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Hamiltonian group actions and the Duistermaat–Heckman Theorem

Master Thesis

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Abstract

Symplectic reduction formalises the classical Noether principle concerning mechanical systems: given a symmetry group of dimension k acting on a system, there appear k preserved quantities in the time evolution of the system that enable to reduce the number of degrees of freedom of the phase space by $2k$. The symplectic reduction of a symplectic manifold under a Hamiltonian group action gives rise to a reduced symplectic structure after taking the quotient of a fiber of the moment map over the action. The Duistermaat–Heckman Theorem assesses the relationship between different reduced spaces for a torus-action: nearby fibers are identified diffeomorphically and the reduced symplectic form of a reduced space depends linearly on the value of the moment map. In cohomology, this linear coefficient is an invariant, characteristic class of the associated torus-bundle. This leads to the fact that the pushforward of the Liouville measure by the moment map is a piecewise polynomial multiple of the Lebesgue measure on the dual of the Lie algebra.

Keywords: symplectic geometry, Coisotropic Embedding Theorem, symplectic reduction, Hamiltonian action, equivariant moment map, coadjoint orbit, canonical form on the cotangent bundle, symplectic toric manifold, Duistermaat–Heckman theorem, Liouville measure, principal bundle, characteristic class.

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Chapter 1

Introduction

The Duistermaat–Heckman Theorem (Theorem 5.15) first appeared in the article 'On the variation in the cohomology of the symplectic form of the reduced phase space' [12] by Johannes Duistermaat and his former PhD student Gert Heckman in 1982. It states that the reduced symplectic structures obtained by taking the Marsden–Weinstein–Meyer quotient of a Hamiltonian space at different values of the moment map is linear in cohomology, and that the linearity coefficient is a characteristic class of the underlying torus-bundle. As they say in the original paper, the result was first thought of in the form of one of its corollaries, the now well-known *Duistermaat–Heckman formula* (Theorem 5.20), which was '*conjectured in some very stimulating discussions with Atiyah and Guillemin, and was the starting point for our paper*'. The formula states that the Duistermaat–Heckman measure (the pushforward of the Liouville measure by the moment map) is a piecewise polynomial multiple of the Lebesgue measure on the dual of the Lie algebra. The other main corollary of the theorem is a so-called *localization formula*, concretely the *exact stationary phase formula*, which computes the inverse Fourier transform of the Duistermaat–Heckman measure in terms of a formula evaluated exclusively at the fixed points of the action. These corollaries were an important discovery in the then still developing theory of symplectic geometry. In fact, the localization formula would be later put within the more general framework of equivariant cohomology [16], independently by Berline and Vergne [4] and by Atiyah and Bott [2]. In combination with results like the Convexity Theorem or Delzant's Classification Theorem about symplectic toric manifolds, they are an example of our better understanding of Hamiltonian torus-actions in comparison to more general actions. The case of non-abelian group-actions remains more obscure. For example, a recent article by Crooks and Weitsman [9] presents a more general formulation of the Duistermaat–Heckman formula for a compact connected Lie group by making use of what they call a *Gelfand–Cetlin datum* on the dual of the Lie algebra of the group.

1.1 Overview

With the objective of setting the stage for the Duistermaat–Heckman Theorem, we begin this thesis with the preliminary Chapter 2, where we briefly review the concepts and results from differential geometry most relevant for this work and we fix the notation and conventions followed thereafter. First, in Section 2.1 we cover Cartan's magic formula and related identities; as well as the basic definitions regarding submanifolds, foliations, and distributions that conclude with the Frobenius Theorem. Secondly, Section 2.2 deals with basic Lie group theory, group actions on manifolds, and the Quotient Theorem; and lastly, Section 2.3 revises the Tubular Neighbourhood Theorem.

On Chapter 3 we introduce the main definitions in symplectic geometry. We start in Section 3.1 with linear symplectic geometry, where we find some easy results that prelude and motivate their non-linear, geometrical versions. In Section 3.2 we cover the key notion of symplectic and Hamiltonian vector fields, in bijection with closed and exact 1-forms respectively via the symplectic form, as well as introducing in Section 3.3 the canonical form on the cotangent bundle of a manifold, both of which are concepts dating back to the origins of symplectic geometry in classical mechanics. We continue including the short Section 3.4 covering the essentials of almost complex structures, which will play a role in the last section of the chapter. The last Section 3.5 is devoted to the so-called 'local form' results, from which the first to be discovered is the famous Darboux Theorem, and we present the now classical approach for its proof due to Moser, i.e., *Moser's trick*. A careful use of this technique in combination with the Tubular Neighbourhood Theorem allows to obtain stronger results localized not around a point but around a compact submanifold. These results, collectively called 'Neighbourhood theorems' are summarized in the main result, the Embedding Theorem 3.47. It exposes the *flexibility* to be found in the symplectic realm: the symplectic structure around a compact submanifold is characterised uniquely (up to symplectomorphism) in terms of the restricted symplectic form and the symplectic normal bundle. In particular, there are no symplectic local invariants, in stark contrast to Riemannian geometry. As a corollary, we obtain the Weinstein Lagrangian Tubular Neighbourhood Theorem, one of the first results in this direction. In our case, the corollary we are most interested in is the equivariant formulation of the *Coisotropic* Embedding Theorem, at the heart of the proof of the Duistermaat–Heckman Theorem. Thus, during this section an effort is made to obtain G-equivariant formulations of every result, under a symplectic action of a compact Lie group.

In Chapter 4 we turn our attention to group actions that preserve the symplectic structure, i.e., symplectic and Hamiltonian actions, dealing first with the easier case of circle actions in Section 4.1. In Section 4.2 we generalise the concept of classical mechanics of the time evolution, i.e. a one-parameter family of symplectomorphisms, being generated by a Hamiltonian function via

Hamilton's equations. The 'generalised Hamiltonian' for a general action is the *moment map* of a Hamiltonian action, an equivariant map taking values on the dual of the Lie algebra of the group under consideration. At this point we include some results elaborating on the relation between moment maps and Lie algebra cohomology, as a side-topic of interest in itself but not essential for the main argument-line of the thesis. After some comments on how to define a canonical symplectic structure on coadjoint orbits in Section 4.3, the second major step in the Duistermaat–Heckman Theorem is addressed in Section 4.4, namely, the technique of *symplectic reduction*, essential to state the theorem itself. It formalises the Noether principle of mechanical systems that observes that whenever a system is acted upon by a symmetry group of dimension k, then there appear k quantities that are preserved in the time evolution; one can then reduce the number of degrees of freedom of the system's phase space by $2k$. The reduction of a symplectic manifold under a Hamiltonian action is given by the Marsden–Weinstein–Meyer quotient: the orbit space of a fiber of the moment map. The mathematical underlying principle is that these fibers are coisotropic submanifolds, and hence foliated by isotropic leaves which coincide with the group orbits. Algebraically we are taking the quotient over the kernel of the symplectic form and hence we obtain a new, well-defined structure. After obtaining some related results, like reduction in stages, or in Section 4.5 applying reduction on the cotangent bundle of a Lie group, we review in Section 4.6 *symplectic toric manifolds* and the main results for the particular case of torus-actions, namely, the Convexity Theorem and Delzant's Classification Theorem. We close the chapter in Section 4.7 with some comments about the generalisation of our results to locally-free actions and the requirement to introduce *orbifolds*.

Lastly, on Chapter 5 we state and prove the Duistermaat–Heckman Theorem. Just as we did in the previous chapter, we first exemplify the arguments for a circle action in Section 5.1. As we say, the Duistermaat–Heckman Theorem assesses the relationship between doing reduction at different values of the moment map, for a torus-action. In Section 5.2 we show that close fibers are diffeomorphic and that the reduced symplectic form depends linearly on the value of the moment map. In cohomology this linear coefficient is a well-defined characteristic class of the underlying torus-bundle, and is hence independent of any diffeomorphism choice. This allows us to obtain in Section 5.3 the Duistermaat–Heckman formula about the pushforward by the moment map of the Liouville measure: it is given, regarding the dual of the Lie algebra as the corresponding affine space, by a piecewise polynomial multiple of the Lebesgue measure. In the particular case of a symplectic toric manifold, this piecewise polynomial is furthermore constant, generalising the fact already known to Archimedes that the area of the 2-sphere strip bounded by two parallels is proportional to the height between the parallels.

In Appendix A we first define in Section A.1 the Fubini-Study structure on the complex projective space, as the archetypal example of a symplectic toric manifold that will also serve as running example of the results and constructions throughout the thesis. We finish the thesis in Section A.2 with a summary of the excellent chapters of [27] about vector bundles, their connections and curvature, characteristic classes, and principal bundles, that permit to prove the invariance of the characteristic classes and are thus the last ingredient of the Duistermaat–Heckman Theorem and its corollaries.

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Chapter 2

Differential geometry preliminaries

In this chapter we make a quick summary of the main definitions and results from elementary differential geometry and topology that we will use, and on the way fixing the notations and conventions that will be followed through. Namely, the main tools of this thesis are differential forms, basic Lie Group Theory and the Quotient Manifold Theorem about homogeneous spaces, and the Tubular Neighbourhood. Some additional tools are developed in the Appendix A.2, essentially vector space-valued forms, connections and curvature on vector bundles, characteristic classes and principal bundles. The main references have been [7, 18, 19, 24].

2.1 Notations and conventions

We consider second countable smooth manifolds of constant dimension and without boundary.

Given a smooth manifolds M, N, we denote by $\mathfrak{X}(M)$ the space of smooth vector fields, i.e., the sections of the tangent bundle TM. We denote by $\mathcal{C}^\infty(M,N)$ the space of smooth functions from M to N, and by $\mathrm{Diff}(M)$ the space of self-diffeomorphisms of M.

We denote by $\Omega^k(M;\mathbb{R})$ the smooth k-forms on M, that is, the sections of the smooth vector bundle $\bigwedge^k T^*M$ (the k-th alternate or exterior product of the cotangent bundle T^*M). We adopt the standard notation $f^*\omega \in \Omega^k(M;\mathbb{R})$ for the pullback of a differential form $\omega \in \Omega^k(N;\mathbb{R})$ by $f \in \mathcal{C}^\infty(M,N)$. We will also use a similar notation for the pushforward of a vector field $X \in \mathfrak{X}(M)$ via a diffeomorphism $f : M \to N$,

$$(f_*X)_q := d_pf(X_{f^{-1}(p)}),$$

for every $q \in N$. That is, $f_*X \in \mathfrak{X}(N)$ is the only f-related vector field to $X \in \mathfrak{X}(M)$. We recall that $Y \in \mathfrak{X}(N)$ is f-related to $X \in \mathfrak{X}(M)$ if, regarding them as derivations,

$$Y(g) \circ f = X(g \circ f)$$

for any $g \in \mathcal{C}^\infty(M, \mathbb{R})$. Furthermore, for $f \in \mathcal{C}^\infty(M, N)$ we also denote by f_* the vector bundle homomorphism between tangent bundles

$$f_* : TM \to TN : (p, v) \mapsto (f(p), d_p f(v)),$$

sometimes also using this notation for the differential at a point,

$$f_*(v) := f_{*,p}(v) := d_p f(v).$$

We take the standard sign convention for the Lie derivative with respect to a vector field $X \in \mathfrak{X}(M)$. That is, if X is the infinitesimal generator of the flow $\varphi_t \in \mathrm{Diff}(M)$ such that

$$\left.\frac{d}{dt}\right|_t \varphi_t = X \circ \varphi_t, \;\; \varphi_0 = \mathrm{id},$$

and $t \mapsto \varphi_t$ is a group homomorphism, then the Lie derivative of the vector field $Y \in X(M)$ with respect to X is

$$\mathfrak{L}_X Y := \left.\frac{d}{dt}\right|_{t=0} \varphi_{-t,*} Y.$$

Lemma 2.1 *The Lie derivative of a vector field coincides with the bracket,*

$$\mathfrak{L}_X Y = [X, Y].$$

Proof We fix some $p \in M$ and regard $\mathfrak{L}_X Y$ as a derivation, that is, we take any $f \in \mathcal{C}^\infty(M, \mathbb{R})$ and compute

$$\begin{aligned}(\mathfrak{L}_X Y)_p(f) &= \left.\frac{d}{dt}\right|_{t=0} \varphi_{-t,*} Y(f)_p \\ &= \lim_{t \to 0} \frac{Y_{\varphi_t(p)}(f \circ \varphi_{-t}) - Y_p(f)}{t}.\end{aligned}$$

By the parameterised Taylor Theorem, we can write $f \circ \varphi_t = f + t g_t$ for some g_t such that $g_0 = \frac{\partial}{\partial t}|_{t=0} f \circ \varphi_t = X(f)$, so that

$$\begin{aligned}(\mathfrak{L}_X Y)_p(f) &= \lim_{t \to 0} \frac{Y_{\varphi_t(p)} - Y_p}{t}(f) - Y_p(g_0) \\ &= X(Y(f))_p - Y(X(f))_p \\ &= [X, Y]_p(f).\end{aligned}$$

$\square$

We will also consider the smooth family of diffeomorphisms $t \mapsto \psi_t \in \mathrm{Diff}(M)$ generated by a time-dependent vector field $X_t \in \mathfrak{X}(M)$, that is

$$\frac{d}{dt}\bigg|_t \psi_t = X_t \circ \psi_t,\ \psi_0 = \mathrm{id}.$$

Notice however that $t \mapsto \psi_t$ need not be a group homomorphism.

Similarly, the Lie derivative of a k-form $\omega \in \Omega^k(M;\mathbb{R})$ with respect to $X \in \mathfrak{X}(M)$ with flow $t \mapsto \varphi_t$ is given by

$$\mathfrak{L}_X\omega := \frac{d}{dt}\bigg|_{t=0} \varphi_t^*\omega.$$

Lemma 2.2 *Consider a smooth family a smooth family of k-forms $t \mapsto \omega_t \in \Omega^k(M;\mathbb{R})$, of diffeomorphisms $t \mapsto \psi_t \in \mathrm{Diff}(M)$, and a time-dependent vector field $X_t \in \mathfrak{X}(M)$ such that*

$$\frac{d}{dt}\bigg|_t \psi_t = X_t \circ \psi_t,\ \psi_0 = \mathrm{id}.$$

Then it holds that
$$\frac{d}{dt}\bigg|_t \psi_t^*\omega_t = \psi_t^*(\mathfrak{L}_{X_t}\omega_t + \frac{d}{dt}\bigg|_t \omega_t).$$

Proof Arguing at a given point $p \in M$ and differentiating at fixed time s we compute, in virtue of Leibniz's differentiation rule for a product,

$$\begin{aligned}
\frac{d}{dt}\bigg|_s (\psi_t^*\omega_t)_p &= \frac{d}{dt}\bigg|_s d_p\psi_t^*(\omega_t)_{\psi_t(p)} \\
&= \frac{d}{dt}\bigg|_s (\psi_t^*\omega_s)_p + \psi_s^*\frac{d}{dt}\bigg|_s (\omega_t)_p \\
&= \psi_s^*(\mathfrak{L}_{X_s}\omega_s + \frac{d}{dt}\bigg|_s \omega_t)_p.
\end{aligned}$$

For the last identity we make the following digression. A time-dependent vector field actually induces different diffeomorphism families depending on the starting time. Given t_0 as initial time and a point $(t,p) \in \mathbb{R}\times M$ in the definition domain starting from t_0 we can define $\psi_{t_0,t}$ as the unique diffeomorphism family such that
$$\frac{d}{dt}\bigg|_t \psi_{t_0,t} = X_t \circ \psi_{t_0,t},\ \psi_{t_0,t_0} = \mathrm{id}.$$

Then the previous diffeomorphisms are just $\psi_t := \psi_{0,t}$. It follows from the uniqueness of solutions that, when all is well defined,

$$\psi_{t_2,t_3} \circ \psi_{t_1,t_2}(p) = \psi_{t_1,t_3}(p).$$

Thus we can compute

$$
\begin{aligned}
\frac{d}{dt}\bigg|_s \psi_t^*\omega &= \frac{d}{dt}\bigg|_0 \psi_{0,s+t}^*\omega \\
&= \frac{d}{dt}\bigg|_0 (\psi_{s,s+t}\circ\psi_{0,s})^*\omega \\
&= \psi_{0,s}^*\frac{d}{dt}\bigg|_0 \psi_{s,t+s}^*\omega \\
&= \psi_s^*\mathfrak{L}_{X_s}\omega.
\end{aligned}
$$

The last identity follows from the observation that the definition of the Lie derivative with respect to the vector field $X_s \in \mathfrak{X}(M)$ (s fixed) only depends on the values of the time derivative of the flow φ_t at $t = 0$ (since it is the identity at time 0), and $\frac{d}{dt}|_{t=0}\psi_{s,s+t} = X_s$, so that $\widetilde{\varphi}_t := \psi_{s,s+t}$ has the same derivative at time $t = 0$ as the flow of X_s regarded as non time-dependent. □

We will denote by $i_X : \Omega^k(M;\mathbb{R}) \to \Omega^{k-1}(M;\mathbb{R})$ the map such that

$$i_X\omega(X_1,\ldots,X_{k-1}) := \omega(X, X_1\ldots,X_{k-1}),$$

and $i_X f = 0$ for $f \in \mathcal{C}^\infty(M,\mathbb{R}) \equiv \Omega^0(M;\mathbb{R})$.

Proposition 2.3 (Cartan's magic formula) *The Lie derivative of a differential k-form $\omega \in \Omega^k(M;\mathbb{R})$ with respect to $X \in \mathfrak{X}(M)$ is given by*

$$\mathfrak{L}_X\omega = di_X\omega + i_X d\omega.$$

Proof One option is to use the flow-box lemma for X and obtain local coordinates x such that $X = \partial_{x_1}$. Then working with a local coordinate expression for ω it becomes a trivial computation. Alternatively, we can argue as follows.

Locally, ω will be a sum of terms of the type $f df_1 \wedge\cdots\wedge df_k$ for some $f, f_i \in \mathcal{C}^\infty(M,\mathbb{R})$, so that it is enough to prove it for such a term. Then, we notice that the formula is trivial for 0-forms, i.e., for smooth functions (since $i_X f = 0$ for $f \in \mathcal{C}^\infty(M,\mathbb{R})$). Furthermore, both sides of Cartan's formula commuting with d, so that the formula also holds for exact 1-forms df_i. Finally, we check that both sides of the formula behave the same way with respect to the exterior product $\wedge$: for $\alpha \in \Omega^k(M;\mathbb{R})$ and $\beta \in \Omega^l(M;\mathbb{R})$ we have

$$
\begin{aligned}
\mathfrak{L}_X(\alpha\wedge\beta) &= (\mathfrak{L}_X\alpha)\wedge\beta + \alpha\wedge(\mathfrak{L}_X\beta), \\
(d\circ i_X + i_X\circ d)(\alpha\wedge\beta) &= ((d\circ i_X + i_X\circ d)\alpha)\wedge\beta \\
&\quad + \alpha\wedge((d\circ i_X + i_X\circ d)\beta).
\end{aligned}
$$

To check the first, we use Leibniz's rule and the anti-commutativity of the exterior product on a local coordinate expression. For the second, we use that

$$
\begin{aligned}
d(\alpha\wedge\beta) &= (d\alpha)\wedge\beta + (-1)^k\alpha\wedge(d\beta), \\
i_X(\alpha\wedge\beta) &= (i_X\alpha)\wedge\beta + (-1)^k\alpha\wedge(i_X\beta).
\end{aligned}
$$

These are checked via local expressions and the anti-commutativity of $\wedge$. $\square$

A last useful computation that we will need later is:

Lemma 2.4 *Given two vector fields $X, Y \in \mathfrak{X}(M)$ and a k-form $\omega \in \Omega^k(M;\mathbb{R})$, then*

$$i_{[X,Y]}\omega = [\mathfrak{L}_X, i_Y]\omega.$$

Proof We compute, for $X, Y \in \mathfrak{X}(M)$, X with flow $\varphi_t \in \mathrm{Diff}(M)$,

$$\begin{aligned} i_{[X,Y]}\omega &= i\left(\frac{d}{dt}\bigg|_{t=0} \varphi_{-t,*}Y\right)\omega \\ &= \frac{d}{dt}\bigg|_{t=0} i(\varphi_{-t,*}Y)\omega \\ &= \frac{d}{dt}\bigg|_{t=0} \varphi_t^*(i_Y\varphi_{-t}^*\omega) \\ &= \mathfrak{L}_X(i_Y\omega) - i_Y(\mathfrak{L}_X\omega) \\ &= [\mathfrak{L}_X, i_Y]\omega. \end{aligned}$$

$\square$

2.1.1 Submanifolds

Let M^m, N^n be smooth manifolds of dimension m, n respectively. We recall that a smooth map in $\mathcal{C}^\infty(M,N)$ is an *immersion*, resp. a *submersion*, if its differential is at every point injective, resp. surjective. Furthermore, it is a smooth *embedding* when it is a topological embedding, that is, a homeomorphism onto its image (with the subspace topology).

An *embedded submanifold of* M is a subset $S \subset M$ such that S is a smooth manifold (according to our convention, i.e., second countable and without boundary) with smooth structure compatible with the subspace topology as a subset of M. We call M the *ambient manifold*, and we say that S has *codimension* equal to $\dim M - \dim S$. Equivalently, an embedded submanifold is a subset $S \subset M$ with a (unique) smooth structure such that the inclusion map $i : S \hookrightarrow M$ is a smooth embedding. It is a *properly embedded submanifold* when the inclusion map is proper, i.e., the preimage of a compact set is also compact. They are sometimes also called *closed submanifolds* since an embedded submanifold is properly embedded if and only if it is a topologically closed subset of the ambient manifold or if and only if the inclusion map is closed.

On the other hand, an *immersed submanifold* is a subset $S \subset M$ endowed with a certain topology (not necessarily the subspace one) with respect to which it is a topological manifold (without boundary), and a compatible smooth structure such that such that the inclusion map $i : S \hookrightarrow M$ is an immersion. It follows that every embedded submanifold is in particular immersed. Since proper maps are closed, an immersed submanifold is properly embedded if and only if the inclusion map $i : S \hookrightarrow M$ is proper or closed. Furthermore, immersed submanifolds are locally embedded as an easy consequence of the inverse function theorem.

For a thorough and excellent discussion see e.g. [18]. For this thesis we define:

Definition 2.5 (Submanifold) A (smooth) *submanifold* N of M is an embedded submanifold of M.

It will also be convenient to fix our nomenclature for *adapted local slice charts*. Given a subset $S \subset M$, a local slice chart around $p \in S$ (adapted to S) is a coordinate chart for M, $\varphi : U \to \mathbb{R}^m$, from an open neighbourhood U of p, such that $\varphi(U \cap S)$ is the intersection of $\varphi(U)$ with a linear subspace. Particularly, if it is a k-dimensional subspace, we can always assume that

$$\varphi(U \cap S) = \{x = (x_1, \ldots, x_m) \in \varphi(U) : x_{k+1} = 0, \ldots, x_m = 0\}.$$

Then, a subset $S \subset M$ is an embedded submanifold if and only if every point has a local slice chart around it (in the sense that an embedded submanifold always has adapted local slice charts, and reciprocally a subset $S \subset M$ with local slice charts around every point is a topological manifold with the subspace topology and has a unique smooth structure making it into an embedded submanifold). More detail can be found for example in [18], §5.

We now briefly recall the setting for the Frobenius Theorem. Let M^m be a smooth manifold. A *distribution on M of rank k* is a smooth rank-k subbundle of the tangent bundle TM. Given a smooth distribution $D \subset TM$, a nonempty immersed submanifold $N \subset M$ is an *integral manifold* of D if $T_pN = D_p$ at every point $p \in N$. Then, a distribution $D \subset TM$ of rank k is called *integrable* if the following conditions are satisfied:

- Through every point $p \in M$ passes an integral submanifold.
- Every point $p \in M$ has a *local flat chart*: a coordinate map $\varphi : U \to \mathbb{R}^m$ around $p \in U$ such that D is spanned by the first k coordinate vector fields $\partial_1, \ldots, \partial_k$ and such that the slices $\{x^{k+1} = c^{k+1}, \ldots, x^m = c^m\}$ for any $c^j \in \mathbb{R}$ are integral manifolds of D.

Lastly, we recall that a distribution is *involutive* if the Lie bracket of any two local smooth sections of D is also a local section of D, i.e., if the subspace of $\mathfrak{X}(M)$ given by sections of D is a Lie-subalgebra, or simply put, if it is closed under the Lie bracket. Hence, every integrable distribution is involutive.

Similarly, let $\mathcal{F}$ be any collection of k-dimensional immersed submanifolds of M. A smooth chart $\varphi : U \to \mathbb{R}^m$ of M is said to be *flat* for $\mathcal{F}$ if each submanifold in $\mathcal{F}$ intersects U in either the empty set or a countable union of k-dimensional slices of the form $\{x^{k+1} = c^{k+1}, \ldots, x^m = c^m\}$. Then, a *foliation* of dimension k on M is a collection $\mathcal{F}$ of disjoint, connected, nonempty, immersed k-dimensional submanifolds of M, called the leaves of the foliation, whose union is M; and such that in a neighborhood of each point $p \in M$ there exists a flat chart for $\mathcal{F}$. It is then easy to show that the tangent bundles of the submanifolds of a foliation form an integrable distribution of M, and hence involutive. The converse is Frobenius Theorem:

Theorem 2.6 (Global Frobenius Theorem) *Let $D \subset TM$ be a distribution on a smooth manifold M. Then, it is involutive if and only if the collection of all maximal connected integral manifolds of D forms a foliation of M.*

An excellent reference for the proof is again [18], §19.

2.2 Lie group theory

We use [18, 24] as our main references.

A *Lie group* is a group G with a manifold structure such that group multiplication is a smooth map $m : G \times G \to G$.

Lemma 2.7 *Given a Lie group G, the map* $\operatorname{inv} : G \to G : g \mapsto g^{-1}$ *is smooth.*

Proof Consider a group G with a smooth structure where m is smooth. Then, left multiplication $L_g : G \to G : h \mapsto gh$ and right multiplication $R_g : G \to G : h \mapsto hg$ by an element $g \in G$ are diffeomorphisms (with inverse $L_{g^{-1}}$, $R_{g^{-1}}$) and hence m is a submersion. Then, $\Delta := m^{-1}(e) \subset G \times G$ for the identity element $e \in G$ is a closed submanifold of dimension $\dim G$ and we consider the smooth bijection $\pi_1| : \Delta \to G$, for the projection $\pi_1 : G \times G \to G : (g, h) \mapsto g$. It follows that $\pi_1|$ has constant rank: fixing $g \in G$ and defining the diffeomorphism $\theta_g : G \times G \to G \times G : (x, y) \mapsto (gx, yg^{-1})$, which restricts to a diffeomorphism $\theta_g| : \Delta \to \Delta$, we have that

$$d_{(g,g^{-1})}\pi_1| = d_e L_g \circ d_{(e,e)}\pi_1| \circ d_{(g,g^{-1})}\theta_{g^{-1}}|.$$

The fact that $\pi_1|$ has constant rank implies that it is a diffeomorphism and hence $\operatorname{inv} = \pi_2 \circ (\pi_1|)^{-1}$ is smooth. $\square$

A *Lie subgroup* $H \subset G$ is a subgroup which is also a submanifold. By standard Lie Group theory (e.g. Theorem 20.12 in [18]), every closed subgroup of a Lie group is a Lie subgroup, i.e., it is embedded. Furthermore, the *homogeneous space* G/H of left or right cosets inherits a unique manifold structure such that the quotient map $G \to G/H$ is a smooth submersion (see Theorem 21.17 of [18]).

A vector field $X \in \mathfrak{X}(G)$ is *left-invariant* if it is L_g-related to itself, for all $g \in G$. Thus, a left-invariant vector field is determined by its value $X_e = \xi$ at the identity element, so that evaluation at e gives a vector space isomorphism

$\mathfrak{X}^L(G) \cong \mathfrak{g} := T_eG$ between all left-invariant vector fields $\mathfrak{X}^L(G)$ and the tangent space at the identity $\mathfrak{g}$. We denote $\xi^L \in \mathfrak{X}^L(G)$ the unique left-invariant vector field such that $\xi^L_e = \xi$, given by $\xi^L_g = L_{g,*}\xi \equiv g\xi$ for every $g \in G$. Given $f \in \mathcal{C}(M, N)$ the Lie bracket $[X', Y']$, for vector fields $X', Y' \in \mathfrak{X}(N)$ that are f-related to $X, Y \in \mathfrak{X}(M)$ (respectively) is f-related to $[X, Y]$. It follows that $\mathfrak{X}^L(G)$ is closed under the Lie bracket operation of vector fields, since a vector is left-invariant if and only if it is L_g-related to itself for all $g \in G$. The space $\mathfrak{g} = T_eG$ with the Lie bracket induced from the bracket of vector fields is called the *Lie algebra of* G. That is, we define the Lie bracket on $\mathfrak{g}$ by

$$[\xi, \eta] := [\xi^L, \eta^L]_e.$$

For matrix Lie groups (i.e., closed subgroups of $GL(n, \mathbb{R})$), the Lie bracket coincides with the commutator of matrices. With this structure, $\mathfrak{g}$ becomes a Lie algebra, which we recall is simply a vector space with a bilinear skew-symmetric form that satisfies the Jacobi identity. Working with the space of right-invariant vector fields, $\mathfrak{X}^R(G)$, composed of the right-invariant vectors $\xi^R_g = R_{g,*}\xi \equiv \xi g$, would have produced the opposite bracket:

$$[\xi^R, \eta^R]_e = -[\xi^L, \eta^L]_e.$$

One way to see it is to notice that the inverse map inv has differential at the identity

$$d_e\mathrm{inv} = -\mathrm{id}$$

so that $\mathrm{inv}_*\xi^L = -\xi^R$, and thus $[\xi^R, \eta^R]$ is inv-related to $[\xi^L, \eta^L]$.

We now let $F^t_\xi : G \to G$ be the flow of ξ^L and define the *exponential map of* G

$$\exp : \mathfrak{g} \to G : \xi \mapsto \exp(\xi) := F^1_\xi(e).$$

The flow of ξ^L is $(g, t) \mapsto g\exp(t\xi)$ and exp is well-defined. Additionally, the flow of ξ^R is then $(g, t) \mapsto \exp(t\xi)g$. The exponential map for matrix groups is just the matrix exponential.

We denote the adjoint map $\mathrm{Ad}_g := d_ec_g \equiv c_{g,*} : \mathfrak{g} \to \mathfrak{g}$, for the conjugation map $c_g : G \to G : h \mapsto ghg^{-1}$, i.e., $\mathrm{Ad}_g = L_{g,*} \circ R_{g^{-1},*}$. Since c_g is a Lie group homomorphism (a smooth group homomorphism), Ad_g is a Lie algebra homomorphism:

$$[\mathrm{Ad}_g\xi, \mathrm{Ad}_g\eta] = \mathrm{Ad}_g[\xi, \eta].$$

Regarding Ad as a linear action $\mathrm{Ad} : G \to GL(\mathfrak{g}) : g \mapsto \mathrm{Ad}_g$, we obtain another Lie algebra homomorphism $\mathrm{ad} := d_e\mathrm{Ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. We denote $\mathrm{ad}_\xi := \mathrm{ad}(\xi) \in \mathfrak{gl}(\mathfrak{g})$, and it turns out that this map is given again by the Lie bracket:

$$\mathrm{ad}_\xi(\eta) = \left.\frac{d}{dt}\right|_0 \mathrm{Ad}_{\exp(t\xi)}(\eta) = [\xi, \eta].$$

This can understood as an alternative definition of the Lie bracket on $\mathfrak{g}$. The fact that ad is a Lie algebra homomorphism is exactly the Jacobi identity for the Lie bracket.

Definition 2.8 (Lie group action) Let G be a Lie group. A *G-action on a manifold M* is a smooth map

$$\psi : G \times M \to M : (g, p) \mapsto \psi_g(p) \equiv gp$$

such that the maps ψ_g are diffeomorphisms and

$$G \to \mathrm{Diff}(M) : g \mapsto \psi_g$$

is a group homomorphism.

That is, the action map $G \times M \to M : (g, p) \mapsto gp := \psi_g(p)$ is smooth, ψ_g is a diffeomorphism for all $g \in G$, and

$$\psi_{gh} = \psi_g \circ \psi_h, \qquad \psi_e = \mathrm{id},$$

for all $g, h \in G$, and the unit element $e \in G$. Thus, we only consider smooth left actions. A manifold M together with a G-action is called a *G-manifold*. A map $F : M_1 \to M_2$ between two G-manifolds is called *G-equivariant* if it intertwines the G-actions, that is

$$gF(p) = F(gp)$$

for every $p \in M_1$.

There are three natural G-actions on itself:

- The action given by left multiplication, $(g, h) \mapsto \psi_g(h) := gh$.
- The left action given by right multiplication, $(g, h) \mapsto \psi_g(h) := hg^{-1}$. We will call this action the *right* G-action, even though it is a left action.
- The conjugation action $(g, h) \mapsto c_g(h) = ghg^{-1}$.

The following property of an action is crucial to take quotients:

Definition 2.9 (Proper action) A continuous G-action on the topological manifold M is *proper*, or G acts *properly* on M, if the map

$$G \times M \to M \times M : (g, p) \mapsto (\psi_g(p), p),$$

is proper, where $G \to \mathrm{Diff}(M) : g \mapsto \psi_g$ denotes the action.

The orbit map at $p \in M$ is

$$j_p : G \to M : g \mapsto \psi_g(p).$$

We will denote the orbit of a point $p \in M$ by $\mathcal{O}_p := G{\cdot}p := \operatorname{im} j_p = \{gp : g \in G\}$, or simply by $\mathcal{O}$. Orbit maps have constant rank and hence their image are immersed submanifolds. If the action is proper and free then they are closed embedded submanifolds of M, $j_p : G \to \mathcal{O}_p \subset M$.

Similarly, we define:

Definition 2.10 (Lie algebra action) Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-action M is a smooth vector bundle map

$$M \times \mathfrak{g} \to TM : (p, \xi) \mapsto \xi_M(p),$$

such that the map $\mathfrak{g} \to \mathfrak{X}(M) : \xi \mapsto \xi_M$ is a Lie algebra anti-homomorphism.

The main example of a $\mathfrak{g}$-action is that of the Lie algebra of G on a G-manifold M, called the *infinitesimal G-action*:

$$\mathfrak{g} \to \mathfrak{X}(M) : \xi \mapsto \xi^{\#}_M, \qquad \xi^{\#}_M(p) := \left.\frac{d}{dt}\right|_{t=0} \psi_{\exp t\xi}(p),\ p \in M.$$

In particular, $\xi^{\#}_M$ has flow $t \mapsto \psi_{\exp t\xi}$. We will omit the subindex M when the G-manifold is clear from context. Equivalently we can write:

$$\xi^{\#}_M(p) = d_e j_p(\xi).$$

Thus, with this notation, the tangent space of an orbit $\mathcal{O}_p$ is

$$T_p\mathcal{O}_p = \operatorname{im} d_e j_p = \{\xi^{\#}_p : \xi \in \mathfrak{g}\},$$

Since the stabilizer of $p \in M$ is the closed subgroup of G given by

$$G_p := \{g \in G : gp = p\},$$

its Lie algebra is given by the kernel of the orbit map,

$$\mathfrak{g}_p := \ker d_e j_p = \{\xi \in \mathfrak{g} : \xi^{\#}_p = 0\}.$$

To see that the infinitesimal G-action is in fact a $\mathfrak{g}$-action we prove:

Lemma 2.11 *The infinitesimal G-action $\xi \mapsto \xi^{\#}$ satisfies*

$$(\mathrm{Ad}_g \xi)^{\#} = \psi_{g,*}\xi^{\#}, \qquad [\xi, \eta]^{\#} = -[\xi^{\#}, \eta^{\#}].$$

In particular, the correspondence $\xi \mapsto \xi^{\#}$ is a Lie algebra anti-homomorphism.

We recall our notation $g\xi := L_{g,*}\xi$ and $\xi g := R_{g,*}\xi$. When it makes it clearer we will write $g\xi g^{-1} \equiv \mathrm{Ad}_g(\xi)$.

Proof We first notice that for all $g, h \in G$, $p \in M$,

$$\psi_g \circ j_p(h) = ghp = ghg^{-1}gp = j_{gp} \circ c_g(h),$$

so that $\psi_{g,*} \circ j_{p,*} = j_{gp,*} \circ \mathrm{Ad}_g$ and hence

$$(\psi_{g,*}\xi^{\#})(gp) = \psi_{g,*} \circ j_{p,*}(\xi) = j_{gp,*} \circ \mathrm{Ad}_g(\xi) = (\mathrm{Ad}_g\xi)^{\#}(gp).$$

Since the orbits gp cover all M, the first identity follows.

The second identity now follows from differentiation of the first: $\xi^{\#}(p) = d_e j_p(\xi)$ is linear in ξ and it follows that

$$[\xi, \eta]^{\#} = (\mathrm{ad}(\xi)(\eta))^{\#} = \frac{d}{dt}\bigg|_{t=0} (\mathrm{Ad}_{\exp(t\xi)}\eta)^{\#} = \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(t\xi),*}\eta^{\#}.$$

Since $\psi_{\exp(t\xi)}$ is by definition the flow of $\xi^{\#}$, the last term is just minus the Lie derivative of $\eta^{\#}$, and we get

$$[\xi^{\#}, \eta^{\#}] = \mathfrak{L}_{\xi^{\#}}(\eta^{\#}) = \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(-t\xi),*}\eta^{\#} = -[\xi, \eta]^{\#}. \qquad \square$$

We could also have argued for the second identity that $\xi^{\#}$ is j_p-related to the right-invariant vector field ξ^R. This implies that $[\xi^{\#}, \eta^{\#}]$ is j_p-related to $[\xi^R, \eta^R]$, and since $[\xi^R, \eta^R]_e = -[\xi, \eta]_e$ we obtain the claim. To see that $\xi^{\#}$ is j_p-related to ξ^R, we just identify in the following equation $L_{\exp(t\xi)}$ as the flow of ξ^R in G:

$$\xi^{\#} \circ j_p(g) = \xi^{\#}(gp) = \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(t\xi),*}(gp) = \frac{d}{dt}\bigg|_{t=0} j_p(\exp(t\xi)g) = j_{p,*}(\xi^R_g).$$

In particular this shows that the infinitesimal G-action is a $\mathfrak{g}$-action. If G is simply connected and M is compact, the converse is true: every $\mathfrak{g}$-action on M integrates to a G-action, but we will not need that here.

A smooth map $F : M_1 \to M_2$ between $\mathfrak{g}$-manifolds is $\mathfrak{g}$-*equivariant* if ξ_{M_2} is F-related to ξ_{M_1} for all $\xi \in \mathfrak{g}$, that is:

$$d_pF(\xi_{M_1}(p)) = \xi_{M_2}(F(p)).$$

Clearly, differentiating the G-equivariance condition for a G-equivariant F we obtain that F is also $\mathfrak{g}$-equivariant for the infinitesimal G-action.

Lastly, we notice that since the left self G-action commutes with the self right action, the generating vector field for the left action is right-invariant and its value at e is ξ. Alternatively, we see that $(g, t) \mapsto \exp(t\xi)g$ as defined for the infinitesimal action of the left action is the flow of ξ^R. In any case, the left action is generated by ξ^R. Similarly the right action is generated by $-\xi^L$, and the adjoint action by $-\xi^L + \xi^R$.

Any G-action on M gives rise to an action on TM and T^*M via $\psi_{g,*}$ and $\widehat{\psi_g}$, where given a diffeomorphism $f : M_1 \to M_2$ we denote its *cotangent lift* by $\widehat{f} := (df^{-1})^*$, i.e.:

$$\widehat{f} : T^*M_1 \to T^*M_2 : (p, v^*) \mapsto (f(p), (d_p f)^{-1,*} v^*).$$

If $p \in M$ is fixed under the action of G, then these actions induce linear G-actions (i.e., representations of G) on T_pM and T_p^*M. In particular, the conjugation action of G on itself induces the left adjoint action Ad on $\mathfrak{g}$ and the left coadjoint action on the dual of the Lie algebra, $\mathfrak{g}^*$, given by

$$G \times \mathfrak{g}^* \to \mathfrak{g}^* : (g, \mu) \mapsto \mathrm{Ad}^*_{g^{-1}}\mu := \mu \circ \mathrm{Ad}_{g^{-1}}.$$

We conclude with the main results about taking the quotient of a G-manifold over the equivalence relationship given by its orbits, i.e., $p, q \in M$ are related if and only if they are in the same orbit.

