empty. Thus the tangent space $L_\varphi(P)$ is a proper subspace of V^* and the annihilator space $W_\varphi(P)$ is non-trivial. On the other hand, at a vertex the tangent space is trivial. This observation allows to see that the set of vertices $\operatorname{Vert}(P)$ of a polyhedron P is always finite:

Corollary 1.3.2. Let $P = \bigcap_{i \in I} H_{(v_i, c_i)}$ be a polyhedron and let $\varphi \in P$ be a point. φ is a vertex of P if and only if $\{v_i\}_{i \in I_\varphi}$ span V i.e. $W_\varphi(P) = V$. In particular, the set of vertices of a polyhedron is always finite, possibly empty.

Proof. The first statement is just Lemma 1.3.1 with $d_\varphi = 0$.

Figure 1.5: Schematic picture of a polyhedron and the tangent space to two points and φ.

For the second, note that for every vertex $\varphi \in P$, we have $|I_\varphi| \geq \dim(V)$ and that I_φ uniquely determines the vertex. Hence, the number of vertices of P does not exceed

$$\binom{|I|}{\dim(V)} = \frac{|I|!}{(|I| - \dim(V))!\,\dim(V)!}$$

and is therefore finite.

Example. Consider again the same polyhedron P as in figure 1.3. See now figure 1.5 for the example of two tangent spaces to this polyhedron at two different points. is a vertex and hence its tangent space is trivial. One might also see that is contained in the two hyperplanes associated to v_2 and v_3, so $I = \{2; 3\}$. Then it follows from the explicit expression in Lemma 1.3.1 that $L(P) = 0$.

The point φ on the other is only contained in the hyperplane associated to v_3 so $I_\varphi = \{3\}$. The tangent space is thus one-dimensional by Lemma 1.3.1 and given as

$$\begin{aligned} L_\varphi(P) &= \partial H_{v_3} \\ &= \{\varphi \in V \mid \langle \varphi; v_3 \rangle = 0\} \\ &= \operatorname{Span}_{\mathbb{R}}((1; 0)) \\ &= \mathbb{R}. \end{aligned}$$

Local Cones and Support Cones

While the tangent space (and thus also the annihilator space) carry some information of the local properties, they do not characterise them sufficiently well for our purposes. This is evident for vertices whose tangent spaces are trivial. This fact can actually in some sense be seen as a local version of the second point of Lemma 1.2.3. Looking at this result also hints at what might be a solution to the problem: Instead of looking at short lines whose midpoint is the chosen point, one might look at short rays emanating from this point.

Definition. 1. The local cone $C_\varphi(P)$ to P at φ is the set

$$C_\varphi(P) = \{\psi \in V \mid \varphi + t\psi \in P \text{ for } t \in [0;\varepsilon] \text{ for some } \varepsilon > 0\}.$$

2. The support cone $S_\varphi(P)$ to P at φ is the set

$$S_\varphi(P) = \{v \in V \mid \langle \psi; v\rangle \geq 0 \text{ for all } \psi \in C_\varphi(P)\}.$$

Just as for the tangent space, it is useful to first establish an explicit expression.

Lemma 1.3.3. Let $P = \bigcap_{i \in I} H_{(v_i;c_i)}$ be a polyhedron and $\varphi \in P$ be a point. Then

1. the local cone to P at φ is given explicitly by

$$C_\varphi(P) = \bigcap_{i \in I_\varphi} H_{v_i}$$

2. and the support cone to P at φ is given explicitly by

$$S_\varphi(P) = \mathrm{co}(\{v_i\}_{i \in I_\varphi})$$

Proof. 1. Suppose first that $\psi \in C_\varphi(P)$. Then note that for $i \in I_\varphi$, there is an $\varepsilon > 0$ such that for all $t \in [0;\varepsilon]$ we have $\varphi + t\psi \in P$. Therefore

$$\begin{aligned}\langle \varphi + t\psi; v_i\rangle &= \langle \varphi; v_i\rangle + t\langle \psi; v_i\rangle \\ &= c_i + t\langle \psi; v_i\rangle \\ &\geq c_i.\end{aligned}$$

It follows that $t\langle \psi; v_i\rangle \geq 0$ and since $t \geq 0$ this implies

$$\langle \psi; v_i\rangle \geq 0 \quad \text{for all } i \in I_\varphi$$

so that $C_\varphi(P) \subseteq \bigcap_{i \in I_\varphi} H_{v_i}$.

Reversely, take $\psi \in \bigcap_{i \in I_\varphi} H_{v_i}$ and treat separately the cases $i \in I_\varphi$ and $i \in I \setminus I_\varphi$. If $i \in I_\varphi$, then as before

$$\begin{aligned}\langle \varphi + t\psi; v_i\rangle &= \langle \varphi; v_i\rangle + t\langle \psi; v_i\rangle \\ &= c_i + t\langle \psi; v_i\rangle \\ &\geq c_i\end{aligned}$$

since $\langle \psi, v_i \rangle \geq 0$ and $t \geq 0$. If $i \in I \setminus I_\varphi$, then we have $c_i - \langle \varphi, v_i \rangle > 0$ and can hence choose an $\epsilon > 0$ sufficiently small such that $\epsilon |\langle \psi, v_i \rangle| < c_i - \langle \varphi, v_i \rangle$. For this choice we get

$$\begin{aligned} \langle \varphi + t\psi, v_i \rangle &= \langle \varphi, v_i \rangle + t \langle \psi, v_i \rangle \\ &< \langle \varphi, v_i \rangle + c_i - \langle \varphi, v_i \rangle \\ &= c_i : \end{aligned}$$

so that we can conclude that $\varphi + t\psi \in P$ for $t \in [0, \epsilon]$. Thus $\psi \in C_\varphi(P)$ and $\bigcap_{i \in I_\varphi} H_{v_i} \subseteq C_\varphi(P)$.

2. Suppose first that $v \in \mathrm{co}(\{v_i\}_{i \in I_\varphi})$, that is, there exist positive real coefficients λ_i such that $v = \sum_{i \in I_\varphi} \lambda_i v_i$. Then for $\psi \in C_\varphi(P)$ we have

$$\langle \psi, v \rangle = \sum_{i \in I_\varphi} \lambda_i \langle \psi, v_i \rangle \geq 0$$

since $\langle \psi, v_i \rangle \geq 0$ as $\psi \in C_\varphi(P) = \bigcap_{i \in I_\varphi} H_{v_i}$. Hence $v \in S_\varphi(P)$ and since $v \in \mathrm{co}(\{v_i\}_{i \in I_\varphi})$ was arbitrary, $\mathrm{co}(\{v_i\}_{i \in I_\varphi}) \subseteq S_\varphi(P)$.

Conversely, suppose $v \notin \mathrm{co}(\{v_i\}_{i \in I_\varphi})$. Since $\mathrm{co}(\{v_i\}_{i \in I_\varphi})$ is a polyhedral cone in V (not in V^* as usual), there exist $\{\psi_j\}_{j \in J}$ such that

$$\mathrm{co}(\{v_i\}_{i \in I_\varphi}) = \bigcap_{j \in J} H_{\psi_j} :$$

We then observe that for all $i \in I_\varphi$ we have $v_i \in \mathrm{co}(\{v_i\}_{i \in I_\varphi})$, so that

$$\langle \psi_j, v_i \rangle \geq 0 \quad \text{for all} \quad j \in J$$

so that $\psi_j \in C_\varphi(P)$ for all $j \in J$. On the other hand, since v is supposed to lie outside of $\mathrm{co}(\{v_i\}_{i \in I_\varphi})$, there must exist a $j \in J$ such that $\langle \psi_j, v \rangle > 0$. As $\psi_j \in C_\varphi(P)$, this implies $v \notin S_\varphi(P)$ and concludes the proof.

Example. Consider the polyhedron P shown in figure 1.6. For the vertex ψ, one sees that $I_\psi = \{1, 2\}$ and so

$$C_\psi(P) = H_{v_1} \cap H_{v_2}$$

as is shown in the figure. For φ on the other hand, we have $I_\varphi = \{2, 3\}$ and so

$$C_\varphi(P) = H_{v_2} \cap H_{v_3} :$$

Figure 1.6: The local cones at two vertices φ and ψ of a polyhedron P.

The name 'support cone' for $S_\varphi(P)$ is no accident: For a given point $\varphi \in P$ in a polyhedron P and $v \in V\setminus\{0\}$, the hyperplane $\partial H_{(v;c)}$ with $c := \langle \varphi; v\rangle$ is a support hyperplane if and only if v is in the support cone. Another way of formulating this may be that there is a 1-1 correspondence between rays in $S_\varphi(P)$ and support hyperplanes at φ. That is because two elements $v; w \in S_\varphi(P)$ define the same support hyperplane if and only if there exists a $\lambda \in \mathbb{R}\setminus\{0\}$ such that $v = \lambda w$.

Lemma 1.3.4. Let $P = \bigcap_{i \in I} H_{(v_i;c_i)}$ be a polyhedron and $\varphi \in \partial P$ be a boundary point. Let $v \in V\setminus\{0\}$ with $c := \langle \varphi; v\rangle$ define an affine hyperplane $\partial H_{(v;c)}$ containing φ. Then $\partial H_{(v;c)}$ is a support hyperplane at φ if and only if $v \in S_\varphi(P)$.

Proof. Take first $v \in S_\varphi(P)$, that is

$$v = \sum_{i \in I_\varphi} \lambda_i v_i \quad \text{with} \quad \lambda_i \in \mathbb{R}_{\geq 0}.$$

Then we note that $c = \sum_{i \in I_\varphi} \lambda_i c_i$ since

$$\begin{aligned} c &= \langle \varphi; v\rangle \\ &= \sum_{i \in I_\varphi} \lambda_i \langle \varphi; v_i\rangle \\ &= \sum_{i \in I_\varphi} \lambda_i c_i. \end{aligned}$$

To check that $P \subseteq H_{(v;c)}$ is almost the same computation: Take $\psi \in P$, then

$$\langle \psi, v \rangle = \sum_{i \in I_\varphi} \lambda_i \langle \psi, v_i \rangle \leq \sum_{i \in I_\varphi} \lambda_i c_i = c$$

where we used that the λ_i are all non-negative.

Reversely, suppose that $\partial H_{(v;c)}$ is a support hyperplane at φ. We show that $v \in S_\varphi(P) = \mathrm{co}(\{v_i\}_{i \in I_\varphi})$. This is a polyhedral cone and we can write

$$S_\varphi(P) = \bigcap_{j \in J} H_{\mu_j}.$$

For all $i \in I_\varphi$ we have $v_i \in \mathrm{co}(\{v_i\}_{i \in I_\varphi})$, so

$$\langle \mu_j, v_i \rangle \leq 0 \quad \text{for all} \quad j \in J.$$

If we had $v \notin S_\varphi(P)$, then there would be a $j \in J$ such that

$$\langle \mu_j, v \rangle > 0.$$

But we can take $\varepsilon > 0$ sufficiently small so that $\varphi + \varepsilon \mu_j \in P$ (since

$$\langle \varphi + \varepsilon \mu_j, v_i \rangle = \langle \varphi, v_i \rangle + \varepsilon \langle \mu_j, v_i \rangle \leq \langle \varphi, v_i \rangle = c_i \quad \text{for all} \quad i \in I_\varphi$$

while for all $i \notin I_\varphi$ we use the same argument as in the second part of the proof of Lemma 1.3.1). But at the same time we have

$$\langle \varphi + \varepsilon \mu_j, v \rangle = \langle \varphi, v \rangle + \varepsilon \langle \mu_j, v \rangle = c + \varepsilon \langle \mu_j, v \rangle > c$$

which contradicts $P \subseteq H_{(v;c)}$.

Faces of Polyhedra

The final aim of this chapter is to give explicit expressions for the faces of a polyhedron similarly to the ones of the tangent space and the local cone.

Lemma 1.3.5. Let $P = \bigcap_{i \in I} H_{(v_i;c_i)}$ be a polyhedron with non-empty interior.

1. Let $\varphi \in \partial P$ be a boundary point. Then

   $$F_{(v_\varphi;c_\varphi)}(P) = \{\psi \in P \mid \langle \psi, v_\varphi \rangle = c_\varphi\} \quad \text{with} \quad v_\varphi = \sum_{i \in I_\varphi} v_i \quad \text{and} \quad c_\varphi = \sum_{i \in I_\varphi} c_i$$

   is a face of P containing φ in its relative interior. In particular, every boundary point lies in the interior of some face.

2. Every proper face F of P can be written as $F = F_{(v_\varphi, c_\varphi)}(P)$ for any $\varphi \in \operatorname{int}(F)$.

Proof. ([4], Theorem 4.15.)

1. By Lemma 1.3.4, $\partial H_{(v_\varphi, c_\varphi)}$ is a support hyperplane to φ. Hence

$$F_{(v_\varphi, c_\varphi)}(P) = P \cap \partial H_{(v_\varphi, c_\varphi)}$$

is a face of P containing φ. The smallest affine subspace containing $F_{(v_\varphi, c_\varphi)}$ is a translate of the tangent space: $\bigcap_{i \in I_\varphi} \partial H_{(v_i, c_i)}$. It is clear that φ itself is contained in this subspace and that every small neighbourhood of φ within this subspace is contained in P and hence also in $F_{(v_\varphi, c_\varphi)}$.

2. F is a face of P and hence there exist $v \in V \setminus \{0\}$ and $c \in \mathbb{R}$ such that $F = F_{(v,c)}(P)$. Since $\partial H_{(v,c)}$ is by assumption a support hyperplane at φ, by Lemma 1.3.4 we can write

$$v = \sum_{i \in I_\varphi} \lambda_i v_i \quad \text{with} \quad \lambda_i \in \mathbb{R}_{\geq 0}.$$

Feeding this to φ we get that $c = \sum_{i \in I_\varphi} \lambda_i c_i$. On the other hand, F is by assumption a non-empty polyhedron and hence its relative interior is non-empty by lemma 1.2.1. For any point $\varphi \in \operatorname{int}(F)$ we then have that $F_{(v_\varphi, c_\varphi)}(P)$ is a face of P containing φ in its interior.

Suppose now that $x \in F_{(v_\varphi, c_\varphi)}(P)$, which implies that $\langle x, v_i \rangle = c_i$ for all $i \in I_\varphi$. Then we see that

$$\begin{aligned} \langle x, v \rangle &= \sum_{i \in I_\varphi} \lambda_i \langle x, v_i \rangle \\ &= \sum_{i \in I_\varphi} \lambda_i c_i \\ &= c \end{aligned}$$

and hence $x \in F$, giving that $F_{(v_\varphi, c_\varphi)}(P) \subseteq F$.

If there existed an $x \in F \setminus F_{(v_\varphi, c_\varphi)}$, then we could find an $i \in I_\varphi$ such that $\langle x, v_i \rangle < c_i$. But this would imply that

$$\tilde{F} = \{ x \in F \mid \langle x, v_i \rangle = c_i \}$$

is a proper face of F. But then $\varphi \in \tilde{F}$ would imply $\varphi \notin \operatorname{int}(F)$ which is a contradiction.

Take now F any non-empty face of a polyhedron P. By the first point of the Lemma 1.3.5, F has an interior point $\varphi \in \operatorname{int}(F)$. By the second point of Lemma 1.3.5, we have $F = F_{(v_\varphi, c_\varphi)}(P)$. However, it is clear that this does not depend upon the interior point we choose. In particular, $I_F := I_\varphi$ is well-defined by the choice of the face. Finally, because

$$\langle \varphi, v_\varphi \rangle = \sum_{i \in I_\varphi} \langle \varphi, v_i \rangle = \sum_{i \in I_\varphi} c_i$$

while $\langle \varphi, v_i \rangle \leq c_i$ for all $i \in I_\varphi$, the equality for the sum can only hold if equality holds for each summand. In conclusion we have arrived at the following explicit expression for an arbitrary proper face:

Corollary 1.3.6. Let $P = \bigcap_{i \in I} H_{(v_i, c_i)}$ be a polyhedron. Every proper face $F \subset P$ is of the form

$$\begin{aligned} F &= \{\varphi \in V \mid \langle \varphi, v_i \rangle = c_i,\ i \in I_F \text{ and } \langle \varphi, v_i \rangle \leq c_i,\ i \in I\} \\ &= \bigcap_{i \in I \setminus I_F} H_{(v_i, c_i)} \cap \bigcap_{i \in I_F} \partial H_{(v_i, c_i)} \end{aligned}$$

where $I_F \subseteq I$ is given as $I_F = I_\varphi$ for any interior point $\varphi \in \operatorname{int}(F)$.

Notation. Together with the Lemmas 1.3.1 and 1.3.3 this shows that the constructions of the tangent space and thereby also the annihilator space as well as the local cone and by consequence also the support cone are well-defined for faces. Instead of indicating individual points, we may henceforth write

$$L_F(P);\ W_F(P);\ C_F(P) \quad \text{or} \quad S_F(P)$$

to designate these objects.

Examples. Let $P = \bigcap_{i \in I} H_{(v_i, c_i)}$ be a polyhedron.

- For the polyhedron P itself (which we recall is considered as a face of full dimension), the tangent space $L_P(P)$ and the local cone $C_P(P)$ are all of V. It follows that the annihilator space $W_P(P)$ and the support cone $S_P(P)$ are trivial.

- For any facet F the tangent space $L_F(P)$ has codimension 1 and hence the annihilator space $W_F(P)$ has dimension 1 and is spanned by a single vector v_i, where $i \in I$. Note however that for general polyhedra other vectors v_j such that $j \in I$, $j \neq i$ might be collinear to v_i and hence also be contained in the annihilator space $W_F(P) = \operatorname{Span}_{\mathbb{R}}(v_i)$. Moreover, for a facet the support cone is a ray and hence the support hyperplane defining the face is unique.

- For any edge e on the other hand, the tangent space $L_e(P)$ is one-dimensional and spanned by a single element. The annihilator space $W_e(P)$ has therefore codimension 1 and in general there might be many ways to find a basis for $W_e(P)$. We will see later that this does not happen in the special case of simple or even unimodular polytopes.

- Finally, for any vertex $'$ of P, the tangent space $L_{'}(P)$ is trivial and hence the annihilator space $W_{'}(P)$ agrees with V. This is just the result seen in the first point of Lemma 1.3.2. However, it is crucial to note that the local cone $C_{'}(P)$ is not trivial. The local cones at vertices of polytopes are actually what we will use to characterise simple and unimodular polytopes.

Chapter 2

Rational Polyhedra

Later we will specialise from the general vector space V to the Lie algebra of a torus. This Lie algebra comes with an additive subgroup called the integer lattice, which will be introduced in detail in Part II, Chapter 6. For the moment, this provides the motivation for studying the interaction of a polyhedron with a lattice in the ambient vector space. We start by establishing some elementary notions for lattices.

2.1 Lattices

There are several definitions of a lattice that can be found in the literature. The following is the one that we deem most fit to the topic of this work.

Definition. A subset $\Lambda \subseteq V$ is called a *lattice* if

1. Λ is an additive subgroup of V i.e. for all $v, w \in \Lambda$, we have $v - w \in \Lambda$,
2. Λ is discrete i.e for every bounded set $B \subseteq V$, the intersection $B \cap \Lambda$ is finite and
3. Λ spans V.

Remark. 1. The second condition can also be formulated as the condition that there is a neighbourhood of the origin that does not contain any point of Λ apart from the origin.

2. Let $W \subseteq V$ be a subspace of V. Then $W \cap \Lambda$ need not be a lattice in W. While it is clearly an additive subgroup and discrete, it may fail the third condition, namely that $\Lambda \cap W$ spans W. However, if W is spanned by lattice points, also this condition holds and $\Lambda_W = \Lambda \cap W$ is a lattice in W.

Since lattices are additive subgroups, for any lattice point $u \in \Lambda$, all integer multiples $ku \in \Lambda$ are again lattice points. Those lattice points which are not the multiple of a lattice point closer to the origin are called primitive:

Figure 2.1: Example of a primitive vector u 2 Z^2 and a lattice vector v 2 Z^2 whcih is not primitive in the lattice Z^2 R^2.

De nition. A lattice vector v 2 is called primitive if there is no integer k 2 Z and no other lattice vector u 2 such that jkj > 1 and v = ku.

Example. Consider the standard lattice Z^2 R^2 and the two lattice vectors shown in gure 2.1.

Primitive vectors are what can be used to build a basis of a lattice. However, the requirement for a set of vectors is stronger that being a basis of the vector space and primitive. Instead we call a set of vectors a basis of the lattice if any lattice point can uniquely be written as integer linear combination in terms of these vectors.

De nition. Let 2 V be a lattice. A set of linearly independent vectors $u_1; :::; u_d$ 2 is called a basis of if every v 2 can be written as an integer linear combination of the $u_1; :::; u_d$:

$$v = \sum_{i=1}^{d} n_i u_i \quad \text{with} \quad n_i \text{ 2 } Z:$$

It is clear from the third condition in the de nition of a lattice above that d = dim(V).

Clearly, any basis of a lattice is a basis of the underlying vector space and consists of primitive vectors but this is only a necessary condition, not a su cient one. We emphasise this here because the distinction is crucial for the unimodular polytopes, the objects that we are aiming to de ne in the next chapter.

Example. For instance the basis of R^2 given by v_1 = (1; 0) and v_2 = (1; 2) consists of two vectors which are primitive in Z^2 but the element u = (1; 1) 2 Z^2

can not be written as an integer linear combination of v_1 and v_2. Instead,

$$u = \frac{1}{2}v_1 + \frac{1}{2}v_2$$

and since v_1 and v_2 form a basis of $\mathbb{R}^2$, the coefficients are unique.

We now aim at proving the existence of such a basis for any lattice. For this, we need some notion of distance between elements of the vector space V. However, the precise definition does not matter and so we assume that V is equipped with an arbitrary inner product $\cdot : V \times V \to \mathbb{R}$. Then there is an induced norm $\|\cdot\| : V \to \mathbb{R}$ and an induced distance $\mathrm{dist} : V \times V \to \mathbb{R}$:

$$\|v\| = \sqrt{v \cdot v} \quad \text{and} \quad \mathrm{dist}(v, w) = \|v - w\|.$$

For any subset $A \subseteq V$, we set

$$\mathrm{dist}(v, A) = \inf_{w \in A} \mathrm{dist}(v, w).$$

The key to showing the existence of a basis is then the following technical lemma.

Lemma 2.1.1. Let $\Lambda \subseteq V$ be a lattice and let $W \subsetneq V$ be a subspace spanned by lattice points, i.e. there exist $v_1, \ldots, v_k \in \Lambda$ such that $W = \mathrm{Span}_{\mathbb{R}}(v_1, \ldots, v_k)$. Then among all the lattice points not contained in W, there is a point which is closest to W. That is, there exists

$$v \in \Lambda \setminus W \quad \text{such that} \quad \mathrm{dist}(v, W) \leq \mathrm{dist}(w, W) \ \forall w \in \Lambda \setminus W$$

Notation. For $\alpha \in \mathbb{R}$, we write $\lfloor \alpha \rfloor$ for the largest integer smaller or equal than α. We write $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ for the fractional part, so that

$$\alpha = \lfloor \alpha \rfloor + \{\alpha\} \quad \text{with} \quad \lfloor \alpha \rfloor \in \mathbb{Z} \quad \text{and} \quad 0 \leq \{\alpha\} < 1.$$

Proof. ([4], Lemma 10.2) Without loss of generality we may assume $v_1, \ldots, v_k$ to be linearly independent so that they form a basis of W. Let then

$$\Pi = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; 0 \leq \lambda_i \leq 1 \quad \text{for} \quad i = 1, \ldots, k \right\} \subseteq W$$

denote the parallelepiped spanned by $v_1, \ldots, v_k$.

Claim. There is a $v \in \Lambda \setminus W$ which is closest to Π.

Proof of Claim. Choose a $\varepsilon > 0$ sufficiently large so that the ε-neighbourhood

$$\Pi_\varepsilon = \{ u \in V \mid \mathrm{dist}(u, \Pi) \leq \varepsilon \}$$

of Π contains a lattice point outside of W. As Λ is discrete, $\Lambda \cap \Pi_\varepsilon$ is finite and we can choose a point v of minimal distance in the finite non-empty set $(\Lambda \setminus W) \cap \Pi_\varepsilon$. This will then clearly satisfy the conditions

$$\mathrm{dist}(v, \Pi) \leq \mathrm{dist}(w, \Pi) \quad \text{for all} \quad w \in \Lambda \setminus W.$$

Claim. For v as in the previous claim we have

$$\operatorname{dist}(v;W) \leq \operatorname{dist}(w;W) \quad \text{for all} \quad w \in \Lambda \setminus W:$$

Proof of Claim. Suppose by contradiction that there was some $u \in \Lambda \setminus W$ such that $\operatorname{dist}(u;W) < \operatorname{dist}(v;W)$. Take then a point $x \in W$ which realises this minimal distance $\operatorname{dist}(u;x) = \operatorname{dist}(u;W)$ and write it as

$$x = \sum_{i=1}^{k} \lambda_i v_i = \underbrace{\sum_{i=1}^{k} \lfloor \lambda_i \rfloor v_i}_{:= y \in \Lambda \cap W} + \underbrace{\sum_{i=1}^{k} \{\lambda_i\} v_i}_{:= z \in \Pi} :$$

But this leads to a contradiction with the first claim since $u - y \in \Lambda \setminus W$ and

$$\begin{aligned} \operatorname{dist}(u - y; \Pi) &\leq \operatorname{dist}(u - y; z) \\ &= \operatorname{dist}(u - y; x - y) \\ &= \operatorname{dist}(u; x) \\ &= \operatorname{dist}(u; W) \\ &< \operatorname{dist}(v; W) \\ &\leq \operatorname{dist}(v; \Pi) \end{aligned}$$

and hence proves the claim and thereby the lemma.

Theorem 2.1.2. Any lattice admits a basis. More precisely, let $\Lambda \subset V$ be a lattice and $W \subseteq V$ be a subspace spanned by lattice points. Then any basis of the lattice $\Lambda_W = \Lambda \cap W$ can be appended to a basis of Λ.

Proof. ([4], Theorem 10.4.) We proceed by induction on $d = \dim(V)$.

For $d = 1$, we can identify $V = \mathbb{R}$ and there is a smallest positive number $a \in \Lambda$. If there was an element $b \in \Lambda$ such that $b \neq ma$ for all $m \in \mathbb{Z}$, then there existed an $n \in \mathbb{Z}$ such that $b - na \in \mathbb{R}$ were smaller than a.

For $d > 1$, choose $d - 1$ linearly independent lattice points and set W to be their span. Hence $\Lambda_W = \Lambda \cap W$ is a lattice of dimension $d - 1$ in W and by the induction hypothesis there is a basis $u_1; \dots; u_{d-1}$ of Λ_W. On the other hand, by Lemma 2.1.1 there is an element $u_d \in \Lambda \setminus W$ closest to W i.e. such that

$$\operatorname{dist}(u_d;W) \leq \operatorname{dist}(u;W) \quad \text{for all} \quad u \in \Lambda \setminus W:$$

Claim. $u_1; \dots; u_d$ is a basis of Λ.

Proof of Claim. It is clear that $u_1; \dots; u_d$ is a basis of V, so we are just left to check that the coefficients of lattice points are integers. Take thus $u \in \Lambda$. Then there exists a unique decomposition

$$u = \sum_{i=1}^{d} \lambda_i u_i \quad \text{for some} \quad \lambda_i \in \mathbb{R}:$$

Suppose by contradiction that $_d$ 2= Z, i.e. f $_d$g 6= 0. Then

$$v = u \quad b \quad {}_d c u_d = f \quad {}_d g u_d + \sum_{i=1}^{d-1} {}_i u_i$$

is a lattice point outside W. But this point v 2 nW is closer to W than u_d as

$$\begin{aligned} \mathrm{dist}(v; W) &= \mathrm{dist}(f \quad {}_d g u_d; W) \\ &= f \quad {}_d g \,\mathrm{dist}(u_d; W) \\ &< \mathrm{dist}(u_d; W) \end{aligned}$$

contradicting minimality of u_d. Hence $_d$ 2 Z. But then

$$w \qquad {}_d u_d = \sum_{i=1}^{d-1} {}_i u_i$$

is in $_W$ and it follows that the $_1; \ldots; \ _{d-1}$ are integers.

Another consequence of Lemma 2.1.1 is the following result:

Corollary 2.1.3. Let V be a lattice and let W V be a subspace spanned by lattice points. Consider the canonical projection $_W$: V ! V=W with kernel W. Then $_{V=W}$:= $_W$( ) is a lattice in V=W.

Proof. (Variation of [4], Corollary 10.3) By linearity $_W$( ) is an additive subgroup. Its elements span V=W as a vector space since $_W$ is surjective. By Lemma 2.1.1 there is a point in nW at minimal distance to W. It follows that there is a non-zero point in $_W$( ) at minimal distance of the origin in V=W. Hence $_W$( ) is discrete and therefore a lattice.

So if W is a subspace which is spanned by lattice points, we get a lattice $_W$ = \ W in W and a lattice $_{V=W}$ = $_W$( ) in the quotient V=W. These lattices will play a major role in Chapter 4.

The Dual Lattice

Any lattice in V de nes a lattice in the dual V by considering all the functionals which take integer values on the lattice. Because we will later work with tori, it is useful to introduce a factor 2 from the start and to work with the following de nition:

De nition. Let V be a lattice. The dual lattice V of is

$$:= \mathrm{Hom}(\ ; 2 \ Z) = f\,' \ 2 \ V \ \ j \ ' (v) \ 2 \ 2 \ Z \quad \text{for all} \quad v \ 2 \quad g:$$

Let now $v_1, \ldots, v_n \in \Lambda$ be a basis of the lattice $\Lambda \subseteq V$. In particular, $v_1, \ldots, v_n$ is thus a basis of V and we get a dual basis of V^*. However, the elements of this dual basis are not contained in Λ^* unless we rescale. Let us thus define the dual lattice basis $\varphi_1, \ldots, \varphi_n$ by

$$\langle \varphi_i, v_j \rangle = 2\delta_{ij}.$$

These elements are now by construction in Λ^* and form a basis of V^*. To justify calling it the dual lattice basis, we show that they form indeed a basis of Λ^*. Take an arbitrary $\lambda \in \Lambda^* \subseteq V^*$ and write it in terms of the V^*-basis $\varphi_1, \ldots, \varphi_n$

$$\lambda = \sum_{i=1}^{n} \lambda_i \varphi_i.$$

Take then $j \in \{1, \ldots, n\}$ also arbitrary and note that

$$\langle \lambda, v_j \rangle = \sum_{i=1}^{n} \lambda_i \langle \varphi_i, v_j \rangle = 2\lambda_j.$$

Since $\lambda \in \Lambda^*$ and $v_j \in \Lambda$, it follows that

$$2\lambda_j \in 2\mathbb{Z} \quad \text{that is,} \quad \lambda_j \in \mathbb{Z}.$$

We conclude that $\{\varphi_1, \ldots, \varphi_n\}$ is indeed a basis of the dual lattice.

The operation of taking the dual of a lattice is inclusion reversing. Moreover, doing it twice yields a lattice in the bidual which can be canonically identified with the lattice itself. More formally, we have the following lemma:

Lemma 2.1.4. Let Λ, Λ' be lattices in V. Then

1. $\Lambda \subseteq \Lambda' \Rightarrow \Lambda'^* \subseteq \Lambda^*$,
2. $(\Lambda^*)^* = \Lambda$ and
3. $\Lambda = \Lambda'$ if and only if $\Lambda^* = \Lambda'^*$.

Proof. 1. Assume that $\varphi \in \Lambda'^*$. Then for any $w \in \Lambda \subseteq \Lambda'$ we must have

$$\varphi(w) \in 2\mathbb{Z}$$

so that $\varphi \in \Lambda^*$. This proves the first point.

2. We use the usual identification of V with $(V^*)^*$:

$$\begin{aligned}(\Lambda^*)^* &= \{f \in (V^*)^* \mid f(\varphi) \in 2\mathbb{Z} \text{ for all } \varphi \in \Lambda^*\} \\ &= \{v \in V \mid \varphi(v) \in 2\mathbb{Z} \text{ for all } \varphi \in \Lambda^*\}.\end{aligned}$$

Assume rst that v 2 . By de nition of , we have

$$' (v) 2 2 Z \quad \text{for all} \quad ' 2 \;;$$

hencev 2 ( ) so that ( ) .

Assume then reversely thatw 2 ( ) V and write it in terms of a basis $u_1; :::; u_d$ 2 of V:

$$w = \sum_{i=1}^{d} {}_i u_i \quad \text{with} \quad u_i \; 2 \; R:$$

Consider then the dual lattice basis $_1; :::; \; _d$ 2 of $u_1; :::; u_d$ and compute

$$_i(w) = \; _i \; _i(u_i) = 2 \; _i:$$

Since $_i$ 2 , this must be in 2 Z and hence $_i$ must be an integer. Thus w 2 which gives the inverse inclusion( ) .

3. Follows directly from the rst two statements.

2.2 Rational Polyhedra

Let now P V be a polyhedron. If V contains a lattice , we can make sense of rational normal vectors and use this to de ne rational polyhedra.

De nition. Let V be a lattice.

1. A vector v 2 V is called -rational or just rational if mv 2 for some positive integer m.
2. A subspaceW V is called -rational if \ W is a lattice in W, that is, if \ W spansW.

Remark. 1. To any rational vector v 2 V we can associate a unique primitive lattice vector u 2 which is parallel to v.

To see this, note that $Span_R(v)$ has by assumption non-trivial intersection with . \ $Span_R(v)$ is a lattice in $Span_R(v)$ and there is a unique smallest element 2 $R_{>0}$ such that u = v 2 .

2. A subspace is rational if and only if it is spanned by rational vectors.

De nition. Let V be a lattice. A polyhedron P V is called -rational if it is of the form

$$P = \bigcap_{i 2 I} H_{(v_i ;c_i)}$$

with all v_i -rational.

Chapter 3

Polytopes

In this short chapter, we will introduce the main object of study in part I: unimodular polytopes. For this we first prove the Weyl-Minkowski theorem that allows us to see polytopes as a special case of polyhedra. This makes it possible to use the formalism for polyhedra established over the previous two chapters. This is important since we will define unimodularity by a condition on the local cones at the vertices.

3.1 Weyl-Minkowski Theorem

We start with the definition of a convex polytope.

Definition. A point $v \in V$ is called a *convex combination* of $v_1, \ldots, v_m \in V$ if

$$v = \sum_{i=1}^{m} \lambda_i v_i \quad \text{with} \quad \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i = 1.$$

The set of all convex combinations of points from a given set $A \subseteq V$ is called the *convex hull* of A and is denoted by $\mathrm{conv}(A)$.

Definition. A *convex polytope* is the convex hull of a finite set of points.

The result known as the Weyl-Minkowski Theorem states that polytopes can be equivalently be characterised as bounded polyhedra. We are primarily interested in seeing polytopes as bounded polyhedra but for the sake of completeness, we also prove the reverse direction. The precise statement is the following:

Theorem 3.1.1 (Weyl-Minkowski)**.**
1. Let P be a polytope. Then P is a bounded polyhedron.
2. Let P be a bounded polyhedron. Then P is the convex hull of the set of its vertices. In particular, P is a polytope.

Proof. 1. ([4], Theorem 4.4.) Let $\Delta = \operatorname{conv}(v_1;\dots;v_m)$ be the convex hull of m points $v_1;\dots;v_m \in V$. To show that this is a polyhedron we exhibit it as the image of a polyhedron under a linear map. The standard simplex

$$\Delta_{\text{simplex}} = \left\{ (\lambda_1;\dots;\lambda_m) : \lambda_i \geq 0 \quad \text{for} \quad i = 1;\dots;m \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

is a bounded polyhedron in $\mathbb{R}^n$ and under the linear map

$$T : \mathbb{R}^m \to V$$
$$(\lambda_1;\dots;\lambda_m) \mapsto \sum_{i=1}^{m} \lambda_i v_i$$

its image is precisely $\operatorname{conv}(v_1;\dots;v_m) = \Delta$. That Δ is bounded is clear since Δ_{simplex} is compact and T is continuous.

