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Contact Toric Manifolds and Their Classification

Master Thesis

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Abstract

In this master thesis, we describe the classification of compact connected contact toric manifolds that is due to Lerman [19] building on previous partial classifications. Compact connected contact toric manifolds are classified according to the image of the moment map and whether the action of the torus is free.

Contents

Introduction

Contact manifolds are odd-dimensional manifolds equipped with a contact structure, that is, a maximally nonintegrable hyperplane field. They can be viewed as odd-dimensional analogues of symplectic manifolds. This relation can be made precise using the notion of symplectization.

A Lie group action on a contact manifold that preserves the contact structure naturally induces a Hamiltonian action on the symplectization of the contact manifold. This allows us to define a contact moment map for the action and investigate the group action using the properties of this moment map.

Analogously to symplectic toric manifolds, if we have an effective action of a torus of dimension $n + 1$ on a contact manifold of dimension $2n + 1$, we call this contact manifold a contact toric manifold.

In this thesis, we are concerned with examples, properties, and classification of contact toric manifolds.

It is a well-known theorem of Delzant [11] that compact connected symplectic toric manifolds are classified by the image of their moment map. This image is a rational polytope with certain properties.

For compact connected contact toric manifolds, the situation is similar. Lerman completed the classification of compact connected toric manifolds [19]. According to Lerman's work, compact connected contact toric manifolds are classified by the image of the contact moment map with exceptional cases occurring in dimensions 3 and 5 depending on whether the action is free.

The main outline of the thesis is as follows. More details about the content of each chapter are given at the beginning of the respective chapter.

Chapter 1 introduces the basic notions about contact and symplectic manifolds. We set the notation and conventions that are used throughout the thesis. We emphasize group actions and the relationship between contact and symplectic manifolds.

In Chapter 2, we describe the local properties of contact toric manifolds and contact moment maps. Then we explain how to deduce global statements from these results.

Finally, in Chapter 3, we state and prove the main classification theorem. We conclude with a discussion about the applications of the classification.

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Chapter 1

Group Actions on Contact and Symplectic Manifolds

In this chapter, we introduce contact and symplectic manifolds and important types of group actions on them.

In Section 1, we define contact and symplectic manifolds. Then, we introduce vector fields that satisfy special conditions which will allow us to define certain properties later on.

Next, in Section 2, we introduce symplectic cones and symplectizations which give us a way to relate symplectic and contact manifolds. These notions allow us to use techniques of symplectic topology to investigate contact manifolds.

In Section 3, we introduce our main objects of consideration, which are group actions and moment maps. Moment maps are first defined for Hamiltonian actions on symplectic manifolds. Then, we define the contact moment map using the lift of contact actions to the symplectization of a contact manifold.

In Section 4, we review symplectic and contact reduction. These constructions will be used in the following chapters to construct certain contact and symplectic manifolds.

Lastly, in Section 5, we define and give examples of contact toric manifolds.

1.1 Contact and Symplectic Manifolds

We start with the definition of contact manifolds:

Definition 1.1 *Let M be a smooth manifold of dimension (2n+1), for $n \geq 0$. A* ***contact structure*** *on M is a maximally nonintegrable hyperplane field $\xi \subseteq TM$. That is, locally $\xi = \ker \alpha$ for a 1-form $\alpha \in \Omega^1(M)$, with $\alpha \wedge (d\alpha)^n \neq 0$.*

Such a 1-form α on M is called a ***contact 1-form****. The pair (M, ξ) is called a* ***contact manifold****.*

Observe that, if α is a contact 1-form on M, then $g\alpha$, where $g : M \to \mathbb{R} - \{0\}$ is a nonvanishing smooth function, is also a contact form defining the same contact structure.

For a contact manifold (M, ξ), let $\xi^0 \subset \mathbb{T}^*M$ be the annihilator of the contact distribution in the cotangent bundle. By definition of a contact manifold, ξ^0 is a line bundle. The contact structure ξ is called **coorientable**, if the line bundle $\xi^0 \cong TM/\xi$ is orientable, that is if it has a nonvanishing global section. The contact structure ξ on a contact manifold M is called **cooriented** if one global section of $\xi^0 - \mathbf{0}$ is chosen, where $\mathbf{0}$ denotes the zero section. In this case, we denote the union of components of $\xi^0 - \mathbf{0}$ in which the image of the chosen global section lies by ξ^0_+ and call it the **coorientation**.

Hence, a contact structure can be defined by a global one form if and only if it is coorientable. For any function $f : M \to \mathbb{R}$ and a contact form α on M, the form $e^f \alpha$ defines the same cooriented contact structure. Conversely, if α and α' define the same cooriented contact structure on M, then $\alpha' = e^f \alpha$ for some function $f : M \to \mathbb{R}$. That is, defining a co-oriented contact structure is the equivalent to giving a conformal class $[e^f \alpha]$ of contact forms on M.

Not all contact structures are coorientable. For an example of a contact structure that is not coorientable on $\mathbb{R}^{n+1} \times \mathbb{R}P^{n+1}$, see [13, Lemma 1.1.1 and Example 2.1.11].

Unless otherwise stated, we will assume the contact structures ξ are cooriented with coorientation ξ^0_+ and a global contact 1-form α, such that $\xi = \ker \alpha$ and $\alpha(M) \subset \xi^0_+$ as a section of the cotangent bundle. We will often write (M, α) or $(M, \xi = \ker \alpha)$ for a (cooriented) contact manifold with the coorientation ξ^0_+ understood.

Closely related to contact manifolds are symplectic manifolds:

Definition 1.2 *A 2-form ω on a smooth manifold X is a **symplectic form** if*

1. *ω is closed (that is, $d\omega = 0$) and,*
2. *ω is nondegenerate for all $p \in X$ (that is, at every point $p \in X$, for any nonzero tangent vector $v \in T_pX$, there is $w \in T_pX$ such that $\omega_p(v, w) \neq 0$).*

*The pair (X, ω) is called a **symplectic manifold**.*

See [21], [9] for more general and detailed discussions about symplectic manifolds.

By the nondegeneracy condition, a symplectic manifold is necessarily of even dimension. If (X, ω) is a symplectic $2n$-manifold, we can reformulate the nondegeneracy condition for ω as $\omega^n \neq 0$. Thus, a symplectic form defines an orientation on X.

For a contact manifold $(M, \xi = \ker \alpha)$ of dimension $2n + 1$, we can also restate the nonintegrability condition $\alpha \wedge (d\alpha)^n \neq 0$ as $d\alpha|_\xi$ is nondegenerate.

By the contact condition, if α is a contact form, $\alpha \wedge (d\alpha)^n$ is a volume form, then a contact manifold M is necessarily orientable.

Here are some examples of contact manifolds:

Example 1.3 *We will define several contact structures on $\mathbb{R}^{2n+1}$ with coordinates $\{(x_1, y_1, \dots, x_n, y_n, z)\}$ for $n \geq 1$:*

First, consider the 1-form

$$\alpha_1 = dz + \sum_{i=1}^{n} x_i dy_i$$

We have,

$$\begin{aligned} \alpha_1 \wedge (d\alpha_1)^n &= (dz + \sum_{i=1}^{n} x_i dy_i) \wedge (\sum_{i=1}^{n} dx_i \wedge dy_i)^n \\ &= n! dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \end{aligned}$$

The form $dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ is a volume form on $\mathbb{R}^{2n+1}$, thus

$$\alpha \wedge d\alpha^n \neq 0$$

Therefore, α_1 is a global contact 1-form on $\mathbb{R}^{2n+1}$ that defines a contact structure

$$\xi_1 = \ker \alpha_1 = span\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, x_1\frac{\partial}{\partial z} - \frac{\partial}{\partial y_1}, \dots, x_n\frac{\partial}{\partial z} - \frac{\partial}{\partial y_n}\}$$

and we get the contact manifold $(\mathbb{R}^{2n+1}, \xi_1 = \ker \alpha_1)$. For the case $2n+1 = 3$, Figure 1.1 describes this contact structure.

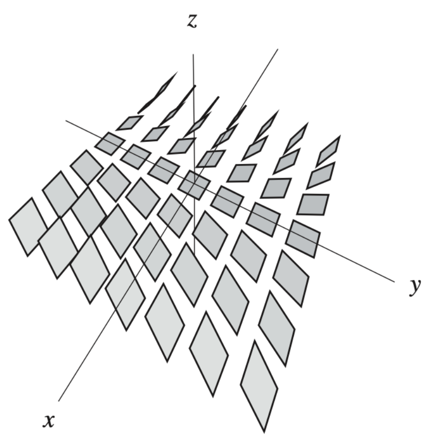

Figure 1.1: The contact structure $\xi_1 = \ker(dz + \sum_{i=1}^{n} x_i dy_i)$, [12].

Similarly, the 1-form

$$\alpha_2 = dz - \sum_{i=1}^{n} y_i dx_i$$

defines a contact manifold $(\mathbb{R}^{2n+1}, \xi_2 = \ker \alpha_2)$.

Lastly, consider the 1-form

$$\alpha_3 = dz + \sum_{i=1}^{n} x_i dy_i - y_i dx_i = dz + \sum_{i=1}^{n} r_i^2 d\varphi_i$$

where (r_i, φ_i) *are the polar coordinates on respective* (x_i, y_i) *planes. Then,*

$$\alpha_3 \wedge (d\alpha_3)^n = 2^n n! dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \neq 0$$

Thus, α_3 *defines a contact structure*

$$\xi_3 = span\{x_1 \frac{\partial}{\partial z} - \frac{\partial}{\partial y_1}, \ldots, x_n \frac{\partial}{\partial z} - \frac{\partial}{\partial y_n}, y_1 \frac{\partial}{\partial z} + \frac{\partial}{\partial x_1}, \ldots, y_n \frac{\partial}{\partial z} + \frac{\partial}{\partial x_n}\}$$

and we get a contact manifold $(\mathbb{R}^{2n+1}, \xi_3 = \ker \alpha_3)$.

Example 1.4 *Consider the unit sphere* $S^{2n+1} \subset \mathbb{R}^{2n+2}$*, and the 1-form*

$$\alpha_{S^{2n+1}} = \sum_{i=1}^{n+1} x_i dy_i - y_i dx_i$$

where we use the coordinates $\{(x_1, y_1, \ldots, x_n, y_n, x_{n+1}, y_{n+1})\}$ *for* $\mathbb{R}^{2n+2}$.

Considering $r^2 = \sum_{i=1}^{n+1} x_i^2 + y_i^2$ *where r is the radial coordinate on* $\mathbb{R}^{2n+2}$*, we get* $rdr \wedge \alpha_{S^{2n+1}} \wedge (d\alpha_{S^{2n+1}})^n \neq 0$ *for* $r \neq 0$*. Thus, since* $S^{2n+1} \subset \mathbb{R}^{2n+2}$ *is the level set* $r = 1$*, the 1-form* $\alpha_{S^{2n+1}} \wedge (d\alpha_{S^{2n+1}})^n$ *is nonzero when restricted to the sphere. The contact structure* $\xi_{S^{2n+1}} = \ker \alpha_{S^{2n+1}}$ *on* S^{2n+1} *is called the* ***standard contact structure on*** S^{2n+1}

Here is an other description of this contact structure on S^{2n+1}*: Consider the smooth map* $f : \mathbb{R}^n \to \mathbb{R}$ *given by*

$$f(x_1, y_1, \ldots, x_n, y_n, x_{n+1}, y_{n+1}) = \sum_{i=1}^{n+1} x_i^2 + y_i^2$$

Then, $S^{2n+1} = f^{-1}(1)$ *and*

$$\begin{aligned} T_p S^{2n+1} &= ker df_p \\ &= ker(2x_1 dx_1 + 2y_1 dy_1 + \cdots + 2x_{n+1} dx_{n+1} + 2y_{n+1} dy_{n+1}) \end{aligned}$$

for $p = (x_1, y_1, \ldots, x_n, y_n, x_{n+1}, y_{n+1}) \in S^{2n+1}$*. We identify* $\mathbb{R}^{2n+2}$ *with* $\mathbb{C}^{n+1}$ *to get a* ***complex structure*** J *on each tangent space, that is a linear map such that* $J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ *and* $J \frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$ *for all* $i = 1, \ldots, n+1$.

One can then check that the contact form is $\alpha_{S^{2n+1}} = -\frac{1}{2} df \circ J|_{S^{2n+1}}$ *and the contact structure is* $(\xi_{S^{2n+1}})_p = T_p S^{2n+1} \cap J(T_p S^{2n+1})$ *at each point* $p \in S^{2n+1}$.

Example 1.5 *Consider the 3-torus* $\mathbb{T}^3 = S^1 \times \mathbb{T}^2$ *with coordinates* (t, θ_1, θ_2)*. Then for each positive integer n, the 1-form*

$$\alpha_n = \sin(nt)d\theta_1 + \cos(nt)d\theta_2$$

induces a contact structure on $\mathbb{T}^3$*. Contact structure is given by*

$$\xi_n = span\{\frac{\partial}{\partial t}, \cos(nt)\frac{\partial}{\partial \theta_1} - \sin(nt)\frac{\partial}{\partial \theta_2}\}$$

The circle $\theta_1 = \theta_2 = constant$ *is tangent to* ξ *and on this circle* ξ *makes n full twists. See Figure 1.2.*

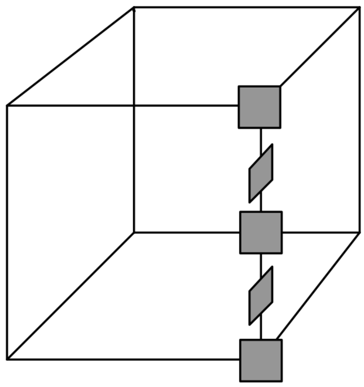

Figure 1.2: The contact structure on 3-torus

See [13], [12] for more examples mostly at dimension 3, and discussions about which manifolds can admit contact structures.

Here are some examples of symplectic manifolds:

Example 1.6 *Consider* $\mathbb{R}^{2n+2}$ *with coordinates* $(x_1, y_1, \ldots, x_{n+1}, y_{n+1})$ *and the symplectic structure*

$$\omega_{st} = \sum_{i=1}^{n+1} dx_i \wedge dy_i$$

Identifying $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ *and* $z_i = x_i + iy_i$ *we get the symplectic structure:*

$$\omega_{st} = \frac{i}{2}\sum_{i=1}^{n+1} dz_i \wedge d\overline{z_i}$$

The next example is especially important for our discussions:

Example 1.7 (Cotangent Bundle of a Manifold) *There is a canonical symplectic structure on the cotangent bundle $M^{2n} = \mathbb{T}^*X$ of a smooth manifold X^n.*

*The **tautological 1-form** λ is defined pointwise for $p = (x, \xi) \in M$ as*

$$\lambda_p = (d\pi_p)^*\xi$$

where $(d\pi_p)^*$ is the dual map of the derivative of the projection $\pi : M \to X$. Equivalently, if $v \in T_pM = T_p(T^*X)$, then

$$\lambda_p(v) = \xi(d\pi_p(v))$$

*The **canonical symplectic 2-form** ω on $M = T^*X$ is then defined as*

$$\omega = d\lambda$$

If (U, x^i) is a local coordinate chart on X and (T^*U, x^i, ξ_i) is the corresponding cotangent coordinates on M, then locally we have

$$\lambda = \sum \xi_i dx^i \text{ and } \omega = \sum d\xi_i \wedge dx^i$$

Now we define the diffeomorphisms between contact manifolds that respect the contact structures:

Definition 1.8 *Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are called **contactomorphic** if there is a diffeomorphism $f : M_1 \to M_2$ such that $df : TM_1 \to TM_2$ maps ξ_1 to ξ_2, that is $df(\xi_1) = \xi_2$. Such an f is called a **contactomorphism**. Equivalently, there exists a nowhere zero function $\lambda : M_1 \to \mathbb{R} - \{0\}$ such that $f^*(\alpha_2) = \lambda\alpha_1$ where α_i is a contact form defining ξ_i for $i = 1, 2$.*

Example 1.9 *The three contact structures on $\mathbb{R}^{2n+1}$ described in Example 1.3 are all contactomorphic. The contact structures ξ_1 and ξ_2 are related by a rotation about the z-axis, and the structures ξ_1 and ξ_3 are related by the map*

$$(\mathbb{R}^{2n+1}, \xi_1) \to (\mathbb{R}^{2n+1}, \xi_3)$$

defined as

$$(x_1, y_1, \ldots, x_n, y_n, z) \mapsto \left(\frac{(x_1 + y_1)}{2}, \frac{(y_1 - x_1)}{2}, \ldots, \frac{(x_n + y_n)}{2}, \frac{(y_n - x_n)}{2}, z + \frac{\sum x_i y_i}{2}\right)$$

*Any of these structures is called the **standard contact structure** ξ_{st} on $\mathbb{R}^{2n+1}$*

Example 1.10 *The contact manifold $(S^{2n+1} - \{p\}, \xi_{S^{2n+1}}|_{S^{2n+1} - \{p\}})$ described in Example 1.4 is contactomorphic to $\mathbb{R}^{2n+1}$ with its standard contact structure (see [13] for constructions of contactomorphisms based on stereographic projection or maps of complex domains). This contactomorphism justifies calling both of these contact structures "standard". Accordingly, we sometimes also use ξ_{st} to denote the standard contact structure on S^{2n+1}.*

Similarly we can define the diffeomorphisms between symplectic manifolds that respect the symplectic structures:

Definition 1.11 *Two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) are called **symplectomorphic** if there is a diffeomorphism $f : M_1 \to M_2$ such that $f^*\omega_2 = \omega_1$. Such an f is called a **symplectomorphism**.*

1.1.1 Contact and Reeb Vector Fields

Now, we will define some vector fields defined on contact manifolds that will be of special importance in the following discussions.

Let $(M, \xi = \ker\alpha)$ be a contact manifold of dimension $2n+1$. Since $d\alpha$ is nondegenerate on ξ by the nonintegrability condition, its kernel defines a unique line field, that is a unique vector field R on M up to scaling which satisfies $\alpha(R) \neq 0$. If we normalize R by the condition $\alpha(R) = 1$, we get a unique vector field associated to a contact form α:

Definition 1.12 *Let $(M, \xi = \ker\alpha)$ be a contact manifold. The **Reeb vector field** R_α is the unique vector field defined by the equations:*

1. $i_R d\alpha = d\alpha(R_\alpha, -) \equiv 0$
2. $\alpha(R_\alpha) = 1$

We also have the notion of vector fields on $(M, \xi = \ker\alpha)$ that preserve the contact structure:

Definition 1.13 *Let $(M, \xi = \ker\alpha)$ be a contact manifold. A vector field v on M is a **contact vector field** if its flow φ_t is a contactomorphism, that is $(\varphi_t)_*\xi = \xi$, for all t. Equivalently, a vector field v is a contact vector field if $\mathcal{L}_v\alpha = g\alpha$ for some function $g : M \to \mathbb{R}$.*

We have $\mathcal{L}_R\alpha = i_{R_\alpha}d\alpha + d(\alpha(R_\alpha)) = 0$. So the Reeb vector field associated to a contact form is, in particular, a contact vector field.