Theorem 2.12 (Quotient Manifold Theorem) *Suppose G is a Lie group acting smoothly, freely, and properly on a smooth manifold M. Then, the* orbit space *M/G given by the quotient of M over the equivalence relation given by the G-orbits is a topological manifold of dimension* $\dim M - \dim G$*, and has a unique smooth structure such that the quotient map $\pi : M \to M/G$ is a smooth submersion. Furthermore, with this smooth structure, $\pi : M \to M/G$ becomes a principal G-bundle.*

Proof We make reference to Theorem 21.10 of [18], where a proof of all except that $\pi : M \to M/G$ is a principal G-bundle can be found, using as a key ingredient the Frobenius Theorem 2.6 and the fact that the G-orbits are the leaves of the foliation associated to the distribution $D_p := \operatorname{im} d_e j_p$. For the interested reader, we deduce that last claim for a left action (the right case is similar). To see that $\pi : M \to M/G$ becomes a principal G-bundle, using the notation introduced in the proof of Theorem 21.10 of [18], we define

$$\Psi : G \times V \to \pi^{-1}(V) : (g, v) \mapsto g\sigma(v),$$

where $V \subset M/G$ is an open subset such that a smooth section $\sigma : V \to \pi^{-1}(V)$ of the projection $\pi : M \to M/G$ is defined. It is then easy to check that Ψ is a G-left-equivariant diffeomorphism that restricts to a diffeomorphism of the fibers of π. An inverse is given by

$$\Psi^{-1} : \pi^{-1}(V) \to G \times V : p \mapsto ((\sigma \circ \pi(p))^{-1}p, \pi(p)). \qquad \square$$

As a consequence one proves:

Theorem 2.13 *Let G be a Lie group and $H \subset G$ a closed subgroup. The left coset space $G/H := \{gH : g \in G\}$ is a topological manifold of dimension* $\dim G - \dim H$*, and has a unique smooth structure such that the quotient map $\pi_H : G \to G/H$ is a smooth submersion. The obvious left action of G on G/H turns G/H into a homogeneous G-space and $\pi_H : G \to G/H$ becomes a principal H-bundle that is G-left-equivariant.*

For a proof we refer to Theorem 21.17 of [18].

As we know, all homogeneous spaces are obtained in this manner. We have the corollary:

Corollary 2.14 *Let G be a Lie group and $H_1 \subset H_2 \subset G$ two closed subgroups. Consider the projection $p : G/H_1 \to G/H_2$ sending the coset gH_1 to gH_2. Then, p is an H_2/H_1-fiber bundle.*

Proof We apply the previous theorem's proof (see again Theorem 21.17 of [18]) to obtain a local trivialization around $g \in G$ of the principal H_2-bundle $\pi_2 : G \to G/H_2$ thanks to a submanifold $g \in Y \subset G$ such that $Y \times H_2 \to YH_2 : (y, h) \mapsto yh$ is a diffeomorphism with an open neighborhood YH_2 of g, and such that $\pi_2|_Y : Y \to \pi_2(Y)$ is a diffeomorphism ($\pi_2(Y)$ is thus an open neighbourhood of gH_2 in G/H_2). Since $H_1 \subset H_2$, $\pi_1|_Y : Y \to \pi_1(Y)$ is also a smooth bijection with smooth inverse given by $(\pi_2|_Y)^{-1} \circ p|_{\pi_1(Y)}$, and thus a diffeomorphism. Then, the map

$$YH_1/H_1 \times H_2/H_1 \to YH_2/H_1 : (yH_1, hH_1) \mapsto (\pi_1|_Y)^{-1}(yH_1)hH_1$$

is a diffeomorphism, and the local trivialization around gH_1 looked for. Its inverse is given by

$$\begin{aligned} YH_2/H_1 &\to YH_1/H_1 \times H_2/H_1 \\ gH_1 &\mapsto ((\pi_2|_Y)^{-1}(gH_2)H_1, ((\pi_2|_Y)^{-1}(gH_2))^{-1}gH_1). \end{aligned} \qquad \square$$

2.3 The Tubular Neighbourhood Theorem

In this final section we briefly introduce the setting for the Tubular Neighbourhood Theorem following [19]. Let (M, g) be a Riemannian manifold and $S \subset M$ an embedded submanifold of codimension k. We define the *normal bundle of S in M with respect to g*, $TS^\perp$, as the k-dimensional smooth vector subbundle of the restriction of TM to S given by

$$TS^\perp := \{(p, v) \in TM : p \in S,\ v \in T_pS^\perp\}.$$

The pointwise restriction of the projection to $TS^\perp$ (for any metric g) induces an isomorphism with the quotient vector bundle given by $NS := TM|_S/TS$, i.e.,

$N_pS := T_pM/T_pS \cong T_pS^{\perp}$, which gives an intrinsic definition independent of a metric. NS is called *the normal bundle*. Furthermore, let E be the restriction of the Riemannian exponential map of (M, g) to NS. A *tubular neighborhood* of S in M is an open subset $U \subset M$ that is the diffeomorphic image under E of an open neighbourhood $V_\delta \subset NS$ of S. Here, $V_\delta \subset NS$, for a smooth positive function $\delta : S \to (0, \infty)$, is defined as

$$V_\delta = \{(p, v) \in NS : ||v||_p < \delta(p)\}.$$

If furthermore the function δ is constant, say $\delta \equiv \varepsilon > 0$, then it is called an *$\varepsilon$-uniform tubular neighborhood.*

We note that any tubular neighborhood V is diffeomorphic to the whole normal bundle NS simply by re-scaling the $\delta(p)$-ball in N_pS to be all of N_pS, for example by

$$V_\delta \to NS : (p, v) \mapsto (p, \frac{v}{\delta(p) - ||v||_p}).$$

Theorem 2.15 (Tubular Neighbourhood Theorem) *Let (M, g) be a Riemannian manifold and an embedded submanifold $S \subset M$. Then, S has a tubular neighborhood in M. If S is compact, then it can be taken to be a uniform tubular neighborhood.*

A proof can be found in Theorem 5.25 of [19].

Corollary 2.16 *Given a smooth manifold M and an embedded submanifold $S \subset M$, there exists a tubular neighborhood $V \subset NS$ of S, an open neighborhood $U \subset M$ of S, and a diffeomorphism $\psi : V \to U$, such that $\psi \circ i = j$ for the inclusions $i : S \hookrightarrow V$ and $j : S \hookrightarrow U$.*

Proof We fix an arbitrary Riemannian metric g on M (this can always be done using partitions of unity) and apply the Tubular Neighbourhood Theorem. □

In particular, we get that the smooth vector bundle structure $p : V \to S$ translates via ψ into a smooth vector bundle structure $\pi : U \to S$. Furthermore, since $p : V \to S$ is a strong deformation retract of $S \subset V$ (via the linear collapse of every fiber), we obtain that $\pi : U \to S$ is a strong deformation retract of $S \subset U$:

Corollary 2.17 *Given a smooth manifold M and an embedded submanifold $S \subset M$, there exists an open neighbourhood $i : S \hookrightarrow U$ and a smooth strong deformation retract $\pi : U \to S$, i.e., there is a smooth homotopy $H : [0, 1] \times U \to U$ such that*

$$H_0 = \mathrm{id}_U, \ \ H_1 = i \circ \pi, \ \ H_t|_S = \mathrm{id}_S \ \forall t.$$

Lastly, we make the following remarks. Given a G-action on a Riemannian manifold (M, g) of a Lie group G, we say that the metric g is G-invariant if $\psi_h^* g = g$ for every $h \in G$, i.e., if G acts by isometries: $G \to \mathrm{Iso}(M, g) : h \mapsto \psi_h$.

If the group G is compact, then we can always obtain a G-invariant metric from any given metric g by averaging over the Haar measure of G, dh (with total mass $dh(G) = 1$):

$$\widehat{g} := \int_G \psi_h^* g dh.$$

Consider thus a G-invariant metric g on M. Then, isometries preserve geodesics and the Riemannian exponential map commutes with the G-action: $\exp_{hp} \circ L_{h,*} = h \exp_p$. Regarding $\exp : W \to M : (p, v) \mapsto \exp_p(v)$ as a map over an open subset $W \subset TM$ of the tangent bundle, we see that exp is G-equivariant with respect to the G-action on TM given by $h(p, v) := (hp, L_{h,*}v)$. Since this map restricted to an adequate set is precisely the diffeomorphism used to prove the Tubular Neighbourhood Theorem 2.15, we get:

Corollary 2.18 (Equivariant Tubular Neighbourhood Theorem) *Let G be a compact group acting on the Riemannian M with a G-invariant metric g, and an embedded, G-invariant submanifold $S \subset M$. Then, S has a G-invariant, tubular neighborhood in M. If S is compact, then it can be taken to be a uniform tubular neighborhood.*

Proof Note that since S is G-invariant, so is $TS \subset TM|_S$ and thus the G-action descends to NS. The only thing we need to check after the previous comments is that, given that G is compact, there always exists a (possibly smaller) G-invariant open neighbourhood of S in M: since the G-orbits $\mathcal{O}$ of S are compact, each one has an open set V such that $\mathcal{O} \subset GV \subset U$; then we cover S by finitely many of these and take the union of the GV. □

Repeating the same arguments for the corollaries of the Tubular Neighbourhood Theorem under a G-action for compact G, in particular choosing a G-invariant metric on M, we obtain equivariant versions thereof; in particular of Corollary 2.17, obtaining a G-equivariant retraction H_t.

Chapter 3

Symplectic geometry

In this chapter we introduce the main relevant definitions and basic results in symplectic geometry, starting with the linear case. After covering symplectic and Hamiltonian vector fields, the standard symplectic structure on cotangent bundles in introduced. Finally, a brief section about almost complex structures is followed by the so called 'local-results', that show how symplectic topology is locally flexible, since as we will see all symplectic manifolds are locally equivalent in the symplectic sense. The contents of this chapter have been predominantly drawn from [6, 7, 22, 23].

3.1 Symplectic linear algebra

We begin by briefly introducing the elemental notions of symplectic structures on vector spaces.

Definition 3.1 (Symplectic vector space) A *symplectic vector space* is a pair (V, ω) consisting of a real finite-dimensional vector space V and a non-degenerate skew-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$. The form ω is called a *symplectic form.*

That is, $\omega(u, v) = -\omega(v, u)$ for any $u, v \in V$, and if $\omega(u, v) = 0$ for all $v \in V$ then $u = 0$. In particular, since any skew-symmetric form has a non-trivial kernel in a odd-dimensional vector space, we see that every symplectic vector space (V, ω) has even dimension $\dim V = 2n$.

Example 3.2 The archetypal example of a symplectic vector space is the Euclidean space $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega_0 := \sum_{i=1}^{n} dx_i \wedge dy_i.$$

(We understand $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$, each copy of $\mathbb{R}^n$ with coordinates x_i and y_i, respectively). Another elementary example is $E \oplus E*$ for any finite-dimensional vector space E and its dual $E*$ with the symplectic form

$$\omega((u,u^*),(v,v^*)) := v^*(u) - u^*(v). \qquad \square$$

Definition 3.3 (Linear symplectomorphism) A linear isomorphism between symplectic vector spaces, $L : (V,\omega) \to (V',\omega')$, is a *linear symplectomorphism* if it preserves the symplectic structure: $L^*\omega' = \omega$. We denote the linear symplectomorphisms of (V,ω) with itself by $\mathrm{Sp}(V,\omega)$, with the notation $\mathrm{Sp}(\mathbb{R}^{2n},\omega_0) := \mathrm{Sp}(2n)$.

Similarly to the orthogonal complement of a linear subspace with respect to a given scalar product, we can define:

Definition 3.4 (Symplectic complement) The *symplectic complement* of a linear subspace $U \subset V$ is the subspace

$$U^\omega := \{w \in V : \omega(w,u) = 0, \ \forall u \in U\}.$$

Clearly, since ω is non-degenerate, it holds that $\dim U + \dim U^\omega = \dim V = 2n$, but U and U^ω need not be transversal. In fact, we define

Definition 3.5 (Lagrangian, symplectic, (co)isotropic subspace) A linear subspace $U \subset V$ is called

- *symplectic*, if $U \cap U^\omega = \{0\}$,
- *isotropic*, if $U \subset U^\omega$,
- *coisotropic*, if $U \supset U^\omega$,
- *Lagrangian*, if $U = U^\omega$.

In particular, U is isotropic if and only if ω vanishes when restricted to U, while it is symplectic if and only if $\omega|_U$ remains non-degenerate. Furthermore, since $U \subset (U^\omega)^\omega$ and since U and U^ω have complementary dimension, both U and $(U^\omega)^\omega$ have the same dimension and hence are equal:

$$U = (U^\omega)^\omega.$$

Thence, U is symplectic if and only if U^ω is symplectic and U is coisotropic if and only if U^ω is isotropic (and vice versa). Furthermore, U is Lagrangian if and only if it has half-dimension n and is either isotropic or coisotropic.

Theorem 3.6 *Let (V,ω) be a $2n$-dimensional symplectic vector space. Then, there exists a basis $\{u_1,\ldots,u_n,v_1,\ldots,v_n\}$ such that*

$$\omega(u_i,u_j) = \omega(v_i,v_j) = 0, \ \omega(u_i,v_j) = \delta_{ij}, \ \forall 0 \le i,j \le n.$$

Such a basis is called a symplectic basis *of (V,ω), or a ω-*standard basis.

In particular, this means that there always exists a linear symplectomorphism $\Psi : \mathbb{R}^{2n} \to V$ such that $\Psi^*\omega = \omega_0$ (sending the $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ to the standard basis).

Proof One possibility is to diagonalize the matrix associated to ω in some initial basis by conjugation, just as for symmetric bilinear forms (i.e., performing the elementary operations of row exchange, multiplication of a row by a scalar, and addition of a scalar multiple of a row to another, simultaneously by rows and columns). Since ω is non-degenerate, we can always obtain the matrix

$$\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}$$

Alternatively, we can argue by induction on n as follows. Since ω is non-degenerate, there exist u_1, v_1 such that $\omega(u_1, v_1) = 1$, so that the subspace U spanned by u_1, v_1 is symplectic. If we then restrict ω to the symplectic complement U^ω of dimension $2n-2$, which is also symplectic, then the induction hypothesis implies that there exists a symplectic basis $\{u_2, \dots, u_n, v_2, \dots, v_n\}$ of U^ω. Since they are in the symplectic complement of W, together they all provide a symplectic basis of V. □

Corollary 3.7 *Given a $2n$-dimensional vector space V and a skew-symmetric bilinear form ω on V, ω is non-degenerate if and only if its n-th wedge power is nonzero,*

$$\omega^n = \omega \wedge \dots \wedge \omega \neq 0.$$

Proof If ω is degenerate, then by choosing $0 \neq v_1 \in V$ in its kernel and completing it to a basis $\{v_1, \dots, v_{2n}\}$, it then follows that $\omega^n(v_1, \dots, v_{2n}) = 0$ and hence $\omega^n = 0$. Reciprocally, if ω is symplectic, then the previous result provides a symplectic basis where the fact that $\omega^n \neq 0$ is apparent, since expressed in terms of this basis we have

$$\frac{1}{n!}\omega_0^n = dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n,$$

which is the standard volume form of $\mathbb{R}^{2n}$. □

This motivates the following definition:

Definition 3.8 (Liouville form) Given a symplectic vector space (V, ω), the *Liouville form* is defined as $\omega^n/n!$.

Thus, the Liouville form associated to the standard symplectic form ω_0 on $\mathbb{R}^{2n}$ coincides with the standard volume form.

A key technique in symplectic geometry is the so-called *symplectic reduction* of a symplectic structure. We now introduce its linear version. Given a coisotropic subspace $U \subset V$, we can quotient it by its symplectic complement. Since we are eliminating precisely the kernel of the symplectic form restricted to U, we obtain a new symplectic space.

Lemma 3.9 (Linear symplectic reduction) *Let (V, ω) be a symplectic vector space and let $U \subset V$ be a subspace. We then have that*

(i) The quotient

$$\overline{U} := U/U \cap U^\omega$$

carries a unique symplectic structure $\overline{\omega}$ such that the restriction $\omega|_U$ agrees with the pullback of $\overline{\omega}$ under the projection $\pi : U \to \overline{U}$,

$$\pi^* \overline{\omega} = \omega|_U .$$

In particular, if U is coisotropic then U/U^ω carries a unique such symplectic structure.

(ii) If $L \subset V$ is a Lagrangian subspace, then the quotient

$$\overline{L} := ((L \cap U) + U \cap U^\omega)/U \cap U^\omega$$

is a Lagrangian subspace of the reduced symplectic vector space $(\overline{U}, \overline{\omega})$.

Proof *(i)* Denote by $[u] = u + u \cap U^\omega$ the class of $u \in U$. Clearly, $U \cap U^\omega$ is an isotropic subspace of U and $\omega(u, v) = 0$ for any $u \in U$, $v \in U \cap U^\omega$ so that $\omega(u_1, u_2)$ does not depend on the representative chosen of $[u_1], [u_2]$. Hence ω descends to a skew-symmetric bilinear form $\overline{\omega}$ in $\overline{U}$. Furthermore, we have that $0 = \overline{\omega}([u], [v]) = \omega(u, v)$ for all $[v] \in \overline{U}$ if and only if $u \in U \cap U^\omega$, that is, if and only if $[u] = 0$. Hence, $\overline{\omega}$ is a symplectic form.

(ii) We first see that the subspace

$$\tilde{L} := (L \cap U) + (U \cap U^\omega)$$

is a Lagrangian subspace of V, as one checks by using De Morgan's elementary identities for the ω-complement and that $L^\omega = L$:

$$\tilde{L}^\omega \cap U = [(L + U^\omega) \cap (U + U^\omega)] \cap U = (L \cap U) + (U \cap U^\omega) = \tilde{L}.$$

Considering now $[v] \in \overline{U}$, then we have that $\overline{\omega}([u], [v]) = 0$ for every $[u] \in \overline{L} = \tilde{L}/(U \cap U^\omega)$ if and only if $v \in \tilde{L}$ and hence if and only if $[v] \in \overline{L}$. $\square$

3.2 Symplectic structures

Let M be a smooth manifold without boundary.

Definition 3.10 (Symplectic structure) A *symplectic form* or a *symplectic structure* on M is a non-degenerate closed 2-form $\omega \in \Omega^2(M; \mathbb{R})$. Non-degenerate means that (T_pM, ω_p) is a symplectic vector space, that is, ω_p is a non-degenerate bilinear form on T_pM, for every $p \in M$.

A *symplectic manifold* is a pair (M, ω) where M is a manifold and ω a symplectic form.

Hence, a symplectic manifold must be of even dimension $2n$, and the n-th exterior product of ω

$$\omega^n = \omega \wedge \cdots \wedge \omega$$

never vanishes. In particular, M has even-dimension and is orientable and in fact oriented by ω^n.

Definition 3.11 (Symplectomorphism) A *symplectomorphism* is a diffeomorphism between symplectic manifolds, $f : (M_1, \omega_1) \to (M_2, \omega_2)$, that preserves the symplectic form, that is, $f^*\omega_2 = \omega_1$. We denote the subgroup of symplectomorphisms from (M, ω) to itself by

$$\mathrm{Symp}(M, \omega) := \{f \in \mathrm{Diff}(M) : f^*\omega = \omega\}.$$

Example 3.12 The canonical example of a symplectic manifold is the real vector space of even dimension, $\mathbb{R}^{2n}$, with the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge y_i. \qquad \square$$

Example 3.13 Another example is the unit 2-sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : ||x|| = 1\}$ with the standard area form

$$(u, v) \in T_x\mathbb{S}^2 \times T_x\mathbb{S}^2 \mapsto \omega_x(u, v) := \langle x, u \times v\rangle$$

for each $x \in \mathbb{S}^2$. The same construction can be extended to any 2-dimensional surface $\Sigma \subset \mathbb{R}^3$. Given a normal vector field $\nu : \Sigma \to \mathbb{S}^2$ such that $\nu(x) \perp T_x\Sigma$ for every $x \in \Sigma$, the standard area form

$$(u, v) \in T_x\mathbb{S}^2 \times T_x\mathbb{S}^2 \mapsto \omega_x(u, v) := \langle \nu(x), u \times v\rangle = \det(\nu(x), u, v),$$

for $x \in \Sigma$, is a symplectic structure on Σ.

More generally, every oriented surface is symplectic and a symplectic structure is just an area form, while a symplectomorphism is an area-preserving diffeomorphism. $\square$

There are two conditions in the definition of symplectic structure. The first is algebraic in nature: non-degeneracy. This establishes a canonical isomorphism between the tangent and cotangent bundles of M:

$$TM \to T^*M : X_p \mapsto (i_X\omega)_p := \omega_p(X, \cdot).$$

This can also be understood as a bijective correspondence

$$\mathfrak{X}(M) \to \Omega^1(M;\mathbb{R}) : X \mapsto i_X\omega.$$

Regarding notation, we will use the alternative notation $i_X\omega \equiv i(X)\omega$ whenever it makes equations more readable.

The second condition in the definition of a symplectic structure is differential: that of closedness. If we compare symplectic to Riemannian geometry, this condition is additional, since in the latter we only ask the algebraic condition of the metric being positive-definite. As we will see in Section 3.5 about Moser's trick, this sharply distinguishes both geometries: symplectic geometry is locally flexible since all symplectic manifolds are locally symplectomorphic, in contrast to Riemannian plentiful non-locally isometric manifolds. In other words, there are no local symplectic invariants.

Another consequence of closedness is that the symplectic form ω represents a cohomology class $a = [\omega] \in H^2(M;\mathbb{R})$. If M is closed and oriented according to ω^n, then the cohomology class $a^n \in H^{2n}(M;\mathbb{R})$ represented by the volume form ω^n has a non-vanishing integral. Hence, a cannot be zero, that is, ω cannot be exact. Equivalently, a has non-zero n-th cup product $a^n = a \cup \cdots \cup a \neq 0$ and represents a non-zero top-cohomology class. In particular, we always have $H^2(M;\mathbb{R}) \neq 0$ for a symplectic, closed manifold. From this we see that there are orientable, even-dimensional closed manifolds that do not admit a symplectic structure, such as all $2n$-spheres with $n \geq 2$, since then $H^2(\mathbb{S}^{2n};\mathbb{R}) = 0$.

As in the previous subsection, we have the same corresponding symplectic substructures associated to a given symplectic structure. We let again (M,ω) be a symplectic manifold of dimension $2n$.

Definition 3.14 (Lagrangian, symplectic, (co)isotropic submanifold) A submanifold $S \subset M$ is *Lagrangian*, *symplectic*, or *(co)isotropic* if for every point $p \in S$ the tangent space T_pS, identified as the corresponding subspace of the symplectic vector space (T_pM, ω_p), is respectively Lagrangian, symplectic, or (co)isotropic.

We recall that we defined a submanifold to be an embedded submanifold, that is, a subset with the subspace topology and a compatible smooth structure such that the inclusion is an embedding, or equivalently, such that it has adapted local slice charts.

Analogously, a distribution (i.e. a subbundle of the tangent bundle) will be called *Lagrangian, symplectic, or (co)isotropic* whenever the subspace at every point is the corresponding subspace of the total tangent space at that point.

Example 3.15 (Lagrangian submanifolds) $\mathbb{RP}^n$ is a Lagrangian submanifold of $\mathbb{CP}^n$ with the Fubini-Study structure ω_{FS} (see Appendix A.1 for a detailed construction). This can be seen directly: $\mathbb{RP}^n$ has half-dimension and is isotropic because ω_1, as defined in Appendix A.1, vanishes when restricted to $\mathbb{R}^{n+1} \setminus \{0\} \subset \mathbb{C}^{n+1} \setminus \{0\}$. This in turn follows from the fact that $dz \wedge d\bar{z} = 0$ when restricted to $\mathbb{R} \subset \mathbb{C}$. $\square$

3.2.1 Symplectic and Hamiltonian vector fields

Symplectic vector fields A foundational notion in symplectic geometry, originated within its roots in classical mechanics, is the interplay between a symplectic structure and the smooth vector fields on the manifold.

Definition 3.16 (Symplectic vector field) A vector field $X \in \mathfrak{X}(M)$ is *symplectic* if the 1-form $i_X\omega$ is closed, that is, $di_X\omega = 0$. We denote the symplectic vector fields of a symplectic manifold by $\mathfrak{X}^{\mathrm{Symp}}(M, \omega)$.

Lemma 3.17 *A vector field $X \in \mathfrak{X}(M)$ is symplectic if and only if it preserves the symplectic form, $\mathfrak{L}_X\omega = 0$.*

Proof It follows from Cartan's formula $\mathfrak{L}_X = i_X d + di_X$ and the fact that $d\omega = 0$, so that $\mathfrak{L}_X\omega = di_X\omega$ for any $\omega \in \Omega^*(M; \mathbb{R})$. $\square$

This observation leads to the following Lemma:

Lemma 3.18 *Suppose $t \mapsto \psi_t \in \mathrm{Diff}(M)$ is a smooth family of diffeomorphisms generated by the time-dependent vector fields $X_t \in \mathfrak{X}(M)$ so that*

$$\left.\frac{d}{dt}\right|_t \psi_t = X_t \circ \psi_t, \;\; \psi_0 = \mathrm{id}.$$

Then, $\psi_t \in \mathrm{Symp}(M, \omega)$ for every t if and only if $X_t \in \mathfrak{X}^{\mathrm{Symp}}(M, \omega)$ for every t.

Proof Lemma 2.2 in the case where $\omega_t \equiv \omega$ implies that

$$\left.\frac{d}{dt}\right|_t \psi_t^*\omega = \psi_t^*\mathfrak{L}_{X_t}\omega.$$

Hence, it follows from the above lemma that $\psi_t^*\omega = \psi_0^*\omega = \omega$ for all t if and only if $\mathfrak{L}_{X_t}\omega = 0$ for all t. $\square$

The next proposition tells us that if two vector fields are symplectic, then its commutator is also symplectic.

Proposition 3.19 *Given two symplectic vector fields $X, Y \in \mathfrak{X}^{\mathrm{Symp}}(M,\omega)$, then*

$$i_{[X,Y]}\omega = d(\omega(Y,X)).$$

In particular, $\mathfrak{X}^{\mathrm{Symp}}(M,\omega) \subset \mathfrak{X}(M)$ is a Lie sub-algebra.

Proof Using Lemma 2.4 we compute

$$i_{[X,Y]}\omega = \mathfrak{L}_X(i_Y\omega) - i_Y\mathfrak{L}_X\omega = di_X(i_Y\omega) = d(\omega(Y,X)),$$

since $\mathfrak{L}_X\omega = di_X\omega = 0$ and $di_Y\omega = 0$. $\square$

Hamiltonian vector fields Given a symplectic vector field $X \in \mathfrak{X}^{\mathrm{Symp}}(M,\omega)$, we can ask ourselves whether the closed form $i_X\omega$ is in fact exact. Hence, we consider a smooth function $H \in \mathcal{C}^\infty(M,\mathbb{R})$. Its differential gives a 1-form dH and via the isomorphism $\mathfrak{X}(M) \cong \Omega^1(M;\mathbb{R})$ induced by the symplectic structure, we get a unique vector field X_H such that

$$i_{X_H}\omega = dH.$$

The vector field X_H is uniquely characterised by this equation. Reciprocally, given a vector field $X \in \mathfrak{X}(M)$ we can ask whether there exists a function H_X such that the above equation holds. This leads to the following notion.

Definition 3.20 (Hamiltonian vector field) A vector field $X \in \mathfrak{X}(M)$ is *Hamiltonian* if the 1-form $i_X\omega$ is exact, that is, there exists some function $H_X \in \mathcal{C}^\infty(M,\mathbb{R})$ such that $i_X\omega = dH_X$. We denote the Hamiltonian vector fields of a symplectic manifold by $\mathfrak{X}^{\mathrm{Ham}}(M,\omega)$. The function H_X is called *a Hamiltonian function* or just a *Hamiltonian* associated to X. Reciprocally, given a function $H \in \mathcal{C}^\infty(M,\mathbb{R})$, we denote the unique Hamiltonian vector field associated to H by X_H.

Notice that given a Hamiltonian vector field X, its Hamiltonian function H_X is uniquely defined only up to a locally constant function.

The map $\mathfrak{X}^{\mathrm{Symp}}(M,\omega) \to H^1(M;\mathbb{R}) : X \mapsto [i_X\omega]$ is surjective, in virtue of the non-degeneracy of ω. Hence, Hamiltonian vector fields complete the short exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{X}^{\mathrm{Ham}}(M,\omega) \longrightarrow \mathfrak{X}^{\mathrm{Symp}}(M,\omega) \longrightarrow H^1(M;\mathbb{R}) \longrightarrow 0,$$

(regarding $H^1(M;\mathbb{R})$ as a commutative Lie algebra). In particular, we get that $\mathfrak{X}^{\mathrm{Ham}}(M,\omega) \subset \mathfrak{X}^{\mathrm{Symp}}(M,\omega)$. The fact that $\mathfrak{X}^{\mathrm{Symp}}(M,\omega) \to H^1(M;\mathbb{R})$ is a Lie algebra homomorphism, or equivalently that $\mathfrak{X}^{\mathrm{Ham}}(M,\omega)$ is a Lie sub-algebra, is just Proposition 3.19: the commutator of any two symplectic vector fields $X, Y \in \mathfrak{X}^{\mathrm{Symp}}(M,\omega)$ is Hamiltonian with Hamiltonian function $\omega(Y,X)$, i.e.:

$$X_{\omega(Y,X)} = [X,Y].$$

In particular, $\mathfrak{X}^{\mathrm{Ham}}(M,\omega) \subset \mathfrak{X}^{\mathrm{Symp}}(M,\omega)$ is an ideal and since additionally

$$[\mathfrak{X}^{\mathrm{Symp}}(M,\omega), \mathfrak{X}^{\mathrm{Symp}}(M,\omega)] \subset \mathfrak{X}^{\mathrm{Ham}}(M,\omega),$$

and the quotient $\mathfrak{X}^{\mathrm{Symp}}(M,\omega)/\mathfrak{X}^{\mathrm{Ham}}(M,\omega) \cong H^1(M;\mathbb{R})$ is abelian.

The obstruction for a symplectic vector field to be Hamiltonian is thus of topological and furthermore global nature. Global because manifolds are locally contractible (euclidean) so that every symplectic vector field is locally Hamiltonian, but not necessarily globally. If however all 1-forms are exact, i.e., $H^1(M;\mathbb{R}) = 0$, then all symplectic vector fields are Hamiltonian.

Summarizing, all the definitions fit together nicely regarded as restrictions of the isomorphism $\mathfrak{X}(M) \to \Omega^1(M;\mathbb{R}) : X \mapsto i_X\omega$:

$$\begin{aligned} \mathfrak{X}(M) &\overset{\cong}{\longleftrightarrow} \Omega^1(M;\mathbb{R}), \\ \mathfrak{X}^{\mathrm{Symp}}(M,\omega) &\overset{\cong}{\longleftrightarrow} Z^1(M), \\ \mathfrak{X}^{\mathrm{Ham}}(M,\omega) &\overset{\cong}{\longleftrightarrow} B^1(M), \end{aligned}$$

for the 1-forms $\Omega^1(M;\mathbb{R})$, closed 1-forms $Z^1(M)$, and exact 1-forms $B^1(M)$.

Given a Hamiltonian vector field X_H with Hamiltonian H, Proposition 3.18 states that the associated flow φ_H^t is a smooth family of symplectomorphisms. If X_H is complete, for example if M is closed, we obtain a 1-parameter group of symplectomorphisms.

Definition 3.21 (Hamiltonian flow) The flow homomorphism $t \mapsto \varphi_H^t$ associated to a complete Hamiltonian vector field X_H such that

$$\left.\frac{d}{dt}\right|_t \varphi_H^t = X_H \circ \varphi_H^t, \;\; \varphi_H^0 = \mathrm{id}.$$

is called the *Hamiltonian flow* associated to H.

If we compute

$$dH(X_H) = i_{X_H}\omega(X_H) = \omega(X_H, X_H) = 0$$

we see that H is preserved by the flow lines of X_H, that is, X_H is tangent to the level sets of H. Hence, the Hamiltonian flow φ_H^t is a family of symplectomorphisms that preserves H.

Example 3.22 Consider the 2-sphere minus the north and south poles $\mathbb{S}^2 \setminus \{(0,0,\pm 1)\}$ and the polar coordinates $\{\theta, x_3\}$ given by radial projection from the vertical axis x_3. It is easy to see that $\omega = d\theta \wedge dx_3$ is the standard area form induced from the euclidean metric with the standard orientation of $\mathbb{S}^2$. In particular, this means that the projection to the cylinder is area preserving.

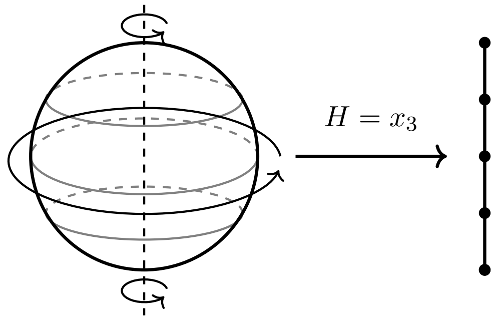

Figure 3.1: Hamiltonian flow on $(\mathbb{S}^2 \setminus \{(0,0,\pm 1)\}, \omega = d\theta \wedge dx_3)$ associated to $H = x_3$.

Thus, ω induces a symplectic structure in $\mathbb{S}^2$. If we take as Hamiltonian function $H = x_3$, then it is immediate that the associated Hamiltonian vector field is the coordinate field of θ, $X_H = \partial_\theta$. Thus the Hamiltonian flow φ_H^t rotates the sphere t radians around the vertical axis (see Figure 3.1).

Definition 3.23 (Poisson bracket) The *Poisson bracket* of the smooth functions $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$ is defined as

$$\{f, g\} := \omega(X_f, X_g),$$

for the Hamiltonian vector fields induced respectively by f and g.

Alternatively, we can express the bracket as

$$\{f, g\} = i_{X_f}\omega(X_g) = df(X_g) = X_g(f) = -X_f(g).$$

Proposition 3.24 *The Poisson bracket induces a Lie algebra structure in $\mathcal{C}^\infty(M, \mathbb{R})$. In particular, the Jacobi identity holds: for all $f, g, h \in \mathcal{C}^\infty(M, \mathbb{R})$, we have*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Proof We only need to prove that the Jacobi identity holds. One way to do it is to argue locally using the standard symplectic coordinates provided by the Darboux Theorem 3.44, where we can assume to be working in $(\mathbb{R}^{2n}, \omega_0)$ and it becomes an easy computation.

Here we will do so by proving a more general formula. Particularly, consider a not necessarily closed, non-degenerate 2-form $\tau \in \Omega^2(M; \mathbb{R})$. Then, the same definitions work for X_f (i.e. $i_{X_f}\tau = df$) and for $\{f, g\} := \tau(X_f, X_g)$. Then we have:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = d\tau(X_f, X_g, X_h).$$

The claim follows for the closed form $\tau := \omega$. To see it, we recall that

$$\begin{aligned} d\tau(X, Y, Z) &= X(\tau(Y, Z)) + Y(\tau(Z, X)) + Z(\tau(X, Y)) \\ &\qquad - \tau([X, Y], Z) - \tau([Y, Z], X) - \tau([Z, X], Y). \end{aligned}$$

From Lemma 2.4 we have $i_{[X,Y]}\tau = [\mathfrak{L}_X, i_Y]\tau$ so that

$$\begin{aligned} i_{[X_f,X_g]}\tau &= \mathfrak{L}_{X_f} \circ i_{X_g}\tau - i_{X_g} \circ \mathfrak{L}_{X_f}\tau \\ &= (i_{X_f} \circ d + d \circ i_{X_f}) \circ i_{X_g}\tau - i_{X_g} \circ (i_{X_f} \circ d + d \circ i_{X_f})\tau \\ &= d \circ i_{X_f} \circ i_{X_g}\tau - i_{X_g} \circ i_{X_f} \circ d\tau \\ &= -d\{f,g\} - i_{X_g} \circ i_{X_f}(d\tau). \end{aligned}$$

Substituting in the previous expression of $d\tau$ and using both the definition of the bracket and the above formula, we get

$$\begin{aligned} d\tau(X_f, X_g, X_h) &= X_f(\{g,h\}) + X_g(\{h,f\}) + X_h(\{f,g\}) \\ &\quad + X_h(\{f,g\}) + d\tau(X_f, X_g, X_h) \\ &\quad + X_f(\{g,h\}) + d\tau(X_g, X_h, X_f) \\ &\quad + X_g(\{h,f\}) + d\tau(X_h, X_f, X_g) \\ &= 2(\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}) + 3d\tau(X_f, X_g, X_h). \end{aligned}$$

□

In fact, $(\mathcal{C}^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$ is a *Poisson algebra*, that is, an algebra with a Lie algebra structure $\{\cdot,\cdot\}$ that satisfies the Leibniz rule for the algebra product, in this case pointwise multiplication. Indeed:

$$\{fg, h\} = X_h(fg) = X_h(f)g + fX_h(g) = f\{g,h\} + g\{f,h\}.$$

Similarly, Proposition 3.19 applied to Hamiltonian vector fields gives

Proposition 3.25 *Given smooth functions $f, g \in \mathcal{C}^\infty(M,\mathbb{R})$, we have that*

$$X_{\{f,g\}} = -[X_f, X_g].$$

An immediate corollary is that if $f, g \in \mathcal{C}^\infty(M,\mathbb{R})$ Poisson commute, that is, if $\{f,g\} = 0$, then their Hamiltonian flows also commute. In particular, the surjective correspondence

$$\mathcal{C}^\infty(M,\mathbb{R}) \to \mathfrak{X}^{\mathrm{Ham}}(M,\omega) : f \mapsto X_f$$

is a Lie algebra anti-homomorphism whose kernel is given by the locally constant functions. We get the short exact sequence

$$0 \longrightarrow Z^0(M) \longrightarrow \Omega^0(M;\mathbb{R}) \longrightarrow \mathfrak{X}^{\mathrm{Ham}}(M,\omega) \longrightarrow 0,$$

where $\Omega^0(M;\mathbb{R}) = \mathcal{C}^\infty(M,\mathbb{R})$ and $Z^0(M)$ denote the closed 0-forms, i.e. locally constant functions.

Proposition 3.26 *Given a symplectomorphism $\psi \in \mathrm{Symp}(M, \omega)$ and $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$, we have that*

1. $X_{f\circ\psi} = \psi_*^{-1} X_f$.
2. $\{f \circ \psi, g \circ \psi\} = \{f, g\} \circ \psi$.

Proof 1. It follows from the computation

$$d(f \circ \psi) = d\psi^* f = \psi^* df = \psi^* i_{X_f}\omega = i_{\psi_*^{-1}X_f}\psi^*\omega = i_{\psi_*^{-1}X_f}\omega,$$

the last identity since ψ is a symplectomorphism so that $\psi^*\omega = \omega$.

2. Using the previous result and that ψ^{-1} is also a symplectomorphism, we compute

$$\begin{aligned}\{f \circ \psi, g \circ \psi\} &= \omega(X_{f\circ\psi}, X_{g\circ\psi}) = \omega(\psi_*^{-1}X_f, \psi_*^{-1}X_g) \\ &= (\psi^{-1,*}\omega(X_f, X_g)) \circ \psi = \{f, g\} \circ \psi.\end{aligned}$$ □

3.3 The cotangent bundle

One of the most important examples of symplectic manifolds are cotangent bundles, also part of the origins of symplectic geometry within classical mechanics (in the shape of phase spaces of position and momentum for a given physical system).

Let M^n be a smooth manifold and T^*M its cotangent bundle, i.e. the vector bundle with fiber equal to the dual of the tangent space,

$$T^*M := \bigcup_{p\in M} \{p\} \times T_pM^*,$$

and coordinate charts defined by, given a chart $\varphi : U \to W \subset \mathbb{R}^n$ for M,

$$\widehat{\varphi} : \pi^{-1}(U) \to W \times \mathbb{R}^m : (p, v^*) \mapsto (\varphi(p), (d_p\varphi)^{-1,*}v^*),$$

where $\pi : T^*M \to M$ is the bundle projection. If φ_i denote the individual coordinates and ∂_{φ_i} are the coordinate vector fields, $\{d\varphi_i\}_i$ is a moving basis of $\pi^{-1}(U)$ dual to $\{\partial_{\varphi_i}\}_i$. Then, the coordinates given by $\widehat{\varphi}$ are precisely $q_i := \varphi_i \circ \pi$ and p_i, where p_i are the coefficients in $\alpha = \sum_i p_i dq_i$ for any section $\alpha : U \to \pi^{-1}(U)$. We will call these *local cotangent coordinates.*

There is a *canonical* 1-form $\theta \in \Omega^1(T^*M; \mathbb{R})$ given as follows: for $m = (p, v^*) \in T^*M$, we define $\theta_m := (d_m\pi)^*v^*$, that is for $X_m = (u, u^*) \in T_m(T^*M)$:

$$\langle\theta_m, X_m\rangle = \langle v^*, d_m\pi(X_m)\rangle = v^*(u).$$

One can characterize θ as follows.