2. ([4], Theorem 4.7.) Consider a non-empty bounded polyhedron $P \subseteq V$ and proceed by induction on $n = \dim(V)$. If $n = \dim(V) = 0$ the result is clear.

Suppose then $n = \dim(V) > 0$ and the result true in all smaller dimensions. Write

$$P = \bigcap_{i \in I} H_{(v_i;c_i)}$$

and consider Vert(P), the set of vertices of P. By convexity of P it is clear that

$$\operatorname{conv}(\operatorname{Vert}(P)) \subseteq P.$$

We are left to prove the opposite inclusion. For this we take an arbitrary point $\varphi \in P$ and distinguish two cases:

- $\varphi \in \partial P$ is a boundary point: By part one of Lemma 1.3.5 φ lies in the interior of some face F. In particular, it lies in a proper affine subspace of V and thus by the induction hypothesis it is a convex combination of the vertices of F. But by the second part of Lemma 1.3.2 the vertices of F are also vertices of P. Hence φ is a convex combination of vertices of P.
- $\varphi \in \operatorname{int}(P)$ is an interior point: Consider a line through φ. Since P is bounded, by Lemma 1.2.3 the rays emanating from φ in both directions have to intersect the boundary of P. We denote these points by φ_1 and φ_2 and note that by the previous case, $\varphi_1; \varphi_2 \in \operatorname{conv}(\operatorname{Vert}(P))$. By convexity, we conclude that $\varphi \in \operatorname{conv}(\operatorname{Vert}(P))$.

In the last part of the proof we have seen a variant of the following fact:

Corollary 3.1.2. *Any edge of a polytope contains exactly two vertices. In particular, any edge of a polytope can be written as the interval with endpoints the two vertices it contains.*

3.2 Simple Polytopes

For a vertex φ of a polytope Δ, $\{v_i\}_{i\in I_\varphi}$ is a spanning set of V^*. In particular this means that $|I_\varphi| \geq \dim(V)$. We will now specialise to the case where we have equality and where $\{v_i\}_{i\in I_\varphi}$ is thus a basis of V^*. If this is the case for any vertex, we call a polytope *simple*. In order to find a good formalisation, it is best to start from the local picture which is given by polyhedral cones.

Definition. Let C be a polyhedral cone. C is called *simple* if it is of the form

$$C = \bigcap_{i\in I} H_{v_i}$$

such that $\{v_i\}_{i\in I}$ is a basis of V^*.

Remark. Note that since the $\{v_i\}_{i\in I}$ span V^*, it follows immediately that $0\in C$ is a vertex and hence that simple cones are pointed. In fact, a polyhedral cone C is simple if and only if it is pointed and has $\dim(C)$ facets.

There is a natural reformulation of the simplicity condition in terms of the edges instead of the normals. The key to this is the following observation:

Lemma 3.2.1. *Let $\{v_i\}_{i\in I}$ be a basis of V^* and let $\{\varphi_i\}_{i\in I}$ be the dual basis of V. Then for any subset $J\subseteq I$ we have*

$$\operatorname{co}(\{\varphi_j\}_{j\in J}) = \bigcap_{j\in J} H_{v_j} \cap U$$

where $U = \operatorname{Span}_{\mathbb{R}}(\{\varphi_j\}_{j\in J}) = \bigcap_{i\in I\setminus J} \partial H_{v_i}$.

Proof. Assume $\lambda \in \bigcap_{j\in J} H_{v_j}\cap U$ and write it as

$$\lambda = \sum_{j\in J} \lambda_j \varphi_j$$

for some (unique) $\lambda_j \in \mathbb{R}$. Since $\lambda \in \bigcap_{j\in J} H_{v_j}\cap U$, we have for all $i\in J$

$$\begin{aligned} 0 &\leq \langle \lambda ; v_i\rangle \\ &= \sum_{j\in J} \lambda_j \langle \varphi_j ; v_i \rangle \\ &= \lambda_i \end{aligned}$$

and hence $\lambda \in \operatorname{co}(\{\varphi_j\}_{j\in J})$.

Reversely, assume $\lambda\in\operatorname{co}(\{\varphi_j\}_{j\in J})$ i.e.

$$\lambda = \sum_{j\in J}\lambda_j\varphi_j \quad \text{with} \quad \lambda_j \geq 0.$$

But then we have for all $i \in J$

$$\begin{aligned}\langle \lambda, v_i \rangle &= \sum_{j \in J} \lambda_j \langle \varphi_j, v_i \rangle \\ &= \lambda_i \\ &\geq 0\end{aligned}$$

and since clearly $\langle \lambda, v_i \rangle = 0$ for $i \in I \setminus J$ we have $\lambda \in \bigcap_{j \in J} H_{v_j} \cap U$.

In the case where $J = I$, this immediately gives the following result:

Corollary 3.2.2. *A polyhedral cone C is simple if and only if it is of the form*

$$C = \mathrm{co}(\{\varphi_i\}_{i \in I})$$

where the $\{\varphi_i\}_{i \in I}$ form a basis of V.

Remark. A polyhedral cone C is thus simple if and only if it is pointed and has exactly $\dim(C)$ edges.

The definition is then naturally extended to polytopes by requiring the local cone at every vertex of the polytope to be simple.

Definition. A polytope Δ is called *simple* if the local cone

$$C_\varphi(\Delta) = \bigcap_{i \in I_\varphi} H_{v_i}$$

at every vertex $\varphi \in \Delta$ is simple.

Example. Consider the simplex Δ in $\mathbb{R}^3$:

$$\Delta = \{(\lambda_1; \lambda_2; \lambda_3) \in \mathbb{R}^3 : \lambda_1; \lambda_2; \lambda_3 \geq 0 \text{ and } \lambda_1 + \lambda_2 + \lambda_3 \leq 1\}.$$

By the Weyl-Minkowski Theorem 3.1.1, this can be seen as a bounded polyhedron. Indeed,

$$\Delta = \bigcap_{i \in I} H_{(v_i; c_i)}$$

with

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

as well as $c_1 = c_2 = c_3 = 0$ and $c_4 = -1$. See also figure 3.1 for a picture. This polytope is simple because at any vertex there are 3 edges meeting. We check the condition for one vertex explicitely, namely the vertex φ specified by $I_\varphi = \{2; 3; 4\}$. At this point the local cone is

$$C_\varphi(\Delta) = H_{v_2} \cap H_{v_3} \cap H_{v_4}$$

and since $\{v_2; v_3; v_4\}$ is clearly a basis of $\mathbb{R}^3$, this cone is simple.

Figure 3.1: The simplex is an example of a simple polytope.

3.3 Unimodular Polytopes

Assume now that V is a lattice. Instead of requiring the normals to be merely a basis of the vector space, we can now require them to be a basis of the lattice. As we saw in section 2.1 this is a stronger requirement than just being a basis of V consisting of primitive lattice vector.

Definition. Let V be a lattice. A polyhedral cone C V is called -unimodular if it is of the form

$$C = \bigcap_{i \in I} H_{v_i}$$

and $\{v_i\}_{i \in I}$ forms a basis of .

Remark. Note that a cone being -unimodular implies that the cone is -rational. Clearly, unimodularity implies that the cone is also simple. However, a -rational simple cone might not be -unimodular.

Just like for simple cones, the condition of unimodularity can be rephrased in terms of the edges instead of the normals.

Corollary 3.3.1. A pointed polyhedral cone C is -unimodular if and only if it is of the form

$$C = \mathrm{co}(\{\varphi_i\}_{i \in I})$$

where $\{\varphi_i\}_{i \in I}$ form a basis of the dual lattice .

Proof. Consider the dual lattice basis instead of the dual basis and note that since this is just a rescaling with positive factors, Lemma 3.2.1 still goes through.

Definition. A polytope is called -unimodular if the local cone $C_\varphi( )$ at every vertex $\varphi \in$ is -unimodular.

Example. The simplex in figure 3.1 is not only simple, but $\mathbb{Z}^3$-unimodular. Indeed, reusing the notation from above, for instance $\{v_2; v_3; v_4\}$ is a lattice basis of $\mathbb{Z}^3$.

Chapter 4

Recursive Aspects

We have already seen that the faces of polyhedra are again polyhedra. However, by construction any proper face F is contained in a proper affine subspace which can be identified with a translate of the tangent space $L_F(P) \subseteq V$. As we are only interested in polyhedra up to translation, we might as well assume that F is contained in $L_F(P)$ itself.

In this chapter, we want to investigate the properties of the face F as a polytope in this affine subspace. The goal is to show that if the original polyhedron was rational, simple or unimodular, then F, considered as a polytope in $L_F(P)$, has the same property.

4.1 Recursive Aspects of Polyhedra

Let $P \subseteq V$ be a polyhedron and consider a proper face $F \subseteq P$. By Lemma 1.3.5, F contains an interior point φ. By translating P by $-\varphi$ we get a copy P' of the polyhedron P such that the copy F' of the face F contains the origin. This has the advantage that F' can be seen as a subset of its tangent space $L_{F'}(P')$ instead of a translate of it. At the same time, translating does not affect the normal vectors $\{v_i\}$ to the defining hyperplanes but only the real coefficients $\{c_i\}$. In particular, the properties of rationality, simplicity and unimodularity are thus not affected by this translation.

Because of this, we will henceforth assume without loss of generality that the polyhedron $P = \bigcap_{i \in I} H_{(v_i, c_i)}$ is such that the face F under consideration contains the origin. In particular, this means that for all $i \in I_F$, we must have $c_i = 0$ and hence the expression for F from Corollary 1.3.6 simplifies to

$$F = \bigcap_{i \in I \setminus I_F} H_{(v_i, c_i)} \cap \bigcap_{i \in I_F} \partial H_{v_i} = \bigcap_{i \in I \setminus I_F} H_{(v_i, c_i)} \cap L_F(P)$$

emphasising again that F is naturally a subset of its tangent space.

However, this is not yet a satisfactory answer to the question with what space we have to work. The formalism established in the previous chapters relies heavily on the use of normals to hyperplanes and if a polyhedron P is in V, then these normals are elements of V^*. The question remaining is thus to find a vector space U such that its dual U^* is precisely the tangent space $L_F(P)$. The main idea for this is to use that $L_F(P) \subseteq V^*$ is the annihilator subspace of $W_F(P) \subseteq V$ and the following result:

Lemma 4.1.1. *Let $W \subseteq V$ be a linear subspace. There is a canonical identification*

$$W^0 \cong \left(\frac{V}{W}\right)^* .$$

Proof. Consider the short exact sequence

$$0 \longrightarrow W \xhookrightarrow{\;i\;} V \xrightarrow{\;pr\;} \frac{V}{W} \longrightarrow 0$$

where $i : W \to V$ is the inclusion and $pr : V \to V/W$ is the canonical projection. Taking the dual we get a second short exact sequence

$$0 \longrightarrow \left(\frac{V}{W}\right)^* \xhookrightarrow{\;pr^*\;} V^* \xrightarrow{\;i^*\;} W^* \longrightarrow 0$$

It follows that $pr^* : (V/W)^* \to V^*$ is injective and thus $(V/W)^*$ is isomorphic to its image under this map. But by exactness we observe that

$$\begin{aligned} \operatorname{Im}(pr^*) &= \ker(i^*) \\ &= \{\varphi \in V^* \mid i^*(\varphi) = 0\} \\ &= \{\varphi \in V^* \mid \varphi \circ i = 0\} \\ &= \{\varphi \in V^* \mid \varphi|_W = 0\} \\ &= W^0 \end{aligned}$$

which concludes the proof.

Using this and the above mentioned fact we conclude that

$$L_F(P) = (W_F(P))^0 \cong \left(\frac{V}{W_F(P)}\right)^* .$$

This now answers the question raised previously since $U = V/W_F(P)$ is a canonical choice. This motivates the following definition:

Definition. Let $F \subseteq P \subseteq V$ be a face of a polyhedron P and $W_F(P)$ be its annihilator space. Then the quotient space

$$\frac{V}{W_F(P)}$$

is called the *effective space* of F.

Remark. By construction (and as seen in the proof of Lemma 4.1.1 just above), the annihilator space and the effective space fit naturally in a short exact sequence

$$0 \longrightarrow W_F(P) \overset{i}{\hookrightarrow} V \overset{pr}{\twoheadrightarrow} V\Big/W_F(P) \longrightarrow 0$$

where again $i : W_F(P) \to V$ is the inclusion and $pr : V \to V/W_F(P)$ is the canonical projection. Taking the dual we get another short exact sequence

$$0 \longrightarrow \left(V\Big/W_F(P)\right)^* \overset{pr^*}{\hookrightarrow} V^* \overset{i^*}{\twoheadrightarrow} (W_F(P))^* \longrightarrow 0$$

showing that $(V/W_F(P))^*$ can be naturally interpreted as a subspace of V^*. This is of course no surprise since $(V/W_F(P))^*$ is canonically identified with $L_F(P)$.

Before, we argued that the face F is contained in the tangent space. We now canonically identified this with the dual of the effective space and the problem is now to understand F as a polyhedron in $(V/W_F(P))^*$ by its halfspaces defined with normals that are elements of the effective space. The projection $pr : V \to V/W_F(P)$ gives a natural way of producing a set of vectors in $V/W_F(P)$ from the normals $\{v_i\}_{i \in I}$ in V. However, this map has a nontrivial kernel, given by $W_F(P)$, so the case $v_i \in W_F(P)$ must be addressed.

By assumption, F and therefore also P contain the origin. Hence for any $i \in I$ we must have $0 \in H_{(v_i;c_i)}$ from where it follows immediately that

$$0 = \langle 0; v_i \rangle \leq c_i:$$

But if $c_i \geq 0$, then the linear inequality defining the halfspace $H_{(0;c_i)}$ is trivially satisfied and so, by convention, $H_{(0;c_i)}$ equals the entire vector space. In other words, the normals $v_i \in W_F(P)$ will give rise to empty conditions. In particular, $\{v_i\}_{i \in I_F}$ are contained in the kernel $W_F(P) = \mathrm{Span}_{\mathbb{R}}(\{v_i\}_{i \in I_F})$ of the projection and are hence mapped to zero. A natural polyhedron in $V/W_F(P)$ that we might consider is thus

$$\tilde{F} = \bigcap_{i \in I \setminus I_F} H_{(pr(v_i);c_i)} \subseteq V/W_F(P) \; ;$$

where all the indices $i \in I_F$ such that $v_i \in W_F(P)$ give rise to empty conditions. Indeed, the inclusion of $\tilde{F}$ into V^* by pr^* is precisely the face F:

Lemma 4.1.2. Let $F \subseteq P \subseteq V^*$ be a proper face and let $pr : V \to V/W_F(P)$ be the corresponding projection. Then

$$\tilde{F} = \bigcap_{i \in I \setminus I_F} H_{(pr(v_i);c_i)}$$

is the unique polyhedron in $(V/W_F(P))^*$ such that $pr^*(\tilde{F}) = F$.

Proof. Assume first that $\varphi \in \mathrm{pr}(\tilde{F})$, that is, there is a $\tilde{\varphi} \in \tilde{F}$ such that $\varphi = \mathrm{pr}(\tilde{\varphi})$. We can then rewrite for any $i \in I$

$$\langle \varphi, v_i \rangle = \langle \mathrm{pr}(\tilde{\varphi}), v_i \rangle = \langle \tilde{\varphi}, \mathrm{pr}(v_i) \rangle$$

and distinguish two cases:

1. If $i \in I \setminus I_F$, we conclude that

$$\langle \varphi, v_i \rangle = \langle \tilde{\varphi}, \mathrm{pr}(v_i) \rangle \geq c_i$$

   since $\tilde{\varphi} \in \tilde{F}$. Hence $\varphi \in H_{(v_i;c_i)}$ for all $i \in I \setminus I_F$.

2. If $i \in I_F$, then $v_i \in W_F(P)$ is in the kernel of pr and so

$$\langle \varphi, v_i \rangle = \langle \tilde{\varphi}, \mathrm{pr}(v_i) \rangle = 0 .$$

   Thus $\varphi \in \partial H_{v_i}$ for all $i \in I_F$.

Hence

$$\varphi \in F = \bigcap_{i \in I \setminus I_F} H_{(v_i;c_i)} \cap \bigcap_{i \in I_F} \partial H_{v_i}$$

and since $\varphi \in \mathrm{pr}(\tilde{F})$ was arbitrary we conclude that $\mathrm{pr}(\tilde{F}) \subseteq F$.

Reversely, assume that $\varphi \in F$. For any $i \in I_F$, we denote by w_i the unique element of $W_F(P)$ such that $\iota(w_i) = v_i$. But since $\varphi \in F$, we have for any $i \in I_F$

$$\begin{aligned} \langle \iota^*(\varphi), w_i \rangle &= \langle \varphi, \iota(w_i) \rangle \\ &= \langle \varphi, v_i \rangle \\ &= 0 \end{aligned}$$

But by Lemma 1.3.1, $\{w_i\}_{i \in I_F}$ span $W_F(P)$ and so $\iota^*(\varphi) = 0$, that is, $\varphi \in \ker(\iota^*)$. By exactness of the sequence above, $\ker(\iota^*) = \mathrm{Im}(\mathrm{pr})$ and therefore there is a $\tilde{\varphi} \in (V/W_F(P))$ such that $\mathrm{pr}(\tilde{\varphi}) = \varphi$. This $\tilde{\varphi}$ is in $\tilde{F}$ since φ is in F: for every $i \in I \setminus I_F$ we compute

$$\begin{aligned} \langle \tilde{\varphi}, \mathrm{pr}(v_i) \rangle &= \langle \mathrm{pr}(\tilde{\varphi}), v_i \rangle \\ &= \langle \varphi, v_i \rangle \\ &\geq c_i . \end{aligned}$$

We conclude that $\varphi \in \mathrm{pr}(\tilde{F})$ and since $\varphi \in F$ was arbitrary, $F \subseteq \mathrm{pr}(\tilde{F})$.

4.1.1 Recursive Aspects of Rational Polyhedra

Let now V be a lattice. If a polyhedron is -rational, the annihilator spaces $W_F(P) = \text{Span}_{\mathbb{R}}(\{v_i\}_{i \in I_F})$ are rational subspaces. This means that $\cap W_F(P)$ is a lattice in the annihilator space $W_F(P)$. From Corollary 2.1.3 it further follows that $pr()$ is a lattice in the effective space $V/W_F(P)$.

Definition. Let P be a -rational polyhedron, F P a face, $W_F(P)$ its annihilator space and $V/W_F(P)$ its effective space. The *annihilator lattice* of the face F is the lattice in the annihilator space $W_F(P)$ given by

$$^0 = \cap W_F(P):$$

The *effective lattice* $_F$ is the lattice in the effective space $V/W_F(P)$ given by

$$_F = pr() :$$

The following result is now an immediate consequence of Lemma 4.1.2:

Proposition 4.1.3. *Let V be a lattice, P V a -rational polyhedron and F P a proper face. Let $pr : V \to V/W_F(P)$ denote the projection on the effective space. Then the unique polyhedron $\tilde{F}$ $(V/W_F(P))$ such that $pr(\tilde{F}) = F$ is $_F$-rational.*

4.2 Recursive Aspects of Polyhedral Cones

The next goal is to establish similar results to the one for the rationality of faces of polyhedra for simplicity and unimodularity of the faces of a polytope. These properties were defined by requiring the local cone at every vertex to be simple or unimodular, so it is natural to first consider the case of polyhedral cones. We begin by showing that faces of simple polyhedral cones are again simple polyhedral cones:

Proposition 4.2.1. *Let C V be a simple polyhedral cone and F C a proper face. Let $pr : V \to V/W_F(C)$ denote the projection on the effective space. Then the unique polyhedral cone $\tilde{C}$ $(V/W_F(C))$ such that $pr(\tilde{C}) = F$ is simple.*

Proof. Since C is simple, it is of the form

$$C = \bigcap_{i \in I} H_{v_i}$$

with $\{v_i\}_{i \in I}$ a basis of V. It follows from Lemma 4.1.2 that

$$\tilde{C} = \bigcap_{i \in I \setminus I_F} H_{pr(v_i)}$$

and we're left to see that $\{pr(v_i)\}_{i \in I \setminus I_F}$ is a basis of $V/W_F(C)$.

By linear independence of $\{v_i\}_{i \in I}$ and Lemma 1.3.1 it is clear that $\dim(V) = |I|$ and $\dim(W_F(C)) = |I_F|$. Hence

$$\dim\left(\frac{W}{W_F(P)}\right) = |I| - |I_F| = |I \setminus I_F|$$

and it is enough to show that $\{pr(v_i)\}_{i \in I \setminus I_F}$ is a spanning set. But this follows directly from surjectivity of the projection $pr : V \to V/W_F(C)$: Any $[v] \in V/W_F(C)$ can be written as $[v] = pr(v)$ for some $v \in V$. This $v \in V$ can be written in terms of the basis $\{v_i\}_{i \in I}$ as $v = \sum_{i \in I} \lambda_i v_i$ where $\lambda_i \in \mathbb{R}$. Recalling that $W_F(C) = \mathrm{Span}_{\mathbb{R}}(\{v_i\}_{i \in I_F})$ is the kernel of pr yields

$$\begin{aligned} [v] &= pr(v) \\ &= pr\left(\sum_{i \in I} \lambda_i v_i\right) \\ &= \sum_{i \in I \setminus I_F} \lambda_i \, pr(v_i). \end{aligned}$$

Since $[v] \in V/W_F(C)$ was arbitrary, this shows that $\{pr(v_i)\}_{i \in I \setminus I_F}$ is indeed a spanning set and thereby concludes the proof.

The extension to unimodularity is now almost immediate:

Proposition 4.2.2. Let $\Lambda \subset V$ be a lattice, $C \subseteq V$ be a Λ-unimodular polyhedral cone and $F \subseteq C$ a proper face. Let $pr : V \to V/W_F(C)$ denote the projection on the effective space and Λ_F be the effective lattice. Then the unique polyhedral cone $\tilde{C} \subseteq (V/W_F(C))$ such that $pr(C) = \tilde{C}$ is Λ_F-unimodular.

Proof. From Proposition 4.2.1 it is clear that

$$\tilde{C} = \bigcap_{i \in I \setminus I_F} H_{pr(v_i)}$$

and that $\{pr(v_i)\}_{i \in I \setminus I_F}$ is a basis of $V/W_F(C)$. It is thus enough to show that if $\{v_i\}_{i \in I}$ is a basis of Λ, then the coefficients of the elements in Λ_F in terms of $\{pr(v_i)\}_{i \in I \setminus I_F}$ are integers. But by definition, $\Lambda_F = pr(\Lambda)$ and so to any element $[u] \in \Lambda_F$ there is an element $u \in \Lambda$ such that $[u] = pr(u)$. Since $\{v_i\}_{i \in I}$ are a basis of Λ, there exist integer coefficients $n_i \in \mathbb{Z}$ such that $u = \sum_{i \in I} n_i v_i$. Hence

$$\begin{aligned} [u] &= pr(u) \\ &= pr\left(\sum_{i \in I} n_i v_i\right) \\ &= \sum_{i \in I \setminus I_F} n_i \, pr(v_i) \end{aligned}$$

which concludes the proof.

Figure 4.1: Illustration of the idea of Lemma 4.3.1.

4.3 Recursive Aspects of Polytopes

The goal of this section is to show that the faces of simple or unimodular polytopes are again simple or unimodular. Since the conditions of simplicity and unimodularity were formulated in terms of local cones, it should not come as a surprise that we can reduce the problem to the case of polyhedral cones. The key idea (illustrated in figure 4.1) is the follwing: Consider a vertex φ of a polytope Π and a face F of Π which contains φ. On one hand, we can first consider $\tilde{F}$, the unique polyhedron in the effective space mapped onto F. One can then consider the local cone at the point $\tilde{\varphi}$ corresponding to φ. On the other hand, one can first look at the local cone $C_\varphi(\Pi)$ at φ and then look at the face C_F of this cone corresponding to F. This face C_F has the same effective space as F and one can then consider $\tilde{C}_F$, the unique polyhedral cone mapped onto C_F. The crucial observation is that these two constructions actually yield the same cone, that is $\tilde{C}_F = C_{\tilde{\varphi}}(\tilde{F})$.

Let $F \subseteq V$ be a face of a polytope Π and consider a vertex φ in F. Since $\varphi \in F$, we have $I_F \subseteq I_\varphi$. Hence, if we consider the local cone,

$$C_\varphi(\Pi) = \bigcap_{i \in I_\varphi} H_{v_i};$$

then for $v = \sum_{i \in I_F} v_i$ we get a face C_F of $C_\varphi(\Pi)$ by putting

$$C_F = C_\varphi(\Pi) \cap H_v = \bigcap_{i \in I_\varphi \setminus I_F} H_{v_i} \cap \bigcap_{i \in I_F} \partial H_{v_i}:$$

We call this the face of the local cone corresponding to the face of the polytope.

In particular, we note that since

$$W_F(\Delta) = \operatorname{Span}_{\mathbb{R}}(\{v_i\}_{i\in I_F}) = W_{C_F}(C_\varphi(\Delta)) \; ;$$

F and C_F have the same effective space. If $\mathrm{pr} : V \to V/W_F(\Delta)$ is the projection, then by Lemma 4.1.2 the unique polyhedral cone $\hat{C}_F \subseteq (V/W_F(\Delta))$ such that $\mathrm{pr}^{-1}(\hat{C}_F) = C_F$ is

$$\hat{C}_F = \bigcap_{i\in I_\varphi \setminus I_{\tilde{F}}} H_{\mathrm{pr}(v_i)} .$$

At the same time, again by Lemma 4.1.2 the unique polytope $\tilde{F} \subseteq (V/W_F(\Delta))$ such that $\mathrm{pr}^{-1}(\tilde{F}) = F$ is

$$\tilde{F} = \bigcap_{i\in I_{\tilde{F}}} H_{(\mathrm{pr}(v_i);c_i)} .$$

Let $\tilde{\varphi}$ be the unique point in $\tilde{F}$ which is mapped onto φ by pr^{-1}. Hence we can look at the local cone to $\tilde{F}$ at $\tilde{\varphi}$ and find

$$C_{\tilde{\varphi}}(\tilde{F}) = \bigcap_{i\in (I_{\tilde{F}})_{\tilde{\varphi}}} H_{\mathrm{pr}(v_i)}$$

with

$$\begin{aligned}
(I_{\tilde{F}})_{\tilde{\varphi}} &= \{ i \in I_{\tilde{F}} \mid \langle \tilde{\varphi} \mid \mathrm{pr}(v_i)\rangle = c_i \} \\
&= \{ i \in I_{\tilde{F}} \mid \langle \mathrm{pr}^{-1}(\tilde{\varphi}); v_i\rangle = c_i \} \\
&= \{ i \in I_{\tilde{F}} \mid \langle \varphi; v_i\rangle = c_i \} \\
&= I_\varphi \setminus I_{\tilde{F}} .
\end{aligned}$$

A comparison of the explicit expressions thus gives

$$C_{\tilde{\varphi}}(\tilde{F}) = \bigcap_{i\in I_\varphi \setminus I_{\tilde{F}}} H_{\mathrm{pr}(v_i)} = \hat{C}_F .$$

To formulate it differently, the operations of taking the local cone and passing to the effective space commute. We summarise this result in a technical lemma:

Lemma 4.3.1. Let Δ be a polytope, $F \subset \Delta$ be a proper face and $\varphi \in F$ a vertex. Let $\mathrm{pr} : V \to V/W_F(\Delta)$ denote the projection on the effective space.

1. Let $\tilde{F} \subseteq (V/W_F(\Delta))$ be the unique polytope such that $\mathrm{pr}^{-1}(\tilde{F}) = F$ and let $\tilde{\varphi} \in V/W_F(\Delta)$ be the unique point such that $\mathrm{pr}^{-1}(\tilde{\varphi}) = \varphi$.
2. Let $C_\varphi(\Delta) = \bigcap_{i\in I_\varphi} H_{v_i}$ be the local cone to Δ at φ and let C_F be the face of $C_\varphi(\Delta)$ corresponding to F. Let $\hat{C}_F \subseteq (V/W_F(\Delta))$ be the unique polyhedral cone such that $\mathrm{pr}^{-1}(\hat{C}_F) = C_F$.

Then

$$\tilde{C}_F = C_{\tilde{\varphi}}(\tilde{F}):$$

Now the work is done and we just have to assemble the pieces:

Theorem 4.3.2. Let V be a simple polyope and F a proper face. Let pr : V ! V=W$_F$(P) denote the projection on the e ective space. Then the unique polytope $\tilde{F}$ (V=W$_F$( )) such that pr ($\tilde{F}$) = F is simple.

Proof. Let $\tilde{\varphi}$ 2 $\tilde{F}$ be a vertex. Then ' := pr ($\tilde{\varphi}$) is a vertex of F. By Lemma 1.3.2 ' is also a vertex of the polytope . Thus by assumption, the local cone C_{φ} ( ) is simple. By Lemma 4.3.1, the local cone $C_{\tilde{\varphi}}(\tilde{F})$ is mapped by pr onto a face C_F of C_{φ} ( ) and is thus simple by Proposition 4.2.1.

Theorem 4.3.3. Let V be a lattice, V a -unimodular polyope and F a proper face. Let pr : V ! V=W$_F$(P) denote the projection on the e ective space and $_F$ be the e ective lattice. Then the unique polytope $\tilde{F}$ (V=W$_F$( )) such that pr ($\tilde{F}$) = F is $_F$-unimodular.

Proof. This is the same proof as for Theorem 4.3.2 except that one has to use Proposition 4.2.2 instead of Proposition 4.2.1.

4.4 The Direct Minkowski-Sum

For any two convex subsets A; B V , the Minkowski sum is de ned as

$$A + B = \{ ' + \; | \; ' \in A; \; \in B \}:$$

Even though this de nition looks easy, already in the case of polyhedra it is very involved. We will not consider it in full generality but only the special case that is of primary concern for symplectic toric manifolds. This is the case where V is the direct sum of two subspaces V_1 and V_2 and each of the polytopes is contiained in one of the summands. The motivation for this is that the Lie algebra of a product of Lie groups is the direct sum of the individual Lie algebras. Before we move on to the concrete results, a word of warning: The main di culty in this section is the notation.

Assume thus that V is the direct sum of two other vector spaces, say V_1 and V_2. Then there is a split short exact sequence

$$0 \longrightarrow V_1 \xrightarrow{i_1} V_1 \; V_2 \xrightarrow{pr_2} V_2 \longrightarrow 0$$

Figure 4.2: Illustration of the direct Minkowski sum of two polytopes Δ_1 and Δ_2.

where $i_{1;2} : V_{1;2} \to V_1 \oplus V_2$ are the natural inclusions and $pr_{1;2} : V_1 \oplus V_2 \to V_{1;2}$ are the natural projections. Taking the dual, we get another split short exact sequence, namely

$$0 \longrightarrow V_2 \xrightarrow{pr_2} V_2 \oplus V_1 \xrightarrow{i_1} V_1 \longrightarrow 0$$

Suppose that $\Delta_1 \subset V_1$ and $\Delta_2 \subset V_2$ are two polytopes in the respective vector spaces V_1 and V_2. We can interpret them as (degenerate) polytopes $pr_1(\Delta_1)$ and $pr_2(\Delta_2)$ in the direct sum $V = V_1 \oplus V_2$ via the inclusions. Then we can consider the Minkowski sum of these polytopes:

$$pr_1(\Delta_1) + pr_2(\Delta_2) = \{\varphi + \psi \mid \varphi \in pr_1(\Delta_1)\,;\ \psi \in pr_2(\Delta_2)\}.$$

Notation. We will adapt the notation $\Delta_1 \oplus \Delta_2 = pr_1(\Delta_1) + pr_2(\Delta_2)$.

Example. See figure 4.2 for a graphical example of the construction.

This construction is considerably simpler than the general Minkowski sum, primarily because there is a rather explicit expression for it:

Proposition 4.4.1. Let $\Delta_1 = \bigcap_{i\in I} H_{(v_i;c_i)} \subset V_1$ and $\Delta_2 = \bigcap_{j\in J} H_{(w_j;c_j)} \subset V_2$ be two polyhedra. Then the direct Minkowski sum of Δ_1 and Δ_2 is given by the explicit expression

$$\Delta_1 \oplus \Delta_2 = \bigcap_{i\in I} H_{(i_1(v_i);c_i)} \cap \bigcap_{j\in J} H_{(i_2(w_j);c_j)}.$$

and is a polytope in $V_1 \oplus V_2$.

Proof. We show

$$\Pi_1 \oplus \Pi_2 = P_1 \cap P_2$$

where

1. $P_1 = \bigcap_{i \in I} H_{(i_1(v_i); c_i)}$ is the extension of Π_1 to $V_1 \times V_2$ and
2. $P_2 = \bigcap_{j \in J} H_{(i_2(w_j); c_j)}$ is the extension of Π_2 to $V_1 \times V_2$.

Take $\varphi \in \Pi_1$ and $\psi \in \Pi_2$. The corresponding point in $\Pi_1 \oplus \Pi_2$ is $pr_1(\varphi) + pr_2(\psi)$. This point lies in P_1 since for any $i \in I$ we have

$$\begin{aligned} \langle pr_1(\varphi) + pr_2(\psi); i_1(v_i)\rangle &= \langle pr_1(\varphi); i_1(v_i)\rangle + \langle pr_2(\psi); i_1(v_i)\rangle \\ &= \Big\langle \varphi; \underbrace{pr_1^* \, i_1}_{=\mathrm{Id}_{V_1}}(v_i)\Big\rangle + \Big\langle \psi; \underbrace{pr_2^* \, i_1}_{=0}(v_i)\Big\rangle \\ &= \langle \varphi; v_i\rangle \geq c_i \end{aligned}$$

as $\varphi \in \Pi_1$. An analogous computation shows that $pr_1(\varphi) + pr_2(\psi) \in P_2$ and hence we conclude that $\Pi_1 \oplus \Pi_2 \subseteq P_1 \cap P_2$.

Reversely, take an arbitrary point $\chi \in P_1 \cap P_2$ and decompose it as $\chi = pr_1(\varphi) + pr_2(\psi) \in V_1 \times V_2$ where $\varphi := i_1^*(\chi) \in V_1$ and $\psi := i_2^*(\chi) \in V_2$. Clearly, $\varphi \in \Pi_1$ as

$$\begin{aligned} \langle \varphi; v_i\rangle &= \langle i_1^*(\chi); v_i\rangle \\ &= \langle \chi; i_1(v_i)\rangle \geq c_i \end{aligned}$$

as $\chi \in P_1$. Analogously, we see that $\psi \in \Pi_2$ which concludes the proof.

One of the simplifications of the direct Minkowski sum with regard to the usual Minkowski sum is that the faces of the sum can be understood completely in terms of the faces of the summands while for the general Minkowski sum this only works in one direction.

Corollary 4.4.2. *Let $\Pi_1 = \bigcap_{i \in I} H_{(v_i; c_i)} \subseteq V_1$ and $\Pi_2 = \bigcap_{j \in J} H_{(w_j; c_j)} \subseteq V_2$ be two polyhedra. Let $F_1 \subseteq \Pi_1$ and $F_2 \subseteq \Pi_2$ be two faces of Π_1 and Π_2 respectively. Then their direct Minkowski sum $F_1 \oplus F_2$ is a face of $\Pi_1 \oplus \Pi_2$.*

Proof. This is immediate from Proposition 4.4.1 if we write the hyperplanes as the intersection of the two corresponding two halfspaces.