Remark 1.14 *While contact vector fields are associated to* contact structures, *Reeb vector fields are associated to* contact forms. *In general, Reeb vector fields that are associated to contact forms that define the same contact structure might be distinct.*

Here are some examples of the Reeb vector fields on the contact manifolds we defined in previous examples:

Example 1.15 *For the standard contact structure α_1 described in Example 1.3, the Reeb vector field is $\frac{\partial}{\partial z}$. Indeed,*

$$\alpha_1(\frac{\partial}{\partial z}) = (dz + \sum_{i=1}^{n} x_i dy_i)(\frac{\partial}{\partial z}) = 1$$

and

$$d\alpha_1(\frac{\partial}{\partial z}, -) = (\sum_{i=1}^{n} dx_i \wedge dy_i)(\frac{\partial}{\partial z}, -) = 0$$

The flow of this vector field is translation along z-coordinate, which preserves the contact structure.

Example 1.16 *For the standard contact structure $\alpha_{S^{2n+1}}$ on the unit sphere described in Example 1.4, the Reeb vector field is*

$$\sum_{i=1}^{n+1}(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i})$$

Given a contact vector field v on M, we can define a function $f : M \to \mathbb{R}$ by $f(p) = \alpha_p(v_p)$.

On the other hand, given any function $f : M \to \mathbb{R}$, we can define a contact vector field v_f by

$$v_f = fR_\alpha + (d\alpha|_\xi)^{-1}(df|_\xi)$$

where $(d\alpha|_\xi)^{-1}(df|_\xi)$ is the unique vector field u such that $i_u(d\alpha|_\xi) = df|_\xi$ and R_α is the Reeb vector field.

This gives a one-to-one correspondence between functions on a contact manifold and contact vector fields, see [13] for more details. In particular, if we take $f \equiv 1$, the contact vector field we get is the Reeb vector field R_α corresponding to α.

1.1.2 Symplectic and Hamiltonian Vector Fields

There is also the corresponding notion of vector fields that preserve the symplectic structure:

Definition 1.17 *Let (X, ω) be a symplectic manifold. A vector field v on M is a* ***symplectic vector field*** *if its flow φ_t is a symplectomorphism, that is $(\varphi_t)^*\omega = \omega$, for all t. Equivalently, a vector field v is a symplectic vector field if $\mathcal{L}_v\omega = 0$.*

Let (X, ω) be a symplectic manifold and let $H : M \to \mathbb{R}$ be a smooth function. Its differential dH is a 1-form. By nondegeneracy of ω, there is a unique vector field X_H on M such that $i_{X_H}\omega = -dH$

Definition 1.18 *A vector field X_H as above is called the Hamiltonian vector field with Hamiltonian function H.*

By these definitions and Cartan's formula, we see that a vector field v on a symplectic manifold (X, ω) is:

- symplectic if and only if $i_v\omega$ is closed,

- Hamiltonian if and only if $i_v\omega$ is exact.

In particular every Hamiltonian vector field is symplectic.

1.2 Symplectic Cones and Symplectization

To define symplectic cones, we need to define Liouville vector fields on symplectic manifolds.

Definition 1.19 *Let (X, ω) be a symplectic manifold. A vector field v on X is a* ***Liouville vector field*** *if $\mathcal{L}_v\omega = \omega$.*

Liouville vector fields allow us to relate symplectic manifolds to contact manifolds:

Lemma 1.20 *Let (X, ω) be a symplectic manifold of dimension $2n+2$ and let v be a Liouville vector field on X. Then, $\alpha = i_v\omega = \omega(v, -)$ is a contact form when restricted to any hypersurface M^{2n+1} transverse to v.*

Proof By Cartan's formula for the Lie derivative, we have

$$\omega = \mathcal{L}_v\omega = i_v d\omega + d(\omega(v, -)) = d(\omega(v, -))$$

because $d\omega = 0$. Therefore,

$$\begin{aligned} \alpha \wedge d\alpha^n &= i_v\omega \wedge d(\omega(v, -))^n \\ &= i_v\omega \wedge \omega^n \\ &= (n+1)^{-1} i_v(\omega^{n+1}) \end{aligned}$$

Thus, by nondegenerecy of ω, if M is transverse to v, then $\alpha \wedge d\alpha^n \neq 0$. □

Such a hypersurface M is called of **contact type** in (X, ω).

For the next definition, recall that a topological group action

$$\psi : G \to \text{Homeo}(X)$$

where we denote the action of $g \in G$ on $x \in X$ by $g \cdot x$ or $\psi_g(x)$, is called **proper** if the map $G \times X \to X \times X$ defined by $(g, x) \longmapsto (x, g \cdot x)$ is a proper map.

Definition 1.21 *A* ***symplectic cone*** *is a triple (X, ω, v) where X is a manifold, ω is a symplectic form on X, and v is a Liouville vector field on (X, ω) generating a smooth proper free action of $\mathbb{R}$.*

Given a cooriented contact manifold $(M, \xi = \ker\alpha)$, we can define a symplectic cone as follows:

Definition 1.22 *Let $(M, \xi = ker\alpha)$ be a contact manifold of dimension 2n-1 for $n \geq 1$. Define $X = \mathbb{R} \times M$ and $\omega = d(e^t \alpha)$ where t is the $\mathbb{R}$ coordinate.*

*The pair (X, ω) is a symplectic manifold, called the **symplectization** of M. The "vertical" vector field $\frac{\partial}{\partial t}$ is a Liouville vector field.*

Remark 1.23 *In this definition, observe the slight abuse of notation. We denote the pullback of α by projection to X again by α. Also, we denote the projection $(t, x) \mapsto t$ by just t.*

Conversely, given a symplectic cone (X, ω, v), we can define a cooriented contact manifold

$$(X/\mathbb{R}, \xi = \ker(s^*(i_v\omega)) = \pi_*(\ker i_v\omega))$$

where $\pi : X \to X/\mathbb{R}$ is the projection to the orbits of the action generated by v and s is a section of this projection.

This gives a one to one correspondence between symplectic cones and a cooriented contact manifolds.

Up to a symplectomorphism, the symplectization of a contact manifold only depends on the contact structure and its coorientation, and not to a contact form. In fact, the symplectization of a cooeriented contact manifold $(M, \xi = \ker \alpha)$ can be defined intrinsically, without reference to a chosen contact form:

By the contact condition, the coorientation ξ^0_+ of $(M, \xi = \ker \alpha)$ is a **symplectic submanifold**[1] of the cotangent bundle $(T^*M, \omega = d\lambda)$ as defined in Example 1.7. Therefore, $(\xi^0_+, \omega|_{\xi^0_+})$ is a symplectic manifold with dimension 1 greater than the dimension of $(M, \xi = \ker \alpha)$.

Moreover, the diffeomorphism

$$h : \mathbb{R} \times M \to \xi^0_+$$

defined by $h(t, x) = e^t \alpha_x$ gives a symplectomorphism between two spaces:

We have, for $v \in T_{(t,x)}(\mathbb{R} \times M)$:

$$\begin{aligned}(h^*\lambda)_{(t,x)}(v) &= \lambda_{e^t\alpha_x}(dh_{(t,x)}(v)) \\ &= e^t\alpha_x(d\pi_{e^t\alpha_x} \circ dh_{(t,x)}(v)) \\ &= e^t\alpha_x(d(\pi \circ h)_{(t,x)}(v))\end{aligned}$$

where we used the definition of the tautological 1-form. The map $\pi \circ h$ is the projection $(t, x) \mapsto x$. Hence, we have $(h^*\lambda)_{(t,x)} = e^t\alpha_x$ (see Remark 1.23).

[1] That is, $\omega|_{\xi^0_+}$ is a symplectic form on ξ^0_+.

Therefore, we have

$$h^*\omega = d(e^t\alpha)$$

Hence, we call both $(\mathbb{R} \times M, d(e^t\alpha))$ and $(\xi^0_+, \omega|_{\xi^0_+})$ *the symplectization* of M.

1.3 Hamiltonian and Contact Actions

From now on we will consider smooth actions

$$\psi : G \to \mathrm{Diff}(M)$$

of Lie groups G on smooth manifolds M. We denote the action of $g \in G$ on $x \in X$ by

$$\psi_g(x) \text{ or } g \cdot x$$

Denote the Lie algebra of G by $\mathfrak{g}$. For each element $X \in \mathfrak{g}$, we call

$$X^\#_x := \frac{d}{dt}|_{t=0}\psi_{\exp tX}(x)$$

the **vector field generated by** X.

Let G be a group acting on a manifold through the action

$$\psi : G \to \mathrm{Diff}(M)$$

If M is symplectic or contact manifold, we might require the action to preserve the respective structure.

Definition 1.24 *Let G be a group acting on a symplectic manifold (M, ω) through the action $\psi : G \to \mathrm{Diff}(M)$. The action ψ is called a* ***symplectic action*** *if $\psi_g = \psi(g)$ is a symplectomorphism for every $g \in G$.*

Similarly, we have:

Definition 1.25 *Let G be a group acting on a contact manifold $(M, \xi = \ker \alpha)$ through the action $\psi : G \to \mathrm{Diff}(M)$. The action ψ is called a* ***contact action*** *if $\psi_g = \psi(g)$ preserves the contact structure and its coorientation for every $g \in G$. That is, $\psi(g)^*\alpha = e^f\alpha$ for a function $f : M \to \mathbb{R}$ for every $g \in G$.*

By the following lemma, we may assume the contact forms are invariant under a proper contact action:

Lemma 1.26 *Suppose a Lie group G acts on a contact manifold $(M, \xi = \ker \alpha)$ by a proper contact action. Then there is a G-invariant contact form α' defining the same cooriented contact structure on M.*

A proof of this fact can be found in [19, Lemma 2.6]. For a compact group G it can be directly proved by averaging α over G. For noncompact G, the proof uses existence of slices. From now on, we will assume that when a Lie group G acts on a contact manifold $(M, \xi = \ker \alpha)$ by a proper contact action, α is G-invariant.

1.3.1 Hamiltonian Actions and Symplectic Moment Map

One can see that vector fields generated by symplectic actions are complete symplectic vector fields. Roughly, we want to define Hamiltonian actions analogously as actions that generate Hamiltonian vector fields. To make this notion precise we define the symplectic moment map and Hamiltanion actions as follows:

Definition 1.27 *Let G be a group acting on a symplectic manifold (M, ω) through the action $\psi : G \to \mathrm{Diff}(M)$*

*The action ψ is called a **Hamiltonian action** if there exists a map $\mu : M \to \mathfrak{g}$ such that:*

- *The component $\mu^X : M \to \mathbb{R}$ of μ along X given by $\mu^X(p) = \langle \mu(p), X \rangle$ is a Hamiltonian function for the vector field $X^\#$:*

$$i_{X^\#}\omega = -d\mu^X$$

- *μ is equivariant with respect to the given action ψ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$:*
$$\mu \circ \psi_g = Ad^*_g \circ \mu$$
for all $g \in G$

*(M, ω, G, μ) is called a **Hamiltonian G-space** and μ is called a **moment map**.*

The following example of a Hamiltonian action of a Torus $\mathbb{T}^n$ on $(\mathbb{C}^n, \omega_{st} = \frac{i}{2}\sum_{i=1}^N dz_i \wedge d\bar{z}_i)$ will be important for our discussion in the following sections:

Example 1.28 *Consider the action of $\mathbb{T}^n$ on $(\mathbb{C}^n, \omega_{st} = \frac{i}{2}\sum_{i=1}^N dz_i \wedge d\bar{z}_i)$ by*

$$[a_1, \ldots, a_n] \cdot (z_1, \ldots, z_n) = (e^{2\pi i a_1} z_1, \ldots, e^{2\pi i a_n} z_n)$$

This action is Hamiltonian with the moment map $\mu : \mathbb{C}^n \to \mathfrak{g} \cong (\mathbb{R}^n)^*$ given by

$$\mu(z_1, \ldots, z_n) = \pi \sum_{i=1}^n |z_i|^2 e_i^*$$

Example 1.29 *Consider a **symplectic vector space** (V, ω), that is a vector space V equipped with a nondegenerate skew symmetric bilinear map ω, and consider the action of the symplectic linear group $Sp(V, \omega)$ by linear transformations [2].*

We may view (V, ω) as a symplectic manifold and canonically identify tangent spaces at any point again with (V, ω).

[2] See [21] for a detailed discussion about symplectic linear group.

The action of $Sp(V,\omega)$ on (V,ω) is Hamiltonian with the moment map

$$\mu : V \to \mathfrak{sp}(V)^*$$

given by

$$v \longmapsto [A \mapsto \frac{1}{2}\omega(Av, v)]$$

where $\mathfrak{sp}(V)$ is the symplectic linear algebra.

Hence, a symplectic representation, that is a homomorphism $\rho : G \to Sp(V,\omega)$, also induces a Hamiltonian action with the moment map

$$\mu_\rho : V \to \mathfrak{g}^*$$

which is the composition with dual of the the induced Lie algebra representation.

1.3.2 Lifted Actions and the Contact Moment Map

Not all group actions defined on a symplectic manifold are hamiltonian, see [9], [21] for conditions for a symplectic action to be a Hamiltonian action in terms of Lie algebra cohomology.

However, if a Lie group acts on a smooth manifold, the action always induces a Hamiltonian action on the cotangent bundle with its canonical symplectic structure:

Let G be a group acting on a manifold M through the action

$$\psi : G \to \text{Diff}(M)$$

Let T^*M be the cotangent bundle of M with its canonical symplectic structure. The action of G on M lifts to an action on the cotangent bundle:

$$\Psi : G \to \text{Diff}(T^*M)$$

where the action is given by

$$\Psi_g(m, \beta) = (\psi_g(m), (\psi_g^{-1})^*\beta)$$

Let $X^\#$ and $\tilde{X}^\#$ be the vector fields generated by $X \in \mathfrak{g}$ on M and T^*M, respectively. The vector field $\tilde{X}^\#$ is a **lift** of $X^\#$. That is, if $\pi : T^*M \to M$ is the natural projection, then $d\pi(\tilde{X}^\#) = X^\#$.

The lifted actions preserve the tautological one form λ on T^*M, see [9, Proposition 2.1]. Thus, we have

$$0 = L_{\tilde{X}^\#}\lambda = d(i_{\tilde{X}^\#}\lambda) + i_{\tilde{X}^\#}d\lambda$$

so $i_{\tilde{X}^\#}\omega = i_{\tilde{X}^\#}d\lambda = -d(i_{\tilde{X}^\#}\lambda)$. That is, the function $H = i_{\tilde{X}^\#}\lambda$ is a Hamiltonian function for the vector field $\tilde{X}^\#$. On the other hand, if we use the definition of the tautological one form, we get:

$$\begin{aligned} H(m,\beta) &= i_{\tilde{X}^\#_{(m,\beta)}}\lambda_{(m,\beta)} \\ &= \lambda_{(m,\beta)}(\tilde{X}^\#_{(m,\beta)}) \\ &= \langle \beta, X^\#_m \rangle \end{aligned}$$

With this previous discussion, we claim that the lifted action is a Hamiltonian action with the moment map $\mu : T^*M \to \mathfrak{g}^*$ given by

$$\langle \mu(m,\beta), X\rangle = \langle \beta, X^\#_m\rangle$$

Indeed, it can also be checked that the map μ is equivariant with respect to the coadjoint action of G, using the definition of Ad and the naturality of the exponential map:

$$\begin{aligned} \langle \mu(\psi_g(m), (\psi_g^{-1})^*\beta), X\rangle &= \langle (\psi_{g^{-1}})^*\beta, X^\#_{\psi_g(m)}\rangle \\ &= \langle \beta, (\psi_{g^{-1}})_* X^\#_{\psi_g(m)}\rangle \\ &= \langle \beta, \frac{d}{dt}|_{t=0}\psi_{g^{-1}\exp tXg}(m)\rangle \\ &= \langle \beta, (Ad_{g^{-1}}X)^\#_m\rangle \\ &= \langle \mu(m,\beta), Ad_{g^{-1}}X\rangle \\ &= \langle Ad_g^* \circ \mu(m,\beta), X\rangle \end{aligned}$$

Therefore, for an action $\psi : G \to \mathrm{Diff}(M)$, we see that T^*M is a Hamiltonian G-space $(T^*M, \omega = d\lambda, G, \mu)$ with the lifted action and the moment map defined as above.

Now consider a contact manifold $(M, \xi = \ker\alpha)$ with a proper contact action of a Lie group G. We may assume α is G-invariant. Then we can consider the restriction of the moment map $\mu : T^*M \to \mathfrak{g}^*$ to the submanifold ξ^0_+ to get the **contact moment map**:

$$\mu : \xi^0_+ \to \mathfrak{g}^*$$

The above discussion can also be described in terms of symplectization: We can lift a proper contact action on $(M, \xi = \ker\alpha)$ to a Hamiltonian action on the symplectization ξ^0_+. The contact moment map is the symplectic moment map for the lifted action.

Given the coorientation of the contact structure, the contact moment map is independent of the contact form. For a fixed G-invariant contact form α, we get the **α-moment map**

$$\mu_\alpha : M \to \mathfrak{g}^*$$

of the action by $\mu_\alpha = \mu \circ \alpha$, that is the map defined by

$$\langle \mu_\alpha(m), X \rangle = \langle \alpha_m, X_m^\# \rangle$$

By the G-invariance of α the map μ_α is again equivariant with respect to the coadjoint action.

In the following discussion we will call both μ_α and μ "moment map" for a contact action where it will not cause confusion.

1.4 Contact and Symplectic Reduction

Reduction gives us a way to get a new contact or symplectic manifold using the moment mapping. By the equivariance condition of Hamiltonian actions, the zero level set $\mu^{-1}(0)$ of the moment map is invariant under the action. The following theorems describe the structures on the orbit space $\mu^{-1}(0)/G$.