Proposition 3.27 *The canonical 1-form is the unique 1-form $\theta \in \Omega^1(T^*M;\mathbb{R})$ such that for every 1-form $\alpha \in \Omega^1(M;\mathbb{R})$ one has*

$$\alpha^*\theta = \alpha,$$

regarding in the left-hand term α as a map $\alpha : M \to T^*M$.

Proof Letting $u \in T_pM$, since α is a section then $d_p\alpha(u)$ projects to u, i.e., $d_{\alpha_p}\pi(d_p\alpha(u)) = u$. Thus

$$\langle(\alpha^*\theta)_p, u\rangle = \langle\theta_{\alpha_p}, d_p\alpha(u)\rangle = \langle\alpha_p, u\rangle.$$

Reciprocally, every non-vertical vector $X_m \in T_m(T^*M)$ (i.e., $d_m\pi(X_m) \neq 0$) can be expressed as $X_m = d_p\alpha(u)$ for $p = \pi(m)$, $u = d_m\pi(X_m)$, and some $\alpha \in \Omega^1(M;\mathbb{R})$. Since non-vertical vectors span $T_m(T^*M)$, uniqueness follows.$\square$

In local cotangent coordinates $(\widehat{\varphi}, U)$, θ takes the form

$$\theta|_{T^*U} = \sum_i p_i dq_i.$$

To see it, we apply the proposition with a section $\alpha = \sum_i \alpha_i dq_i : U \to T^*U$. Since $q_i(\alpha) = q_i$, $p_i(\alpha) = \alpha_i$, (omitting π), $\alpha^* \sum_i p_i dq_i = \sum_i \alpha_i dq_i = \alpha$.

Theorem 3.28 *The pair (T^*M, ω) for $\omega := -d\theta$ is a symplectic manifold.*

Proof In local cotangent coordinates $(\widehat{\varphi}, U)$, we have $w|_{T^*U} = \sum_i dq_i \wedge dp_i$.$\square$

We denote $\omega := -d\theta$ the *standard form* on the cotangent bundle T^*M of a manifold M.

Example 3.29 The graph of a closed form $\alpha \in \Omega^1(M;\mathbb{R})$ in $(T^*M, \omega = -d\theta)$ is a Lagrangian submanifold, since

$$\alpha^*\omega = -\alpha^*d\theta = -d\alpha^*\theta = -d\alpha = 0. \qquad \square$$

There is a natural way of obtaining symplectomorphisms and Hamiltonian vector fields on cotangent bundles. Let $f : M_1 \to M_2$ be a diffeomorphism and denote its *cotangent lift* by $\widehat{f} := (df^{-1})^*$, i.e.:

$$\widehat{f} : T^*M_1 \to T^*M_2 : (p, v^*) \mapsto (f(p), (d_pf)^{-1,*}v^*).$$

By definition, for $\alpha \in \Omega^1(M_1;\mathbb{R})$ we have the commuting diagram

$$\begin{array}{ccc} T^*M_1 & \xrightarrow{\widehat{f}} & T^*M_2 \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle(f^{-1})^*\alpha} \\ M_1 & \xrightarrow{f} & M_2. \end{array}$$

Proposition 3.30 (Naturality of the canonical 1-form) *Let $\widehat{f} : T^*M_1 \to T^*M_2$ be the cotangent lift of $f : M_1 \to M_2$. Then $\widehat{f}$ preserves the canonical 1-forms: $\widehat{f}^*\theta_2 = \theta_1$, and hence f^* is a symplectomorphism, $\widehat{f}^*\omega_2 = \omega_1$.*

Proof For $\alpha \in \Omega^1(M_1;\mathbb{R})$, we compute

$$\alpha^*(\widehat{f}^*\theta_2) = (\widehat{f}\circ\alpha)^*\theta_2 = ((f^{-1,*}\alpha)\circ f)^*\theta_2 = f^*((f^{-1,*}\alpha)^*\theta_2) = f^*(f^{-1,*}\alpha) = \alpha.$$

□

Since $\widehat{f \circ g} = \widehat{f} \circ \widehat{g}$, we have found a natural group homomorphism

$$\mathrm{Diff}(M) \to \mathrm{Symp}(T^*M, \omega) : f \mapsto \widehat{f}.$$

We will not deal with infinite dimensional Lie groups, but we will just say that this is a symplectic action on (T^*M, ω) of the infinite dimensional Lie group $\mathrm{Diff}(M)$.

Similarly, given a vector field $X \in \mathfrak{X}(M)$, we can lift it to $\widehat{X} \in \mathfrak{X}(T^*M)$ by first lifting its flow $t \mapsto \varphi^t_X \in \mathrm{Diff}(M)$ to $t \mapsto \widehat{\varphi^t_X} \in \mathrm{Diff}(T^*M)$ and then defining $\widehat{X}$ to be the generator of the lifted flow,

$$\varphi^t_{\widehat{X}} := \widehat{\varphi^t_X}, \quad \widehat{X} := \frac{d}{dt}\bigg|_{t=0} \widehat{\varphi^t_X}.$$

We call $\widehat{X}$ the *cotangent lift of* $X \in \mathfrak{X}(M)$. It can also be described as follows.

Proposition 3.31 *$\widehat{X}$ is the unique vector field in $\mathfrak{X}(T^*M)$ that is π-related to $X \in \mathfrak{X}(M)$ and that preserves the canonical 1-form, $\mathfrak{L}_{\widehat{X}}\theta = 0$.*

Proof By definition we have $\varphi^t_{\widehat{X}} = \widehat{\varphi^t_X}$, so that $\pi \circ \varphi^t_{\widehat{X}} = \varphi^t_X \circ \pi$, i.e., $\widehat{X}$ is π-related to X. Additionally, since $\widehat{f}^*\theta = \theta$ for any $f \in \mathrm{Diff}(M)$, the flow of $\widehat{X}$ preserves θ and thus $\mathfrak{L}_{\widehat{X}}\theta = 0$.

Reciprocally, if $Y \in \mathfrak{X}(T^*M)$ is π-related to $X \in \mathfrak{X}(M)$ then $\pi \circ \varphi^t_Y = \varphi^t_X \circ \pi$ so that Y is completely determined by the condition $\mathfrak{L}_Y\theta = 0$:

$$0 = \mathfrak{L}_Y\theta = (d \circ i_Y + i_Y \circ d)\theta = d(i_Y\theta) - i_Y\omega$$

is equivalent to $i_Y\omega = d(i_Y\theta)$. The function $i_Y\theta$ is given by $(p, v^*) \mapsto \theta_{(p,v^*)}(Y_{(p,v^*)}) = v^* \circ d_{(p,v^*)}\pi(Y_{(p,v^*)}) = v^*(X_p)$ and hence is completely determined by X. Since ω is symplectic, this in turn uniquely fixes Y. □

Again we will not handle this here but $X \mapsto \widehat{X}$ would correspond to the (minus) infinitesimal $\mathrm{Diff}(M)$-action on (T^*M, ω). However, we can check:

Proposition 3.32 *For all $X \in \mathfrak{X}(M)$, $\widehat{X}$ is a Hamiltonian vector field with Hamiltonian function $i_{\widehat{X}}\theta$ and the map*

$$\mathfrak{X}(M) \to \mathfrak{X}^{\mathrm{Ham}}(T^*M, \omega) : X \mapsto \widehat{X}$$

is a Lie algebra homomorphism.

Proof Firstly, by the same computation above, we see that $i_{\widehat{X}}\omega = d(i_{\widehat{X}}\theta)$, i.e., $\widehat{X}$ is the Hamiltonian vector field associated to the smooth function

$$i_{\widehat{X}}\theta : T^*M \to \mathbb{R} : (p, v^*) \mapsto v^*(X_p).$$

Secondly, we have that $\widehat{[X,Y]} = [\widehat{X}, \widehat{Y}]$. To see it we use the previous proposition. It is clear that $[\widehat{X}, \widehat{Y}]$ is π-related to $[X, Y]$ because both $\widehat{X}, \widehat{Y}$ are to X, Y respectively. Hence we only need to check that $[\widehat{X}, \widehat{Y}]$ preserves θ given that $\widehat{X}, \widehat{Y}$ do so, and indeed using Cartan's formula 2.3 and Lemma 2.4:

$$\begin{aligned}
\mathfrak{L}_{[\widehat{X},\widehat{Y}]}\theta &= (d \circ i_{[\widehat{X},\widehat{Y}]} + i_{[\widehat{X},\widehat{Y}]} \circ d)\theta \\
&= (d \circ [\mathfrak{L}_{\widehat{X}}, i_{\widehat{Y}}] + [\mathfrak{L}_{\widehat{X}}, i_{\widehat{Y}}] \circ d)\theta \\
&= d \circ \mathfrak{L}_{\widehat{X}} \circ i_{\widehat{Y}}\theta + \mathfrak{L}_{\widehat{X}} \circ i_{\widehat{Y}} \circ d\theta \\
&= \mathfrak{L}_{\widehat{X}} \circ (d \circ i_{\widehat{Y}}\theta + i_{\widehat{Y}} \circ d)\theta \\
&= \mathfrak{L}_{\widehat{X}} \circ \mathfrak{L}_{\widehat{Y}}\theta \\
&= 0.
\end{aligned}$$ □

In local coordinates, if $X|_U = \sum_i X_i \partial_{\varphi_i}$, then $i_{\widehat{X}}\theta(q,p) = \sum_i p_i X_i(q)$. Thus, from the local expression of ω we get that

$$\widehat{X}|_U = \sum_i X_i \partial_{q_i} - \sum_{ij} p_i \frac{\partial X_i}{\partial q_j} \partial_{p_j}.$$

Corollary 3.33 *The map*

$$\mathfrak{X}(M) \to \mathcal{C}^\infty(T^*M, \mathbb{R}) : X \mapsto i_{\widehat{X}}\theta$$

is a Lie algebra anti-homomorphism (for the Poisson structure on $\mathcal{C}^\infty(T^*M, \mathbb{R})$).

Proof Since $X \mapsto \widehat{X}$ preserves the Lie bracket, we only have to check that $\widehat{X} \mapsto i_{\widehat{X}}\theta$ is a Lie algebra anti-homomorphism. We compute:

$$\begin{aligned}
\{i_{\widehat{X}}\theta, i_{\widehat{Y}}\theta\} &= -d\theta(\widehat{X}, \widehat{Y}) \\
&= -\mathfrak{L}_{\widehat{X}}(\theta(\widehat{Y})) + \mathfrak{L}_{\widehat{Y}}(\theta(\widehat{X})) + \theta([\widehat{X}, \widehat{Y}]) \\
&= -\theta([\widehat{X}, \widehat{Y}]) + \theta([\widehat{Y}, \widehat{X}]) + \theta([\widehat{X}, \widehat{Y}] \\
&= \theta([\widehat{Y}, \widehat{X}]) \\
&= -i_{[\widehat{X},\widehat{Y}]}\theta.
\end{aligned}$$

We have used the definition of the Poisson bracket for $\omega = -d\theta$, the expression for the differential of a form, and that $\mathfrak{L}_{\widehat{X}}\theta = \mathfrak{L}_{\widehat{Y}}\theta = 0$ □

We finish with one more useful property.

Proposition 3.34 *For any diffeomorphism $f \in \mathrm{Diff}(M)$ and a vector field $X \in \mathfrak{X}(M)$, we have*

$$\widehat{f}_* \widehat{X} = \widehat{f_* X}.$$

Proof We just check that the flows of both vector fields coincide:

$$\varphi^t_{\widehat{f}_* \widehat{X}} = \widehat{f} \circ \varphi^t_{\widehat{X}} \circ \widehat{f}^{-1} = \widehat{f} \circ \widehat{\varphi^t_X} \circ \widehat{f}^{-1} = (f \circ \widehat{\varphi^t_X \circ f^{-1}}) = \widehat{\varphi^t_{f_* X}}. \qquad \square$$

3.4 Almost complex structures

For this section we follow the brief introduction found in [24]. We recall that a complex structure on a real vector space V is an automorphism $J : V \to V$ such that $J^2 = -\mathrm{id}_V$.

Definition 3.35 (Compatible complex structure) A complex structure $J : V \to V$ on the symplectic vector field (V, ω) is ω-*compatible* if

$$g_J(u, v) := \omega(u, Jv)$$

defines a positive definite inner product on V. We will denote all ω-compatible complex structures on V by

$$\mathcal{J}(V, \omega).$$

This means we are asking two conditions on J. Firstly, we ask that $\omega(\cdot, J\cdot)$ is symmetric, which is equivalent to J being a symplectomorphism, since

$$J^*\omega(u, v) = g_J(Ju, v), \qquad g_J(v, Ju) = \omega(v, J^2 u) = \omega(u, v).$$

It is clearly also equivalent to $J^* g_J = g_J$, and thus is equivalent to J being skew-adjoint with respect to the bilinear form g_J, $J^T = -J$. In particular, J is a unitary symplectomorphism whenever g_J defines an inner product. Secondly, we ask that $g_J(u, u) \geq 0$ for every $u \in V$ and that $g_J(u, u) = 0$ only if $u = 0$.

We equip $\mathcal{J}(V, \omega)$ with the subspace topology of $\mathrm{End}(V)$ (with the compact open topology).

Example 3.36 A compatible complex structure for $(\mathbb{R}^{2n}, \omega_0)$ is given by

$$J_0 e_i = f_i, \qquad J_0 f_i = -e_i.$$

This identifies $(\mathbb{R}^{2n}, J_0)$ with $\mathbb{C}^{2n}$ and its natural complex structure given multiplication by the complex unit $\sqrt{-1}$. $\square$

An ω-compatible complex structure $J : V \to V$ makes V into a Hermitian complex vector space with the Hermitian product

$$h_J(u, v) := g_J(u, v) + \sqrt{-1}\omega(u, v).$$

Clearly, h_J is complex-linear in the first variable and complex-antilinear in the second, for

$$h_J(u, Jv) = \sqrt{-1}h_J(u, v), \qquad h_J(Ju, v) = -\sqrt{-1}h_J(u, v),$$

and it is definite positive since $h_J(u, u) = g_J(u, u)$. Reciprocally, J will be compatible if this formula defines a Hermitian product.

The generalisation to smooth manifolds is that of almost complex structures:

Definition 3.37 (Almost complex structure) An *almost complex structure* on a smooth manifold M is an automorphism $J : TM \to TM$ of the tangent bundle such that $J^2 = -\mathrm{id}$, i.e., if $J_p : T_pM \to T_pM$ is a complex structure at every point $p \in M$. Consider further a smooth 2-form $\omega \in \Omega^2(M; \mathbb{R})$ on M. An almost complex structure J is *ω-compatible* if J_p is ω_p-compatible for every $p \in M$. We will denote all ω-compatible almost complex structures on M by

$$\mathcal{J}(M, \omega).$$

In particular, the smooth bilinear form g_J defined on the tangent bundle by

$$g_J(u, v) := \omega(u, Jv)$$

defines a Riemannian metric on M.

The next Theorem gives a convenient and canonical method for constructing compatible almost complex structures. We denote by $\mathrm{Riem}(V)$ the set of all inner products of V, i.e. of all symmetric, definite positive bilinear forms.

Theorem 3.38 *Let (V, ω) be a symplectic vector space. There is a canonical continuous and surjective map*

$$F : \mathrm{Riem}(V) \to \mathcal{J}(V, \omega).$$

Furthermore, the map $G : \mathcal{J}(V, \omega) \to \mathrm{Riem}(V) : J \mapsto g_J$ is a section of F, i.e. $F \circ G(J) = J$.

Proof Consider a scalar product $k \in \mathrm{Riem}(V)$ and let $A \in GL(V)$ be uniquely defined by the equation

$$k(u, v) = \omega(u, Av).$$

Since ω is skew symmetric, A is skew-adjoint with respect to k:

$$k(Au, v) = \omega(Au, Av) = -\omega(Av, Au) = -k(Av, u) = k(u, -Av).$$

It then follows that in the polar decomposition $A = J|A|$ for $|A| := (A^TA)^{1/2} = (-A^2)^{1/2}$ (note that A^TA is symmetric, positive semi-definite and hence diagonalizable with non-negative eigenvalues, and that J is unitary), J commutes with $|A|$. Thus, J is a complex structure: $J^2 = A(-A^2)^{-1/2}A(-A^2)^{-1/2} = -A^2A^{-2} = -\mathrm{id}_V$. Furthermore, the equation

$$\omega(u, Jv) = \omega(u, |A|^{-1}Av) = k(|A|^{-1/2}v, |A|^{-1/2}u)$$

shows that g_J defines a positive definite inner product, so that J is ω-compatible. Since $J := A(-A^2)^{-1/2}$ is continuous in A and A depends continuously on k (as can be checked arguing with concrete coordinates), we conclude that such a map F exists. Lastly, if $k = g_J$ is induced by some $J \in \mathcal{J}(V, \omega)$, then we see that by construction $A = J$, so that $A(-A^2)^{-1/2} = A = J$ and thus $F \circ G(J) = F(g_J) = J$. □

In particular, this theorem allows to canonically choose compatible complex structures that will maintain any continuous dependence. In fact, $\mathcal{J}(V, \omega)$ has the structure of a smooth manifold and F is smooth. Before proving this, we state another important consequence:

Corollary 3.39 *The space of ω-compatible complex structures $\mathcal{J}(V, \omega)$ is contractible.*

Proof Let $X = \mathrm{Riem}(V)$ and $Y = \mathcal{J}(V, \omega)$. Then X can be regarded as a convex subset of $GL(V)$, so that it is contractible. If we choose such a contraction $\varphi : I \times X \to X$ with $\varphi_0 = \mathrm{id}_X$ and φ_1 being constant, equal to some point of X, then we can define the contraction of Y given by $\psi := F \circ \varphi \circ (\mathrm{id}_I \times G)$, i.e.

$$\psi : I \times Y \to Y : (t, y) \mapsto F \circ \varphi_t \circ G(y).$$

Since $F \circ G = \mathrm{id}_{\mathcal{J}(V,\omega)}$, $\psi_0 = \mathrm{id}_{\mathcal{J}(V,\omega)}$ while ψ_1 is constant. □

We will prove that $\mathcal{J}(V, \omega)$ has a smooth structure by identifying it as a homogeneous space. To do so, let us fix some compatible complex structure $J \in \mathcal{J}(V, \omega)$, and fix the Hermitian product h_J. Then, the unitary maps are symplectomorphisms, $U(V, h_J) \subset \mathrm{Sp}(V, \omega)$, since in particular they will preserve the complex part of h_J. This can also be seen by taking any h_J-orthonormal complex basis $\{e_i^n\}$ and noting that together with $\{f_i := Je_i\}_i^n$ they are a symplectic basis of (V, ω) since

$$\omega(e_i, f_j) = \omega(e_i, Je_j) = g_J(e_i, e_j) = \delta_{ij},$$

and similarly $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$. Since a map in $U(V, h_J)$ takes an orthonormal basis to another orthonormal basis, it is also taking this symplectic basis to a symplectic basis and is thus symplectic.

Consider now another compatible complex structure $J' \in \mathcal{J}(V, \omega)$ and let $A : V \to V$ be the map sending an orthonormal basis with respect to h_J into one for $h_{J'}$. This again sends a symplectic basis to a symplectic basis and thus

$A \in \mathrm{Sp}(V,\omega)$ and satisfies $A^*J' := A^{-1}J'A = J$. This shows that the action of $\mathrm{Sp}(V,\omega)$ on $\mathcal{J}(V,\omega)$ given by

$$\mathrm{Sp}(V,\omega) \times \mathcal{J}(V,\omega) \to \mathcal{J}(V,\omega) : (\psi, J) \mapsto \psi^*J := \psi^{-1}J\psi$$

is transitive. Furthermore, the stabilizer of J is given by all ψ such that $\psi^*J = J$, or equivalently such that $g_{\psi^*J} = g_J$. Since $g_{\psi^*J}(\cdot,\cdot) = \omega(\cdot, \psi^{-1}J\psi\cdot) = \omega(\psi\cdot, J\psi\cdot) = \psi^*g_J$ we conclude that $\psi^*J = J$ if and only if $\psi \in U(V, h_J)$. We thus have obtained:

Corollary 3.40 *The action of the symplectic group* $\mathrm{Sp}(V,\omega)$ *on the space* $\mathcal{J}(V,\omega)$ *of* ω*-compatible complex structures is transitive, with stabilizer at* $J \in \mathcal{J}(V,\omega)$ *equal to the unitary group* $U(V,h_J)$*. In particular,* $\mathcal{J}(V,\omega)$ *may be viewed as a homogeneous space*

$$\mathcal{J}(V,\omega) = \mathrm{Sp}(V,\omega)/U(V,h_J).$$

This shows in particular that $\mathrm{Sp}(V,\omega)$ is connected and that $\mathcal{J}(V,\omega)$ is a non-compact smooth manifold of dimension $(2n^2+n) - n^2 = n^2 - n$ if $\dim V = 2n$ (note that $U(V,h_J)$ is a compact subgroup and hence the quotient has a smooth structure). Notice that the quotient topology coincides with the subspace topology induced from $\mathcal{J}(V,\omega) \subset \mathrm{End}(V)$. In particular, Theorem 3.38 can now be strengthened to state that the map F is smooth: $F(k) := J$ with $J := A(-A^2)^{-1/2}$ for $k(\cdot,\cdot) = \omega(\cdot, A\cdot)$ is smooth since all eigenvalues of $-A^2$ are strictly positive (A is invertible).

Additionally, we will make use of this theorem with the additional requirement:

Corollary 3.41 *Let* (V,ω) *be a symplectic vector space and consider a finite number* k *of symplectic subspaces* $\{U_i\}_i^k$ *in direct sum, i.e.* $\bigoplus_i^k U_i = V$*. Then there is a canonical smooth map*

$$\widetilde{F} : \bigoplus_i^k \mathrm{Riem}(U_i) \to \mathcal{J}(V,\omega).$$

Furthermore, the map $G : \mathcal{J}(V,\omega) \to \mathrm{Riem}(V) : J \mapsto g_J$ *is a section of* $\widetilde{F}$*, i.e.* $\widetilde{F} \circ G(J) = J$*.*

Proof We simply apply Theorem 3.38 to each subspace U_i and consider the direct sum of $F_i(g_i)$ for the inner product g_i on every subspace U_i. $\square$

We will make use of this result in the next section in the following way. We consider a smooth vector bundle over a manifold such that each fiber has a symplectic structure depending smoothly on the base point (this will be defined later as a *symplectic vector bundle*). Then, given smooth *symplectic subbundles* that intersect pairwise in the zero section (if necessary we add the subbundle given by the symplectic complementary to the sum of the rest so that their sum is direct), we can choose smooth metrics on each subspace (using partitions of unity) and apply the corollary to obtain an ω-compatible almost complex structure that *preserves* these subbundles.

3.5 Local forms and Moser's trick

A fundamental technique in symplectic geometry is the so-called Moser's trick. It allows the construction of isotopies of a symplectic manifold that provide strong results about the local flexibility of symplectic structures. Particularly, Darboux Theorem states that all symplectic manifolds are locally symplectomorphic. This is in stark contrast to Riemannian geometry, where curvature is an intrinsic invariant that distinguishes locally isometric manifolds. Thence, as we said, in symplectic geometry there are no non-trivial local invariants; the emphasis lies instead in finding global invariants.

Here, local is interpreted in slightly different ways. Localization around a point yields Darboux Theorem 3.44, but it is also possible to localize around a submanifold and obtain similar symplectomorphic equivalence results as we will see in Lemma 3.42. All of these revolve around the following technique due to Moser in [25].

Moser's trick In a nutshell, Moser's argument considers a smooth family of symplectic forms $\omega_t \in \Omega^2(M;\mathbb{R})$ such that its time derivative is exact, that is, the differential of a smooth family $\sigma_t \in \Omega^1(M;\mathbb{R})$:

$$\left.\frac{d}{dt}\right|_t \omega_t = d\sigma_t.$$

It then obtains a family of diffeomorphisms $\psi_t \in \mathrm{Diff}(M)$ such that

$$\psi_t^* \omega_t = w_0, \ \forall t.$$

To do so, we consider the flow ψ_t of a time-dependent vector field X_t, i.e.,

$$\left.\frac{d}{dt}\right|_t \psi_t = X_t \circ \psi_t, \ \psi_0 = \mathrm{id}.$$

Then, we choose X_t so that $\psi_t^*\omega_t$ is constant. Since ω_t is closed for all t, using Lemma 2.2 and the hypothesis about the time-derivative of ω_t, the time derivative of $\psi_t^*\omega_t$ is given by

$$\left.\frac{d}{dt}\right|_t \psi_t^*\omega_t = \psi_t^*(\mathfrak{L}_{X_t}\omega_t + \left.\frac{d}{dt}\right|_t \omega_t) = \psi_t^*(i_{X_t} \circ d\omega_t + d \circ i_{X_t}\omega_t + d\sigma_t) = \psi_t^* d(i_{X_t}\omega_t + \sigma_t).$$

It is then enough to ask

$$i_{X_t}\omega_t + \sigma_t = 0. \tag{3.1}$$

The non-degeneracy of ω_t for all t determines a unique smooth vector field X_t for each family σ_t. The only remaining argument is to ensure that we can integrate X_t to obtain the flow ψ_t. If M is closed, then this is always possible. This will be the setup in everything that follows. If M is not closed, then we need to verify that the solutions of the differential equation exist for the required time interval. The difficulty in applying Moser's trick generally appears while verifying that one can in fact find the smooth family σ_t.

Lemma 3.42 (Moser isotopy) *Let M be a $2n$-dimensional manifold and $S \subset M$ a compact submanifold. Suppose we have two closed 2-forms $\omega_0, \omega_1 \in \Omega^2(M;\mathbb{R})$ such that at every point $p \in S$ the forms ω_0 and ω_1 are equal and non-degenerate on T_qM.Then, there exist open neighbourhoods U_0, U_1 of S and a diffeomorphism $\psi : U_0 \to U_1$ such that*

$$\psi^*\omega_1 = \omega_0,\ \ \psi|_S = \mathrm{id}_S.$$

Proof After the previous comments on Moser's argument, it will be enough to find a 1-form $\sigma \in \Omega^1(U_0;\mathbb{R})$ in an open neighbourhood $S \subset U_0$ such that

$$(\omega_1 - \omega_0)|_{U_0} = d\sigma,\ \ \sigma|_{T_SM} = 0, \tag{3.2}$$

where T_SM is the pullback of the tangent bundle TM via the inclusion $S \subset M$. If we assume this, we may consider in U_0

$$\omega_t := \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + td\sigma,\ \ t \in [0,1].$$

Since $(d\sigma)_p = 0$ at any $p \in S$ and ω_0 is non-degenerate in T_SM, we may reduce U_0 so that ω_t is non-degenerate for all $t \in [0,1]$. By construction $\frac{d}{dt}\omega_t = d\sigma$, and Moser's trick provides us with the vector field X_t such that (3.1) holds. Finally, integrating this vector field provides the desired flow ψ_t such that $\psi^*\omega_t = \omega_0$ and we take $\psi := \psi_1$ after reducing again U_0 so that all solutions are defined in $t \in [0,1]$, given the compactness of S and the fact that solutions on S are defined for all $t \in \mathbb{R}$. This is a consequence of $X_t|_S \equiv 0$, that in turn follows from $\sigma|_{T_SM} = 0$. Finally this also ensures that $\psi_t|_S = \mathrm{id}_S$ for all t.

Hence, we only have to find $\sigma \in \Omega^1(U_0;\mathbb{R})$ satisfying (3.2). For this, we will use the homotopy operator used to prove Poincaré's Lemma in combination with a tubular neighborhood U of S in M and the strong deformation retract $H : [0,1] \times U \to U$ given by Corollary 2.17. For this, since H_t is a diffeomorphism for $t > 0$ we let $v_t(p) := (\frac{d}{dt}|_t H_t)(H_t^{-1}(p))$ for $t > 0$ (note that for each $t > 0$, v_t is only defined in $H_t(U)$), $v_0 \equiv 0$, $i : S \hookrightarrow U$ and $\pi := H_0 : U \to S$. We define the *Poincaré operator*

$$Q : \Omega^k(U;\mathbb{R}) \to \Omega^{k-1}(U;\mathbb{R}) : \omega \mapsto \int_0^1 H_t^*(i_{v_t}\omega)dt.$$

It is well defined since for all $t > 0$ we are only computing i_{v_t} on $H_t(U)$, within the domain of definition of v_t. Direct computation shows that it is a *homotopy operator* between $H_0 = \mathrm{id}_U$ and $H_1 = i \circ \pi$:

$$\begin{aligned}(Q \circ d + d \circ Q)\omega &= \int_0^1 H_t^*(i_{v_t} d\omega)dt + d\int_0^1 H_t^*(i_{v_t}\omega)dt \\ &= \int_0^1 H_t^*(i_{v_t} d + d \circ i_{v_t})\omega dt \\ &= \int_0^1 H_t^* \mathfrak{L}_{v_t}\omega dt \\ &= \int_0^1 \frac{d}{dt}\bigg|_t H_t^*\omega dt \\ &= (\mathrm{id}_U - i \circ \pi)^*\omega.\end{aligned}$$

In the second to last identity we have used Lemma 2.2 noting that H_t is the flow of v_t for $t > 0$ in $H_t(U)$, and hence $H_t^*\mathfrak{L}_{v_t}\omega = \frac{d}{dt}|_t H_t^*\omega$ for $t > 0$. The same identity at $t = 0$ follows then from continuity.

Restricting further so that $U_0 \subset U$ we get, since $d\omega_i = 0$ and $i^*(\omega_1 - \omega_0) = 0$,

$$\omega_1 - \omega_0 = ((i \circ \pi)^* + Q \circ d + d \circ Q)(\omega_1 - \omega_0) = d\sigma,$$

for $\sigma := Q(\omega_1 - \omega_0)$. Lastly, it is clear that $\sigma|_{T_SM} = 0$ since $(\omega_1 - \omega_0)|_{T_SM} = 0$.□

As we did for the equivariant version of the Tubular Neighbourhood Theorem, Theorem 2.18, we observe that a G-equivariant version of Moser isotopy Theorem 3.42 follows given a compact group G acting on M. Firstly, Moser's trick will yield a G-equivariant family of symplectomorphisms $t \mapsto \psi_t$ whenever the generating vector fields $t \mapsto X_t$ are G-invariant. In turn this follows if both ω_t and σ_t are G-invariant. Hence, in Moser isotopy's proof we just need to check that the $\sigma \in \Omega^1(U_0; \mathbb{R})$ we obtain such that $\omega = d\sigma$ is G-invariant (U_0 can be chosen G invariant arguing as in 2.18). To do this, we just note that the Poincaré operator is G-equivariant (i.e. $\psi_g^* Q = Q\psi_g^*$) in virtue of the equivariance of the homotopy H_t provided by the Equivariant Tubular Neighbourhood Theorem 2.18. Thus, if ω_i are G-invariant, so will $\sigma = Q(\omega_1 - \omega_0)$ be. We have just proved that:

Lemma 3.43 (Equivariant Moser isotopy) *Let G be a compact Lie group acting over a $2n$-dimensional manifold M and $S \subset M$ a compact, G-invariant submanifold. Suppose we have two closed, G-invariant 2-forms $\omega_0, \omega_1 \in \Omega^2(M; \mathbb{R})$ such that at every point $p \in S$ the forms ω_0 and ω_1 are equal and non-degenerate on T_pM. Then, there exist open G-invariant neighbourhoods U_0, U_1 of S and a G-equivariant diffeomorphism $\psi : U_0 \to U_1$ such that*

$$\psi^*\omega_1 = \omega_0, \;\; \psi|_S = \mathrm{id}_S.$$

Our first application of Moser's isotopy Theorem is the Darboux Theorem, the geometric version of Proposition 3.6 about symplectic vector spaces all being symplectomorphic.

Theorem 3.44 (Darboux, [10]) *Any symplectic structure (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic structure.*

Proof Consider any $p \in M$ and local coordinates $\varphi : U \to \mathbb{R}^{2n}$ around $p \in U$. Then, we have two symplectic structures in U, ω and the pullback of the standard symplectic structure $\varphi^*\omega_0$. Furthermore, composing with a linear isomorphism we can suppose that $\omega|_p = \varphi^*\omega_0|_p$. We now apply the Moser isotopy Lemma to the case where $S = \{p\}$ is a single point to obtain a diffeomorphism ψ of U (after possibly reducing it) such that

$$\psi^*\omega = \varphi^*\omega_0.$$

Taking $\varphi \circ \psi^{-1}$ we get a symplectomorphism between (U, ω) and an open neighbourhood of 0 with the canonical symplectic structure ω_0. □

In particular, if we choose the coordinates given by the previous diffeomorphism, we can write

$$\omega = \sum_i^n dx_i \wedge dy_i.$$

3.5.1 Neighbourhood theorems

Darboux's Theorem can be strengthened to provide local statements not just around points but around compact submanifolds. The first versions of these theorems were proven by Weinstein, such as the Weinstein Lagrangian Embedding Theorem [28]. We will prove a more general statement due to Marle [21], following [24].

We start with the next theorem, making the required setup for the Moser isotopy Lemma 3.42 more flexible. Before it we briefly introduce the concept of:

Definition 3.45 (Symplectic vector bundle) A *symplectic vector bundle* over a manifold M is a pair (E, ω) consisting of a smooth real vector bundle $\pi : E \to M$ and a family of symplectic forms $\omega_p : E_p \times E_p \to \mathbb{R}$ over the fibers $E_p := \pi^{-1}(p)$ for each $p \in M$ that varies smoothly over $p \in M$.

Two symplectic vector bundles (E_j, ω_j), $j = 0, 1$ are *isomorphic* if there exists a vector bundle isomorphism $\Psi : E_0 \to E_1$ such that $\Psi^*\omega_1 = \omega_0$.

Here smooth over $p \in M$ means that $p \mapsto \omega_p(X_p, Y_p)$ is smooth when evaluated over smooth sections X, Y of E. Hence, $(\omega_p)_p$ fit together to give a smooth section of the exterior power $E^* \wedge E^*$ for the dual bundle $E^* = \mathrm{Hom}(E, \mathbb{R})$.

Theorem 3.46 *Let (M_j, ω_j), $j = 0, 1$ be two symplectic manifolds and $i_j : S_j \hookrightarrow M_j$ two compact submanifolds. Suppose there exists a diffeomorphism $\psi : S_0 \to S_1$ covered by a symplectic vector bundle isomorphism*

$$\widehat{\psi} : TM_0|_{S_0} \to TM_1|_{S_1}$$

such that $\widehat{\psi}$ restricts to the tangent map $\psi_* : TS_0 \to TS_1$. Then, $\widehat{\psi}$ extends to a symplectomorphism $\Psi : U_0 \to U_1$ from an open neighborhood U_0 of S_0 in M_0 to an open neighborhood U_1 of S_1 in M_1.

Proof In virtue of the Tubular Neighbourhood Theorem 2.15, for each $j = 0, 1$ there are tubular neighbourhoods $V_j := V_{\delta_j} \subset TS_j^{\perp}$ of the zero section $S_j \times \{0\} \subset TS_j^{\perp}$ (for some smooth functions $\delta_j : S_j \to (0, \infty)$), and open neighbourhoods U_j of S_j on M_j such that

$$f_j : V_j \to U_j$$

is a diffeomorphism such that $f_j \circ k_j = i_j$, for the inclusions $k_j : S_j \times \{0\} \hookrightarrow TS_j^{\perp}$. For each $j = 0, 1$, lifting these maps to the tangent bundles we obtain the commutative diagram

$$\begin{tikzcd} T(TS_j^{\perp}) & \supset & TV_j \arrow[r, "\widehat{f}", "\cong"'] & TU_j & \subset & TM_j \\ & & TS_j. \arrow[u, hook, "\widehat{k}_j"] \arrow[ur, hook, "\widehat{i}_j"'] & & & \end{tikzcd}$$

Since we are arguing locally, we can assume that $M_j = U_j$, and further that $M_j = V_j$ via the f_j, after the key observation that the new induced bundle isomorphism $\widehat{\psi}' : TV_0|_{S_0} \to TV_1|_{S_1}$ still restricts to the tangent map $\psi_* : TS_0 \to TS_1$. To see it, we just note that we get another commutative diagram

$$\begin{tikzcd} TV_1|_{S_1} \arrow[r, "\widehat{\psi}'", "\cong"'] \arrow[d, "\widehat{f}_0"', "\cong"] & TV_1|_{S_1} \arrow[d, "\widehat{f}_1"', "\cong"] \\ TM_0|_{S_0} \arrow[r, "\widehat{\psi}", "\cong"'] & TM_1|_{S_1} \\ TS_0 \arrow[r, "\psi_*", "\cong"'] \arrow[u, hook, "\widehat{i}_0"] \arrow[uu, hook, bend left=60, "\widehat{k}_0"] & TS_1 \arrow[u, hook, "\widehat{i}_1"'] \arrow[uu, hook, bend right=60, "\widehat{k}_1"'] \end{tikzcd}$$

Thus, from $\widehat{\psi} \circ \widehat{i_0} = \widehat{i_1} \circ \psi_*$ we get that $\widehat{\psi}' \circ \widehat{k_0} = \widehat{k_1} \circ \psi_*$. All in all, we may assume that each S_j is the zero section of the vector bundle $V_j \to S_j$, and that we have an isomorphism

$$\widehat{\psi} : V_0 \to V_1$$

such that its restriction to the zero sections is a diffeomorphism preserving the symplectic forms. We can finally pullback ω_1 via $\widehat{\psi}$ and work only with the submanifold $S := S_0 \subset M := V_0$ and two symplectic structures ω_0 and $\omega_1 := \widehat{\psi}^*\omega_1$ that coincide at S, $\omega_0|_S = \omega_1|_S$. The Moser isotopy Lemma 3.42 now furnishes the result. $\square$

Just as before, the corresponding G-equivariant version for a compact group action follows through with no change in the proof other than noting that G-equivariance or invariance is maintained throughout (here we would ask for a G-equivariant symplectic bundle isomorphism $\widehat{\psi}$, and that the G-action acts by symplectic bundle isomorphisms, to obtain a G-equivariant symplectomorphism Ψ).

For the next result, we introduce here the *symplectic normal bundle* $TS^\omega/(TS\cap TS^\omega)$ of a submanifold $i : S \hookrightarrow M$, given by

$$p \mapsto T_pS^{\omega_p}/(T_pS \cap T_pS^{\omega_p}).$$

Similarly, TS^ω denotes the symplectic complement subbundle $p \mapsto (T_pS)^{\omega_p}$.

Theorem 3.47 (Embedding Theorem in symplectic geometry) *Let (M_j, ω_j), $j = 0, 1$ be two symplectic manifolds and $i_j : S_j \hookrightarrow M_j$ two compact submanifolds. Let*

$$F_j := TS_j^{\omega_j}/(TS_j \cap TS_j^{\omega_j})$$

be their symplectic normal bundles. Suppose further that there exists a symplectic bundle isomorphism

$$\widehat{\psi} : F_0 \to F_1$$

covering a diffeomorphism $\psi : S_0 \to S_1$ such that

$$\psi^* i_1^* \omega_1 = i_0^* \omega_0.$$

Then, ψ extends to a symplectomorphism $\Psi : U_0 \to U_1$ from an open neighborhood U_0 of S_0 in M_0 to an open neighborhood U_1 of S_1 in M_1, such that Ψ induces $\widehat{\psi}$.