Lemma 4.4.3. *Let $\Pi_1 \subseteq V_1$ and $\Pi_2 \subseteq V_2$ be two polyhedra and let F be a face of $\Pi_1 \oplus \Pi_2$. Then $i_1^*(F)$ is a face of Π_1 and $i_2^*(F)$ is a face of Π_2. Moreover, F corresponds to the direct Minkowski sum of $i_1^*(F)$ and $i_2^*(F)$:*

$$F = i_1^*(F) \oplus i_2^*(F).$$

Proof. If $\sigma_1 = \bigcap_{i\in I} H_{(v_i;c_i)}$ and $\sigma_2 = \bigcap_{j\in J} H_{(w_j;c_j)}$, then by Proposition 4.4.1

$$\sigma_1 \times \sigma_2 = \bigcap_{i\in I} H_{(i_1(v_i);c_i)} \cap \bigcap_{j\in J} H_{(i_2(w_j);c_j)}.$$

The face F is characterised by a set $(I_F \cup J_F) \subseteq (I \cup J)$ of indices which are active on F:

$$F = \bigcap_{i\in I} H_{(i_1(v_i);c_i)} \cap \bigcap_{i\in I_F} \partial H_{(i_1(v_i);c_i)} \cap \bigcap_{j\in J} H_{(i_2(w_j);c_j)} \cap \bigcap_{j\in J_F} \partial H_{(i_2(w_j);c_j)}.$$

We show that $i_1(F)$ and $i_2(F)$ are the faces characterised by I_F and J_F respectively:

$$i_1(F) = \bigcap_{i\in I} H_{(v_i;c_i)} \cap \bigcap_{i\in I_F} \partial H_{(v_i;c_i)}$$
$$i_2(F) = \bigcap_{j\in J} H_{(w_j;c_j)} \cap \bigcap_{j\in J_F} \partial H_{(w_j;c_j)}.$$

Assume first that $\xi \in F$ and note that for all $i \in I$

$$\langle i_1(\xi); v_i\rangle = \langle \xi; i_1(v_i)\rangle \geq c_i$$

with equality if and only if $i \in I_F \subseteq I$. Hence

$$i_1(F) \subseteq \left(\bigcap_{i\in I} H_{(v_i;c_i)} \cap \bigcap_{i\in I_F} \partial H_{(v_i;c_i)} \right).$$

To get the reverse inclusion, take $\varphi \in \bigcap_{i\in I} H_{(v_i;c_i)} \cap \bigcap_{i\in I_F} \partial H_{(v_i;c_i)}$. Then, choose any point $\psi \in F$ and put $\chi = i_2(\psi)$ and $\xi := pr_1(\varphi) + pr_2(\chi)$. By construction

$$\begin{aligned} i_1(\xi) &= i_1(pr_1(\varphi) + pr_2(\chi)) \\ &= (pr_1 \circ i_1)(\varphi) + (pr_2 \circ i_1)(\chi) \\ &= \varphi \end{aligned}$$

and we are left to show that $\xi \in F$. But on one hand we have for all $i \in I$

$$\begin{aligned} \langle \xi; i_1(v_i)\rangle &= \langle pr_1(\varphi); i_1(v_i)\rangle + \langle pr_2(\chi); i_1(v_i)\rangle \\ &= \langle \varphi; pr_1 \circ i_1(v_i)\rangle + \langle \chi; pr_2 \circ i_1(v_i)\rangle \\ &= \langle \varphi; v_i\rangle \\ &\geq c_i \end{aligned}$$

with equality if and only if $i \in I_F$. On the other hand, for any $j \in J$ we have

$$\begin{aligned}
\langle \psi, i_2(w_j)\rangle &= \langle pr_1(\varphi), i_2(w_j)\rangle + \langle pr_2(\psi), i_2(\omega_j)\rangle \\
&= \langle \psi, pr_1 \circ i_2(w_j)\rangle + \langle \psi, pr_2 \circ i_2(w_j)\rangle \\
&= \langle \psi, w_j\rangle \\
&= \langle i_2(\psi), w_j\rangle \\
&= \langle \psi, i_2(w_j)\rangle \\
&\leq c_j
\end{aligned}$$

with equality if $j \in J_F$ since $\psi \in F$. Hence we found $\psi \in F$ such that $i_1(\psi)$. Repeating the same argument for $i_2(F)$ concludes the proof of the first part. The second part, $i_1(F) \oplus i_2(F) = F$ follows again directly from the explicit expression in Proposition 4.4.1.

Putting the last two results together, we conclude this section by the following result:

Proposition 4.4.4. Let P_1 and P_2 be two polyhedra. Then there is a one-one correspondence between faces of their direct Minkowski sum $P_1 \oplus P_2$ and direct Minkowski sums of faces of P_1 and P_2.

4.5 Cutting Polytopes

Now we turn our attention to hyperplanes which go through a polytope instead of being laid on the surface. What we mean by "going through" can already be defined for general convex sets:

Definition. Let A be a convex set. A hyperplane $H_{(v;c)}$ is said to split A if both open halfspaces determined by $H_{(v;c)}$ contain points of A i.e. there exist $\varphi, \psi \in A$ such that $\langle \varphi, v\rangle > c$ and $\langle \psi, v\rangle < c$.

The goal is to understand the "cutting" of a polytope, that means the intersection of a polytope with an affine halfspace such that the boundary hyperplane splits the polytope. More precisely, let $P = \bigcap_{i \in I} H_{(v_i;c_i)}$ be a polytope and $H_{(v;c)}$ be a hyperplane splitting P, that is, there exist $\varphi_1, \varphi_2 \in P$ such that

$$\langle \varphi_1, v\rangle > 0 \quad \text{and} \quad \langle \varphi_2, v\rangle < 0.$$

We then investigate the properties of the polytope

$$P_{cut} = P \cap H_{(v;c)} = \bigcap_{i \in I} H_{(v_i;c_i)} \cap H_{(v;c)}$$

under the familiar assumptions that is rational, simple or unimodular. The goal is to establish criteria on $H_{(v;c)}$ such that $_{cut}$ is again rational, simple or unimodular. These properties can then immediately be extended to

$$_{slice} = \cap \partial H_{(v;c)}.$$

This is clear since $_{slice} = {}_{cut} \cap \partial H_{(v;c)}$ is a face of $_{cut}$ and so we can apply Proposition 4.1.3, Theorem 4.3.2 or Theorem 4.3.3.

We will treat these problems in order of increasing complexity, starting with the almost trivial case of rationality. First however, let us set some useful notation:

Notation. We will denote by J the index set of $_{cut}$, that is, the one obtained from

$$I \cup \{j\} \quad \text{with} \quad v_j = v \quad \text{and} \quad c_j = c.$$

by omitting the redundant indices. Note that since $\partial H_{(v;c)} \cap \mathrm{int}() \neq \emptyset$, the index j is not redundant.

Let us start now with the easiest case:

Proposition 4.5.1. Let $\subset V$ be a lattice, $\subset V$ a -rational polytope and $H_{(v;c)}$ be a -rational hyperplane which splits . Then the polytope

$$_{cut} = \cap H_{(v;c)}$$

is again -rational.

Proof. This is clear since

$$_{cut} = \bigcap_{i \in J} H_{(v_i;c_i)}$$

and all the $\{v_i\}_{i \in J}$ (including $v_j = v$) are -rational by assumption.

Next, we turn our attention to simple polytopes. Let thus $= \bigcap_{i \in I} H_{(v_i;c_i)}$ be a simple polytope and consider

$$_{cut} = \bigcap_{i \in I} H_{(v_i;c_i)} \cap H_{(v;c)} = \bigcap_{i \in J} H_{(v_i;c_i)}$$

where $\partial H_{(v;c)}$ is an affine hyperplane splitting . Let be a vertex of $_{cut}$ which lies in $\partial H_{(v;c)}$. Note that since is a vertex, $|J| \geq n = \dim(V)$ and the local cone $C(\,{}_{cut})$ is simple exactly if $|J| = n$. There are two main cases to distinguish:

1. If was not a vertex of , we must have $|I| = n - 1$. It follows that $|J| = |I \cup \{ \}| = n$ and hence the local cone
$$C(\ _{cut}) = \bigcap_{i \in J} H_{v_i}$$
is simple for any v.

2. If however, is already a vertex of , then we have by simplicity of that $|I| = n$. Recalling that J is the index set obtained from $I \cup \{ \}$ by omitting the redundant indices, it follows that $C(\ _{cut})$ is simple exactly if precisely one index in I becomes redundant for $_{cut}$.

   An index $k \in I$ is redundant for $_{cut}$ if and only if $\partial H_{(v_k;c_k)}$ is a hyperplane isolating $\bigcap_{i \in I \setminus \{k\}} H_{(v_i;c_i)} \cap H_{(v;c)}$. But the hyperplane $\partial H_{(v_k;c_k)}$ contains and hence $k \in I$ is redundant for $_{cut}$ if and only if $\partial H_{(v_k;c_k)}$ is a support hyperplane to $\bigcap_{i \in I \setminus \{k\}} H_{(v_i;c_i)} \cap H_{(v;c)}$ at . This in turn is the case if and only if v_k is in the support cone $S(\bigcap_{i \in I \setminus \{k\}} H_{(v_i;c_i)} \cap H_{(v;c)})$ by Lemma 1.3.4. By Lemma 1.3.3 this support cone can explicitly be written as
$$S(\ _{cut}) = \mathrm{cone}(\{v_i\}_{i \in I \setminus \{k\}} \cup \{ v\}):$$
This means that v_k is contained in the support cone if and only if there exist positive real numbers $_i;\ \geq 0$ such that
$$v_k = \sum_{i \in I \setminus \{k\}} {}_i v_i + \ v:$$
By linear independence of $\{v_i\}_{i \in I}$ it is clear that $\neq 0$ and hence
$$v = \frac{1}{\ } v_j - \sum_{i \in I \setminus \{k\}} \frac{\ _i}{\ } v_i:$$
The conclusion is thus that $C(\ _{cut})$ is simple if and only if the expression of v in terms of the basis $\{v_i\}_{i \in I}$ of V is of the form $v = \sum_{i \in I} \ _i v_i$ with exactly one $_i$ strictly positive.

Putting this second, more involved case aside, we conclude: A sufficient, but not necessary condition for
$$_{cut} = \ \cap H_{(v;c)}$$
to be simple is that $\mathrm{Vert}(\ ) \cap H_{(v;c)} = \emptyset$, where we recall that $\mathrm{Vert}(\ )$ denotes the set of vertices of . For the remainder of these notes, we will restrict our attention to this case:

Definition. Let be a polytope and $H_{(v;c)}$ be a hyperplane which splits . $H_{(v;c)}$ is said to *cut* if no vertex of lies in $H_{(v;c)}$ i.e.
$$\mathrm{Vert}(\ ) \cap H_{(v;c)} = \emptyset:$$

Proposition 4.5.2. *Let be a simple polytope and $H_{(v;c)}$ a hyperplane which cuts . Then the polytopes*

$$_{cut} = \cap H_{(v;c)}$$

is again simple.

Consider then a unimodular polytope and a hyperplane $H_{(v;c)}$ which cuts an edge e in a point which is not a vertex of . The question that we need to address is then whether the unimodularity condition at the point satisfied?

Unimodular polytopes are in particular simple and for simple polytopes the local cone at is

$$C(\, _{cut}) = \bigcap_{i \in J} H_{v_i}$$

where $J = I \cup \{j\}$. But now by assumption, $\{v_i\}_{i \in I}$ are part of a lattice basis. Indeed the edge e contains contains two vertices of , say φ_1 and φ_2 so that $I_e \subset I_{\varphi_1}, I_{\varphi_2}$ and $\{v_i\}_{i \in I_{\varphi_1}}$, $\{v_i\}_{i \in I_{\varphi_2}}$ are by assumption lattice bases. The question therefore reduces to understanding in what ways one element of a lattice basis can be replaced by another.

In the unimodular case both $\{v_i\}_{i \in I_{\varphi_1}}$ and $\{v_i\}_{i \in I_{\varphi_2}}$ form a basis of a lattice . Let i_1 be the unique index in I_{φ_1} not in I_e and similarly, let i_2 be the unique index in I_{φ_2} not in I_e. The corresponding vectors can be written in terms of the other basis as

$$v_{i_1} = \sum_{i \in I_{\varphi_2}} m_i v_i \quad \text{and} \quad v_{i_2} = \sum_{i \in I_{\varphi_1}} n_i v_i$$

for some integer coefficients $m_i; n_i \in \mathbb{Z}$. By invariance under translation, we might assume that φ_1 is the origin. It follows that $c_i = 0$ for all $i \in I_{\varphi_1}$, so in particular $c_i = 0$ if $i \in I_e$. Also $0 \in$ implies that $0 = \langle 0; v_{i_2} \rangle < c_{i_2}$, where the inequality is strict because $0 \notin \partial H_{(v_{i_2}; c_{i_2})}$. Using these simplifications we can compute

$$\begin{aligned} \langle \varphi_2; v_{i_1} \rangle &= \sum_{i \in I_{\varphi_2}} m_i \langle \varphi_2; v_i \rangle \\ &= \sum_{i \in I_{\varphi_2}} m_i c_i \\ &= m_{i_2} c_{i_2} \end{aligned}$$

and from there we compute that

$$\begin{aligned} c_{i_2} &= \langle \varphi_2; v_{i_2}\rangle \\ &= \sum_{i \in I_{\varphi_1}} n_i \langle \varphi_2; v_i\rangle \\ &= n_{i_1}\langle \varphi_2; v_{i_1}\rangle \\ &= n_{i_1} m_{i_2} c_{i_2}. \end{aligned}$$

Note first that because $\varphi_2 \in \Delta$, we have

$$m_{i_2} c_{i_2} = \langle \varphi_2; v_{i_1}\rangle \geq c_{i_1} = 0$$

and since $c_{i_2} > 0$, this implies $m_{i_2} \geq 0$. On the other hand, also since $c_{i_2} > 0$ we have that

$$c_{i_2} = n_{i_1} m_{i_2} c_{i_2}$$

implies

$$n_{i_1} m_{i_2} = 1.$$

Finally we use that n_{i_1} and m_{i_2} are integers so that the above equation only allows for the solutions $n_{i_1} = \pm 1 = m_{i_2}$. But since we already established that $m_{i_2} \leq 0$, the case with $+1$ is not possible so that finally we get

$$m_{i_2} = -1 = n_{i_1}.$$

In order to interpret this result properly, we need to zoom out again. We have just shown that the coefficients of v_{i_1} and v_{i_2} in the expression of the relative other basis is -1, for instance

$$v_{i_2} = -v_{i_1} + \sum_{i \in I_e} n_i v_i.$$

But $\{v_i\}_{i \in I_e}$ forms a basis of the annihilator lattice $\Lambda^0 = \Lambda \cap W_e(\Delta)$ and hence the above expression in the form

$$v_{i_2} + v_{i_1} = \sum_{i \in I_e} n_i v_i$$

means nothing but that the sum of v_{i_1} and v_{i_2} is contained in the annihilator lattice Λ^0. We summarise this again in a Lemma:

Lemma 4.5.3. Let $\Lambda \subset V$ be a lattice, Δ be a Λ-unimodular polytope, e an edge in Δ and $\varphi_0; \varphi_1$ the two vertices contained in the edge e. Then

1. there exist unique indices $i_1 \in I_{\varphi_1}$ and $i_2 \in I_{\varphi_2}$ such that

$$I_{\varphi_1} \setminus \{i_1\} = I_e = I_{\varphi_2} \setminus \{i_2\};$$

and

2. the sum of the corresponding vectors v_{i_1} and v_{i_2} is in the annihilator lattice $^0 = \ \backslash\ W_e(\ )$ of the edge:

$$v_{i_1} + v_{i_2}\ 2\ \ ^0:$$

The natural follow up question is now whether this last condition is also a su cient condition in the following sense: Suppose that e is an edge containing the vertices $'_1$ and of $_{cut}$. We adapt the notation from the Lemma but with $i_2 = j$ and $v_{i_2} = v$ and assume that $f v_i g_{i2I_{'_1}}$ is a basis of . Is the condition that $v_{i_1} + v\ 2\ \ ^0$ enough to assure that $f v_i g_{i2J}$ is a basis of as well?

The answer is yes: Since $f v_i g_{i2I_{'_1}}$ is a basis of , any lattice point $u\ 2$ can be written as linear combination with integer coe cients

$$u = \sum_{i2I_{'_1}} n_i v_i \quad \text{with} \quad n_i\ 2\ Z:$$

But if $v_{i_1} + v\ 2\ \ ^0$, then we can write

$$v_{i_1} = \ v + \sum_{i2I_e} m_i v_i \quad \text{with} \quad m_i\ 2\ Z$$

and substitute this in the above:

$$u = \sum_{i2I_e} n_i v_i + n_{i_1} \left(v + \sum_{i2I_e} m_i v_i \right)$$
$$= \sum_{i2I_e} (n_i + n_{i_1} m_i) v_i \quad n_{i_1} v$$

where clearly $n_i + n_{i_1} m_i\ 2\ Z$ and $n_{i_1}\ 2\ Z$. This shows that any lattice point $u\ 2$ is an integer linear combination of elements in $f v_i g_{i2I}$ and thereby that $f v_i g_{i2I}$ is a basis of . We have shown the following result:

Lemma 4.5.4. Let V be a lattice, V a simple polytope, e an edge in and $'_1$; the two vertices contained in e. Suppose that $f v_i g_{i2I_{'_1}}$ is a lattice basis of and write i_1 for the unique index in $I_{'_1} n I_e$ and j for the unique index in $I\ n I_e$. Then $f v_i g_{i2I}$ is a basis of if and only if $v_{i_1} + v\ 2\ \ ^0$ where $^0 = \ \backslash\ W_e(\ )$ is the annihilator lattice of the edge e.

De nition. Let V be a lattice and V a -unimodular polytope.

1. Let $H_{(v;c)}$ be a -rational hyperplane cutting an edge e. Let $'$ be the vertex contained in e such that $'\ 2\ H_{(v;c)}$ and j the unique index in $I_{'} n I_e$. The hyperplane is called *compatible* with the edge e if $v_j + v$ is contained in the annihilator lattice of the edge e.

Figure 4.3: Example of a polytope cut by a hyperplane $H_{(v;c)}$. This hyperplane is a reduction level for .

2. A hyperplane $H_{(v;c)}$ is called a reduction level for if it cuts and is compatible with all edges that it cuts.

This de nition is precisely what we need for the next theorem to hold:

Theorem 4.5.5. Let be a -unimodular polytope and $H_{(v;c)}$ a hyperplane which cuts . Then

$$_{cut} = \cap H_{(v;c)}$$

is again -unimodular if and only if $H_{(v;c)}$ is a reduction level for .

Remark. Consider a -unimodular polytope and take an edge e of . Let $'_1$ and $'_2$ be the two vertices contained in e, write i_1 for the unique index in $I_{'_1} n I_e$ and i_2 for the unique index in $I_{'_2} n I_e$. By unimodularity of we have $v_{i_1} + v_{i_2} 2 \; ^0 = \; \cap W_e( )$. Then the trivial observation

$$^0 3 \; v_{i_1} + v_{i_2} = (v_{i_1} + v) + (v_{i_2} \quad v)$$

shows that $v_{i_1} + v \; 2 \; ^0$ if and only if $(v_{i_2} \quad v) \; 2 \; ^0$.

What this means is that if a hyperplane is a reduction level for , then both sides of the hyperplane will be unimodular polytopes. This is because the halfspace on the other side of the hyperplane is precisely $H_{(v; c)}$, in particular it is described by the annihilating vector v.

Example. Consider the polytope R^2 shown in gure 4.3. We see that $I_{'} = f1;2g$ and that the hyperplane $H_{(v;c)}$ cuts the two edges e_1 and e_2. Clearly, $I_{e_1} = f1g$ so that $i_1 = f2g$ is the unique index in $I_{'} n I_{e_1}$. The annihilator space of e_1 is $W_{e_1}(\ ) = \ \mathrm{Span}_R(v_1)$ and the annihilator lattice is $\ {}^0 = \mathrm{Span}_Z(v_1)$. The hyperplane $H_{(v;c)}$ is compatible with e_1 if $v_{i_1} + v = v_2 + v\ 2\ \ {}^0$. Taking the explicit expressions

$$v_1 = \begin{matrix} 1 \\ 0 \end{matrix}\ ; \qquad v_2 = \begin{matrix} 0 \\ 1 \end{matrix} \quad \text{and} \quad v = \begin{matrix} 1 \\ 1 \end{matrix}$$

we get

$$v_2 + v = \begin{matrix} 1 \\ 0 \end{matrix} \ = \ v_1\ 2\ \ {}^0:$$

Repeating this argument for e_2 shows that $H_{(v;c)}$ is a reduction level.

Part II

Symplectic Toric Manifolds

Chapter 5

Hamiltonian Actions

In this chapter we introduce Hamiltonian actions on symplectic manifolds. In the Appendix, Chapter A, the reader can find the most important definitions and properties of general Lie group actions on smooth manifolds. This is why we will focus here only on the symplectic and Hamiltonian aspects of it. After a short exhibition of the basic definitions, a more detailed account of recursive aspects is given.

5.1 Symplectic Actions

A smooth action of a Lie group G on a smooth manifold M is a Lie group homomorphism $\psi : G \to \mathrm{Diff}(M)$. If M is equipped with a symplectic form ω, then the diffeomorphisms of M which preserve ω, the so-called symplectomorphisms, form a subgroup $\mathrm{Sympl}(M;\omega)$. The action of a group G is symplectic if every group element preserves the symplectic structure.

Definition. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group on a symplectic manifold $(M;\omega)$. ψ is called a *symplectic action* if it acts by symplectomorphisms i.e.

$$\psi : G \to \mathrm{Sympl}(M) \subset \mathrm{Diff}(M):$$

If G is a Lie group which acts on a smooth manifold M via $\psi : G \to \mathrm{Diff}(M)$, the fundamental vector field $X^\sharp \in \mathfrak{X}(M)$ associated to an element $X \in \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G is defined pointwise by

$$X^\sharp_p = \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(tX)}(p):$$

For a more complete introduction of the fundamental vector field and some of its basic properties, we refer the reader to the appendix, section A.2.

If $\psi : G \to \mathrm{Diff}(M)$ is a symplectic action, then the fundamental vector fields are symplectic, that is, $d\iota_{X^\sharp}\omega = 0$ for any $X \in \mathfrak{g}$. This follows from closedness of the symplectic form and Cartan's magic formula:

$$\begin{aligned} d\iota_{X^\sharp}\omega &= d\iota_{X^\sharp}\omega + \iota_{X^\sharp} d\omega \\ &= L_{X^\sharp}\omega \\ &= \frac{d}{dt}\Big|_{t=0} \psi_{\exp(tX)}^*\omega \\ &= \frac{d}{dt}\Big|_{t=0} \omega \\ &= 0 \end{aligned}$$

where we used that $\psi_{\exp(tX)}^*\omega = \omega$ since $\psi_{\exp(tX)}$ is by assumption a symplectomorphism.

If the Lie group G is compact, and $p \in M$ is a fixed point of the action, then a neighbourhood of p is characterised by the isotropy representation: If $\psi : G \to \mathrm{Diff}(M)$ denotes the action of G on M, then the isotropy representation is given by

$$\begin{aligned} \rho : G &\to GL(T_pM) \\ g &\mapsto d(\psi_g)_p . \end{aligned}$$

If now again the action is symplectic, then also the isotropy representation is symplectic.

Proposition 5.1.1. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth symplectic action of a Lie group on a symplectic manifold M and let $p \in M$ be a fixed point. Then the isotropy representation

$$\begin{aligned} \rho : G &\to GL(T_pM) \\ g &\mapsto d(\psi_g)_p \end{aligned}$$

is a symplectic representation i.e. for any $g \in G$ we have $\rho_g \in Sp(T_pM; \omega_p)$.

In section A.4 in the appendix, it is deduced from the slice theorem that the connected components of the set of points which are fixed by a group action $\psi : G \to \mathrm{Diff}(M)$ are smooth submanifolds of M. Morally, this follows from the fact that a neighbourhood of a fixed point looks like the isotropy representation and the elements which are fixed by a given representation form a vector subspace. But if the action ψ is symplectic, then the isotropy representation is symplectic as well and its set of fixed points actually forms a symplectic subspace:

Lemma 5.1.2. Let $\rho : G \to Sp(V; \omega)$ be a symplectic representation of a compact Lie group G on a symplectic vector space $(V; \omega)$. Then

$$V^G = \{ v \in V \mid \rho_g(v) = v \quad \text{for all} \quad g \in G \}$$

is a symplectic subspace of $(V; \omega)$.

Proof. Since G is compact, we can choose an ω-compatible almost complex structure J on V which is G-invariant. V^G is an almost complex subspace and therefore also a symplectic subspace.

This then immediately translates to the connected components of the fixed point set being symplectic submanifolds. It is useful to consider the extension of this result to the fixed point set under the action of a subgroup.

Proposition 5.1.3. *Let $(M;\omega)$ be a symplectic manifold and equipped with a symplectic action of Lie group G. If $H \subset G$ is a compact Lie subgroup, then the connected components of the set M^H of points which are fixed by H are symplectic submanifolds of M.*

Proof. We only show that the submanifolds are symplectic. This follows from the identification $T_p(M^H) = (T_pM)^H$ if $p \in M^H$ and that by Lemma 5.1.2 the subspace $(T_pM)^H$ is a symplectic subspace.

If $X \in T_p(M^H)$, then there exists a smooth curve $\gamma : \mathbb{R} \to M^H$ so that $\gamma(0) = p$ and $\dot\gamma(0) = X$. But then for any $h \in H$

$$
\begin{aligned}
d(\varphi_h)_p[X] &= d(\varphi_h)_p \circ d\gamma_0 \left(\frac{\partial}{\partial t}\bigg|_0\right) \\
&= d(\varphi_h \circ \gamma)_0 \left(\frac{\partial}{\partial t}\bigg|_0\right) \\
&= d(\gamma)_0 \left(\frac{\partial}{\partial t}\bigg|_0\right) \\
&= X
\end{aligned}
$$

and hence $X \in (T_pM)^H$ giving $T_p(M^H) \subset (T_pM)^H$.

To see the reverse inclusion, choose an H-invariant Riemannian metric on M, which is possible since H is compact and hence can be averaged over. For any $h \in H$, there is a commutative diagram for the Riemannian exponential map

$$
\begin{array}{ccc}
T_pM & \xrightarrow{\exp_p} & M \\
\downarrow{\scriptstyle d(\varphi_h)_p} & & \downarrow{\scriptstyle \varphi_h} \\
T_pM & \xrightarrow{\exp_p} & M
\end{array}
$$

Hence for $Y \in (T_pM)^H$, the curve $\gamma(t) = \exp_p(tY)$ lies in M^H as

$$
\begin{aligned}
\varphi_h(\exp_p(tY)) &= \exp_p(d(\varphi_h)_p[tY]) \\
&= \exp\left(t\, d(\varphi_h)_p[Y]\right) \\
&= \exp_p(tY)
\end{aligned}
$$

and is such that $\gamma(0) = p$ and $\dot\gamma(0) = Y$. Hence $Y \in T_p(M^H)$ which implies $(T_pM)^H \subset T_p(M^H)$ and concludes the proof.

5.2 Hamiltonian Actions

In the previous section we established that the fundamental vector fields induced by a symplectic action are symplectic vector fields, meaning that $\iota_{X^\sharp}\omega$ is closed. The core idea in the definition of a Hamiltonian action is now to require the fundamental vector fields to be Hamiltonian, meaning that $\iota_{X^\sharp}\omega$ is exact. To this end, we will introduce a generalisation of Hamiltonian functions called the moment map. Since we will require this moment map to be equivariant, we first have to recall the definition of the coadjoint representation.

Recall. 1. Let $c_g : G \to G$ be conjugation with the element $g \in G$. The derivative at the identity is a linear map

$$d(c_g)_e : T_eG \to T_{c_g(e)}G = T_eG$$

that is, identifying T_eG with the Lie algebra $\mathfrak{g}$ of G, we get a map $\mathrm{Ad} : G \to GL(\mathfrak{g})$ which is a homomorphism by the chain rule. This is called the adjoint representation of G on $\mathfrak{g}$:

$$\begin{aligned} \mathrm{Ad} : G &\to GL(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}_g := d(c_g)_e : \end{aligned}$$

2. The coadjoint representation of G on the dual $\mathfrak{g}^*$ is defined as the dual map of $\mathrm{Ad}_{g^{-1}}$ i.e.

$$\begin{aligned} \mathrm{Ad}^* : G &\to GL(\mathfrak{g}^*) \\ g &\mapsto \mathrm{Ad}^*_g := (\mathrm{Ad}_{g^{-1}})^* \end{aligned}$$

that is, for all $X \in \mathfrak{g}$ and $\varphi \in \mathfrak{g}^*$ we have

$$\langle \mathrm{Ad}^*_g[\varphi], X \rangle = \langle \varphi, \mathrm{Ad}_{g^{-1}}[X] \rangle :$$

3. Let G_1, G_2 be Lie groups which act on smooth manifolds M_1, M_2 via $\psi_1 : G_1 \to \mathrm{Diff}(M_1)$ and $\psi_2 : G_2 \to \mathrm{Diff}(M_2)$ respectively. Let further $\alpha : G_1 \to G_2$ be a Lie group homomorphism and $\varphi : M_1 \to M_2$ a smooth map. We say that φ is (ψ_1, ψ_2)-equivariant with respect to α if the following diagram commutes for all $g \in G_1$:

$$\begin{array}{ccc} M_1 & \xrightarrow{\ \varphi\ } & M_2 \\ \downarrow{\scriptstyle \psi_{1,g}} & & \downarrow{\scriptstyle \psi_{2,\alpha(g)}} \\ M_1 & \xrightarrow{\ \varphi\ } & M_2 \end{array}$$

Definition. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group on a symplectic manifold (M, ω). ψ is called a *Hamiltonian action* if there exists a map

$$\mu : M \to \mathfrak{g}^*$$

satisfying the following two conditions:

- For each element $X \in \mathfrak{g}$, the component of μ along X given by

$$\begin{aligned} \mu^X : M &\to \mathbb{R} \\ p &\mapsto \mu^X(p) := \langle \mu(p), X \rangle \end{aligned}$$

  is a Hamiltonian function for the fundamental vector field $X^\sharp$:

$$d\mu^X = \iota_{X^\sharp}\omega.$$

- μ is (ψ, Ad^*)-equivariant i.e. the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ \downarrow{\scriptstyle \psi_g} & & \downarrow{\scriptstyle \mathrm{Ad}^*_g} \\ M & \xrightarrow{\mu} & \mathfrak{g}^* \end{array}$$

(M, ω, G, μ) is called a **Hamiltonian G-space** and μ is called a **moment map**.

Example. Let G be a Lie group and (V, ω) a symplectic vector space. Recall that a symplectic representation is a group homomorphism $\rho : G \to \mathrm{Sp}(V, \omega)$. This induces a Lie algebra representation

$$\begin{aligned} \hat{\rho} : \mathfrak{g} &\to \mathfrak{sp}(V, \omega) \\ X &\mapsto \frac{d}{dt}\Big|_{t=0} \rho\big(\exp(tX)\big) \end{aligned}$$

so that for any $X \in \mathfrak{g}$ and any $v \in V$ we have

$$\hat{\rho}_X(v) = \frac{d}{dt}\Big|_{t=0} \rho\big(\exp(tX)\big)(v) = X^\sharp_v$$

where we identify $T_vV = V$. We put

$$\begin{aligned} \mu : V &\to \mathfrak{g}^* \\ v &\mapsto \Big(\mu(v) : X \mapsto \frac{1}{2}\omega\big(\hat{\rho}_X(v), v\big)\Big). \end{aligned}$$

and verify that this is a moment map:

1. First, we check (ρ, Ad^*)-equivariance by using Lemma A.2.3 in the appendix. Also, we identify $T_vV = V$ and linear maps with their differentials.

Then for any $X \in \mathfrak{g}$ and any $v \in V$ we have

$$\begin{aligned}
\langle \mu(\psi_g(v)), X \rangle &= \frac{1}{2}\omega\left(\hat{\psi}_X(\psi_g(v)), \psi_g(v)\right) \\
&= \frac{1}{2}\omega\left(X^\sharp_{\psi_g(v)}, \psi_g(v)\right) \\
&= \frac{1}{2}\omega\left([\mathrm{Ad}_g(\mathrm{Ad}_{g^{-1}}(X))]^\sharp_{\psi_g(v)}, \psi_g(v)\right) \\
&= \frac{1}{2}\omega\left(d(\psi_g)_v[\mathrm{Ad}_{g^{-1}}(X)]^\sharp_v, d(\psi_g)_v(v)\right) \\
&= \frac{1}{2}\omega\left([\mathrm{Ad}_{g^{-1}}(X)]^\sharp_v, v\right) \\
&= \frac{1}{2}\omega\left(\hat{\psi}_{\mathrm{Ad}_{g^{-1}}(X)}(v), v\right) \\
&= \left\langle \mu(v), \mathrm{Ad}_{g^{-1}}(X) \right\rangle \\
&= \left\langle \mathrm{Ad}^*_g \mu(v), X \right\rangle .
\end{aligned}$$

2. Secondly, we check that μ^X is the Hamiltonian function for the vector field $X^\sharp$. For any $u, v \in V$ we compute

$$\begin{aligned}
d\mu^X_v[u] &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0} \omega\left(\hat{\psi}_X(v+tu), v+tu\right) \\
&= \frac{1}{2}\left(\omega\left(\frac{d}{dt}\Big|_{t=0}\hat{\psi}_X(v+tu), v\right) + \omega\left(\hat{\psi}_X(v), \frac{d}{dt}\Big|_{t=0}(v+tu)\right)\right) \\
&= \frac{1}{2}\left(\omega(\hat{\psi}_X(u), v) + \omega(\hat{\psi}_X(v), u)\right) \\
&= \omega(\hat{\psi}_X(v), u) \\
&= \omega(X^\sharp_v, u) \\
&= \iota_{X^\sharp_v}\omega(u)
\end{aligned}$$

where we used that since $\hat{\psi}$ is a symplectic Lie algebra representation, it holds that

$$\omega(\hat{\psi}_X(u), v) + \omega(u, \hat{\psi}_X(v)) = 0 .$$

This proves that μ is indeed a moment map.

Just as for Hamiltonians, level sets of moment maps are of particular interest. By the implicit function theorem, the level sets of regular values are manifolds in their own right. In order to find which points are regular, one has to investigate the differential of the moment map.

Recall. 1. If $\psi : G \to \mathrm{Diff}(M)$ is a smooth action of a Lie group G on a smooth manifold M and $p \in M$ a point, the elements in G fixing p form a Lie subgroup of G called the *stabiliser subgroup*

$$G_p = \{ g \in G \mid \psi_g(p) = p \}.$$

The Lie algebra of G_p is called the *stabiliser algebra* and is the following subspace of $\mathfrak{g}$ (see Proposition A.3.3 in the appendix):

$$\mathfrak{g}_p = \left\{ X \in \mathfrak{g} \mid X^\sharp_p = 0 \right\}.$$

If the stabiliser subgroup G_p is trivial for a all points $p \in M$, the action is called *free*. If the stabiliser algebra $\mathfrak{g}_p$ is trivial for all points $p \in M$, then the action is called *locally free*. By a slight abuse of language we say that G acts freely/locally freely at a point $p \in M$, if the stabiliser subroup/algebra is trivial at p.

The *orbit* of p under G is the set of points in M that p can be mapped onto by an element of G:

$$O_p = \{ \psi_g(p) \mid g \in G \}.$$

The action $\psi : G \to \mathrm{Diff}(M)$ is called *transitive* if M consists of a single orbit.

2. If (V, ω) is a symplectic vector space and $W \subset V$ is a linear subspace, then its *symplectic orthocomplement* is the vector supspace

$$U^\omega = \{ v \in V \mid \omega(v, u) = 0 \quad \text{for all} \quad u \in U \}$$

and has dimension $\dim(V) - \dim(U)$.

Lemma 5.2.1. *Let (M, ω, G, μ) be a hamiltonian G-space. Then for every $p \in M$ we have*

$$\ker(d\mu_p) = (T_p(O_p))^\omega \quad \text{and} \quad \mathrm{im}(d\mu_p) = (\mathfrak{g}_p)^0.$$

Proof. By the first condition in the definition of a moment map we have

$$\langle d\mu_p[v], X \rangle = \omega_p(v, X^\sharp_p) \quad \text{for all} \quad X \in \mathfrak{g},\ v \in T_pM.$$

Claim. It holds that

$$d\mu_p[v] = 0 \quad \Longleftrightarrow \quad \omega_p(v, X^\sharp_p) = 0\,; \ \forall X \in \mathfrak{g}.$$

Proof of Claim. If $d\mu_p[v] = 0$, then for all $X \in \mathfrak{g}$

$$\omega_p(v, X^\sharp_p) = \langle d\mu_p[v], X \rangle = 0.$$

Reversely, assume that $\omega_p(v, X^\sharp_p) = 0$ for all $X \in \mathfrak{g}$. Then this holds in particular for the elements of any basis for $\mathfrak{g}$. If $d\mu_p[v]$ vanishes on all basis elements, it is identically zero.