We start by symplectic reduction:

Theorem 1.30 (Marsden, Weinstein, Meyer, [20], [22]) *Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \to M$ be the inclusion map.*

Assume G acts freely on $\mu^{-1}(0)$. Then

- *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a manifold,*
- *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle, and*
- *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{red}$.*

The pair (M_{red}, ω_{red}) is called the ***reduction*** *of (M, ω, G, μ) with respect to G, μ.*

Remark 1.31 *If another Lie group H with Lie algeba $\mathfrak{h}$ acts through a Hamiltonian action on (M, ω, G, μ) with the moment map $\Psi : M \to \mathfrak{h}^*$ and the actions of H and G commute, we get an induced Hamiltonian action of H on (M_{red}, ω_{red}) with the moment map $\Psi_{red} : M_{red} \to \mathfrak{h}^*$ such that*

$$\Psi \circ i = \Psi_{red} \circ \pi$$

In Chapter 3, we will be using symplectic reduction to construct contact manifolds with a given image of the contact moment map.

We also have the analogous result of contact reduction for contact manifolds:

Theorem 1.32 (Albert, [3], Geiges, [13]) *Suppose a Lie group G acts on a contact manifold $(M, \xi = \ker \alpha)$ preserving the contact form α. Let $\mu_\alpha : M \to \mathfrak{g}^*$ denote the corresponding α-moment map.*

Assume $\mu_\alpha^{-1}(0)$ is a manifold and assume that G acts freely and properly on $\mu_\alpha^{-1}(0)$. Then α descends to a contact form α_0 on $M_0 := \mu_\alpha^{-1}(0)/G$ such that

$$\alpha|_{\mu_\alpha^{-1}(0)} = \pi^*\alpha_0$$

where π is the projection to the orbit space.

The contact structure on M_0 defined by α_0 depends only on the contact structure defined by α and not on the form α itself.

We will be using contact reduction to prove Lemma 2.7 which describes a contact action of a torus locally.

In fact, in certain cases, symplectic reduction can be done at levels other than 0. However, at nonzero level sets, a contact form does not descend to a contact form in the reduced space.

Remark 1.33 *Symplectization and reduction of contact manifolds commute in the following sense:*

Suppose a Lie group G acts on a contact manifold (M, α) preserving the contact form α and satisfying the assumptions of contact reduction. Let

$$\mu_\alpha : M \to \mathfrak{g}^*$$

denote the corresponding α-moment map.

Then, we have the reduced contact manifold $(\mu_\alpha^{-1}(0)/G, \alpha_0)$ with $i^*\alpha = \pi^*\alpha_0$ where $i : \mu_\alpha^{-1}(0) \to M$ is the inclusion.

The symplectization of the reduced contact manifold is $(\mathbb{R} \times (\mu_\alpha^{-1}(0)/G), d(e^t\alpha_0))$.

On the other hand, consider the symplectization $(\mathbb{R} \times M, d(e^t\alpha)))$ of (M, α). The action of G lifts to the action defined by $g \cdot (t, x) = (t, g \cdot x)$ for all $g \in G$ and $(t, x) \in \mathbb{R} \times M$.

This action is Hamiltonian with the moment map $\mu : \mathbb{R} \times M \to \mathfrak{g}^*$ given by $\mu(t, x) = e^t\mu_\alpha(x)$. In fact, this is the moment map of the action given by the action lifted to $\xi_+^0 \cong \mathbb{R} \times M$ where identification is by the map h as defined before.

We have that $\mu^{-1}(0) = \mathbb{R} \times \mu_\alpha^{-1}(0)$ and G acts freely on $\mu^{-1}(0)$. Therefore, we get the reduced space $\mu^{-1}(0)/G = (\mathbb{R} \times \mu_\alpha^{-1}(0))/G \cong \mathbb{R} \times (\mu_\alpha^{-1}(0)/G)$ where the identification follows from the definition of the action on the symplectization.

Moreover, we have

$$\begin{aligned} i^*d(e^t\alpha) &= d(e^t i^*\alpha) \\ &= d(e^t\pi^*\alpha_0) \\ &= \pi^*d(e^t\alpha_0) \end{aligned}$$

where i is the inclusion of the respective 0-levels, π is the respective projections to the reduced spaces, and α_0 is as above.

Therefore, the symplectic form on the reduced space is $\omega_{red} = d(e^t\alpha_0)$ and the reduction of the symplectization of (M, α) is again $(\mathbb{R} \times (\mu_\alpha^{-1}(0)/G), d(e^t\alpha_0))$.

In particular, as in Remark 1.31, if the action of H commutes with G, it induces a contact action and a contact moment map on the reduced space.

1.5 Contact Toric Manifolds

Recall that an action of the group G on a manifold M is called **effective** if the only element that fixes all the points of M is the identity $id \in G$.

Definition 1.34 *A **contact toric** G**-manifold** is a co-oriented contact manifold $(M, \xi = \ker\alpha)$ with an effective action of a torus G preserving the contact structure and its co-orientation (i.e. an effective contact action of a torus G), such that $2\dim G = \dim M + 1$.*

Remark 1.35 *When we consider a contact toric G-manifold, we will consider it as a triple $(M, \xi = \ker\alpha, \mu : \xi_+^\circ \to \mathfrak{g}^*)$ or $(M, \xi = \ker\alpha, \mu_\alpha : M \to \mathfrak{g}^*)$.*

We will first state a lemma concerning the image of the moment map of a contact toric G-manifold:

Lemma 1.36 *Suppose $(M, \xi = \ker\alpha, \mu : \xi_+^0 \to \mathfrak{g}^*)$ is a contact toric G-manifold. Then zero is not in the image of the contact moment map $\mu : \xi_+^0 \to \mathfrak{g}^*$.*

The proof of this lemma uses a representation theoretic argument based on the dimension and can be found in [19, Lemma 2.12].

We may fix an inner product on $\mathfrak{g}$ and hence on $\mathfrak{g}^*$. Then there exists a unique G-invariant contact form preserving ξ and its co-orientation such that $\|\mu_\alpha(x)\| = 1$ for all $x \in M$. This can be done by taking any G-invariant contact form α' defining ξ and setting $\alpha_x = \frac{1}{\|\mu'_\alpha(x)\|}\alpha'_x$ which is possible by the previous lemma.

To describe the action of the torus on a contact toric manifold, we will consider the image of the moment map. To this end we have the following definition:

Definition 1.37 *Let $(M, \xi = \ker\alpha)$ be a co-oriented contact manifold with a contact action of a Lie group G and let $\mu : \xi_+^\circ \to \mathfrak{g}^*$ denote the moment map.*

The set

$$C(\mu) := \mu(\xi_+^\circ) \cup \{0\}.$$

*is called the **moment cone**.*

Note that if the contact form α is G-invariant, we have

$$C(\mu) = \{t\eta : \eta \in \mu_\alpha(M), t \in [0, \infty)\}$$

We have the following notion of equivalence of contact toric manifolds:

Definition 1.38 *Two contact toric G-manifolds* (M, α, μ_α) *and* $(M_0, \alpha_0, \mu_{\alpha_0})$ *are* ***isomorphic*** *if there exists a G-equivariant coorientation preserving contactomorphism* $\varphi : M \to M_0$*. The map* φ *is called an* ***isomorphism****. We denote the group of isomorphisms of* (M, α, μ_α) *by* $Iso(M, \alpha, \mu_\alpha)$ *or* $Iso(M)$

Here are some examples of contact toric G-manifolds and their moment cones:

Example 1.39 *Let* $S^3 = \{(x_0, y_0, x_1, y_1) \in \mathbb{R}^4 : (x_0^2 + y_0^2) + (x_1^2 + y_1^2) = 1\}$ *be the standard 3-sphere with the contact form*

$$\alpha = (x_0 dy_0 - y_0 dx_0) + (x_1 dy_1 - y_1 dx_1)$$

The 2-torus $\mathbb{T}^2 = S^1 \times S^1$*, where we denote points with coordinates* $\{(\theta_0, \theta_1)\} \in \mathbb{T}^2$*, acts on* S^3 *by rotation of* (x_0, y_0)*- and* (x_1, y_1)*-planes by* θ_0 *and* θ_1*, respectively. That is,* $\mathbb{T}^2$ *action on* S^3 *is generated by the vector fields* $H_i = (x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i})$ *for* $i = 0, 1$ *on* S^3*. This action preserves the contact structure. In addition, fixing all points in* S^3 *implies the action of a point* $(\theta_0, \theta_1) \in \mathbb{T}^2$ *does not rotate the points on* (x_0, y_0)*- and* (x_1, y_1)*-planes, which is only the case for the identity element in this action by* $\mathbb{T}^2$*. Thus, the action is also effective. Therefore,* S^3 *is a contact toric* $\mathbb{T}^2$*-manifold.*

Now consider the α*-moment map* $\mu_\alpha : S^3 \to \mathfrak{g}^*$*, where we identify the lie algebra of the torus as* $\mathfrak{g}^* \cong (\mathbb{R}^2)^* = span\{e_0^*, e_1^*\}$ *with* $e_i \in \mathfrak{g}$ *generating* H_i*. We have:*

$$\begin{aligned}\mu_\alpha(x_0, y_0, x_1, y_1)(e_i) &= \alpha_{(x_0, y_0, x_1, y_1)}((e_i)^\#((x_0, y_0, x_1, y_1)) \\ &= \alpha_{(x_0, y_0, x_1, y_1)}((x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i})) \\ &= (x_i^2 + y_i^2)\end{aligned}$$

From this, we have $\mu_\alpha(x_0, y_0, x_1, y_1) = (x_0^2 + y_0^2)e_0^* + (x_1^2 + y_1^2)e_1^*$*. As*

$$(x_0^2 + y_0^2) + (x_1^2 + y_1^2) = 1$$

this shows that the image of the α*-moment map is*

$$\mu_\alpha(S^3) = \{t_0 e_0^* + t_1 e_1^* : t_0 + t_1 = 1, t_i \geq 0\}$$

That is, $\mu_\alpha(S^3)$ *is the standard 1-simplex in* $\mathfrak{g}^*$*. Then, the moment cone is*

$$C(\mu) = \{s_0 e_0^* + s_1 e_1^* : s_i \geq 0\}$$

That is, the first quadrant $(\mathbb{R}^2)^*_{\geq 0}$ *in* $\mathfrak{g}^* \cong (\mathbb{R}^2)^* \cong \mathbb{R}^2$*.*

Example 1.40 *We may extend the above example to higher dimensions immediately. The torus* $\mathbb{T}^{n+1}$ *acts on the sphere* S^{2n+1}*, through the rotations generated by the vector fields* $H_i = (x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i})$ *for* $i = 0, 1, \dots, n$ *on* S^{2n+1}*. With this action,* S^{2n+1} *is a contact toric* $\mathbb{T}^{n+1}$*-manifold.*

By a similar argument as above, the image of the α*-moment map is*

$$\mu_\alpha(S^{2n+1}) = \{\sum_{i=0}^{n} t_i e_i^* : \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}$$

That is, $\mu_\alpha(S^{2n+1})$ *is the standard n-simplex in* $\mathfrak{g}^* \cong (\mathbb{R}^n)^*$*. Then, the moment cone is* $C(\mu) = (\mathbb{R}^{n+1})^*_{\geq 0}$

Example 1.41 *Consider the manifold* $M = S^1 \times \mathbb{T}^2$ *with coordinates* $\{(t, \theta_1, \theta_2)\}$ *with the contact form*

$$\alpha = \cos t d\theta_1 + \sin t d\theta_2$$

The 2-torus $\mathbb{T}^2$ *with coordinates* $\{(\varphi_0, \varphi_1)\}$ *acts on points* $(t, \theta_1, \theta_2) \in Y = S^1 \times \mathbb{T}^2$ *by termwise addition on the second factor. This action is free and preserves the contact structure. Therefore,* $S^1 \times \mathbb{T}^2$ *is a contact toric* $\mathbb{T}^2$*-manifold.*

Now consider the α*-moment map* $\mu_\alpha : S^1 \times \mathbb{T}^2 \to \mathfrak{g}^* \cong span\{d\varphi_0|_0, d\varphi_1|_0\}$*. For* $\frac{\partial}{\partial \varphi_0}|_0 \in \mathfrak{g} = span\{\frac{\partial}{\partial \varphi_0}|_0, \frac{\partial}{\partial \varphi_1}|_0\}$*, we have* $\exp(t\frac{\partial}{\partial \varphi_0}|_0) = (t, 0) \in \mathbb{T}^2$*. Therefore:*

$$\begin{aligned}
\mu_\alpha(t, \theta_1, \theta_2)(\frac{\partial}{\partial \varphi_0}|_0) &= \alpha_{(t,\theta_1,\theta_2)}((\frac{\partial}{\partial \varphi_0}|_0)^\#(t, \theta_1, \theta_2)) \\
&= \alpha_{(t,\theta_1,\theta_2)}(\frac{d}{ds}|_{t=0} \exp(t\frac{\partial}{\partial \varphi_0}|_0) \cdot (t, \theta_1, \theta_2)) \\
&= \alpha_{(t,\theta_1,\theta_2)}(\frac{d}{ds}|_{t=0}(t, \theta_1 + t, \theta_2)) \\
&= \alpha_{(t,\theta_1,\theta_2)}(\frac{\partial}{\partial \theta_1}|_{(t,\theta_1,\theta_2)}) \\
&= \cos t
\end{aligned}$$

By a similar calculation, $\mu_\alpha(t, \theta_1, \theta_2)(\frac{\partial}{\partial \varphi_1}|_0) = \sin t$*. Therefore, we have*

$$\mu_\alpha(t, \theta_1, \theta_2) = \cos t d\varphi_0|_0 + \sin t d\varphi_1|_0$$

This shows that the image of the α*-moment map is*

$$\mu_\alpha(S^1 \times \mathbb{T}^2) = \{\cos t d\varphi_0|_0 + \sin t d\varphi_1|_0 : t \in S^1\}$$

From this, we see that the moment cone is

$$\begin{aligned}
C(\mu) &= \{s \cos t d\varphi_0|_0 + s \sin t d\varphi_1|_0 : t \in S^1, s \geq 0\} \\
&= \{s_0 d\varphi_0|_0 + s_1 d\varphi_1|_0 : s_0, s_1 \in \mathbb{R}\} \\
&= \mathfrak{g}^* \cong (\mathbb{R}^2)^* \cong \mathbb{R}^2
\end{aligned}$$

We can generalize this argument to the same manifold $M = S^1 \times \mathbb{T}^2$ *with coordinates* $\{(t, \theta_1, \theta_2)\}$ *with the contact form*

$$\alpha_n = \cos(nt)d\theta_1 + \sin(nt)d\theta_2$$

with n *a positive integer. The moment cone is again* $C(\mu) = \mathfrak{g}^* \cong \mathbb{R}^2$.

We will see later that the contact toric manifolds described in Example 1.41 are the only compact connected 3-dimensional contact toric manifolds with a free torus action.

Chapter 2

Contact Toric Manifolds

In this chapter, we describe the properties of contact toric manifolds. We closely follow the descriptions and results of Lerman [19].

In Sections 1 and 2, we describe the local structure of contact toric manifolds. We give local forms for moment maps of contact toric manifolds and describe the images and fibers of moment maps.

In Section 3, we use sheaf cohomology to obtain information about the isomorphism classes of contact toric manifolds from the local considerations of the previous sections.

For a torus $G \cong \mathbb{T}^n$ with Lie algebra $\mathfrak{g} \cong \mathbb{R}^n$, we denote the integral lattice

$$\ker\{\exp : \mathfrak{g} \to G\}$$

by $\mathbb{Z}_G$ and identify it with $\mathbb{Z}^n$.

For the sake of brevity, we will abbreviate compact connected contact toric G-manifolds as c.c.c.t G-manifolds.

2.1 Local Structure of Contact Toric Manifolds

We will consider embedded submanifolds as subsets of a manifold. Thus, if $i : N \to M$ is an embedding, we identify N with the image $i(N)$ and the tangent space T_nN with $di_n(T_nN)$.

For the discussion of this section, we need the following definition:

Definition 2.1 *A* ***symplectic vector bundle*** *over a manifold M is a pair (E, ω) consisting of a real vector bundle $\pi : E \to M$ and a family of nondegenerate skew-symmetric bilinear $\omega_q : E_q \times E_q \to R$ on the fibers $E_q = \pi^{-1}(q)$ of the vector bundle that vary smoothly with $q \in M$. That is, $q \mapsto \omega_q$ is a smooth section of the vector bundle $\bigwedge^2 E^*$.*

Two symplectic vector bundles (E_1, ω_1) *and* (E_2, ω_2) *are called* ***isomorphic*** *if there exists a vector bundle isomorphism* $F : E_1 \to E_2$ *such that* $F^*\omega_2 = \omega_1$*. We denote isomorphic symplectic vector bundles as* $(E_1, \omega_1) \cong (E_2, \omega_2)$*.*

A ***conformal symplectic vector bundle*** *over a manifold* M *is a pair* $(E, [\omega])$ *where* (E, ω) *is a symplectic vector bundle and* $[\omega]$ *is the conformal class of* ω*, that is the collection* $\{e^f \omega : f : M \to \mathbb{R}_+\}$ *of symplectic forms.*

Note that, for any contact manifold $(M, \xi = \ker \alpha)$, the pair $(\xi, \omega = d\alpha|_\xi)$ is a symplectic vector bundle over M by the contact condition.

Definition 2.2 *Let* $(M, \xi = \ker \alpha)$ *be a cooriented contact manifold. An embedded submanifold* $N \subseteq M$ *is* ***preisotropic*** *if*

1. *N is transverse to ξ, and*
2. *the distribution $\zeta = TN \cap \xi$ is isotropic in the symplectic vector bundle $(\xi, \omega = d\alpha|_\xi)$*[1]*.*

The inclusion $i : N \to M$ *is called a* ***preisotropic embedding****.*

For a preisotropic embedding $i : N \to M$*, we define the distribution*

$$\zeta = TN \cap \xi = \ker i^*\alpha$$

to be the ***characteristic distribution*** *of* i*.*

We define the ***conformal symplectic normal bundle*** $(E, [\omega_E])$ *of the embedding* i *by* $E = \zeta^\omega / \zeta$ *where* ζ^ω *is the symplectic orthogonal to* ζ *in the symplectic vector bundle* $(\xi, d\alpha|_\xi)$ *and* $[\omega_E]$ *is the conformal class of symplectic structure induced on the vector bundle* E *by* $[d\alpha|_\xi]$*.*

A pre-isotropic embedding is uniquely determined by its characteristic distribution and its conformal symplectic normal bundle:

Theorem 2.3 (Uniqueness of Preisotropic Embeddings) *Let* $(M_j, \xi_j = \ker \alpha_j)$*, for* $j = 1, 2$*, be two contact manifolds and suppose* $i_j : N \to M_j$*, for* $j = 1, 2$ *are two preisotropic embeddings such that*

$$i_1^*\alpha_1 = e^f i_2^*\alpha_2$$

and

$$(E_1, \omega_1) \cong (E_2, \omega_2)$$

as symplectic vector bundles, where $f, h \in C^\infty(N)$ *are two functions and* $(E_1, [\omega_1])$ *and* $(E_2, [\omega_2])$ *are the conformal symplectic normal bundles of the embeddings.*

Then there exist neighborhoods U_j *of* $i_j(N) \subseteq M_j$ *and a diffeomorphism*

$$\varphi : U_1 \to U_2$$

[1]That is, $\omega|_\zeta \equiv 0$.

such that $i_2 = \varphi \circ i_1$ *and* $\varphi^* \alpha_2 = e^g \alpha_1$ *for some function* $g \in C^\infty(U_1)$.