Proof Consider first some compact submanifold $i : S \hookrightarrow M$ of a symplectic manifold (M, ω). We have three natural symplectic vector bundles over S:

$$\begin{aligned}
E &:= TS/(TS \cap TS^\omega) \\
F &:= TS^\omega/(TS \cap TS^\omega) \\
G &:= (TS \cap TS^\omega) \oplus (TS \cap TS^\omega)^*,
\end{aligned}$$

where in E and F we have the symplectic forms induced on each quotient by ω and in G we have the standard symplectic form on $V \times V^*$ defined in Example 3.2, that is:

$$((u, u^*), (v, v^*)) \mapsto v^*(u) - u^*(v).$$

We can identify E and F with subbundles intersecting in the zero section and in turn jointly complementary to $(TS \cap TS^\omega)$ in $TS + TS^\omega$; i.e., we fix a choice for a splitting of the short exact sequence

$$0 \longrightarrow TS \cap TS^\omega \hookrightarrow TS + TS^\omega \xrightarrow{\pi_E \oplus \pi_F} E \oplus F \longrightarrow 0.$$

Here, the projections π_E, π_F are

$$\pi_E : TS + TS^\omega \to (TS + TS^\omega)/TS^\omega \cong TS/(TS \cap TS^\omega) = E,$$

for the natural isomorphism $TS/(TS \cap TS^\omega) \to (TS + TS^\omega)/TS^\omega : [u] \mapsto [u]$; and similarly for F. To choose the splitting, we proceed as follows: we fix a Riemannian metric on $TS+TS^\omega$, we identify F with the orthogonal complement of TS, and then we identify E with the orthogonal complement of $TS \cap TS^\omega$ within TS. By doing so, we fix a right-inverse $f : TS + TS^\omega \to TS \cap TS^\omega$ to the inclusion $TS \cap TS^\omega \hookrightarrow TS+TS^\omega$ and a left-inverse $h : E \oplus F \to TS+TS^\omega : (a, b) \mapsto h_E(a) + h_F(b)$ to the projection $\pi_E \oplus \pi_F$. This allows us to write

$$TS + TS^\omega \cong E \oplus F \oplus (TS \cap TS^\omega) : u \mapsto (\pi_E(u), \pi_F(u), f(u)),$$

with inverse

$$E \oplus F \oplus (TS \cap TS^\omega) \cong TS + TS^\omega : (a, b, v) \mapsto h_E(a) + h_F(b) + v,$$

as an isomorphism of bundles preserving the 2-forms defined on each of them (notice that ω_p vanishes in $T_pS \cap T_pS^{\omega_p}$).

By Corollary 3.41 and its following comment, there exists some ω-compatible almost complex structure J on $TM|_S$ preserving the two symplectic vector bundles E and F. Then, the isotropic subbundle $J(TS \cap TS^\omega) \subset TM|_S$ is a complement to $TS + TS^\omega$ (it is the orthogonal complement with respect to g_J), which is in turn identified via g_J with $(TS \cap TS^\omega)^*$:

$$\begin{array}{ccccc} J_p(T_pS \cap T_pS^{\omega_p}) & \longrightarrow & T_pS \cap T_pS^{\omega_p} & \longrightarrow & (T_pS \cap T_pS^{\omega_p})^* \\ & & & & \\ J_pu & \longmapsto & u & \longmapsto & g_{J,p}(u, \cdot) = (v \mapsto g_{J,p}(u, v)), \end{array}$$

i.e. given that $g_{J,p}(u, v) = \omega_p(-J_pu, v))$, it is the natural isomorphism $\beta : u \mapsto -i_u\omega_p$. This finally shows that $TM|_S \cong E \oplus F \oplus G$ via

$$\begin{aligned} \alpha : (TS + TS^\omega) \oplus J(TS + TS^\omega) &\to E \oplus F \oplus G \\ u + v &\mapsto (\pi_E(u), \pi_F(u), (f(u), -i_v\omega)), \end{aligned}$$

naturally and as symplectic vector bundles. In particular, it gives a symplectic isomorphism between $(TS \cap TS^{\omega}) \oplus J(TS \cap TS^{\omega})$ and G since for $(u_1, u_2), (v_1, v_2) \in (T_pS \cap T_pS^{\omega_p}) \oplus J_p(T_pS \cap T_pS^{\omega_p})$ we have:

$$\begin{aligned}\omega_p(u_1 + u_2, v_1 + v_2) &= \omega_p(u_1, v_2) + \omega_p(u_2, v_1) = -i_{v_2}\omega_p(u_1) + i_{u_2}\omega_p(v_1) \\ &= \beta(v_2)(u_1) - \beta(u_2)(v_1),\end{aligned}$$

(i.e., the symplectic structure defined on $V \oplus V^*$ for any vector space V in Example 3.2). In order to conclude, we choose such isomorphisms $\alpha_j : TM_j|_{S_j} \to E_j \oplus F_j \oplus G_j$ for each $j = 0, 1$. By hypothesis we also have a symplectic bundle isomorphism $\widehat{\psi} : F_0 \to F_1$ covering a diffeomorphism $\psi : S_0 \to S_1$ such that $\psi^* i_1^* \omega_1 = i_0^* \omega_0$. This last identity implies that $\psi_* : TS_0 \to TS_1$ induces bundle isomorphisms $E_0 \cong E_1 : [u] \mapsto [\psi_*(u)]$ and $G_0 \cong G_1 : (u, u^*) \mapsto (\psi_*(u), u^* \circ \psi_*^{-1})$, both clearly symplectic. We denote the direct sum of both by $\overline{\psi_*} : E_0 \oplus G_0 \cong E_1 \oplus G_1$. We define the symplectic bundle isomorphism

$$\widehat{\varphi} : E_0 \oplus F_0 \oplus G_0 \to E_1 \oplus F_1 \oplus G_1 : (u, v, w) \mapsto (\overline{\psi_*}(u), \widehat{\psi}(v), \overline{\psi_*}(w)),$$

and note that it induces a symplectic bundle isomorphism $\widehat{\varphi}' : TM_0|_{S_0} \to TM_1|_{S_1}$ after completing the upper rectangle of the diagram

$$\begin{tikzcd}
E_0 \oplus F_0 \oplus G_0 \arrow[r, "\widehat{\varphi}", "\cong"'] & E_1 \oplus F_1 \oplus G_1 \\
TM_0|_{S_0} \arrow[u, "\alpha_0", "\cong"'] \arrow[r, "\widehat{\varphi}'", "\cong"'] & TM_1|_{S_1} \arrow[u, "\alpha_1", "\cong"'] \\
TS_0 \arrow[u, hook] \arrow[uu, hook, bend left] \arrow[r, "\psi_*", "\cong"'] & TS_1. \arrow[u, hook] \arrow[uu, hook, bend right]
\end{tikzcd}$$

We define the inclusion maps $TS_j \hookrightarrow E_j \oplus F_j \oplus G_j$ to commute with α_j and the natural inclusions $TS_j \subset TMj|_{S_j}$. Furthermore, the outer rectangle of maps is also commutative, and thus the whole diagram is as well. To see it, we just notice that $f_j|_{TS_j \cap TS_j^{\omega}} = \mathrm{id}_{TS_j \cap TS_j^{\omega}}$ since it is a left-inverse to the inclusion, and hence $f_1 \circ \psi_*|_{TS_0 \cap TS_0^{\omega}} = \psi_* \circ f_0|_{TS_0 \cap TS_0^{\omega}}$. Similarly, the projections π_{E_j} commute with ψ_* because ψ_* preserves the symplectic form and thus commutes with taking the quotient over TS^{ω}, i.e., $\pi_{E_1}|_{TS_1} \circ \psi_* = \psi_* \circ \pi_{E_0}|_{TS_0}$. Since $\alpha_j|_{TS_j}$ only acts via f_j and π_{E_j} (recall our specific choice of splitting), we conclude that $\alpha_1|_{TS_1} \circ \psi_* = \psi_* \circ \alpha_0|_{TS_0}$.

Finally, the diagram is commutative and the map $\widehat{\varphi}' : TM_0|_{S_0} \to TM_1|_{S_1}$ restricts to the tangent map $\psi_* : TS_0 \to TS_1$. Theorem 3.46 now completes the proof. □

This result states that a neighborhood of a compact submanifold $i : S \hookrightarrow M$ is characterized up to symplectomorphism by $i^*\omega$ together with the symplectic normal bundle TS^ω. In particular, if N is coisotropic, a neighborhood is completely determined by $i^*\omega$. Again, the same comments about a G-equivariant formulation apply.

We finally obtain the following result. In Chapter 5 we will state and prove its corresponding equivariant version that we will then use to prove the Duistermaat–Heckman Theorem. We postpone that version, Theorem 5.3, until after we have introduced the concept of a moment map and Hamiltonian actions.

Theorem 3.48 (Coisotropic embedding, [13]) *Let (M_j, ω_j), $j = 0, 1$ be two symplectic manifolds of the same dimension and consider a common compact submanifold S with coisotropic embeddings $i_j : S \hookrightarrow M_j$ such that $i_0^*\omega_0 = i_1^*\omega_1$. Then, there exist open neighborhoods U_j of $i_j(S)$ in M_j, and a symplectomorphism $\psi : U_0 \to U_1$ such that $i_1 = \psi \circ i_0$.*

Proof Taking $S_0 := S_1 := S$ with $\psi := \mathrm{id}_S$, and noting that for a coisotropic submanifold the symplectic normal bundle is trivial, Theorem 3.47 furnishes the result. $\square$

Similar versions can be obtained analogously for Lagrangian or isotropic embeddings, the Lagrangian case as an immediate application of Theorem 3.48. In particular, the Lagrangian case admits a stronger formulation called the Weinstein Lagrangian Tubular Neighbourhood Theorem:

Theorem 3.49 (Weinstein Lagrangian Tubular Neighbourhood) *Let (M, ω) be a symplectic manifold and $i : L \hookrightarrow M$ a compact Lagrangian submanifold. There exists a neighborhood U_0 of L in M, a neighborhood U_1 of L in T^*L (with the standard symplectic structure of a cotangent bundle), and a symplectomorphism $\psi : U_0 \to U_1$ fixing L.*

Proof In virtue of Theorem 3.46, it is enough to find a symplectic bundle isomorphism $TM|_L \cong T(T^*L)|_L$ covering the identity map $\mathrm{id}_{TL} \equiv (\mathrm{id}_L)_*$. To do so, we choose a compatible almost complex structure J on M thanks to Corollary 3.41, so that $J(TM|_L) \subset TM|_L$ is a Lagrangian subbundle complementary to TL in $TM|_L$, and is thus isomorphic (by means of the symplectic form, as we did in the previous proof with $(TS \cap TS^\omega) \oplus J(TS \cap TS^\omega)$) to the dual bundle T^*L. It follows that

$$TM|_L \cong TL \oplus T^*L$$

as a symplectic vector bundle. The same argument applies to M replaced with T^*L, so that $TM|_L \cong TL \oplus T^*L \cong T(T^*L)|_L$, and this covers the identity map $(\mathrm{id}_L)_*$ since both isomorphisms do. $\square$

Chapter 4

Symplectic actions

In this chapter we study in more detail symplectic actions and the conditions under which the associated symplectomorphisms are Hamiltonian flows. This will lead to the notion of a moment map, a generalisation of the Hamiltonian function that formalizes the Noether principle: associated to every k-dimensional group symmetry of a mechanical system, there are k quantities that are preserved in the dynamical evolution of the system that can be used to reduce the number of degrees of freedom of the system by $2k$. The technique of symplectic reduction, cornerstone result in this thesis, formalises this phenomenon.

We first restrict ourselves to the example case of circle actions in order to develop an intuition and motivate the definitions. After introducing the general definition of a Hamiltonian action and obtaining the main results and properties of a moment map, we introduce coadjoint orbits as an important tool in obtaining the Normal Form Theorem in the next chapter. Symplectic reduction is then proven and another important tool introduced, namely symplectic reduction on cotangent bundles of Lie groups. Finally, we have included a section about torus-actions as one of the best understood examples of Hamiltonian actions. There, the key results of the Convexity Theorem and the Delzant Classification Theorem are presented. We conclude with some comments on orbifold singularities. We follow mainly [7, 3, 23, 24].

4.1 Circle actions

Definition 4.1 (Hamiltonian $\mathbb{S}^1$-action) A *Hamiltonian action* of $\mathbb{S}^1$ on (M, ω) is a one-parameter subgroup

$$\mathbb{R} \to \mathrm{Symp}(M, \omega) : t \mapsto \psi_t$$

of symplectomorphisms of M that is 2π-periodic, i.e., $\psi_{2\pi} = \mathrm{id}$, and which is the integral of a Hamiltonian vector field X_H. The Hamiltonian function

$H : M \to \mathbb{R}$ is called the *moment map* of the action.

We note again that H is only defined up to adding a constant. In particular, if M is compact it may be normalized so that $\int_M H\omega^n = 0$.

Lemma 4.2 *Suppose that we have a Hamiltonian $\mathbb{S}^1$ action on (M, ω) such that $\mathbb{S}^1$ acts freely on a regular level set $H^{-1}(\lambda) \subset M$ of the moment map H. Then, the quotient orbit space manifold*

$$B_\lambda := H^{-1}(\lambda)/\mathbb{S}^1$$

carries a canonical symplectic structure τ_λ which is characterized by the property that its pullback to $H^{-1}(\lambda)$ is the restriction $\omega|_{H^{-1}(\lambda)}$. The symplectic manifold $(B_\lambda, \tau_\lambda)$ is called the symplectic quotient *of (M, ω) at $\lambda \in \mathbb{R}$.*

Proof Consider the hypersurface $S = H^{-1}(\lambda)$. Since it has codimension 1, it is a coisotropic submanifold, and its tangent space at $p \in S$ has symplectic complement

$$(T_pS)^\omega = \{v \in T_pM : \omega(v, u) = 0,\ \forall u \in T_pS = \ker d_pH\}.$$

This subspace has dimension 1 and contains X_H (non zero since $dH \neq 0$ for λ is a regular value) because $d_pH(X_H) = \omega_p(X_H, X_H) = 0$, so that $(T_pS)^\omega$ is the line bundle generated by X_H. Therefore, Lemma 3.9 says that at each point $p \in S$, ω_p descends to the quotient $T_pS/(T_pS)^\omega$ and gives rise to a non-degenerate 2-form $\overline{\omega}_p$. Since the $\mathbb{S}^1$-action is Hamiltonian, we have that $\psi_t^*(\omega_{\psi_t(p)}) = \omega_p$ for all $p \in S$ and $t \in \mathbb{S}^1$, so that ω also descends to the quotient manifold $B_\lambda = S/\mathbb{S}^1$ (well-defined in virtue of Theorem 2.12). Now, $\omega = \pi^*\overline{\omega}$, with the pullback of the projection $\pi : S \to S/\mathbb{S}^1$, is closed, and hence so is $\overline{\omega}_p$ (notice π is a submersion so π^* is injective). □

The manifold thus obtained $(B_\lambda, \omega_\lambda)$ is called the *symplectic quotient* or *reduced space* of (M, ω) at the value of the moment map $\lambda \in \mathbb{R}$.

Example 4.3 Coming back to the Example 3.22 about $\mathbb{S}^2$, we see that the Hamiltonian flow associated to $H = x_3$ provides an $\mathbb{S}^1$-action on the fibers of H. However, in this case this action is not only free but transitive and hence the symplectic quotient is just a single point for all $\lambda \in (-1, 1)$. □

Example 4.4 (Circle action on $\mathbb{C}^{n+1}$) Consider $\mathbb{C}^{n+1}$ with the standard symplectic form of $\mathbb{R}^{2n+2}$

$$\omega_0 = \sum_i dx_i \wedge dy_i = \sum_i r_i dr_i \wedge d\theta_i = \frac{i}{2}\sum_i dz_i \wedge \overline{z}_i,$$

depending on whether we use standard, polar (of course only valid away from 0), or complex coordinates for each factor $\mathbb{C}$. We have the smooth $\mathbb{S}^1$-action given by the diagonal multiplication by the elements of $\mathbb{S}^1 \subset \mathbb{C}$, i.e.:

$$\mathbb{S}^1 \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} : (\xi, z) \mapsto \xi z = (\xi z_0, \dots, \xi z_n).$$

In particular consider

$$\mathbb{R} \to \mathrm{Diff}(\mathbb{C}^{n+1}) : t \mapsto (\psi_t : z \mapsto e^{ti}z).$$

Then, ψ_t is the flow associated to the vector field $\sum_i \partial_{\theta_i}$ (notice that in $\mathbb{R}^2 \equiv \mathbb{C}$ the vector ∂_θ at z coincides, as an element of $\mathbb{C}$, with iz). This vector field is the Hamiltonian vector field associated to the Hamiltonian function

$$H = -\frac{1}{2}||z||^2 + c,$$

for any constant $c \in \mathbb{R}$. Hence, this $\mathbb{S}^1$-action is Hamiltonian on $(\mathbb{C}^{n+1}, \omega_0)$. The codomain of such an H is $(-\infty, c]$, with $\lambda < c$ a regular value and level set given by the $(2n+1)$-sphere of radius $\sqrt{2(c-\lambda)}$. Fixing $c := 0$, the reduced space at $\lambda = -\frac{1}{2}$ is given by

$$H^{-1}(-\frac{1}{2})/\mathbb{S}^1 = \mathbb{S}^{2n+1}/\mathbb{S}^1 \cong \mathbb{CP}^n.$$

The reduced symplectic structure at $\lambda = -\frac{1}{2}$ is the unique form $\overline{\omega}$ such that

$$\mathrm{pr}_2^*\overline{\omega} = \omega_0|_{\mathbb{S}^{2n+1}},$$

where the map

$$\mathrm{pr}_2 : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$$

is the *Hopf fibration* describing $\mathbb{CP}^n$ as a quotient of the sphere (see also Appendix A.1). In virtue of Proposition A.1 we get the Fubini-Study symplectic structure on $\mathbb{CP}^n$, so that:

$$(\mathbb{C}^{n+1})_{-\frac{1}{2}} \cong (\mathbb{CP}^n, \omega_{FS})$$

Furthermore, if we instead reduce at some other regular value $\lambda < 0$ we have the $\mathbb{S}^1$-equivariant diffeomorphism

$$\psi : H^{-1}(\lambda) = \sqrt{2(-\lambda)}\mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1} : z \mapsto z/\sqrt{2(-\lambda)},$$

so that $(H^{-1}(\lambda), \omega_0|_{H^{-1}(\lambda)}) \cong (\mathbb{S}^{2n+1}, \psi^{-1,*}\omega_0|_{(H^{-1}(\lambda)})$, with $\psi^{-1,*}\omega_0|_{(H^{-1}(\lambda)} = -2\lambda\omega_0|_{\mathbb{S}^{2n+1}}$. Taking again the quotient, we see that the reduction at λ is symplectomorphic to

$$(\mathbb{C}^{n+1})_\lambda \cong (\mathbb{CP}^n, -2\lambda\omega_{FS}).$$

□

4.2 Hamiltonian actions and moment maps

In this section we generalise the definitions and results of the Hamiltonian circle actions to actions of general Lie groups. Where before we only had a family indexed by a one dimensional set, $t \mapsto \psi_t$, now for every $\xi \in \mathfrak{g}$ the family $t \mapsto \psi_{\exp t\xi}$ is a one-parameter group of symplectomorphisms and it is unclear how to find the Hamiltonian function for all. The moment map will provide the adequate generalization of the Hamiltonian function H.

4.2.1 Symplectic and Hamiltonian actions

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let (M, ω) be a symplectic G-manifold.

Definition 4.5 (Symplectic action) A G-action on (M, ω) is *symplectic* if it acts by symplectomorphisms, i.e., if there is a group homomorphism

$$G \to \mathrm{Symp}(M, \omega) : g \mapsto \psi_g.$$

In virtue of Proposition 3.18, this is equivalent to the infinitesimal G-action being such that

$$\mathfrak{g} \to \mathfrak{X}^{\mathrm{Symp}}(M, \omega) : \xi \mapsto \xi^{\#},$$

i.e., the 1-form $i_{\xi^{\#}}\omega$ is closed for all $\xi \in \mathfrak{g}$. Lemma 2.11 states that the correspondence $\mathfrak{g} \to \mathfrak{X}^{\mathrm{Symp}}(M, \omega)$ is a Lie algebra anti-homomorphism.

As a first application, using the equivariant version of the Moser isotopy Lemma 3.43, we get an equivariant Darboux Theorem:

Theorem 4.6 (Equivariant Darboux) *Consider a symplectic G-action on (M^{2n}, ω). Then, given a point $p \in M$ fixed by G, the local symplectomorphism with $(\mathbb{R}^{2n}, \omega_0)$ around p can be chosen G-equivariant.*

Proof Retracing the arguments used to prove the standard version of the Darboux Theorem 3.44, one sees that equivariance is preserved throughout.□

In the same way symplectic vector fields can be Hamiltonian, we define:

Definition 4.7 (Weakly Hamiltonian action) A G-action on (M, ω) is *weakly Hamiltonian* if $\xi^{\#}$ is Hamiltonian for each $\xi \in \mathfrak{g}$.

That is, the action is weakly Hamiltonian if we have a Lie algebra anti-homomorphism

$$\mathfrak{g} \to \mathfrak{X}^{\mathrm{Ham}}(M, \omega) : \xi \mapsto \xi^{\#}.$$

Equivalently, if $i_{\xi^{\#}}\omega$ is exact for every $\xi \in \mathfrak{g}$ so that there exists a Hamiltonian function H^{ξ} such that

$$i_{\xi^{\#}}\omega = dH^{\xi}.$$

Recall that the Hamiltonian functions H^{ξ} are only fixed up to a locally constant function.

Lemma 4.8 *We can choose H^{ξ} for each $\xi \in \mathfrak{g}$ such that $\xi \mapsto H^{\xi}$ is a linear map.*

Proof Let's denote a new choice of Hamiltonian functions $\tilde{H}^{\xi} = H^{\xi} + a_{\xi}$ for locally constant functions a_{ξ} for each $\xi \in \mathfrak{g}$. We fix a basis $\{\xi_i\}_i$ of $\mathfrak{g}$ and put $a_0 := -H^0$, so that $\tilde{H}^0 = 0$, and arbitrarily fix a_{ξ_i} for each i. Then for $\xi = \sum_i \lambda_i \xi_i$ we have that $H^\xi - \sum_i \lambda_i H^{\xi_i}$ is locally constant, and we can fix $a_\xi := -H^\xi + \sum_i \lambda_i H^{\xi_i} + \sum_i \lambda_i a_{\xi_i}$ so that $\tilde{H}^\xi - \sum_i \lambda_i \tilde{H}^{\xi_i} = 0$ and hence $\xi \mapsto \tilde{H}^\xi$ is linear. □

We will say that the correspondence $\xi \mapsto H^\xi$ is weakly Hamiltonian if it is linear and satisfies $i_{\xi^\#}\omega = dH^\xi$.

Definition 4.9 (Hamiltonian action) A G-action on (M,ω) is *Hamiltonian* if the map

$$\mathfrak{g} \to \mathcal{C}^\infty(M,\mathbb{R}) : \xi \mapsto H^\xi$$

can be chosen to be linear and G-equivariant with respect to the adjoint action on $\mathfrak{g}$,

$$G \times \mathfrak{g} \to \mathfrak{g} : (g,\xi) \mapsto \mathrm{Ad}_g(\xi),$$

and to the G-action on $\mathcal{C}^\infty(M,\mathbb{R})$ induced by the pullback of the action,

$$G \times \mathcal{C}^\infty(M,\mathbb{R}) \to \mathcal{C}^\infty(M,\mathbb{R}) : (g,f) \mapsto \psi^*_{g^{-1}} f = f \circ \psi_{g^{-1}}.$$

This translates into $\xi \mapsto H^\xi$ being linear and such that

$$H^{\mathrm{Ad}_g(\xi)} = H^\xi \circ \psi_{g^{-1}}.$$

Example 4.10 If G is abelian, we only need to check the weakly Hamiltonian condition, $i_{\xi^\#}\omega = dH^\xi$, and that H is constant under the G-action. □

The next lemma characterizes Hamiltonian actions.

Lemma 4.11 *Let $G \to \mathrm{Symp}(M,\omega) : g \mapsto \psi_g$ be a weakly Hamiltonian action, $\mathfrak{g} \to \mathfrak{X}^{\mathrm{Ham}}(M,\omega) : \xi \mapsto \xi^\#$ the infinitesimal action and $\mathfrak{g} \to \mathcal{C}^\infty(M,\mathbb{R}) : \xi \mapsto H^\xi$ a linear map such that $i_{\xi^\#}\omega = dH^\xi$ for every $\xi \in \mathfrak{g}$. Consider the following assertions:*

(i) The map $\xi \mapsto H^\xi$ is G-equivariant, i.e., the action is Hamiltonian:

$$H^{\mathrm{Ad}_g \xi} = H^\xi \circ \psi_{g^{-1}}$$

for all $\xi \in \mathfrak{g}, g \in G$.

(ii) The map $\xi \mapsto H^\xi$ is a Lie algebra homomorphism between $\mathfrak{g}$ and the Poisson structure on $\mathcal{C}^\infty(M,\mathbb{R})$:

$$H^{[\xi,\eta]} = \{H^\xi, H^\eta\}.$$

for all $\xi, \eta \in \mathfrak{g}$.

Then (i) implies (ii). If G is connected, then (ii) also implies (i).

Proof The fact that *(i)* implies *(ii)* follows from differentiation of the identity

$$H^{\mathrm{Ad}_{\exp(-t\xi)}(\eta)} = H^\eta \circ \psi_{\exp(t\xi)}$$

at $t = 0$ together with the linearity of $\xi \mapsto H^\xi$:

$$\begin{aligned} H^{[\xi,\eta]} &= H^{\frac{d}{dt}|_{t=0}\mathrm{Ad}_{\exp(t\xi)}(\eta)} = -\frac{d}{dt}\Big|_{t=0} H^{\mathrm{Ad}_{\exp(-t\xi)}(\eta)} = -\frac{d}{dt}\Big|_{t=0} H^\eta \circ \psi_{\exp(t\xi)} \\ &= -dH^\eta(\xi^\#) = -\omega(\eta^\#, \xi^\#) = \{H^\xi, H^\eta\}. \end{aligned}$$

To see that *(ii)* implies *(i)* assume that G is connected and take any $\xi, \eta \in \mathfrak{g}, g \in G$. Then from Lemma 2.11 we get that $(g\xi g^{-1})^\# = \psi_{g,*}\xi^\#$ so that $H^{g\xi g^{-1}}$ and $H^\xi \circ \psi_{g^{-1}} = \psi^*_{g^{-1}} H^\xi$ generate the same vector (recall from the first part of Lemma 3.26 that $X_{f\circ\psi} = \psi_*^{-1} X_f$) and hence their difference is locally constant. This in turn implies, using *(ii)* twice, that

$$\begin{aligned} H^{g^{-1}[\xi,\eta]g} &= H^{[g^{-1}\xi g, g^{-1}\eta g]} \\ &= \{H^{g^{-1}\xi g}, H^{g^{-1}\eta g}\} \\ &= \{\psi_g^* H^\xi, \psi_g^* H^\eta\} \\ &= \psi_g^*\{H^\xi, H^\eta\} \\ &= \psi_g^* H^{[\xi,\eta]}. \end{aligned} \tag{4.1}$$

For the second to last identity we have used that $\{f \circ \psi, g \circ \psi\} = \{f, g\} \circ \psi$ (2. in Lemma 3.26). We now take $g_1 \in G$ and a smooth path $g : [0,1] \to G$ from e to g_1 and denote $\eta(t) := \dot g(t) g(t)^{-1}$, where $\dot g(t)$ is the time derivative at time t. We have that:

$$\frac{d}{dt}\Big|_t \psi_{g(t)} = \eta(t)^\# \circ \psi_{g(t)}, \tag{4.2}$$

$$\frac{d}{dt}\Big|_t g(t)^{-1}\xi g(t) = \mathrm{Ad}_{g(t)^{-1}}([\xi, \eta(t)]). \tag{4.3}$$

On the one hand, to obtain (4.2) we use that differentiation of $j_p \circ R_g = j_{gp}$ gives $d_g j_p = d_e j_{gp} \circ d_g R_{g^{-1}}$, and hence

$$\begin{aligned} \frac{d}{dt}\Big|_t \psi_{g(t)}(p) &= \frac{d}{dt}\Big|_t j_p(g(t)) = d_{g(t)} j_p(\dot g(t)) = d_e j_{g(t)p}(\dot g(t) g(t)^{-1}) \\ &= \eta(t)^\#(g(t)p) = \eta(t)^\# \circ \psi_{g(t)}(p). \end{aligned}$$

On the other hand, we get similarly (4.3) by first computing

$$\begin{aligned} \frac{d}{dt}\Big|_t g(t)\xi g(t)^{-1} &= \frac{d}{dt}\Big|_t \mathrm{Ad}_{g(t)}(\xi) = d_{g(t)}\mathrm{Ad}(\dot g(t))(\xi) \\ &= ([\dot g(t) g(t)^{-1}, \mathrm{Ad}_{g(t)}\xi]) = \mathrm{Ad}_{g(t)}([g(t)^{-1}\dot g(t), \xi]). \end{aligned}$$

Here we have used that Ad is a Lie algebra homomorphism and that the differential of Ad at $g \in G$ of $\dot{g} \in T_gG$, with $\eta := \dot{g}g^{-1}$, is given by

$$d_g\mathrm{Ad}(\dot{g})(\xi) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(t\eta)g}(\xi) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(t\eta)} \circ \mathrm{Ad}_g(\xi) = [\eta, \mathrm{Ad}_g(\xi)].$$

To obtain (4.3), we now compose the above time-derivative with the differential of the inverse map $\mathrm{inv} : G \to G : h \mapsto h^{-1}$, given by:

$$d_g\mathrm{inv} = d_eR_{g^{-1}} \circ d_e\mathrm{inv} \circ d_gl_{g^{-1}} = -d_eR_{g^{-1}} \circ d_gl_{g^{-1}},$$

(recall that $d_e\mathrm{inv} = -\mathrm{id}$), to get

$$\begin{aligned}\frac{d}{dt}\bigg|_t g(t)^{-1}\xi g(t) &= \mathrm{Ad}_{g(t)^{-1}}([g(t)d_{g(t)}\mathrm{inv}(\dot{g}(t)), \xi]) \\ &= -\mathrm{Ad}_{g(t)^{-1}}([\dot{g}(t)g(t)^{-1}, \xi]) = \mathrm{Ad}_{g(t)^{-1}}([\xi, \eta(t)]).\end{aligned}$$

Putting together equations (4.2) and (4.3) we get:

$$\begin{aligned}\frac{d}{dt}\bigg|_t (H^\xi \circ \psi_{g(t)} - H^{g(t)^{-1}\xi g(t)}) &= d_{\psi_{g(t)}}H^\xi(\eta^\# \circ \psi_{g(t)}) - H^{\mathrm{Ad}_{g(t)^{-1}}([\xi,\eta(t)])} \\ &= d_{g(t)}(H^\xi \circ \psi_{g(t)})(\psi_{g(t)^{-1},*}\eta(t)^\#) - \psi_{g(t)}^*H^{[\xi,\eta(t)]}.\end{aligned}$$

In the last identity we have used equation (4.1). Now from the first part of Proposition 3.26 we get that $\psi_{g(t)^{-1},*}\eta(t)^\# = X_{H^{\eta(t)}\circ\psi_{g(t)}}$ so that

$$\begin{aligned}\frac{d}{dt}\bigg|_t (H^\xi \circ \psi_{g(t)} - H^{g(t)^{-1}\xi g(t)}) &= \omega(X_{H^\xi\circ\psi_{g(t)}}, X_{H^{\eta(t)}\circ\psi_{g(t)}}) - \psi_{g(t)}^*H^{[\xi,\eta(t)]} \\ &= \{H^\xi \circ \psi_{g(t)}, H^{\eta(t)} \circ \psi_{g(t)}\} - \psi_{g(t)}^*H^{[\xi,\eta(t)]} \\ &= \psi_{g(t)}^*(\{H^\xi, H^{\eta(t)}\} - H^{[\xi,\eta(t)]}) \\ &= 0,\end{aligned}$$

using again that $\{f \circ \psi, g \circ \psi\} = \{f, g\} \circ \psi$ for the third identity and finishing with *(ii)*. Since this expression is zero at time $t = 0$, it is identically zero. Evaluating at $t = 1$ gives that $H^\xi \circ \psi_{g_1} = H^{g_1^{-1}\xi g_1}$ for arbitrary $g_1 \in G$. □

Thus, one can interpret the equivariance condition as follows. For a weakly Hamiltonian action, the infinitesimal action vector fields define a Lie algebra anti-homomorphism, $\mathfrak{g} \to \mathfrak{X}^{\mathrm{Ham}}(M, \omega) : \xi \mapsto \xi^\#$, which can always be lifted to a linear map $\mathfrak{g} \mapsto \mathcal{C}^\infty(M, \mathbb{R}) : \xi \mapsto H^\xi$. Then, the action is Hamiltonian if and only if this lift can be chosen G-equivariant. For a connected group, this in turn means that it can be lifted to a Lie algebra homomorphism, completing the following commutative Lie (anti-)homomorphism diagram:

$$\begin{tikzcd} \mathcal{C}^\infty(M,\mathbb{R}) \arrow[rr, "f\mapsto X_f"] & & \mathfrak{X}^{\mathrm{Ham}}(M,\omega) \\ & \mathfrak{g}. \arrow[ul, "\xi\mapsto H^\xi"] \arrow[ur, "\xi\mapsto\xi^\#"'] & \end{tikzcd}$$

Remark 4.12 In the previous section about Hamiltonian $\mathbb{S}^1$-actions, there was no need to introduce weakly Hamiltonian actions since they are equivalent to Hamiltonian ones: $\mathbb{S}^1$ is connected so that we can check condition *(ii)* of the previous Lemma. Since $\mathbb{S}^1$ is 1-dimensional, we trivially have $H^{[\xi,\xi]} = 0 = \{H^\xi, H^\xi\}$. $\square$

4.2.2 Moment maps

Consider a weakly Hamiltonian G-action on (M,ω) and choose a linear map $\xi \mapsto H^\xi$. Linearity means that at every point $p \in M$ we obtain a linear map $H(p) : \mathfrak{g} \to \mathbb{R} : \xi \mapsto H^\xi(p)$, that is, an element of $\mathfrak{g}^*$. Thus, an alternative way to understand the map $\xi \mapsto H^\xi$ is as a map $M \to \mathfrak{g}^* : p \to H(p)$. Then, the G-equivariance condition is translated into equivariance with respect to the G-action on M and the coadjoint action on $\mathfrak{g}^*$. This leads to the following definition:

Definition 4.13 (Moment map) A *moment map* for a G-action on (M,ω) is a G-equivariant smooth map

$$\Phi : M \to \mathfrak{g}^*$$

such that the smooth map

$$\Phi^\xi : M \to \mathbb{R} : p \mapsto \langle \Phi(p), \xi\rangle,$$

is a Hamiltonian function for $\xi^\#$, for every $\xi \in \mathfrak{g}$. We then say that the triple (M,ω,Φ) is a *Hamiltonian G-space.*

Here, we denote by $\langle\cdot,\cdot\rangle$ the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and the action of G on $\mathfrak{g}^*$ is the left dual of adjoint action:

$$G \times \mathfrak{g}^* \to \mathfrak{g}^* : (g,\mu) \mapsto \mathrm{Ad}^*_{g^{-1}}(\mu) = \mu \circ \mathrm{Ad}_{g^{-1}}.$$

We are thus asking two conditions. The Hamiltonian function condition can be expressed as

$$\langle d_p\Phi(v), \xi\rangle \equiv d_p\Phi^\xi(v) = \omega_p(\xi^\#_p, v), \tag{4.4}$$

for every $\xi \in \mathfrak{g}$, $p \in M$, $v \in T_pM$. The equivariance condition is expressed as

$$\langle \Phi \circ \psi_g, \xi\rangle = \langle (\mathrm{Ad}_{g^{-1}})^*\Phi, \xi\rangle \equiv \langle \Phi, \mathrm{Ad}_{g^{-1}}\xi\rangle,$$

for every $\xi \in \mathfrak{g}$. As we said, this is equivalent to the linear map $\xi \mapsto \Phi^\xi = \langle\Phi,\xi\rangle$ being G-equivariant with respect to the adjoint action on $\mathfrak{g}$ and the G-action on $\mathcal{C}^\infty(M,\mathbb{R})$ given by $\psi^*_{g^{-1}}$, as in Definition 4.9. In view of Lemma 4.11, this implies that it is a Lie algebra homomorphism,

$$\langle \Phi, [\xi,\eta]\rangle = \{\langle \Phi,\xi\rangle, \langle \Phi,\eta\rangle\} \tag{4.5}$$

for all $\eta,\xi \in \mathfrak{g}$. When G is connected, this in turns suffices for Φ to be G-equivariant. A last useful relation comes from differentiating the equivariance relation of Φ:

$$d_p\Phi(\xi^\#_p) = -\Phi(p)\circ \mathrm{ad}_\xi. \tag{4.6}$$

Example 4.14 If a group G acts in a Hamiltonian way via $g \mapsto \psi_g$ on (M,ω), and if $f : H \to G$ is a Lie group homomorphism (e.g. inclusion of a subgroup), then the action of H via $h \mapsto \psi_{f(h)}$ is Hamiltonian and the moment map is the composition $f^*\circ\Phi$ of the G-moment map Φ with the dual map $f^* : \mathfrak{g}^* \to \mathfrak{h}^*$ induced by the Lie algebra homomorphism $f_* : \mathfrak{h} \to \mathfrak{g}$. To see it we just notice that for $\xi \in \mathfrak{h}$, $\xi^{\#_H} = (f_*\xi)^{\#_G}$ so that $i_{\xi^{\#_H}}\omega = d\Phi^{f_*\xi}$ and that $f^*\circ\Phi$ is equivariant since f_* is. In the case of an inclusion $i : H \hookrightarrow G$, the moment map for the H-action is just the restriction $i^*\Phi$. □

From the point of view of dynamical systems, the significance of the moment map comes from the following. Given a Hamiltonian function $H \in \mathcal{C}^\infty(M,\mathbb{R})$ that is invariant under the action of G, i.e., if $H\circ\psi_g = H$ for every $g\in G$, then $\mathfrak{L}_{\xi^\#}H = 0$ and hence

$$0 = dH(\xi^\#) = \omega(X_H,\xi^\#) = \{H, H^\xi\}.$$

This implies that not only is H preserved by G, but reciprocally so is the moment map of the G-action preserved by the Hamiltonian flow generated by H. If H describes the time evolution of a system and G describes some symmetry of the system (through the fact that $H\circ\psi_g = H$), then the moment map is conserved in the evolution of the system.

As another important example, we recall that on a cotangent bundle (T^*M,ω) with $\omega = -d\theta$ for θ the canonical form, the cotangent lift $\widehat{X} \in \mathfrak{X}(T^*M)$ of a vector field $X \in \mathfrak{X}(M)$ is Hamiltonian associated to $i_{\widehat{X}}\theta$. Thus, if we consider a G-action on M, it lifts to a weakly Hamiltonian action on T^*M via $G \to \mathrm{Ham}(T^*M,\omega) : g \mapsto \widehat{\psi_g}$. Furthermore, if we denote the infinitesimal generator of this action by $\widehat{\xi} := \xi^\#_{TM}$ for any $\xi\in\mathfrak{g}$, then by construction $\widehat{\xi}$ is the cotangent lift of $\xi^\#_M$. It turns out that this action is Hamiltonian:

Proposition 4.15 (Cotangent lift of an action) *The cotangent lift of a G-action is Hamiltonian with moment map*

$$\Phi : T^*M \to \mathfrak{g}^* : p \mapsto (\xi \mapsto i_{\widehat{\xi}}\,\theta(p)).$$

Proof The infinitesimal G-action on T^*M is given by composing that of M, $\xi \mapsto \xi_M^\#$, with the cotangent lift $X \mapsto \widehat{X}$, i.e., by $\xi \mapsto \widehat{\xi}$. In turn, Corollary 3.33 states that the map $\mathfrak{X}(M) \to \mathcal{C}^\infty(T^*M, \mathbb{R}) : X \mapsto i_{\widehat{X}}\theta$, is a Lie algebra anti-homomorphism. The map

$$\mathfrak{g} \to \mathcal{C}^\infty(T^*M, \mathbb{R}) : \xi \mapsto i_{\widehat{\xi}}\theta,$$

is the composition of $\mathfrak{g} \to \mathfrak{X}(M) : \xi \mapsto \xi_M^\#$ with the previous map $X \mapsto i_{\widehat{X}}\theta$, both Lie algebra anti-homomorphisms, and thus is a Lie algebra homomorphism. In particular, Φ is also a Lie algebra homomorphism.