Using this claim and that $T_p(\mathcal{O}_p)$ is spanned by the fundamental vector fields (see second point of Proposition A.3.5 in the appendix) we conclude

$$\ker(d\mu_p) = \left\{ v \in T_pM \mid \omega_p(v, X^\sharp_p) = 0 \ \forall X \in \mathfrak{g} \right\} = (T_p(\mathcal{O}_p))^{\omega_p}.$$

Next, we observe that

$$\begin{aligned} \dim(\ker(d\mu_p)) &= \dim((T_p\mathcal{O})^{\omega_p}) \\ &= \dim(T_pM) - \dim(T_p\mathcal{O}) \\ &= \dim(M) - \dim(\mathcal{O}) \\ &= \dim(M) - (\dim(G) - \dim(G_p)) \end{aligned}$$

so that

$$\begin{aligned} \dim(\mathrm{im}(d\mu_p)) &= \dim(T_pM) - \dim(\ker(d\mu_p)) \\ &= \dim(G) - \dim(G_p) \\ &= \dim(\mathfrak{g}) - \dim(\mathfrak{g}_p) \\ &= \dim\left((\mathfrak{g}_p)^0\right). \end{aligned}$$

Hence to prove the second result it suffices to show one inclusion. But it is clear that $\mathrm{im}(d\mu_p) \subseteq (\mathfrak{g}_p)^0$ since for $v \in T_pM$ and any $X \in \mathfrak{g}_p$ we observe

$$\langle d\mu_p[v], X \rangle = \omega_p(v, X^\sharp_p) = \omega_p(v, 0) = 0$$

as $\mathfrak{g}_p = \{X \in \mathfrak{g} \mid X^\sharp_p = 0\}$.

Corollary 5.2.2. Let (M, ω, G, μ) be a hamiltonian G-space and let $p \in M$ be a point. Then

1. p is a regular point of μ if and only if G acts locally freely at p and
2. the orbit $\mathcal{O}_p$ through p is open if and only if $d\mu_p$ is injective.

Proof. 1. Using Lemma 5.2.1 we get

$$\begin{aligned} \mathfrak{g}_p = 0 \quad &\iff \quad (\mathfrak{g}_p)^0 = \mathfrak{g}^* \\ &\iff \quad d\mu_p \text{ surjective.} \\ &\iff \quad p \text{ regular point of } \mu \end{aligned}$$

2. Similarly, by non-degeneracy of ω_p

$$\begin{aligned} d\mu_p \text{ injective} \quad &\iff \quad (T_p(\mathcal{O}_p))^{\omega_p} = 0 \\ &\iff \quad T_p(\mathcal{O}_p) = T_pM \\ &\iff \quad \dim(\mathcal{O}_p) = \dim(M) \\ &\iff \quad \mathcal{O}_p \text{ open.} \end{aligned}$$

An element $\varphi \in \mathfrak{g}^*$ is a regular value of the moment map if $\mu^{-1}(\varphi)$ only contains regular points. By the Corollary 5.2.2, this is the case if and only if G acts locally freely on $\mu^{-1}(\varphi)$. It then follows by the implicit function that $\mu^{-1}(\varphi)$ is an embedded submanifold of dimension $\dim(M) - \dim(G)$. However, this submanifold must not be invariant under the action. Indeed, by equivariance of the moment map, we see that for any $p \in \mu^{-1}(\varphi)$ and any $g \in G$, we have

$$\mu \circ \psi_g(p) = \mathrm{Ad}^*_g \circ \mu(p) = \mathrm{Ad}^*_g(\varphi)$$

so that $\psi_g(p)$ is in $\mu^{-1}(\varphi)$ if and only if $\mathrm{Ad}^*_g(\varphi) = \varphi$. Hence the submanifold $\mu^{-1}(\varphi)$ is invariant if and only if φ is a fixed point of the coadjoint representation. There are two important cases where this is guaranteed: The first is that $\varphi = 0$ and the second that G is Abelian, implying that the coadjoint representation is trivial.

5.3 Recursive Aspects of Hamiltonian Spaces

If a submanifold N of a Hamiltonian G-space (M, ω, G, μ), such as for instance $\mu^{-1}(0)$ explained above, is invariant under the G-action, the latter can restricted to this submanifold. In this section, we show that N naturally inherits the structure of a Hamiltonian G-space and a moment map. Similarly, we will see that M can also be seen as a Hamiltonian H-space for any Lie subgroup $H < G$ and we will investigate product constructions. At the end, the constructions of symplectic reduction and symplectic cutting are briefly presented and the question whether the resulting manifolds inherit Hamiltonian actions will be addressed in detail.

Before delving into the first case, we establish some technicalities that will be of use for several of these recursive aspects.

Lemma 5.3.1. *Let $\rho : G \to H$ be a Lie group homomorphism and let $\mathrm{Ad}^G : G \to GL(\mathfrak{g})$ and $\mathrm{Ad}^H : H \to GL(\mathfrak{h})$ be the two associated adjoint representations, $(\mathrm{Ad}^G)^* : G \to GL(\mathfrak{g}^*)$ and $(\mathrm{Ad}^H)^* : H \to GL(\mathfrak{h}^*)$ be the two associated coadjoint representations. Then the following two diagrams commute for all $g \in G$:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d_e\rho} & \mathfrak{h} \\ \big\downarrow \mathrm{Ad}^G_g & & \big\downarrow \mathrm{Ad}^H_{\rho(g)} \\ \mathfrak{g} & \xrightarrow{d_e\rho} & \mathfrak{h} \end{array} \qquad \begin{array}{ccc} \mathfrak{h}^* & \xrightarrow{d_e\rho^*} & \mathfrak{g}^* \\ \big\downarrow (\mathrm{Ad}^H)^*_{\rho(g)} & & \big\downarrow (\mathrm{Ad}^G)^*_g \\ \mathfrak{h}^* & \xrightarrow{d_e\rho^*} & \mathfrak{g}^* \end{array}$$

Proof. Denote the two conjugations by $c^G_g : G \to G$ and $c^H_h : H \to H$ respectively. Observe that for any $\tilde{g} \in G$ we have

$$\begin{aligned} \rho \circ c^G_g(\tilde{g}) &= \rho(g\tilde{g}g^{-1}) \\ &= \rho(g)\,\rho(\tilde{g})\,\rho(g)^{-1} \\ &= c^H_{\rho(g)} \circ \rho(\tilde{g}). \end{aligned}$$

Taking the differential at $e \in G$ we obtain the first commutative diagram and taking the dual yields the second.

Recall. 1. Let $\varphi : M \to N$ be a smooth map between smooth manifolds, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be vector fields. Then we say that X and Y are φ-related if

$$d\varphi_p[X_p] = Y_{\varphi(p)} \qquad \forall p \in M.$$

2. Let $\Phi : G \to \mathrm{Diff}(M)$ be an action of a Lie group on a smooth manifold M. An alternative description of the fundamental vector is via the orbit map of a given point $p \in M$

$$\begin{aligned} \Phi^p : G &\to M \\ g &\mapsto \Phi_g(p) \end{aligned}$$

as

$$X^\sharp_p = d(\Phi^p)_e[X].$$

Lemma 5.3.2. Let G_1, G_2 be Lie groups which act on smooth manifolds M_1, M_2 via $\Phi : G_1 \to \mathrm{Diff}(M_1)$ and $\Psi : G_2 \to \mathrm{Diff}(M_2)$ respectively. Let further $\rho : G_1 \to G_2$ be a Lie group homomorphism and $\varphi : M_1 \to M_2$ a smooth map which is (Φ, Ψ)-equivariant with respect to ρ. Then for every $X \in \mathfrak{g}_1$, $X^\sharp \in \mathfrak{X}(M_1)$ and $(d\rho_e[X])^\sharp \in \mathfrak{X}(M_2)$ are φ-related i.e. for all $p \in M_1$ we have

$$d\varphi_p[X^\sharp_p] = (d\rho_e[X])^\sharp_{\varphi(p)}.$$

Proof. First we observe that for every $p \in M_1$ and $g \in G_1$ we have

$$\begin{aligned} \varphi \circ \Phi^p(g) &= \varphi \circ \Phi_g(p) \\ &= \Psi_{\rho(g)} \circ \varphi(p) \\ &= \Psi^{\varphi(p)} \circ \rho(g) \end{aligned}$$

so that $\varphi \circ \Phi^p = \Psi^{\varphi(p)} \circ \rho$ for any $p \in M_1$. Using this we get

$$\begin{aligned} d\varphi_p[X^\sharp_p] &= d\varphi_{\Phi^p(e)} \circ d(\Phi^p)_e[X] \\ &= d(\varphi \circ \Phi^p)_e[X] \\ &= d\left(\Psi^{\varphi(p)} \circ \rho\right)_e[X] \\ &= d\left(\Psi^{\varphi(p)}\right)_e \circ d\rho_e[X] \\ &= (d\rho_e[X])^\sharp_{\varphi(p)} \end{aligned}$$

for any $p \in M_1$ which concludes the proof.

Lemma 5.3.3. Let M, N be smooth manifolds and $f : M \to N$ be a smooth map. If $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are f-related, then for all $\alpha \in \Omega(N)$ we have

$$f^*(\iota_Y \alpha) = \iota_X f^*\alpha$$

Proof. This is a straightforward computation: Take $p \in M$ and $\xi_1, \ldots, \xi_n \in T_pM$ and note that

$$\begin{aligned}
[f^*(\iota_Y \alpha)]_p(\xi_1, \ldots, \xi_n) &= (\iota_Y \alpha)_{f(p)}(df_p[\xi_1], \ldots, df_p[\xi_n]) \\
&= \alpha_{f(p)}\left(Y_{f(p)}, df_p[\xi_1], \ldots, df_p[\xi_n]\right) \\
&= \alpha_{f(p)}(df_p[X_p], df_p[\xi_1], \ldots, df_p[\xi_n]) \\
&= (f^*\alpha)_p(Y_p, \xi_1, \ldots, \xi_n) \\
&= (\iota_Y(f^*\alpha)_p)(\xi_1, \ldots, \xi_n).
\end{aligned}$$

We conclude since p and $\xi_1, \ldots, \xi_n \in T_pM$ were arbitrary.

5.3.1 Subgroup

Let now (M, ω, G, μ) be a Hamiltonian G-space and denote the action by $\psi : G \to \mathrm{Diff}(M)$. If $H < G$ is a Lie subgroup, then the composition of the inclusion $i : H \hookrightarrow G$ with ψ is an action of H on M. Differently put, there exists a unique action $\phi : H \to \mathrm{Diff}(M)$ such that the following diagram commutes for all $h \in H$:

$$\begin{array}{ccc}
M & \xrightarrow{\mathrm{Id}_M} & M \\
\downarrow{\scriptstyle \phi_h} & & \downarrow{\scriptstyle \psi_{i(h)}} \\
M & \xrightarrow{\mathrm{Id}_M} & M
\end{array}$$

or in words, the identity Id_M is (ϕ, ψ)-equivariant with respect to the inclusion $i : H \hookrightarrow G$.

Proposition 5.3.4. Let (M, ω, G, μ) be a Hamiltonian G-space. Let H be a Lie subgroup, $i : H \hookrightarrow G$ denote the inclusion and $di_e : \mathfrak{g} \to \mathfrak{h}$ the inclusion of the respective Lie algebras and $di_e^* : \mathfrak{g}^* \to \mathfrak{h}^*$ the map dual to its differential. Then $(M, \omega, H, di_e^* \circ \mu)$ is a Hamiltonian H-space.

Proof. It follows by Lemma 5.3.2 and equivariance of the identity that $X^\sharp = (di_e[X])^\sharp$. On the other hand, we note that $(di_e^* \circ \mu)^X = \mu^{di_e[X]}$ for all $X \in \mathfrak{h}$ since for any $p \in M$

$$\begin{aligned}
(di_e^* \circ \mu)^X(p) &= \langle di_e^* \mu(p), X \rangle \\
&= \langle \mu(p), di_e[X] \rangle \\
&= \mu^{di_e[X]}(p)
\end{aligned}$$

Combining those two facts we get

$$\begin{aligned} d(di_e \quad)^X &= d \quad {}^{di_e[X]} \\ &= \quad {}_{(di_e[X])]} ! \\ &= \quad {}_{X]} !: \end{aligned}$$

Using Lemma 5.3.1 and equivariance of we compute

$$\begin{aligned} (di_e \quad) \quad {}_h &= d\,i_e \qquad {}_{i(h)} \\ &= d\,i_e \quad Ad_{i(h)} \\ &= Ad_h \quad (di_e \quad) \end{aligned}$$

which shows(; Ad)-equivariance and thereby concludes the proof.

Considering the special case of the Lie subgroup de ned as the connected component containing the identity, we can formalise the intuitive fact that the moment map only sees the identity component of G:

Corollary 5.3.5. Let $(M; !; G; \ )$ be a hamiltonian G-space. Then the image of the moment map is completely determined by the action of the identity component G^0.

Proof. The identity component G^0 is a closed normal subgroup of G and so we can apply the Proposition 5.3.4: If $i : G^0 \ ,! \ G$ is the inclusion, then $(M; !; G^0; di_e \quad)$ is a hamiltonian G^0-space. However, since G^0 is the identity component, $di_e : g ! \ g$ is the identity. Hence also di_e is the identity and the result follows.

5.3.2 Submanifold

If a submanifold N M is G-invariant i.e. $_g(p)$ 2 N for all p 2 N and all g 2 G, the restriction $_g j_N$ is a smooth action of G on N. Another way of putting this is that there exists a unique action : G ! Di (N) of G on N such that the following diagram commutes for all g 2 G

$$\begin{array}{ccc} N & \overset{i}{\hookrightarrow} & M \\ \downarrow {\scriptstyle g} & & \downarrow {\scriptstyle g} \\ N & \overset{i}{\hookrightarrow} & M \end{array}$$

where $i : N \ ,! \ M$ is the inclusion. In words, the inclusion is(;)-equivariant.

Proposition 5.3.6. Let $(M; !; G; \ )$ be a hamiltonian G-space and let N be a G-invariant symplectic submanifold of M. Denote $i : N \ ,! \ M$ the inclusion map. Then $(N; i \ !; G; i \quad)$ is a hamiltonian G-space.

Proof. We check that $i^*\mu = \mu \circ i : N \to \mathfrak{g}^*$ is a moment map for $\tilde{\psi}$:

1. $i^*\mu$ is $(\tilde{\psi}; \mathrm{Ad}^*)$-equivariant since the inclusion is $(\tilde{\psi}; \psi)$-equivariant and μ is $(\psi; \mathrm{Ad}^*)$-equivariant: for all $g \in G$ we have

$$\begin{aligned} i^*\mu \circ \tilde{\psi}_g &= \mu \circ i \circ \tilde{\psi}_g \\ &= \mu \circ \psi_g \circ i \\ &= \mathrm{Ad}^*_g \circ \mu \circ i \\ &= \mathrm{Ad}^*_g \circ i^*\mu . \end{aligned}$$

2. For the second condition of a moment map, for any $X \in \mathfrak{g}$ first observe that $(i^*\mu)^X = i^*\mu^X$ as

$$\begin{aligned} (i^*\mu)^X(p) :&= \langle i^*\mu(p), X \rangle \\ &= \langle \mu(i(p)), X \rangle \\ &= (\mu^X)(i(p)) \\ &= i^*\mu^X(p). \end{aligned}$$

   Then, using again Lemmas 5.3.3 and 5.3.2 one finds

$$\begin{aligned} d(i^*\mu)^X &= d\, i^*\mu^X \\ &= i^*\, d\mu^X \\ &= i^*\iota_{X^\sharp}\omega \\ &= \iota_{X^\sharp} i^*\omega \end{aligned}$$

   which is exactly what is needed since $i^*\omega$ is the symplectic form on N.

5.3.3 Products

Recall. Let $(M_1; \omega_1)$ and $(M_2; \omega_2)$ be two symplectic manifolds. Consider the product $M_1 \times M_2$ together with the two projections

$$\begin{aligned} \mathrm{pr}_1 : M_1 \times M_2 &\to M_1 \\ (p; q) &\mapsto p \end{aligned}$$

and

$$\begin{aligned} \mathrm{pr}_2 : M_1 \times M_2 &\to M_2 \\ (p; q) &\mapsto q. \end{aligned}$$

Then for any $\lambda_1; \lambda_2 \in \mathbb{R}\setminus\{0\}$, $(M_1 \times M_2; \lambda_1 \mathrm{pr}_1^*\omega_1 + \lambda_2 \mathrm{pr}_2^*\omega_2)$ is again a symplectic manifold.

Indeed, closedness is clear since exterior derivation is linear and commutes with the pullbacks and nondegeneracy can be seen from the following computation (where we write $m_1 = \dim(M_1)$ and $m_2 = \dim(M_2)$):

$$(\;_1 pr_1^* \omega_1 + \;_2 pr_2^* \omega_2)^{m_1+m_2} = \;_1^{m_1} \;_2^{m_2} \binom{m_1+m_2}{m_1} ((pr_1^* \omega_1)^{m_1} \wedge (pr_2^* \omega_2)^{m_2}) \neq 0 .$$

The first case that one might consider is if one group G acts on both M_1 and M_2. Suppose thus that $\;: G \to \mathrm{Diff}(M_1)$ and $\;: G \to \mathrm{Diff}(M_2)$ are two actions of a single Lie group G on M_1 and M_2 respectively. Then there is an induced action of G on the product $M_1 \; M_2$ as

$$\begin{aligned} &: G \to \mathrm{Diff}(M_1 \; M_2) \\ &\quad g \mapsto (\;_g : (p_1; p_2) \mapsto (\;_g(p_1); \;_g(p_2)) . \end{aligned}$$

For each $g \in G$ the diagrams

$$\begin{array}{ccc} M_1 \; M_2 & \xrightarrow{pr_1} & M_1 \\ \downarrow \;_g & & \downarrow \;_g \\ M_1 \; M_2 & \xrightarrow{pr_1} & M_1 \end{array} \qquad \begin{array}{ccc} M_1 \; M_2 & \xrightarrow{pr_2} & M_2 \\ \downarrow \;_g & & \downarrow \;_g \\ M_1 \; M_2 & \xrightarrow{pr_2} & M_2 \end{array}$$

commute and it follows that if and are symplectic actions, then is a symplectic action as well:

$$\begin{aligned} \;_g^*(\;_1 pr_1^* \omega_1 + \;_2 pr_2^* \omega_2) &= \;_1 \;_g^*(pr_1^* \omega_1) + \;_2 \;_g^*(pr_2^* \omega_2) \\ &= \;_1(pr_1 \; \;_g)^* \omega_1 + \;_2(pr_2 \; \;_g)^* \omega_2 \\ &= \;_1(\;_g \; pr_1)^* \omega_1 + \;_2(\;_g \; pr_2)^* \omega_2 \\ &= \;_1 pr_1^*(\;_g^* \omega_1) + \;_2 pr_2^*(\;_g^* \omega_2) \\ &= \;_1 pr_1^* \omega_1 + \;_2 pr_2^* \omega_2 . \end{aligned}$$

Lemma 5.3.7. *Let $(M_1; \omega_1; G; \;_1)$ and $(M_2; \omega_2; G; \;_2)$ be two hamiltonian G-spaces. Then*

$$(M_1 \; M_2; pr_1^* \omega_1 + pr_2^* \omega_2; G; \;_1 \; pr_1 + \;_2 \; pr_2)$$

is also a Hamiltonian G-space.

Proof. We check that $\; := \;_1 \; pr_1 + \;_2 \; pr_2$ is indeed a moment map:

1. $(Ad^* ; \;)$-equivariance is straightforward by equivariance of $\;_1$ and $\;_2$ as well as the two commutative diagrams for $\;_g$ above: For any $g \in G$ one has

$$\begin{aligned} \; \;_g &= \;_1 \; pr_1 \; \;_g + \;_2 \; pr_2 \; \;_g \\ &= \;_1 \; \;_g \; pr_1 + \;_2 \; \;_g \; pr_2 \\ &= Ad_g^* \; \;_1 \; pr_1 + Ad_g^* \; \;_2 \; pr_2 \\ &= Ad_g^* \; . \end{aligned}$$

2. First, we observe that for any $X \in \mathfrak{g}$ it holds that $\mu^X = \mu_1^X \circ pr_1 + \mu_2^X \circ pr_2$ since

$$\begin{aligned}
\mu^X(p_1;p_2) &= \langle \mu(p_1;p_2); X\rangle \\
&= \langle \mu_1(p_1) + \mu_2(p_2); X\rangle \\
&= \langle \mu_1(p_1); X\rangle + \langle \mu_2(p_2); X\rangle \\
&= \mu_1^X(p_1) + \mu_2^X(p_2) \\
&= (\mu_1^X \circ pr_1 + \mu_2^X \circ pr_2)(p_1;p_2)
\end{aligned}$$

for any $(p_1;p_2) \in M_1 \times M_2$. Now the result follows by the chain rule, Lemma 5.3.2 and Lemma 5.3.3:

$$\begin{aligned}
d\mu^X &= d\left(\mu_1^X \circ pr_1 + \mu_2^X \circ pr_2\right) \\
&= d\mu_1^X \circ dpr_1 + d\mu_2^X \circ dpr_2 \\
&= -\iota_{X^\sharp}\omega_1 \circ dpr_1 - \iota_{X^\sharp}\omega_2 \circ dpr_2 \\
&= -pr_1^*(\iota_{X^\sharp}\omega_1) - pr_2^*(\iota_{X^\sharp}\omega_2) \\
&= -\iota_{X^\sharp}(pr_1^*\omega_1) - \iota_{X^\sharp}(pr_2^*\omega_2) \\
&= -\iota_{X^\sharp}(pr_1^*\omega_1 + pr_2^*\omega_2):
\end{aligned}$$

Consider now the reverse situation and suppose that for a given smooth manifold, two different Lie groups G and H act via $\psi: G \to \mathrm{Diff}(M)$ and $\phi: H \to \mathrm{Diff}(M)$. If the two actions commute, that is, for all $g \in G$ and all $h \in H$ the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\psi_g} & M \\
\downarrow{\scriptstyle \phi_h} & & \downarrow{\scriptstyle \phi_h} \\
M & \xrightarrow{\psi_g} & M
\end{array}$$

commutes, then the product group $G \times H$ acts naturally on M by

$$\begin{aligned}
\theta: G \times H &\to \mathrm{Diff}(M) \\
(g;h) &\mapsto \psi_g \circ \phi_h:
\end{aligned}$$

This is a well-defined action as

$$\begin{aligned}
\theta_{(\tilde g;\tilde h)} \circ \theta_{(g;h)} &= \psi_{\tilde g} \circ \phi_{\tilde h} \circ \psi_g \circ \phi_h \\
&= \psi_{\tilde g} \circ \psi_g \circ \phi_{\tilde h} \circ \phi_h \\
&= \psi_{\tilde g g} \circ \phi_{\tilde h h} \\
&= \theta_{(\tilde g g;\tilde h h)} \\
&= \theta_{(\tilde g;\tilde h)\,(g;h)}:
\end{aligned}$$

Again, if $(M;\omega)$ is a symplectic manifold and and are symplectic actions, so is . This is clear since

$$\begin{aligned} {}_{(g;h)}\omega &= (\quad {}_g \quad {}_h) \ \omega \\ &= {}_h \quad {}_g\omega \\ &= {}_h\omega \\ &= \omega: \end{aligned}$$

Assuming that both and are hamiltonian, with moment maps $_G$ and $_H$ respectively, is as defined above hamiltonian as well? The natural candidate for the moment map is

$$\begin{aligned} : M &\to g \quad h \\ p &\mapsto d(pr_G)_{(e;e)} \quad {}_G(p) + d(\ pr_H)_{(e;e)} \quad {}_H(p) \end{aligned}$$

where $pr_{G;H} : G \quad H \to G;H$ are the projections so that $d(pr_{G;H})_{(e;e)}$ just correspond to the inclusions $g\ ;h \hookrightarrow g \quad h$. We start by noticing that for all $p \in M$ and all $X \in g \quad h$ we have

$$\begin{aligned} {}^X(p) &= (d(\ pr_G)_{(e;e)} \quad {}_G + d(\ pr_H)_{(e;e)} \quad {}_H)^X(p) \\ &= \Big\langle d(pr_G)_{(e;e)} \quad {}_G(p) + d(\ pr_H)_{(e;e)} \quad {}_H(p); X \Big\rangle \\ &= \Big\langle {}_G(p); d(pr_G)_{(e;e)}[X] \Big\rangle + \Big\langle {}_H(p); d(pr_H)_{(e;e)}[X] \Big\rangle \\ &= \quad {}_G^{d(pr_G)_{(e;e)}[X]} + {}_H^{d(pr_H)_{(e;e)}[X]} \quad (p) \end{aligned}$$

so that

$$\begin{aligned} d \ {}^X &= d \ {}_G^{d(pr_G)_{(e;e)}[X]} + d \ {}_H^{d(pr_H)_{(e;e)}[X]} \\ &= \quad (d(\ pr_G)_{(e;e)}[X])^{]}\ \omega \qquad (d(\ pr_H)_{(e;e)}[X])^{]}\ \omega \\ &= \quad (d(\ pr_G)_{(e;e)}[X])^{]} + (d(\ pr_H)_{(e;e)}[X])^{]}\ \omega: \end{aligned}$$

On the other hand, we see that the fundamental vector field of the product action at a point $p \in M$ is

$$\begin{aligned} X^{]}_p &= d(\ {}^p)_{(e;e)}[X] \\ &= d_1(\ {}^p)_{(e;e)}[X] + d_2(\ {}^p)_{(e;e)}[X] \end{aligned}$$

where d_1 denotes the derivative with regard to the first entry while leaving the second fixed at e and similarly for d_2. By the definition of the product action , for any g 2 G_1 it holds that

$$
\begin{aligned}
{}^{p}(g;e) &= \ {}_{(g;e)}(p) \\
&= \ {}_{g} \quad {}_{e}(p) \\
&= \ {}_{g}(p) \\
&= \ {}^{p}(g) \\
&= \ {}^{p} \quad pr_{G}(g;e)
\end{aligned}
$$

and analogously for the second entry. It follows that the fundamental vector eld of the product action can be written as the sum of the fundamental vector elds of the two factors:

$$
\begin{aligned}
X^{]}_{p} &= d_{1}(\ {}^{p} \quad pr_{G})_{(e;e)}[X] + d_{2}(\ {}^{p} \quad pr_{H})_{(e;e)}[X] \\
&= d(\ {}^{p})_{e} \quad d(pr_{G})_{(e;e)}[X] + d(\ {}^{p})_{e} \quad d(pr_{H})_{(e;e)}[X] \\
&= (d(\ pr_{G})_{(e;e)}[X])^{]}_{p} + (d(\ pr_{H})_{(e;e)}[X])^{]}_{p}:
\end{aligned}
$$

Combining this with the above gives

$$
d \ {}^{X} = \ {}_{X^{]}}!:
$$

In order to get (; Ad)-equivariance, an additional hypothesis is needed. Namely, we have to require that ${}_{G}$ is -invariant and that reversely ${}_{H}$ is -invariant. Assuming this and using Lemma 5.3.1 we can compute that

$$
\begin{aligned}
{}_{(g;h)} &= (d(\ pr_{G})_{(e;e)} \quad {}_{G} + d(\ pr_{H})_{(e;e)} \quad {}_{H}) \quad {}_{g} \quad {}_{h} \\
&= d(\ pr_{G})_{(e;e)} \quad {}_{G} \quad {}_{g} + d(\ pr_{H})_{(e;e)} \quad {}_{H} \quad {}_{h} \\
&= d(\ pr_{G})_{(e;e)} \quad (Ad^{G}_{g}) \quad {}_{G} + d(\ pr_{H})_{(e;e)} \quad (Ad^{H}_{h}) \quad {}_{H} \\
&= (\ Ad^{G \ H}_{(g;h)})
\end{aligned}
$$

showing that these additional assumptions are indeed su cient. We have thus shown the following

Lemma 5.3.8. Let G and H be two Lie groups which act in a Hamiltonian way on a symplectic manifold (M; !) such that the actions commute and the moment maps ${}_{G}$: M ! g and ${}_{H}$: M ! h are invariant under the respective other action. If $pr_{G;H}$: G H ! G; H denote the projections on the factors, then

$$
(M; !; G \quad H; d(pr_{G})_{(e;e)} \quad {}_{G} + d(\ pr_{H})_{(e;e)} \quad {}_{H})
$$

is a Hamiltonian G H-space.

Putting the two last results together one gets the following consequence:

Proposition 5.3.9. Let $(M_1, \omega_1, G, \mu_1)$ be a Hamiltonian G-space, $(M_2, \omega_2, H, \mu_2)$ a Hamiltonian H-space and let $pr_{1,2} : M_1 \times M_2 \to M_{1,2}$ and $pr_{G,H} : G \times H \to G, H$ denote the projections on the factors. Then

$$(M_1 \times M_2, pr_1^*\omega_1 + pr_2^*\omega_2, G \times H, d(pr_G)^*_{(e,e)} \circ \mu_1 \circ pr_1 + d(pr_H)^*_{(e,e)} \circ \mu_2 \circ pr_2)$$

is a Hamiltonian $G \times H$-space.

Proof. G acts on M_2 by the trivial action in a Hamiltonian way and the corresponding moment map is zero. The two actions on M_2 hence clearly satisfy the hypotheses of Lemma 5.3.8 and we get a Hamiltonian action of $G \times H$ on M_2 whose moment map is

$$\tilde{\mu}_2 = d(pr_H)^*_{(e,e)} \circ \mu_2.$$

Similarly, $G \times H$ acts in a Hamiltonian way on M_1 and the moment map is given by

$$\tilde{\mu}_1 = d(pr_G)^*_{(e,e)} \circ \mu_1.$$

The result now follows directly from Lemma 5.3.7.

Remark. Note that for making the notation not heavier than it already is, we stuck to the case $\lambda_1 = \lambda_2 = 1$. By linearity of all the constructions involved, the same arguments also go through for all other choices of $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$. In particular, one could use the *twisted product form* that is obtained by choosing $\lambda_1 = 1$ and $\lambda_2 = -1$ so that we get another Hamiltonian $G \times H$-space

$$(M_1 \times M_2, pr_1^*\omega_1 - pr_2^*\omega_2, G \times H, d(pr_G)^*_{(e,e)} \circ \mu_1 \circ pr_1 - d(pr_H)^*_{(e,e)} \circ \mu_2 \circ pr_2).$$

5.3.4 Symplectic Reduction

Let us now come back to the level sets of the moment map. In section 5.2 it was established that $\mu^{-1}(\varphi)$ was a smooth submanifold if the action restricted to $\mu^{-1}(\varphi)$ was locally free. Moreover, it was argued that $\mu^{-1}(\varphi)$ is invariant under the action if the acting group G is Abelian or if $\varphi = 0$. One might be tempted to deduce from Proposition 5.3.6 that the restriction of the action to $\mu^{-1}(\varphi)$ is also a Hamiltonian action, but this is in general not possible since $\mu^{-1}(\varphi)$ does not have to be a symplectic submanifold. Indeed, the tangent space to a point $p \in \mu^{-1}(\varphi)$ can be identified as

$$T_p(\mu^{-1}(\varphi)) = \ker(d\mu_p) = (T_p(\mathcal{O}_p))^{\omega_p}$$

using Lemma 5.2.1. But if $\varphi = 0$ or if G is Abelian, then equivariance becomes invariance on $\mu^{-1}(\varphi)$, meaning in particular that μ is constant on the orbits. It follows that $d\mu_p : T_pM \to \mathfrak{g}^*$ maps $T_p(\mathcal{O}_p)$ to zero i.e. $T_p(\mathcal{O}_p) \subset \ker(d\mu_p) = (T_p(\mathcal{O}_p))^{\omega_p}$. Hence $T_p(\mathcal{O}_p) \subset T_pM$ is an isotropic subspace. The key to defining a symplectic structure is then following lemma from symplectic linear algebra:

Lemma 5.3.10. *Let (V, ω) be a symplectic vector space and suppose that $U \subset V$ is an isotropic subspace i.e. $U \subset U^{\omega}$ or equivalently $\omega|_{U \times U} = 0$. Then ω induces a canonical symplectic form $\tilde{\omega}$ on U^{ω}/U.*

Proof. ([5], Lemma 23.3.) Let $v, w \in U^{\omega}$ and write $[v], [w] \in U^{\omega}/U$ for their equivalence classes. Define then a 2-form on the quotient as

$$\begin{aligned} \tilde{\omega} : U^{\omega}/U \times U^{\omega}/U &\to \mathbb{R} \\ ([v], [w]) &\mapsto \omega(v, w) \end{aligned}$$

and check that this is

1. well-defined since for any $u, u' \in U$

$$\omega(v + u, w + u') = \omega(v, w) + \underbrace{\omega(v, u')}_{=0} + \underbrace{\omega(u, w)}_{=0} + \underbrace{\omega(u, u')}_{=0} = \omega(v, w)$$

by definition of the symplectic orthocomplement and

2. non-degenerate: If $v \in U^{\omega}$ is such that $\omega(v, w) = 0$ for all $w \in U^{\omega}$, then $v \in (U^{\omega})^{\omega} = U$ so that $[v] = 0$ in the quotient.

The idea is thus to get the quotient of $T_p(\mu^{-1}(\varphi)) = (T_p(\mathcal{O}_p))^{\omega_p}$ by $T_p(\mathcal{O}_p)$ as tangent space. It would therefore be natural to consider the orbit space $\mu^{-1}(\varphi)/G$. However, for this to be a manifold, we need a stronger hypothesis: If we assume that the action restricted to $\mu^{-1}(\varphi)$ is free, not only locally free, the quotient $\mu^{-1}(\varphi)/G$ is actually a manifold. Assembling all those arguments, one can then prove the following theorem:

Theorem 5.3.11 (Marsden-Weinstein, Meyer)**.** *Let (M, ω, G, μ) be a hamiltonian G-space for a compact Lie group G and take $\varphi \in \mu(M)$. Assume that $\varphi = 0$ or that G is Abelian. Let $i : \mu^{-1}(\varphi) \hookrightarrow M$ denote the inclusion map and assume further that G acts freely on $\mu^{-1}(\varphi)$. Then*

1. *the orbit space $M_{red} = \mu^{-1}(\varphi)/G$ is a manifold,*

2. *$\pi : \mu^{-1}(\varphi) \to M_{red}$ is a principal G-bundle and*

3. *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{red}$.* ∎

Definition. The pair (M_{red}, ω_{red}) is called the *symplectic reduction* or the *symplectic quotient* of (M, ω) by G and φ.

Residual Symmetries

Assume now that we are in this situation and that (M, ω, H, μ_H) is a hamiltonian H-space and that (M_{red}, ω_{red}) is its symplectic quotient by H and μ_H. Suppose then further that there is another Lie group G that acts on M via $\psi : G \to \mathrm{Diff}(M)$. Under what conditions does the action ψ descend to the symplectic quotient M_{red}?

The first thing one needs to check is that $\mu_H^{-1}(\varphi)$ is G-invariant. The natural condition to ensure this, is requiring the moment map μ_H to be ψ-invariant i.e. for all $g \in G$:

$$\mu_H \circ \psi_g = \mu_H.$$

If this holds, one can restrict ψ_g to $\mu_H^{-1}(\varphi)$ and gets a commutative diagram

$$\begin{array}{ccc} \mu_H^{-1}(\varphi) & \overset{i}{\hookrightarrow} & M \\ \downarrow \tilde{\psi}_g & & \downarrow \psi_g \\ \mu_H^{-1}(\varphi) & \overset{i}{\hookrightarrow} & M \end{array}$$

Next, one has to check whether the action ψ restricted to $\mu_H^{-1}(\varphi)$ descends to the quotient $\mu_H^{-1}(\varphi)/H$. Again there is a natural hypothesis to ensure this, namely that the actions of H and G on M commute: for any $g \in G$ and any $h \in H$ we require that

$$\psi_g \circ \psi_h = \psi_h \circ \psi_g.$$

Using this, we can define the action

$$\begin{aligned} \bar{\psi} : G &\to \mathrm{Diff}(M_{red}) \\ g &\mapsto \bar{\psi}_g : \mathcal{O}_p \mapsto \mathcal{O}_{\psi_g(p)}. \end{aligned}$$

Indeed, this is well defined: If $q = \psi_h(p)$ is another representative of the orbit $\mathcal{O}_p$, then

$$\psi_g(q) = \psi_g(\psi_h(p)) = \psi_h(\psi_g(p))$$

shows that $\mathcal{O}_{\psi_g(p)} = \mathcal{O}_{\psi_g(q)}$. Note also that this is the unique action such that the bundle map $\pi : \mu_H^{-1}(\varphi) \to M_{red}$ is $(\tilde{\psi}, \bar{\psi})$-equivariant.