Moreover, if a Lie group G *acts properly on* N, M_1, M_2 *making the embeddings* i_j *G-equivariant and if the action on* M_j *are contact, then we may choose the neighborhoods* U_1, U_2 *to be G-invariant and the map* φ *to be G-equivariant.*

The proof of Theorem 2.3 can be found in [19, Theorem 3.5, Theorem 3.6]. The proof is done by choosing a compatible almost complex structure on $(\xi, d\alpha|_\xi)$ and uses the equivariant version of the Darboux Theorem for contact manifolds.

Lemma 2.4 *Let* $\mu_\alpha : M \to \mathfrak{g}^*$ *be the* α*-moment map for a contact action of a torus* G *on a contact manifold* $(M, \xi = \ker \alpha)$*. Suppose for some point* $x \in M$ *we have* $\mu_\alpha(x) \neq 0$*. Then the orbit* $G \cdot x$ *is preisotropic in* $(M, \xi = \ker \alpha)$*.*

Proof Suppose $\mu_\alpha(x) \neq 0$ for some point $x \in M$. Then,

$$\langle \mu_\alpha(x), X \rangle = \langle \alpha_x, X^\#_x \rangle \neq 0$$

for some $X \in \mathfrak{g}$. That is, $X^\#_x \notin \xi_x$

On the other hand, by definition, $X^\#_x = \frac{d}{dt}|_{t=0} \exp tX \cdot x$. Therefore $X^\#_x \in T_x(G \cdot x)$.

Since ξ is a codimension 1 distribution, this shows $G \cdot x$ is transverse to ξ at the point x. By G invariance, this proves that $G \cdot x$ is transverse to ξ.

Next, consider a fiber ζ_x of the characteristic distribution ζ at a point x:

$$\zeta_x = T_x(G \cdot x) \cap \xi_x = \{X^\#_x \in T_x(G \cdot x) : \langle \alpha_x, X^\#_x \rangle = 0\}$$

Consider the set $\mathfrak{k} = \{X \in \mathfrak{g} : \langle \alpha_x, X^\#_x \rangle = 0\} \subseteq \mathfrak{g}$. The set $\mathfrak{k}$ is closed under the Lie bracket as $\mathfrak{g}$ is abelian. Hence, the distribution $\zeta \subseteq \xi$ is closed under the Lie bracket. Then, for vector fields $X, Y \in \zeta$ we have

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) = 0$$

That is, ζ is isotropic in $(\xi, d\alpha|_\xi)$. Therefore, $G \cdot x$ is preisotropic in M. □

Note that, by Lemmas 1.36 and 2.4 the orbits in a contact toric manifold are preisotropic.

Definition 2.5 *Let* $\mu_\alpha : M \to \mathfrak{g}^*$ *be the* α*-moment map for a contact action*

$$\psi : G \to \mathrm{Diff}(M)$$

of a torus G *on a contact manifold* $(M, \xi = \ker \alpha)$*. Suppose for some point* $x \in M$ *we have* $\mu_\alpha(x) \neq 0$*. Let* ζ_x *denote the fiber at* x *of the characteristic distribution of the preisotropic embedding* $G \cdot x \to (M, \xi)$ *and let* ζ^ω_x *denote its symplectic perpendicular in* $(\xi, \omega = d\alpha|_\xi)$*.*

*We define the **symplectic slice** at x for the action of G on (M, ξ) to be the symplectic vector space $V = \zeta_x^{\omega}/\zeta_x$ with the symplectic structure ω_V induced by ω.*

We refer to the symplectic representation of the isotropy group G_x on (V, ω_V) induced by

$$\psi_g \mapsto d(\psi_g)_x$$

*as the **symplectic slice representation**. We denote the moment map induced by this representation by*

$$\Phi_V : V \to \mathfrak{g}^*$$

See Example 1.29 to recall how this moment map is defined.

*We define the **characteristic subalgebra** of the preisotropic embedding*

$$G \cdot x \to (M, \xi)$$

to be

$$\mathfrak{k} := (\mathbb{R}\mu_\alpha(x))^0 \cong \{X \in \mathfrak{g} : \langle \alpha_x, X_x^{\#} \rangle = 0\}$$

where the identification is the canonical identification with the double dual.

Note that $\mathfrak{g}_x \subseteq \mathfrak{k}$, where $\mathfrak{g}_x$ is the Lie algebra of G_x, since for $X \in \mathfrak{g}_x$ generate $X_x^{\#} = 0$. Moreover, $\zeta_x \cong \mathfrak{k}/\mathfrak{g}_x$ by definition of ζ and $\mathfrak{k}$. Also, notice that $\mathfrak{k}$ is co-oriented.

Lemma 2.6 *Let $(M_j, \xi_j = \ker \alpha_j)$, for $j = 1, 2$, be two contact manifolds with actions ψ^j of the torus G preserving the contact forms α_j with the corresponding moment maps $\mu_{\alpha_j} : M_j \to \mathfrak{g}$.*

Suppose $x_j \in M_j$, $j = 1, 2$ are two points such that

1. *$0 \neq \mu_{\alpha_1}(x_1) = \lambda \mu_{\alpha_2}(x_2)$ for some $\lambda > 0$: that is, the characteristic subalgebras agree as co-oriented subspaces of $\mathfrak{g}$;*
2. *the isotropy groups are equal : $G_{x_1} = G_{x_2}$;*
3. *the symplectic slice representations at x_1 and x_2 are isomorphic as symplectic representation up to a conformal factor: that is, there exists an isomorphism $L : V_1 \to V_2$ such that for every $v \in V_1$*

   $$L(d(\psi_g^1)_{x_1}(v)) = C d(\psi_g^2)_{x_2}(L(v))$$

   for some $C > 0$, where V_j the symplectic slice at x_j for the action of G on $(M_j, \xi_j = \ker \alpha_j)$.

Then there exist G-invariant neighborhoods U_j of $G \cdot x_j$ in M_j, $j = 1, 2$, and a G-equivariant diffeomorphism $\varphi : U_1 \to U_2$ such that $\varphi^* \alpha_2 = e^f \alpha_1$ for some function $f : M_1 \to \mathbb{R}$.

Proof By (1), we have the characteristic subalgebras are

$$\mathfrak{k} = (\mathbb{R}\mu_{\alpha_1}(x_1))^0 = (\mathbb{R}\mu_{\alpha_2}(x_2))^0$$

Then, we have the characteristic distributions and conformal symplectic normal bundles of the preisotropic embeddings $i_i : G \cdot x_j \to M_j$ as

$$\zeta_j \cong G \times_{G_{x_j}} \mathfrak{k}/\mathfrak{g}_{x_j} \text{ and } E_j \cong G \times_{G_{x_j}} V_j$$

By the assumptions, the maps i_j satisfy the requirements for Theorem 2.3 and we get the desired diffeomorphism. □

Lemma 2.7 *Let $(M, \xi = \ker \alpha)$ be a contact manifold with an action of a torus G preserving the contact form α. Suppose $x \in M$ is such that $\mu_\alpha(x) \neq 0$.*

Let $\mathfrak{k} = (\mathbb{R}\mu_\alpha(x))^0$ be the characteristic subalgebra and $G_x \to Sp(V, \omega_V)$ the symplectic slice representation. Choose splittings

$$\begin{aligned} \mathfrak{g}_x^0 &\cong (\mathfrak{k}/\mathfrak{g}_x)^* \oplus \mathbb{R}\mu_\alpha(x) \\ \mathfrak{g}^* &\cong \mathfrak{g}_x^0 \oplus \mathfrak{g}_x^* \end{aligned}$$

and thereby a splitting

$$\mathfrak{g}^* \cong (\mathfrak{k}/\mathfrak{g}_x)^* \oplus \mathbb{R}\mu_\alpha(x) \oplus \mathfrak{g}_x^*.$$

Let $i : \mathfrak{g}_x^* \hookrightarrow \mathfrak{g}^*$, $j : (\mathfrak{k}/\mathfrak{g}_x)^* \hookrightarrow \mathfrak{g}^*$ be the corresponding embeddings.

There exists a G-invariant neighborhood U of the zero section $G \cdot [1, 0, 0]$ in

$$N = G \times_{G_x} ((\mathfrak{k}/\mathfrak{g}_x)^* \oplus V)$$

and an open G-equivariant embedding $\varphi : U \hookrightarrow M$ with $\varphi([1, 0, 0]) = x$ and a G-invariant 1-form α_N on N such that

1. *$\varphi^*\alpha = e^f \alpha_N$ for some function $f \in C^\infty(U)$ and*
2. *the α_N-moment map μ_{α_N} is given by*

$$\mu_{\alpha_N}([a, \eta, v]) = \mu_\alpha(x) + j(\eta) + i(\Phi_V(v))$$

where $\Phi_V : V \to \mathfrak{g}^*$ is the moment map for the slice representation.

Consequently,

$$\begin{aligned} \mu_\alpha \circ \varphi([a, \eta, v]) &= (e^f \mu_{\alpha_N})([a, \eta, v]) \\ &= e^{f([a,\eta,v])}(\mu_\alpha(x) + j(\eta) + i(\Phi_V(v))) \end{aligned}$$

for some G-invariant function f on N.

We give a rough sketch of the proof. For the details of the full proof see [19].

Proof (Sketch) Identify $T^*G \cong G \times \mathfrak{g}^*$ using trivialization of the cotangent bundle by translations. Consider the hypersurface

$$\Sigma = G \times (\mu_\alpha(x) + j((\mathfrak{k}/\mathfrak{g}_x)^*) + i(\mathfrak{g}_x)) \subseteq G \times \mathfrak{g}^* \cong T^*G$$

Consider the Liouville vector field v on $T^*G \cong G \times \mathfrak{g}^*$ generated by the flow $(t, g, \nu) \mapsto (g, e^t\nu)$. As, $\mu_\alpha(x) \neq 0$, the vector field v is transverse to Σ. Therefore, Σ is a hypersurface of contact type with contact form given by the restriction of the tautological 1-form λ_{T^*G}.

Consider the actions $g \cdot (a, \nu) = (ga, \nu)$ of G and $h \cdot (a, \nu) = (ah^{-1}, \nu)$ of G_x on $G \times \mathfrak{g}^* \cong T^*G$. These actions preserve the hypersurface Σ, the Liouville vector field v, and the 1-form λ_{T^*G}.

On the other hand, consider the action of G_x on V given by the slice representation. This action preserves the 1-form $\alpha_V = i_R\omega_V$ where R is the radial vector field in V

The product $\Sigma \times V$ is a contact manifold with the contact form $\lambda_{T^*G} \oplus \alpha_V$ where G_x acts preserving the contact form through the diagonal action where actions are defined as above in each component.

The corresponding moment map $\mu : \Sigma \times V \to \mathfrak{g}_x^*$ is given by

$$\mu((g, \mu_\alpha(x) + j(\eta) + i(\beta)), v) = -\beta + \Phi_V(v)$$

Thus, $\mu^{-1}(0) = \{(g, \eta, \beta, v) : \beta = \Phi_V(v)\}$ and by contact reduction, we have

$$N := \mu^{-1}(0)/G_x \cong G \times_{G_x} ((\mathfrak{k}/\mathfrak{g}_x)^* \oplus V)$$

and $\lambda_{T^*G} \oplus \alpha_V$ descends to a G-invariant contact form α_N on N. The moment map for the action of G on $\Sigma \times V$ descends to the desired moment map on N.

By Lemma 2.6, a neighborhood in N embeds into M as desired. □

We will state the following two representation theoretic results without proof. For the proofs, see [11], [19].

Lemma 2.8 *If $\rho : H \to Sp(V, \omega)$ is a faithful symplectic representation of a compact abelian group H and if $2 \dim H = \dim V$ then H is connected and the weights of ρ form a basis of the weight lattice $\mathbb{Z}_H^*$ of H.*

By the construction of the moment map Φ_V, it is homogeneous and its image is a cone. The next lemma gives a detailed description of the image $\Phi_V(V)$

Lemma 2.9 *Let $(M, \alpha, \mu_\alpha : M \to \mathfrak{g}^*)$ be a contact toric G-manifold. For any point $x \in M$ the symplectic slice representation $\rho : G_x \to Sp(V)$ is faithful and $\dim G_x = \frac{1}{2} \dim V$. Consequently the isotropy group G_x is connected.*

Also the image of the moment map $\Phi_V(V)$ for the slice representation ρ has the following properties:

- *the cone $\Phi_V(V)$ has $d = \dim G_x$ edges;*
- *each edge is spanned by a weight of G_x;*
- *these weights form a basis of the integral lattice of G_x.*

Hence the cone $\Phi_V(V)$ completely determines the slice representation ρ.

By the previous lemma, in the special case of c.c.c.t. G-manifolds, we can describe this local form as follows:

Theorem 2.10 *Let $(M, \alpha, \mu_\alpha : M \to \mathfrak{g}^*)$ be a c.c.c.t. G-manifold normalized so that $\mu_\alpha(M) \subseteq S(\mathfrak{g}^*) = \{\eta \in \mathfrak{g}^* \mid ||\eta|| = 1\}$.*

Let $x \in M$ be a point, $G_x = \{g \in G : g \cdot x = x\}$ be its isotropy group (which is connected). Let $\rho : G_x \to Sp(V, \omega_V)$ denote the symplectic slice representation, $\Phi_V : V \to \mathfrak{g}^*$ denote the corresponding moment map, and let $\mathfrak{k} = (\mathbb{R}\mu_\alpha(x))^0$ be the characteristic subalgebra. Choose the embeddings

$$i : \mathfrak{g}_x^* \to \mathfrak{g}^*$$

and

$$j : (\mathfrak{k}/\mathfrak{g}_x)^* \to \mathfrak{g}^*$$

as in Lemma 2.7.

There exists an open embedding φ from a neighborhood of the orbit $G/G_x \times \{0\} \times \{0\}$ in $G/G_x \times (\mathfrak{k}/\mathfrak{g}_x)^* \times V$ into M such that

$$(\mu_\alpha \circ \varphi)(aG_x, \eta, v) = \frac{\mu_\alpha(x) + j(\eta) + i(\Phi_V(v))}{||\mu_\alpha(x) + j(\eta) + i(\Phi_V(v))||}. \tag{2.1}$$

Proof Since the group G_x is connected, the sequence

$$0 \longrightarrow G_x \longrightarrow G \longrightarrow G/G_x \longrightarrow 0$$

splits. Therefore, we have

$$\begin{aligned} G \times_{G_x} ((\mathfrak{k}/\mathfrak{g}_x)^* \oplus V) &= G \times ((\mathfrak{k}/\mathfrak{g}_x)^* \oplus V)/G_x \\ &= G/G_x \times G_x \times ((\mathfrak{k}/\mathfrak{g}_x)^* \oplus V)/G_x \\ &\cong G/G_x \times (\mathfrak{k}/\mathfrak{g}_x)^* \times V \end{aligned}$$

The theorem follows from Lemma 2.7. □

The previous results can be used to prove the following result which will be useful to construct locally isomorphic contact toric manifolds.

Lemma 2.11 *Let $(M, \alpha, \mu_\alpha : M \to \mathfrak{g}^*)$ be a compact connected contact toric G-manifold. Then*

1. *The connected components of the fibers of μ_α are G-orbits.*
2. *For any point $x \in M$ and any sufficiently small G-invariant neighborhood U of x in M, the sets $\mathbb{R}^+\mu_\alpha(x) = \{t\mu_\alpha(x) : t > 0\}$ and $C(\mu_\alpha|_U))$ determine the contact toric manifold $(U, \alpha|_U, \mu_\alpha|_U = \mu_{\alpha|_U})$.*

   (Recall that
   $$C(\mu_\alpha|_U)) = \{t\mu_\alpha(x) \mid t \in [0,\infty),\, x \in U\}$$
   is the moment cone of μ_α).

For the proof, see [19]. It proceeds by using the local form in a neighborhood to investigate the fibers and determine the c.c.c.t. G-manifold using the established properties of the isotropy group and the slice representation.

2.2 Properties of Contact Moment Maps

To describe the properties of moment maps we introduce the orbital moment map. The orbital moment map will be our main tool to describe and classify c.c.c.t. G-manifolds M using the orbit space M/G:

Definition 2.12 *Let $\mu_\alpha : M \to \mathfrak{g}^*$ be the moment map for an action of a torus G on a manifold M preserving a contact form α. We define* ***orbital moment map*** *to be the induced map $\mu_\alpha : M/G \to \mathfrak{g}^*$ on the orbit space.*

To describe the image of the moment map we need to define rational polyhedral cones:

Definition 2.13 *Let $\mathfrak{g}^*$ be the dual of the Lie algebra of a torus G. A subset $C \subseteq \mathfrak{g}^*$ is a* ***rational polyhedral cone*** *if there exists a finite set of vectors $\{v_i\}$ in the integral lattice $\mathbb{Z}_G$ of G such that*

$$C = \bigcap \{\eta \in \mathfrak{g}^* \mid \langle \eta, v_i \rangle \geq 0\}.$$

Without loss of generality, we will assume that the set $\{v_i\}$ is ***minimal****, i.e., that for any index j*

$$C \neq \bigcap_{i \neq j} \{\eta \in \mathfrak{g}^* \mid \langle \eta, v_i \rangle \geq 0\},$$

and that each vector v_i is ***primitive****, i.e., $sv_i \notin \mathbb{Z}_G$ for $s \in (0,1)$.*

The following theorem gives the convexity and the connectedness properties of the contact moment map. It is the analogue of the Atiyah-Guillemin-Sternberg Convexity Theorem for symplectic toric manifolds. In our presentation, it is formulated in the "convexity package" of Karshon and Chiang [10]:

Theorem 2.14 *Let $(M, \xi = \ker \alpha, \mu : \xi^0_+ \to \mathfrak{g}^*)$ be a c.c.c.t. G-manifold. Assume the torus G has dimension greater than 2. Then*

1. *The moment map μ is open as a map to its image.*
2. *The moment cone $C(\mu)$ is a convex polyhedral cone.*
3. *The nonzero level sets, $\mu^{-1}(y)$, for $y \neq 0$, are connected.*
4. *Let A be a convex subset of $\mathfrak{g}^*$.*
   - *If the action is transverse, suppose that $0 \notin A$.*
   - *If the action is not transverse, suppose that $A \neq \{0\}$.*

   Then the preimage $\mu^{-1}(A)$ is connected.