To see that it is also G-equivariant, we can either use the explicit expression

$$i_{\widehat{\xi}}\theta : T^*M \to \mathbb{R} : (p, v^*) \mapsto \langle v^*, \xi_p^\# \rangle,$$

or note that by Proposition 2.11 in combination with Proposition 3.34, $(\widehat{\psi_g})_* \widehat{\xi} = \widehat{\mathrm{Ad}_g \xi}$, and since $(\widehat{\psi_g})^* \theta = \theta$,

$$(\widehat{\psi_g})^* i_{\widehat{\xi}}\theta = i_{(\widehat{\psi_g})_*^{-1}\widehat{\xi}}\,\theta = i_{\widehat{\mathrm{Ad}_{g^{-1}}\xi}}\theta. \qquad \square$$

In the following, we will see that weakly Hamiltonian actions are generally not Hamiltonian. However, one has:

Lemma 4.16 *A weakly Hamiltonian G-action on (M, ω) is Hamiltonian if*

1. *G is compact, or*
2. *M is compact.*

Proof Consider a linear map $\xi \mapsto H^\xi$ such that $i_{\xi^\#}\omega = H^\xi$, not necessarily equivariant. We have to choose an equivariant Φ. Define

$$g \cdot H := \mathrm{Ad}_{g^{-1}}^* H \circ \psi_{g^{-1}}.$$

Then, clearly Φ is equivariant if and only if $g \cdot \Phi = \Phi$ for all $g \in G$. First, we claim that $g \cdot H$ is also weakly Hamiltonian for the G-action:

$$\begin{aligned}
d\langle g \cdot H, \xi \rangle &= \psi_{g^{-1}}^* d\langle H, \mathrm{Ad}_{g^{-1}}\xi \rangle \\
&= \psi_{g^{-1}}^* (i_{(\mathrm{Ad}_{g^{-1}}\xi)^\#}\omega) \\
&= \psi_{g^{-1}}^* (i_{\psi_{g^{-1},*}\xi^\#}\omega) \\
&= i_{\xi^\#}(\psi_{g^{-1}}^*\omega) \\
&= i_{\xi^\#}\omega,
\end{aligned}$$

where in the second to last identity we have used Lemma 2.11 and in the last identity that $\psi_{g^{-1}}^*\omega = \omega$. If G is compact, we obtain a G-equivariant moment map by averaging over the group Haar measure.

If M is compact, we can normalize H by the condition $\int_M H\omega^n = 0$ (assuming M connected arguing on each component). Then, H chosen in such a way is G-invariant because $g \cdot H$ is also weakly Hamiltonian and has the same normalization $\int_M g \cdot H\omega^n = 0$ (since ψ_g preserves the Liouville form $\omega^n/n!$), so that it must hold that $g \cdot H = H$ for all $g \in G$. $\square$

The next results show that there is an obstruction for a symplectic action to be weakly Hamiltonian or Hamiltonian in terms of the Lie algebra cohomology of G. We first briefly introduce this notion. It will also be related to the uniqueness of moment maps.

Let $\mathfrak{g}$ be a Lie-algebra. Define for every integer $k > 0$

$$\Omega^k(\mathfrak{g}; \mathbb{R}) := \Lambda^k(\mathfrak{g}^*)$$

as the k-th alternate sum of the dual of $\mathfrak{g}$, i.e., the alternating k-cochains $\mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$. Define the coboundary map $\partial^k : \Omega^k(\mathfrak{g}; \mathbb{R}) \to \Omega^{k+1}(\mathfrak{g}; \mathbb{R})$ by the formula

$$\partial^k \omega(X_0, \ldots, X_k) := \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_n),$$

where the hat denotes that a given argument is not present. We also put $\Omega^0(\mathfrak{g}; \mathbb{R}) := \mathbb{R}$ and $\partial^0 := 0$. It is then easy to check that $\partial^{k+1} \circ \partial^k = 0$, so that we get a cochain complex $(\Omega^k(\mathfrak{g}; \mathbb{R}), \partial^k)$.

Definition 4.17 (Lie algebra cohomology) The k-th *Lie algebra cohomology group or Chevalley cohomology group* is defined as the k-th cohomology group of $(\Omega^k(\mathfrak{g}; \mathbb{R}), \partial^k)$,

$$H^k(\mathfrak{g}; \mathbb{R}) := \ker \partial^k / \operatorname{im} \partial^{k-1}.$$

Proposition 4.18 *Given a connected Lie group G, the moment map of a Hamiltonian G-action is determined up to a cocycle $c \in \Omega^1(\mathfrak{g}; \mathbb{R})$. In particular, it is unique if and only if $H^1(\mathfrak{g}; \mathbb{R}) = 0$.*

Proof Consider two different moment maps $\Phi_1, \Phi_2 : M \to \mathfrak{g}^*$. Since both provide Hamiltonian functions for the same vector field $\xi^\#$, their difference is locally constant, hence constant:

$$\langle \Phi_1 - \Phi_2, \xi \rangle = c(\xi)$$

for some fixed $c \in \Omega^1(\mathfrak{g}; \mathbb{R})$. Since both Φ_i satisfy equation (4.5), c is a cocycle:

$$c([\xi, \eta]) = \langle \Phi_1 - \Phi_2, [\xi, \eta] \rangle = \omega(\xi^\#, \eta^\#) - \omega(\xi^\#, \eta^\#) = 0. \qquad \square$$

In short, moment maps are unique up to elements of $H^1(\mathfrak{g}; \mathbb{R}) = \ker \partial^1$. Since $\partial^1 c(\xi, \eta) = c([\xi, \eta])$, if we denote $[\mathfrak{g}, \mathfrak{g}] := \{[\xi, \eta] : \xi, \eta \in \mathfrak{g}\}$, then

$H^1(\mathfrak{g};\mathbb{R}) = [\mathfrak{g},\mathfrak{g}]^0$. Thus, two extreme cases are, on the one hand, a semisimple group (i.e., $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$) with unique maps, and on the other hand, an abelian group where moment maps are only defined up to an element of the dual Lie algebra $\mathfrak{g}^*$.

Lemma 4.19 *Consider a weakly Hamiltonian action of a connected Lie group and choose a linear map $\mathfrak{g} \to \mathcal{C}^\infty(M,\mathbb{R}) : \xi \mapsto H^\xi$ such that $X_\xi = X_{H^\xi}$ for all $\xi \in \mathfrak{g}$. Then, there exists a unique 2-form $\tau \in \Omega^2(\mathfrak{g};\mathbb{R})$, $\tau : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$, such that for all $\xi, \eta \in \mathfrak{g}$*

$$\{H^\xi, H^\eta\} - H^{[\xi,\eta]} = \tau(\xi,\eta).$$

The form τ is a cocycle and hence determines a class $[\tau] \in H^2(\mathfrak{g};\mathbb{R})$, i.e., for all $\xi,\eta,\zeta \in \mathfrak{g}$ we have

$$\tau([\xi,\eta],\zeta) + \tau([\eta,\zeta],\xi) + \tau([\zeta,\xi],\eta) = 0.$$

Proof We just note, using the fact that $\xi \mapsto \xi^\#$ and $f \mapsto X_f$ are both Lie algebra anti-homomorphisms (the latter by Proposition 3.25), that

$$X_{H^{[\xi,\eta]}} = [\xi,\eta]^\# = -[\xi^\#,\eta^\#] = -[X_{H^\xi}, X_{H^\eta}] = X_{\{H^\xi,H^\eta\}}.$$

Thus, $H^{[\xi,\eta]} - \{H^\xi, H^\eta\}$ is a locally constant and hence constant function for every $\xi,\eta \in \mathfrak{g}$. Since this correspondence is also bilinear, we get the existence of such a $\tau \in \Omega^2(\mathfrak{g};\mathbb{R})$. Furthermore, this implies that

$$\{H^{[\xi,\eta]}, H^\zeta\} = \{\{H^\xi, H^\eta\}, H^\zeta\},$$

so that the cocycle identity for τ follows from the Jacobi identities for the Poisson bracket and for the Lie bracket. $\square$

The class $[\tau] \in H^2(\mathfrak{g};\mathbb{R})$ is determined only up to a coboundary $\partial^1\sigma$, that is, up to a form of the type $(\xi,\eta) \mapsto \sigma([\xi,\eta])$ for some linear map $\sigma : \mathfrak{g} \to \mathbb{R}$. This class $[\tau]$ is the zero class if and only if it itself is of this type. For a connected Lie group action, this is precisely the case of a Hamiltonian action:

Corollary 4.20 *Consider a weakly Hamiltonian action of a connected Lie group with linear map $\mathfrak{g} \to \mathcal{C}^\infty(M,\mathbb{R}) : \xi \mapsto H^\xi$ and $\tau \in \Omega^2(\mathfrak{g};\mathbb{R})$ defined as above. Then, the action is Hamiltonian if and only if $[\tau] = 0$, i.e., if there exists linear map $\sigma : \mathfrak{g} \to \mathbb{R}$ such that*

$$\{H^\xi, H^\eta\} - H^{[\xi,\eta]} = \sigma([\xi,\eta]).$$

Proof If the action is Hamiltonian it holds for $\sigma = 0$. Reciprocally, given such a σ, we choose $\tilde{H}_\xi := H^\xi + \sigma(\xi)$ and hence

$$\{\tilde{H}_\xi, \tilde{H}_\eta\} - \tilde{H}_{[\xi,\eta]} = \{H^\xi, H^\eta\} - H^{[\xi,\eta]} - \sigma([\xi,\eta]) = 0. \qquad \square$$

Corollary 4.21 *Consider a smooth G-action on a symplectic manifold (M, ω) for a connected Lie group.*

- *If $H^1(\mathfrak{g}; \mathbb{R}) = 0$, then every symplectic action is weakly Hamiltonian.*
- *If $H^2(\mathfrak{g}; \mathbb{R}) = 0$, then every weakly Hamiltonian action is Hamiltonian.*

Proof For the first claim, we observe that $H^1(\mathfrak{g}; \mathbb{R}) = [\mathfrak{g}, \mathfrak{g}]^0$ means that $H^1(\mathfrak{g}; \mathbb{R}) = 0$ if and only if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Since the commutator of any two symplectic vector fields is Hamiltonian, we obtain that for every $\xi \in \mathfrak{g}$, $\xi^\#$ is Hamiltonian with Hamiltonian function $-\omega(\eta^\#, \zeta^\#)$ for any $\eta, \zeta \in \mathfrak{g}$ such that $\xi = [\eta, \zeta]$. The second part follows from the previous corollary. □

The next Theorem puts this corollary into a clearer light.

Theorem 4.22 *Given a compact, connected Lie group G with Lie algebra $\mathfrak{g}$, then the de Rham cohomology of G coincides with the Lie algebra cohomology of $\mathfrak{g}$:*

$$H^k_{dR}(G; \mathbb{R}) \cong H^k(\mathfrak{g}; \mathbb{R}).$$

Proof We introduce the subspace of left-invariant k-forms, $\Omega^k_l(G; \mathbb{R})$, of those forms ω such that $L_h^* \omega = \omega$ for any $h \in G$. Since being left-invariant is preserved by the differential boundary map, we get a sub-cochain complex $(\Omega^k_l(G; \mathbb{R}), d)$. Since G is compact, we can choose a finite, bi-invariant measure dg with mass 1 (i.e., the Haar measure) and we define the averaging map

$$I : \Omega^k(G; \mathbb{R}) \to \Omega^k_l(G; \mathbb{R}) : \omega \mapsto \int_G L_g^* \omega dg,$$

producing a left-invariant form $I(\omega)$. Clearly, the map I is a cochain map, since the differential boundary map d commutes with both L_g^* and the integral. We get a map

$$H^k(I) : H^k(G; \mathbb{R}) \to H^k_l(G; \mathbb{R}) : [\omega] \mapsto [I(\omega)],$$

where H_l^* denotes the cohomology of left-invariant forms. Further, denoting $i_l : \Omega^k_l(G; \mathbb{R}) \hookrightarrow \Omega^k(G; \mathbb{R})$ the inclusion, then clearly $I \circ i_l = \mathrm{id}_{\Omega^k_l(G;\mathbb{R})}$, and hence $H^k(I)$ is injective.

We now show that $H^k(I)$ is also surjective. To see it, we use the corollary of de Rham's Theorem (see e.g. Theorem 18.14 in [18]) stating that a closed k-form is exact if and only if its integral over any smooth singular k-cycle is zero, i.e., if $\int_z \omega = 0$ for every smooth singular k-chain such that $\partial z = 0$, for the singular boundary map ∂. Given such a cycle z we compute

$$\int_z I(\omega) = \int_z \int_G L_g^* \omega dg = \int_G \int_z L_g^* \omega dg = \int_G \int_{L_g(z)} \omega dg = \int_G \int_z \omega dg = \int_z \omega,$$

where we have used that $L_g(z)$ is homotopy equivalent to z via $L_{g(t)}(z)$ for any smooth path between e and g, in virtue of G being connected, and thus ω integrates zero over the boundary cycle $L_g(z) - z$. Since $\int_z (I(\omega) - \omega) = 0$ for every cycle z, $I(\omega) - \omega$ is exact and hence $I(\omega)$ represents the same cohomology class as ω. In this way, $H^k(I)$ is also surjective.

Thus, $H^k(I)$ is an isomorphism with inverse the inclusion $H^k(i_l)$. We can then compute de Rham's cohomology only using left-invariant forms. Since left-invariant forms are determined by their value at the identity element e, we obtain an isomorphism for every k

$$\Omega^k_l(G;\mathbb{R}) \to \Omega^k(\mathfrak{g};\mathbb{R}) : \omega \mapsto \omega_e.$$

To finish, we only need to check that the boundary map is the same for both cochain complexes, and this follows from the formula for the differential of a form $\omega \in \Omega^k(M;\mathbb{R})$,

$$\begin{aligned} d\omega(X_0,\dots,X_k) = &\sum_i (-1)^i \mathfrak{L}_{X_i}(\omega(X_0,\dots,\widehat{X_i},\dots,X_n)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_n). \end{aligned}$$

Since left-invariant vector fields are a moving basis for TG, it is enough to check what we want for them. But left-invariant forms evaluated on left-invariant vector fields are constant, and hence the Lie derivatives in the formula above are zero and we are done. □

Remark 4.23 Coming back to Hamiltonian $\mathbb{S}^1$-actions, an alternative way to see that there is no distinction between weakly Hamiltonian and Hamiltonian actions is due to the fact that $H^2(\mathbb{S}^1;\mathbb{R}) = 0$. □

We include the following result.

Theorem 4.24 (Whitehead Lemmas) *A compact Lie group G is semisimple if and only if $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$.*

For the interested reader, a proof can be found in pages 93-95 of [17].

Hence, if G is compact, connected and semisimple, any symplectic action is Hamiltonian and the corresponding moment map unique.

Example 4.25 (Torus action on $\mathbb{C}^n$) We consider again $\mathbb{C}^n$ with the standard symplectic form of $\mathbb{R}^{2n}$,

$$\omega_0 = \sum_i dx_i \wedge dy_i = \sum_i r_i dr_i \wedge d\theta_i = \frac{i}{2}\sum_i dz_i \wedge \overline{z}_i.$$

We have the smooth T-action of the torus $T = \mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ given by the $\mathbb{S}^1$ complex multiplication action in each coordinate:

$$\mathbb{T}^n \times \mathbb{C}^n \to \mathbb{C}^n : (\lambda, z) \mapsto (\lambda_1 z_1, \dots, \lambda_n z_n).$$

Since this action is by elements of $U(n)$, it preserves ω_0 and is thus symplectic. It is in fact Hamiltonian. To see it, let $\mathfrak{t} \cong \mathbb{R}^n$ be its Lie algebra, where the identification with $\mathbb{R}^n$ is given by sending to the standard basis of $\mathbb{R}^n$, in the obvious way, the vectors ξ_i such that the i-th coordinate of the Lie exponential $\exp(t\xi_i) \in T$ is $\exp(it) \in \mathbb{S}^1$ (and the other coordinates fixed at 1). Then, the infinitesimal generators are

$$(\xi_i)^\#_z = \frac{d}{dt}\bigg|_{t=0} (z_1, \ldots, \exp(it)z_i, \ldots, z_n) = (0, \ldots, iz_i, \ldots, 0) = \partial_{\theta_i},$$

for the partial derivative with respect to the angular polar coordinate of the i-th complex plane factor in $\mathbb{C}^n$, ∂_{θ_i}. Thus:

$$i_{\xi_i^\#}\omega = -r_i dr_i = d(-\frac{1}{2}|z_i|^2),$$

and the map

$$\Phi : \mathbb{C}^n \to \mathfrak{t}^* : z \mapsto -\frac{1}{2}(|z_i|^2)_i$$

is T-invariant. Here we identify $\mathfrak{t}^* \cong (\mathbb{R}^n)^*$ thanks to the identification $\mathfrak{t} \cong \mathbb{R}^n$ above. For a general $\xi = \sum_i \lambda_i \xi_i \in \mathfrak{t}$ we have $\xi^\# = \sum_i \lambda_i \partial_{\theta_i}$, so that

$$i_{\xi^\#}\omega = d\sum_i -\frac{1}{2}\lambda_i |z_i|^2 = d\langle \Phi(z), \xi \rangle.$$

Thus, the T-action is Hamiltonian with moment map Φ.

If instead we consider the torus-action given by

$$\mathbb{T}^n \times \mathbb{C}^n \to \mathbb{C}^n : (\lambda, z) \mapsto (\lambda_1^{k_1} z_1, \ldots, \lambda_n^{k_n} z_n).$$

where $\{k_i\}_i^n \subset \mathbb{Z}$ are some integer exponents, then we have $\xi_i^\# = k_i \partial_{\theta_i}$ and we obtain a Hamiltonian action with moment map

$$\Phi : \mathbb{C}^n \to \mathfrak{t} : z \mapsto -\frac{1}{2}(k_i |z_i|^2)_i. \qquad \square$$

4.3 Coadjoint orbits

We remind that we distinguish the G-manifold M via the subindex $\xi \mapsto \xi_M^\#$ in the infinitesimal action. In the following we have G-actions on M and on $\mathfrak{g}^*$, and we write $\xi_M^\#$ versus $\xi_{\mathcal{O}}^\#$, or more simply $\xi_p^\#$ versus $\xi_\mu^\#$ for $p \in M$ and $\mu \in \mathfrak{g}^*$.

We begin by noting that for any Hamiltonian G-space (M, ω, Φ), the moment map determines the pullback of ω to any G-orbit. Indeed, since $\xi_M^\#$ is the Hamiltonian vector field for Φ^ξ, we have

$$\omega(\xi_M^\#, \eta_M^\#) = \{\Phi^\xi, \Phi^\eta\} = \Phi^{[\xi,\eta]}.$$

In particular, if the action is transistive, i.e., M is a homogeneous Hamiltonian G-manifold, ω is completely determined by Φ, as we will see shortly. First we show how the above equation can be used as motivation to define a canonical symplectic structure on coadjoint orbits:

Theorem 4.26 (Kirillov–Kostant–Souriau) *Let $\mathcal{O} \subset \mathfrak{g}^*$ be an orbit for the coadjoint G-action on $\mathfrak{g}^*$. Then, there exists a unique symplectic form on $\mathcal{O}$ for which the coadjoint action is Hamiltonian and the moment map is the inclusion $\Phi : \mathcal{O} \hookrightarrow \mathfrak{g}^*$.*

Proof At a point $\mu \in \mathcal{O}$, the coadjoint infinitesimal action is given by $\xi^\#_\mu = -\mathrm{ad}^*_\xi \mu \equiv -\mu \circ \mathrm{ad}_\xi$, so that the tangent space to the orbit is

$$T_\mu\mathcal{O} = \{-\mathrm{ad}^*_\xi\mu : \xi \in \mathfrak{g}\} \subset \mathfrak{g}^*.$$

For such $\mu \in \mathcal{O}$ consider the skew-symmetric bilinear form on $\mathfrak{g}$,

$$B_\mu(\xi, \eta) := \langle \mu, [\xi, \eta]\rangle.$$

Writing $B_\mu(\xi, \cdot) = \mathrm{ad}^*_\xi\mu$ we see that $\ker B_\mu$ consists of all $\xi \in \mathfrak{g}$ such that $\mathrm{ad}^*_\xi\mu = 0$, i.e., $\xi^\#_\mu = 0$. It follows that the skew-symmetric form, for vectors $\xi^\#_\mu = -\mathrm{ad}^*_\xi\mu$, $\eta^\#_\mu = -\mathrm{ad}^*_\eta\mu \in T_\mu\mathcal{O}$, given by

$$\omega_\mu(-\mathrm{ad}^*_\xi\mu, -\mathrm{ad}^*_\eta\mu) := \langle \mu, [\xi, \eta]\rangle,$$

is a well-defined symplectic 2-form on $T_\mu\mathcal{O}$. Together $\mu \mapsto \omega_\mu$ define a smooth G-invariant 2-form on $\mathcal{O}$. Smoothness follows from G-invariance, since then its pullback with the action map gives $\omega_{g\mu} = d_{g\mu}\psi^*_{g^{-1}}\omega_\mu$ for any fixed $\mu \in \mathcal{O}$.

To see that it is G-invariant, we take vectors $u = \mathrm{ad}^*_\xi\mu$, $v = \mathrm{ad}^*_\eta\mu \in T_\mu\mathcal{O}$ and notice that $gu = \mathrm{Ad}^*_{g^{-1}}\mathrm{ad}^*_\xi\mu = \mathrm{ad}^*_{\mathrm{Ad}_g\xi}\mathrm{Ad}^*_{g^{-1}}\mu = \mathrm{ad}^*_{\mathrm{Ad}_g\xi}(g\mu)$, so that

$$\begin{aligned}
\omega_{g\mu}(gu, gv) &= \langle g\mu, [\mathrm{Ad}_g\xi, \mathrm{Ad}_g\eta]\rangle \\
&= \langle \mathrm{Ad}^*_{g^{-1}}\mu, \mathrm{Ad}_g([\xi, \eta])\rangle \\
&= \langle \mu, [\xi, \eta]\rangle \\
&= \omega_\mu(u, v).
\end{aligned}$$

Furthermore, for any $v = \mathrm{ad}^*_\xi\mu \in T_\mu\mathcal{O}$ and $\eta \in \mathfrak{g}$, the inclusion map $\Phi : \mathcal{O} \hookrightarrow \mathfrak{g}^*$ satisfies

$$\begin{aligned}
d_\mu\langle \Phi, \eta\rangle(v) &= \langle v, \eta\rangle \\
&= \langle \mathrm{ad}^*_\xi\mu, \eta\rangle \\
&= \langle \mu, [\xi, \eta]\rangle \\
&= \omega_\mu(\mathrm{ad}^*_\xi\mu, \mathrm{ad}^*_\eta\mu) \\
&= \omega_\mu(v, -\eta^\#_\mu) \\
&= \omega_\mu(\eta^\#_\mu, v) \\
&= (i_{\eta^\#_\mathcal{O}}\omega)_\mu(v),
\end{aligned}$$

that is, $d\langle\Phi,\eta\rangle = i_{\eta_{\mathcal{O}}^{\#}}\omega$. This allows to conclude two things. First, that ω is closed and thus a symplectic structure: since $i_{\xi_{\mathcal{O}}^{\#}}\omega$ is closed and ω is G-invariant, then, for any $\xi \in \mathfrak{g}$,

$$i_{\xi_{\mathcal{O}}^{\#}} \circ d\omega = \mathfrak{L}_{\xi_{\mathcal{O}}^{\#}}\omega - d \circ i_{\xi_{\mathcal{O}}^{\#}}\omega = 0 - 0 = 0.$$

The fact that the $\xi_{\mathcal{O}}^{\#}$ span $T_\mu\mathcal{O}$ implies $d\omega = 0$. Second, the action is weakly Hamiltonian with Hamiltonian function $\langle\Phi,\xi\rangle$ associated to $\xi_{\mathcal{O}}^{\#}$. Since Φ is obviously G-equivariant, it is the moment map for the coadjoint action of G on $\mathcal{O}$, and $(\mathcal{O},\omega,\Phi)$ is a Hamiltonian G-space. Uniqueness of ω follows from the previous remark about ω being determined by the moment map. $\square$

We will call this symplectic form the *KKS form* and denote it by $\omega_{\mathcal{O}}$. The following theorem will be important to prove the non-abelian version of the normal form Theorem 5.12.

Theorem 4.27 (Kostant–Souriau) *Let (M,ω,Φ) be a Hamiltonian G-space on which G acts transitively. Then $\Phi : M \to \Phi(M)$ is a covering space of a coadjoint orbit, with 2-form obtained by pullback of the KKS form on $\mathcal{O}$.*

Proof Let $\mathcal{O} = \Phi(M)$ be the corresponding orbit (since the action is transitive and Φ equivariant). It is clear that the map $\Phi : M \to \mathcal{O}$ is a submersion (it is surjective and of constant rank, again due to equivariance and transitiveness). We have seen that the 2-form on M is determined by the moment map, and the formula

$$\omega_p(\xi_p^{\#},\eta_p^{\#}) = \langle\Phi(p),[\xi,\eta]\rangle = \omega_{\mathcal{O},\Phi(p)}(\xi_{\Phi(p)}^{\#},\eta_{\Phi(p)}^{\#})$$

shows precisely that $\Phi^*\omega_{\mathcal{O}} = \omega$ (notice that $d_p\Phi(\xi_p^{\#}) = \xi_{\Phi(p)}^{\#}$ by equivariance). Hence, Φ actually has bijective differential at every point and is thus a local diffeomorphism.

To see that Φ is furthermore a covering map, we have to use some theory of homogeneous spaces. If we identify $M \cong G/G_p$ and $\mathcal{O} \cong G/G_{\Phi(p)}$ for some $p \in M$ (each identification induced from the corresponding orbit maps), we get a map $\pi : G/G_p \to G/G_{\Phi(p)}$ induced by Φ. Again because Φ is G-equivariant, $G_p \subset G_{\Phi(p)}$, and π is the projection $gG_p \mapsto gG_{\Phi(p)}$ (notice that π is a G-equivariant map sending the coset G_p to $G_{\Phi(p)}$). Then, Corollary 2.14 states that $\Phi : M \to \mathcal{O}$ is a $G_{\Phi(p)}/G_p$-fiber bundle. Since Φ is also a local diffeomorphism, the fiber must be discrete and it is a covering space. $\square$

In particular, we prove that $G_{\Phi(p)}/G_p$ must be discrete. Hence, non-trivial coverings can be obtained only if the stabilizer $G_{\Phi(p)}$ is disconnected. If we know that this cannot occur, then the result can be strengthened to say that the moment map is a symplectomorphism. This is the case if the stabilizer groups of the coadjoint action G_μ are all connected, and this occurs for compact, connected Lie groups (see e.g. Theorem 4.5 of [24]). We will make use of this result in Theorem 4.43 for the cotangent lift of the self G-action, where in fact we will have that $G_p = G_{\Phi(p)}$.

4.4 Symplectic reduction

The technique of symplectic reduction is an important technique in symplectic geometry and it will be the basis for stating and proving the Duistermaat–Heckman Theorem. As we said, it is a formalization of a classical phenomenon of mechanical systems: whenever there is a symmetry group of dimension k acting on a system, the number of degrees of freedom of the phase space may be reduced by $2k$.

The mathematical underlying principle is that every coisotropic submanifold is foliated by isotropic leaves, and whenever the quotient space of leaves is a manifold (of codimension twice the original codimension), it then inherits the symplectic structure. The main example is the case where the coisotropic submanifold is the zero level set of the moment map of a Hamiltonian group action, leading to the Marsden–Weinstein–Meyer quotient where the isotropic leaves coincide with the orbits and hence the symplectic quotient endows the orbit space with a symplectic structure.

We follow §5.4 in [23] and §5 in [24], carrying out the simplest construction of symplectic reduction for the trivial coadjoint orbit and then developing a more general treatment for reduction at other elements of $\mathfrak{g}^*$, i.e., at other level sets of the moment map. We begin studying the *null distribution* of an integrable distribution of a symplectic manifold (M, ω).

Proposition 4.28 *Let $D \subset T_pM$ be an integrable distribution of (M, ω) and let $D^\omega \cap D$ be the* null distribution *given by $p \mapsto D_p^\omega \cap D_p$. Then $D^\omega \cap D$ is also integrable.*

Proof By Frobenius Theorem 2.6, we only need to check that $D^\omega \cap D$ is also involutive. Thus, we take two local sections $X, Y \in \mathfrak{X}(M)$ of $D^\omega \cap D$, and we also take a local section $Z \in \mathfrak{X}(M)$ of D. Then, using the formula for the differential of a 2-form and the closedness of ω, we get:

$$\begin{aligned} 0 = d\omega(X,Y,Z) &= \mathfrak{L}_X(\omega(Y,Z)) - \mathfrak{L}_Y(\omega(Z,X)) + \mathfrak{L}_Z(\omega(X,Y)) \\ &\quad - \omega([X,Y],Z) + \omega([Z,X],Y) - \omega([Y,Z],X) \\ &= \omega([X,Y],Z). \end{aligned}$$

We have used that all first three terms vanish, since $\omega(X_1, X_2) = 0$ whenever X_1 is a section of D and X_2 of D^ω. Similarly, the last two terms vanish since the Lie bracket is closed in D in virtue of its integrability. Since Z was an arbitrary section of D, we conclude that $[X, Y]$ is also a section of D^ω and hence of the null distribution $D^\omega \cap D$. $\square$

The foliation associated to the null distribution is called the *null foliation*. Actually, the same proof allows to prove that the kernel $\ker\omega := \{(p,v) \in TM : i_v\omega_p = 0\}$ of a smooth form $\omega \in \Omega^k(M;\mathbb{R})$ is an integrable distribution. Note that this is a smooth distribution, in particular it is of constant rank thanks to ω's non-degeneracy.

If D is a coisotropic distribution, then $D^\omega \cap D = D^\omega$ and we get:

Corollary 4.29 *Let $D \subset T_pM$ be a coisotropic integrable distribution and let D^ω be the isotropic distribution given by the symplectic complement $p \mapsto D^\omega_p \subset D_p$. Then D^ω is also integrable.*

Proposition 4.28 will be applied to the tangent subbundle of an embedded submanifold $S \subset M$. As promised, we see that the null distribution $p \mapsto \ker\omega_p|_{T_pS} = T_pS \cap T_pS^\omega$ is integrable and comes from the null foliation. To this end, we define the equivalence relation $\sim$ on S by $p_0 \sim p_1$ if they both lie in the same leaf, i.e., if there exists a smooth path $\gamma : [0,1] \to S$ such that $\gamma(0) = p_0$, $\gamma(1) = p_1$ and $\gamma'(t) \in T_{\gamma(t)}S \cap T_{\gamma(t)}S^\omega$ for every $t \in [0,1]$. Let $[p]$ be the equivalence class of $p \in S$. Following §5.4 of [23], we will call an embedded submanifold S *regular* if the following condition is satisfied:

Definition 4.30 (Regular submanifold) A submanifold $S \subset M$ is *regular* if for every $p \in S$ there exists a submanifold $\Sigma \subset S$ containing p (called a local slice through p) that intersects every leaf of the null foliation of S at most once and such that $T_pS = T_p\Sigma \oplus T_pS \cap T_pS^\omega$ for every $p \in \Sigma$. Moreover, the quotient space $\overline{S} := S/\sim$ is Hausdorff.

The main point of the proof of the Quotient Manifold Theorem 2.12 is proving precisely that the manifold M satisfies itself the same regularity conditions with respect to the foliation given by the orbits of a proper, free action. With this condition, we can define a smooth structure on the Hausdorff, second countable space $\overline{S}$ in terms of the local homeomorphisms $\pi|_\Sigma : \Sigma \to \pi(\Sigma)$, that we define to be diffeomorphisms. The fact that each Σ only intersects leaves once implies that $\pi|_\Sigma$ is a bijection, and it is easy to check that the unique leaf-preserving map between two such submanifolds is a diffeomorphism (as it is shown in the next proof).

Proposition 4.31 *Given a regular submanifold $S \subset M$, the quotient space $\overline{S} = S/\sim$ has a unique smooth structure such that $\pi : S \to \overline{S}$ is a submersion, and a unique symplectic form $\overline{\omega}$ whose pullback under π is the restriction of ω:*

$$\pi^*\overline{\omega} = \omega|_S.$$

Proof Consider $0 \le k \le n$ such that $\dim TS \cap TS^\omega = k$ and a point $p_0 \in S$. Frobenius Theorem 2.6 provides a local flat chart $\varphi : U \to \mathbb{R}^{2n-k}$ for S on an open neighbourhood $U \subset S$ of p_0 such that $\varphi(p_0) = 0$, $\varphi(U) \cap \{c\} \times \mathbb{R}^k$

is a connected integral submanifold for every $c \in \mathbb{R}^{2n-2k}$, and such that $v \in T_pS \cap T_pS^\omega$ if and only if $d_p\varphi(v) \in \{0\} \times \mathbb{R}^k$ for $v \in T_pS$ and $p \in U$.

Consider the submanifold $\Sigma_0 \subset S$ through p_0 of the definition of regular submanifold. Shrinking U, we may assume that there are $\varepsilon, \delta > 0$ such that

$$W := \varphi(U) = B_\delta^{2n-2k} \times B_\varepsilon^k$$

(for the ball $B_r^l \subset \mathbb{R}^l$ of radius $r > 0$) and such that $\varphi(U \cap \Sigma_0)$ is the graph of a function $f_0 : B_\delta^{2n-2k} \to B_\varepsilon^k$ with $f(p_0) = 0$, as a consequence of $T_pS = T_p\Sigma_0 \oplus T_pS \cap T_pS^\omega$ and that Σ_0 intersects each leaf at most once. This translated via φ means that $\varphi(U \cap \Sigma_0)$ is a submanifold of $\varphi(U)$ such that the coordinate projection $\pi : \mathbb{R}^{2n-k} \to \mathbb{R}^{2n-2k} : (c, x) \mapsto c$ restricted to $\varphi(U \cap \Sigma_0)$ is a diffeomorphism.

We consider now the closed 2-form on $W = \varphi(U)$

$$\tau := (\varphi^{-1})^*\omega \in \Omega^2(W; \mathbb{R})$$

so that for any $q \in W$ and $w \in \mathbb{R}^{2n-k}$ we have $\tau_q(w, \cdot) = 0$ if and only if $w \in \{0\} \times \mathbb{R}^k$. In particular, the restriction of τ to any $W \cap \{c\} \times \mathbb{R}^k$ is zero, while its restriction to $\varphi(U \cap \Sigma_0)$ is a symplectic form.

If we consider now another local slice $\Sigma_1 \subset S$ through $p_1 \in U \cap [p_0]$, and let $\varphi(U \cap \Sigma_1)$ be the graph of $f_1 : B_\delta^{2n-2k} \to B_\varepsilon^k$, we then have a unique diffeomorphism $\psi : U \cap \Sigma_0 \to U \cap \Sigma_1$ preserving the leaves, namely, such that

$$\varphi \circ \psi \circ \varphi^{-1}(x, f_0(x)) = (x, f_1(x)), \; x \in B_\delta^{2n-2k}.$$

The fact that $\tau_q(w, \cdot) = 0$ if $w \in \{0\} \times \mathbb{R}^k$ for $(\varphi^{-1})^*\omega = \tau$ implies that $(\varphi \circ \psi \circ \varphi^{-1})^*\tau|_{\varphi(U\cap\Sigma_1)} = \tau|_{\varphi(U\cap\Sigma_0)}$: the map $\varphi \circ \psi \circ \varphi^{-1}$ is of the shape $(x, y) \mapsto (x, \theta(x))$ for some smooth θ. Additionally, τ is closed and thus for any $X \in \mathfrak{X}(W)$ of type $X_p = (0, Y_p) \in \mathbb{R}^{2n-2k} \times \mathbb{R}^k$, $i_X\tau = 0$ so that $\mathfrak{L}_X\tau = 0$ and $\tau_{(x,y)} = \tau_{(x,y')}$ for every $y, y' \in B_\varepsilon^k$. If we take now $w_i = (u_i, v_i) \in \mathbb{R}^{2n-2k} \times \mathbb{R}^k$ for $i = 0, 1$, we get

$$(\varphi \circ \psi \circ \varphi^{-1})^*\tau_{(x,y)}(w_1, w_2) = \tau_{(x,\theta(x))}((u_1, 0), (u_2, 0)) = \tau_{(x,y)}(w_1, w_2),$$

using also that $\tau_{(x,y)}(w_1, w_2) = \tau_{(x,y)}((u_1, 0), (u_2, 0))$. We obtain that

$$\psi^*\omega|_{U\cap\Sigma_1} = \omega|_{U\cap\Sigma_0}.$$

That is, the transition map ψ induced when changing from nearby local slices Σ_0 to Σ_1 is a symplectomorphism. Thence, given any two points whose local charts thus defined intersect, we have a chain of symplectomorphic transition maps connecting them, so that the transition map is symplectomorphic. A change of local slice chart φ to φ' also leads to symplectomorphic transition maps, since by construction each of them is symplectomorphic. Crucially, this shows that there is a unique well-defined symplectic structure $\overline{\omega}$ on the quotient $\overline{S}$ such that

$$(\pi|_\Sigma)^*\overline{\omega}|_{\pi(\Sigma)} = \omega|_\Sigma$$

for every submanifold Σ. The uniqueness of the smooth structure follows from the standard fact that surjective submersions uniquely determine the smooth structure of the quotient, see for example Theorem 4.31 in [18]. □

Alternatively, once we have established the smooth structure of $\overline{S}$ such that π is a smooth submersion, we can define ω as the unique 2-form such that

$$\pi^*\overline{\omega} = \omega,$$

i.e, as the unique 2-form on the spaces

$$T_{[p]}\overline{S} \equiv T_pS/T_pS \cap T_pS^\omega$$

for $p \in S$ such that

$$d_p\pi^*\overline{\omega}_{[p]} = \omega_p.$$

In virtue of Lemma 3.9, part *(i)*, $\overline{\omega}$ is a symplectic form.

Remark 4.32 In Proposition 5.4.7 of [23] it can be seen how to generalise the second part of Lemma 3.9 to the case of Lagrangian submanifolds. It turns out that as in the linear case, we obtain that the projection of a Lagrangian submanifold is again a Lagrangian immersed submanifold in the quotient, if however only under enough regularity hypothesis. □

Consider now a Hamiltonian G-space (M, ω, Φ). Clearly, 0 is a fixed point of the G-action on its dual Lie algebra and hence is itself an orbit. In particular, it is G-invariant and the level set $S := \Phi^{-1}(0)$ is also G-invariant, i.e., it contains all its orbits. If furthermore 0 is a regular value of the moment map Φ, then S is an (embedded) submanifold of M and a G-manifold itself. We will argue for a general $\mu \in \mathfrak{g}^*$, but the case $\mu = 0$ can be taken as reference.

In fact, the properties of Φ turn $S_\mu := \Phi^{-1}(\mu)$ into a G_μ-submanifold whose null foliation is given by the G_μ-orbits. For the zero level set $S = \Phi^{-1}(0)$ this means that S is a coisotropic submanifold whose isotropic leaves are the G-orbits.

Recall that the map assigning the infinitesimal G-action at p for every $\xi \in \mathfrak{g}$ is given by the differential at $e \in G$ of the orbit map $j_p : G \to \mathcal{O}_p$, where $\mathcal{O}_p = \{gp : g \in G\}$, that is,

$$d_e j_p : \mathfrak{g} \to T_pM : \xi \mapsto \xi_p^\#.$$

Recall also that orbits are immersed submanifolds with tangent space $T_p\mathcal{O}_p = \operatorname{im} d_e j_p$, and that we denote the stabilizer of $p \in M$ by $G_p = \{g \in G : gp = p\}$, with Lie algebra $\mathfrak{g}_p = \ker d_e j_p$.

Lemma 4.33 *Consider a Hamiltonian G-space (M, ω, Φ) and $p \in M$. Then:*

(i) The symplectic complement of the kernel of $d_p\Phi$ is the tangent space of the orbit at p, i.e.,

$$(\ker d_p\Phi)^\omega = \operatorname{im} d_e j_p.$$

(ii) The image of $d_p\Phi$ is the annihilator of the Lie algebra of the stabilizer at p, i.e.,

$$\operatorname{im} d_p\Phi = (\ker d_e j_p)^0.$$

Proof To obtain part *(i)*, we notice that equation (4.4), $d_p\Phi^\xi(v) = \omega_p(\xi_p^\#, v)$ for $v \in T_pM$, implies that $v \in \ker d_p\Phi$ if and only if $\langle d_p\Phi(v), \xi\rangle = 0$ for all $\xi \in \mathfrak{g}$, if and only if $\omega_p(\xi_p^\#, v) = 0$ for all $\xi \in \mathfrak{g}$, i.e., $v \in (\operatorname{im} d_e j_p)^\omega$.