There is another convenient way of writing this action $\bar{\psi}$ which also makes explicit that it is smooth. Consider for this a point $\mathcal{O}_p \in M_{red}$ and an open neighbourhood $U_{\mathcal{O}_p} \subset M_{red}$ of it together with any smooth local section $s : U_{\mathcal{O}_p} \to \mu_H^{-1}(\varphi)$ of the principal bundle $\pi : \mu_H^{-1}(\varphi) \to M_{red}$. Then we can define locally

$$\bar{\psi}_g = \pi \circ \tilde{\psi}_g \circ s.$$

This does not depend on the choice of section since H acts transitively on the fibres of π: The images of two different sections are mapped onto each other by

$_h$ for some $h \in H$. Hence these local definitions can be glued together to define $_g$ on all of M_{red} in this way.

Claim. If $: G \to \mathrm{Diff}(M)$ is symplectic, then so is $: G \to \mathrm{Diff}(M_{red})$.

Proof of Claim. Using the local sections, we see that for any $g \in G$

$$
\begin{aligned}
{}_g \omega_{red} &= (\quad \tilde{}_g \quad s)\ \omega_{red} \\
&= s \quad (\tilde{}_g) \qquad \omega_{red} \\
&= s \quad (\tilde{}_g) \quad i\ \omega \\
&= s \quad (i \quad \tilde{}_g)\ \omega \\
&= s\ (\ {}_g \quad i)\ \omega \\
&= s \quad i \qquad {}_g \omega \\
&= s \quad i\ \omega \\
&= s \qquad \omega_{red} \\
&= (\qquad s)\ \omega_{red} \\
&= \omega_{red}
\end{aligned}
$$

and hence that is a symplectic action on $(M_{red}; \omega_{red})$.

Claim. If $: G \to \mathrm{Diff}(M)$ is Hamiltonian with a H-invariant moment map $_G : M \to g$ i.e. $_G \quad _h = \ _G$ for all $h \in H$, then $: G \to \mathrm{Diff}(M_{red})$ is Hamiltonian with a moment map $_G : M_{red} \to g$ satisfying $_G \quad = \quad _G \quad i$. Moreover, this condition completely determines the moment map $_G$.

Proof of Claim. By H-invariance $_G$ descends to a map

$$
\begin{aligned}
{}_G : M_{red} &\to g \\
O_p &\to \quad {}_G(i(p))
\end{aligned}
$$

or again, using a local sections we can write $_G = \ _G \quad i \quad s$. It is clear that this satisfies $_G \quad = \quad _G \quad i$. Since is surjective, this property completely characterises the moment map. We can then verify that this is indeed a Hamiltonian moment map:

1. $(\ ; \mathrm{Ad} \ )$-equivariance holds since $_G$ is $(\ ; \mathrm{Ad} \ )$-equivariant: For any $g \in G$ we observe

$$
\begin{aligned}
{}_G \quad {}_g &= {}_G \qquad \tilde{}_g \quad s \\
&= {}_G \quad i \quad \tilde{}_g \quad s \\
&= {}_G \quad {}_g \quad i \quad s \\
&= \mathrm{Ad}_g \quad {}_G \quad i \quad s \\
&= \mathrm{Ad}_g \quad {}_G \qquad s \\
&= \mathrm{Ad}_g \quad {}_G :
\end{aligned}
$$

2. To check the Hamiltonian property, we need to verify that

$$\left\langle d(\bar{\mu}_G)_{O_p}[v], X\right\rangle = (\omega_{red})_{O_p}(v, X^\sharp_{O_p})$$

for any $X \in \mathfrak{g}$ and any $v \in T_{O_p}(M_{red})$. Since π is submersion, for any $v \in T_{O_p}(M_{red})$ there is a $w \in T_p(\mu_H^{-1}(\varphi))$ such that $d\pi_p[w] = v$. Using this we can compute

$$\begin{aligned}
\left\langle d(\bar{\mu}_G)_{O_p}[v], X\right\rangle &= \left\langle d(\bar{\mu}_G)_{O_p} \circ d\pi_p[w], X\right\rangle \\
&= \left\langle d(\bar{\mu}_G \circ \pi)_p[w], X\right\rangle \\
&= \left\langle d(\mu_G \circ i)_p[w], X\right\rangle \\
&= \left\langle d(\mu_G)_{i(p)} \circ di_p[w], X\right\rangle \\
&= \omega_{i(p)}(di_p[w], X^\sharp_{i(p)}) \\
&= \omega_{i(p)}(di_p[w], di_p[X^\sharp_{i(p)}]) \\
&= (i^*\omega)_p(w, X^\sharp_p) \\
&= (\pi^*\omega_{red})_p(w, X^\sharp_p) \\
&= (\omega_{red})_{\pi(p)}(d\pi_p[w], d\pi_p[X^\sharp_p]) \\
&= (\omega_{red})_{O_p}(v, X^\sharp_{O_p})
\end{aligned}$$

where we used repeatedly the chain rule and Lemma 5.3.2 as well as the symplectic reduction theorem 5.3.11.

We summarise this discussion:

Proposition 5.3.12. Let (M, ω, H, μ_H) be a Hamiltonian H-space where the group action is denoted as $\psi : H \to \mathrm{Diff}(M)$. Assume that there is another Hamiltonian group action $\tau : G \to \mathrm{Diff}(M)$ by a second Lie group G which commutes with ψ and such that μ_H is G-invariant and μ_G is H-invariant. Then the symplectic quotient $(M_{red}, \omega_{red}, G, \bar{\mu}_G)$ with $M_{red} = \mu_H^{-1}(\varphi)/H$, ω_{red} and $\bar{\mu}_G$ defined by satisfying $\bar{\mu}_G \circ \pi = \mu_G \circ i$, is a Hamiltonian G-space.

5.3.5 Symplectic Cutting

This presentation of the basic construction of symplectic cutting follows [8].

Convention. For this section we will identify the dual $\mathfrak{s}^*$ of the Lie algebra $\mathfrak{s}$ of S^1 with $\mathbb{R}$ in the following way: Let $v \in \mathfrak{s} \setminus \{0\}$ be a fixed non-zero vector. Each element $\varphi \in \mathfrak{s}^*$ gives a unique real number c by evaluation at v, that is, $\langle \varphi, v\rangle = c \in \mathbb{R}$. Reversely, since $\mathfrak{s}$ is one-dimenstional and $\{v\}$ is hence a basis, any value $c \in \mathbb{R}$ defines an element $\varphi \in \mathfrak{s}^*$ by setting $\langle \varphi, v\rangle = c$ and extending linearly.

Let $(M, \omega, S^1, \mu_{S^1})$ be a Hamiltonian S^1-space. Parametrise S^1 by $\theta \in [0, 2\pi[$ and let $\psi : S^1 \to \mathrm{Diff}(M)$ denote the action. S^1 also acts on the complex plane $\mathbb{C}$ with the standard symplectic structure $\omega_0 = \frac{i}{2} dz \wedge d\bar{z}$ by multiplication

$$\phi : S^1 \to \mathrm{Diff}(\mathbb{C})$$
$$\theta \mapsto \phi_\theta : z \mapsto e^{-i\theta} z.$$

This action is Hamiltonian and the corresponding moment map is

$$\mu_{\mathbb{C}} : \mathbb{C} \to \mathbb{R} = \mathfrak{s}^*$$
$$z \mapsto \frac{1}{2}|z|^2.$$

By Lemma 5.3.7, the product $M \times \mathbb{C}$ is again a Hamiltonian S^1-space:

$$(M \times \mathbb{C}, \mathrm{pr}_M^* \omega + \mathrm{pr}_{\mathbb{C}}^* \omega_0, S^1, \mu_{S^1} \circ \mathrm{pr}_M + \mu_{\mathbb{C}} \circ \mathrm{pr}_{\mathbb{C}})$$

where the moment map is given explicitly as

$$\mu : M \times \mathbb{C} \to \mathbb{R}$$
$$(p, z) \mapsto \mu_{S^1}(p) + \frac{1}{2}|z|^2.$$

For an arbitrary $c \in \mathbb{R}$, the level set is

$$\mu^{-1}(c) = \left(\mu_{S^1}^{-1}(c) \times \{0\}\right) \sqcup \bigsqcup_{r>0} \left(\mu_{S^1}^{-1}(c-r) \times \left\{\frac{1}{2}|z|^2 = r\right\}\right).$$

In other words, for each $c \in \mathbb{R}$, the preimage $\mu^{-1}(c)$ is a disjoint union of two S^1-invariant (S^1 is Abelian) subsets where the first can be identified with

$$\mu_{S^1}^{-1}(c) \times \{0\} = \mu_{S^1}^{-1}(c) = \{p \in M \mid \mu_{S^1}(p) = c\}$$

and the second with

$$\bigsqcup_{r>0} \left(\mu_{S^1}^{-1}(c-r) \times \left\{\frac{1}{2}|z|^2 = r\right\}\right) = \{p \in M \mid \mu_{S^1}(p) < c\} \times S^1$$

where we identify $\{\frac{1}{2}|z|^2 = r\} = \{\sqrt{2r}e^{i\varphi} \mid \varphi \in [0, 2\pi[\} = S^1$.

If S^1 acts freely on $\mu_{S^1}^{-1}(c)$, then it also acts freely on $\mu^{-1}(c)$. Indeed, it also acts freely on the second part since the S^1 action on $\mathbb{C}$ is free except at the origin. It follows that one can find a symplectic quotient by S^1 and μ as in Theorem 5.3.11 at such a value $c \in \mathbb{R}$.

Definition. We will denote the resulting space by (M_c, ω_c) and call it the symplectic cut of M below c with respect to μ_{S^1}.

Because the decomposition of $\mu^{-1}(c)$ into two disjoint components above was S^1-invariant, also $M_{\leq c}$ is the union of two disjoint components: The first one can be identified with the symplectic quotient of M at $c \in \mathbb{R}$

$$\mu_{S^1}^{-1}(c) \big/ S^1 := M_c$$

and the second with

$$\{p \in M \mid \mu_{S^1}(p) > c\} \times S^1 \big/ S^1 = \{p \in M \mid \mu_{S^1}(p) > c\} := M_{<c}.$$

This identification is given explicitly by

$$\begin{aligned} \phi : \left. M_{<c} \times S^1 \right/ S^1 &\to M_{<c} \\ O_{(p,\varphi)} &\mapsto \psi_{-\varphi}(p) \end{aligned}$$

where $\psi : S^1 \to \mathrm{Diff}(M)$ still denotes the S^1-action and where $O_{(p,\varphi)}$ denotes the S^1-orbit of $(p,\varphi) \in M_{<c} \times S^1$. Note that this is well-defined since any $\theta \in S^1$ acts on $(p,\varphi) \in M \times S^1$ by mapping it onto $(\psi_\theta(p), \varphi + \theta)$ and that

$$\phi(O_{(\psi_\theta(p),\varphi+\theta)}) = \psi_{-\varphi-\theta}(\psi_\theta(p)) = \psi_{-\varphi}(p).$$

The inverse is given explicitly by

$$\begin{aligned} \chi : M_{<c} &\to \left. M_{<c} \times S^1 \right/ S^1 \\ p &\mapsto O_{(p,0)} \end{aligned}$$

and where we recall that S^1 is still parametrised by the angle so that the 0 in this equation does not correspond to the origin of the complex plane but the identity element in S^1. Indeed one can check that ϕ and χ are reciprocally inverse as

$$\begin{aligned} \chi(\phi(O_{(p,\varphi)})) &= \chi(\psi_{-\varphi}(p)) \\ &= O_{(\psi_{-\varphi}(p),0)} \\ &= O_{(p,\varphi)} \end{aligned}$$

where the last step uses that acting with φ maps $(\psi_{-\varphi}(p), 0)$ onto $(\psi_\varphi \psi_{-\varphi}(p), \varphi) = (p, \varphi)$. Similarly,

$$\begin{aligned} \phi(\chi(p)) &= \phi(O_{(p,0)}) \\ &= \psi_0(p) \\ &= p. \end{aligned}$$

In conclusion we constructed a new symplectic manifold $M_{\leq c}$ which is made up of the two disjoint components:

$$M_{\leq c} = M_c \sqcup M_{<c}.$$

Remark. A similar construction but with the twisted product symplectic manifold $(M \times C, pr_M^* \omega - pr_C^* \omega_0)$ and the the corresponding moment map yields the symplectic cut $M_{\geq c} = M_c \sqcup M_{>c}$ of M above c with respect to μ_{S^1}. These two cut spaces can be glued together along the submanifold M_c to recover the original symplectic manifold (M, ω).

Residual Symmetries

Take now (M, ω, G, μ_G) a Hamiltonian G-space, where G is a compact Abelian Lie group and denote the action as $\psi : G \to \mathrm{Diff}(M)$. Assume then that H is some S^1-subgroup of G and note that by Proposition 5.3.4 it also acts in a Hamiltonian way on M: This action is given by $\varphi := \psi \circ j$ with moment map $\mu_H := dj_e^* \circ \mu_G$ where $j : H \hookrightarrow G$ denotes the inclusion. We then carry out the above construction with regard to the subgroup $H = S^1$ and investigate whether the symplectic cut manifold $M_{\leq c}$ inherits a G-action:

1. Extend the action of G to the product space $M \times C$ by letting it act trivially on the second component. We denote this action by

$$\begin{aligned} \psi^1 : G &\to \mathrm{Diff}(M \times C) \\ g &\mapsto (\psi^1)_g = \psi_g \times \mathrm{Id}_C. \end{aligned}$$

   By Lemma 5.3.7 this action is Hamiltonian and its moment map corresponds to $\mu_G \circ pr_M$.

2. Let $H = S^1$ act on the product space $M \times C$ by

$$\begin{aligned} \Phi : H &\to \mathrm{Diff}(M \times C) \\ H &\mapsto (\Phi)_h = \varphi_h \times \sigma_h. \end{aligned}$$

   i.e. by the action φ induced by G on the first factor and by the multiplication σ described above on the second factor. By Lemma 5.3.7 also this action is Hamiltonian with moment map $\mu_{S^1} := dj_e^* \circ \mu_G \circ pr_M + \mu_C \circ pr_C$.

3. These two actions commute since the group G is Abelian:

$$\begin{aligned} (\psi^1)_g \circ (\Phi)_h &= (\psi_g \circ \varphi_h) \times (1_g \circ \sigma_h) \\ &= (\psi_g \circ \psi_{j(h)}) \times (\sigma_h \circ 1_g) \\ &= (\psi_{gj(h)}) \times (\sigma_h \circ 1_g) \\ &= (\psi_{j(h)g}) \times (\sigma_h \circ 1_g) \\ &= (\psi_{j(h)} \circ \psi_g) \times (\sigma_h \circ 1_g) \\ &= (\varphi_h \circ \psi_g) \times (\sigma_h \circ 1_g) \\ &= (\Phi)_h \circ (\psi^1)_g. \end{aligned}$$

For the same reason the two moment maps are invariant under the respective other action:

$$
\begin{aligned}
({}_G \quad pr_M) \quad (\qquad)_h &= {}_G \quad {}_h \quad pr_M \\
&= {}_G \quad {}_{j(h)} \quad pr_M \\
&= {}_G \quad pr_M
\end{aligned}
$$

and

$$
\begin{aligned}
{}_{S^1} \quad (\qquad 1)_g &= (d\, j_e \quad {}_G \quad pr_M + \quad {}_C \quad pr_C) \quad (\qquad 1)_g \\
&= d\, j_e \quad {}_G \quad {}_g \quad pr_M + \quad {}_C \quad 1_g \quad pr_C \\
&= d\, j_e \quad {}_G \quad pr_M + \quad {}_C \quad pr_C \\
&= \quad {}_{S^1} :
\end{aligned}
$$

4. Hence we conclude by Proposition 5.3.12 that $M_c := {}_{S^1}^{1}(c)=H$ inherits a Hamiltonian G-action. Explicitly, if $O_{(p;z)}$ is the H-orbit of $(p; z) \; 2 \quad {}_{S^1}^{1}(c)$, then this action is given by

$$
\begin{aligned}
&: G \,! \quad Di \quad (M \;\; {}_c) \\
&\;\; g \; 7! \quad {}_g : O_{(p;z)} \; 7! \; O_{(\;\; {}_g(p);z)} :
\end{aligned}
$$

Similarly, the moment map is given by

$$
\begin{aligned}
{}_G : M \;\; {}_c \,! \quad g \\
O_{(p;z)} \; 7! \; (\;\; {}_G \quad pr_M) \quad i\,(p; z) = \quad {}_G(p)
\end{aligned}
$$

where $i : \; {}_{S^1}^{1}(c) \,! \quad M \qquad C$ is the inclusion.

5. Untangling the de nitions, one sees that the action of G on $M_{<c}$ is just given by the restriction of the G-action on M . Indeed, using the explicit identi cations and from above, for any $p \; 2 \; M_{<c}$ and any $g \; 2 \; G$ we see that the action corresponds to

$$
\begin{aligned}
{}_g(p) &= \qquad {}_g \qquad (p) \\
&= \qquad {}_g(O_{(p;0)}) \\
&= \;\; (O_{(\;\; {}_g;0)}) \\
&= \;\; {}_0 \qquad {}_g(p) \\
&= \;\; {}_g(p);
\end{aligned}
$$

It follows in the same way that the moment map of this action just corresponds to the restriction of the moment map ${}_G$ to $M_{<c}$.

6. Similarly, by the trivial identification

$$M_c = \mu_H^{-1}(c)/H = (\mu_H^{-1}(c) \times \{0\})/H$$

we observe that G acts on $\mu_H^{-1}(c)$ just by restriction and that the moment map is thus once again just the restriction of μ_G. As G is Abelian, the moment map μ_G is constant on the whole G-orbits, so in particular it is constant on the H-orbits and descends to a moment map $\bar{\mu}_G$ on $M_c = \mu_H^{-1}(c)/H$ i.e. if $\mathcal{O}_p$ is the H-orbit of $p \in \mu_H^{-1}(c)$, then $\bar{\mu}_G(\mathcal{O}_p) = \mu_G(p)$.

In conclusion, if symplectic cutting is carried out with regard to an S^1-subgroup H of a compact Abelian Lie group G, then the resulting cut space $M_{\leq c}$ inherits a Hamiltonian G action: On $M_{<c} = \{p \in M \mid \mu_H(p) < c\}$ this action is just the restriction and also the moment map can be identified with the restriction. Because G is Abelian, the G-action also descends to an action on $M_c = M_{red} = \mu_H^{-1}(c)/H$ and so does the moment map.

We will now conclude this chapter by studying how the construction of symplectic cutting affects the image of the moment map. This serves both as an introduction to the final two chapters of this work and as a motivation for the name *symplectic cut*. We will proceed in three steps:

1. Since H is an S^1-subgroup of G, its Lie algebra is generated by a single vector $v \in \mathfrak{g}$. We will write $\tilde{v}$ for the vector v but as an element of $\mathfrak{h}$, that is, $\tilde{v} \in \mathfrak{h}$ is the unique element in the Lie algebra of H such that $dj_e[\tilde{v}] = v$. We then choose this vector to identify $\mathfrak{h}^*$ with $\mathbb{R}$, that is, to the value $c \in \mathbb{R}$, we associate

$$\begin{aligned} \varphi : \mathfrak{h} &\to \mathbb{R} \\ \tilde{v} &\mapsto c. \end{aligned}$$

2. We start by M_c and as we argued above, $\bar{\mu}_G : M_c \to \mathfrak{g}^*$ is induced by the restriction of μ_G to $\mu_H^{-1}(c)$ so that

$$\bar{\mu}_G(M_c) = \mu_G(\mu_H^{-1}(c)).$$

But as $\mu_H = dj_e^* \circ \mu_G$ it holds that

$$\mu_H^{-1}(c) = \mu_G^{-1}\left((dj_e^*)^{-1}(c)\right).$$

However, with the identification $c \cong \varphi : \tilde{v} \mapsto c$ from above, we quickly see that for $\xi \in \mathfrak{g}^*$,

$$\begin{aligned} \xi \in (dj_e^*)^{-1}(\varphi) &\iff dj_e^*[\xi] = \varphi \\ &\iff \langle dj_e^*[\xi], \tilde{v}\rangle = \langle \varphi, \tilde{v}\rangle \\ &\iff \langle \xi, dj_e[\tilde{v}]\rangle = c \\ &\iff \langle \xi, v\rangle = c \\ &\iff \xi \in \partial H_{(v,c)}. \end{aligned}$$

We conclude that

$$\mu_G(M_c) = \mu_G(M) \cap \partial H_{(v;c)}.$$

3. Also for $M_{<c}$, the moment map just corresponds to the restriction so that we see

$$\mu_G(M_{<c}) = \mu_G(\{p \in M \mid \mu_H(p) < c\})$$

But with the same arguments as above (still identifying $\varphi \simeq c$),

$$\begin{aligned}
\mu_H(p) < \varphi &\iff dj_e^* \mu_G(p) < \varphi \\
&\iff \langle dj_e^* \mu_G(p), \partial_t \rangle < \langle \varphi, \partial_t \rangle \\
&\iff \langle \mu_G(p), dj_e[\partial_t] \rangle < c \\
&\iff \langle \mu_G(p), v \rangle < c \\
&\iff \mu_G(p) \in \operatorname{int} H_{(v;c)}
\end{aligned}$$

so that we can conclude

$$\mu_G(M_{<c}) = \mu_G(M) \cap \operatorname{int} H_{(v;c)}.$$

Finally, combining the results of step 2 and step 3, we see that

$$\mu_G(M_{\leq c}) = \mu_G(M) \cap H_{(v;c)}$$

which in view of section 4.5 motivates the name symplectic cut.

Chapter 6

Hamiltonian Torus Actions

In the last part of the previous chapter, it was established that for compact Abelian Lie groups the cutting construction can be carried out with regard to a subgroup so that the resulting space inherits an action. For this result both compactness and commutativity were crucial. In this chapter, we will address the question of what additional properties a moment map has, if one requires the acting Lie group to be compact Abelian. Since the image of the moment map is entirely characterised by the action of the identity component, we will further reduce to considering connected compact Abelian Lie groups. We start this chapter by investigating the relation between abstract connected compact Abelian Lie groups and tori.

6.1 Toric Framework

The first part of this section is inspired by [9].

For connected compact Abelian Lie groups, the exponential map allows to interpret the Lie group as a quotient of its Lie algebra by a lattice. The first step towards this result is the following observation:

Proposition 6.1.1. Let G be a connected Lie group. The exponential map $\exp : \mathfrak{g} \to G$ is a homomorphism if and only if G is Abelian.

Proof. Assume first that $\exp : \mathfrak{g} \to G$ is a group homomorphism. As a Lie group, G is in particular a connected topological group and therefore generated by any neighbourhood of the identity. Since $\exp : \mathfrak{g} \to G$ is a diffeomorphism on a neighbourhood of the identity, it follows that any $g \in G$ can be written as a product of elements in the image of $\exp$ i.e.

$$g = \prod_i \exp(X_i) \qquad \text{for some } X_i \in \mathfrak{g}$$

Doing the same for $h \in G$ we can write $h = \prod_j \exp(Z_j)$ for some $Z_j \in \mathfrak{g}$. Finally, using that exp is assumed to be a homomorphism we get

$$
\begin{aligned}
gh &= \prod_i \exp(X_i) \prod_j \exp(Z_j) \\
&= \exp\left(\sum_i X_i + \sum_j Z_j \right) \\
&= \exp\left(\sum_j Z_j + \sum_i X_i \right) \\
&= \prod_j \exp(Z_j) \prod_i \exp(X_i) \\
&= hg
\end{aligned}
$$

showing that indeed, G is Abelian.

Conversely, if G is Abelian, then the multiplication $m : G \times G \to G$ is a homomorphism since

$$
\begin{aligned}
m((g_1, h_1) \cdot (g_2, h_2)) &= m(g_1 g_2, h_1 h_2) \\
&= g_1 g_2 h_1 h_2 \\
&= g_1 h_1 g_2 h_2 \\
&= m(g_1, h_1) m(g_2, h_2),
\end{aligned}
$$

where for readability we used $\cdot$ to denote multiplication in $G \times G$. The statement now follows from the naturality of the exponential map since the following diagram commutes:

$$
\begin{array}{ccc}
\mathfrak{g} \times \mathfrak{g} & \xrightarrow{dm_{(e,e)}} & \mathfrak{g} \\
\downarrow \exp_G \times \exp_G & & \downarrow \exp_G \\
G \times G & \xrightarrow{m} & G
\end{array}
$$

For particular elements $X, Z \in \mathfrak{g}$ this becomes

$$
\begin{array}{ccc}
(X, Z) & \xmapsto{dm_{(e,e)}} & X + Z \\
\downarrow \exp_G \times \exp_G & & \downarrow \exp_G \\
(\exp_G(X), \exp_G(Z)) & \xmapsto{m} & \exp_G(X)\exp_G(Z) = \exp_G(X + Z)
\end{array}
$$

where we used that the differential of the multiplication at the identity is

$$
\begin{aligned}
dm_{(e,e)} : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\
(X, Z) &\mapsto X + Z.
\end{aligned}
$$

Actually, the argument in the beginning of this proof also tells us that the exponential map of a connected Lie group is surjective if it is a homomorphism. Since this is the case if the Lie group is Abelian, the following result holds:

Corollary 6.1.2. *For connected Abelian Lie Groups the exponential map is a surjective homomorphism.*

If the exponential map is a surjective homomorphism, one can apply the first isomorphism theorem and interpret the group as the quotient of its Lie algebra by the kernel of the exponential map. This kernel is a discrete subgroup of the Lie algebra as the exponential map is a local diffeomorphism.

Proposition 6.1.3. *If G is a connected Abelian Lie group, then*

$$G \cong \mathfrak{g} / \mathfrak{g}_{\mathbb{Z}}$$

with $\mathfrak{g}_{\mathbb{Z}} = \ker(\exp)$ a discrete subgroup of $\mathfrak{g}$.

Definition. The discrete subgroup $\mathfrak{g}_{\mathbb{Z}} = \ker(\exp)$ of $\mathfrak{g}$ is called the *integral lattice*.

Note however that this might not be a lattice as defined in section 2.1: While it is a discrete additive subgroup, it may not span the Lie algebra $\mathfrak{g}$. However, if $\mathfrak{g}_{\mathbb{Z}}$ does not span $\mathfrak{g}$, then it is contained in a proper subspace. This reasoning can be repeated until a smallest subspace is found and in this subspace $\mathfrak{g}_{\mathbb{Z}}$ is indeed a lattice as defined in section 2.1. This motivates the name *integral lattice* and together with Theorem 2.1.2, which allows to choose a basis for $\mathfrak{g}_{\mathbb{Z}}$, gives the following result:

Proposition 6.1.4. *$\mathfrak{g}_{\mathbb{Z}}$ is generated over $\mathbb{Z}$ by linearly independent vectors $v_1, \dots, v_k \in \mathfrak{g}$ i.e.*

$$\mathfrak{g}_{\mathbb{Z}} = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_k.$$

Putting the different ingredients together gives a classification of connected Abelian Lie groups, in particular establishing the link between the standard torus $T^k = \mathbb{R}^k / \mathbb{Z}^k$ and connected compact Abelian Lie groups:

Theorem 6.1.5. *Any connected Abelian Lie group G is isomorphic to a product of the standard torus T^k and $\mathbb{R}^s$: $G \cong T^k \times \mathbb{R}^s$. In particular,*

- *any simply connected Abelian Lie group G is isomorphic to $\mathbb{R}^s$: $G \cong \mathbb{R}^s$ and*
- *any compact connected Abelian Lie group G is isomorphic to the standard torus T^k: $G \cong T^k$.*

Proof. By the above we already have

1. $\exp : \mathfrak{g} \to G$ is a surjective homomorphism,
2. $\ker(\exp)$ is a discrete subgroup of $\mathfrak{g}$ and hence
3. $\mathfrak{g}_{\mathbb{Z}}$ is generated by linearly independent vectors $v_1, \ldots, v_k \in \mathfrak{g}$.

We can then find $v_{k+1}, \ldots, v_n \in \mathfrak{g}$ (where $n = \dim(G)$) so that $v_1, \ldots, v_n$ form a basis of $\mathfrak{g}$. This determines an isomorphism $\mathfrak{g} \cong \mathbb{R}^n$ such that

$$\mathfrak{g}_{\mathbb{Z}} \cong \mathbb{Z}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} \cong \mathbb{R}^n.$$

This immediately gives

$$\begin{aligned} G &\cong \mathfrak{g} \big/ \mathfrak{g}_{\mathbb{Z}} \\ &\cong \mathbb{R}^n \big/ \mathbb{Z}^k \times \{0\} \\ &\cong \mathbb{R}^k \big/ \mathbb{Z}^k \times \mathbb{R}^{n-k} \\ &\cong T^k \times \mathbb{R}^{n-k}. \end{aligned}$$

The other two statements follow immediately.

Therefore, for the remainder of this text we will adapt the following definition of a torus:

Definition. A torus is a compact connected Abelian Lie group.

Notation. We will often denote elements of the torus by their equivalence classes in the quotient of the Lie algebra by the integral lattice e.g. $[\xi] \in G = \mathfrak{g}/\mathfrak{g}_{\mathbb{Z}}$ is the equivalence class of $\xi \in \mathfrak{g}$ in $\mathfrak{g}/\mathfrak{g}_{\mathbb{Z}}$.

6.1.1 Subtori

Let G be a torus. Consider a Lie subgroup $S < G$ which is itself a torus. Indeed as any Lie subgroup is closed, compactness is inherited from G. Moreover, as a subgroup of an Abelian group, S is also automatically Abelian so that it actually suffices to require S to be connected in order to be a torus in its own right:

Definition. Let G be a torus. A *subtorus* of G is a connected Lie subgroup $S < G$.

Let then $i : S \hookrightarrow G$ be the inclusion of a subtorus S in a torus G. By naturality of the exponential map this gives a commutative diagram:

$$\begin{array}{ccc} \mathfrak{s} & \overset{di_e}{\hookrightarrow} & \mathfrak{g} \\ \downarrow{\scriptstyle \exp_S} & & \downarrow{\scriptstyle \exp_G} \\ S & \overset{i}{\hookrightarrow} & G \end{array}$$

where $di_e : \mathfrak{s} \hookrightarrow \mathfrak{g}$ is just the inclusion of the Lie algebras. Denote $\mathfrak{s}_{\mathbb{Z}} = \ker(\exp_S)$ the integral lattice of S and consider an arbitrary element $v \in \mathfrak{s}_{\mathbb{Z}}$. Then it follows from the diagram that $di_e[v]$ (which is v interpreted as an element of $\mathfrak{g}$) is an element of the integral lattice $\mathfrak{g}_{\mathbb{Z}} = \ker(\exp_G)$ of G:

$$\exp_G \circ di_e[v] = i \circ \exp_S(v) = e_G.$$

Reversely, assume that $v \in \mathfrak{s}$ is such that $di_e[v] \in \mathfrak{g}_{\mathbb{Z}}$. Then again by commutativity it follows that

$$e_G = \exp_G \circ di_e[v] = i \circ \exp_S(v).$$

Since i is injective, this implies that $\exp_S(v) = e_S$ and therefore that $v \in \mathfrak{s}_{\mathbb{Z}} = \ker(\exp_S)$. The conclusion is thus that

$$di_e[\mathfrak{s}_{\mathbb{Z}}] = d\,i_e[\mathfrak{s}] \cap \mathfrak{g}_{\mathbb{Z}}$$

Moreover, as $\mathfrak{s}_{\mathbb{Z}} \subset \mathfrak{s}$ is a lattice, so is $di_e[\mathfrak{s}_{\mathbb{Z}}] \subset di_e[\mathfrak{s}]$ and therefore $di_e[\mathfrak{s}]$ is a $\mathfrak{g}_{\mathbb{Z}}$-rational subspace of $\mathfrak{g}$.

Reversely, assume that $\mathfrak{s}$ is a $\mathfrak{g}_{\mathbb{Z}}$-rational subspace of $\mathfrak{g}$. By this we formally mean that, if $di_e : \mathfrak{s} \hookrightarrow \mathfrak{g}$ denotes the inclusion, then $di_e[\mathfrak{s}] \cap \mathfrak{g}_{\mathbb{Z}}$ is a lattice in $di_e[\mathfrak{s}]$. It is then clear that

$$\mathfrak{s}_{\mathbb{Z}} := d\,i_e^{-1}(di_e[\mathfrak{s}] \cap \mathfrak{g}_{\mathbb{Z}})$$

is a lattice in $\mathfrak{s}$. It follows that $S := \mathfrak{s}/\mathfrak{s}_{\mathbb{Z}}$ is a torus and for (hopefully) obvious reasons we will denote the projection as $\exp_S : \mathfrak{s} \to \mathfrak{s}/\mathfrak{s}_{\mathbb{Z}} = S$. By construction, $\mathfrak{s}_{\mathbb{Z}} \subset \ker(\exp_G \circ di_e)$ and therefore this map factors through $S = \mathfrak{s}/\mathfrak{s}_{\mathbb{Z}}$. This means that there exists a homomorphism $i : S \to G$ such that the diagram

$$\begin{array}{ccc} \mathfrak{s} & \overset{di_e}{\hookrightarrow} & \mathfrak{g} \\ \downarrow{\scriptstyle \exp_S} & & \downarrow{\scriptstyle \exp_G} \\ S & \overset{i}{\longrightarrow} & G \end{array}$$

commutes. Assume then that $[v] \in \ker(i)$. By surjectivity of $\exp_S$ there is a $v \in \mathfrak{s}$ such that $[v] = \exp_S(v)$ and therefore

$$\begin{aligned} e_G &= i([v]) \\ &= i \circ \exp_S(v) \\ &= \exp_G \circ di_e[v] \end{aligned}$$

so that $di_e[v] \in \mathfrak{g}_\mathbb{Z}$. Hence $v \in \mathfrak{s}_\mathbb{Z}$ and thus $[v] = 0$ so that $i : S \hookrightarrow G$ is injective. This shows that S can actually be interpreted as a subtorus of G via the injective homomorphism $i : S \hookrightarrow G$.

6.1.2 Quotient Tori

Let now G be a torus and S be a subtorus. Let $i : S \hookrightarrow G$ be the inclusion and $di_e : \mathfrak{s} \hookrightarrow \mathfrak{g}$ be its differential. This induces a short exact sequence

$$0 \longrightarrow \mathfrak{s} \xhookrightarrow{di_e} \mathfrak{g} \xrightarrow{d\pi_e} \mathfrak{g}/\mathfrak{s} \longrightarrow 0$$

where $d\pi_e : \mathfrak{g} \to \mathfrak{g}/\mathfrak{s}$ is just the projection. For better readability, we set $\mathfrak{t} := \mathfrak{g}/\mathfrak{s}$. If $\mathfrak{g}_\mathbb{Z}$ is the integral lattice of G, then it follows by Corollary 2.1.3 that also $\mathfrak{t}_\mathbb{Z} := d\pi_e[\mathfrak{g}_\mathbb{Z}] \subset \mathfrak{t}$ is a lattice. It follows that $T = \mathfrak{t}/\mathfrak{t}_\mathbb{Z}$ is another torus and we will denote the projection as $\exp_T : \mathfrak{t} \to \mathfrak{t}/\mathfrak{t}_\mathbb{Z} = T$.