See Karshon and Chiang's work [10] for the proof. This convexity result is a generalization of the result proved by Lerman [18]. The proofs use techniques of length spaces and convexity.

By the Lemma 1.36, we get the convexity and connectedness result for c.c.c.t. G-manifolds, originally due to Banyaga and Molino [5],[6]:

Corollary 2.15 *Suppose $(M, \xi = \ker \alpha, \mu : \xi^0_+ \to \mathfrak{g}^*)$ is a c.c.c.t. G-manifold with $\dim M > 3$. Then, the fibers, $\mu^{-1}(y)$ are connected and the moment cone $C(\mu)$ is a convex rational polyhedral cone.*

The following lemma will be useful to describe the structure of the orbit space M/G:

Lemma 2.16 *Suppose a Lie group G acts on a manifold M preserving a contact form α. Let $\mu_\alpha : M \to \mathfrak{g}^*$ denote the corresponding moment map. Suppose the action of G at a point x is free and the value η of the moment map at x is non-zero. Then $\pi_\eta \circ d(\mu_\alpha)_x : T_xM \to \mathfrak{g}^*/\mathbb{R}\eta$ is onto. Here $\pi_\eta : \mathfrak{g}^* \to \mathfrak{g}^*/\mathbb{R}\eta$ is the quotient projection.*

Proof We will show that $(\pi_\eta \circ d(\mu_\alpha)_x)^* : (\mathfrak{g}^*/\mathbb{R}\eta)^* \to T^*_xM$ is injective to show that $\pi_\eta \circ d(\mu_\alpha)_x$ is onto. We will show for any nonzero

$$X \in (\mathfrak{g}^*/\mathbb{R}\eta)^* \cong^2 \ker \eta$$

we have $(\pi_\eta \circ d(\mu_\alpha)_x)^*(X) \not\equiv 0$. That is, we will show that there is $v \in T_xM$ such that

$$(\pi_\eta \circ d(\mu_\alpha)_x)^*(X)(v) = \langle d(\mu_\alpha)_x(v), X\rangle \neq 0$$

Since the action of G is free at x, for any $X \neq 0$ in $\mathfrak{g}$, we have $X^\#_x \neq 0$. Since the action of G preserves α, we have

$$0 = \mathcal{L}_{X^\#}\alpha = i_{X^\#}d\alpha + d(i_{X^\#}\alpha)$$

[2] Identification is the one induced by the identification with double dual via evaluation.

It follows that, for any $v \in T_x M$, we have

$$\begin{aligned} \langle d(\mu_\alpha)_x(v), X\rangle &= d\langle \mu_\alpha, X\rangle_x(v) \\ &= d(i_{X^\#}\alpha)_x(v) \\ &= -d\alpha_x(X^\#_x, v) = d\alpha_x(v, X^\#_x) \end{aligned}$$

Now

$$\begin{aligned} (\mathfrak{g}^*/\mathbb{R}\eta)^* \cong \ker\eta &= \{X \in \mathfrak{g} : \langle \mu_\alpha(x), X\rangle = 0\} \\ &= \{X \in \mathfrak{g} : \langle \alpha_x, X^\#_x\rangle = 0\} \\ &= \{X \in \mathfrak{g} : X^\#_x \in \xi\} \end{aligned}$$

By the contact condition $d\alpha_x$ is nondegenerate on ξ. Hence, if $X^\#_x \in \xi$ is nonzero, there is a $v \in \xi$ such that $d\alpha_x(v, X^\#_x) \neq 0$, proving the lemma. □

In higher dimensions, the fibers of the moment map are exactly the orbits of the torus G and the orbital moment map is an embedding, as we will see in the next lemma. While for the case $\dim M = 3$ this is not true, the orbital moment map is locally an embedding.

The following lemma describes the structure of the orbit space M/G for a c.c.c.t. G-manifold via the moment map μ_α.

Lemma 2.17 (Sructure of the Orbit Space) *Let $(M, \alpha, \mu_\alpha : M \to \mathfrak{g}^*)$ be a c.c.c.t. G-manifold normalized so that $\mu_\alpha(M) \subseteq S(\mathfrak{g}^*)$*

1. *For any $G \cdot x \in M/G$ there is a neighborhood U of $G \cdot x$ in M/G such that the restriction of the orbital moment map $\overline{\mu}_\alpha$ to U is an embedding into $S(\mathfrak{g}^*)$.*
2. *If the action of G is free, then the moment map $\mu_\alpha : M \to S(\mathfrak{g}^*)$ is a submersion.*
3. *If $\dim M > 3$, then the fibers of the moment map μ_α are G-orbits. Consequently the orbital moment map $\overline{\mu}_\alpha : M/G \to S(\mathfrak{g}^*)$ is a (topological) embedding.*
4. *If $\dim M > 3$ and the moment map $\mu_\alpha : M \to S(\mathfrak{g}^*)$ is onto, then the action of G on M is free, hence $\mu_\alpha : M \to S(\mathfrak{g}^*)$ is a principal G-bundle.*

Proof 1. This is a consequence of the local normal form theorem. By Theorem 2.10, there is a G-equivariant embedding φ of a G-invariant neighborhood of the orbit $G/G_x \times \{0\} \times \{0\} \subseteq G/G_x \times (\mathfrak{k}/\mathfrak{g}_x)^* \times V$ into a neighborhood of $x \in M$ such that $\varphi(G/G_x \times 0 \times 0) = G \cdot x$.

Let U be the image of this neighborhood in M/G. Then, by the normal form of the map $\mu_\alpha \circ \varphi$, the orbital moment map gives an embedding of U into $S(\mathfrak{g}^*)$.

2. By the preceding lemma, the differential of the map $\mu_\alpha : M \to S(\mathfrak{g}^*)$ is surjective at every point.

   The image $\mu_\alpha(M)$ is open. On the other hand, as M is compact so the image $\mu_\alpha(M)$ is closed. By connectedness of $S(\mathfrak{g}^*)$, the image is the whole sphere.

3. By 2.11, the connected components of the fibers of μ_α are G-orbits. On the other hand, by Corollary 2.15 the fibers of μ_α are connected. Therefore, the fibers are G-orbits.

   Since the fibers are G-orbits, the orbital moment map $\overline{\mu}_\alpha : M/G \to S(\mathfrak{g}^*)$ is an injective map from a compact space, to a Hausdorff space. Therefore, it is a topological embedding.

4. By part (3), $\overline{\mu_\alpha}$ is a homeomorphism onto its image. Hence, for any orbit $G \cdot x$ we may find a G-invariant neighborhood U of $G \cdot x$ and an open neighborhood W of $\mu_\alpha(x) \in S(\mathfrak{g}^*)$ such that

   $$\mu_\alpha(U) = W \cap \mu_\alpha(M).$$

   Moreover, by Theorem 2.10, we may choose U,W such that

   $$\mu_\alpha(U) = W \cap \left\{ \frac{\mu_\alpha(x) + j(\eta) + i(\Phi_V(v))}{||\mu_\alpha(x) + j(\eta) + i(\Phi_V(v))||} : \eta \in (\mathfrak{k}/\mathfrak{g}_x)^*, v \in V \right\}$$

   where we use the notation of the previous section.

   Assume the action of G is not free. Then, G_x is at least 1-dimensional for some point $x \in M$ and the slice representation V at x is at least 2-dimensional by Lemma 2.9. The cone $\Phi_V(V)$ has G_x edges and each edge is spanned by a weight of G_x. Therefore, $\Phi_V(V)$ is a proper cone in $\mathfrak{g}_x^*$. Hence, by the local form above

   $$W \cap \mu_\alpha(M) \neq W$$

   contradicting the fact that $\mu_\alpha : M \to S(\mathfrak{g}^*)$ is onto. Therefore, the action of G must be free.

   Again, by part (3), the fibers of μ_α are G orbits and G acts freely. Therefore, $\mu_\alpha : M \to S(\mathfrak{g}^*)$ is a principal G-bundle. □

The parts (3) and (4) of the lemma are not true for the case that $\dim M = 3$.

As a counterexample, if we consider $(S^1 \times T^2, \cos(nt)d\theta_1 + \sin(nt)d\theta_2)$ for $n > 1$, then the orbital moment map $t \mapsto (\cos(nt), \sin(nt))$ is not an embedding into S^1.

The following notion of local equivalence will be our main tool to investigate contact toric manifolds via the local properties of their orbit spaces:

Definition 2.18 *Two contact toric G-manifolds* $(M_1, \alpha_1, \mu_{\alpha_1} : M \to \mathfrak{g}^*)$ *and* $(M_2, \alpha_2, \mu_{\alpha_2} : M \to \mathfrak{g}^*)$ *are called* ***locally isomorphic*** *if*

- *there is a homeomorphism* $f : M_1/G \to M_2/G$, *and*
- *for any point* $x \in M_1/G$, *there is a neighborhood* $U \subseteq M/G$ *of* x *and an isomorphism of contact toric manifolds* $f_U : \pi_1^{-1}(U) \to (\pi_2)^{-1}(f(U))$ *such that* $\pi_2 \circ f_U = f \circ \pi_1$

where $\pi_i : M_i \to M_i/G$ *are orbit quotient maps.*

The following theorem will allow us to construct local isomorphisms between c.c.c.t. G-manifolds, given a homeomorphism of orbit spaces:

Theorem 2.19 *Let* $(M_1, \alpha_1, \mu_{\alpha_1} : M \to \mathfrak{g}^*)$ *and* $(M_2, \alpha_2, \mu_{\alpha_2} : M \to \mathfrak{g}^*)$ *be two c.c.c.t. G-manifolds normalized so that* $\mu_{\alpha_i}(M)$ *lies in* $S(\mathfrak{g}^*)$, *for* $i = 1, 2$.

Suppose there is a homeomorphism

$$f : M_1/G \to M_2/G$$

so that $\overline{\mu_{\alpha_2}} \circ f = \overline{\mu_{\alpha_1}}$ *where* $\overline{\mu_{\alpha_i}} : M_i/G \to S(\mathfrak{g}^*)$ *are orbital moment maps.*

Then, $(M_1, \alpha_1, \mu_{\alpha_1} : M \to \mathfrak{g}^*)$ *and* $(M_2, \alpha_2, \mu_{\alpha_2} : M \to \mathfrak{g}^*)$ *are locally isomorphic.*

Proof Denote the projections to the orbit space by $\pi_i : M_i \to M_i/G$. We will construct local contactomorphisms that induce f.

Let $G \cdot x_1 \in M_1/G$ be an orbit and pick $x_2 \in \pi_2^{-1}(f(G \cdot x_1))$. By the assumption $\overline{\mu_{\alpha_2}} \circ f = \overline{\mu_{\alpha_1}}$, we have

$$\mu_{\alpha_2}(x_2) = \overline{\mu_{\alpha_2}}(G \cdot x_2) = \overline{\mu_{\alpha_1}}(G \cdot x_1) = \mu_{\alpha_1}(x_1)$$

Let U_1 be a G invariant neighborhood of $x_1 \in M_1$, and $U_2 = \pi_2^{-1}(f(\pi(U_1)))$ a G invariant neighborhood of $x_2 \in M_2$. Then, we have

$$\mu_{\alpha_2}(U_2) = \overline{\mu_{\alpha_2}}(f(\pi(U_1)) = \overline{\mu_{\alpha_1}}(\pi(U_1)) = \mu_{\alpha_1}(U_1)$$

Therefore, by the Lemma 2.11, if U_1 and consequently U_2 are small enough, we have an isomorphism of contact toric manifolds:

$$\varphi_U : (U_1, \alpha_1, \mu_{\alpha_1}) \to (U_2, \alpha_2, \mu_{\alpha_2})$$

By Lemma 2.17, if U_1 and consequently U_2 are small enough, the orbital moment maps are embeddings into $S(\mathfrak{g}^*)$. Consider the map

$$\overline{\varphi_U} : U_1/G \to U_2/G$$

induced by φ_U. From $\mu_{\alpha_2}(\varphi(U_1)) = \mu_{\alpha_1}(U_1)$, we have

$$\overline{\mu_{\alpha_2}} \circ \overline{\varphi_U} = \overline{\mu_{\alpha_1}}$$

Therefore,
$$\overline{\varphi_U} = \overline{\mu_{\alpha_2}}^{-1} \circ \overline{\mu_{\alpha_1}} = f$$
where we suppressed restrictions.

Hence, $(M_1, \alpha_1, \mu_{\alpha_1})$ and $(M_2, \alpha_2, \mu_{\alpha_2})$ are locally isomorphic. □

2.3 Cohomology Classification of Local Isomorphisms

In this section, we sketch the sheaf theoretic results that will give us the tools to investigate isomorphism classes of contact toric manifolds using the local results we described in the previous sections.

Recall the correspondence between smooth functions and contact vector fields that we described in Section 1.1.1.

Assume G acts through a contact action on $(M, \xi = \ker \alpha)$. In that correspondence, if we require α to be a G-invariant contact form, then we get a correspondence between G-invariant functions f and G-invariant contact vector fields v_f.

The following lemma describes how such a v_f behaves on the orbits of the action:

Lemma 2.20 *Suppose $(M, \xi = \ker \alpha, \mu_\alpha)$ is a contact toric G-manifold. For any G-invariant function f, the flow φ_t^f of the corresponding contact vector field v_f preserves the contact form α and induces the identity map on the orbit space M/G. In particular, v_f is tangent to G-orbits.*

Proof We will start by considering the Reeb vector field associated to α and then use this special case to prove the lemma.

First, consider the Reeb vector field, that is $f = 1$ and $v_f = R_\alpha$.

We have
$$\mathcal{L}_{R_\alpha}\alpha = d(\alpha(R_\alpha)) + i_{R_\alpha} d\alpha = d(1) + 0 = 0$$
by the definition of the Reeb vector field. Hence, the flow of R_α preserves α.

Moreover, since R_α is unique and α is assumed to be G-invariant, R_α is also G-invariant. Then, for $X \in \mathfrak{g}$ we have
$$\mathcal{L}_{X^\#} R_\alpha = -\mathcal{L}_{R_\alpha} X^\# = 0$$
by definition of the Lie derivative of vector fields, as $X^\#$ is induced by the action of G.

Combining these, we get
$$\begin{aligned}\mathcal{L}_{R_\alpha}\langle \mu_\alpha, X\rangle &= \mathcal{L}_{R_\alpha}(\alpha(X^\#)) \\ &= (\mathcal{L}_{R_\alpha}\alpha)(X^\#) + \alpha(\mathcal{L}_{R_\alpha} X^\#) = 0\end{aligned}$$

Therefore, the flow of R_α preserves $\langle \mu_\alpha, X \rangle$ for all $X \in \mathfrak{g}$. Hence, R_α is tangent to the fibers of μ_α.

Since, by Lemma 2.11, connected components of the fibers of μ_α are G-orbits, R_α is tangent to G-orbits.

Now for the general case, let f be a G-invariant function and consider the corresponding G-invariant contact vector field

$$v_f = fR_\alpha + (d\alpha|_\xi)^{-1}(df|_\xi)$$

Again, we have

$$\begin{aligned} \mathcal{L}_{v_f} \langle \mu_\alpha, X \rangle &= \mathcal{L}_{v_f}(\alpha(X^\#)) \\ &= (\mathcal{L}_{v_f}\alpha)(X^\#) + \alpha(\mathcal{L}_{v_f} X^\#) \end{aligned}$$

We have $\mathcal{L}_{v_f}\alpha = R_\alpha(f)\alpha$ by the correspondence of functions and contact vector fields. Moreover, since v_f is G-invariant, for $X \in \mathfrak{g}$ we have

$$\mathcal{L}_{X^\#} v_f = -\mathcal{L}_{v_f} X^\# = 0$$

by a similar argument as before. Thus

$$\mathcal{L}_{v_f} \langle \mu_\alpha, X \rangle = (\mathcal{L}_{v_f}\alpha)(X^\#) + \alpha(\mathcal{L}_{v_f} X^\#) = \mathcal{L}_{v_f}\alpha = R_\alpha(f)\alpha$$

Since, R_α is tangent to G-orbits, its integral curves $\rho_t(x)$ are contained in G-orbits. Hence for every t, $f(\rho_t(x)) = f(g \cdot x)$ for some $g \in G$. Hence, we have

$$R_\alpha(f) = \mathcal{L}_{R_\alpha} f = 0$$

for a G-invariant function f.

Therefore, from the previous discussion

$$\mathcal{L}_{v_f}\alpha = 0$$

so v_f preserves the contact form. Moreover,

$$\mathcal{L}_{v_f} \langle \mu_\alpha, X \rangle = 0$$

and by the argument as in the case of the Reeb vector field, v_f is tangent to G-orbits and its flow φ_t^f induces identity on M/G. □

By the following theorem, the isomorphism classes of contact toric manifolds that are locally isomorphic to a given one are determined by elements of the first cohomology group with coefficients in a given sheaf:

Proposition 2.21 *For a fixed torus G, the isomorphism classes of contact toric G-manifold locally isomorphic to a given contact toric G-manifold* $(M, \xi = \ker\alpha, \mu_\alpha)$ *are in one-to-one correspondence with the elements of the first Čech cohomology group* $H^1(M/G, \mathcal{S})$ *where* $\mathcal{S}$ *is the sheaf of groups on the orbit space* M/G *defined by*

$$\mathcal{S}(U) = Iso(\pi^{-1}(U))$$

where $Iso(\pi^{-1}(U))$ *is the group of isomorphisms of the contact toric manifold*

$$(\pi^{-1}(U), \alpha|_{\pi^{-1}(U)}, \mu_\alpha|_{\pi^{-1}(U)})$$

and $\pi : M \to M/G$ *is the orbit quotient map.*

Proof (Sketch) The argument is similar to the one from [15, Theorem 4.2].