Similarly, for part *(ii)* we note that $d_p\Phi^\xi = i_{\xi_p^\#}\omega_p$ implies that $\operatorname{im} d_p\Phi \subset (\ker d_e j_p)^0$, and that from part *(i)* both subspaces have the same dimension:

$$\begin{aligned}\dim \operatorname{im} d_p\Phi &= \operatorname{codim} \ker d_p\Phi = \dim(\ker d_p\Phi)^\omega \\ &= \dim \operatorname{im} d_e j_p = \operatorname{codim} \ker d_e j_p = \dim(\ker d_e j_p)^0. \qquad \square\end{aligned}$$

This allows us to prove:

Proposition 4.34 *A point $\mu \in \mathfrak{g}^*$ is a regular value of Φ if and only if for all $p \in \Phi^{-1}(\mu)$ the stabilizer G_p is discrete. In that case, $\Phi^{-1}(\mu)$ is an embedded submanifold, the leaf of the null foliation through $p \in \Phi^{-1}(\mu)$ is the orbit $G_\mu \cdot p$, and $\dim G_\mu \cdot p = \dim G_\mu$.*

Proof A point $\mu \in \mathfrak{g}$ is a regular value if and only if $\operatorname{im} d_p\Phi = \mathfrak{g}^*$ for every $p \in \Phi^{-1}(\mu)$. The identity $\operatorname{im} d_p\Phi = (\ker d_e j_p)^0$ implies that $\operatorname{im} d_p\Phi = \mathfrak{g}^*$ if and only if $\mathfrak{g}_p = \ker d_e j_p = 0$, which in turn is equivalent to G_p being discrete (i.e., that the action is locally free at p).

Consider now the inclusion $i_\mu : \Phi^{-1}(\mu) \hookrightarrow M$ and $p \in \Phi^{-1}(\mu)$. On the one hand, we have that $(T_p\Phi^{-1}(\mu))^\omega = (\ker d_p\Phi)^\omega = \operatorname{im} d_e j_p = T_pG \cdot p$. On the other hand, we have that $T_pG_\mu \cdot p = d_e j_p^G(\mathfrak{g}_\mu) = d_e j_p(\ker d_e j_\mu^{\mathfrak{g}^*})$ ($j^G, j^{\mathfrak{g}^*}$ are the corresponding orbit maps). The G-equivariance of Φ can be written as $\Phi \circ j_p^G = j_{\Phi(p)}^{\mathfrak{g}^*} = j_\mu^{\mathfrak{g}^*}$, and thus $\ker d_e j_\mu^{\mathfrak{g}^*} = (d_e j_p^G)^{-1}(\ker d_p\Phi)$, i.e.

$$T_pG_\mu \cdot p = d_e j_p^G(\ker d_e j_\mu^{\mathfrak{g}^*}) = \ker d_p\Phi \cap \operatorname{im} d_e j_p^G = T_p\Phi^{-1}(\mu) \cap T_pG \cdot p.$$

With this, we have found that

$$\ker i_\mu^*\omega|_p = T_p\Phi^{-1}(\mu) \cap (T_p\Phi^{-1}(\mu))^\omega = T_p\Phi^{-1}(\mu) \cap T_pG \cdot p = T_pG_\mu \cdot p.$$

This means that the null foliation is given by the G_μ-orbits. The last claim about dimensions follows from the fact that $d_e j_p$ is injective if G_p is discrete. $\square$

As we said, we will address first the case $\mu = 0$. It is a regular value if the stabilizers of every point in $\Phi^{-1}(0)$ are discrete, and since $G_0 = G$, the null foliation is then given by the G-orbits. In particular, $\Phi^{-1}(0)$ is coisotropic because it is G-invariant and thus $(\ker d_p\Phi)^\omega = \operatorname{im} d_e j_p \subset T_p\Phi^{-1}(0) = \ker d_p\Phi$. Thus, in order to apply Proposition 4.31 we only need to check regularity. This we will do by asking for a stronger condition: that G acts properly (in particular if G is compact) and freely on the level set $\Phi^{-1}(0)$. As we know, this implies that the orbits are embedded submanifolds. However, it is not enough that 0 is a regular value Φ for the quotient to be a manifold.

Theorem 4.35 (The Marsden–Weinstein–Meyer Theorem) *Consider a Hamiltonian G-space (M, ω, Φ). Suppose further that G acts properly and freely on the level set $S := \Phi^{-1}(0)$. Then, 0 is a regular value of Φ and S is a regular coisotropic submanifold of M whose isotropic leaves are the G-orbits. The projection to the orbit quotient space*

$$\pi : \Phi^{-1}(0) \to \Phi^{-1}(0)/G$$

is a principal G-bundle and the quotient $\overline{S} := \Phi^{-1}(0)/G$ is a symplectic manifold of dimension

$$\dim \overline{S} = \dim M - 2\dim G.$$

The symplectic structure $\overline{\omega}$ is uniquely characterised by

$$\pi^*\overline{\omega} = i_S^*\omega$$

for the inclusion $i_S : S \hookrightarrow M$. The symplectic quotient manifold $(\overline{S}, \overline{\omega})$ is called the reduction*, the* symplectic quotient*, or the* Marsden–Weistein quotient *of (M, ω) with respect to G, Φ, denoted by*

$$M /\!\!/ G := (\overline{S}, \overline{\omega}).$$

Proof We have already done most of the work. Since the action is free in S, 0 is a regular value of Φ and S is a coisotropic manifold whose isotropic leaves are the G-orbits, and we conclude by proving that S is a regular coisotropic manifold. First, it is a standard result that the orbit space of a proper action is Hausdorff: the equivalence relation, regarded as the subset of $S \times S$ of related points, is precisely the image of the map $G \times X \to X \times X : (g, x) \mapsto (x, gx)$. Since this map is proper and X is locally compact Hausdorff, it is a closed map so that so is its image, and hence the quotient $S/\sim$ is Hausdorff (alternatively, since S is first-countable, we can argue with sequences). Second, the local slice theorem for proper free actions (used in the proof of Theorem 2.12, see again Theorem 21.10 in [18]) provides the regularity condition. Thus, the theorem follows from Proposition 4.31 except for the claim about $\pi : \Phi^{-1}(0) \to \Phi^{-1}(0)/G$ being a principal G-bundle, which follows from Theorem 2.12. There in particular we proved precisely this last claim; in fact, the smooth section $\sigma : V \to \pi^{-1}(V)$ inverse to π on $V \subset S/G$ can here be taken to be $\pi|_\Sigma^{-1}$ for Σ as in the definition of regular submanifold and V given by $\pi^{-1}(\pi(\Sigma))$. $\square$

Remark 4.36 Since $\Phi^{-1}(0)$ is always G-invariant, we always have the following chain complex for each $p \in \Phi^{-1}(0)$,

$$0 \longrightarrow \mathfrak{g} \xrightarrow{d_e j_p} T_p M \xrightarrow{d_p \Phi} \mathfrak{g}^* \longrightarrow 0.$$

Then, 0 is a regular value of Φ if and only if $d_e j_p$ is injective, if and only if $d_p\Phi$ is surjective, i.e., if the chain complex is exact at $\mathfrak{g}$ and $\mathfrak{g}^*$. Its homology at T_pM is then precisely the tangent space at $[p]$ of $\overline{S} = \Phi^{-1}(0)/G$:

$$T_{[p]}\overline{S} = \ker d_p\Phi / \operatorname{im} d_e j_p.$$

Lemma 3.9 says that the symplectic form ω_p on T_pM descends to a symplectic form on the quotient $\overline{\omega}_p$, and Theorem 4.35 says that whenever the action is proper and free on $\Phi^{-1}(0)$, then these forms all fit together on the quotient manifold $\overline{S}$ endowing it with a symplectic structure. $\square$

We now do symplectic reduction at a level set $\Phi^{-1}(\mu)$ different than $\mu = 0$, with analogous arguments.

Theorem 4.37 *Consider a Hamiltonian G-space (M, ω, Φ). Suppose that G_μ acts properly and freely on the level set $S_\mu := \Phi^{-1}(\mu)$. Then, μ is a regular value of Φ and S_μ is a regular submanifold of M whose null foliation leaves are the G_μ-orbits. The projection to the orbit quotient space*

$$\pi_\mu : \Phi^{-1}(\mu) \to \Phi^{-1}(\mu)/G$$

is a principal G_μ-bundle and the quotient $\overline{S}_\mu := \Phi^{-1}(\mu)/G$ is a symplectic manifold of dimension

$$\dim \overline{S}_\mu = \dim M - \dim G - \dim G_\mu.$$

The symplectic structure $\overline{\omega}_\mu$ is uniquely characterised by

$$\pi_\mu^* \overline{\omega}_\mu = i_\mu^* \omega$$

for the inclusion $i_\mu : S_\mu \hookrightarrow M$. The quotient manifold $(\overline{S}_\mu, \overline{\omega}_\mu)$, abbreviated M_μ, is called the symplectic reduction *of (M, ω) with respect to G, Φ at μ.*

Proof The only difference with the previous proof is that now S need not be coisotropic and instead Proposition 4.34 provides the corresponding null foliation in terms of G_μ-orbits. However this does not affect any of the arguments. $\square$

In fact, the symplectic quotients $(\overline{S}_\mu, \overline{\omega}_\mu)$ depend only on the coadjoint orbit $\mathcal{O} = G \cdot \mu$, i.e., reduced spaces at values of Φ in the same G-orbit are symplectomorphic. To see it, we consider the model space

$$M \times \mathcal{O}^-, \qquad \widetilde{\omega} := \omega \oplus -\omega_{\mathcal{O}}$$

with the diagonal G-action, where $\mathcal{O}^-$ denotes the orbit $\mathcal{O}$ with minus the KKS form, $-\omega_{\mathcal{O}}$. Clearly, $(M \times \mathcal{O}^-, \widetilde{\omega}, \widetilde{\Phi})$ is a Hamiltonian G-space with moment map

$$\widetilde{\Phi}(p, \mu) := \Phi(p) - \mu$$

for $p \in M$, $\mu \in \mathcal{O}$. Furthermore, the following allows to generalise the symplectic reduction construction to the preimage of any coadjoint orbit. It turns out that one recovers in this way the reduced spaces at non-zero levels.

Theorem 4.38 *Consider a Hamiltonian G-space (M, ω, Φ) and $\mu \in \mathfrak{g}^*$. Then, $\mu \in \mathfrak{g}^*$ is a regular value of Φ if and only if 0 is a regular value of $\widetilde{\Phi}$, and the G_μ-action on $\Phi^{-1}(\mu)$ is free if and only if the G-action on $\widetilde{\Phi}^{-1}(0)$ is free. If G acts properly and freely on $S_{\mathcal{O}} := \Phi^{-1}(\mathcal{O})$, then $\overline{S}_{\mathcal{O}} := \Phi^{-1}(\mathcal{O})/G$ is a manifold of dimension*

$$\dim \overline{S}_{\mathcal{O}} = \dim M - \dim G - \dim G_\mu$$

and

$$\pi_{\mathcal{O}} : S_{\mathcal{O}} \to \overline{S}_{\mathcal{O}}$$

is a principal G-bundle. It has a unique symplectic structure $\overline{\omega}_{\mathcal{O}}$ such that

$$\pi_{\mathcal{O}}^* \overline{\omega}_{\mathcal{O}} = i_{\mathcal{O}}^* \omega,$$

for $i_{\mathcal{O}} : S_{\mathcal{O}} \hookrightarrow M$, that is given by

$$\begin{aligned} \overline{\omega}_{[p]}([v_1], [v_2]) : &= \omega_p(v_1, v_2) - \langle \Phi(p), [\xi_1, \xi_2] \rangle \\ &= \omega_p(v_1 - \xi_{1,p}^\#, v_2 - \xi_{2,p}^\#), \end{aligned}$$

for $p \in S_{\mathcal{O}}$, $v_1, v_2 \in T_p S_{\mathcal{O}}$ and $\xi_1, \xi_2 \in \mathfrak{g}$ chosen such that

$$d_p\Phi(v_i) + \mathrm{ad}_{\xi_i}^* \Phi(p) = d_p\Phi(v_i - \xi_{i,p}^\#) = 0, \ \ i = 1, 2.$$

Furthermore, there are canonical symplectomorphisms

$$(\overline{S}_\mu, \overline{\omega}_\mu) \cong (\overline{S}_{\mathcal{O}}, \overline{\omega}_{\mathcal{O}}) \cong (M \times \mathcal{O}^-) /\!\!/ G.$$

Proof We have the G-equivariant diffeomorphism

$$\Psi : \Phi^{-1}(\mathcal{O}) \to \widetilde{\Phi}^{-1}(0) : p \mapsto (p, \Phi(p)),$$

so that the G-action on $\Phi^{-1}(\mathcal{O})$ is (locally) free if and only if the G-action on $\widetilde{\Phi}^{-1}(0)$ is. Since $\Phi^{-1}(\mathcal{O}) = G \cdot \Phi^{-1}(\mu)$, the G_μ-action on $\Phi^{-1}(\mu)$ is (locally) free if and only if the G-action on $\Phi^{-1}(\mathcal{O})$ is, and the first two claims follow.

The map $M \to M \times \mathcal{O}^- : p \mapsto (p, \mu)$ pulls back $\widetilde{\omega}$ to ω, and hence so does for the corresponding restricted forms its restriction $\Phi^{-1}(\mu) \to \widetilde{\Phi}^{-1}(0) : p \mapsto (p, \mu)$, coinciding with the restriction $\Psi|$ of Ψ to $\Phi^{-1}(\mu)$. If G acts properly and freely on $= \Phi^{-1}(\mathcal{O})$, from Ψ's G-equivariance we get that G acts properly and freely on $\widetilde{\Phi}^{-1}(0)$, and similarly G_μ acts properly and freely on $\Phi^{-1}(\mu)$ and thus also on $\Phi^{-1}(0) \cap M \times \{\mu\}$. Theorem 4.37 states that we can take quotients to obtain:

$$\begin{array}{ccc} \overline{S}_\mu = \Phi^{-1}(\mu)/G_\mu & \xrightarrow{\overline{\Psi|}} & \widetilde{\Phi}^{-1}(0) \cap M \times \{\mu\}/G_\mu \\ \big\downarrow & & \big\downarrow \\ \overline{S}_\mathcal{O} = \Phi^{-1}(\mathcal{O})/G & \xrightarrow{\overline{\Psi}} & M \times \mathcal{O}^- /\!\!/ G = \widetilde{\Phi}^{-1}(0)/G. \end{array}$$

These maps are furthermore all symplectomorphisms. The vertical maps, induced by the corresponding inclusions (before taking quotients), are diffeomorphisms because the set $\Phi^{-1}(\mu)$ intersects all G-orbits in $\Phi^{-1}(\mathcal{O})$, and analogously for $\widetilde{\Phi}^{-1}(0) \cap M \times \{\mu\}$ in $M \times \mathcal{O}^-$. They are symplectomorphic by construction. Since $\Psi|$ is preserves the 2-forms, it induces a symplectomorphism $\overline{\Psi|}$, and finally by commutativity so is the map $\overline{\Psi}$ induced by Ψ.

Theorem 4.35 furnishes the rest of the Theorem: $\widetilde{\Phi}^{-1}(0)$ is a regular coisotropic manifold whose isotropic leaves are the G-orbits, and the quotient $\widetilde{\Phi}^{-1}(0)/G$ is a manifold inheriting the symplectic structure $\widetilde{\omega}$ of $\widetilde{M}$ restricted to the submanifold

$$\widetilde{S} := \widetilde{\Phi}^{-1}(0) = \{(p, \mu) \in M \times \mathcal{O}^- : \mu = \Phi(p)\} = \{(p, \Phi(p)) : p \in M,\ \Phi(p) \in \mathcal{O}\}.$$

Then, for $\widetilde{p} = (p, \Phi(p)) \in \widetilde{\Phi}^{-1}(0)$ the tangent space is given by

$$T_{\widetilde{p}}\widetilde{S} = \ker d_{\widetilde{p}}\widetilde{\Phi} = \{(v, -\mathrm{ad}^*_\xi \Phi(p)) \in T_pM \times T_{\Phi(p)}\mathcal{O} : -\mathrm{ad}^*_\xi \Phi(p) = d_p\Phi(v)\}.$$

We notice that since $d_p\Phi(\xi^\#_p) = -\mathrm{ad}^*_\xi \Phi(p)$, $-\mathrm{ad}^*_\xi \Phi(p) = d_p\Phi(v)$ if and only if $d_p\Phi(v - \xi^\#_p) = 0$ so that

$$T_{\widetilde{p}}\widetilde{S} = \{(v, d_p\Phi(v)) : v \in T_pM,\ \exists \xi \in \mathfrak{g} \text{ s.t. } d_p\Phi(v - \xi^\#_p) = 0\}.$$

This translates via Ψ into $\Phi^{-1}(\mathcal{O})$ being a G-invariant embedded submanifold of M with tangent space at $p \in S_\mathcal{O}$ given by

$$T_pS_\mathcal{O} = \ker d_p\Phi + \mathrm{im}\, d_e j_p,$$

all whose points are regular points of Φ, and whose quotient $\Phi^{-1}(\mathcal{O})/G$ is a symplectic manifold. Its symplectic structure is given by

$$\begin{aligned}\overline{\omega}_{[p]}([v_1],[v_2]) :&= \widetilde{\omega}_{(p,\Phi(p))}((v_1, d_p\Phi(v_1)),(v_2, d_p\Phi(v_2)))\\ &= \omega_p(v_1,v_2) - \omega_{\mathcal{O},\Phi(p)}(\mathrm{ad}^*_{\xi_2}\Phi(p), \mathrm{ad}^*_{\xi_1}\Phi(p))\\ &= \omega_p(v_1,v_2) - \langle \Phi(p), [\xi_1,\xi_2]\rangle,\end{aligned}$$

for $v_1, v_2 \in T_pS_{\mathcal{O}}$ and $\xi_1,\xi_2 \in \mathfrak{g}$ chosen such that $d_p\Phi(v_i - \xi^{\#}_{i,p}) = 0,\ i = 1,2$. The alternative expression

$$\overline{\omega}_{[p]}([v_1],[v_2]) = \omega_p(v_1 - \xi^{\#}_{1,p}, v_2 - \xi^{\#}_{2,p})$$

comes from choosing as representatives of $[v_i]$ the vectors $v_i - \xi^{\#}_{i,p}$. It can also be checked directly using the properties of Φ. $\square$

4.4.1 Reduction in stages and partial reduction

Reduction can be done in stages, i.e., doing reduction first with respect to a subgroup. We will treat the most basic case, where we have G- and H-actions on (M,ω), both groups compact, such that the action map ψ^G of G commutes with the action map ψ^H of H, $[\psi^G_g, \psi^H_h] = 0$ for any $g \in G$, $h \in H$. This is equivalent to the fact that the G- and H-actions combine into a $(G\times H)$-action. For connected groups one has furthermore

$$[\psi^G_{\exp(t\xi)}, \psi^H_{\exp(t\eta)}] = 0 \iff [\xi^{\#}, \eta^{\#}] = 0$$

for any $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{h}$. We thus consider a $(G \times H)$-action on (M,ω). Then, the $G \times H$ is Hamiltonian if and only if each individual action is Hamiltonian, and the moment map $\widetilde{\Phi}$ of the $(G \times H)$-action is the direct sum $\widetilde{\Phi} = \Phi^G \oplus \Phi^H$ of the individual moment maps.

Clearly, given a $(G\times H)$-Hamiltonian space $(M,\omega,\widetilde{\Phi})$ we then have commuting Hamiltonian G- and H-actions with moment maps $\Phi^G = \pi^*_G\widetilde{\Phi}$, $\Phi^H = \pi^*_H\widetilde{\Phi}$ (for $\pi_G : G \times H \to G$, $\pi_H : G \times H \to H$), i.e., the components $\widetilde{\Phi} = \Phi^G \oplus \Phi^H$ in $(\mathfrak{g},\mathfrak{h})^* \cong \mathfrak{g}^* \oplus \mathfrak{h}^*$.

To see the reciprocal, we notice that in the $(G \times H)$-Hamiltonian space case, Φ^G is H-invariant and vice versa (since the adjoint action of G is trivial in $\mathfrak{h}^*$). Thus, given compact groups G and H, and Hamiltonian, commuting, G- and H-actions, we first make the moment maps Φ^G, Φ^H invariant under the other group action by averaging each over the other's Haar measure. Since the actions are symplectic and commute, averaging over H preserves the identity $i_{\xi^{\#}}\omega = d\Phi^G$ for any $\xi \in \mathfrak{g}$ as well as the G-equivariance $\Phi^G \circ \psi_g = \mathrm{Ad}^*_{g^{-1}}\Phi^G$ for $g \in G$. Once this is done and Φ^G, Φ^H are invariant, it then immediately follows that $\widetilde{\Phi} := \Phi^G \oplus \Phi^H$ defines a moment map for the induced Hamiltonian $(G \times H)$-action.

In this setting, we consider a regular value $\mu \in \mathfrak{g}^*$ of Φ^G. Since Φ^H is G-invariant, it descends to the symplectic quotient of (M, ω, Φ^G) at μ as a map $\Phi^H_\mu : \overline{S}_\mu \to \mathfrak{h}^*$. Analogously, the H-action descends to an H-action on $\overline{S}_\mu$, after noting that a level set of Φ^G is H-invariant and the H-action restricts to $(\Phi^G)^{-1}(\mu)$. As one expects, this defines a new Hamiltonian H-space:

Proposition 4.39 (Reduction in stages) *Consider compact Lie groups G, H, and a $(G \times H)$-Hamiltonian space $(M, \omega, \Phi^G \oplus \Phi^H)$. Suppose that $\mu \in \mathfrak{g}^*$ is a regular value of Φ^G and $(\mu, \nu) \in \mathfrak{g}^* \oplus \mathfrak{h}^*$ a regular value of $\Phi^G \oplus \Phi^H$. Then, ν is a regular value of Φ^H_μ. If G_μ acts freely on $(\Phi^G)^{-1}(\mu)$ and $G_\mu \times H_\nu$ acts freely on $(\Phi^G)^{-1}(\mu) \cap (\Phi^H)^{-1}(\nu)$, then H_ν acts freely on $(\Phi^H_\mu)^{-1}(\nu)$, and there is natural symplectomorphism*

$$(M_\mu)_\nu \cong M_{(\mu,\nu)}.$$

Proof We note that if $G_\mu \times H_\nu$ acts with finite (respectively trivial) stabilizers on $(\Phi^G)^{-1}(\mu) \cap (\Phi^H)^{-1}(\nu)$, then the same holds true for the H_ν-action on $(\Phi^H_\mu)^{-1}(\nu)$. Since a level set having discrete stabilizers is equivalent to the value being regular for the moment map, by Proposition 4.34, we obtain the first part. Now if additionally G_μ acts freely on $(\Phi^G)^{-1}(\mu)$, we can take its reduction M_μ. The second part then follows because the natural identification

$$(M_\mu)_\nu = (\Phi^H_\mu)^{-1}(\nu)/H_\nu \cong (\Phi^G)^{-1}(\mu) \cap (\Phi^H)^{-1}(\nu)/(G_\mu \times H_\nu) = M_{(\mu,\nu)}$$

preserves 2-forms. This in turn follows from the fact that both quotient manifolds are described by the same quotient, namely the submersion

$$(\Phi^G)^{-1}(\mu) \cap (\Phi^H)^{-1}(\nu) \to (\Phi^G)^{-1}(\mu) \cap (\Phi^H)^{-1}(\nu)/(G_\mu \times H_\nu),$$

which of course factors first through

$$(\Phi^G)^{-1}(\mu) \cap (\Phi^H)^{-1}(\nu) \to (\Phi^G)^{-1}(\mu) \cap (\Phi^H)^{-1}(\nu)/G_\mu = (\Phi^H_\mu)^{-1}(\nu)$$

followed by

$$(\Phi^G)^{-1}(\mu) \cap (\Phi^H)^{-1}(\nu)/G_\mu \to (\Phi^G)^{-1}(\mu) \cap (\Phi^H)^{-1}(\nu)/(G_\mu \times H_\nu). \qquad \square$$

We now obtain a similar result for a partial reduction when taking first reduction over a normal subgroup. For simplicity we argue for a torus. We recall from Example 4.14 that given Hamiltonian G-space (M, ω, Φ), and $i_H : H \hookrightarrow G$ a subgroup, then H acts by restriction in a Hamiltonian way with moment map $i^*_\mathfrak{h} \circ \Phi$, composing with the projection $i^*_\mathfrak{h} : \mathfrak{g}^* \to \mathfrak{h}^*$, dual to $i_\mathfrak{h} : \mathfrak{h} \hookrightarrow \mathfrak{g}$.

Proposition 4.40 (Partial reduction) *Consider a torus $T = \mathbb{T}^k$ and a T-Hamiltonian space (M, ω, Φ). Suppose that $H \subset T$ is a closed subgroup acting* *freely on the level set $\Phi_H^{-1}(0)$ for $\Phi_H = i_{\mathfrak{h}}^* \circ \Phi$. Then, the reduced space of (M, ω, Φ_H) at 0 is a (T/H)-Hamiltonian space with moment map $\overline{\Phi}_0$ such that*

$$\overline{\Phi}_0 \circ \pi_0^H = \Phi|_{\Phi_H^{-1}(0)},$$

where $\pi_0^H : \Phi_H^{-1}(0) \to \Phi_H^{-1}(0)/H$ is the projection.

Proof The first observation is that $Lie(T/H) \cong \mathfrak{t}/\mathfrak{h}$, so that the dual of the Lie algebra of T/H is $\pi_{\mathfrak{h}}^* : (\mathfrak{t}/\mathfrak{h})^* \cong \mathfrak{h}^0 \equiv \ker i_{\mathfrak{h}}^*$, the annihilator of $\mathfrak{h}$ in $\mathfrak{t}^*$, given the exact short sequence

$$0 \longrightarrow \mathfrak{h} \xrightarrow{i_{\mathfrak{h}}} \mathfrak{t} \xrightarrow{\pi_{\mathfrak{h}}} \mathfrak{t}/\mathfrak{h} \longrightarrow 0,$$

and its dual

$$0 \longrightarrow (\mathfrak{t}/\mathfrak{h})^* \xrightarrow{\pi_{\mathfrak{h}}^*} \mathfrak{t}^* \xrightarrow{i_{\mathfrak{h}}^*} \mathfrak{h}^* \longrightarrow 0.$$

Thus, the restriction $\Phi|_{\Phi_H^{-1}(0)} : \Phi_H^{-1}(0) \to \mathfrak{t}^*$ takes values in $\ker i_{\mathfrak{h}}^* = \mathfrak{h}^0 \cong (\mathfrak{t}/\mathfrak{h})^*$. It is T-invariant for the restriction of the T-action on $\Phi_H^{-1}(0)$ (this set is T-invariant since $\ker i_{\mathfrak{h}}^*$ is), since Φ itself is invariant (for the coadjoint action is trivial). For that reason, $\Phi|_{\Phi_H^{-1}(0)}$ factors uniquely through an induced map $\overline{\Phi}_0 : \Phi_H^{-1}(0)/H \to \mathfrak{h}^0$ such that

$$\overline{\Phi}_0 \circ \pi_0^H = \Phi|_{\Phi_H^{-1}(0)}.$$

To conclude, we just note that the T-action on $\Phi_H^{-1}(0)$ induces a symplectic T-action on the reduced space $(\Phi_H^{-1}(0)/H, \overline{\omega}_0^H)$, and this in turns factors through an induced T/H-action. This action is Hamiltonian with moment map $\overline{\Phi}_0$. $\overline{\Phi}_0$ is T/H-invariant because Φ was T-invariant, and for the weakly Hamiltonian condition we use the fact that π_0^H is a submersion and thus $\pi_0^{H,*}$ injective. Then we conclude from the identity

$$\begin{aligned}\pi_0^{H,*} \circ d\overline{\Phi}_0 = d\Phi|_{\Phi_H^{-1}(0)} = (i_{\xi^\#}\omega)|_{\Phi_H^{-1}(0)} = i_{\xi^\#_{\Phi_H^{-1}(0)}}(\pi_0^{H,*}\overline{\omega}_0^H)\\ = \pi_0^{H,*}(i_{\xi^\#_{\Phi_H^{-1}(0)/H}}\overline{\omega}_0^H),\end{aligned}$$

where $\pi_{0,*}^H \xi^\#_{\Phi_H^{-1}(0)} = \xi^\#_{\Phi_H^{-1}(0)/H}$. □

Example 4.41 (Torus action on $\mathbb{CP}^n$) Consider the $\mathbb{T}^{n+1}$-action on $\mathbb{C}^{n+1}$ described in Example 4.25. In Example 4.4 we took the quotient over the diagonal $\mathbb{S}^1$ subgroup and obtained $\mathbb{CP}^n$. That is, given the torus $T = \mathbb{T}^{n+1}$ acting on $(\mathbb{C}^{n+1}, \omega_0)$, we are doing partial reduction with respect to the closed subgroup $H \subset T$ given by the diagonal $\mathbb{S}^1 \subset \mathbb{T}^{n+1}$. In this case, we choose as moment map

$$\Phi : \mathbb{C}^{n+1} \to \mathfrak{t} : z \mapsto -\frac{1}{2}(|z_i|^2)_i + c,$$

for any $c \in \mathbb{R}^{n+1}$ such that $\sum_i c_i = 1/2$, so that the induced moment map $\Phi_H = i_{\mathfrak{h}}^* \Phi$ is now

$$\Phi_H(z) = -\frac{1}{2}||z||^2 + \frac{1}{2},$$

and as we checked in Example 4.25, the reduced space at 0 is $(\mathbb{CP}^n, \omega_{FS})$. To see it, one just checks that $i_{\mathfrak{h}} : \mathfrak{h} \cong \mathbb{R} \hookrightarrow \mathfrak{t} \cong \mathbb{R}^{n+1} : t \mapsto (t, \dots, t)$, so that the dual map is $i_{\mathfrak{h}}^* : \mathfrak{t}^* \cong \mathbb{R}^{n+1} \hookrightarrow \mathfrak{h}^* \cong \mathbb{R} : (\mu_0, \dots, \mu_n) \mapsto \sum_i \mu_i$. By the previous proposition, $(\mathbb{CP}^n, \omega_{FS})$ is a Hamiltonian T/H-space with moment map $\overline{\Phi} : \mathbb{CP}^n \to \mathfrak{h}^*$ such that

$$\overline{\Phi} \circ \mathrm{pr}_2 = \Phi|_{\mathbb{S}^{2n+1}},$$

for $\mathrm{pr}_2 : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ the projection, keeping with the notation of Appendix A.1. Under the identification $\mathfrak{t} \cong \mathbb{R}^{n+1}$, $\mathfrak{h}^* = \ker i_{\mathfrak{h}}^* \subset \mathfrak{t}^* \cong \mathbb{R}^{n+1}$ corresponds with $\{\mu \in \mathbb{R}^{n+1} : \sum_i \mu_i = 0\}$. If we pre-compose $\overline{\Phi} \circ \mathrm{pr}_2$ with the projection

$$\mathrm{pr}_1 := \varphi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{S}^{2n+1} : z \mapsto \frac{z}{|z|},$$

we obtain the pullback of the moment map $\overline{\Phi}$ to $\mathbb{C}^{n+1} \setminus \{0\}$ via $\mathrm{pr} = \mathrm{pr}_1 \circ \mathrm{pr}_2 : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. We obtain:

$$\overline{\Phi} \circ \mathrm{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathfrak{h}^0 : z \mapsto -\frac{(|z_i|^2)_i}{2||z||^2} - c,$$

where of course we can freely choose the constant c (as long as $i_{\mathfrak{h}}^*(c) = 1/2$, this is only due to the fact we are viewing the dual of the Lie algebra of T/H as a subset of $\mathbb{R}^{n+1}$). $\square$

4.5 The cotangent bundle of a Lie group

We now introduce the last ingredient needed to obtain the normal form Theorem 5.12 in the next chapter. We let G be a compact Lie group and consider its tangent and cotangent bundles. We recall that we work with left actions, so that we have the natural G-self-action on the left, and the left action given by right multiplication, $R_{g^{-1}} : G \to G : h \mapsto hg^{-1}$, that we call the right action. Clearly, the right and left actions commute, and thus, as in the previous subsection, we can think of them as a $(G \times G)$-action.

We will call *left-trivialization of* TG to the vector bundle isomorphism

$$l_* : G \times \mathfrak{g} \to TG : (g, \xi) \mapsto (g, g\xi).$$

Similarly, the *left-trivialization of* T^*G is the vector bundle isomorphism

$$l^* : G \times \mathfrak{g}^* \to TG : (g, \mu) \mapsto (g, g^{-1,*}\mu).$$

From now onwards, we will identify TG and T^*G with $G \times \mathfrak{g}$ and $G \times \mathfrak{g}^*$ in this way and treat them indistinctly. In particular, we can apply the cotangent bundle constructions of Section 3.3 to T^*G and obtain an induced symplectic structure in $G \times \mathfrak{g}^*$ from left-trivialization.

The left-invariant vector field ξ^L becomes the constant section ξ in $G \times \mathfrak{g}$ under left-trivialization, while the right-invariant vector field ξ^R becomes $\mathrm{Ad}_{g^{-1}}\xi$. Similarly, we can consider the maps induced in $G \times \mathfrak{g}$ and $G \times \mathfrak{g}^*$ by the left and right actions, that is:

Lemma 4.42 *Under left-trivialization of* TG*, the tangent maps to inversion* $\mathrm{inv} : G \to G : g \mapsto g^{-1}$*, left action* $L_h : G \to G : g \mapsto hg$*, and right action* $R_{h^{-1}} : G \to G : h \mapsto gh^{-1}$ *are given by*

$$\begin{aligned}
\mathrm{inv}_*(g, \xi) &= (g^{-1}, -\mathrm{Ad}_g\xi),\\
(L_h)_*(g, \xi) &= (hg, \xi),\\
(R_{h^{-1}})_*(g, \xi) &= (gh^{-1}, \mathrm{Ad}_h\xi).
\end{aligned}$$

Similarly, the respective cotangent lifts are given by

$$\begin{aligned}
\widehat{\mathrm{inv}}(g, \mu) &= (g^{-1}, -\mathrm{Ad}^*_{g^{-1}}\mu),\\
\widehat{L_h}(g, \mu) &= (hg, \mu),\\
\widehat{R_{h^{-1}}}(g, \mu) &= (gh^{-1}, \mathrm{Ad}^*_{h^{-1}}\mu).
\end{aligned}$$

Proof We compute

$$\begin{aligned}
l_*^{-1} \circ \mathrm{inv}_* \circ l_*(g, \xi) &= l_*^{-1}(g^{-1}, d_g\mathrm{inv}(g\xi)) = l_*^{-1}(g^{-1}, (d_e\mathrm{inv}\xi)g^{-1})\\
&= (g^{-1}, g(-\xi g^{-1})) = (g^{-1}, -\mathrm{Ad}_g\xi).
\end{aligned}$$

For the left action we have

$$l_*^{-1} \circ (L_h)_* \circ l_*(g, \xi) = l_*^{-1}(hg, (hg)\xi) = (hg, \xi),$$

while for the right action

$$l_*^{-1} \circ (R_{h^{-1}})_* \circ l_*(g, \xi) = l_*^{-1}(gh^{-1}, g\xi h^{-1}) = (gh^{-1}, h\xi h^{-1}) = (gh^{-1}, \mathrm{Ad}_h\xi).$$

The computations for the cotangent bundle are entirely similar. □

Since the generating vector fields for the left and right action in G are ξ^R and $-\xi^L$ respectively, according to Proposition 4.15 we find that the moment maps for these two actions on T^*G are $\Phi_L^\xi(g,\mu_g) = \langle \mu_g, \xi^R\rangle$ and $\Phi_R^\xi(g,\mu_g) = \langle \mu_g, -\xi^L\rangle$. Translated to $G\times\mathfrak{g}^*$ they become:

$$\Phi_L(g,\mu) = \mathrm{Ad}_{g^{-1}}\mu, \qquad \Phi_R(g,\nu) = -\nu.$$

Note that the cotangent lifts of both the left action and the right action are free, since both actions are free on G. In particular, every $\mu\in\mathfrak{g}^*$ is a regular value for both moment maps.

Theorem 4.43 *Let G be a compact group. Then, the symplectic reduction $(T^*G)_\nu$ by the right action, with the Hamiltonian action inherited from the left action, is identified via the induced moment map $\Phi_{L,\nu}$ with the coadjoint orbit $G\cdot(-\nu)$.*

Analogously, the symplectic reduction $(T^*G)_\mu$ by the left action, with the Hamiltonian action inherited from the right action, is identified via the induced moment map $\Phi_{R,\mu}$ with the coadjoint orbit $G\cdot(-\mu)$.

Proof We note that all groups and spaces are compact, and that both actions commute and are free. Thus, Proposition 4.39 allows us to take symplectic reduction in stages and any order. Secondly, since both actions are transitive in G, in particular the induced left action on $(T^*G)_\nu \equiv (G\times\mathfrak{g}^*)_\nu = G\times\{-\nu\}/G_\nu$ is transitive, with $G_\nu = \{h : \mathrm{Ad}^*_h\nu = \nu\}$. Thus, we can apply Theorem 4.27, and we obtain that the induced moment map $\Phi_{L,\nu} : (T^*G)_\nu \to G\cdot(-\nu)$ is not just a covering space but a diffeomorphism, since now the stabilizer at a point $[(g,-\nu)]\in(G\times\mathfrak{g}^*)_\nu$ coincides with the stabilizer for the coadjoint action, G_ν.

For the second part, we either use the same arguments interchanging the roles of the left and right actions after noting that the right action is transitive on $(G\times\mathfrak{g}^*)_\mu = \{(g,\nu) : \mathrm{Ad}_{g^{-1}}\nu = \mu\}/G_\mu$, or the fact that $\widehat{\mathrm{inv}}$ is a symplectomorphism that exchanges the left and right actions. □

Notice that in particular, if $\mu = 0$ or $\nu = 0$, we obtain trivial spaces after reduction, i.e., $T^*G /\!\!/ G \cong \{0\}$ under either the left or right actions.

Theorem 4.44 *Let (M,ω,Φ) be a Hamiltonian G-space. Let G act diagonally on $T^*G\times M$ as a Hamiltonian G-space, where the G-action in the T^*G factor is the right action. Consider the reduced space at 0 as a Hamiltonian G-space $(T^*G\times M) /\!\!/ G$, where the G-action is induced from the left action on T^*G. Then, there is a canonical isomorphism of Hamiltonian G-spaces*

$$(T^*G\times M) /\!\!/ G \cong M.$$

Proof We work with $G\times\mathfrak{g}^*$ under left-trivialization. The map

$$\theta : (G\times\mathfrak{g}^*)\times M \to (G\times\mathfrak{g}^*)\times M : (g,\mu,p)\mapsto(g,\mu,gp)$$

is symplectic, and takes the diagonal action to the right action on the first factor:

$$\theta\circ((R_{h^{-1}})_*\times\psi_h)\circ\theta^{-1}(g,\mu,p)=\theta(gh^{-1},\mathrm{Ad}^*_{h^{-1}}\mu,hg^{-1}p)=(gh^{-1},\mathrm{Ad}^*_{h^{-1}}\mu,p),$$

while it takes the left action on T^*G to the diagonal action induced by the left action on T^*G and the G-action on M:

$$\theta\circ((L_h)_*\times\mathrm{id}_M)\circ\theta^{-1}(g,\mu,p)=\theta(hg,\mu,g^{-1}p)=(hg,\mu,hp).$$

Thus, we have a G-equivariant symplectomorphism of Hamiltonian G-spaces

$$(G\times\mathfrak{g}^*)\times M /\!\!/ G\cong(G\times\mathfrak{g}^* /\!\!/ G)\times M\cong\{0\}\times M\equiv M$$

that intertwines the action $L_h\times\mathrm{id}_M$ on $(G\times\mathfrak{g}^*)\times M$ with the G-action on M. $\square$

4.6 Torus-actions

As we have already seen, an important case of Hamiltonian actions is that of compact and abelian group actions, i.e., torus-actions. In this section we present an overview of the main definitions and results regarding Hamiltonian torus-actions, as one of the best understood cases. The key results are the Convexity Theorem and the Delzant Classification Theorem.