Consider then the map $\exp_T \circ d\pi_e : \mathfrak{g} \to T$ and note that since $\ker(\exp_T) = \mathfrak{t}_\mathbb{Z} = d\pi_e[\mathfrak{g}_\mathbb{Z}]$, the integral lattice $\mathfrak{g}_\mathbb{Z}$ is mapped to zero. Therefore this map factors through $\mathfrak{g}/\mathfrak{g}_\mathbb{Z} = G$ and hence there exists a homomorphism $\varphi : G \to T$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\pi_e} & \mathfrak{t} \\ \downarrow{\scriptstyle \exp_G} & & \downarrow{\scriptstyle \exp_T} \\ G & \xrightarrow{\varphi} & T \end{array}$$

Let then $[v] \in \mathfrak{t}/\mathfrak{t}_\mathbb{Z} = T$ be an arbitrary element. Since $\exp_T$ is surjective, there exists $v \in \mathfrak{t}$ such that $\exp_T(v) = [v]$. Since also $d\pi_e$ is surjective, there exists $w \in \mathfrak{g}$ such that $d\pi_e[w] = v$. But then

$$\begin{aligned} \varphi \circ \exp_G(w) &= \exp_T \circ d\pi_e[w] \\ &= \exp_T(v) \\ &= [v] \end{aligned}$$

and as $[v] \in T$ was arbitrary, this shows that φ is surjective. Together with the commutative diagram for the subtorus S we have thus found a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{s} & \xhookrightarrow{di_e} & \mathfrak{g} & \xrightarrow{d\pi_e} & \mathfrak{t} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \exp_S} & & \downarrow{\scriptstyle \exp_G} & & \downarrow{\scriptstyle \exp_T} & & \\ 1 & \longrightarrow & S & \xhookrightarrow{i} & G & \xrightarrow{\varphi} & T & \longrightarrow & 1 \end{array}$$

where the top row is exact. We verify that also the bottom row is exact by a diagram chase:

1. Injectivity of $i : S \to G$ and surjectivity $\pi : G \to T$ has already been shown.
2. Assume $[v] \in i(S)$ i.e. there is a $[w] \in S$ such that $[v] = i([w])$. Since $\exp_S$ is surjective, there is a $w \in \mathfrak{s}$ mapped onto $[w]$ by the exponential map. Then by commutativity of the diagram

$$\begin{aligned} \pi([v]) &= \pi \circ i([w]) \\ &= \pi \circ i \circ \exp_S(w) \\ &= \exp_T \circ d\pi_e \circ di_e(w) \\ &= 0 \end{aligned}$$

   where the last step is by exactness of the first row. Hence $\operatorname{Im}(i) \subseteq \ker(\pi)$.
3. Take now $[v] \in \ker(\pi)$. By surjectivity of $\exp_G$ there is a $v \in \mathfrak{g}$ such that $[v] = \exp_G(v)$. Then

$$\begin{aligned} e &= \pi([v]) \\ &= \pi \circ \exp_G(v) \\ &= \exp_T \circ d\pi_e[v] \end{aligned}$$

   implies that $d\pi_e[v] \in \ker(\exp_T) = \mathfrak{t}_Z = d\pi_e[\mathfrak{g}_Z]$. That means that there exists a $u \in \mathfrak{g}_Z$ such that $d\pi_e[v] = d\pi_e[u]$ or differently put $v - u \in \ker(d\pi_e)$. Since $u \in \mathfrak{g}_Z = \ker(\exp_G)$, we have $\exp_G(v - u) = \exp_G(v) = [v]$ and by exactness of the top row there is a $w \in \mathfrak{s}$ such that $di_e[w] = v - u$. This gives

$$\begin{aligned} [v] &= \exp_G(v - u) \\ &= \exp_G \circ di_e[w] \\ &= i \circ \exp_S(w) \end{aligned}$$

   showing that $[v] \in \operatorname{Im}(i)$. Since $[v] \in \ker(\pi)$ was arbitrary, it follows that $\ker(\pi) \subseteq \operatorname{Im}(i)$.

This establishes exactness of the bottom row and we identify T as the quotient torus by the first isomorphism theorem

$$T \cong G \big/ S.$$

Note that the construction above gives that the integral lattice of the quotient torus T is given by $d\pi_e[\mathfrak{g}_Z]$.

Remark. Before going back into the symplectic world, we invite the reader to compare this section to section 4.1.1. There should be an evident parallel between Λ and $\mathfrak{g}_Z$, the annihilator lattice Λ^0 and $\mathfrak{s}_Z$ as well as the effective lattice Λ_F and $\mathfrak{t}_Z$. This is of course not at all a coincidence and will soon be important.

6.2 Local Forms

In the beginning of Chapter 5 we argued repeatedly that the neighbourhood of a fixed point is characterised by the isotropy representation. The goal of this section is to make this statement rigorous for Hamiltonian actions of tori. Our starting point is the Darboux-Weinstein theorem, which will only be used in the special case where $N = \{p\}$ is a single point.

Theorem 6.2.1 (Darboux-Weinstein Theorem). Let M be a smooth manifold, $N \subset M$ a submanifold and $i : N \hookrightarrow M$ be the inclusion. Assume that ω_0 and ω_1 are two symplectic forms on M which agree on N i.e. $(\omega_0)_p = (\omega_1)_p$ for all $p \in N$. Then there exist neighbourhoods U_0 and U_1 of N in M together with a symplectomorphism $\varphi : (U_0; \omega_0|_{U_0}) \to (U_1; \omega_1|_{U_1})$.

Remark. The proof is an application of the Moser trick and the tubular neighbourhood theorem B.5.1. We skip the detailed derivation of this result and refer the interested reader to Chapter 7 of [5] for a sound derivation.

Although this is the classical version, we will need a slightly stronger result: In order to use this to study the local form of an action, the diffeomorphism has to respect the group action. This can indeed be achieved if the acting group G is compact. Compactness is needed so that one can render the objects involved in the proof G-equivariant by averaging them over the group G.

Theorem 6.2.2 (Equivariant Darboux-Weinstein Theorem). Let M be a smooth manifold and $N \subset M$ a submanifold. Assume that ω_0 and ω_1 are two symplectic forms on M which agree on N i.e. $(\omega_0)_p = (\omega_1)_p$ for all $p \in N$. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth action of a compact Lie group G which is symplectic with regard to ω_0 and ω_1. Then there exist G-invariant neighbourhoods U_0 and U_1 of N in M together with a G-equivariant symplectomorphism $\varphi : (U_0; \omega_0|_{U_0}) \to (U_1; \omega_1|_{U_1})$.

Let now $(M; \omega; T; \mu)$ a Hamiltonian T-space with T a torus. We will apply this in the special case where $N = \{p\}$ is a single point which is fixed by the T-action. One of the two symplectic forms will just be the symplectic form ω of M. For the second, one considers the isotropy representation on the tangent space T_pM and chooses a basis which is well-adapted to this symplectic representation together with a symplectic form. Choosing a Riemannian metric and consequently a Riemannian exponential map this can be used to define the second symplectic form on a neighbourhood of p. First, we make precise what is meant with a basis that is well-adapted.

Definition. Let T be a torus, $\mathfrak{t}$ its Lie algebra and $\mathfrak{t}_{\mathbb{Z}}$ its integral lattice. The dual lattice of the integral lattice is called the weight lattice and denoted as $\mathfrak{t}^*_{\mathbb{Z}} \subset \mathfrak{t}^*$.

Lemma 6.2.3. There is a bijective correspondence between isomorphism classes of $2n$-dimensional symplectic representations of a torus T and unordered n-tuples of elements (possibly with repetition) of the weight lattice $\mathfrak{t}^*_{\mathbb{Z}} \subset \mathfrak{t}^*$ of T.

Let (V, ω) be a $2n$-dimensional symplectic vector space. Let $\rho : T \to Sp(V, \omega)$ be a symplectic representation with weights $\alpha^{(1)}, \dots, \alpha^{(n)} \in \mathfrak{t}^*_{\mathbb{Z}}$. There exists a decomposition

$$(V, \omega) = \bigoplus_{i=1}^{n} (V_i, \omega_i)$$

into invariant mutually perpendicular two-dimensional symplectic subspaces which can each be identified with $\mathbb{C}$, equipped with the standard symplectic form $\omega_0 = \frac{i}{2} dz \wedge d\bar{z}$. T acts on the factor V_i by

$$\begin{aligned} \rho_i : T &\to Sp(\mathbb{C}, \omega_0) \\ [\theta] &\mapsto (\rho_i)_{[\theta]} : z \mapsto e^{i\langle \alpha^{(i)}, \theta \rangle} z \end{aligned}$$

and the moment map is

$$\begin{aligned} \mu : V = \bigoplus_{i=1}^{n} V_i &\to \mathfrak{t}^* \\ (z_1, \dots, z_n) &\mapsto \frac{1}{2} \sum_{i=1}^{n} |z_i|^2 \alpha^{(i)} \end{aligned}$$

Proof. See [10], Appendix A.

We can now state and sketch the proof of the main result of this section:

Theorem 6.2.4 (Toric Darboux Theorem). Let (M, ω, T, μ) be a hamiltonian T-space, where T is a $2n$-dimensional torus. Let $p \in M^T$ be a fixed point and denote by $\alpha^{(1)}, \dots, \alpha^{(n)} \in \mathfrak{t}^*_{\mathbb{Z}}$ the weights of the isotropy representation. Then there exists a T-invariant neighbourhood U of p in M together with coordinate functions $(z_1, \dots, z_n)$ centered at p such that

a) the symplectic form is in Darboux-form

$$\omega|_U = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i;$$

b) the T-action is given by the weights of the isotropy representation

$$\begin{aligned} e^{i[\theta]} \cdot (z_1, \dots, z_n) &= (e^{i\langle \alpha^{(1)}, \theta \rangle} z_1, \dots, e^{i\langle \alpha^{(n)}, \theta \rangle} z_n) \\ e^{i[\theta]} \cdot (\bar{z}_1, \dots, \bar{z}_n) &= (e^{-i\langle \alpha^{(1)}, \theta \rangle} \bar{z}_1, \dots, e^{-i\langle \alpha^{(n)}, \theta \rangle} \bar{z}_n) \end{aligned}$$

where $[\theta] \in \mathfrak{t}/\mathfrak{t}_{\mathbb{Z}} = T$ and

c) the moment map becomes

$$\mu|_U = \mu(p) + \frac{1}{2}\sum_{i=1}^{n} |z_i|^2 \lambda^{(i)}.$$

Sketch of Proof. ([6], Theorem 1.6.6.) Choose a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of T_pM such that the isotropy representation ρ has the form described in Lemma 6.2.3. Choose a T-invariant Riemannian metric on M and consider the corresponding exponential map $\exp_p : T_pM \to U'$ for some neighbourhood U' of p. Using this we can find coordinates $(z'_1, \dots, z'_n, \bar{z}'_1, \dots, \bar{z}'_n)$ on U' centered at p and such that

$$\left.\frac{\partial}{\partial z'_i}\right|_p = e_i \quad \text{and} \quad \left.\frac{\partial}{\partial \bar{z}'_i}\right|_p = f_i.$$

By construction, the symplectic form ω has Darboux form at p

$$\omega_p = \frac{i}{2}\sum_{i=1}^{n} dz'_i|_p \wedge d\bar{z}'_i|_p.$$

The exponential map is (ψ, ρ)-equivariant as the metric is T-invariant. Therefore T acts on $(z'_1, \dots, z'_n, \bar{z}'_1, \dots, \bar{z}'_n)$ as described in the statement.

We then put

$$\omega_1 = \frac{i}{2}\sum_{i=1}^{n} dz'_i \wedge d\bar{z}'_i$$

and note that this coincides with ω at p. Furthermore this is T-invariant as can be seen from the explicit description of the action. By the equivariant Darboux-Weinstein Theorem, there are hence T-invariant neighbourhoods U_0 and U_1 of p in U' together with a T-equivariant symplectomorphism $\varphi : (U_0, \omega|_{U_0}) \to (U_1, \omega_1|_{U_1})$ fixing p. We then set $U = U_0$ and $z_i = z'_i \circ \varphi$ as well as $\bar{z}_i = \bar{z}'_i \circ \varphi$. Note that T acts on $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$ as it did on $(z'_1, \dots, z'_n, \bar{z}'_1, \dots, \bar{z}'_n)$ since φ is an equivariant symplectomorphism. The same reasoning applies to the moment map.

6.2.1 Morse-Bott Functions from Moment Maps

The toric Darboux Theorem can be used to prove that component of moment maps are Morse-Bott functions. Since this is a property that will be used in the proof of the main result in the next chapter, we briefly recall the essential notions and prove the exact statement.

Definition. Let M be a smooth manifold and $f \in C^\infty(M)$ a smooth function. A point $p \in M$ is called a *critical point* of f if $df_p = 0$. A point $x \in \mathbb{R}$ is called a *critical value* of f if it is the image under f of a critical point. We denote the subset of M given by all points $p \in M$, which are critical points of f by $\mathrm{Crit}(f) \subset M$.

Remark. If c is a critical value, not all points in $f^{-1}(c)$ must be critical points. It suffices that $f^{-1}(c)$ contains a single critical point.

Remark. A smooth function $f \in C^\infty(M)$ defines a section of the cotangent bundle via

$$\begin{aligned} s_f : M &\to T^*M \\ p &\mapsto (p, df_p). \end{aligned}$$

On other hand, T^*M contains M as an embedded submanifold, that is, the image of the zero-section

$$\begin{aligned} s_0 : M &\to T^*M \\ p &\mapsto (p, 0). \end{aligned}$$

Writing $M := \mathrm{Im}(s_0)$ and $\Gamma_f := \mathrm{Im}(s_f)$ we then have by definition that

$$\mathrm{Crit}(f) = M \cap \Gamma_f .$$

Definition. Let M be a smooth manifold and $f \in C^\infty(M)$ a smooth function. The Hessian $\mathrm{Hess}_p(f)$ of f at a critical point $p \in M$ is the symmetric bilinear form given by

$$[\mathrm{Hess}_p(f)](X, Y) = (L_X L_Y(f))(p) = X_p(Y(f))$$

for any $X, Y \in \mathfrak{X}(M)$.

Remark. Note that this is symmetric since $p \in M$ is a critical point:

$$X_p(Y(f)) - Y_p(X(f)) = [X, Y]_p(f) = df_p([X, Y]) = 0 .$$

Definition. Let M be a smooth manifold and $f \in C^\infty(M)$ be a smooth function. A critical point $p \in \mathrm{Crit}(f)$ is called *nondegenerate* if $\mathrm{Hess}_p(f) : T_pM \times T_pM \to \mathbb{R}$ is nondegenerate or equivalently if s_f intersects s_0 transversally at $(p, 0)$.

Definition. Let M be a smooth manifold. A smooth function $f \in C^\infty(M)$ is called a *Morse function* if all its critical points are nondegenerate or equivalently if s_f intersects s_0 transversally.

Definition. Let M be a smooth manifold and $N \subset M$ a compact connected submanifold. N is called a *nondegenerate critical submanifold* of $f \in C^\infty(M)$ if

1. $N \subset \mathrm{Crit}(f)$ and
2. the Hessian $\mathrm{Hess}_p(f)|_{\nu_p N}$ restricted to the normal bundle (see section B.5) is nondegenerate for each point $p \in N$.

If $\mathrm{Crit}(f)$ consists of nondegenerate critical submanifolds, then f is called a Morse-Bott function.

Theorem 6.2.5. Let G be a compact Lie group and let (M, ω, G, μ) a hamiltonian G-space. For any $X \in \mathfrak{g}$, the component of μ along X given by

$$\begin{aligned} \mu^X : M &\to \mathbb{R} \\ p &\mapsto \mu^X(p) = \langle \mu(p), X \rangle \end{aligned}$$

is a Morse-Bott function and the critical manifolds are symplectic submanifolds.

Proof. (Inspired by [11], Theorem 2.2) $X \in \mathfrak{g}$ generates a one-parameter subgroup of G. Its closure is connected, compact since G is and Abelian. Thus it is a torus $T < G$ of some dimension. By definition of the moment map, $d\mu^X = \iota_{X^\sharp}\omega$ so that

$$d\mu^X_p = 0 \iff \iota_{X^\sharp_p}\omega_p = 0$$

However, by nondegeneracy of ω_p, this is the case exactly if $X^\sharp_p = 0$. But the fundamental vector fields span $T_p(\mathcal{O}_p)$ and therefore the orbit of p is discrete. Since T is connected this implies that p is a fixed point and it follows that

$$\mathrm{Crit}\, \mu^X = M^T.$$

We conclude by Proposition 5.1.3 that the connected components are symplectic submanifolds.

To see the non-degeneracy, we look at the moment map in a neighbourhood of a point $p \in M^T$ in its local form of the Toric Darboux Theorem 6.2.4

$$\mu = \mu(p) + \frac{1}{2}\sum_{i=1}^{n} |z_i|^2 \alpha^{(i)}.$$

At the same time, just as in the proof of Proposition 5.1.3 we have the identification $T_p(M^T) = (T_pM)^T$. It follows from the explicit expression in Lemma 6.2.3 that $(T_pM)^T$ corresponds to the zero weight space. Thus on $\nu_p(M^T) = T_pM/T_p(M^T) = T_pM/(T_pM)^T$ the map

$$\mu^X = \langle \mu(p), X \rangle + \left\langle \frac{1}{2}\sum_{i=1}^{n} |z_i|^2 \alpha^{(i)}, X \right\rangle$$

is non-degenerate. This shows that $\mu^X : M \to \mathbb{R}$ is a Morse-Bott function.

6.3 The Convexity Theorem and Moment Polytopes

That the components of moment maps are Morse-Bott functions is also used in the proof of the convexity theorem. This is the first result providing a link between the two parts of this work by stating for a Hamiltonian torus action, the image of the moment map is actually a polytope given that the manifold is compact and connected.

Theorem 6.3.1 (Atiyah, Guillemin-Sternberg)**.** *Let (M, ω, T, μ) be a compact connected Hamiltonian T-space for a torus T. Then:*

1. *the levels of μ are connected;*
2. *the image of μ is convex;*
3. *the image of μ is the convex hull of a finite number of points, that are images of the fixed points of the action.*

Proof. See e.g. the paper by Atiyah [1].

Definition. In this case, the image $\mu(M)$ of the moment map is called the *moment polytope*.

Consider a compact connected Hamiltonian T-space (M, ω, T, μ) for a torus T and let $\mu(M)$ be its moment polytope. Let $p \in M$ be a fixed point and apply the toric Darboux theorem. This yields that on some neighbourhood U of p, the moment map is given by

$$\mu|_U = \mu(p) + \frac{1}{2}\sum_{i=1}^{n} |z_i|^2 \alpha^{(i)}.$$

As $|z_i|^2 \geq 0$, $\mu_U(U)$ is the translation by $\mu(p)$ of a neighbourhood of the origin of the polyhedral cone $\mathrm{co}(\{\alpha^{(i)}\}_{i=1,\ldots,n})$. By construction, this cone corresponds is the local cone $C_{\mu(p)}(\mu(M))$ of the moment polytope $\mu(M)$ at the point $\mu(p)$. In particular, this means that $\mu(p)$ is a vertex of the moment polytope if and only if the cone $\mathrm{co}(\{\alpha^{(i)}\}_{i=1,\ldots,n})$ is pointed.

6.3.1 Effective Hamiltonian Torus Actions

We start with the following observation:

Proposition 6.3.2. *Let $\psi : T \to \mathrm{Diff}(M)$ be a smooth effective action of a torus T on a connected smooth manifold M. Let $p \in M$ be a fixed point of this action. Then the isotropy representation $\rho : T \to GL(T_pM)$ at p is faithful.*

Sketch of proof. This is a special case of Lemma A.4.7 in the appendix. We sketch the idea for fixed points.

Since T is compact, one can choose a T-invariant metric and a corresponding (ρ, ψ)-equivariant exponential map $\exp_p : T_pM \to M$. This is a local diffeomorphism such that $\exp_p(0) = p$, that is, there is an open neighbourhood V of the origin in T_pM and an open neigbourhood U of p in M such that $\exp_p|_V : V \to U$ is a diffeomorphism.

We assume by contradiction that is not faithful and that there exists a non trivial element [ ] 2 ker( ). Consider then any q 2 U and note that there is a unique v 2 V such that q = $\exp_p$(v). By (;)-equivariance we have

$$
\begin{aligned}
{}_{[\ ]}(q) &= {}_{[\ ]} \quad \exp_p(v) \\
&= \exp_p \quad {}_{[\ ]}(v) \\
&= \exp_p(v) \\
&= q:
\end{aligned}
$$

Hence [ ] xes the whole open neighbourhood U. It follows that the set of points xed by [ ] is open. But by continuity of , it is also closed. Since M is connected, it follows that [ ] xes all of M which contradicts being e ective.

The question is then how this translates to the local cones at those points. This is the content of the next, representation theoretic result:

Proposition 6.3.3. Let : T ! Sp(V; !) be a 2n-dimensional representation of an m-dimensional torus T. Then the weights $^{(1)}$; :::; $^{(n)}$ 2 t_Z Z-span the weight lattice t_Z if and only if is faithful.

Proof. We rst argue that if the weights do not span t over R, then cannot be faithful. Assume thus that

$$W = \mathrm{Span}_R \ f \ ^{(i)} g_{i\,2I}$$

has dimension k < m. Then its annihilator space

$$
\begin{aligned}
W^0 &= f f \ 2 \ (t\ ) \ \ j \ f(' \ ) = 0 \quad \text{for all} \ \ ' \ 2 \ W g \\
&= f X \ 2 \ t \ j \ ' \ (X\ ) = 0 \quad \text{for all} \ \ ' \ 2 \ W g \\
&= f X \ 2 \ t \ j \ \ ^{(i)}(X\ ) = 0 \quad \text{for} \ \ i \ 2 \ I g :
\end{aligned}
$$

has strictly positive dimension m k. Hence we can choose a nonzero element X 2 tnt$_Z$ such that $^{(i)}$(X) = 0 for all i 2 I . However, the corresponding nontrivial element of T would then act trivially on V showing that can not be faithful.

Therefore, if is faithful, the weights must R-span t and := Span$_Z$ f $^{(i)}$g$_{i=1\,;:::;n}$ is a sublattice of the weight lattice. Taking the dual we get (t$_Z$) ( ) or (by Lemma 2.1.4) equivalently t$_Z$ with

$$
\begin{aligned}
&= (\quad) \\
&= f f \ 2 \ (t\ ) \ \ j \ f(' \ ) \ 2 \ 2 \ Z \quad \text{for all} \ \ ' \ 2 \quad g \\
&= f X \ 2 \ t \ j \ ' \ (X\ ) \ 2 \ 2 \ Z \quad \text{for all} \ \ ' \ 2 \quad g \\
&= f X \ 2 \ t \ j \ \ ^{(i)}(X\ ) \ 2 \ 2 \ Z \quad \text{for all} \ \ i = 1\,; :::; n g:
\end{aligned}
$$

The weights determine a map $\alpha = \alpha^{(1)} \oplus \cdots \oplus \alpha^{(n)} : \mathfrak{t} \to \mathbb{R}^n$. Composing this with the projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/(2\pi\mathbb{Z})^n$, we get a map whose kernel contains exactly those elements of $\mathfrak{t}$ that act trivially

$$\tilde{\rho} := \pi \circ \alpha : \mathfrak{t} \to \mathbb{R}^n/(2\pi\mathbb{Z})^n$$
$$v \mapsto \pi(\alpha^{(1)}(v), \ldots, \alpha^{(n)}(v)).$$

Clearly, this kernel contains the integral lattice. On the other hand, from the explicit expression we get that

$$\begin{aligned}
\ker(\tilde{\rho}) &= \{X \in \mathfrak{t} \mid \pi \circ \alpha(X) = 0\} \\
&= \{X \in \mathfrak{t} \mid \alpha(X) \in \ker(\pi) = (2\pi\mathbb{Z})^n\} \\
&= \{X \in \mathfrak{t} \mid \alpha^{(i)}(X) \in 2\pi\mathbb{Z} \quad \text{for all} \quad i = 1, \ldots, n\} \\
&= \Lambda.
\end{aligned}$$

We can now conclude since

$$\begin{aligned}
\rho \text{ faithful} \quad &\iff \quad \ker(\tilde{\rho}) = \mathfrak{t}_\mathbb{Z} \\
&\iff \quad \Lambda = \mathfrak{t}_\mathbb{Z} \\
&\iff \quad \Lambda^* = \mathfrak{t}_\mathbb{Z}^* \\
&\iff \quad \mathrm{Span}_\mathbb{Z}\{\alpha^{(i)}\}_{i=1,\ldots,n} = \mathfrak{t}_\mathbb{Z}^*.
\end{aligned}$$

Therefore, for a faithful symplectic representation of a torus, the weights span the weight lattice. The number of weights must thus be equal or bigger than the dimension of the vector space. In the case of equality, the weights not only span the weight lattice over $\mathbb{Z}$ but they are also linearly independent. This means that they form a basis of the weight lattice.

Corollary 6.3.4. Suppose that $\rho : T \to Sp(V, \omega)$ is a symplectic representation of a torus T on a symplectic vector space (V, ω). If ρ is faithful, then

$$\dim(T) \leq \frac{1}{2}\dim(V).$$

If equality holds, the set of weights for ρ forms a basis of the weight lattice.

Since the dimension of the isotropy representations at fixed points coincides with the dimension of the manifold, this immediately carries over. In the case of equality, the weights of the isotropy representations form a basis for each fixed point and therefore by Lemma 3.3.1 the local cones at every fixed point are $\mathfrak{t}_\mathbb{Z}^*$-unimodular. Since any vertex is the image under a fixed point, the local cone at every vertex is $\mathfrak{t}_\mathbb{Z}^*$-unimodular and hence the moment polytope is $\mathfrak{t}_\mathbb{Z}^*$-unimodular. Moreover, since the local cones at fixed points are $\mathfrak{t}_\mathbb{Z}^*$-unimodular, they are pointed and $\mu(p)$ is a vertex for any fixed point p of the T-action.

Corollary 6.3.5. Suppose that (M, ω, T, μ) is a compact connected Hamiltonian T-space for a torus T. If the T-action is faithful, then

$$\dim(T) \leq \frac{1}{2}\dim(M).$$

If equality holds, then the image $\mu(p)$ of any point $p \in M$ which is fixed by the T-action is a vertex of the $t_{\mathbb{Z}}$-unimodular moment polytope $\mu(M)$.

Chapter 7

Symplectic Toric Manifolds

In the last chapter, the focus is now on symplectic manifolds which are equipped with an effective torus action such that the dimension boundary established at the end of the previous chapter is saturated. Such manifolds are referred to as symplectic toric manifolds and the image of their moment maps are unimodular polytopes. It is a result by Delzant that symplectic toric manifolds are classified by unimodular polytopes that are their moment polytopes. The goal of this last chapter is to investigate recursive aspects of this correspondence. On the polytope side, this has already been done in Chapter 4. Now, we will investigate the corresponding properties on the side of symplectic toric manifolds. Our starting point will be the local picture given by symplectic toric representations.

7.1 Symplectic Toric Representations

Definition. A symplectic toric T-representation is a symplectic representation $\rho : T \to Sp(V;\omega)$ of a torus T on a symplectic vector space $(V;\omega)$ such that for $T_e = T/\ker(\rho)$ we have

$$\dim(T_e) = \frac{1}{2}\dim(V).$$

Assume that ρ is a faithful representation and hence that Corollary 6.3.4 applies. Since by assumption we are in the case of equality, we conclude that the weights $\{\alpha^{(i)}\}_{i\in I}$ of the representation form a basis of the weight lattice $\mathfrak{t}^*_{\mathbb{Z}}$ and by Lemma 6.2.3 that the image of the moment map is

$$\mu(V) = \operatorname{co}(\{\alpha^{(i)}\}_{i\in I}).$$

By Corollary 3.3.1 this is a $\mathfrak{t}_{\mathbb{Z}}$-unimodular polyhedral cone.

Reversely, given a pointed $\mathfrak{t}_{\mathbb{Z}}$-unimodular cone, we get a basis $\{v_i\}_{i\in I}$ of the integral lattice $\mathfrak{t}_{\mathbb{Z}}$. Taking the dual lattice basis of the weight lattice $\mathfrak{t}^*_{\mathbb{Z}}$ then determines a faithful representation as described in Lemma 6.2.3.

It is clear that these two constructions are reciprocally inverse to each other and we can conclude that faithful symplectic toric representations are classified by $\mathfrak{t}_\mathbb{Z}$-unimodular polyhedral cones in $\mathfrak{t}^*$. This can be seen as the local version of the Delzant correspondence.

Remark. Note that this also covers the more general notion of a symplectic toric representation above. The representation ρ of T induces a faithful representation ρ_e of T_e and then the classification applies to ρ_e.

What is the preimage of a given point φ in the moment cone of a faithful symplectic toric representation $\rho : T \to Sp(V;\omega)$? To answer this question, we first set the notation and write

$$\Delta(V) = co(\{\alpha^{(i)}\}_{i\in I}) = \bigcap_{i\in I} H_{v_i}$$

where $\{\alpha^{(i)}\}_{i\in I}$ and $\{v_i\}_{i\in I}$ are dual bases of the weight lattice $\mathfrak{t}^*_\mathbb{Z}$ and the integral lattice $\mathfrak{t}_\mathbb{Z}$ respectively. From the explicit expression in Lemma 6.2.3 we see that we can write

$$\varphi = \sum_{i\in I} \lambda_i \alpha^{(i)} = \frac{1}{2}\sum_{i\in I} |z_i|^2 \alpha^{(i)}$$

and by linear independence of the $\{\alpha^{(i)}\}_{i\in I}$ we get for all $i \in I$

$$\lambda_i = \frac{1}{2}|z_i|^2 \iff |z_i| = \sqrt{2\lambda_i}.$$

Thus we get that

$$\mu^{-1}(\varphi) = \left\{ (S_{\sqrt{2\lambda_1}}, \dots, S_{\sqrt{2\lambda_n}}) \in V = \bigoplus_{i=1}^{n} V_i \right\}$$

where, using again the identification of V_i with $\mathbb{C}$ from Lemma 6.2.3, we see that

$$\begin{aligned} S_{\sqrt{2\lambda_i}} &= \{ z \in \mathbb{C} \mid |z| = \sqrt{2\lambda_i} \} \\ &= \{ \sqrt{2\lambda_i}\, e^{2\pi i t} \mid t \in [0, 2\pi[\} \\ &= \{ \sqrt{2\lambda_i}\, e^{i\langle \alpha^{(i)}, \xi\rangle} \mid [\xi] \in T \} \end{aligned}$$

In other words, $S_{\sqrt{2\lambda_i}}$ is the orbit of $\sqrt{2\lambda_i}$ under the S^1-subgroup generated by v_i in the subrepresentation V_i. Combining the different orbits, one sees that $\mu^{-1}(\varphi)$ is the T-orbit of $(\sqrt{2\lambda_1}, \dots, \sqrt{2\lambda_n}) \in \bigoplus_{i=1}^{n} V_i$ under the representation ρ.

We then notice that for any $j \in I$

$$\langle \varphi, v_j \rangle = \sum_{i\in I} \lambda_j \left\langle \alpha^{(i)}, v_j \right\rangle = 2\lambda_j$$

so that

$$I_\varphi = \{ i \in I \mid \lambda_i = 0 \}.$$

Thus for all $i \in I_\varphi$, the circle reduces to a point and the dimension of the orbit $\mu^{-1}(\varphi)$ is thus

$$\dim \mu^{-1}(\varphi) = |I| - |I_\varphi| = \dim(L_\varphi(\mu(V))) .$$

Reversely, an element $[\xi] \in \mathfrak{t}/\mathfrak{t}_Z = T$ will act trivially on $\mu^{-1}(\varphi)$ precisely if only the components corresponding to the S^1-subgroups generated by $\{v_i\}_{i \in I_\varphi}$ are non-vanishing. Hence the stabiliser of $\mu^{-1}(\varphi)$ is the image of the annihilator space $W_\varphi(\mu(V))$ under the projection $\mathfrak{t} \to \mathfrak{t}/\mathfrak{t}_Z$, or in other words, it is the subtorus whose Lie algebra is $W_\varphi(\mu(V))$.

We summarise this discussion in the following proposition:

Proposition 7.1.1. *For $\rho : T \to Sp(V;\omega)$ a symplectic toric representation, the moment cone $C = \mu(V)$ is the orbit space and the moment map is the point-orbit map. In particular, for any $\varphi \in \mu(V)$ we have that*

1. *$\mu^{-1}(\varphi)$ is a single T-orbit;*
2. *the dimension of this orbit is equal to the dimension of the tangent space to C at φ i.e.*
$$\dim \mu^{-1}(\varphi) = \dim(L_\varphi(C))$$
and
3. *the stabiliser of $\mu^{-1}(\varphi)$ is the subtorus whose Lie algebra is the annihilator space to C at φ i.e*
$$\mathfrak{t}_{\mu^{-1}(\varphi)} = W_\varphi(C).$$

Remark. This is again a local version of a result by Delzant ([7], Lemma 2.2 or Theorem 7.3.4 in this master's thesis).

7.2 Recursive Aspects of Symplectic Toric Representations

Mirroring the approach of Chapter 4, we now investigate recursive aspects of symplectic toric representations. The natural object to investigate are symplectic subrepresentations:

Definition. A *symplectic toric subrepresentation* of a symplectic toric representation $(V;\omega)$ is a T-invariant symplectic subspace $W \subset V$.

While it is clear that these are symplectic representations in their own right, one must show that the dimension requirement is met.

Proposition 7.2.1. A symplectic toric subrepresentation is itself a symplectic toric representation.

Proof. Let $\rho : T \to Sp(V; \omega)$ denote the representation and $\sigma : T \to Sp(W; \omega|_\omega)$ be the induced representation on $W \subset V$. Let further $T_W = \ker \sigma$ and $T_V = \ker \rho$ be the subtori of T acting trivially on W and V respectively and note that $T_V < T_W$. Clearly, the representation of T/T_W on W induced by σ is faithful and therefore by Corollary 6.3.4

$$\dim \left(T \big/ T_W \right) \geq \frac{1}{2} \dim(W).$$

Since W is a symplectic subspace of V, it holds that $V = W \oplus W^\omega$. The representation of T_W/T_V on V is by construction faithful and acts trivially on W. Hence it must act faithfully on the subrepresentation W^ω giving again by Corollary 6.3.4

$$\dim \left(T_W \big/ T_V \right) \geq \frac{1}{2} \dim(W^\omega) = \frac{1}{2}(\dim(V) - \dim(W)).$$

Hence

$$\begin{aligned} \dim \left(T \big/ T_W \right) &= \dim \left(T/T_V \big/ T_W/T_V \right) \\ &= \underbrace{\dim \left(T \big/ T_V \right)}_{= \frac{1}{2}\dim(V)} - \underbrace{\dim \left(T_W \big/ T_V \right)}_{\geq \frac{1}{2}(\dim(V) - \dim(W))} \\ &\leq \frac{1}{2}\dim(V) - \frac{1}{2}(\dim(V) - \dim(W)) \\ &= \frac{1}{2}\dim(W) \end{aligned}$$

so that indeed

$$\dim \left(T \big/ T_W \right) = \frac{1}{2}\dim(W)$$

and we conclude that W is symplectic toric representation.

Proposition 7.2.2. Symplectic toric subrepresentations are in one-one correspondence with the faces of the moment cone.

Proof. Let $\rho : T \to (V; \omega)$ be a symplectic toric representation and let

$$C = co(\{\alpha^{(i)}\}_{i \in I}) = \bigcap_{i \in I} H_{v_i}$$

be its moment cone, where $\{\alpha^{(i)}\}_{i\in I}$ and $\{v_i\}_{i\in I}$ are dual lattice bases of $\mathfrak{t}^*_{\mathbb{Z}}$ and $\mathfrak{t}_{\mathbb{Z}}$ respectively. We use the notation from Lemma 6.2.3.

Claim. Symplectic toric subrepresentations are in one-one correspondence with subsets $J \subseteq I$ of the index set.

Proof of Claim. Assume $v \in V_i$ is in a subrepresentation W. Then the whole of V_i is contained in W since any point of V_i can be reached via scaling and rotating. Hence there is a subset $J \subseteq I$ such that $W = \bigoplus_{j\in J} V_j$. Reversely, for any subset $J \subseteq I$, we get a symplectic toric subrepresentation by putting $W = \bigoplus_{j\in J} V_j$. This shows the claim.