Suppose $(M_0, \xi_0 = \ker\alpha_0, \mu_{\alpha_0})$ is a contact toric G-manifold locally isomorphic to $(M, \xi = \ker\alpha, \mu_\alpha)$.

Fix a homeomorphism $g : M/G \to M_0/G$. Choose an open cover V_i of M/G such that for each i there is a diffeomorphism $s_i : \pi^{-1}(V_i) \to (\pi_0)^{-1}(g(V_i))$ inducing g on V_i where $\pi_0 : M_0 \to M_0/G$ is the orbit map. This is possible by being locally isomorphic. Set

$$f_{ij} = s_i^{-1} \circ s_j|_{V_i \cap V_j}$$

The collection of maps $\{f_{ij} : \pi^{-1}(V_i \cap V_j) \to \pi^{-1}(V_i \cap V_j)\}$ is a Čech 1-cocycle whose cohomology class in $H^1(M/G, \mathcal{S})$ is independent of the choices made to define it.

Conversely, given an element of $H^1(M/G, \mathcal{S})$ we can represent it by a Čech cocycle

$$f_{ij} = \pi^{-1}(V_i \cap V_j) \to \pi^{-1}(V_i \cap V_j)$$

We construct the corresponding contact toric G-manifold by taking the disjoint union of the manifolds $(\pi^{-1}(V_i), \alpha|_{\pi^{-1}(V_i)}, \mu_\alpha|_{\pi^{-1}(V_i)})$ and gluing using f_{ij}. The cocycle condition guarantees that the gluing is consistent.

Now, let $\{f_{ij}\}$ be the collection obtained by $(M_0, \xi_0 = \ker\alpha_0, \mu_{\alpha_0})$ which is locally isomorphic to $(M, \xi = \ker\alpha, \mu_\alpha)$. The map

$$\widetilde{F} : \coprod \pi^{-1}(V_i) \to M_0$$

defined by $\widetilde{F}|_{\pi^{-1}(V_i)} = s_i$ induces an isomorphism

$$F : (\coprod \pi^{-1}(V_i))/\sim \to M_0$$

The converse correspondence follows from the fact that cohomologous cochains induce the same gluing. □

For the next proposition, let $C^\infty(\pi^{-1}(U))^G$ denote G-invariant smooth functions on $\pi^{-1}(U)$ and similarly, $C^\infty(\pi^{-1}(U), \mathbb{Z}_G)^G$ denote G-invariant smooth functions on $\pi^{-1}(U)$ with values in the lattice $\mathbb{Z}_G$

Proposition 2.22 *Let (M, α, μ_α) be a contact toric G-manifold. Let $\pi : M \to M/G$ denote the orbit map, and let $\mathbb{Z}_G := \ker\{\exp : \mathfrak{g} \to G\}$ denote the integral lattice of the torus G. There exists a short exact sequence of sheaves of groups*

$$0 \longrightarrow \underline{\mathbb{Z}_G} \xrightarrow{\ j\ } \mathcal{C} \xrightarrow{\ \Lambda\ } \mathcal{S} \longrightarrow 0$$

where for a sufficiently small open subset U of the orbit space M/G

1. $\underline{\mathbb{Z}_G}(U) := C^\infty(\pi^{-1}(U), \mathbb{Z}_G)^G$;
2. $\mathcal{C}(U) := C^\infty(\pi^{-1}(U))^G$;
3. $\mathcal{S}(U) := Iso(\pi^{-1}(U))$ *is the sheaf defined in Proposition 2.21.*

Hence $\mathcal{S}$ is a sheaf of abelian groups and the cohomology groups $H^i(M/G, \mathcal{S})$ are defined for all indices $i \geq 0$.

The maps of the exact sequence are defined as follows:

Let $f \in \mathcal{C}(U)$. Then, by Lemma 2.20, the flow φ_t^f induces identity on $U \subseteq M/G$. Hence, $\varphi_t^f \in \mathcal{S}(U)$ for all t. We define

$$\Lambda(f) = \varphi_1^f$$

Let $X \in \mathbb{Z}_G$. As $\mathbb{Z}_G$ is discrete, we identify the constant function with value X on connected U with the element $X \in \mathbb{Z}_G$ and extend accordingly to locally constant functions. We define

$$j(X) = \langle \mu_\alpha, X \rangle$$

For the proof of the proposition, see [19, Proposition 5.3]. The proof uses the arguments of [15] and the notion of basic forms from [16] to show that the sequence is exact.

Corollary 2.23 *Under the hypotheses of the proposition above,*

$$H^i(M/G, \mathcal{S}) \cong H^{i+1}(M/G, \underline{\mathbb{Z}_G})$$

for all $i > 0$.

Proof From the fact that M admits a partition of unity, the sheaf $\mathcal{C}$ is a fine sheaf. Hence $H^i(M/G, \mathcal{C}) \cong 0)$ for all $i > 0$.

Therefore, from the long exact sequence of cohomology groups induced by the sequence in Proposition 2.22, we get the isomorphism

$$H^i(M/G, \mathcal{S}) \cong H^{i+1}(M/G, \underline{\mathbb{Z}_G})$$

for all $i > 0$. □

Chapter 3

Classification of Compact Connected Contact Toric Manifolds

In this chapter, we will discuss the classification of compact connected contact toric (shortly, c.c.c.t) G-manifolds $(M^{2n+1}, \alpha, \mu_\alpha)$ where $G = \mathbb{T}^{n+1}$. We will conclude with some applications of the classification theorem.

The complete classification we will describe in this chapter is due to Lerman [19], building on the previous partial classification results due to Banyaga - Molino [6], [7], [5] and Boyer - Galicki [8].

3.1 Statement of the Classification Theorem

To state the classification theorem for c.c.c.t. G-manifolds, we need to define the notion of "good cones". These cones are analogous to Delzant (or unimodular) polytopes that appear in the classification of symplectic toric manifolds. See [11] and [9] for Delzant's Theorem.

Definition 3.1 (Good Cones) *A rational polyhedral cone*

$$C = \bigcap_{i=1}^{N} \{\eta \in \mathfrak{g}^* : \langle \eta, v_i \rangle \geq 0\}$$

where $\{v_i\} \subseteq \mathbb{Z}_G$ and $N \geq n+1$ with non-empty interior is ***good*** *if the annihilator of a linear span of the vectors in a face F of codimension k, where $0 < k < \dim G$, is the Lie algebra of a subtorus H of G and the normals to the face form a basis of the integral lattice $\mathbb{Z}_H = \mathbb{Z}_G \cap \mathfrak{h}$ of H, where $\mathfrak{h}$ is the lie algebra of H.*

That is, if

$$F = C \cap \bigcap_{j=1}^{k} \{\eta \in \mathfrak{g}^* : \langle \eta, v_{i_j} \rangle = 0\}$$

is a face of codimension k of C , where $0 < k < \dim G$, for $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$, then

$$\mathbb{Z}_G \cap \{\sum_{j=1}^{k} a_j v_{i_j} : a_j \in \mathbb{R}\} = \{\sum_{j=1}^{k} m_j v_{i_j} : m_j \in \mathbb{Z}\} = \mathbb{Z}_H$$

and $\{v_{i_j}\}$ is independent over $\mathbb{Z}$.

The condition for a rational polyhedral cone to be good can be also stated as any codimension k face, where $0 < k < \dim G$, is the intersection of exactly k facets whose set of normals $\{v_{i_j}\}$ can be completed to a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1} \cong \mathbb{Z}_G$.

With the definition of good cones, c.c.c.t. G-manifolds are classified as follows:

Theorem 3.2 (Lerman [19]) *Compact connected contact toric G-manifolds*

$$(M^{2n+1}, \alpha, \mu_\alpha : M \to \mathfrak{g}^*)$$

where $G = \mathbb{T}^{n+1}$, are classified as follows:

1. *Suppose $\dim M = 3$ and the action of $G = \mathbb{T}^2$ is free. Then, M is a principal $\mathbb{T}^2$-bundle over S^1, hence is diffeomorphic to $\mathbb{T}^3 = S^1 \times \mathbb{T}^2$. The contact structure on M is given by the form*

$$\alpha = \cos(nt)d\theta_1 + \sin(nt)d\theta_2$$

   for some positive integer n, where $(t, \theta_1, \theta_2) \in S^1 \times \mathbb{T}^2$. The moment cone is $\mathfrak{g}^* \cong (\mathbb{R}^2)^*$.

2. *Suppose $\dim M = 3$ and the action of $G = \mathbb{T}^2$ is not free. Then M is diffeomorphic to a lens space (this includes S^3 and $S^1 \times S^2$) equipped with one of the various contact structures. As a c.c.c.t. G-manifold, (M, α, μ_α) is classified by two rational numbers r, q with $0 \leq r < 1$ and $r < q$.*

3. *Suppose $\dim M = 2n + 1 > 3$ and the action of $G = \mathbb{T}^{n+1}$ is free. Then M is a principal $\mathbb{T}^{n+1}$-bundle over S^n. Moreover, each principal $\mathbb{T}^{n+1}$-bundle over S^n has a unique $\mathbb{T}^{n+1}$-invariant contact structure making it a compact connected contact toric manifold. The moment cone is $\mathfrak{g}^* \cong (\mathbb{R}^{n+1})^*$.*

4. *Suppose $\dim M = 2n + 1 > 3$ and the action of $G = \mathbb{T}^{n+1}$ is not free. Then the moment cone of (M, α, μ_α) is a good cone. Conversely, given a good cone $C \subseteq \mathfrak{g}^*$, there is a unique c.c.c.t. G-manifold (M, α, μ_α) with moment cone C.*

By a theorem of Giroux [14], for distinct integers, the contact manifolds that appear in (1) are distinct.

Principal $\mathbb{T}^{n+1}$-bundles over S^n are in one-to-one correspondence with the second cohomology classes of S^n with $\mathbb{Z}^{n+1}$ coefficients, see [15]. Since the cohomology groups are $H^2(S^n, \mathbb{Z}^{n+1}) = 0$ for $n \neq 2$, it follows that for the case that $\dim M > 5$ in part (3), this bundle M is the trivial bundle $\mathbb{T}^{n+1} \times S^n$.

In the case of $\dim M = 5$ in part (3), we have $H^2(S^2, \mathbb{Z}^3) = \mathbb{Z}^3$, and each of these principal $\mathbb{T}^3$-bundles over S^2 that are indexed by $\mathbb{Z}^3$ carries a unique $\mathbb{T}^3$-invariant contact structure.

3.2 Proof of the Classification Theorem

We will now describe the proof of Theorem 3.2.

In the cases (1) and (3) of free actions, the proofs proceed by viewing the c.c.c.t. G-manifolds (M, α, μ_α) as principal torus bundles over spheres that are locally isomorphic to S^*G, and showing the isomorphisms using the cohomology classification of isomorphism classes.

In the case (2) of non-free action in dimension 3, we show that the image of the moment map is given by two rational numbers and it determines the c.c.c.t. G-manifolds. Then, we construct c.c.c.t. manifolds by "contact cutting".

Lastly, in the case (4) of non-free action in higher dimensions, we construct c.c.c.t. G-manifolds by adapting the construction of Delzant from [11]. Conversely, we show that the contact cone is a good cone by using the local properties of c.c.c.t. manifolds described in Chapter 2.

3.2.1 Free Actions in Dimension 3

Suppose $(M^3, \alpha, \mu_\alpha)$ is a c.c.c.t. G-manifold, where $G = \mathbb{T}^2$, normalised such that $\mu_\alpha(M)$ lies in $S(\mathfrak{g}^*) = S((\mathbb{R}^2)^*) \cong S^1$ and suppose the action of $\mathbb{T}^2$ is free. Then the orbit space $M/\mathbb{T}^2$ is a 1-dimensional compact connected manifold without boundary, hence it is a circle S^1. Moreover, the projection

$$M \to M/\mathbb{T}^2 \cong S^1$$

is a principal $\mathbb{T}^2$-bundle. Since any principal $\mathbb{T}^2$-bundle over S^1 is trivial, M is diffeomorphic to $S^1 \times \mathbb{T}^2 = \mathbb{T}^3$.

It remains to show that

$$(M, \alpha, \mu_\alpha) \cong (S^1 \times \mathbb{T}^2, \alpha_n, \mu_{\alpha_n})$$

where

$$\alpha_n = cos(nt)d\theta_1 + sin(nt)d\theta_2$$

for some positive integer n, as contact toric manifolds.

By Corollary 2.17, the map $\mu_\alpha : M \to S((\mathbb{R}^2)^*)$ is a submersion because the action of $\mathbb{T}^2$ is free. Hence, the induced orbital moment map

$$\overline{\mu_\alpha} : M/G \cong S^1 \to S(\mathfrak{g}^*) \cong S^1$$

is also a submersion and, by equality of dimensions, a local diffeomorphism. Therefore, by compactness of S^1, the map $\overline{\mu_\alpha} : S^1 \to S^1$ is a covering map.

Let n be the number of sheets of the covering map $\overline{\mu_\alpha} : S^1 \to S^1$ of the circle. By a similar argument as above, $\overline{\mu_{\alpha_n}} : S^1 \to S^1$ is also an n-sheeted covering map. Therefore, by the equivalence of covering maps, there exists a homeomorphism

$$f : S^1 \cong M/G \to S^1 \cong (S^1 \times \mathbb{T}^2)/\mathbb{T}^2$$

such that $\overline{\mu_{\alpha_n}} \circ f = \overline{\mu_\alpha}$. Then, by Theorem 2.19, c.c.c.t. $\mathbb{T}^2$-manifolds (M, α, μ_α) and $(S^1 \times \mathbb{T}^2, \alpha_n, \mu_{\alpha_n})$ are locally isomorphic.

By Proposition 2.21, isomorphism classes of c.c.c.t. G-manifolds (M, α, μ_α) locally isomorphic to $(S^1 \times \mathbb{T}^2, \alpha_n, \mu_{\alpha_n})$ are in one-to-one correspondence with the elements of the first Čech cohomology group $H^1((S^1 \times \mathbb{T}^2)/\mathbb{T}^2, \mathcal{S})$ where $\mathcal{S}$ is defined as in Proposition 2.21. However, by Corollary 2.23, we have

$$H^1((S^1 \times \mathbb{T}^2)/\mathbb{T}^2, \mathcal{S}) = H^2(S^1, \mathbb{Z}^2) = 0$$

Therefore,

$$(M, \alpha, \mu_\alpha : M \to (\mathbb{R}^2)^*) \cong (S^1 \times \mathbb{T}^2, \alpha_n, \mu_{\alpha_n} : S^1 \times \mathbb{T}^2 \to (\mathbb{R}^2)^*)$$

as c.c.c.t. $\mathbb{T}^2$-manifolds and the moment cone is $C(\mu_\alpha) = C(\mu_{\alpha_n}) = (\mathbb{R}^2)^*$.

3.2.2 Non-free Actions in Dimension 3

We first describe the orbit space and the image of the orbital moment map:

Proposition 3.3 *Let (M, α, μ_α) is a c.c.c.t. G-manifold normalised such that $\mu_\alpha(M)$ lies in $S(\mathfrak{g}^*)$ and suppose the action of $G = \mathbb{T}^2$ is not free. Then*

1. *The orbit space M/G is homeomorphic to the interval $[0, 1]$.*
2. *The orbital moment map $\overline{\mu_\alpha} : M/G \to S(\mathfrak{g}^*) \cong S^1$ lifts to an embedding $\widetilde{\mu_\alpha} : M/G \to \mathbb{R}$ so that $p \circ \widetilde{\mu_\alpha} = \overline{\mu_\alpha}$ where $p : \mathbb{R} \to S^1$ is the covering map $p(t) = (cost, sint)$.*
3. *$\widetilde{\mu_\alpha}(M/G) = [t_1, t_2]$, and $\tan t_1, \tan t_2$ are rational numbers.*
4. *If $(M_0, \alpha_0, \mu_{\alpha_0})$ is another such c.c.c.t. G-manifold with*

$$\widetilde{\mu_{\alpha_0}}(M_0/G) = [t_1, t_2] = \widetilde{\mu_\alpha}(M/G)$$

then $(M_0, \alpha_0, \mu_{\alpha_0})$ is isomorphic to (M, α, μ_α).

Proof Since $0 \notin \mu_\alpha(M)$ by Lemma 1.36, we have that for any point $x \in M$, there is some $X \in \mathfrak{g}$ such that $0 \neq \langle \mu_\alpha(x), X \rangle = \alpha_x(X_x^\#)$. In particular, $exp(tX)$ does not fix x, and hence G does not fix x. Therefore, the action of G on M has no fixed points. By Lemma 2.9, all the isotropy groups G_x are connected. By the dimension, G_x are either trivial or circles.

Assume G_x is a circle S^1. Then, $\mathfrak{g}_x$ is equal to the characteristic algebra $\mathfrak{k}$ by dimension. On the other hand, G_x is a subtorus of G. Therefore, $\mu_\alpha(x)$ is a multiple of a weight $\eta \in \mathbb{Z}_G^*$. Also, in this case, the dimension of the symplectic slice is 2 and the symplectic slice representation is isomorphic to the standard action of S^1 on $\mathbb{C}$ by multiplication. By Theorem 2.10, a neighbourhood of $x \in M$ is diffeomorphic to $S^1 \times \mathbb{C}$ and a neighbourhood of $G \cdot x \in M$ is diffeomorphic to $\mathbb{C}/S^1 \cong [0, \infty)$.

Thus, if G_x is a circle S^1, then then a neighbourhood of $G \cdot x \in M/G$ is diffeomorphic to $[0, \infty)$. Therefore, M/G is a compact connected topological 1-manifold and hence:

- M/G is homeomorphic to $[0, 1]$;
- there are exactly two orbits $G \cdot x_1$, $G \cdot x_2$ (that are mapped to $0, 1$) such that G_{x_1} and G_{x_2} are isomorphic to the circle S^1.
- At x_j for $j = 1, 2$, $\mu_\alpha(x_j) = (\cos t_j, \sin t_j) \in \mathbb{Z}^2 \cong \mathbb{Z}_G^*$

By the last point $\tan t_j$ is rational.

Since $M/G \cong [0, 1]$ is contractible, the orbital moment map $\overline{\mu_\alpha}$ lifts to a map

$$\widetilde{\mu_\alpha} : M/G \to \mathbb{R}$$

such that $p \circ \widetilde{\mu_\alpha} = \overline{\mu_\alpha}$. By the preceeding discussion,

$$\widetilde{\mu_\alpha}(M/G) = [\widetilde{\mu_\alpha}(G_{x_2}), \widetilde{\mu_\alpha}(G_{x_2})] = [t_1, t_2]$$

where we may assume $0 \leq t_1 < 2\pi$ and we have that $\tan t_j$ is rational.