4.6.1 The Convexity Theorem

A central result discovered independently and simultaneously by Atiyah [1] and by Guillemin–Sternberg [15] is the fact that the image of a moment map of a Hamiltonian torus-action over a (connected) closed manifold is convex:

Theorem 4.45 (Convexity Theorem, [1], [15]) *Let (M,ω) be a compact, connected symplectic manifold and let $T=\mathbb{T}^k$ be a torus. Suppose that we have a Hamiltonian action $\psi:\mathbb{T}^k\to\mathrm{Symp}(M,\omega)$ with moment map $\Phi:M\to\mathfrak{t}\cong\mathbb{R}^k$. Then:*

1. *The level sets $\Phi^{-1}(\mu)$ are connected, for any $\mu\in\mathfrak{t}$.*
2. *The image $\Phi(M)$ is the convex hull of the fixed points of the action.*

The image $\Phi(M)$ of the moment map is called the *moment polytope.*

Proof Other than the original papers, a good reference is [23], §5.5, where everything is self-contained except for the results concerning dynamical systems.

The proof is by induction on the dimension k of the torus acting on M, and at each inductive step, convexity follows from first obtaining that the level sets are connected. To do so, one needs to introduce the notion of a *Morse–Bott* function, a generalisation of Morse functions where the critical set may be a submanifold of any dimension (i.e., not just of dimension 0) with the caveat that $\ker d^2 f = T_x\mathrm{Crit}(f)$ for any $x \in \mathrm{Crit}(f) := \{x \in M : d_x f = 0\}$ ($d^2 f$ is the symmetric bilinear form induced in $T_x M$ at a critical point x). One then proves that the functions Φ^ξ for the moment map of a Hamiltonian torus-action are Morse–Bott. It follows from dynamical systems theory that the level sets of these are connected (and hence convex, as they are 1-dimensional), and thus the case $k = 1$ of an $\mathbb{S}^1$-action is settled.

It is interesting to note that for any $\xi \in \mathfrak{t}$, the critical set is given by

$$\mathrm{Crit}(\Phi^\xi) = \bigcap_{t \in T_\xi} \mathrm{Fix}(\psi_t),$$

where $T_\xi := \exp_T(\mathbb{R}\xi) \subset T$ (with Lie algebra $\mathfrak{t}_\xi \subset \mathfrak{t}$), since $p \in \mathrm{Crit}(\Phi^\xi)$ if and only if $d_p \Phi^\xi = 0$ and by Lemma 4.33 applied to the restriction of the T-action to a T_ξ-action, this is equivalent to $\ker d_e j_p|_{T_\xi} = \mathfrak{t}_\xi$, i.e., $d_e j_p|_{T_\xi} = 0$. By connection of T_ξ, this condition is equivalent to p being fixed by all elements of T_ξ. Furthermore, this subset $\mathrm{Crit}(\Phi^\xi) \subset M$ turns out to be a symplectic, embedded submanifold. To see it, we note that there exists an almost complex structure J on M that is compatible with ω and T-invariant, i.e., $\psi_{t,*} \circ J = J \circ \psi_t$ (fixing a Riemannian metric g_0 on M and averaging it over T, we obtain an invariant metric g. The image of g under the map $F : \mathrm{Riem}(V) \to \mathcal{J}(V, \omega)$ of Theorem 3.38 is a T-invariant almost complex structure compatible with ω.)

Then, one shows that for any subgroup $H \subset T$, the fixed point set

$$\mathrm{Fix}(H) = \bigcap_{t \in H} \mathrm{Fix}(\psi_t),$$

is a symplectic submanifold of M. If we let $x \in \mathrm{Fix}(G)$ and $t \in H$, then the differential $d_x\psi_t : T_x M \to T_x M$ provides a unitary map of $T_x M$, i.e., $t \mapsto d_x \psi_t$ is a unitary G-action on the complex symplectic vector space $(T_x M, \omega_x, J_x)$. Now consider the Riemannian exponential map $\exp_x : T_x M \to M$; by construction it is equivariant, i.e.

$$\exp_x(d_x\psi_t(u)) = \psi_t(\exp_x(u)).$$

Hence, the fixed points of H near x correspond to the fixed points of H on the tangent space $T_x M$. In other words,

$$T_x\mathrm{Fix}(H) = \bigcap_{t \in H} \ker(\mathrm{id} - d_x\psi_t).$$

Since the linear maps $d_x\psi_t$ are unitary transformations of $T_x M$, it follows that the eigenspace associated to 1 is invariant under J_x and therefore a symplectic subspace.

To see the full argument we again refer to [23], §5.5. $\square$

Example 4.46 In Example 4.41 we obtained that $(\mathbb{CP}^n, \omega_{FS}, \Phi)$ is a Hamiltonian T/H-space with the moment map

$$\Phi : \mathbb{CP}^n \to \mathfrak{h}^0 : [z] \mapsto -\frac{(|z_i|^2)_i}{2||z||^2} - c,$$

for $c \in \mathbb{R}^{n+1}$ such that $\sum_i c_i = 1/2$. Here, $T = \mathbb{T}^{n+1}$ and $H \cong \mathbb{S}^1$ is the diagonal group. Thus, $T/H \cong \mathbb{T}^n$, i.e., it is an n-torus action. Clearly, its image is the convex hull of the image of the points $q_i = [e_i] \in \mathbb{CP}^n$, where $\{e_i\}_i \subset \mathbb{R}^{n+1}$ is the standard basis. These points are of course the fixed points of the action. □

4.6.2 Symplectic toric manifolds

The Hamiltonian $\mathbb{T}^n$-action on $\mathbb{CP}^n$ is the paradigmatic example of a *symplectic toric manifold*. Before introducing them, we recall that:

Definition 4.47 (Faithful action) A G-action on a manifold M is *faithful* if the map $\psi : G \to \mathrm{Diff}(M)$ is injective, i.e., if each group element $g \in G$ moves at least one point of M.

Alternatively, this means that $\cap_{p\in M} G_p = \{e\}$ for the stabilizers G_p of $p \in M$. Clearly, given a torus-action on a manifold M of T, we can always assume that it is faithful simply by taking the quotient of T over the closed group $H := \cap_{p\in M} T_p$ (we need T to be abelian to ensure that H is a normal subgroup). Since the action of H is trivial, this induces a T/H-action on M with the same properties as the original action (i.e., symplectic or Hamiltonian), where of course T/H is another torus of possibly smaller dimension than T.

Definition 4.48 (Symplectic toric manifold) A *symplectic toric manifold* is a connected, closed, symplectic manifold (M, ω) of dimension $2n$ together with a Hamiltonian and faithful T-action of a torus $T = \mathbb{T}^n$ of dimension n. We will sometimes denote a symplectic toric manifold as (M, ω, Φ, T).

Two symplectic toric manifolds, $(M_i, \omega_i, \Phi_i, T_i)$, $i = 1, 2$, are *equivalent* if there exists an isomorphism $\lambda : T_1 \to T_2$ and a λ-equivariant symplectomorphism $\psi : M_1 \to M_2$ such that $\Phi_2 \circ \psi = \Phi_1$

Example 4.49 Our other recurring example, the $\mathbb{S}^1$-action on $\mathbb{S}^2$ by rotation as in Example 4.3, is also a symplectic toric manifold. □

Remark 4.50 Given a symplectic toric manifold (M, ω) with moment map $\Phi : M \to \mathfrak{t}$, a generic fibre $\Phi^{-1}(\mu)$ of the moment is a T-orbit and hence is a Lagrangian embedded torus. □

The following results use the fact that a faithful torus-action of $T = \mathbb{T}^k$ always has orbits of dimension k. This result can be found for example in [5].

Corollary 4.51 *Let (M, ω) be a compact, connected symplectic manifold and let $T = \mathbb{T}^k$ be a k-torus acting faithfully and in a Hamiltonian way on M. Then there must be at least $k+1$ fixed points of the T-action.*

Proof Consider a point $p \in M$ of a k-dimensional orbit; then the moment map Φ is a submersion at p, that is, the components $d_p\Phi^j$ are linearly independent and $\Phi(p)$ is an interior point of the moment polytope $\Phi(M)$. Thus, $\Phi(M)$ is a non-degenerate convex polytope in $\mathbb{R}^k$ which must then have at least $k+1$ vertices. □

Corollary 4.52 *Let (M^{2n}, ω, Φ) be a Hamiltonian $\mathbb{T}^k$-space. If the $\mathbb{T}^k$-action is faithful, then $k \leq n$.*

Proof Consider again some k-dimensional orbit $\mathcal{O}$. Since the moment map Φ is constant on $\mathcal{O}$, given $p \in \mathcal{O}$ the differential $d_p\Phi : T_pM \to \mathfrak{t}^*$ maps $T_p\mathcal{O}$ to 0 and as in Section 4.4, $T_p\mathcal{O} \subset \ker d_p\Phi = (T_{\mathcal{O}})^\omega$, that is, $\mathcal{O}$ is an isotropic submanifold of M. In particular, by symplectic linear algebra, we must have $k = \dim \mathcal{O} \leq n$. □

Thus, symplectic toric manifolds represent the case of a Hamiltonian, faithful, torus-action of maximal dimension, i.e., half of the dimension of M. Since a torus-action can be assumed to be faithful, the key of the definition lies within this dimensionality.

4.6.3 Delzant's Classification Theorem

It turns out that if on top of a Hamiltonian, faithful, torus-action we have a symplectic toric manifold, i.e., M is closed and connected and the dimension of the torus is maximal and equal to the half-dimension of M, then the Convexity Theorem can be significantly strengthened in the following sense. The moment polytope of a symplectic toric manifold must satisfy strict conditions and furthermore symplectic toric manifolds are classified up to equivariant symplectomorphism by their moment polytopes. This is the content of the Delzant Classification Theorem. We begin by specifying what these conditions for such a polytope are:

Definition 4.53 (Delzant polytope) A *Delzant polytope* $\Delta \subset \mathbb{R}^n$ (or also an *n-Delzant polytope*) is a polytope[1] that satisfies:

(1) *simplicity*: there are n edges meeting at each vertex;

(2) *rationality*: the edges meeting at the vertex p are rational in the sense that each edge is of the form $p + tu_i$, $t \geq 0$ and $u_i \in \mathbb{Z}^n$;

[1] We recall that a polytope is simply the convex hull of a finite number of points in the affine space $\mathbb{R}^n$. We call these generating points vertices if removing one of them changes the polytope. We call edge any 1-dimensional segment between vertices that lies in the boundary of the polytope, and face the intersection of the polytope with the affine hyperspace defined by any $n-1$-edges if such intersection is contained in the boundary.

(3) *smoothness*: for each vertex p, the corresponding vectors u_i can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

For example, an isosceles triangle whose equal sides have length 1 and are parallel to the standard basis of $\mathbb{R}^2$ is a 2-Delzant polytope, however changing the length of one of the equal sides (of course such that the slope is still rational) will only be again Delzant if the change is an increase of length.

Delzant's Theorem classifies the equivariant-symplectomorphism classes of symplectic toric manifolds in terms of the combinatorial data encoded in their Delzant polytopes:

Theorem 4.54 (Delzant, [11]) *Symplectic toric manifolds are classified up to equivariant symplectomorphism by their Delzant polytopes. In particular, there is a bijective correspondence given by the image of the moment map between the sets:*

$$\left\{ \begin{array}{c} \textit{symplectic toric mflds.} \\ \textit{up to equiv.-symp.} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Delzant polytopes} \\ \textit{up to translation} \end{array} \right\}$$
$$(M, \omega, \Phi, T) \longmapsto \Phi(M).$$

Proof For the well-definedness and uniqueness statement, we refer to the original paper [11]. The surjectivity statement is however easy and constructive and we present it here, where we are closely following [6].

Given a Delzant polytope $\Delta \subset \mathbb{R}^n$, one can construct a symplectic toric manifold $(M_\Delta, \omega_\Delta, \Phi_\Delta, T_\Delta)$ such that $\Phi_\Delta(M_\Delta) = \Delta$ as follows. We let $v_i \in \mathbb{Z}^n$ be the primitive[2] outward-pointing normal vectors to the $d \geq n+1$ faces of Δ. Then, for some $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, d$, we can write $\Delta = \{\mu \in \mathbb{R}^n : \langle \mu, v_i \rangle \leq \lambda_i,\ \forall i\}$. Then, if $\{e_i\}_i^d$ is the standard basis of $\mathbb{R}^d$, we define the map

$$\begin{aligned} \pi : \mathbb{R}^d &\to \mathbb{R}^n, \\ e_i &\mapsto v_i. \end{aligned}$$

By the simplicity, rationality and smoothness of Δ, π restricts to a surjective map $\pi|_{\mathbb{Z}^d} : \mathbb{Z}^d \to \mathbb{Z}^n$. Thus, we can take the quotient of each affine space over the integer lattice and π induces a well-defined surjective map

$$\pi : \mathbb{R}^d/\mathbb{Z}^d \equiv \mathbb{T}^d \to \mathbb{R}^n/\mathbb{Z}^n \equiv \mathbb{T}^n$$

(we abuse the notation and also call π to the induced map between tori). The kernel of this map is a closed torus $i : N \hookrightarrow \mathbb{T}^d$ of dimension $n - d$ with Lie algebra $i_* : \mathfrak{n} \hookrightarrow \mathbb{R}^d$. Our candidate for $(M_\Delta, \omega_\Delta, \Phi_\Delta, T_\Delta)$ is going to be the partial symplectic reduction (Proposition 4.40) of the standard $\mathbb{T}^d$-action on

[2] An integer vector $v \in \mathbb{Z}^n$ is primitive if it cannot be expressed as $v = ku$ for $u \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$ with $|k| > 1$.

$(\mathbb{C}^d, \omega_0)$ with respect to $N \subset \mathbb{T}^d$, thus taking $T_\Delta := \mathbb{T}^d/N$. For the construction to work, we need to fix the constant of the moment map so that it is given by

$$\Phi : \mathbb{C}^d \to \mathbb{R}^d : z \mapsto (-\frac{|z_i|^2}{2} + \lambda_i)_i.$$

We start by noticing that for the dual map $i^* : \mathbb{R}^d \to \mathfrak{n}^*$, the moment map of the restricted N-action is $\varphi := i^* \circ \Phi : M \to \mathfrak{n}^*$. Then, the Convexity Theorem 4.45 states that $Z := \varphi^{-1}(0) \subset \mathbb{C}^d$ is connected, but it is actually compact and thus φ proper. One just notices that $\Phi(Z) = \pi^*(\Delta)$ since for a given $y = \pi^*(x)$, for some $x \in (\mathbb{R}^n)^*$, we have (using that $\pi(e_i) = v_i$):

$$\begin{aligned} y \in \operatorname{im} \Phi &\iff \langle y, e_i \rangle \leq \lambda_i, \; i = 1, \dots, d \\ &\iff \langle x, v_i \rangle \leq \lambda_i, \; i = 1, \dots, d \\ &\iff x \in \Delta. \end{aligned}$$

From this together with the fact that $\operatorname{im} \pi^* = \ker i^*$, as one sees from the short exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \longrightarrow 0.$$

we see that $\Phi(Z) = \operatorname{im} \Phi \cap \ker i^* = \operatorname{im} \Phi \cap \operatorname{im} \pi^*$ and hence $\Phi(Z) = \pi^*(\Delta)$. In particular, $\varphi(Z) = i^* \circ \Phi(Z)$ is compact, and since φ is continuous, it is proper.

The next thing to check is that N acts freely on Z and this is also easily seen. The stabilizer of a point $z \in \mathbb{C}^d$ under the $\mathbb{T}^d$-action will be the subtorus generated by the standard basis vectors $e_i \in \mathbb{R}^d$ associated to the zero coordinates z_i of z. The worst possible case is the following: we pick a vertex $p \in \Delta$ and let $I = \{i_j\}_{j=1}^n$ be the set of indices i_j for the n facets meeting at p. Given $z \in Z$ such that $\Phi(z) = \pi^*(p)$, since p is given by the equations $\langle p, v_i \rangle = \lambda_i$, $i \in I$, we see that:

$$\begin{aligned} \langle p, v_i \rangle = \lambda_i &\iff \langle \pi^*(p), e_i \rangle = \lambda_i \\ &\iff \langle \Phi(z), e_i \rangle = \lambda_i \\ &\iff -\frac{|z_i|^2}{2} + \lambda_i = \lambda_i \\ &\iff z_i = 0. \end{aligned}$$

I.e., those points z in the preimage of a vertex p are points whose coordinates in the set I are zero, and whose other coordinates are nonzero. Without loss of generality, we can assume that $I = \{1, \dots, n\}$ so that the stabilizer of z is

$$T_z = \mathbb{T}^n \times \{1\} \times \cdots \times \{1\} \subset \mathbb{T}^d.$$

Then the restriction $\pi| : T_z \to \mathbb{R}^n$ is bijective, since it maps the independent set $\{v_i\}_{i=1}^n$ to the standard basis of $\mathbb{R}^n$. In particular, $N \cap T_z = \{1\}$, and all

N-stabilizers at points mapping to vertices are trivial. Since this is the worst case, the N-action restricts to a free action on Z.

The claim now follows from Proposition 4.40 after choosing the following canonical representation of T/N as a standard torus $\mathbb{T}^n$: the short exact sequence

$$0 \longrightarrow N \xhookrightarrow{i} T \xrightarrow{\pi} \mathbb{T}^n \longrightarrow 0,$$

induces a Lie group isomorphism

$$\overline{\pi} : T/N \overset{\cong}{\to} \mathbb{T}^n : [t] \mapsto \pi(t),$$

The desired isomorphism is $\overline{\pi}^{-1} : \mathbb{T}^n \cong T/N$, since now the image of the induced moment map $(\overline{\pi}^{-1})^* \circ \overline{\Phi} : M_\Delta \to (\mathbb{T}^n)^*$ is (with the notation $\overline{\Phi}$ of the moment map induced by Φ introduced in Proposition 4.40):

$$\operatorname{im}((\overline{\pi}^{-1})^* \circ \overline{\Phi}) = (\overline{\pi}^{-1})^*(\operatorname{im}\overline{\Phi}) = (\overline{\pi}^{-1})^*(\Phi(Z)) = (\overline{\pi}^{-1})^* \circ \pi^*(\Delta) = \Delta,$$

where at the end we use that $\pi \circ \overline{\pi}^{-1} = \mathrm{id}_{\mathbb{T}^n}$ and hence $(\overline{\pi}^{-1})^* \circ \pi^* = \mathrm{id}_{\mathbb{R}^n}$. □

4.7 Orbifold singularities

In this section, we briefly comment on how to generalise symplectic reduction to the case when the G-action is not free on a level set but only locally free, that is, when the stabilizers are discrete. It turns out that the same construction can be carried out with no modifications, except for the fact that the quotient space need not be a manifold. The issue at hand are the discrete stabilizers that will further quotient an otherwise locally euclidean space. This can be seen when we apply the Quotient Theorem 2.12, where the action must be free so that a slice chart of the type described can be found, i.e., one where different slices belong to different orbits. This is no longer the case for a discrete stabilizer, but instead finitely many slices may belong to the same orbit for arbitrarily small slice neighbourhoods.

The solution to this problem is the generalisation of the notion of a smooth manifold due to Satake in [26]: orbifolds. These are singular manifolds where the singularities are modelled on the quotient of the euclidean space $\mathbb{R}^m$ by some finite group Γ acting smoothly.

Definition 4.55 (Orbifold chart) Let M be a Hausdorff, second countable topological space. An *orbifold chart* on M is a triple (V, Γ, φ) where Γ is a finite group acting faithfully and smoothly on $\mathbb{R}^m$, $V \subset \mathbb{R}^m$ is a Γ-invariant domain (connected open subset) such that the set of points where the Γ-action is not free has codimension at least two, and $\varphi : V \to M$ is a Γ-invariant map that induces a homeomorphism over its image, $V/\Gamma \to \varphi(V) := U \subset M$. Two orbifold charts on M $(V_i, \Gamma_i, \varphi_i)$, $i = 1, 2$, are *compatible* if whenever $U_1 \subset U_2$ then there exists an injective homomorphism $\lambda : \Gamma_1 \to \Gamma_2$ called the *gluing map*, and a λ-equivariant open embedding $\psi : V_1 \to V_2$, such that $\varphi_1 = \varphi_2 \circ \psi$.

An *orbifold atlas* is a family of pairwise-compatible orbifold charts on M whose images $\{U\}$ provide a basis of open sets and such that the gluing maps are unique up to composition with group elements, i.e., given $\psi', \psi : V_1 \to V_2$ then $\psi' = g\psi$ for a unique $g \in \Gamma_2$. Two orbifold atlases are *equivalent* if their union is also an orbifold atlas.

Thus, our definition of orbifold is:

Definition 4.56 (Orbifold) An *m-dimensional orbifold* is a Hausdorff, second countable topological space M together with an equivalence class of orbifold atlases on M.

Of course, ordinary manifolds are the particular case of orbifolds where the groups Γ are all trivial. It turns out that for a fixed $p \in M$, one can always choose an orbifold chart (V, Γ, φ) around it such that $\varphi^{-1}(p)$ is a single point, or alternatively such that $\Gamma_q = \Gamma$ for any $q \in \varphi^{-1}(p)$. Such a chart is called a *structure chart for p*. In that case, one can define the tangent space at $p \in M$ as

$$T_pM := T_{\varphi^{-1}(p)}V/\Gamma^p_*,$$

where $\Gamma^p_* := \{d_{\varphi^{-1}(p)}\psi_\gamma : \gamma \in \Gamma\} \subset GL(T_{\varphi^{-1}(p)}V)$. In general, most of the differential calculus and geometry carries over to orbifolds (for example, *Riemannian orbifolds* arise from orbifold charts together with a Γ-invariant metric on V). One can even define *symplectic toric orbifolds* via the same definition for orbifolds. In fact, they have also been classified in a generalisation of Delzant's Theorem due to Lerman and Tolman [20]: a symplectic toric orbifold is characterised up to equivariant symplectomorphism by its moment polytope in combination with a positive integer label attached to each face of the polytope. These polytopes are more general than the Delzant polytopes, in that only simplicity and rationality are required; the edge vectors need only form a *rational* basis of $\mathbb{Z}^n$.

Orbifolds are relevant in our case due to the following result:

Theorem 4.57 *Consider a proper, faithful and locally free G-action on a smooth manifold M. Then, the quotient space M/G has a natural orbifold structure.*

A proof can be found in e.g. Proposition 1.5.1. in page 17 of [8].

Thus, all our previous results of symplectic reduction can be generalised to the case of locally free actions and we will obtain *symplectic orbifolds* as reduced spaces.

We will not delve more in this direction but only comment how this observation leads to a slightly stronger version of the Duistermaat–Heckman Theorem and its corollaries.

Chapter 5

The Duistermaat–Heckman Theorem

In this final chapter of the thesis, we prove the Duistermaat–Heckman Theorem for torus-actions and on of its main corollaries, the Duistermaat–Heckman formula about the pushforward of the Liouville measure via the moment map. We follow the same scheme as in the previous chapter and obtain first the results for the simpler case of circle actions. Before proving the general statement of Duistermaat–Heckman, a so-called normal form result for non-abelian groups is obtained. The main references for this chapter are [7, 12, 14].

Let (M, ω, Φ) be a Hamiltonian G-space. We will assume throughout this section that M is connected, G is a connected compact Lie group, and that the moment map Φ is proper. We know that $\Phi : M \to \mathfrak{g}^*$ is a smooth G-equivariant map such that $p \mapsto \langle \Phi(p), \xi \rangle$ is a Hamiltonian function for $\xi^\# \in \mathfrak{X}^{\text{Ham}}(M, \omega)$. Furthermore, Proposition 4.18 tells us that Φ is uniquely defined up to a constant cocycle: up to some $\mu \in \mathfrak{g}^*$ such that $\mu \in [\mathfrak{g}, \mathfrak{g}]^0$, i.e., $\mu([\xi, \eta]) = 0$ for any $\xi, \eta \in \mathfrak{g}$. If G is abelian, then $[\mathfrak{g}, \mathfrak{g}]^0 = \mathfrak{g}^*$ and hence Φ is uniquely defined up to any constant $c \in \mathfrak{g}^*$. Thus, in the abelian case, any fiber $\Phi^{-1}(\mu)$ can be treated as the zero fiber $\Phi^{-1}(0)$ just by choosing $\Phi' := \Phi - \mu$. Alternatively, one can use Theorem 4.37 and notice that for an abelian group $G_\mu = G$ (since the coadjoint action is trivial), the construction is equivalent.

The question addressed in this chapter is to find the relation between the symplectic quotients obtained after applying the symplectic reduction to different fibers. We will start by studying fibers close to the zero level set, i.e., at values $\mu \in \mathfrak{g}^*$ close to 0.

Since the G-action is proper, in order to apply Theorem 4.35 on the fibers, we only need to check that G acts freely on each fiber. As we are interested on fibers close to 0, it is in fact enough to ask that G acts freely only on $\Phi^{-1}(0)$:

Lemma 5.1 *Consider a Hamiltonian, proper G-action on (M, ω) with proper moment map Φ. Suppose that G acts freely on $\Phi^{-1}(0)$. Then, there exists an open neighbourhood U of 0 in $\mathfrak{g}^*$ such that G acts freely on $\Phi^{-1}(U)$.*

Proof Suppose that it is not true. Then, there exists a decreasing sequence of relatively compact open neighbourhoods $\{U_i\}_i$ of 0 converging to 0 such that G does not act freely on $\Phi(U_i)$ for each i. Thus, there is a sequence $\{x_i\}_i \subset M$ with $\Phi(x_i) \in U_i$ and a sequence $\{g_i\}_i \subset G \setminus \{e\}$ such that $g_i x_i = x_i$. Since Φ is proper and the U_i relatively compact, we can assume that $x_i \to x$ for some $x \in M$ such that $\Phi(x) = 0$ by continuity. Thanks to the G-action being proper we can also assume that $g_i \to g$ for some $g \in G$. Hence, $gx = x$ in the limit, and since by hypothesis G acts freely on $\Phi^{-1}(0)$, we conclude that $g = e$. For the same reason, as we argued in the symplectic reduction Theorem 4.35, the orbit map $d_e j_p : \mathfrak{g} \to T_x M : \xi \mapsto \xi_x^\#$ is injective. Thus, the differential of the map

$$\Psi : G \times M \to M \times M : (g, x) \mapsto (x, gx)$$

at the point (e, x), given by

$$d_{(e,x)}\Psi : \mathfrak{g} \times T_x M \to T_x M \times T_x M : (\xi, v) \mapsto (v, v + \xi_x^\#),$$

is injective, and hence Ψ is a local diffeomorphism over its image (note that it is closed since it is proper). This means that $\Psi(g_i, x_i) = (x_i, g_i x_i) = (x_i, x_i) = \Psi(e, x_i)$ for points $(g_i, x_i), (e, x_i)$ accumulating at (e, x), and thus it must be $g_i = e$, contradicting our initial assumption. $\square$

From now onwards, we consider a convex neighbourhood U of 0 in $\mathfrak{g}^*$ such that G acts freely on $\Phi^{-1}(U)$. Theorem 4.37 states that U is composed of regular values of Φ, and, applied to each $\mu \in U$, it says that the level set

$$S_\mu = \Phi^{-1}(\mu)$$

is an embedded submanifold whose null foliation is given by the G_μ-orbits, that the orbit projection

$$\pi_\mu : S_\mu \to \overline{S}_\mu := S_\mu / G_\mu$$

is a principal G_μ-bundle, and that $\overline{S}_\mu$ is a symplectic manifold of dimension $\dim M - \dim G - \dim G_\mu$ with symplectic structure $\overline{\omega}_\mu$ such that

$$\pi_\mu^* \overline{\omega}_\mu = i_\mu^* \omega,$$

for the inclusion $i_\mu : S_\mu \hookrightarrow M$. In the abelian case, the same is true for $G_\mu = G$, so that now S_μ are coisotropic submanifolds foliated by the isotropic G-orbits.

In order to compare the spaces $(\overline{S}_\mu, \overline{\omega}_\mu)$ for different $\mu \in U$, we will obtain a normal form expression from which the Duistermaat–Heckman Theorem will follow as an application to the abelian case. On the torus case, after adequately identifying (via G-equivariant diffeomorphisms) the different $(\overline{S}_\mu, \overline{\omega}_\mu)$ with a fixed $(\overline{S}_{\mu_0}, \overline{\omega}_{\mu_0})$, Duistermaat–Heckamn states that the relation between the symplectic forms $\overline{\omega}_\mu$ is linear in μ, in the sense that in de Rham cohomology $H^2(\overline{S}_{\mu_0};\mathbb{R})$ we have

$$[\overline{\omega}_\mu] = [\overline{\omega}_{\mu_0}] + \langle \mu - \mu_0, c\rangle$$

for a fixed coefficient $c \in H^2(\overline{S}_{\mu_0};\mathfrak{t})$. In fact, Duistermaat–Heckman states that the coefficient is the *first characteristic class* of the torus-bundle $\pi_0 : S_0 \to \overline{S}_0$, which in particular does not depend on the level set μ since as we said, the fibers are G-equivariantly diffeomorphic. In the Appendix section A.2 we review the main notions about Lie algebra-valued forms and characteristic classes; particularly, the first characteristic class of a torus-bundle is defined at the end of the Appendix as an ad hoc, non-standard convention used in this thesis.

We note that $\overline{S}_\mu$ is compact since the level set $\Phi^{-1}(\mu)$ also is, as a consequence of Φ being proper. We begin by studying the simpler case $G = \mathbb{T}^1 \equiv \mathbb{S}^1$, where the main argument is seen with more clarity.

5.1 Duistermaat–Heckman for the circle

Since $G = \mathbb{S}^1$ is abelian, we don't need to work around 0. Nonetheless, since the moment map is only defined up a constant element of $\mathfrak{g}^*$, we can always assume to do so and take as reference the compact, $\mathbb{S}^1$-invariant fiber $S_0 = \Phi^{-1}(0)$ and the principal $\mathbb{S}^1$-bundle $\pi_0 : S_0 \to \overline{S}_0$. By Lemma A.18 we can fix some Ehresmann connection α on S_0, i.e., $\alpha \in \Omega^1(S_0;\mathfrak{g})$ such that $\mathfrak{L}_{\xi^\#}\alpha = 0$ and $\alpha(\xi^\#) = \xi$ for the infinitesimal action vector field $\xi^\#$ generated by the basis element $\xi \equiv i \in \mathfrak{g} \equiv i\mathbb{R} \cong \mathbb{R}$ (that is, we are identifying the circle Lie algebra via the differential of the Lie group exponential map $t \mapsto \exp(it)$). From here onwards, for the circle $G = \mathbb{S}^1$, we identify $\mathfrak{g} \cong \mathbb{R}$ in this way and similarly $\mathfrak{g}^* \cong \mathbb{R}$ via the standard product of $\mathbb{R}$. From α we construct the following closed 2-form defined on the product manifold $X := S_0 \times \mathbb{R}$,

$$\sigma := \mathrm{pr}^*_{S_0} i_0^*\omega - d(x\,\mathrm{pr}^*_{S_0}\alpha) \equiv i_0^*\omega - d(x\alpha),$$

where $\mathrm{pr}_{S_0} : S_0 \times \mathbb{R} \to S_0$ is the projection (whose pullback we drop henceforth from the notation unless unclear) and x denotes the coordinate for the $\mathbb{R}$ factor, $x : S_0 \times \mathbb{R} \to \mathbb{R}$. We consider the $\mathbb{S}^1$-action on X given by the restricted $\mathbb{S}^1$-action on the factor S_0.

Lemma 5.2 *The 2-form σ is symplectic on a neighbourhood U of S_0 in X and satisfies the equation*

$$i_{\xi^\#}\sigma = dx.$$

Thus, $(U, \sigma|_U, x)$ is a Hamiltonian $\mathbb{S}^1$-space with the restricted $\mathbb{S}^1$-action on S_0.

Proof Clearly, σ is closed since $i_0^*\omega$ is. Moreover, there exists a neighborhood of S_0 where σ is non-degenerate and hence symplectic. To see it, we note that

$$\sigma|_{x=0} = i_0^*\omega + \alpha \wedge dx$$

satisfies

$$\sigma|_{x=0}(\xi^{\#}, \partial_x) = 1$$

and since $\ker i_0^*\omega_p$ is generated by the vector $\xi_p^{\#}$ at every point $p \in M$, σ is non-degenerate for every point with $x = 0$. By continuity, there exists an open neighbourhood $U \subset \mathbb{R}$ of 0 such that $(U, \sigma|_U)$ is a symplectic manifold.

A similar computation shows that

$$i_{\xi^{\#}}\sigma = i_{\xi^{\#}}(i_0^*\omega) - i_{\xi^{\#}} \circ d(x\alpha) = 0 + (d \circ i_{\xi^{\#}} - \mathfrak{L}_{\xi^{\#}})(x\alpha) = d \circ i_{\xi^{\#}}(x\alpha) = dx,$$

so that x is a Hamiltonian function of X in U. Since $x|_U$ is $\mathbb{S}^1$-invariant, it follows that $x|_U$ is a moment map for the Hamiltonian $\mathbb{S}^1$-action on U, after identifying $\mathfrak{g}^* \cong \mathfrak{g} \equiv \mathbb{R}$ via the standard scalar product of $\mathbb{R}$. $\square$

Furthermore, we can choose $U = S_0 \times (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ using the fact that S_0 is compact since Φ is proper. Thus, we have a model $(S_0 \times (-\varepsilon, \varepsilon), \sigma|_{S_0 \times (-\varepsilon,\varepsilon)}, x|_{S_0 \times (-\varepsilon,\varepsilon})$ and we now want to translate the problem from the proper, $\mathbb{S}^1$-equivariant submersion $\Phi : \Phi^{-1}(-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ to the one we have just defined given by $x : S_0 \times (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$. There are several ways to do this, and therein lies the heart of the different approaches to the Duistermaat–Heckman proof. We will present how to do this by using the equivariant version of Theorem 3.48:

Theorem 5.3 (Equivariant coisotropic embedding) *Let (M_j, ω_j), $j = 0, 1$ be two symplectic manifolds of the same dimension and consider Hamiltonian proper G-actions on both with moment maps $\Phi_j : M_j \to \mathfrak{g}^*$, $j = 0, 1$. Consider further a common compact submanifold S with coisotropic, G-equivariant embeddings $i_j : S \hookrightarrow M_j$. Suppose that $i_0^*\omega_0 = i_1^*\omega_1$ and $\Phi_0 \circ i_0 = \Phi_1 \circ i_1$. Then, there exist G-invariant open neighborhoods U_j of $i_j(S)$ in M_j, and a G-equivariant symplectomorphism $\psi : U_0 \to U_1$ such that $i_1 = \psi \circ i_0$ and $\Phi_1 \circ \psi = \Phi_0$.*

Proof The coisotropic embedding Theorem 3.48 in combination with the comments of that section on how to obtain a G-equivariant formulation gives us the result except for the claim about the moment maps Φ_j. To see that $\Phi_1 \circ \psi = \Phi_0$, we just notice that since ψ is a G-equivariant symplectomorphism, $\widetilde{\Phi}_0 := \Phi_1 \circ \psi$ is another moment map for the Hamiltonian G-action on U_0. Furthermore, by hypothesis $\Phi_0 \circ i_0 = \Phi_1 \circ i_1$, so that

$$\widetilde{\Phi}_0 \circ i_0 = \Phi_1 \circ \psi \circ i_0 = \Phi_1 \circ i_1 = \Phi_0 \circ i_0.$$

Since by Proposition 4.18 the moment map of a Hamiltonian action is unique up to a locally constant cocycle, and since we can assume that all connected components of U_0 intersect $i_0(S_0)$, it follows that $\Phi_1 \circ \psi = \widetilde{\Phi}_0 = \Phi_0$. $\square$

With this result it is now easy to translate the problem from $\Phi : \Phi^{-1}(-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ to $x : S_0 \times (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ (we abbreviate $\sigma| := \sigma|_{S_0 \times (-\varepsilon,\varepsilon)}$):

Proposition 5.4 *There exists a G-equivariant symplectomorphism ψ between an open neighborhood of S_0 in (M, ω) and an open neighborhood $(S_0 \times (-\varepsilon, \varepsilon), \sigma|)$ of $S_0 \times \{0\}$ restricting to the zero section inclusion map $\psi|_{S_0} \equiv j_0 : S_0 \hookrightarrow S_0 \times (-\varepsilon, \varepsilon) : p \mapsto (p, 0)$ and such that $x \circ \psi = \Phi$.*

Proof We consider the two inclusions $i_0 : S_0 \hookrightarrow M$ and $j_0 : S_0 \hookrightarrow S_0 \times (-\varepsilon, \varepsilon)$ and note that $i_0^*\omega = j_0^*\sigma$ (since $\mathrm{pr}_{S_0} \circ j_0 = \mathrm{id}_{S_0}$), and furthermore that $\Phi|_{S_0} \equiv 0 \equiv x|_{S_0 \times \{0\}}$. Since S_0 is a compact, coisotropic submanifold of both symplectic spaces, the Proposition follows from Theorem 5.3. $\square$

The symplectomorphism ψ is thus a G-equivariant *local trivialization* for the fiber-bundle $\Phi : \Phi^{-1}(-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$. We note that it is at this is point where we must ask for a proper moment map, in order to ensure the compactness of S_0. This reminisces for example of the Ehresmann Theorem, i.e., that a proper submersion between connected manifolds is a fiber-bundle, where again properness plays a key role.

This proposition allows to study, instead of $(\overline{S}_t = \Phi^{-1}(t)/\mathbb{S}^1, \overline{\omega}_t)$, the reduced spaces of $(S_0 \times (-\varepsilon, \varepsilon), \sigma|)$ with $\sigma = i_0^*\omega - d(x\alpha)$, given by $x^{-1}(t)/\mathbb{S}^1 = S_0/\mathbb{S}^1 \times \{t\}$, diffeomorphic to $\overline{S}_0$ for every t. In fact, all reduced spaces $\overline{S}_{t_0}, \overline{S}_{t_1}$ at values t_0, t_1 that can be connected through values t of Φ such that $\mathbb{S}^1$ acts freely on the level set $\Phi^{-1}(t)$ are diffeomorphic by a concatenation of such locally trivializing symplectomorphisms ψ. This works also for a $\mathbb{T}^k$-action and we will now make concrete this general case.

First however we define, for convenience:

Definition 5.5 (Free value of the moment map) Given a Hamiltonian G-space (M, ω, Φ), we say that $\mu \in \mathfrak{g}^*$ is a *free value of* Φ if G acts freely on the level set $\Phi^{-1}(\mu)$.

Suppose $\Phi : M \to \mathfrak{t}^*$ is a fiber-bundle (i.e., we restrict M to the corresponding component of free values). We fix a trivialization around S_{μ_0}, $\psi : \Phi^{-1}(V) \to S_{\mu_0} \times V$, choosing $V \subset \mathfrak{t}^*$ convex, and such that $\psi|_{S_{\mu_0}} = \mathrm{id}_{S_{\mu_0}}$, for the fiber $S_{\mu_0} := \Phi^{-1}(\mu_0)$. For each $\mu \in V$, we obtain a diffeomorphism

$$\varphi_\mu^{\mu_0} : S_{\mu_0} \to S_\mu : q \mapsto \psi^{-1}(q, \mu).$$

Given a chain of connected open sets V_i and trivializations ψ_i around V_i, $i = 1, \ldots, n$, connecting $\mu_0 \in V_1$ with $\mu_n \in V_n$ via points $\mu_i \in V_i \cap V_{i+1}$, $i = 1, \ldots, n-1$, we have the concatenation of diffeomorphisms

$$\varphi_{\mu_n}^{\mu_{n-1}} \circ \cdots \circ \varphi_{\mu_1}^{\mu_0} : S_{\mu_0} \to S_{\mu_n}.$$

The key observation is that, even though the choice of symplectomorphisms ψ_i at each step of the concatenation of compositions is not uniquely defined, all choices lead to the same map in cohomology for a given chain of trivializations.

This is due to the fact that each local trivialization $\psi : \Phi^{-1}(V) \to S_{\mu_0} \times V$ is chosen over a convex set $V \subset \mathfrak{t}^*$, in particular contractible. Thus, given some other $\psi' : \Phi^{-1}(V) \cong S_{\mu_0} \times V$, then $\psi' \circ \psi^{-1} : S_{\mu_0} \times V \to S_{\mu_0} \times V : (p, \mu) \mapsto (f_\mu(p), \mu)$ is homotopic to the identity via $H_t(p, \mu) := (f_{(1-t)\mu_0 + t\mu}(p), \mu)$. Fixing a prescribed chain of local trivializations to follow for each point in this connected component, we can compare all fibers of the component.