But then we note that if $i : W \hookrightarrow V$ is the inclusion, then

$$
\begin{aligned}
i^*\mu(W) &= \mu(i(W)) \\
&= \mu\Big(i\Big(\bigoplus_{j\in J} V_j\Big)\Big) \\
&= \mathrm{cone}(\{\alpha^{(j)}\}_{j\in J}) \\
&= \bigcap_{j\in J} H_{v_j} \cap \mathrm{Span}_{\mathbb{R}}\{\alpha^{(j)}\}_{j\in J} \\
&= \bigcap_{j\in J} H_{v_j} \cap \bigcap_{i\in I\setminus J} \partial H_{v_i}
\end{aligned}
$$

using Lemma 3.2.1. This shows that $i^*\mu(W)$ is the face of C associated to $I_F = I\setminus J$.

Reversely, from any face F we get an index set $J = I\setminus I_F$ and therefore a symplectic toric subrepresentation $W = \bigoplus_{j\in J} V_j$ whose image under the moment map is exactly F.

7.3 Symplectic Toric Manifolds

Finally, we introduce the main object of interest, symplectic toric manifolds. Note however, that we give a slightly different definition than what is currently standard in the literature. While the definition given here also includes the standard one as a special case, we would argue that the following definition is better suited for the investigation of recursive aspects:

Definition. A symplectic toric T-manifold is a compact connected symplectic manifold (M, ω) equipped with

- a Hamiltonian action $\rho : T \to \mathrm{Sympl}(M)$ such that for $T_e = T/\ker(\rho)$ we have

$$\dim(T_e) = \frac{1}{2}\dim(M);$$

ˆ and a choice of a moment map : M ! t .

Around a xed point, a symplectic toric manifold looks like a symplectic toric representation:

Proposition 7.3.1. Let (M; !; T;) be a symplectic toric manifold. Then the isotropy representation at a xed point p 2 M is a symplectic toric representation.

Proof. Consider the induced e ective action : T_e ! Di (M). By Lemma 5.1.1, the isotropy representation is symplectic. By Proposition 6.3.2 it is faithful and so the result follows.

The next goal is to state Delzant's classi cation theorem. For this, one must rst decide when two symplectic toric manifolds are to be considered the same.

De nition. Let M_1 be a symplectic toric T_1-manifold and M_2 be a symplectic toric T_2-manifold. Denote the actions by : T_1 ! Di (M_1) and : T_2 ! Di (M_2) respectively. M_1 and M_2 are calledisomorphic or equivalentif there exists a group homomorphism : T_1 ! T_2 and a symplectomorphism' : M_1 ! M_2 which is (;)-equivariant with respect to , that is, the following diagram commutes for all t 2 T_1

$$\begin{array}{ccc} M_1 & \xrightarrow{\ '\ } & M_2 \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle (t)} \\ M_1 & \xrightarrow{\ '\ } & M_2 \end{array}$$

Proposition 7.3.2. Let M_1 and M_2 be isomorphic symplectic toric manifolds. Then, using the notation from the de nition above,

$$_1 = (d \quad _e) \quad _2 \quad ' \ +$$

where 2 t_1 is a constant.

Proof. Let X 2 t_1 be an arbitrary element of the Lie algebra ofT_1. Then consider the map

$$\begin{aligned} {}^X_1 : M_1 & ! \quad R \\ p & 7! \ h \ _1(p); X i \end{aligned}$$

and compute its di erential:

$$\begin{aligned} d \ {}^X_1 &= \quad _{X\,]} !_1 \\ &= \quad _{X\,]} ' \quad !_2 \\ &= \quad ' \ (\ _{(d \quad _e[X])\,]} !_2) \\ &= ' \ (d \ {}^d_2 \ {}_e[X]) \\ &= d \ ' \quad {}^d_2 \ {}_e[X] \quad : \end{aligned}$$

But observing that for any $p \in M_1$ we have

$$\begin{aligned} \varphi^* \mu_2^{d\psi_e[X]}(p) &= \mu_2^{d\psi_e[X]}(\varphi(p)) \\ &= \langle \mu_2(\varphi(p)), d\psi_e[X] \rangle \\ &= \langle d\psi_e^* \circ \mu_2 \circ \varphi(p), X \rangle \\ &= (d\psi_e^* \circ \mu_2 \circ \varphi)^X(p) \end{aligned}$$

and as $X \in \mathfrak{t}_1$ was arbitrary, the differentials on both sides of the equation in the statement coincide. This concludes the proof since M_1 is connected.

Remark. Consider the composition of $\psi : T_1 \to T_2$ with the projection $\pi_2 : T_2 \to T_2/\ker(\rho_2)$. By equivariance it follows that if $t \in \ker(\rho_1)$, then $\psi(t) \in \ker(\rho_2)$ and hence $\ker(\rho_1) \subseteq \ker(\pi_2 \circ \psi)$. It follows that $\pi_2 \circ \psi$ descends to the quotient and we get a Lie group homomorphism $\bar\psi : T_1/\ker(\rho_1) \to T_2/\ker(\rho_2)$. Moreover, if $\bar\rho_1 : T_1/\ker(\rho_1) \to \mathrm{Diff}(M_1)$ and $\bar\rho_2 : T_2/\ker(\rho_2) \to \mathrm{Diff}(M_2)$ are the induced effective actions, then φ is $(\bar\rho_1, \bar\rho_2)$-equivariant with respect to $\bar\psi$ i.e. the following diagram commutes for all $t \in T_1/\ker(\rho_1)$:

$$\begin{array}{ccc} M_1 & \xrightarrow{\ \varphi\ } & M_2 \\ \big\downarrow \bar\rho_1(t) & & \big\downarrow \bar\rho_2(\bar\psi(t)) \\ M_1 & \xrightarrow{\ \varphi\ } & M_2 \end{array}$$

By effectiveness of the two actions in this diagram, $\bar\psi$ has to be injective. Indeed, if $t \in \ker(\bar\psi)$, then $\bar\rho_2(\bar\psi(t))$ acts trivially and therefore so does $\bar\rho_1(t)$. Since $\bar\rho_1$ is effective, this implies that $t = e$ and thereby that $\bar\psi$ is injective. Because $\bar\psi$ as a Lie group homomorphism has constant rank, it follows that $d\bar\psi_e : (\mathfrak{t}_1)_e \to (\mathfrak{t}_2)_e$ is injective. By equality of dimensions, $d\bar\psi_e$ is an isomorphism.

Using this, the interpretation of the above Proposition 7.3.2 is that the moment polytopes, when considered as polytopes in $(\mathfrak{t}_1)_e^*$ and $(\mathfrak{t}_2)_e^*$, are translates of each other.

This was the last ingredient needed to state Delzant's Theorem. We will assume for this that the actions are directly effective and that the polytopes are not contained in any affine subspace.

Theorem 7.3.3 (Delzant). Symplectic toric manifolds are classified up to equivalence by unimodular polytopes up to translation. More specifically, the bijective correspondence between these two sets is given by the moment map:

$$\{\text{symplectic toric manifolds}\}\big/_{\text{equiv.}} \xleftrightarrow{\ 1-1\ } \{\text{unimodular polytopes}\}\big/_{\text{transl.}}$$

$$(M, \omega, T, \mu) \longmapsto \mu(M)$$

Proof. See the original paper [7] or Chapter 29 of [5] for an English version.

The arguments made prior to Proposition 7.1.1 can be adapted to Delzants construction. This yields the following result:

Theorem 7.3.4 (Delzant). For a symplectic toric manifold $(M; \omega; T; \mu)$, the moment polytope $\Delta = \mu(M)$ is the orbit space and the moment map is the point-orbit map. In particular, for any $\varphi \in \mu(M)$ we have that:

1. $\mu^{-1}(\varphi)$ is a single T-orbit;
2. the dimension of this orbit is equal to the dimension of the tangent space to Δ at φ i.e.
$$\dim \mu^{-1}(\varphi) = \dim(L_\varphi(\Delta))$$
and
3. the stabiliser of μ^{-1} is the subtorus whose Lie algebra is the annihilator space to Δ at φ i.e.
$$\mathfrak{t}_{\mu^{-1}(\varphi)} = W_\varphi(\Delta) :$$

Proof. See [7], Lemma 2.2. or [6], Theorem 2.4.5.

Corollary 7.3.5. Let $(M; \omega; T; \mu)$ be a symplectic toric manifold. Then the fixed points in M are in one-to-one correspondence with the vertices of the moment polytope $\mu(M)$.

Proof. Consider a vertex $\varphi \in \mu(M)$ of the moment polytope. It follows by the first two points of Theorem 7.3.4 that $\mu^{-1}(\varphi)$ is a zero-dimensional T-orbit. Since T is a connected Lie group which acts smoothly on M, this implies that $\mu^{-1}(\varphi)$ consists of a single point which is hence a fixed point of the action.

Reversely, suppose that p is a fixed point and look at $\varphi = \mu(p)$. Assume the contraposition i.e. that φ is not a vertex and deduce from Theorem 7.3.4 that then $\mu^{-1}(\varphi)$ would be a single orbit of strictly positive dimension. But this gives a contradiction since by construction $p \in \mu^{-1}(\varphi)$ and the orbit of p has dimension zero since p is assumed to be a fixed point.

7.4 Recursive Aspects of Symplectic Toric Manifolds

7.4.1 Symplectic Toric Submanifolds

The next goal is to define the global version of a symplectic toric subrepresentation. The goal to have in mind is that this should be a symplectic toric manifold in its own right. Hence it should surely be invariant under the action, so the latter can be restricted to it. Moreover, it should be a closed connected symplectic manifold in its own right. This suggests the following definition:

Definition. A symplectic toric submanifold N of a a symplectic toric T-manifold M is a T-invariant closed connected symplectic submanifold N.

Just as for symplectic toric subrepresentations, one has to check the dimension requirement. However, this can be reduced to the case of symplectic toric representations.

Proposition 7.4.1. A symplectic toric submanifold N of M is itself a symplectic toric T-manifold.

Proof. Let $i : N \hookrightarrow M$ be the inclusion. By Lemma 5.3.6 it is clear that $(N, i^*\omega, T, i^*\Phi)$ is a hamiltonian T-space.

Claim. There exists a fixed point $p \in N$.

Proof of Claim. Since N is closed and M is compact, N is itself compact. By assumption it is also connected and so it satisfies the hypotheses of the convexity theorem 6.3.1. Since N is non-empty, so is its moment polytope which hence contains a vertex. Again by Theorem 6.3.1 this vertex is the image of a fixed point $p \in N$. Since the action of T on N is just the restriction of the action on M, this shows the claim.

Consider the isotropy representation $\rho : T \to Sp(T_pM, \omega_p)$ at $p \in N$, which by Lemma 7.3.1 is a symplectic toric representation. Since N is a symplectic submanifold T_pN is a symplectic subspace of T_pM. Since N is T-invariant, T_pN is a symplectic subrepresentation. Hence T_pN is a symplectic toric subrepresentation and the statement follows by Proposition 7.2.1.

Theorem 7.4.2. Symplectic toric submanifolds are in one-one correspondence with the faces of the moment polytope.

Proof. Let (M, ω, T, Φ) be a symplectic toric manifold and $\Delta = \Phi(M)$ be its moment polytope. We first show that the image of a symplectic toric submanifold N is a face of

$$\Phi(M) = \Delta = \bigcap_{i \in I} H_{(v_i, c_i)}.$$

Denoting $j : N \hookrightarrow M$ the inclusion, proposition 7.4.1 gives that $(N, j^*\omega, T, j^*\Phi)$ is again a symplectic toric manifold. Hence, by Corollary 7.3.5 $j^*\Phi(N) = \Phi(j(N))$ is the convex hull of the images of the fixed points of the action $\rho : T \to \mathrm{Diff}(M)$ contained in N. Using Corollary 1.2.2 this shows that $\Phi(j(N))$ is the convex hull of a subset of vertices of $\Phi(M)$.

Consider then an arbitrary fixed point $p \in N$. By Proposition 7.3.1, the isotropy representation at this point is a symplectic toric representation. The image under its moment map is the local cone

$$C_{\ell}(\Delta) = \bigcap_{i \in I_{\ell}} H_{v_i}$$

at $\varphi := \mu(p)$. Since N is a symplectic toric submanifold, T_pN is a symplectic toric subrepresentation. By Proposition 7.2.1, T_pN corresponds to a face, meaning that the image under its moment map, given by the local cone $C_\varphi(\mu(j(N)))$, is a face of $C_\varphi(\Delta)$. Hence there exists a subset $I_F \subset I_\varphi$ and a non-zero vector $v = \sum_{i \in I_F} v_i$ such that

$$C_\varphi(\mu(j(N))) = C_\varphi(\Delta) \cap \partial H_v.$$

Claim. For $c := \langle \varphi, v\rangle$, we have

$$\mu(j(N)) = \mu(M) \cap \partial H_{(v,c)} := F.$$

Proof of Claim. Take any point $q \in N$ and consider $\psi = \mu(j(q))$, its image under μ. By convexity, the entire interval $[\varphi, \psi]$ connecting φ and ψ is contained in $\mu(j(N))$. Hence, $\psi - \varphi \in C_\varphi(\mu(j(N)))$ which yields that

$$\langle \psi, v\rangle = \langle \varphi, v\rangle = c.$$

Hence $\psi \in F$. Since $q \in N$ was arbitrary, this shows $\mu(j(N)) \subset F$.

To show the reverse inclusion, it is enough to show that any vertex of the face F is contained in $\mu(j(N))$. To see this we start by noticing that both F and $\mu(j(N))$ are unimodular polytopes of dimension $\dim(N)/2$. In particular, at any vertex there are $\dim(N)/2$ edges meeting. If not all vertices of F were contained in $\mu(j(N))$, then there existed an edge $e \subset F$ connecting a vertex $\varphi \in \mu(j(N))$ and a vertex $\psi \in F \setminus \mu(j(N))$. But the existence of such an edge would imply that in $\mu(j(N))$ there is another edge $e' \subset \mu(j(N))$, not collinear to e, ending at φ. By linear independence of the edges of $\mu(M)$ meeting at φ, this implies that $C_\varphi(\mu(j(N)))$ can not be a face of $C_\varphi(\mu(M))$. This is a contradiction with N being a symplectic toric submanifold and therefore the claim is proven.

Reversely, let F be a face of

$$\mu(M) = \Delta = \bigcap_{i \in I} H_{(v_i, c_i)}$$

and consider $N := \mu^{-1}(F)$. Since μ is continuous and F is closed, N is closed. Furthermore, $\mu^{-1}(F)$ is made up of the orbits corresponding to the points in F so it is clearly T-invariant.

Consider then a point φ in the interior of F and note that

$$F = \Delta \cap H_{(v,c)} \quad \text{where} \quad v = \sum_{i \in I_\varphi} v_i \quad \text{and} \quad c = \sum_{i \in I_\varphi} c_i$$

In particular, since all the v_i are t_Z-rational, v is also t_Z-rational. It follows that v determines an S^1-subgroup of T which by Proposition 5.3.4 acts in a Hamiltonian way on M: If $i : S^1 \hookrightarrow T$ is the inclusion, the moment map is given by $di_e : M \to s$ where s denotes the Lie algebra of the S^1-subgroup. Note in particular that by construction $v \in s$ and $di_e : s \hookrightarrow t$ is just the inclusion. Also, since s is one-dimensional and $v \neq 0$, we note that v is a basis of s. Hence for any $p \in M$, $di_e (p)$ is completely determined by its action on v. Identifying s with its image under di_e we get

$$\langle di_e (p); v\rangle = \langle (p); di_e[v]\rangle = \langle (p); v\rangle :$$

But $(p) \in$ so that

$$\langle (p); v\rangle = c \iff (p) \in F \iff p \in N$$

which shows that N is a level set of di_e. Therefore, by the convexity Theorem 6.3.1 N is connected.

By Theorem 6.2.5 we know that $(di_e)^v$ is a Morse-Bott function and the connected components of $Crit((di_e)^v)$ are thus symplectic submanifolds. However, by construction N is such a critical submanifold of $(di_e)^v$. This shows that it is a symplectic submanifold of M and hence concludes the proof.

7.4.2 Products of Symplectic Toric Manifolds

We saw in Proposition 5.3.9 that the product of two Hamiltonian spaces is again Hamiltonian. We now want to see whether we get a symplectic toric manifold if we start out with two symplectic toric manifolds.

We thus suppose that $(M_1; !_1; G; _1; _1)$ and $(M_2; !_2; H; _2; _2)$ are two symplectic toric manifolds. By Proposition 5.3.9 (we also recover the notation from this result) the product manifold

$$(M_1 \quad M_2; pr_1 !_1 + pr_2 !_2; G \quad H; d(pr_G)_{(e;e)} \quad _1 \quad pr_1 + d(pr_H)_{(e;e)} \quad _2 \quad pr_2)$$

is a hamiltonian $G \quad H$-space. The action is given by

$$\begin{aligned} ' : G \quad H &\to Di (M_1 \quad M_2) \\ (g; h) &\mapsto {}_g \quad {}_h : \end{aligned}$$

and so we see that $\ker(') = \ker( ) \quad \ker( )$. It follows that the dimension requirement is met as

$$\begin{aligned} \dim \frac{G \quad H}{\ker(')} &= \dim \frac{G \quad H}{\ker( ) \quad \ker( )} \\ &= \dim \frac{G}{\ker( )} \quad \frac{H}{\ker( )} \\ &= \frac{1}{2}\dim(M_1) + \frac{1}{2}\dim(M_2) \\ &= \frac{1}{2}\dim(M_1 \quad M_2) : \end{aligned}$$

In conclusion, we have thus shown the following proposition:

Proposition 7.4.3. Let $(M_1, \omega_1, G, \rho_1, \mu_1)$ and $(M_2, \omega_2, H, \rho_2, \mu_2)$ be two symplectic toric manifolds. Let $pr_{1,2} : M_1 \times M_2 \to M_{1,2}$ and $pr_{G,H} : G \times H \to G, H$ denote the projections. Then $(M_1 \times M_2, \omega, G \times H, \rho, \mu)$ with

- symplectic form $\omega = pr_1^* \omega_1 + pr_2^* \omega_2$ and
- moment map $\mu = d(pr_G)^*_{(e,e)} \circ \mu_1 \circ pr_1 + d(pr_H)^*_{(e,e)} \circ \mu_2 \circ pr_2$

is a symplectic toric manifold.

This moment map is easier than it looks. To see why, first observe that it has image in $(\mathfrak{g} \times \mathfrak{h})^* = \mathfrak{g}^* \times \mathfrak{h}^*$ and that $d(pr_G)^*_{(e,e)} : \mathfrak{g}^* \hookrightarrow \mathfrak{g}^* \times \mathfrak{h}^*$ and $d(pr_H)^*_{(e,e)} : \mathfrak{h}^* \hookrightarrow \mathfrak{g}^* \times \mathfrak{h}^*$ just correspond to the obvious inclusions. Omitting them from the notation we could thus just write

$$\mu(p, q) = \mu_1(p) + \mu_2(q).$$

It follows that the moment polytope of $M_1 \times M_2$ corresponds to the direct Minkowski sum of the moment polytopes of M_1 and M_2:

$$\mu(M_1 \times M_2) = \mu_1(M_1) \oplus \mu_2(M_2).$$

This observation allows to transfer all the results about the Minkowski sum to the product of symplectic toric manifolds. Consider for instance Proposition 4.4.4 in combination with Theorem 7.4.2:

Corollary 7.4.4. Let $(M_1, \omega_1, G, \rho_1, \mu_1)$ and $(M_2, \omega_2, H, \rho_2, \mu_2)$ be two symplectic toric manifolds. Then the symplectic toric submanifolds of $(M_1 \times M_2, \omega, G \times H, \rho, \mu)$ are in one-one correspondence with products of symplectic toric submanifolds of M_1 and M_2 respectively.

Example. Consider $M_1 \times M_2$ as a symplectic toric manifold as described above. Take $q \in M_2$ and consider the injective map

$$\begin{aligned} i_q : M_1 &\hookrightarrow M_1 \times M_2 \\ p &\mapsto (p, q). \end{aligned}$$

The goal is to interpret M_1 as a symplectic toric submanifold of $M_1 \times M_2$ using this injection. It is clear that $i_q(M_1)$ is a connected closed symplectic submanifold. However, it is only $G \times H$-invariant if $q \in M_2$ is a fixed point under the H-action. Hence for any fixed point $q \in M_2$, there is a symplectic toric submanifold $i_q(M_1)$ which is equivalent to M_1.

On the level of the moment polytopes, the corresponding picture is that the direct Minkowski sum of $\mu_1(M_1)$ with a vertex φ of $\mu_2(M_2)$ is equivalent to $\mu(M_1)$:

$$\mu_1(M_1) \oplus \{\varphi\} \cong \mu_1(M_1).$$

7.4.3 Symplectic Cutting and Symplectic Reduction

We close this master's thesis with a few words about the symplectic cutting of symplectic toric manifolds. At the end of chapter 5, it has been shown that

$$(M_c) = (M) \cap H_{(v;c)}:$$

In the case of a symplectic toric manifolds, (M) is unimodular polytope. If $\partial H_{(v;c)}$ is a reduction level of (M), then also (M_c) is a unimodular polytope. However, one has to be careful here. On the side of the symplectic toric manifolds, one has to require that the S^1 subgroup of the torus acts freely on $^{1}(c)$. If this is the case, then M_c will be another symplectic toric manifold and (M_c) the corresponding unimodular polytope.

The converse is not yet entirely understood. If $\partial H_{(v;c)}$ is a reduction level for (M), it is clear by the Delzant classification that (M_c) defines another symplectic toric manifold. But in order to show that it is indeed obtained by cutting M, the requirement that the S^1-subgroup acts freely on $^{1}(c)$ has to be checked. In the conclusion below, we sketch a possible answer to this issue.

Conclusion

In Part I, we have introduced unimodular polytopes and studied their recursive properties. We have seen that the faces are again unimodular polytopes, how the direct Minkowski-sum makes a new unimodular polytope of two old ones and saw how one can cut a polytope preserving unimodularity.

In Part II, we have presented Hamiltonian spaces with a focus on recursive aspects. We have then specialised to Hamiltonian torus actions and their moment polytopes. We defined symplectic toric manifolds and argued why their moment polytopes are unimodular. It was shown that the product of symplectic toric manifolds corresponds to a special case of the Minkowski-sum which we called the direct Minkowski-sum. The main result was that the symplectic toric submanifolds of a symplectic toric manifold are classified by the faces of its moment polytope.

We have also approached the subject of symplectic cutting. For the corresponding operation on the unimodular polytopes, we found a necessary and sufficient criterion for a cut to yield unimodular polytopes. On the side of the symplectic toric manifolds however, we are left with the assumption that the action on the level set has to be free. This issue could be accounted for by passing through orbifolds. In order to obtain an orbifold, it would be sufficient that the action on the level set is locally free. This in turn is always the case for the cuts considered. However, instead of unimodular polytopes, orbifolds give rise to simple rational polytopes with labels. The advantage of this process would be that one could establish criteria for this to be unimodular based only on the cutting hyperplane. It would then follow that the cut symplectic toric orbifold is in fact a symplectic toric manifold.

Another direction that could be investigated is the splitting of unimodular polytopes with a hyperplane containing a vertex. In some specific cases, this yields again a unimodular polytope on one side. However, the usual construction of symplectic cutting can not be carried out as usual since the level set contains a fixed point and hence the quotient might have singularities that even can not be addressed with the orbifold formalism.

Finally, there are situations, not covered by the special case seen in this work, where the Minkowski sum of two unimodular polytopes yields another unimodular polytope. It could prove interesting to understand this in more detail and investigate to what operations this corresponds on the level of the symplectic toric manifolds.

Bibliography

[1] Michael Francis Atiyah, *Convexity and commuting hamiltonians*, Bulletin of the London Mathematical Society **14** (1982), no. 1, 1 15.

[2] Michele Audin, *The topology of torus actions on symplectic manifolds*, Vol. 93, Birkhäuser, 2012.

[3] Alexander Barvinok, *A course in convexity*, Vol. 54, American Mathematical Society, 2002.

[4] Alexander Barvinok, *Integer points in polyhedra*, Vol. 452, European Mathematical Society, 2008.

[5] Ana Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2004.

[6] Ana Cannas da Silva, *Seminar on symplectic toric manifolds*, Lecture Notes for Seminar (2019).

[7] Thomas Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bulletin de la Société mathématique de France **116** (1988), no. 3, 315 339.

[8] Shubham Dwivedi, Jonathan Herman, Lisa C Je rey, and Theo Van den Hurk, *Hamiltonian group actions and equivariant cohomology*, Springer, 2019.

[9] Alessandra Iozzi and Robert Zimmer, *Introduction to lie groups*, Lecture Notes, ETH Zürich (2016).

[10] Eugene Lerman and Susan Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Transactions of the American Mathematical Society **349** (1997), no. 10, 4201 4230.

[11] Eckhard Meinrenken, *Symplectic geometry*, Lecture Notes, University of Toronto (2000).

[12] Eckhard Meinrenken, *Group actions on manifolds*, Lecture Notes, University of Toronto (2003).

[13] Will J. Merry, *Di erential geometry*, Lecture Notes for Course at ETH Zürich (2021).

Appendix A

Lie Group Actions

This content of this chapter is taken from [12], [13] and [2].

A.1 Basic Definitions

Definition. An smooth action of a Lie group G on a smooth manifold M is a group homomorphism

$$
\begin{aligned}
: G &\to \mathrm{Diff}(M) \\
g &\mapsto {}_g
\end{aligned}
$$

such that the associated evaluation map

$$
\begin{aligned}
ev : G \times M &\to M \\
(g; p) &\mapsto {}_g(p)
\end{aligned}
$$

is smooth.

Example. Let G be a Lie group. G acts canonically on itself in three different ways which will be used later in this work:

1. left translation is given by

$$
\begin{aligned}
L : G &\to \mathrm{Diff}(G) \\
g &\mapsto (L_g : h \mapsto gh);
\end{aligned}
$$

2. right translation is given by

$$
\begin{aligned}
R : G &\to \mathrm{Diff}(G) \\
g &\mapsto (R_g : h \mapsto hg^{-1})
\end{aligned}
$$

3. and conjugation is the map

$$\begin{aligned} c : G &\to \mathrm{Diff}(G) \\ g &\mapsto (c_g : h \mapsto ghg^{-1}) \end{aligned}$$

which is nothing but the composition of the two translations:

$$c_g = L_g \circ R_g = R_g \circ L_g \quad \text{for all} \quad g \in G.$$

Definition. Let $G_1; G_2$ be Lie groups which act on smooth manifolds $M_1; M_2$ via $: G_1 \to \mathrm{Diff}(M_1)$ and $: G_2 \to \mathrm{Diff}(M_2)$ respectively. Let further $: G_1 \to G_2$ be a Lie group homomorphism and $' : M_1 \to M_2$ a smooth map. We say that $'$ is $(;)$-equivariant with respect to if the following diagram commutes for all $g \in G_1$:

$$\begin{array}{ccc} M_1 & \xrightarrow{\ '\ } & M_2 \\ \big\downarrow {\scriptstyle g} & & \big\downarrow {\scriptstyle (g)} \\ M_1 & \xrightarrow{\ '\ } & M_2 \end{array}$$

Remark. Let $: G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group G on a smooth manifold M.

1. If a submanifold $N \subset M$ is G-invariant i.e. $_g(p) \in N$ for all $p \in N$ and all $g \in G$, the restriction $_g|_N$ is a smooth action on N. Another way of putting this is that there exists a unique action $: G \to \mathrm{Diff}(N)$ of G on N such that the following diagram commutes for all $g \in G$

$$\begin{array}{ccc} N & \xhookrightarrow{\ i\ } & M \\ \big\downarrow {\scriptstyle g} & & \big\downarrow {\scriptstyle g} \\ N & \xhookrightarrow{\ i\ } & M \end{array}$$

   where $i : N \hookrightarrow M$ is the inclusion. In words, the inclusion is $(;)$-equivariant.

2. If $H < G$ is a Lie subgroup, then the composition of the inclusion $i : H \hookrightarrow G$ with is an action of H on M. More generally, if $: H \to G$ is a Lie group homomorphism, then the composition with yields an action of H on M such that the following diagram commutes for all $h \in H$:

$$\begin{array}{ccc} M & \xrightarrow{\ \mathrm{Id}_M\ } & M \\ \big\downarrow {\scriptstyle h} & & \big\downarrow {\scriptstyle (h)} \\ M & \xrightarrow{\ \mathrm{Id}_M\ } & M \end{array}$$

   Again, this just means that the identity Id_M is $(;)$-equivariant with respect to .

A.2 Fundamental Vector Fields

Definition. Let G be a Lie group which acts on a smooth manifold M via $\Phi : G \to \mathrm{Diff}(M)$. The fundamental vector field $X^\sharp$ associated to $X \in \mathfrak{g}$ is defined pointwise by

$$X^\sharp_p = \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(tX)}(p).$$

Remark. An alternative description is obtained using the orbit map of a given point $p \in M$

$$\begin{aligned} \Phi^p : G &\to M \\ g &\mapsto \Phi_g(p) \end{aligned}$$

as

$$X^\sharp_p = d(\Phi^p)_e[X].$$

Indeed, writing

$$\begin{aligned} \gamma : \mathbb{R} &\to G \\ t &\mapsto \exp(tX) \end{aligned}$$

and observing that that $\gamma(0) = e$ and $\dot\gamma(0) = X$ we compute

$$\begin{aligned} d(\Phi^p)_e(X) &= d(\Phi^p)_{\gamma(0)}[\dot\gamma(0)] \\ &= d(\Phi^p)_{\gamma(0)} \circ d\gamma_0 \left(\left.\frac{\partial}{\partial t}\right|_0\right) \\ &= d(\Phi^p \circ \gamma)_0 \left(\left.\frac{\partial}{\partial t}\right|_0\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Phi^p(\gamma(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Phi_{\gamma(t)}(p) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(tX)}(p) \\ &= X^\sharp_p \end{aligned}$$

Recall. Let $f : M \to N$ be a smooth map between smooth manifolds, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be vector fields. Then we say that X and Y are f-related if

$$df_p[X_p] = Y_{f(p)} \quad \forall p \in M.$$

Lemma A.2.1. Let G_1, G_2 be Lie groups which act on smooth manifolds M_1, M_2 via $\Phi : G_1 \to \mathrm{Diff}(M_1)$ and $\Psi : G_2 \to \mathrm{Diff}(M_2)$ respectively. Let further

$\psi : G_1 \to G_2$ be a Lie group homomorphism and $\varphi : M_1 \to M_2$ a smooth map which is $(\alpha;\beta)$-equivariant with respect to ψ. Then for every $X \in \mathfrak{g}_1$, $X^\sharp \in \mathfrak{X}(M_1)$ and $(d\psi_e[X])^\sharp \in \mathfrak{X}(M_2)$ are φ-related i.e. for all $p \in M_1$ we have

$$d\varphi_p[X^\sharp_p] = (d\psi_e[X])^\sharp_{\varphi(p)}.$$

Proof. First we observe that for every $p \in M_1$ and $g \in G_1$ we have

$$\varphi \circ \alpha^p(g) = \varphi \circ \alpha_g(p) = \beta_{\psi(g)} \circ \varphi(p) = \beta^{\varphi(p)} \circ \psi(g)$$

so that $\varphi \circ \alpha^p = \beta^{\varphi(p)} \circ \psi$ for any $p \in M_1$. Using this we get

$$\begin{aligned} d\varphi_p[X^\sharp_p] &= d\varphi_{\alpha^p(e)} \circ d(\alpha^p)_e[X] \\ &= d(\varphi \circ \alpha^p)_e[X] \\ &= d\left(\beta^{\varphi(p)} \circ \psi\right)_e[X] \\ &= d\left(\beta^{\varphi(p)}\right)_e \circ d\psi_e[X] \\ &= (d\psi_e[X])^\sharp_{\varphi(p)} \end{aligned}$$

for any $p \in M_1$ which concludes the proof.

Proposition A.2.2. Let $\alpha : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group G on a smooth manifold M. The map

$$\begin{aligned} \sharp : \mathfrak{g} &\to \mathfrak{X}(M) \\ X &\mapsto X^\sharp \end{aligned}$$

is a Lie algebra homomorphism i.e.

$$[X;Y]^\sharp = [X^\sharp;Y^\sharp] \quad \text{for all} \quad X;Y \in \mathfrak{g}.$$

Proof. We start by considering the map

$$\begin{aligned} F : G \times M &\to M \\ (g;p) &\mapsto \alpha_{g^{-1}}(p) \end{aligned}$$

which is clearly surjective. G acts on the product $G \times M$ via $\gamma := R \times \mathrm{Id}_M$ and on M via α. As for any $g \in G$ we have

$$\begin{aligned} F \circ \gamma_g(h;p) &= F(hg^{-1};p) \\ &= \alpha_{(hg^{-1})^{-1}}(p) \\ &= \alpha_{gh^{-1}}(p) \\ &= \alpha_g \circ \alpha_{h^{-1}}(p) \\ &= \alpha_g \circ F(h;p) \end{aligned}$$

the map F is $(\; ; \;)$-equivariant. Using Lemma 5.3.2 with $\; = \mathrm{Id}_G$ we obtain that the fundamental vector fields on $G \times M$ and those on M are F-related i.e. for all $X \in \mathfrak{g}$ we have

$$dF_{(h;p)}[X^{]}_{(h;p)}] = X^{]}_{F(h;p)}:$$

Recall then that Lie brackets of pairs of F-related vector fields are again F-related, that is, dF is a Lie algebra homomorphism on the set of F-related vector fields. Since F is surjective, it is thus enough to show that $^{]}$ is a Lie algebra homomorphism. But since $\; = R \times \mathrm{Id}_M$ is just the identity on the second factor, it even suffices to check that $R^{]}$ is a Lie algebra homomorphism.

For this case, we observe first that $L_g \circ R^h(k) = L_g \circ R_k(h) = R_k \circ L_g(h) = R^{L_g(h)}(k)$ for any $g; h; k \in G$ and thus

$$\begin{aligned} d(L_g)_h [X^{]}_h] &= d(L_g)_h \circ d\left(R^h\right)_e [X] \\ &= d\left(L_g \circ R^h\right)_e [X] \\ &= d\left(R^{L_g(h)}\right)_e [X] \\ &= X^{]}_{L_g(h)} \end{aligned}$$

or put in words, the fundamental vector fields are left-invariant. Since also

$$X^{]}_e = d(R^e)_e [X] = d(\mathrm{Id}_G)_e [X] = X$$

the fundamental vector fields generated by R are actually equal to the unique left-invariant vector fields. The result now follows by the definition of the Lie bracket.

Recall. Let $c_g : G \to G$ be conjugation with the element $g \in G$. The derivative at the identity of this map is a linear map

$$d(c_g)_e : T_eG \to T_{c_g(e)}G = T_eG$$

that is, identifying T_eG with the Lie algebra $\mathfrak{g}$ of G, we get a map $\mathrm{Ad} : G \to GL(\mathfrak{g})$ which is a homomorphism by the chain rule. This is called the *adjoint representation* of G on $\mathfrak{g}$:

$$\begin{aligned} \mathrm{Ad} : G &\to GL(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}_g := d(c_g)_e: \end{aligned}$$

Lemma A.2.3. *Let $\; : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group on a smooth manifold M. For every $X \in \mathfrak{g}$, $g \in G$ and $p \in M$ we have*

$$d(\;_g)_p [X^{]}_p] = [\mathrm{Ad}_g(X)]^{]}_{\;_g(p)}:$$

Proof. We start with a preliminary computation: Take $h \in G$ arbitrary and note

$$\begin{aligned}
\Phi_g \circ \Phi^p(h) &= \Phi_g \circ \Phi_h(p) \\
&= \Phi_{gh}(p) \\
&= \Phi_{ghg^{-1}g}(p) \\
&= \Phi_{ghg^{-1}}(\Phi_g(p)) \\
&= \Phi^{\Phi_g(p)}(ghg^{-1}) \\
&= \Phi^{\Phi_g(p)} \circ c_g(h)
\end{aligned}$$

so that $\Phi_g \circ \Phi^p = \Phi^{\Phi_g(p)} \circ c_g$ where $c_g : G \to G$ again denotes conjugation with the element g. Using this we can now compute

$$\begin{aligned}
d(\Phi_g)_p[X^\sharp_p] &= d(\Phi_g)_p \circ d(\Phi^p)_e[X] \\
&= d(\Phi_g \circ \Phi^p)_e[X] \\
&= d\left(\Phi^{\Phi_g(p)} \circ c_g\right)_e[X] \\
&= d\left(\Phi^{\Phi_g(p)}\right)_e \circ d(c_g)_e[X] \\
&= d\left(\Phi^{\Phi_g(p)}\right)_e[\mathrm{Ad}_g(X)] \\
&= [\mathrm{Ad}_g(X)]^\sharp_{\Phi_g(p)}
\end{aligned}$$

which concludes the proof.