It remains to show that $\widetilde{\mu_\alpha}$ is an embedding. By Lemma 2.17, $\overline{\mu_\alpha}$ is locally an embedding and it lifts to an injective local embedding $\widetilde{\mu_\alpha} : M/G \cong [0, 1] \mapsto [t_1, t_2]$. Therefore, $\widetilde{\mu_\alpha} : M/G \mapsto [t_1, t_2]$ is a homeomorphism.

Now assume $(M_0, \alpha_0, \mu_{\alpha_0})$ is another such c.c.c.t. G-manifold as described in part (4). Then, for the homeomorphism $f = \widetilde{\mu_{\alpha_0}}^{-1} \circ \widetilde{\mu_\alpha}$ we have $\overline{\mu_{\alpha_0}} \circ f = \overline{\mu_\alpha}$. Hence, by Proposition 2.19, $(M_0, \alpha_0, \mu_{\alpha_0})$ is localy isomorphic to (M, α, μ_α).

On the other hand, by previous parts M/G is contractible. Hence, by 2.23, $H^1(M/G, \mathcal{S}) = H^2(M/G, \mathbb{Z}^2) = 0$. Therefore, $(M_0, \alpha_0, \mu_{\alpha_0})$ is isomorphic to (M, α, μ_α) as c.c.c.t. G-manifolds, by 2.21. □

It remains to show the converse statement. We will need the following equivariant version of contact cutting to prove the existence of the desired c.c.c.t. $\mathbb{T}^2$-manifold:

Theorem 3.4 *Suppose $(\widetilde{M}, \alpha)$ is a contact manifold, M is a manifold with boundary of the same dimension as $\widetilde{M}$ embedded in $\widetilde{M}$. Suppose further that there is a neighborhood U in $\widetilde{M}$ of the boundary ∂M and a free S^1 action on U preserving α such that the corresponding moment map $f : U \to \mathbb{R}$ satisfies*

1. *$f^{-1}(0) = \partial M$ and*
2. *$f^{-1}([0, \infty)) = U \cap M$.*

Let $M_{cut} = M/\sim$, where, for $m \neq m'$, $m \sim m'$ if and only if

1. *$m, m' \in \partial M$ and*
2. *$m = \lambda \cdot m'$ for some $\lambda \in S^1$,*

Then M_{cut} is a contact manifold, $\partial M/S^1$ is a contact submanifold of M_{cut}, and $M_{cut} \setminus (\partial M/S^1)$ is contactomorphic to $M \setminus \partial M$.

Moreover, if there is an action of a Lie group G on $\widetilde{M}$ preserving M, α and commuting with the action of S^1 on U, then there is an induced action of G on M_{cut} preserving the induced contact structure.

For the proof and further discussion about contact cutting, see Lerman, [17].

With this result, we can show the existence of a c.c.c.t manifold given two rational numbers.

Proposition 3.5 *Given $t_1, t_2 \in \mathbb{R}$ with $0 \leq t_1 < 2\pi, t_1 < t_2$ and $\tan t_1, \tan t_2$ are rational numbers, there is a c.c.c.t. G-manifold (M, α, μ_α) with $\widetilde{\mu_\alpha}(M/G) = [t_1, t_2]$.*

Proof Suppose we are given $t_1, t_2 \in \mathbb{R}$ with $0 \leq t_1 < 2\pi, t_1 < t_2$ and $\tan t_1, \tan t_2$ are rational numbers. Then for each $i = 1, 2$ there is a $(m_i, n_i) \in \mathbb{Z}_2$ such that $(\cos t_i, \sin t_i)$ lies on the ray through (m_i, n_i).

Choose $\varepsilon > 0$ sufficiently small so that $f_1(t) = -n_1 \cos t + m_1 \sin t$ is non-negative on $[t_1, t_1 + \varepsilon)$ and $f_2(t) = n_2 \cos t - m_2 \sin t$ is non-negative on $[t_2, t_2 + \varepsilon)$.

Consider $\mathbb{R} \times S^1 \times S^1$ with the contact form

$$\alpha = \cos t d\theta_1 + \sin t d\theta_2$$

for $(t, \theta_1, \theta_2) \in \mathbb{R} \times S^1 \times S^1$.

To apply contact cutting, let

$$M = [t_1, t_2] \times S^1 \times S^1$$

and

$$U = ((t_1 - \varepsilon, t_1 + \varepsilon) \cup (t_2 - \varepsilon, t_2 + \varepsilon)) \times S^1 \times S^1$$

Consider $f : U \to \mathbb{R}$ given as

$$f(t, \theta_1, \theta_2) = \begin{cases} f_1(t) & t \in (t_1 - \varepsilon, t_1 + \varepsilon) \\ f_2(t) & t \in (t_2 - \varepsilon, t_2 + \varepsilon) \end{cases}$$

The function f is a moment map for a circle action on U generated by the vector field $-n_1 \frac{\partial}{\partial \theta_1} + m_1 \frac{\partial}{\partial \theta_2}$ on $(t_1 - \varepsilon, t_1 + \varepsilon) \times S^1 \times S^1$ and by the vector field $n_2 \frac{\partial}{\partial \theta_1} - m_2 \frac{\partial}{\partial \theta_2}$ on $(t_2 - \varepsilon, t_2 + \varepsilon) \times S^1 \times S^1$.

Note that M, U, and the above circle action with moment map f satisfies the assumptions of contact cutting. Moreover, the action of $\mathbb{T}^2$ on $\mathbb{R} \times S^1 \times S^1$ by multiplication in the $S^1 \times S^1$ factor preserves M, α, U and commutes with the action of S^1. Therefore we may apply Proposition 3.4 to obtain c.c.c.t. G-manifold M_{cut} with the moment map lifting to a map with image $[t_1, t_2]$.□

The cut space can be described as $[t_1, c] \times S^1 \times S^1 / \sim$ and $[c, t_2] \times S^1 \times S^1 / \sim$ each of which is a solid torus, glued at their boundary $\{c\} \times S^1 \times S^1$. See [23] for descriptions of the lens spaces as two solid tori glued at their boundary.

3.2.3 Free Actions in Higher Dimensions

Let (M, α, μ_α) be a c.c.c.t. G-manifold normalised such that $\mu_\alpha(M)$ lies in $S(\mathfrak{g}^*)$ with $\dim M = 2n + 1 > 3$. Suppose the action of G is free.

By Corollary 2.17, the map $\mu_\alpha : M \to S(\mathfrak{g}^*)$ is a principal G-bundle. Therefore, the orbital moment map $\overline{\mu_\alpha} : M/G \to S(\mathfrak{g}^*)$ gives a diffeomorphism.

On the other hand, consider the unit cosphere bundle $S^*G \cong G \times S(\mathfrak{g}^*)$ with its standard contact structure induced by λ and the action of G by multiplication in G factor, making it a c.c.c.t G-manifold with moment map μ_λ. Then, again the orbital moment map $S^*G/G \to S(\mathfrak{g}^*)$ gives a diffeomorphism.

As we have $S^*G/G \cong S(\mathfrak{g}^*) \cong M/G$, by Proposition 2.19, (M, α, μ_α) is locally isomorphic to $(S^*G, \lambda, \mu_\lambda)$. Therefore, by Proposition 2.21, (M, α, μ_α) corresponds to a cohomology class in $H^1(S(\mathfrak{g}^*), \mathcal{S})$.

For a group H we will denote the sheaf given by $U \mapsto C^\infty(U, H)$ and the function restrictions by $\underline{H}$. Recall that, principal G-bundles over $S(\mathfrak{g}^*)$ are classified by the classes in the Čech cohomology group $H^1(S(\mathfrak{g}^*), \underline{G})$.

We will construct an isomorphism $H^1(S(\mathfrak{g}^*), \mathcal{S}) \to H^1(S(\mathfrak{g}^*), \underline{G})$ to show that every principal G-bundle over $S(\mathfrak{g}^*)$ has a unique invariant contact structure making it a c.c.c.t. G-manifold.

Recall that we have the following exact sequence of sheaves on $M/G \cong S(\mathfrak{g}^*)$ induced by the exact sequence of groups:

$$0 \longrightarrow \underline{\mathbb{Z}_G} \longrightarrow \underline{\mathfrak{g}} \xrightarrow{\exp} \underline{G} \longrightarrow 0$$

On the other hand, we have the following short exact sequence of sheaves by Proposition 2.22:

$$0 \longrightarrow \underline{\mathbb{Z}_G} \xrightarrow{j} \mathcal{C} \xrightarrow{\Lambda} \mathcal{S} \longrightarrow 0$$

We will construct maps a and b such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathbb{Z}_G} & \xrightarrow{j} & \mathcal{C} & \xrightarrow{\Lambda} & \mathcal{S} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & \underline{\mathbb{Z}_G} & \longrightarrow & \underline{\mathfrak{g}} & \xrightarrow{\exp} & \underline{G} & \longrightarrow & 0 \end{array}$$

Denote the bundle projection by $\pi : M \to S(\mathfrak{g}^*)$, and by choosing an open set $U \subseteq S(\mathfrak{g}^*)$ small enough, identify $\pi^{-1}(U) \cong U \times G$.

Let $f \in \mathcal{C}$. Then f defines a G-invariant contact vector field v_f. By Lemma 2.20, v_f is tangent to the G orbits, that is, to the fibers of π. So for any $x \in U$, we have a unique element $X(x) \in \mathfrak{g}$ such that

$$v_f(m) = (X(x)^{\#})(m)$$

for all $m \in \pi^{-1}(x)$. See [19] for more details on how $X(x) \in \mathfrak{g}$ is constructed. Define $a(f) : U \to \mathfrak{g}$ to be

$$a(f)(x) = X(x)$$

By definition $\varphi \in \mathcal{S}(U)$ is a G-equivariant diffeomorphism of $U \times G$ onto itself, and is determined by its values on $U \times \{1\}$. Define $b(\varphi) : U \to G$ to be

$$b(\varphi)(x) = proj_G \circ \varphi(x, 1)$$

where $proj_G : U \times G \to G$ is the projection.

The left-hand square commutes by the fact that the function $\langle \mu_\alpha, X \rangle$ generates the contact vector field $X^{\#}$ for $X \in \mathfrak{g}$.

For the right-hand square, note that $\Lambda(f) = \varphi_1^f$ is the time one flow of v_f. That is, by definition of exponential map,

$$\Lambda(f)(m) = exp(v_f(m)) \cdot m = exp((X(x)^{\#})(m)) \cdot m = exp(a(x)) \cdot m$$

for $m \in \pi^{-1}(U)$ and $x = \pi(m)$. Thus,

$$b(\Lambda(f))(x) = proj_G \circ \Lambda(f)(x, 1) = proj_G(exp(a(x)) \cdot (x, 1)) = exp(a(x))$$

where we ued the identification $\pi^{-1}(U) \cong U \times G$. Therefore, $b \circ \Lambda = exp \circ a$ as desired.

As the diagram above commutes and the sheaves $\mathcal{C}$ and $\underline{\mathfrak{g}}$ are fine sheaves, we have the following diagram from the long exact sequence of cohomology groups:

$$\begin{array}{ccc} H^2(S(\mathfrak{g}^*), \underline{\mathbb{Z}_G}) & \xleftarrow[\cong]{} & H^1(S(\mathfrak{g}^*), \mathcal{S}) \\ \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle b^*} \\ H^2(S(\mathfrak{g}^*), \underline{\mathbb{Z}_G}) & \xleftarrow[\cong]{} & H^1(S(\mathfrak{g}^*), \underline{G}) \end{array}$$

Therefore the map induced by b is an isomorphism.

As in the arguments of [15, Section 4], the map b allows us to glue c.c.c.t. G-manifolds locally isomorphic to S^*G and identify them with isomorphism classes of principle G-bundles over the sphere S^*G.

3.2.4 Non-free Actions in Higher Dimensions

First, we will construct a symplectic cone (W, ω, X) with an action of $G = \mathbb{T}^{n+1}$ such that the moment cone is C. We will start with the standard action of a torus $\mathbb{T}^N$ on $(\mathbb{C}^N, \frac{i}{2}\sum_{i=1}^N dz_i \wedge d\bar{z}_i)$ and use symplectic reduction by a suitable subgroup of $K \subset \mathbb{T}^N$ to obtain the desired symplectic cone. From this cone, we will obtain the desired compact connected contact toric $\mathbb{T}^{n+1}$-manifold (M, α, μ_α) with the moment cone C.

The construction is adapted from Delzant's construction [11] of compact symplectic toric manifolds.

Theorem 3.6 *Let n be greater than 1 and $C \subseteq \mathfrak{g}^* \cong (\mathbb{R}^{n+1})^*$ be a good cone. Then, there exists a compact connected contact toric $\mathbb{T}^{n+1}$-manifold $(M^{2n+1}, \alpha, \mu_\alpha)$ such that the moment cone is C.*

Proof Let $C = \bigcap_{i=1}^N \{\eta \in (\mathbb{R}^{n+1})^* : \langle \eta, v_i \rangle \geq 0\}$ be the given cone, where $v_i \in \mathbb{Z}^N$ for $i = 1, \dots, N$ are a minimal, primitive set of normals.

Define the map $\varphi : \mathbb{Z}^N \to \mathbb{Z}^{n+1}$ by $\varphi(e_i) = v_i$, and extend it linearly to $\varphi : \mathbb{R}^N \to \mathbb{R}^{n+1}$. Since C is a good cone, v_i span $\mathbb{Z}^{n+1}$. Thus, the map φ maps $\mathbb{Z}^N$ onto $\mathbb{Z}^{n+1}$ and induces a map of tori

$$\varphi : \mathbb{T}^N = (\mathbb{R}^N/\mathbb{Z}^N) \longrightarrow \mathbb{T}^{n+1} = (\mathbb{R}^{n+1}/\mathbb{Z}^{n+1})$$

We will denote all three of these maps by φ. In particular, φ denotes both the map of Lie groups $\mathbb{T}^k$ and the induced map of their Lie algebras $\mathbb{R}^k$.

We will denote the kernel

$$\{[a] : \sum_{i=1}^N a_i v_i \in \mathbb{Z}^{n+1}\}$$

of the map $\varphi : \mathbb{T}^N \to \mathbb{T}^{n+1}$ by K, which is a closed subgroup of $\mathbb{T}^N$.

From the map φ, we get the short exact sequences Lie groups, Lie algebras, and the dual spaces:

$$0 \longrightarrow K \xrightarrow{i} \mathbb{T}^N \xrightarrow{\varphi} \mathbb{T}^{n+1} \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{k} \xrightarrow{i} \mathbb{R}^N \xrightarrow{\varphi} \mathbb{R}^{n+1} \longrightarrow 0$$

$$0 \longrightarrow (\mathbb{R}^{n+1})^* \xrightarrow{\varphi^*} (\mathbb{R}^N)^* \xrightarrow{i^*} \mathfrak{k}^* \longrightarrow 0$$

where i are the corresponding inclusion maps.

Consider the Hamiltonian action of $\mathbb{T}^N$ on $(\mathbb{C}^N, \omega_{st} = \frac{i}{2}\sum_{i=1}^N dz_i \wedge d\bar{z}_i)$ by

$$[a_1, \dots, a_N] \cdot (z_1, \dots, z_N) = (e^{2\pi i a_1} z_1, \dots, e^{2\pi i a_N} z_N)$$

with the moment map $\mu : \mathbb{C}^N \to (\mathbb{R}^N)^*$ given by

$$\mu(z_1, \dots, z_N) = \pi \sum_{i=1}^N |z_i|^2 e_i^*$$

where e_i^* is the dual basis to the standard basis of $\mathbb{R}^N$. Then, the map

$$i^* \circ \mu : \mathbb{C}^N \to \mathfrak{k}^*$$

is the moment map of the action of the kernel K on $\mathbb{C}^N$.

Observe that $\beta \in \mu(\mathbb{C}^N)$ if and only if $\langle \beta, e_i \rangle \geq 0$ for all $i = 1, \dots, N$, since for suitable $z \in \mathbb{C}^N$, the expression $|z_i|^2$ can take any value in $\mathbb{R}_{\geq 0}$.

We aim to show that the action of K on the $(i^* \circ \mu)^{-1}(0)$ is free. Then, we can use symplectic reduction to obtain the symplectic manifold $(i^* \circ \mu)^{-1}(0)/K$

First, we claim that $(i^* \circ \mu)^{-1}(0) = \mu^{-1}(\varphi^*(C))$. We have $z \in (i^* \circ \mu)^{-1}(0)$ exactly when $\mu(z) \in \ker i^* \cap \mu(\mathbb{C}^N)$. On the other hand, by exactness of the third sequence above and definitions of φ and C, we have

$$\begin{aligned} \ker i^* \cap \mu(\mathbb{C}^N) &= \operatorname{im} \mu^* \cap \mu(\mathbb{C}^N) \\ &= \{\varphi^*\eta : \eta \in (\mathbb{R}^{n+1})^* \text{ and } \langle \varphi^*\eta, e_i \rangle \geq 0 \text{ for all } i = 1, \dots, N\} \\ &= \{\varphi^*\eta : \eta \in (\mathbb{R}^{n+1})^* \text{ and } \langle \eta, \varphi(e_i) \rangle \geq 0 \text{ for all } i = 1, \dots, N\} \\ &= \{\varphi^*\eta : \eta \in (\mathbb{R}^{n+1})^* \text{ and } \langle \eta, v_i \rangle \geq 0 \text{ for all } i = 1, \dots, N\} \\ &= \varphi^*(C) \end{aligned}$$

Therefore, $z \in (i^* \circ \mu)^{-1}(0)$ if and only if $\mu(z) \in \mu^*(C)$. That is,

$$(i^* \circ \mu)^{-1}(0) = \mu^{-1}(\varphi^*(C))$$

as we claimed.

To show that the action of K is free, we need to show that the stabilizer $K_z = K \cap \mathbb{T}^N_z$ of a point $z \in (i^* \circ \mu)^{-1}(0)$ under the action of K is trivial.