Remark. There are two interesting special cases for this result, both of them will be of importance later on:

- If the Lie group G is Abelian, then conjugation c_g and hence also the adjoint representation Ad_g are the identity and we get

$$d(\Phi_g)_p[X^\sharp_p] = X^\sharp_{\Phi_g(p)}.$$

- If g fixes the point $p \in M$, then $d(\Phi_g)_p : T_pM \to T_{\Phi_g(p)}M = T_pM$ is an element of $GL(T_pM)$. In particular, the lemma gives

$$d(\Phi_g)_p[X^\sharp_p] = [\mathrm{Ad}_g(X)]^\sharp_p$$

  and hence that the set of fundamental vector fields is preserved by the isotropy representation. This will become important in a moment since we can interpret this set as the tangent space to an orbit.

Finally, note that if both cases apply, i.e. $g \in G_p$ and G is Abelian, then $d(\Phi_g)_e$ sends fundamental vector fields onto themselves:

$$d(\Phi_g)_e[X^\sharp_p] = X^\sharp_p.$$

A.3 Basic Properties of Lie Group Actions

Definition. Let $\Phi : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group G on a smooth manifold M.

1. Let $p \in M$ be a point. The *orbit* $\mathcal{O}_p$ through p is the set
$$\mathcal{O}_p = \{\Phi_g(p) \mid g \in G\}.$$
Sometimes, if the point p is either clear from the context or not relevant, we will omit it and just write $\mathcal{O}$ instead of $\mathcal{O}_p$.

2. The *stabiliser subgroup* G_p of p is
$$G_p = \{g \in G \mid \Phi_g(p) = p\}.$$

3. Φ is called *free* if all stabiliser subgroups are trivial i.e. $G_p = \{e\}$ for all $p \in M$.

4. Φ is called *locally free* if all stabiliser subgroups are discrete i.e. G_p is discrete for all $p \in M$.

5. Φ is called *effective* if every non-trivial group element moves at least one point i.e. $\ker(\Phi) = \{e\}$.

6. Φ is called *transitive* if $\mathcal{O}_p = M$ for all $p \in M$.

Lemma A.3.1. *Let $\Phi : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group G on a smooth manifold M. For any $g \in G$, the stabiliser subgroups of p and $\Phi_g(p)$ are related by conjugation with g i.e.*
$$G_{\Phi_g(p)} = g G_p g^{-1}.$$

Proof. Assume first that $h \in G_p$ and compute
$$\begin{aligned}\Phi_{ghg^{-1}}(\Phi_g(p)) &= \Phi_{gh}(p)\\ &= \Phi_g(\Phi_h(p))\\ &= \Phi_g(p).\end{aligned}$$
It follows that $ghg^{-1} \in G_{\Phi_g(p)}$ and hence that $g G_p g^{-1} \subseteq G_{\Phi_g(p)}$.

Reversely, assume that $h \in G_{\Phi_g(p)}$ and note that hence
$$\Phi_h(\Phi_g(p)) = \Phi_g(p)$$
so that
$$\Phi_{g^{-1}hg}(p) = p.$$
Hence $g^{-1}hg \in G_p$ and thus $h \in g G_p g^{-1}$ which gives $G_{\Phi_g(p)} \subseteq g G_p g^{-1}$.

Remark. 1. To each orbit $\mathcal{O}$ we can hence associate a conjugation class of subgroups of G which we call the *type of the orbit*. More precisely, we introduce an equivalence relation on the set of subgroups of G by

$$H \sim H^0 \quad \text{if} \quad \exists g \in G : gHg^{-1} = H^0$$

and Lemma A.3.1 shows that all stabiliser subgroups of points in a given orbit are equivalent to one another. In particular, the type of the orbit through a given point $p \in M$ is given by (G_p) where (G_p) denotes the equivalence class of G_p in G.

In the case of primary interest for this paper, the groups will be Abelian and hence conjugation is trivial. It follows that $(H) = \{H\}$ for each subgroup $H < G$.

2. Suppose a Lie group G acts on two smooth manifolds M and N via ψ and ξ respectively. If $\varphi : M \to N$ is a $(\psi; \xi)$-equivariant map then we observe for $p \in M$ and $g \in G_p$ that

$$\xi_g(\varphi(p)) = \varphi(\psi_g(p)) = \varphi(p):$$

Hence $g \in G_{\varphi(p)}$ and hence $G_p < G_{\varphi(p)}$.

Definition. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group G on a smooth manifold M. For any Le subgroup $H < G$ we define

$$M^H = \{p \in M \mid H < G_p\}$$
$$M^{(H)} = \{p \in M \mid (H) < (G_p)\}$$
$$M_H = \{p \in M \mid H = G_p\}$$
$$M_{(H)} = \{p \in M \mid (H) = (G_p)\}$$

where $(H) < (G_p)$ means that H is equivalent to a subgroup of G_p. The set M^H is the *fixed point set* of H and $M_{(H)}$ is the set of points of *orbit type* (H).

Remark. 1. Again, the definition here is given in all generality but in the present work we will only be concerned with Abelian groups in which case $M^H = M^{(H)}$ and $M_H = M_{(H)}$.

2. Note that in the extreme case $H = G$, all the above sets coincide and correspond to the set of points which are fixed under the action of the whole group via ψ.

3. Suppose a Lie group G acts on two smooth manifolds M and N via ψ and ξ respectively. If $\varphi : M \to N$ is a $(\psi; \xi)$-equivariant map, then by the previous remark, we have $G_p < G_{\varphi(p)}$ and hence for any subgroup $H < G$

it follows that

$$\begin{aligned}\varphi(N^H) &= \{\varphi(p) \in M \mid H < G_p\}\\ &\subset \{q \in M \mid H < G_q\}\\ &= M^H;\end{aligned}$$

that is, if p is fixed by H, then $\varphi(p)$ is fixed by H as well.

Proposition A.3.2. Let $\Phi : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group on a smooth manifold M. Then the subgroup

$$N = \ker(\Phi) = \bigcap_{p \in M} G_p$$

is closed and normal in G. Moreover, Φ induces an effective action of the quotient group G/N on M.

Proof. N is the intersection of closed subgroups, hence it is itself a closed subgroup. N is normal since it is the kernel of a group homomorphism. Finally, we get an induced action by

$$\begin{aligned}\bar\Phi : G/N &\to \mathrm{Diff}(M)\\ gN &\mapsto \Phi_g\end{aligned}$$

which is well-defined since $N = \ker(\Phi)$ and effective by construction.

Proposition A.3.3. Let $\Phi : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group G on a smooth manifold M and $p \in M$ be a point. Then the Lie algebra $\mathfrak{g}_p$ (called the stabiliser algebra) of the stabiliser subgroup G_p is given by

$$\mathfrak{g}_p = \left\{ X \in \mathfrak{g} \mid X^\sharp_p = 0 \right\}.$$

Proof. Assume $X \in \mathfrak{g}_p$. Then for all $t \in \mathbb{R}$ we have $\exp(tX) \in G_p$ and hence $\Phi_{\exp(tX)}(p) = p$. Taking the derivative at $t = 0$ we get

$$0 = \frac{d}{dt}\Big|_{t=0} \Phi_{\exp(tX)}(p) = X^\sharp_p.$$

Reversely, assume that $X \in \mathfrak{g}$ is such that $X^\sharp_p = 0$. Note then that the flow of $X^\sharp$ is given by $\Phi_{\exp(tX)}$ and that hence $\Phi_{\exp(tX)}(p) = p$ since $X^\sharp_p = 0$. It follows that $\exp(tX) \in G_p$ and thus $X \in \mathfrak{g}_p$.

Theorem A.3.4. Let $\Phi : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group G on a smooth manifold M. Let $p \in M$ be a point and G_p its stabiliser subgroup. Then the orbit map $\Phi^p : G \to M$ descends to an injective immersion

$$\begin{aligned}\bar\Phi^p : G/G_p &\to M\\ gG_p &\mapsto \Phi_g(p).\end{aligned}$$

Proof. First we check that this is indeed injective. Take $g, g' \in G$ such that

$$ {}^p(gG_p) = {}^p(g'G_p) $$

and note that this means by definition that

$$ {}_g(p) = {}_{g'}(p). $$

It follows that

$$ {}_{g'^{-1}g}(p) = p $$

and hence that $g'^{-1}g \in G_p$ giving $g'G_p = gG_p$ and showing injectivity.

To check that this is a smooth immersion we look at the differential of the orbit map

$$ d({}^p)_g : T_gG \to T_{{}_g(p)}M. $$

By invariance the rank is constant and it is thus sufficient to look at the case $g = e$. But in this case Proposition A.3.3 gives that the kernel is given by $\mathfrak{g}_p$ which gives the result.

Proposition A.3.5. *Let $: G \to M$ be a smooth action of a Lie group on a smooth manifold M. Then*

1. *is locally free if and only if all the stabiliser algebras are trivial i.e. if $\mathfrak{g}_p = \{0\}$ for all $p \in M$,*
2. *if is transitive, then the map*

$$ \begin{aligned} d({}^p)_e : \mathfrak{g} &\to T_pM \\ X &\mapsto X^\sharp_p \end{aligned} $$

   is surjective for every $p \in M$ and

3. *if is effective, then the kernel of the map*

$$ \begin{aligned} {}^\sharp : \mathfrak{g} &\to \mathfrak{X}(M) \\ X &\mapsto X^\sharp \end{aligned} $$

   is trivial.

Proof. 1. This follows directly from Proposition A.3.3.

2. Assume first that is transitive. Take $\in T_pM$ and a curve $: \mathbb{R} \to M$ such that $(0) = p$ and $\dot{}(0) =$. Since is transitive and smooth, we can find a smooth curve $g : \mathbb{R} \to G$ such that $g(0) = e$ and

$$ {}_{g(t)}(p) = (t). $$

Consider then $X := \underline{g}(0) \in \mathfrak{g}$ and compute

$$
\begin{aligned}
X_p^\sharp &= d(\ ^p)_e[X] \\
&= d(\ ^p)_{g(0)}[\underline{g}(0)] \\
&= d(\ ^p)_{g(0)} \quad dg_0 \ \frac{\partial}{\partial t}_0 \\
&= d(\ ^p \ \ g)_0 \ \frac{\partial}{\partial t}_0 \\
&= \frac{d}{dt}_{t=0} (\ ^p \ \ g)(t) \\
&= \frac{d}{dt}_{t=0} \ _{g(t)}(p) \\
&= \dot{\ }(0) \\
&=
\end{aligned}
$$

showing that $d(\ ^p)_e$ is indeed surjective. We conclude since $p \in M$ was arbitrary.

3. Assume $X \in \mathfrak{g}$ is such that $X^\sharp = 0$. Hence

$$\frac{d}{dt}_{t=0} \ _{\exp(tX)}(p) = X_p^\sharp = 0 \quad \text{for all} \quad p \in M$$

and therefore $_{\exp(tX)}(p)$ is constant with value p around zero (since the value at $t = 0$ is p). Because this holds for any $p \in M$ and is effective, we must have $\exp(tX) = \mathrm{Id}$. Since exp is a local diffeomorphism it then follows that $X = 0$.

Definition. Let $: G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group on a smooth manifold M. is called *proper* if the map

$$
\begin{aligned}
G \ \ M &\to M \ \ M \\
(g; p) &\mapsto (p; \ _g(p))
\end{aligned}
$$

is proper i.e. if pre-images of compact sets are compact.

Remark. This case is of interest for the present work since actions of compact groups are always proper.

Corollary A.3.6. Let $: G \to \mathrm{Diff}(M)$ be a proper smooth action of a Lie group G on a smooth manifold M. Then the orbits O are embedded, closed submanifolds with

$$T_p(O) = \{ X_p^\sharp \mid X \in \mathfrak{g} \}:$$

Proof. That the orbits are embedded submanifolds follows from Theorem A.3.4 since proper injective immersions are embeddings. The explicit description for the tangent space is then obvious from point 2 of Proposition A.3.5 and the trivial observation that G acts transitively on the orbits.

Corollary A.3.7. *Let $\Phi : G \times M \to M$ be a proper smooth action of a Lie group on a connected smooth manifold M. Then Φ is transitive if and only if the map*

$$d(\Phi^p)_e : \mathfrak{g} \to T_pM$$
$$X \mapsto X^\sharp_p$$

is surjective for every $p \in M$.

Proof. One direction has already been proved in Proposition A.3.5. For the reverse direction, note that if $d(\Phi^p)_e$ is surjective, the orbit O_p through p is open. Since it is also closed by Corollary A.3.6 and M is connected, it is either empty or all of M. Since $p \in O_p$ only the latter is possible and hence Φ is transitive.

A.4 The Slice Theorem and its Applications

Corollary A.4.1. *Let $\varphi : G \to \mathrm{Diff}(M)$ be a smooth proper action of a Lie group G on a smooth manifold M. Let $p \in M$ be a point, O_p the G-orbit through p and G_p the stabiliser subgroup of p. Then the isotropy representation given by*

$$\rho : G_p \to GL(T_pM)$$
$$g \mapsto d(\varphi_g)_p$$

preserves the tangent space $T_p(O_p)$ to the orbit at p. Moreover, if G is Abelian, ρ is the identity on $T_p(O_p)$.

Proof. This is just the combination of Corollary A.3.6 and the special cases of Lemma A.2.3.

Hence the isotropy representation descends to a representation on the quotient $\nu_pO := T_pM/(T_pO)$ which is called the *normal space* to O at p:

$$\sigma : G_p \to GL(\nu_pO)$$
$$g \mapsto d(\varphi_g)_p.$$

Definition. $\sigma : G_p \to GL(\nu_pO)$ is called the *slice representation* at p.

Consider then $\pi_{\nu_pO} : G \times_{G_p} \nu_pO \to G/G_p$ which is the associated bundle obtained from the homogeneous bundle $G_p \to G \to G/G_p$ and the representation described above. Then we might

1. interpret G/G_p as a submanifold of $G \times_{G_p} \nu_p O$ by the zero section or
2. interpret G/G_p as the G-orbit of p in M by the map $\iota^p : G/G_p \to M$ from Theorem A.3.4 which is an embedded submanifold if we assume G to be compact.

Theorem A.4.2 (The Slice Theorem). Let G be a compact connected Lie group acting on a manifold M and let $p \in M$ be a point. There exists an invariant open neighbourhood $U_{O_p} \subset M$ of O_p and an invariant open neighbourhood $U_0 \subset G \times_{G_p} \nu_p(O_p)$ of the zero section G/G_p together with a G-equivariant diffeomorphism $f_p : U_0 \to U_{O_p}$ which sends the zero section G/G_p onto the orbit O_p by ι_p, that is, such that the following diagram commutes:

$$\begin{array}{ccc} G/G_p & \hookrightarrow & U_0 \subset G \times_{G_p} \nu_p(O_p) \\ \downarrow \iota_p & & \downarrow f_p \\ O_p & \hookrightarrow & U_{O_p} \subset M. \end{array}$$

Remark. Recalling that the G-action on $G \times \nu_p(O_p)$ commutes with the G_p action, we observe that

$$\left(G \times_{G_p} \nu_p(O_p)\right)/G \cong \left((G \times \nu_p(O_p))/G_p\right)/G \cong \left((G \times \nu_p(O_p))/G\right)/G_p \cong \nu_p(O_p)/G_p.$$

The slice theorem now gives that near $p \in M$ the quotient manifold M/G looks (only locally of course) like the quotient $(G \times_{G_p} \nu_p(O_p))/G = \nu_p(O_p)/G_p$. In other words, quotients of manifolds by actions of compact Lie groups are modeled on quotients of the normal spaces by the slice representation.

Corollary A.4.3. Let G be be a compact connected Lie group acting on a smooth manifold M and let $p \in M$ be a point. Then there exists a neighbourhood U_{O_p} of the G-orbit O_p through p such that the stabiliser subgroup G_q of any point $q \in U_{O_p}$ is conjugate to a subgroup of G_p.

Proof. By the slice theorem, we can identify a neighbourhood U_{O_p} with a neighbourhood U_0 of the zero section in $G \times_{G_p} \nu_p(O_p)$. The action of $\tilde{g} \in G$ on a point $[g;\xi] \in U_0$ is given by

$$\tilde{g} \cdot [g;\xi] = [\tilde{g}g;\xi].$$

By the definition of the associated bundle, we have $[\tilde{g}g;\xi] = [g;\xi]$ exactly if there exists an $h \in G_p$ such that

$$(\tilde{g}g;\xi) = (gh^{-1};\rho_h(\xi)).$$

But this means that $h \in (G_p)_\xi$, where $(G_p)_\xi < G_p$ is the stabiliser subgroup of ξ in G_p, and $\tilde{g} = gh^{-1}g^{-1}$. Hence

$$G_{[g;\xi]} = g(G_p)_\xi\, g^{-1}$$

which proves the statement.

Let's continue the line of reasoning of this proof and note that $G_{[g;\xi]} = gG_pg^{-1}$ if and only if $(G_p)_\xi = G_p$, that is, if ξ is fixed by G_p meaning that $\xi \in \nu_p(O_p)^{G_p}$. Hence $[g;\xi] \in (G \times_{G_p} \nu_p(O_p))_{(G_p)}$ if and only if $\xi \in (\nu_p(O_p))^{G_p}$ which shows that

$$(G \times_{G_p} \nu_p(O_p))_{(G_p)} = G \times_{G_p} (\nu_p(O_p))^{G_p}.$$

As $(\nu_p(O_p))^{G_p}$ is by construction a G_p-invariant subspace of $\nu_p(O_p)$, this is a smooth vector subbundle of $G \times_{G_p} \nu_p(O_p)$. Actually, noting that by construction G_p acts trivially on $(\nu_p(O_p))^{G_p}$, we have

$$G \times_{G_p} (\nu_p(O_p))^{G_p} = \frac{G \times (\nu_p(O_p))^{G_p}}{G_p} = G/G_p \times (\nu_p(O_p))^{G_p}.$$

Proposition A.4.4. Let G be a compact connected Lie group acting on a smooth manifold M. Let $H < G$ be a closed subgroup of G. Then the connected components of $M_{(H)}$ are smooth submanifolds of M.

Proof. Let $p \in M_{(H)}$ be a point and note that as $p \in M_{(H)}$, there exists $g \in G$ such that $gG_pg^{-1} = H$. Hence the point $\psi_g(p) \in O_p \subset M_{(H)}$ has the stabiliser subgroup

$$G_{\psi_g(p)} = gG_pg^{-1} = gg^{-1}Hgg^{-1} = H.$$

It follows by the above that we can find a neighbourhood U_{O_p} of the orbit O_p such that

$$(U_{O_p})_{(H)} = (G \times_H (\nu_p(O_p))^H) = G/H \times (\nu_p(O_p))^H.$$

This shows the statement since, again by the above, this is a smooth vector subbundle, so in particular a smooth submanifold.

Corollary A.4.5. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group G on a smooth manifold M. Then the connected components of the set M^G of fixed points under this action are smooth submanifolds of M.

Proof. This follows directly from Proposition A.4.4 by noting that $M^G = M_{(G)}$.

Corollary A.4.6. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth action of a Lie group G on a smooth manifold M and let $H < G$ be a Lie subgroup. Then the connected components of the fixed point set M^H of H are smooth submanifolds of M.

Proof. Consider the action of H on M which is induced by ψ and apply Corollary A.4.5.

Lemma A.4.7. Suppose that G is a compact Abelian group acting effectively on a connected manifold M. Then every slice representation is faithful.

Proof. By the slice theorem A.4.2 we may assume that an open neighbourhood U_{O_p} of O_p is in the homogeneous bundle $G \times_{G_p} V_p$ and denote it again by U_0. Assume then that the slice representation ρ of G_p on V_p is not faithful i.e. there exists a non-trivial $h \in G_p$ such that $\rho_h(v) = v$ for all $v \in V_p$. But then for any $[g; v] \in U_0$ we get (remember that G is Abelian)

$$\begin{aligned} h \cdot [g; v] &= [hg; v] \\ &= [gh; v] \\ &= [g; \rho_{h^{-1}}(v)] \\ &= [g; v] \end{aligned}$$

and conclude that h fixes the whole neighbourhood $U_0 = U_{O_p}$. Differently put, if h fixes a point p, then it also fixes an open neighbourhood of p and hence the set of elements of M fixed by h is open. But it is also closed and hence since M is connected, it is equal to M. Hence h is non-trivial but fixes all of M which contradicts the fact that G acts effectively on M.

Appendix B

Bundle Theory

This content of this chapter is taken from [13] and [5].

B.1 Fibre Bundles

Definition. Let E, M and F be smooth manifolds and suppose that $\pi : E \to M$ is a smooth surjective map. We say that $\pi : E \to M$ is a *fibre bundle* over M with *fibre* F if for every point $p \in M$ there exists a neighbourhood U of p together with a smooth map $\varphi : \pi^{-1}(U) \to F$ such that

$$\hat{\varphi} := (\pi, \varphi) : \pi^{-1}(U) \to U \times F$$

is a diffeomorphism. We call

- (U, φ) a *bundle chart* for E,
- $\hat{\varphi}$ a *local trivialisation* of E,
- E the *total space* of the bundle,
- M the *base space* of the bundle and
- F the *fibre*.

A *bundle atlas* on E is an open cover $\{U_a\}_{a \in A}$ of M together with corresponding bundle charts $\varphi_a : \pi^{-1}(U_a) \to F$.

Notation. We write $F \to E \to M$ to denote a fibre bundle E over M with fibre F.

Definition. Let $F \to E \to M$ be a fibre bundle. The *fibre over* $p \in M$ is $E_p := \pi^{-1}(p)$.

Lemma B.1.1. *Let $F \to E \to M$ be a fibre bundle. Then π is a surjective submersion and each fibre E_p is an embedded submanifold of E which is diffeomorphic to the fibre F.*

Definition. Let $F_{1;2} \to E_{1;2} \xrightarrow{\pi_{1;2}} M_{1;2}$ be two fibre bundles. A *fibre bundle morphism along a smooth map* $\varphi : M_1 \to M_2$ is a smooth map $\Phi : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

If $(U; \psi)$ is a bundle chart on E, then for $p \in U$ the bundle chart map restricts to a diffeomorphism

$$\psi_p := \psi|_{E_p} : E_p \to F.$$

Definition. Let $F \to E \to M$ be a fibre bundle and $\{(U_a; \psi_a) \mid a \in A\}$ a bundle atlas. If $U_a \cap U_b \neq \emptyset$, then there is a well-defined map

$$\begin{aligned} \psi_{ab} : U_a \cap U_b &\to \mathrm{Diff}(F) \\ p &\mapsto \psi_a|_{E_p} \circ (\psi_b|_{E_p})^{-1} \end{aligned}$$

called the *transition function* from the bundle chart $(U_a; \psi_a)$ to the bundle chart $(U_b; \psi_b)$.

Recall. $\rho : G \to \mathrm{Diff}(F)$ is an *effective action* if ρ is injective. If this is the case we may regard G as a subgroup of $\mathrm{Diff}(F)$.

Definition. Suppose that $L \to E \to M$ is a fibre bundle and that $\rho : G \to \mathrm{Diff}(F)$ is an effective action of a Lie group G on the fibre F. A bundle atlas $\{(U_a; \psi_a) \mid a \in A\}$ is said to be a *$(G; \rho)$-bundle atlas* if all its transition functions take values in $\rho(G)$. If such an atrals exists, we say that E is a $(G; \rho)$-fibre bundle and call G the *structure group* of the bundle.

B.2 Vector Bundles

Definition. Let M be a smooth manifold. A *vector bundle* over M is a $(GL(V); \rho_{can})$-fibre bundle $V \to E \to M$, where V is a vector space and ρ_{can} is the canonical action of $GL(V)$ on V.

Proposition B.2.1. Let $V \to E \to M$ be a fibre bundle with fibre a vector space V. Then E is a vector bundle if and only if it is possible to endow each fibre E_p with a vector space structure and find a bundle atlas $\{(U_a; \psi_a) \mid a \in A\}$ such that for any $p \in U_a$ the map $\psi_a|_{E_p} : E_p \to V$ is an isomorphism.

Definition. Let $V_{1;2} \to E_{1;2} \xrightarrow{\pi_{1;2}} M_{1;2}$ be two vector bundles. A *vector bundle morphism along a smooth map* $\varphi : M_1 \to M_2$ is a fibre bundle morphism $\Phi : E_1 \to E_2$ along φ which restricts to linear maps on the fibres i.e. for each $p \in M$, the map $\Phi|_{E_p} : E_p \to F_{\varphi(p)}$ is linear.

B.3 Principal Bundles

Definition. Let M be a smooth manifold and G a Lie group. A *G-principal bundle* over M is a G-fibre bundle $G \to P \to M$ where G acts on itself via left translation

$$\begin{aligned} L : G &\to \mathrm{Diff}(G) \\ g &\mapsto (L_g : h \mapsto gh) \end{aligned}$$

Proposition B.3.1. *Let $G \to P \to M$ be a fibre bundle with fibre a Lie group G. Then the following are equivalent:*

1. *P is a G-principal bundle;*
2. *There exists a smooth free action $\ : G \to \mathrm{Di}\ (P)$ which is fibre preserving together with a bundle atlas $\{(U_a;\ _a) \mid a \in A\}$ such that the bundle chart maps $_a$ are $(\ ; L)$-equivariant. In other words, for each $a \in A$ and every $g \in G$ the following diagrams commute*

$$\begin{array}{ccc} {}^{1}(U_a) & \xrightarrow{\ g\ } & {}^{1}(U_a) \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} \\ G & \xrightarrow{\ L_g\ } & G \end{array} \qquad\qquad \begin{array}{ccc} P & \xrightarrow{\ g\ } & P \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

3. *There exists a smooth free action $\ : G \to \mathrm{Di}\ (P)$ which is fibre preserving i.e. $\ _g = \$ for any $g \in G$ and transitive on the fibres i.e $O_u = P_{(u)} = {}^{1}(\ (u))$ for any $u \in P$.*

Corollary B.3.2. *Let $\ : G \to \mathrm{Di}\ (P)$ be a proper free action of a Lie group G on a smooth manifold P. Then $\ : P \to P/G$ is a G-principal bundle. In particular, a homogeneous space G/H can be seen as the base space of of an H-principal bundle $H \to G \to G/H$.*

Definition. Let $G_{1;2} \to P_{1;2} \xrightarrow{1;2} M_{1;2}$ be two principal $G_{1;2}$-bundles and let $\ : G_1 \to G_2$ be a Lie group homomorphism. A *principal bundle morphism* along a smooth map $' : M_1 \to M_2$ with respect to is a smooth map $\ : P_1 \to P_2$ which is $(\ _1;\ _2)$-equivariant i.e. for any $g \in G_1$, the following diagram commutes:

$$\begin{array}{ccc} P_1 & \longrightarrow & P_2 \\ \downarrow{\scriptstyle (\ _1)_g} & & \downarrow{\scriptstyle (\ _2)\ (g)} \\ P_1 & \longrightarrow & P_2 \end{array}$$

In particular, if $G_1 = G_2 = G$ we say that $\ : P_1 \to P_2$ is a *principal G-bundle morphism* if it is a principal bundle morphism with respect to Id_G.

B.4 Associated Bundles

Definition. Let $G \to P \xrightarrow{\pi} M$ be a principal G-bundle and let ψ be a smooth action of G on another smooth manifold F. Define then an action of G on $P \times G$ by

$$\begin{aligned} P \times F &\to P \times F \\ (u,q) &\mapsto (R_g(u), \psi_g(q)) \end{aligned}$$

and put

$$P \times_G F := \frac{P \times F}{G}.$$

Writing $[u,q] \in P \times_G F$ for the equivalence class of $(u,q) \in P \times F$ we define a map

$$\begin{aligned} \pi_F : P \times_G F &\to M \\ [u,q] &\mapsto \pi(u) \end{aligned}$$

and call $\pi_F : P \times_G F \to M$ an *associated bundle* of P.

Theorem B.4.1 (The Associated Bundle Theorem)**.** *Let $G \to P \xrightarrow{\pi} M$ be a principal G-bundle and let ψ be a smooth action of G on another smooth manifold F.*

1. *The associated bundle $\pi_F : P \times_G F \to M$ is a (G,ψ)-fibre bundle with fibre F.*
2. *The quotient map*

$$\begin{aligned} p : P \times F &\to P \times_G F \\ (u,q) &\mapsto [u,q] \end{aligned}$$

   is a principal G-bundle.

3. *The first projection*

$$\begin{aligned} pr_1 : P \times F &\to P \\ (u,q) &\mapsto u \end{aligned}$$

   is a principal G-bundle morphism along π_F, in particular, the following diagram commutes

$$\begin{array}{ccc} P \times F & \xrightarrow{pr_1} & P \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ P \times_G F & \xrightarrow{\pi_F} & M \end{array}$$

4. *For each $u \in P$, the map*

$$
\begin{aligned}
{}_u : F &\to P \quad {}_G F \\
q &\mapsto [u, q]
\end{aligned}
$$

is a di eomorphism from F to the fibre $(P \quad {}_G F)_{\ (u)} := \ {}_F^{\ 1}(\ (u))$ in the associated bundle over $\ (u)$.

Corollary B.4.2. *Let $G \to P \to M$ be a principal G-bundle and suppose that is a representation of G on a vector space V. Then the associated bundle $P \quad {}_G V$ is a vector bundle and the map*

$$
\begin{aligned}
{}_u : V &\to P \quad {}_G V \\
v &\mapsto [u, q]
\end{aligned}
$$

from the associated bundle theorem is a linear isomorphism.

Example. We will spell this out for the case of the principal bundle $H \to G \to G/H$ where G is a Lie group and H a closed subgroup. First, we note that $\ : G \to G/H$ is just the usual projection $g \mapsto gH$ where gH is the coset. The closed subgroup H acts on G by

$$
\begin{aligned}
: H &\to \mathrm{Diff}(G) \\
h &\mapsto (R_{h^{\ 1}} : g \mapsto gh^{\ 1})
\end{aligned}
$$

which is fibre preserving since

$$
\begin{aligned}
{}_h(g) &= \ (gh^{\ 1}) \\
&= gh^{\ 1}H \\
&= gH \\
&= \ (g)
\end{aligned}
$$

and transitive on the fibres as

$$
\begin{aligned}
O_g &= \{\ {}_h(g) \mid h \in H\} \\
&= \{gh^{\ 1} \mid h \in H\} \\
&= \{gh \mid h \in H\} \\
&= \ ^{1}(gH).
\end{aligned}
$$

Now suppose that V is a vector space and that

$$
\begin{aligned}
: H &\to GL(V) \\
h &\mapsto \ {}_h
\end{aligned}
$$

is a representation of H on V. Then the action of $h \in H$ on $G \times V$ from the definition of the associated bundle is simply

$$(g, v) \mapsto (r_h(g), \rho_h(v)) = (gh^{-1}, \rho_h(v)).$$

The associated bundle is then the orbit space

$$G \times_H V := G \times V \big/ H$$

and let

$$\begin{aligned} p : G \times V &\to G \times_H V \\ (g, v) &\mapsto [g, v] \end{aligned}$$

where $[g, v]$ denotes the equivalence class/the orbit of (g, v) be the projection map. The bundle map is given by

$$\begin{aligned} \pi_V : G \times_H V &\to G/H \\ [g, v] &\mapsto gH. \end{aligned}$$

Then Corollary B.4.2 gives that $\pi_V : G \times_H V \to G/H$ is indeed a vector bundle. Moreover, we get a commutative diagram

$$\begin{array}{ccc} G \times V & \xrightarrow{pr_1} & G \\ \downarrow p & & \downarrow \\ G \times_H V & \xrightarrow{\pi_V} & G/H \end{array}$$

where $pr_1 : G \times V \to G$ is just the projection on the first factor. Finally, for each $g \in G$, the map

$$\begin{aligned} \iota_g : V &\to G \times_H V \\ v &\mapsto [g, v] \end{aligned}$$

is a linear isomorphism from V to $(G \times_H V)_{gH}$.

However, in this special case we can even go a bit further than the associated bundle theorem: Also $\tilde{g} \in G$ acts naturally on $G \times V$ by

$$(g, v) \mapsto (\tilde{g}g, v).$$

This action clearly commutes with the action of H on $G \times V$ and hence descends to an action on $G \times_H V$:

$$\begin{aligned} \tau : G &\to \mathrm{Diff}(G \times_H V) \\ \tilde{g} &\mapsto (\tau_{\tilde{g}} : [g, v] \mapsto [\tilde{g}g, v]). \end{aligned}$$

But G also acts transitively on G/H ($\bar{g} \in G$ sends $gH \in G/H$ onto $\bar{g}gH$) an we observe that

$$\begin{aligned} \pi_V \circ \bar{g}([g,v]) &= \pi_V([\bar{g}g, v]) \\ &= \bar{g}gH \\ &= \bar{g} \cdot gH \\ &= \bar{g} \cdot \pi(g) \\ &= \bar{g} \cdot \pi_V([g,v]) \end{aligned}$$

that is, for each $\bar{g} \in G$ the following diagram commutes:

$$\begin{array}{ccc} G \times_H V & \xrightarrow{\pi_V} & G/H \\ \downarrow \bar{g} & & \downarrow \bar{g} \\ G \times_H V & \xrightarrow{\pi_V} & G/H \end{array}$$

Finally, as for any vector bundle, we can interpret G/H as a submanifold of $G \times_H V$ using the zero section:

$$G/H = \{[g,0] \mid g \in G\} \subset G \times_H V.$$

B.5 Normal Bundles

Recall. Let M be a smooth manifold and N be an embedded submanifold with $i : N \hookrightarrow M$ the inclusion. For each $p \in M$, the differential $\mathrm{d}i_p : T_pN \to T_{i(p)}M$ is injective and allows to interpret T_pN as a subspace of T_pM where we also write $p = i(p)$.

Definition. Let M be a smooth manifold and N an embedded submanifold. The quotient

$$\nu_p N = T_pM \Big/ T_pN$$

is an $(\dim(M) - \dim(N))$-dimensional vector space called the *normal space* to N at $p \in N$. The set

$$\nu N = \bigsqcup_{p \in N} \nu_p N$$

together with the natural projection $\pi : \nu N \to N$ is called the *normal bundle of* N and has the structure of a vector bundle over N of rank $(\dim(M) - \dim(N))$.

Remark. 1. Note that the normal bundle fits perfectly in a short exact sequence of vector bundles:

$$0 \to TN \to TM|_N \to \nu N \to 0.$$

2. As a manifold N has dimension $\dim(N) + (\dim(M) \quad \dim(N)) = \dim(M)$.

Recall. 1. The zero section of N given as

$$\begin{aligned} i_0 : N &\hookrightarrow N \\ p &\mapsto 0_p \end{aligned}$$

embeds N as a closed submanifold in N.

2. A neighbourhood U_0 of the zero section N in N is called convex if the intersection with all fibres is convex i.e. if $U_0 \cap {}_pN$ is convex for each $p \in N$.

Theorem B.5.1 (The Tubular Neighbourhood Theorem)**.** *Let M be a smooth manifold and N and embedded submanifold. There exists a convex neighbourhood U_0 of N in N, a neighbourhood U of N in M and a di eormorphism $' : U_0 \to U$ such that N N is sent to N M i.e. such that the following diagram commutes:*

$$\begin{array}{ccccc} N & U_0 & \xrightarrow{\quad ' \quad} & U & M \\ & \nwarrow{\scriptstyle i_0} & & \nearrow{\scriptstyle i} & \\ & & M_2 & & \end{array}$$