To this end, consider the stabilizer $\mathbb{T}^N_z$ of a point $z \in \mathbb{C}^N$ under the action of $\mathbb{T}^N$. Let $[a] \in \mathbb{T}^N$. We have,

$$\begin{aligned} [a] \cdot z = z &\Leftrightarrow e^{2\pi i a_i} z_i = z_i \text{ for all } i \\ &\Leftrightarrow a_i \in \mathbb{Z} \text{ for all } i \text{ such that } z_i \neq 0 \end{aligned}$$

That is,

$$\mathbb{T}^N_z = \{[a] \in \mathbb{T}^N : a_i \in \mathbb{Z} \text{ for all } i \text{ such that } z_i \neq 0\}$$

and

$$\begin{aligned} K_z &= K \cap \mathbb{T}^N_z \\ &= \{[a] \in \mathbb{T}^N : \sum_{i=1}^N a_i v_i \in \mathbb{Z}^{n+1} \text{ and } a_i \in \mathbb{Z} \text{ for all } i \text{ such that } z_i \neq 0\} \\ &= \{[a] \in \mathbb{T}^N : \sum_i a_i v_i \in \mathbb{Z}^{n+1} \text{ for all } i \text{ such that } z_i = 0, \\ &\qquad \text{and } a_i \in \mathbb{Z} \text{ for all } i \text{ such that } z_i \neq 0\} \end{aligned}$$

for a point $z \in \mathbb{C}^N$.

On the other hand, $z \in (i^* \circ \mu)^{-1}(0)$ if and only if $\mu(z) = \varphi^*(\eta)$ for some $\eta \in C$. We have $\langle \mu(z), e_i \rangle = \pi |z_i|^2$. Hence,

$$\begin{aligned} z_j = 0 &\Leftrightarrow |z_j|^2 = 0 \\ &\Leftrightarrow \langle \mu(z), e_i \rangle = 0 \\ &\Leftrightarrow \langle \varphi^*(\eta), e_i \rangle = 0 \\ &\Leftrightarrow \langle \eta, v_i \rangle = 0 \end{aligned}$$

By the condition of being a good cone, the set of v_i's such that $\langle \eta, v_i \rangle = 0$ are $\mathbb{Z}$-independent. In particular, if $\sum_i a_i v_i \in \mathbb{Z}^{n+1}$ for $\{i : \langle \eta, v_i \rangle = 0\}$, then a_i is in $\mathbb{Z}$. Hence, for $z \in (i^* \circ \mu)^{-1}(0)$, we have

$$K_z = \{[a] \in \mathbb{T}^N : a_i \in \mathbb{Z}\} = \{0\}$$

That is, K acts freely on $(i^* \circ \mu)^{-1}(0)$. Therefore, we get the reduced space $(i^* \circ \mu)^{-1}(0)/K$ with the symplectic form ω such that

$$proj^* \omega = \omega_{st}|_{(i^* \circ \mu)^{-1}(0)}$$

Now observe, by the fact that $\{v_i\}_{i=1}^N$ is a minimal set that defines the good cone, the map

$$s : \mathbb{T}^{n+1} \to \mathbb{T}^N$$

defined by $s(\sum_{i=1}^{n+1} a_i v_i) = \sum_{i=1}^{n+1} a_i e_i$ gives a splitting of

$$0 \longrightarrow K \xrightarrow{i} \mathbb{T}^N \xrightarrow{\mu} \mathbb{T}^{n+1} \longrightarrow 0$$

and we can view $T^N \cong T^{n+1} \times K$. Thus, we can view the Hamiltonian actions of $T^{n+1} = T^N/K$ and K as commuting actions to get the induced Hamiltonian action of $T^{n+1} = T^N/K$ on $(i^* \circ \mu)^{-1}(0)/K$ where the moment map

$$\mu_0 : (i^* \circ \mu)^{-1}(0)/K \to (\mathbb{R}^{n+1})^*$$

is given by $\mu|_{(i^* \circ \mu)^{-1}(0)} = \mu_0 \circ proj$. Also,

$$\mu_0((i^* \circ \mu)^{-1}(0)/K) = \mu((i^* \circ \mu)^{-1}(0)) = \mu^*(C) \cong C$$

The action of K restricts to an action on $((i^* \circ \mu)^{-1}(0) - \{0\})$ and the action of $\mathbb{R}$ on $\mathbb{C}^N$ given by $t \cdot z = e^t z$ commutes with the action of T^N. Thus,

$$(W, \omega) = (((i^* \circ \mu)^{-1}(0) - \{0\})/K, \omega)$$

is a symplectic cone with the action of $\mathbb{R}$ induced by the action of $\mathbb{R}$ on $\mathbb{C}^N$.

Moreover, as the unit sphere S^{2n+1} in the symplectic cone $\mathbb{C}^N - \{0\}$ is a $\mathbb{T}^N$ invariant compact hypersurface of contact type, we have that

$$\begin{aligned} M &= (((i^* \circ \mu)^{-1}(0) - \{0\})/\mathbb{R})/K \\ &= (i^* \circ \mu)^{-1}(0) \cap S^{2n+1}/K \end{aligned}$$

is a compact contact toric $\mathbb{T}^{n+1}$-manifold with the corresponding moment map

$$\mu|_M : M \to (\mathbb{R}^{n+1})^*$$

with the moment cone $\mu^*(C) \cong C$

Lastly, we need to check the connectedness of M. The cone $\mu^*(C) \cong C$ is connected. On the other hand, the fiber $\mu^{-1}(\xi)$ is a (possibly degenerate) product of circles, and hence path connected.

We had established that $(i^* \circ \mu)^{-1}(0) = \mu^{-1}(\varphi^*(C))$. Suppose

$$(i^* \circ \mu)^{-1}(0) = \mu^{-1}(\varphi^*(C)) \subseteq U_1 \cup U_2$$

where U_i are disjoint open sets in $\mathbb{C}^N$. Since fibers are connected, we have $\mu^{-1}(\xi) \subseteq U_i$ for some i. Thus, we have $\varphi^*(C) = A_1 \cup A_2$ where A_i are disjoint and defined as

$$A_i = \{\xi \in \varphi^*(C) : \mu^{-1}(\xi) \in U_i\} = \varphi^*(C) \cap \mu(U_i)$$

Observe that the moment map $\mu : \mathbb{C}^N \to (\mathbb{R}^N)^*$ given by

$$\mu(z_1, \ldots, z_N) = \pi \sum_{i=1}^{N} |z_i|^2 e_i^*$$

is an open map. Therefore, A_i are disjoint relatively open sets in $\varphi^*(C)$. By connectedness of $\varphi^*(C)$, one of the A_i is empty. Hence, $(i^* \circ \mu)^{-1}(0)$ is contained in one U_i, showing that $(i^* \circ \mu)^{-1}(0)$ is connected.

By the condition $2n + 1 > 3$ we have the manifold $(i^* \circ \mu)^{-1}(0) - \{0\}$ is also connected and as a continuous quotient of a connected manifold, M is connected. Therefore,

$$(M^{2n+1}, \alpha, \mu_\alpha)$$

is a c.c.c.t. $\mathbb{T}^{n+1}$-manifold with the good moment cone $\mu^*(C) \cong C$ □

Now we prove the converse statement. That is, we show that the moment cone of a c.c.c.t. manifold is a good cone.

Theorem 3.7 *Suppose (M, α, μ_α) is a c.c.c.t. G-manifold where $G = \mathbb{T}^{n+1}$, $\dim M > 3$ and the action of G is not free. Then the moment cone $C(\mu)$ is a good rational polyhedral cone.*

Before the proof, we state a fact about cones that will be needed in the proof:

Let $C \in \mathfrak{g}^*$ be a polyhedral cone and let $F \subseteq C$ be a face. Denote the subspace spanned by the vectors in F by $\langle F \rangle$ and let $\pi_F : \mathfrak{g}^* \to \mathfrak{g}^* / \langle F \rangle$ be the projection.

We have $\pi_F^{-1}(\pi_F(C)) \cong \pi_F(C) \times \langle F \rangle$ and for any point q in the interior of F there is an open neighbourhood W of q in $\mathfrak{g}^*$ such that

$$C \cap W = \pi_F^{-1}(\pi_F(C)) \cap W$$

Now we can sketch the argument from [19]:

Proof (Sketch) By Theorem 2.14 we have that

$$C := C(\mu) = \bigcap_{i=1}^{N} \{\eta \in \mathfrak{g}^* : \langle \eta, v_i \rangle \geq 0\}$$

is convex rational polyhedral cone, for a minimal primitive set of vectors $\{v_i\} \subseteq \mathbb{Z}_G$.

Moreover, by Lemma 2.17, since the action is not free, $\mu_\alpha : M \to S(\mathfrak{g}^*)$ is not onto. Therefore, C is a proper cone in $\mathfrak{g}^*$.

Let F be a codimension k face of C, with $0 < k < \dim G$. Then,

$$\langle F \rangle = \bigcap_{j=1}^{k} \{\eta \in \mathfrak{g}^* : \langle \eta, v_{i_j} \rangle = 0\}$$

for a subset $\{v_{i_j}\} \subseteq \{v_i\} \subseteq \mathbb{Z}_G$.

Let $x \in M$ be a point such that $\mu_\alpha(x)$ is in the interior of F. Therefore, $\dim \mathfrak{g}_x = \dim G_x = k$.

By Lemma 2.17, the orbital moment map gives a topological embedding of the orbit space M/G into $S(\mathfrak{g}^*)$. Thus, for any open and G-invariant neighbourhood U of the orbit $G \cdot x$, there is an open neighbourhood $\widetilde{W}$ in the sphere $S(\mathfrak{g}^*)$ such that

$$\mu_\alpha(U) = \mu_\alpha(M) \cap \widetilde{W}$$

Then, we have

$$C(\mu|_U) = C(\mu) \cap W$$

for the cones, where $W = \mathbb{R}^+ \widetilde{W} \cup \{0\}$.

By Theorem 2.10, using the same notation, locally we have

$$C(\mu) = \mathbb{R}^+(\mu_\alpha(x) + j((\mathfrak{k}/\mathfrak{g}_x)^*) + i(\Phi_V(V))) \cup \{0\}$$

and

$$C(\mu|_U) = (\mathbb{R}^+(\mu_\alpha(x) + j((\mathfrak{k}/\mathfrak{g}_x)^*) + i(\Phi_V(V))) \cup \{0\}) \cap W$$

By identifying the dual of the quotient and the annihilator, for $\widetilde{W}$ sufficiently small, we have

$$\mathbb{R}^+(\mu_\alpha(x) + j((\mathfrak{k}/\mathfrak{g}_x)^*)) \cap W = \mathfrak{g}_x^0 \cap W$$

From this, we get

$$C(\mu|_U) = (\mathfrak{g}_x^0 + i(\Phi_V(V))) \cap W = proj_x^{-1}(\Phi_V(V)) \cap W$$

where $proj_x : \mathfrak{g}^* \to \mathfrak{g}_x^*$ is the natural projection of $\mathfrak{g}^* \cong \mathfrak{g}_x^0 + \mathfrak{g}_x^*$ onto $\mathfrak{g}_x^*$.

Note that, $\langle F \rangle = \mathfrak{g}_x^0$ and the map $proj_x$ can be identified with π_F. Moreover, if we identify $\mathfrak{g}/\mathfrak{g}_x^0 \cong \mathfrak{g}_x^*$, then $\pi_F(C) \cong \Phi_V(V)$ and

$$\pi_F^{-1}(\pi_F(C)) = proj_x^{-1}(\Phi_V(V))$$

Also, $\{v_{i_j}\} \subset \mathfrak{g}_x \cong (\mathfrak{g}_x^0)^0$. Therefore, $\{v_{i_j}\} \subseteq \mathbb{Z}_{G_x} = \mathbb{Z}_G \cap \mathfrak{g}_x$.

Hence, for a small enough W, we have

$$\begin{aligned} C(\mu) \cap W = C(\mu|_U) &= W \cap proj_x^{-1}(\Phi_V(V)) \\ &= W \cap \pi_F^{-1}(\pi_F(C)) \\ &= W \cap \bigcap_{j=1}^{k} \{\eta \in \mathfrak{g}^* : \langle \eta, v_{i_j} \rangle \geq 0\} \end{aligned}$$

Therefore, $proj_x^{-1}(\Phi_V(V)) = \bigcap_{j=1}^k \{\eta \in \mathfrak{g}^* : \langle \eta, v_{i_j} \rangle \geq 0\}$ and

$$\Phi_V(V) = \bigcap_{j=1}^{k} \{\eta \in \mathfrak{g}_x^* : \langle \eta, v_{i_j} \rangle \geq 0\}$$

where $\{v_{i_j}\}$ is a minimal primitive set with this property.

On the other hand, by Lemma 2.9,

$$\Phi_V(V) = \{\sum_{j=1}^{k} a_j \nu_j : a_j \geq 0\}$$

for some basis of $\mathbb{Z}_{G_x}^*$.

The dual set $\{v_{i_j}^*\} \subseteq \mathbb{Z}_{G_x}^*$ of $\{v_{i_j}\} \subseteq \mathbb{Z}_{G_x}$ spans the cone $\Phi_V(V)$ and by the minimality it is also a basis of $\mathbb{Z}_{G_x}^*$. Therefore, $\{v_{i_j}\}$ is a basis of $\subseteq \mathbb{Z}_{G_x}$ and C is a good cone. □

It remains to show that there is a one-to-one correspondence between good cones and c.c.c.t G-manifolds.

For this, let (M, α, μ_α) be a c.c.c.t. G-manifold normalised such that $\mu_\alpha(M_0)$ lies in $S(\mathfrak{g}^*)$ with the moment cone C.

Now assume, $(M_0, \alpha_0, \mu_{\alpha_0})$ is another such c.c.c.t. G-manifold normalised such that $\mu_\alpha(M_0)$ lies in $S(\mathfrak{g}^*)$, with the same moment cone C. Then, we have

$$\mu_{\alpha_0}(M_0) = \overline{\mu_{\alpha_0}}(M_0/G) = \overline{\mu_\alpha}(M/G) = \mu_\alpha(M)$$

By Corollary 2.19, the c.c.c.t $(M_0, \alpha_0, \mu_{\alpha_0})$ is locally isomorphic to (M, α, μ_α). We want to show that they are isomorphic as c.c.c.t. G-manifolds.

By Theorem 2.23, we have

$$H^1(M/G, \mathcal{S}) = H^2(M/G, \mathbb{Z}^{n+1})$$

On the other hand, $M/G \cong \mu_\alpha(M)$ by Proposition 2.17.

The image of the moment map is contractible and $H^2(M/G, \mathbb{Z}^{n+1}) = 0$. Therefore, by 2.21, $(M_0, \alpha_0, \mu_{\alpha_0})$ is isomorphic to (M, α, μ_α) as a c.c.c.t G-manifold.

3.3 Applications of the Classification

Consider the cosphere bundle $S^*\mathbb{T}^n := (T^*\mathbb{T}^n - \mathbf{0})/\mathbb{R}$ of the n-torus where the action of $t \in \mathbb{R}$ is given by the dilation $(p, q) \mapsto (p, e^t q)$. Then $S^*\mathbb{T}^n$ has a natural contact structure induced by the symplectic structure on $T^*\mathbb{T}^n$.

Similarly, consider the cosphere bundle $S^*S^2 := (T^*S^2\backslash\mathbf{0})/\mathbb{R}$ of the sphere. Then S^*S^2 has a natural contact structure induced by the symplectic structure on T^*S^2. See [21, Example 3.5.7] for details of these contact structures which are constructed in a similar manner to Example 1.4.

Using the classification theorem for compact connected contact toric manifolds, we can prove the following two theorems (see [19, Section 1]) regarding the contact actions on $S^*\mathbb{T}^n$ and S^*S^2:

Theorem 3.8 *Up to isomorphism, there is only one effective $\mathbb{T}^n$-action on $S^*\mathbb{T}^n$ making it a contact toric manifold.*

Theorem 3.9 *Up to isomorphism, there is only one effective $\mathbb{T}^2$-action on S^*S^2 making it a contact toric manifold.*

As a consequence of Theorem 3.9 and the classification theorem, we see that any effective contact $\mathbb{T}^n$-action on $S^*\mathbb{T}^n$ must be free. Using this result and the relation of the symplectic structure on $T^*T^n\backslash 0$ and the contact structure it induces on S^*T^n, we can prove:

Proposition 3.10 *Any effective $\mathbb{T}^n$-action on $T^*\mathbb{T}^n\backslash 0$ which preserves the symplectic form and commutes with dilations, is free.*

Through the work of Toth and Zeldich [24] this proposition implies that certain classes of metrics (called ”toric integrable”, see [19]) on tori are flat. In fact, this problem about the toric integrable metrics was the main motivation for the classification of c.c.c.t. manifolds. See [19] and [4] for further discussions about this problem and its relation to contact toric manifolds.

More recently, in [1], [2], Abreu and Macarini proved that for a contact toric G-manifold which falls under the case (4) of the Theorem 3.2 of Lerman, if in addition the first Chern class of the symplectization vanishes, then normals that define the moment cone can be used to define a polytope $D \in \mathbb{R}^n$ where $n = \dim G - 1$. They then used the combinatorial properties such as the volume of this polytope to describe the invariants of contact toric manifolds. In particular, they calculated the contact homology, mean Euler characteristic, and number of closed orbits in the flow of Reeb vector fields for certain examples of contact toric manifolds.

3.4 Submanifolds of Contact Toric Manifolds

In this last section, we will point out some possible directions of investigation about G-invariant compact connected contact submanifolds of a c.c.c.t. G-manifolds.

Let N be a G-invariant compact connected codimension 2 contact submanifold of a c.c.c.t. G-manifold M with the embedding $i : N \to M$. That is, a contact manifold (N, ξ') embedded in (M, ξ) with the embedding $i : N \to M$, such that $di(\xi') = \xi \cap di(TN)$ and action of G on M preserves N and ξ'.

The action is necessarily non-effective by dimension considerations. However, by taking the quotient by the kernel of the restricted action on N, we may view N as a c.c.c.t. manifold as the quotient group is connected.

In the case of a non-free action of $G = \mathbb{T}^{n+1}$ on M, in higher dimensions where $2n+1 > 5$, the classification indicates that these submanifolds N correspond to the faces of the moment cone. In the case that $2n+1 = 5$ and $\dim N = 3$, by the classification, N needs to be a certain lens space embedded in M as a contact submanifold.

In the case of a free action of $G = \mathbb{T}^{n+1}$ on M, in higher dimensions where $2n+1 > 7$, the classification implies that these submanifolds N correspond to the subspaces of $\mathfrak{g}^*$ and are embeddings of $\mathbb{T}^n \times S^{n-1}$ into $\mathbb{T}^{n+1} \times S^{n+1}$ as contact manifolds. In the case of $2n+1 = 7$, by the classification, N needs to be a (not necessarily trivial) principal $\mathbb{T}^3$ bundle over S^2 embedded in $\mathbb{T}^4 \times S^3$ as a contact submanifold.

The existence and properties of such embeddings are possible directions that remain to be investigated.

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