Proposition 5.6 *Let $t \in \mathbb{R}$ be in the same component of free values of Φ as the free value 0. Then, the reduced space $M_t = (\overline{S}_t, \overline{\omega}_t)$ is symplectomorphic to*

$$(\overline{S}_0, \overline{\omega}_0 - tC),$$

where C is the curvature form of an Ehresmann connection on the $\mathbb{S}^1$-bundle $\pi_0 : S_0 \to \overline{S}_0$ and thus a representative of its first characteristic class.

Proof Working directly on $(S_0 \times (-\varepsilon, \varepsilon), \sigma|)$, we see that σ restricts via the inclusion $j_t : S_0 \hookrightarrow S_0 \times (-\varepsilon, \varepsilon)$ at t to the form

$$j_t^* \sigma = j_t^*(i_0^* \omega - d(x\alpha)) = i_0^* \omega - t d\alpha.$$

In virtue of Theorem A.32, we have $\Omega = \pi_0^* C$ for the curvature Ω of α and, by Definition A.33, C is a representative of the first characteristic class of the $\mathbb{S}^1$-bundle (note that the G-action on $S_0 \times \{t\}$ is just the G-action on S_0, and we are abusing the notation by not distinguishing these two spaces, for example denoting also π_0 the orbit quotient of $S_0 \times \{t\}$). Since the group is abelian, $\Omega = d\alpha + \frac{1}{2}[\alpha, \alpha] = d\alpha$ (alternatively, using Theorem A.20 we see that $\Omega = (d\alpha)^h = d\alpha$, since $d\alpha$ is already horizontal, for $0 = \mathfrak{L}_{\xi^\#}\alpha = i_{\xi^\#} d\alpha + d i_{\xi^\#}\alpha = i_{\xi^\#} d\alpha$). Thus, $j_t^* \sigma = \pi_0^*(\overline{\omega}_0 - tC)$, and in virtue of Theorem 4.35, we see that $\overline{\omega}_0 - tC$ is the reduced form on the reduction of $(S_0 \times (-\varepsilon, \varepsilon), \sigma|)$ at t. We conclude using the equivariant symplectomorphism ψ of the previous proposition to obtain an induced symplectomorphism between this space and the reduced space M_t. $\square$

Theorem 5.7 (Duistermaat–Heckman for $\mathbb{S}^1$) *Consider a Hamiltonian $\mathbb{S}^1$-action over a symplectic manifold (M, ω) such that the moment map $\Phi : M \to \mathfrak{g}^* \cong \mathbb{R}$ is proper. Then, given $t, t' \in \mathbb{R}$ in the same component of free values of Φ, we have*

$$[\overline{\omega}_{t'}] = [\overline{\omega}_t] - (t' - t)c,$$

where $c \in H^2(M_t; \mathbb{R})$ is the first characteristic class of the $\mathbb{S}^1$-bundle $\pi_t : S_t \to \overline{S}_t$, and we are using the canonical identification described above between the different level sets $S_{t'}$.

Proof The statement follows from the previous proposition in combination with the invariance of the cohomology class of the first characteristic class, noting that a G-equivariant diffeomorphism between principal G-bundles preserves characteristic classes, since its pullback sends Ehresmann connections to Ehresmann connections. $\square$

Remark 5.8 If we use the more general version of symplectic reduction that we briefly discussed in Section 4.7 and accept locally free actions, and thus orbifolds as reduced spaces, then the same proof provides a version of Duistermaat–Heckman where we obtain the same conclusion for values $t, t' \in \mathbb{R}$ in the same component of *regular* values of Φ, and not just of *free* values of Φ. □

Example 5.9 In Example 4.4 we showed that $\mathbb{C}^{n+1}$ with the standard symplectic form ω_0 becomes a Hamiltonian $\mathbb{S}^1$-space under the diagonal $\mathbb{S}^1$-action with Hamiltonian proper moment map

$$H = -\frac{1}{2}||z||^2,$$

and that for any regular value $\lambda < 0$ the reduction at λ is symplectomorphic to

$$(\mathbb{C}^{n+1})_\lambda \cong (\mathbb{CP}^n, -2\lambda\omega_{FS}).$$

Clearly, any two $\lambda, \lambda' < 0$ are in the same free value region. Furthermore, the diffeomorphic identifications between fiber can be all taken to be the adequate scaling to the fiber at $\lambda = -\frac{1}{2}$, that is, the $\mathbb{S}^1$-equivariant diffeomorphisms

$$\psi : H^{-1}(\lambda) = \sqrt{2(-\lambda)}\mathbb{S}^{2n+1} \to H^{-1}(-\frac{1}{2}) = \mathbb{S}^{2n+1} : z \mapsto z/\sqrt{2(-\lambda)}.$$

Under this identifications, the Duistermaat–Heckman Theorem states that

$$-2\lambda'[\omega_{FS}] = -2\lambda[\omega_{FS}] - (\lambda' - \lambda)c,$$

for the first characteristic class $c \in H^2(\mathbb{S}^{2n+1}; \mathbb{R})$ of the $\mathbb{S}^1$-bundle $\mathrm{pr}_2 : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$. We conclude that

$$[\omega_{FS}] = \frac{1}{2}c.$$

This can be checked directly as follows. We consider the Ehresmann connection $\alpha \in \Omega^1(\mathbb{S}^{2n+1}; \mathbb{R})$ such that α_z for $z \in \mathbb{S}^{2n+1}$ is given by the $\langle \cdot, \cdot \rangle$-orthogonal component parallel to iz (here $\langle \cdot, \cdot \rangle$ is the standard hermitian inner product in $\mathbb{C}^{n+1}$):

$$\alpha_z(v) := \langle iz, v \rangle,$$

that is,

$$\alpha = \langle iz, dz \rangle = \sum_j -i\bar{z}_j dz_j.$$

Since $\langle \cdot, \cdot \rangle$ is $U(n+1)$-invariant, this form is $\mathbb{S}^1$-invariant and satisfies $\alpha(\xi^\#) = 1$ for the infinitesimal generator $\xi^\#$ of the $\mathbb{S}^1$-action, normalised to have period 2π, i.e., $\xi^\#(z) \equiv iz$. Thus, the curvature of the connection is

$$d\alpha = \langle idz, dz \rangle = i \sum_j dz_j \wedge d\bar{z}_j = 2\omega_0|_{\mathbb{S}^{2n+1}}.$$

This identity of basic forms translates into $2\omega_{FS}$ being a representative of the first characteristic class of $\mathrm{pr}_2 : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$. □

5.2 General case

Let (M, ω, Φ) be a Hamiltonian G-space for a compact Lie group G and proper map Φ. The approach will be the same as in the previous section: we will find a simpler model and use Theorem 5.3 to translate the problem from M to this model. This produces a *normal form* that allows one to localize around the zero level set of the moment map and obtain a standard expression for the reduced spaces at values close to zero in terms of coadjoint orbits and the zero level set. From the normal form result, Theorem 5.12, the Duistermaat–Heckman Theorem 5.15 then follows as a particularization to the abelian case, i.e., where $G = \mathbb{T}^k$ is a torus.

5.2.1 Normal form

As we saw in Lemma 5.1, if G acts freely on the zero level set, then so it does on level sets of nearby values $\mu \in U \subset \mathfrak{g}^*$ for an open neighbourhood V of 0. For each $\mu \in V$ we have a reduced space $M_\mu = (\overline{S}_\mu, \overline{\omega}_\mu)$ given by the orbit quotient

$$\pi_\mu : S_\mu = \Phi^{-1}(\mu) \to \overline{S}_\mu := S_\mu / G_\mu$$

with the unique symplectic structure $\overline{\omega}_\mu$ such that

$$\pi_\mu^* \overline{\omega}_\mu = i_\mu^* \omega,$$

for the inclusion $i_\mu : S_\mu \hookrightarrow M$.

Consider $S_0 = \Phi^{-1}(0)$ and the principal G-bundle $\pi_0 : S_0 \to \overline{S}_0$; by Lemma A.18 there exists an Ehresmann connection $\alpha \in \Omega^1(S_0; \mathfrak{g})$. That is, α satisfies $g^*\alpha = \mathrm{Ad}_g \circ \alpha$ (i.e., it is G-equivariant) and $\alpha(\xi^\#) = \xi$ for every vector $\xi \in \mathfrak{g}$. Our model space is the product $X := S_0 \times \mathfrak{g}^*$ with the diagonal action given by the G-action on M and the coadjoint action on $\mathfrak{g}^*$. Let $\mathrm{pr}_{S_0}, \mathrm{pr}_{\mathfrak{g}^*}$ be the projections from $X = S_0 \times \mathfrak{g}^*$ to the first and second factors. The symplectic structure will be given by the adequate restriction of the 2-form $\sigma \in \Omega^2(X; \mathbb{R})$,

$$\sigma := \mathrm{pr}_{S_0}^*(i_0^*\omega) - d\langle \mathrm{pr}_{\mathfrak{g}^*}, \mathrm{pr}_{S_0}^* \alpha \rangle.$$

We will omit in the notation the pullback $\mathrm{pr}_{S_0}^*$ and write $\sigma \equiv i_0^*\omega - d\langle \mathrm{pr}_{\mathfrak{g}^*}, \alpha \rangle$.

Lemma 5.10 *The 2-form σ is symplectic on a neighbourhood U of $S_0 \times \{0\}$ in X and satisfies the equation*

$$i_{\xi^\#}\sigma = d\langle \mathrm{pr}_{\mathfrak{g}^*}, \xi \rangle,$$

for all $\xi \in \mathfrak{g}$. Thus, $(U, \sigma|_U, \mathrm{pr}_{\mathfrak{g}^*})$ is a Hamiltonian G-space.

Proof Clearly, σ is closed since ω is and thus also $i_0^*\omega$. To find a neighborhood where σ is also non-degenerate, we note that on points of $S_0 \times \{0\}$ we have

$$\sigma|_{S_0 \times \{0\}} = i_0^*\omega - \langle d\mathrm{pr}_{\mathfrak{g}^*}, \alpha \rangle.$$

On the one hand, $\ker \mathrm{pr}_{S_0}^* i_0^*\omega = T_pG\cdot p\times\mathfrak{g}^*$ with $T_pG\cdot p$ the tangent space to the orbit at $p \in S_0$. On the other hand, $\ker\langle d\mathrm{pr}_{\mathfrak{g}^*}, \alpha\rangle = \ker\alpha \times \{0\}$ is a horizontal distribution complementary to $T_pG\cdot p\times\mathfrak{g}^*$. Thus, σ is non-degenerate on these points and we can choose a neighborhood U of $S_0 \times \{0\}$ where σ is symplectic.

We then compute

$$i_{\xi^\#}\sigma = i_{\xi^\#} i_0^*\omega - i_{\xi^\#} \circ d\langle \mathrm{pr}_{\mathfrak{g}^*}, \alpha\rangle = 0 + d \circ i_{\xi^\#}\langle \mathrm{pr}_{\mathfrak{g}^*}, \alpha\rangle = d\langle \mathrm{pr}_{\mathfrak{g}^*}, \xi\rangle,$$

where we have used that $\ker i_0^*\omega = T_pG\cdot p$ and the fact that $\mathfrak{L}_{\xi^\#}\langle \mathrm{pr}_{\mathfrak{g}^*}, \alpha\rangle = 0$ since $g^*\langle \mathrm{pr}_{\mathfrak{g}^*}, \alpha\rangle = \langle \mathrm{Ad}_{g^{-1}}\mathrm{pr}_{\mathfrak{g}^*}, g^*\alpha\rangle = \langle \mathrm{pr}_{\mathfrak{g}^*}, \alpha\rangle$. Since $\mathrm{pr}_{\mathfrak{g}^*}$ is obviously G-equivariant, $(U, \sigma|_U, \mathrm{pr}_{\mathfrak{g}^*})$ is a Hamiltonian G-space. □

As before, we can choose $U = S_0 \times V$ for some open set $0 \in V \subset \mathfrak{g}^*$ because S_0 is compact using that Φ is proper.

Proposition 5.11 *There exists a G-equivariant symplectomorphism ψ between an open neighborhood of S_0 in (M,ω) and an open neighborhood $(S_0 \times V, \sigma|)$ of $S_0\times\{0\}$ for some open $V \subset \mathfrak{g}^*$, restricting to the zero section inclusion map $\psi|_{S_0} \equiv j_0 : S_0 \hookrightarrow S_0 \times \{0\}$ and such that $\mathrm{pr}_{\mathfrak{g}^*} \circ \psi = \Phi$.*

Proof We consider the two inclusions $i_0 : S_0 \hookrightarrow M$ and $j_0 : S_0 \hookrightarrow S_0 \times V$ and note that $i_0^*\omega = j_0^*\sigma$ (since $\mathrm{pr}_{S_0} \circ j_0 = \mathrm{id}_{S_0}$), and furthermore that $\Phi|_{S_0} \equiv 0 \equiv \mathrm{pr}_{\mathfrak{g}^*}|_{S_0\times\{0\}}$. The result follows from Theorem 5.3. □

Theorem 5.12 (Normal form) *Consider a compact Lie group G and a Hamiltonian G-space (M, ω, Φ) with proper moment map Φ. Suppose that G acts freely on the zero level set $\Phi^{-1}(0)$. Then, there exists an open neighborhood $V \subset \mathfrak{g}^*$ of 0 such that G acts freely on $\Phi^{-1}(\mu)$ for each $\mu \in V$ and the reduced space $\overline{S}_\mu$ is diffeomorphic to*

$$\overline{S}_\mu \cong (S_0 \times G \cdot (-\mu))/G,$$

the orbit space associated to the G-space $S_0 \times G \cdot (-\mu)$ with the diagonal G-action. In particular, the reduced spaces fiber over $\overline{S}_0$ with fiber coadjoint orbits $G \cdot (-\mu)$. If we let $\Psi : G \cdot (-\mu) \hookrightarrow \mathfrak{g}^*$ be the inclusion, then the pullback of the 2-form $\overline{\omega}_\mu \in \Omega^2(\overline{S}_\mu; \mathbb{R})$ to $S_0 \times G \cdot (-\mu)$ is given by

$$i_0^*\omega + d\langle \Psi, \alpha\rangle.$$

Remark 5.13 The minus sign $G \cdot (-\mu)$ will appear naturally as the moment map of the right G-action on T^*G is $\Phi_R = -\mathrm{pr}_{\mathfrak{g}^*}$. Of course, the theorem is equivalent to having

$$\overline{S}_\mu \cong (S_0 \times G \cdot \mu)/G,$$

together with

$$i_0^*\omega - d\langle \Psi, \alpha\rangle.$$

□

Proof In virtue of the previous proposition, we work directly with the Hamiltonian G-space $(S_0 \times V, \sigma|, \mathrm{pr}_{\mathfrak{g}^*})$, for $\sigma = i_0^*\omega - d\langle \mathrm{pr}_{\mathfrak{g}^*}, \alpha\rangle$.

In order to simplify the quotient over G_μ needed to obtain each reduced space, we identify $S_0 \times \mathfrak{g}^*$ with the quotient space $(S_0 \times T^*G)/G$ of $S_0 \times T^*G$ under the diagonal G-action with the right action on the T^*G factor. Again, left-trivialization permits to identify $T^*G \cong G \times \mathfrak{g}^*$, and we will work directly with $(S_0 \times G \times \mathfrak{g}^*)/G \cong S_0 \times \mathfrak{g}^*$. To see that this is a diffeomorphism, we denote

$$\widetilde{\varphi} : S_0 \times \mathfrak{g}^* \to S_0 \times G \times \mathfrak{g}^* : (p, \nu) \mapsto (p, e, \nu).$$

so that composing with the projection

$$\pi_G : S_0 \times G \times \mathfrak{g}^* \to (S_0 \times G \times \mathfrak{g}^*)/G$$

(where $/G$ will always represent henceforth the quotient over this diagonal action) we have a diffeomorphism

$$\varphi := \pi_G \circ \widetilde{\varphi} : S_0 \times \mathfrak{g}^* \to (S_0 \times G \times \mathfrak{g}^*)/G : (p, \nu) \mapsto [(p, e, \nu)].$$

It is injective since given $(p, \nu), (p', \nu') \in S_0 \times \mathfrak{g}^*$ such that there exists $g \in G$ with $(gp, g^{-1}, \mathrm{Ad}^*_{g^{-1}}\nu) = g(p, e, \nu) = (p', e, \nu')$ then clearly $g = e$ and $(p, \nu) = (p', \nu')$. To prove surjectiveness we observe that $\widetilde{\varphi}$ satisfies

$$\widetilde{\varphi}(g(p, \nu)) = \widetilde{\varphi}(gp, \mathrm{Ad}^*_{g^{-1}}\nu) = (gp, e, \mathrm{Ad}^*_{g^{-1}}\nu) = g(p, g, \nu).$$

In particular, φ then satisfies

$$\varphi(g(p, \nu)) = \varphi(gp, \mathrm{Ad}^*_{g^{-1}}\nu) = [(p, g, \nu)],$$

and thus is surjective. The inverse map

$$\varphi^{-1} : (S_0 \times G \times \mathfrak{g}^*)/G \to S_0 \times \mathfrak{g}^* : [(p, g, \nu)] \mapsto g(p, \nu) = (gp, \mathrm{Ad}^*_{g^{-1}}\nu)$$

is also smooth (as it is checked pre-composing with π_G) and thus φ is a diffeomorphism. The above identity also says that φ is G-equivariant for the G-action on $(S_0 \times G \times \mathfrak{g}^*)/G$ induced by the left action on the $G \times \mathfrak{g}^*$ factor of $S_0 \times G \times \mathfrak{g}^*$. Thus, we can work on $S_0 \times G \times \mathfrak{g}^*$ and consider these two commuting G-actions:

- on the one hand, the left action on the $G \times \mathfrak{g}^*$ factor of $S_0 \times G \times \mathfrak{g}^*$;
- on the other hand, the diagonal action on $S_0 \times G \times \mathfrak{g}^*$ induced from the G-action on S_0 and the right action on $G \times \mathfrak{g}^*$.

If we take the quotient over the second action, we obtain the identification $S_0 \times \mathfrak{g}^* \cong (S_0 \times G \times \mathfrak{g}^*)/G$ via φ, and this sends the original diagonal action on $S_0 \times \mathfrak{g}^*$ to the first action, i.e., the left action on the $G \times \mathfrak{g}^*$ factor. It should be noted that there is no symplectic structure on $S_0 \times G \times \mathfrak{g}^*$. However, we do have the closed 2-form $(\varphi^{-1} \circ \pi_G)^*\sigma$.

The strategy now is to use the fact that the two G-actions on $S_0 \times G \times \mathfrak{g}^*$ commute and to take the quotients in the opposite order. That is, the symplectic quotient of $(S_0 \times V, \sigma|, \mathrm{pr}_{\mathfrak{g}^*})$ at μ is $\mathrm{pr}_{\mathfrak{g}^*}^{-1}(\mu)/G_\mu = S_0/G_\mu \times \{\mu\}$, with $S_0 \times V \subset S_0 \times \mathfrak{g}^* \cong (S_0 \times G \times \mathfrak{g}^*)/G$. The moment map $\mathrm{pr}_{\mathfrak{g}^*}$ translates via φ into the map $\overline{\Phi} := \mathrm{pr}_{\mathfrak{g}^*} \circ \varphi^{-1} : (S_0 \times G \times \mathfrak{g}^*)/G \to \mathfrak{g}^*$, and we have

$$\overline{\Phi} \circ \pi_G(p, g, \nu) = \overline{\Phi}([(p, g, \nu)]) = \mathrm{Ad}^*_{g^{-1}}\nu = \Phi_L \circ \mathrm{pr}_{G\times\mathfrak{g}^*}(p, g, \nu)$$

for the moment map $\Phi_L(p, \nu) = \mathrm{Ad}^*_{g^{-1}}\nu$ of the left action on $G \times \mathfrak{g}^*$ with its cotangent symplectic structure, so that $(\overline{\Phi} \circ \pi_G)^{-1}(\mu) = \pi_G^{-1}(\varphi(\mathrm{pr}_{\mathfrak{g}^*}^{-1}(\mu))) = S_0 \times \Phi_L^{-1}(\mu)$. The final step comes from the fact that since φ translates the action on $S_0 \times \mathfrak{g}^*$ to the left action on the factor $G \times \mathfrak{g}^*$ within $(S_0 \times G \times \mathfrak{g}^*)/G$, taking in $S_0 \times G \times \mathfrak{g}^*$ first the quotient of $S_0 \times \Phi_L^{-1}(\mu)$ over G_μ induces, by Theorem 4.43, a diffeomorphism $\mathrm{id}_{S_0} \times \Phi_{R,\mu} : S_0 \times (\Phi_L^{-1}(\mu)/G_\mu) \cong S_0 \times G\cdot(-\mu)$, where $\Phi_{R,\mu} : \Phi_L^{-1}(\mu)/G_\mu \to G \cdot (-\mu)$ is the map induced by the moment map of the right action on $G \times \mathfrak{g}^*$, $\Phi_R(p, \nu) = -\nu$. In particular, $\mathrm{id}_{S_0} \times \Phi_R| : S_0 \times \Phi_L^{-1}(\mu) \to S_0 \times G \cdot (-\mu)$ describes the symplectic reduction quotient $S_0 \times (\Phi_L^{-1}(\mu)/G_\mu)$. We obtain a commuting diagram:

$$\begin{tikzcd}
S_0 \times G \times \mathfrak{g}^* \arrow[d, "\pi_G"] & S_0 \times \Phi_L^{-1}(\mu) \arrow[l, hook', "j_\mu"'] \arrow[r, "\mathrm{id}_{S_0}\times\Phi_R|"] \arrow[d, "\pi_G|"] & S_0 \times G(-\mu) \arrow[d, "\overline{\pi}_G"] \\
(S_0 \times G \times \mathfrak{g}^*)/G & (S_0 \times \Phi_L^{-1}(\mu))/G \arrow[l, hook'] \arrow[r, "\overline{\pi}_\mu"] & (S_0 \times G(-\mu))/G.
\end{tikzcd}$$

Thus, the reduced spaces satisfy $\overline{S}_\mu \cong (S_0 \times G(-\mu))/G$ and tracing back the maps, the diffeomorphism is given by $[(p, \nu)] \mapsto [(p, -\nu)]$ where of course ν can be taken to be μ. The only thing left to check is that the induced form $\overline{\sigma}_\mu$ on $(S_0 \times G(-\mu))/G$ such that $\overline{\pi}_\mu^*\overline{\sigma}_\mu = \varphi^{-1,*}\sigma|_{(S_0\times\Phi_L^{-1}(\mu))/G}$ pulls back via $\overline{\pi}_G$ to the form in $S_0 \times G(-\mu)$ with the claimed formula. To do so, we define the map $\Lambda := \varphi^{-1} \circ \pi_G$ that, using the previous formulas, is given by

$$\Lambda : S_0 \times G \times \mathfrak{g}^* \to S_0 \times \mathfrak{g}^* : (p, g, \nu) \mapsto g(p, \nu).$$

The reduced symplectic form induced in $(S_0 \times G(-\mu))/G$ pulls back via $\overline{\pi}_\mu$ to the restriction of $\varphi^{-1,*}\sigma$, which in turn pulls back via $\pi_G|$ to the restriction of $\Lambda^*\sigma$. Thus, to conclude we only need to check that $\Lambda^*\sigma$ restricted to $S_0 \times \Phi_L^{-1}(\mu)$ coincides with the pullback via $\mathrm{id}_{S_0} \times \Phi_R|$ of the proposed form (by the commutativity of the diagram and the fact that $(\mathrm{id}_{S_0} \times \Phi_R|)^*$ is injective, then both forms would be equal). We first note that

$$(\mathrm{id}_{S_0} \times \Phi_R|)^*(i_0^*\omega + d\langle\Psi, \alpha\rangle) = i_0^*\omega - d\langle\mathrm{pr}_{\mathfrak{g}^*}, \alpha\rangle.$$

Then, we compute the restriction $\Lambda^*\sigma$ noting that $\Lambda(p, g, \nu) = (gp, \mu)$ for any $(p, g, \nu) \in S_0 \times \Phi_L^{-1}(\mu)$. On the one hand, we have $\Lambda^*(i_0^*\omega) = g^*(i_0^*\omega) = i_0^*\omega$ since the action is symplectic. On the other hand, for $(p, g, \nu) \in S_0 \times \Phi_L^{-1}(\mu)$, we see that

$$\begin{aligned} \Lambda^* \langle \mathrm{pr}_{\mathfrak{g}^*}, \alpha \rangle|_{(p,g,\nu)} &= \langle \mu, (g^*\alpha)_p \rangle \\ &= \langle \mu, \mathrm{Ad}_g \circ \alpha_p \rangle \\ &= \langle \mathrm{Ad}_g^* \mu, \circ \alpha_p \rangle \\ &= \langle \nu, \alpha_p \rangle. \end{aligned}$$

Thus we finally get

$$\Lambda^*\sigma|_{S_0 \times \Phi_L^{-1}(\mu)} = i_0^*\omega - d\langle \mathrm{pr}_{\mathfrak{g}^*}, \alpha \rangle = (\mathrm{id}_{S_0} \times \Phi_R|)^*(i_0^*\omega + d\langle \Psi, \alpha \rangle). \qquad \square$$

5.2.2 The Duistermaat–Heckman Theorem

Consider now the case of a torus-action, i.e., where $G := T = \mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$. Let us denote its Lie algebra $\mathfrak{t} \cong \mathbb{R}^k$, identified naturally via its exponential map, and its dual $\mathfrak{t}^* \cong \mathfrak{t}$ again naturally via the standard scalar product on $\mathbb{R}^k$. Since the group is abelian, the coadjoint action is trivial and the discussion of the previous subsection can be applied around any $\mu_0 \in \mathfrak{t}^*$ as discussed in the introduction of the chapter. With this in mind and for a proper moment map $\Phi : M \to \mathfrak{t}^*$, the normal form Theorem 5.12 states that, if T acts freely on the level set $\Phi^{-1}(\mu_0)$, then there exists an open neighborhood $V \subset \mathfrak{t}^*$ of μ_0 such that T acts freely on $\Phi^{-1}(\mu)$ for each $\mu \in V$ and the reduced space $\overline{S}_\mu$ is diffeomorphic to

$$\overline{S}_\mu \cong (S_{\mu_0} \times \{\mu\})/T = \overline{S}_{\mu_0} \times \{\mu\}.$$

As in the previous section, by chaining the above diffeomorphisms we obtain diffeomorphisms between any fixed fiber, say S_{μ_0}, and all fibers within the same component of free values. The computation of the reduced form shows, since now $\Psi : T \cdot \mu \hookrightarrow \mathfrak{t}^*$ is constant, that the restriction of σ to $S_{\mu_0} \times \{\mu\}$ is given by

$$j_\mu^*\sigma = j_{\mu_0}^*\sigma - \langle \mu - \mu_0, d\alpha \rangle.$$

Remark 5.14 In reality, to obtain this conclusion under a torus action, Proposition 5.11 is enough, since the normal form follows immediately after noting that the coadjoint action is trivial, since then $T_\mu = T$ for any $\mu \in \mathfrak{t}^*$. $\square$

To finish, we argue as in Proposition 5.6: Theorem A.32 states that the curvature of α is $\Omega = \pi_{\mu_0}^* C$ and, by Definition A.33, C is a representative of the first characteristic class of the torus-bundle $\pi_{\mu_0} : S_{\mu_0} \to \overline{S}_{\mu_0}$. Since the group is abelian, $\Omega = d\alpha$ and the above equation becomes

$$j_\mu^*\sigma = j_{\mu_0}^*\sigma - \langle \mu - \mu_0, d\alpha \rangle = \pi_{\mu_0}^*(\overline{\omega}_{\mu_0} - \langle \mu - \mu_0, C \rangle).$$

Since the first characteristic class of a torus bundle is preserved by an equivariant diffeomorphism, we have proven:

Theorem 5.15 (Duistermaat–Heckman Theorem, [12]) *Consider a Hamiltonian T-space (M, ω, Φ) for $T = \mathbb{T}^k$ such that the moment map $\Phi : M \to \mathfrak{t}^*$ is proper. Then, given $\mu, \mu' \in \mathfrak{g}^*$ within the same connected component of free values of Φ, we have*

$$[\overline{\omega}_\mu] = [\overline{\omega}_{\mu_0}] + \langle \mu - \mu_0, c \rangle,$$

where $c \in H^2(\mu_0; \mathbb{R})$ is the first characteristic class of the torus-bundle $S_{\mu_0} \to \overline{S}_{\mu_0}$, and we are using the prescribed identification between fibers of Φ.

Remark 5.16 We make the same remark as in the previous section about locally free actions and orbifolds as reduced spaces. We can strengthen the Duistermaat–Heckman Theorem and obtain the same conclusion for values $\mu, \mu' \in \mathbb{R}$ in the same component of *regular* values of Φ, and not just of *free* values of Φ, relating the induced symplectic structures on the quotient orbifolds. □

5.3 The pushforward of the Liouville Measure

Definition 3.8 of the Liouville form of a symplectic vector space naturally carries over to manifolds:

Definition 5.17 (Liouville form) Given a symplectic manifold (M, ω), the *Liouville form* is defined as

$$\frac{1}{n!}\omega^n.$$

Clearly, it is a volume form, sometimes called the *symplectic volume form*. In this way, if M is compact we can define the *symplectic volume* of (M, ω) as

$$\mathrm{vol}_\omega(M) := \int_M \frac{1}{n!}\omega^n.$$

Analogously, one defines the *Liouville measure* m_ω as the measure induced by the Liouville form on the Borel sets of M, that is:

$$m_\omega(U) := \int_U \frac{1}{n!}\omega^n,$$

for any Borel set $U \subset M$, so that for an integrable function $f \in \mathcal{L}^1(M)$ we define

$$\int_M f dm_\omega := \int_M f \frac{1}{n!}\omega^n.$$

Here of course we choose the orientation of M induced by ω^n and the integral of M compatible with this orientation, so that $\int_M f dm_\omega > 0$ for any positive function.

Definition 5.18 (Duistermaat–Heckman measure) Given a Hamiltonian T-space (M, ω, Φ) for a torus $T = \mathbb{T}^k$, the *Duistermaat–Heckman measure*, ρ, is the pushforward of m_ω by $\Phi : M \to \mathfrak{g}^*$. That is, given a Borel set $V \subset \mathfrak{t}^*$, we have

$$\rho(V) := \Phi_* m_\omega(V) = m_\omega(\Phi^{-1}(V)) = \int_{\Phi^{-1}(V)} \frac{1}{n!}\omega^n.$$

For an integrable function $f \in \mathcal{L}^1(\mathfrak{t}^*)$, we then have

$$\int_{\mathfrak{t}^*} f d\rho := \int_M f \circ \Phi dm_\omega = \int_M f \circ \Phi \frac{1}{n!}\omega^n.$$

Regarding $\mathfrak{t}^* \cong \mathbb{R}^k$ as an affine space, we also have the Lebesgue measure on $\mathfrak{t}^*$, which we denote $d\mu$. It turns out that under the conditions for the Duistermaat–Heckman Theorem, i.e., the moment map Φ is proper and the T-action is free, the relation between $d\mu$ and ρ is very simple. Before stating what it is, we recall that a measurable function $\lambda : \mathfrak{t}^* \to \mathbb{R}$ is called (when it exists) the *Radon–Nikodym* derivative of ρ with respect to $d\mu$, denoted as

$$\lambda = \frac{d\rho}{d\mu},$$

if for any integrable $f \in \mathcal{L}^1(\mathfrak{t}^*)$ we have

$$\int_{\mathfrak{t}^*} f d\rho = \int_{\mathfrak{t}^*} f \lambda d\mu.$$

Proposition 5.19 *Consider a Hamiltonian T-space (M, ω, Φ) with $T = \mathbb{T}^k$ a torus and a proper moment map Φ, and suppose given $\mu \in \mathfrak{g}^*$ within the same connected component of free values of Φ as 0. Then, the Radon–Nikodym derivative of the Duistermaat–Heckman measure with respect to the Lebesgue measure, $\lambda : \mathfrak{t}^* \to \mathbb{R}$, is given by $(2\pi)^k$ times the symplectic volume of the reduced space at μ:*

$$\lambda(\mu) = (2\pi)^k \mathrm{vol}_{\overline{\omega}_\mu}(\overline{S}_\mu).$$

Proof By the discussion of the previous section, it is enough to argue locally around the 0 level set $S_0 = \Phi^{-1}(0)$. In virtue of Proposition 5.11, we have a symplectomorphism $\psi : U \to S_0 \times V$ from an open neighbourhood $U \subset M$ of S_0 to the model space $S_0 \times V$ for an open set $V \subset \mathfrak{t}^*$, the latter with the symplectic form

$$\sigma = i_0^* \omega - \langle d\mathrm{pr}_{\mathfrak{t}^*}, \alpha \rangle - \langle \mathrm{pr}_{\mathfrak{t}^*}, d\alpha \rangle.$$

Here we are using the notation

$$\langle d\mathrm{pr}_{\mathfrak{t}^*}, \alpha \rangle = \sum_{\ell=1}^{k} d\mu_\ell \wedge \alpha^\ell,$$

where $\{\mu_\ell\}_\ell^k$ are coordinates in terms of the standard dual basis $\{\xi_\ell^*\}_\ell^k$ of $\mathfrak{t}^*$, dual to the standard basis $\{\xi_\ell\}_\ell^k$ of $\mathfrak{t} \cong \mathbb{R}^k$. We normalise these vectors so that each infinitesimal T-action vector field $(\xi_\ell)^\#$ has period 2π. In particular, $\{\alpha^\ell\}_\ell^k$ are the components of α in the basis $\{\xi_\ell\}_\ell^k$, i.e., $\alpha = \sum_\ell \alpha^\ell \xi_\ell$. Thus, we can also write

$$\langle \mathrm{pr}_{\mathfrak{t}^*}, d\alpha\rangle = \sum_{\ell=1}^{k} \mu_\ell d\alpha^\ell.$$

We consider some function $f \in \mathcal{C}^\infty(\mathfrak{t}^*, \mathbb{R})$ with compact support in V and compute

$$\begin{aligned}\int_{\mathfrak{t}^*} f d\rho &= \int_M f \circ \Phi \frac{\sigma^n}{n!} = \int_U f \circ \Phi \frac{\omega^n}{n!} = \int_{S_0 \times V} \psi^{-1,*}(f \circ \Phi \frac{\omega^n}{n!}) \\ &= \int_{S_0 \times V} f \circ \mathrm{pr}_{\mathfrak{t}^*} \frac{\sigma^n}{n!},\end{aligned}$$

using that $\Phi \circ \psi^{-1} = \mathrm{pr}_{\mathfrak{g}^*}$ and that $\psi^{-1,*}\omega = \sigma$. Now we expand the power of σ as follows, noting that 2-forms commute:

$$\begin{aligned}\frac{\sigma^n}{n!} &= \frac{1}{n!}(i_0^*\omega - \langle d\mathrm{pr}_{\mathfrak{t}^*}, \alpha\rangle - \langle \mathrm{pr}_{\mathfrak{t}^*}, d\alpha\rangle)^n \\ &= \frac{1}{n!}\sum_{\ell=0}^{n}(-1)^\ell \frac{n!}{\ell!(n-\ell)!}(i_0^*\omega - \langle \mathrm{pr}_{\mathfrak{t}^*}, d\alpha\rangle)^{n-\ell} \wedge (\langle d\mathrm{pr}_{\mathfrak{t}^*}, \alpha\rangle)^\ell.\end{aligned}$$

However, we notice that at any point $(p,\mu) \in S_0 \times V$ we can always take a basis of $T_{(p,\mu)}S_0 \times V \equiv T_pS_0 \times \mathfrak{t}^*$ given by the vectors $\{(0, \xi_\ell^*)_{\ell=1}^k \cup \{((\xi_\ell)_p^\#, 0)\}_{\ell=1}^k$ completed with some other $2(n-k)$ vectors of the type $(v, 0) \in T_pS_0 \times \mathfrak{t}^*$. But the factors $\langle d\mathrm{pr}_{\mathfrak{t}^*}, \alpha\rangle$ are the only ones that can be non zero when evaluated on the vectors $\{(0, \xi_\ell^*)_{\ell=1}^k$. Furthermore, each factor $\langle d\mathrm{pr}_{\mathfrak{t}^*}, \alpha\rangle$ can take at most one such vector and be non zero, so that in the above sum all terms with $\ell < k$ vanish. Reciprocally, all terms with $\ell > k$ also vanish because then we are bound to repeat some term $d\mu_\ell$ when taking the ℓ-th exterior power. Thus we have, at a point $(p,\mu) \in S_0 \times V$:

$$\begin{aligned}\left.\frac{\sigma^n}{n!}\right|_{(p,\mu)} &= \frac{(-1)^k}{k!(n-k)!}(i_0^*\omega - \langle \mu, d\alpha_p\rangle)^{n-k} \wedge (\langle d\mathrm{pr}_{\mathfrak{t}^*}, \alpha_p\rangle)^k \\ &= \frac{(-1)^k}{k!(n-k)!}(j_\mu^*\sigma)_p^{n-k} \wedge \left(\sum_{\ell=1}^{k} d\mu_\ell \wedge \alpha_p^\ell\right)^k,\end{aligned}$$

where we have noticed that $i_0^*\omega - \langle \mu, d\alpha_p\rangle = j_\mu^*\sigma$. The last factor can be written as

$$\left(\sum_{\ell=1}^{k} d\mu_\ell \wedge \alpha^\ell\right)^k = k!(-1)^{k(k-1)/2} d\mu_1 \wedge \cdots \wedge d\mu_k \wedge \alpha^1 \wedge \cdots \wedge \alpha^k,$$

since again only the summand with exactly k distinct factors $d\mu_\ell$ does not identically vanish. The sign comes from reordering the factors. We can notice that $d\mu_1\wedge\cdots\wedge d\mu_k$ is the standard volume form of $\mathfrak{t}^* \cong \mathbb{R}^k$ and thus corresponds to the Lebesgue measure $d\mu$ on $\mathfrak{g}^*$. Since we defined $(\xi_\ell)^\#$ to have period 2π, this corresponds with the measure provided by the standard Riemannian metric on $\mathbb{T}^k$, in which $\mathrm{vol}(\mathbb{T}^k) = (2\pi)^k$. Similarly we abbreviate $\overline{\alpha} := \alpha^1 \wedge \cdots \wedge \alpha^k$. This k-form at p on S_0 will compute the determinant with respect to the vectors $\{(\xi_\ell)^\#_p\}_{\ell=1}^k \subset T_pS_0$, and is in particular a volume form of the orbits of S_0 satisfying:

$$\overline{\alpha}_p((\xi_1)^\#_p, \ldots, (\xi_k)^\#_p) = 1.$$

Combining everything, we get

$$\begin{aligned}\int_{\mathfrak{t}^*} f d\rho &= (-1)^{k(k-1)/2} \int_{S_0\times V} f \circ \mathrm{pr}_{\mathfrak{t}^*} \frac{(j_\mu^*\sigma)^{n-k}}{(n-k)!} \wedge \overline{\alpha} \wedge d\mu \\ &= (-1)^{k(k-1)/2} \int_V d\mu f(\mu) \int_{S_0} \frac{(j_\mu^*\sigma)^{n-k}}{(n-k)!} \wedge \overline{\alpha},\end{aligned}$$

in virtue of Fubini's Theorem. We conclude that the Radon–Nikodym derivative of ρ with respect to $d\mu$ is given by, up to sign,

$$\lambda(\mu) = \int_{S_0} \frac{(j_\mu^*\sigma)^{n-k}}{(n-k)!} \wedge \overline{\alpha}.$$

To finish, we will express this integral in terms of the symplectic volume of the reduced space at μ. To do so, we choose local trivializations $\varphi_i : W_i \to T \times \overline{W}_i$ of the principal bundle $\pi_0 : S_0 \to \overline{S}_0$, where $\overline{W}_i := W_i/T$ provide an open covering of $\overline{S}_0$, and we choose a partition of unity $\{\theta_i\}_i$ of $\overline{S}_0$ subordinate to $\{\overline{W}_i\}_i$. We have

$$\begin{aligned}\lambda(\mu) &= \sum_i \int_{W_i} \theta_i \circ \pi_0 \frac{(j_\mu^*\sigma)^{n-k}}{(n-k)!} \wedge \overline{\alpha} \\ &= \sum_i \int_{T\times\overline{W}_i} \theta_i \circ \pi_{\overline{W}_i} \varphi_i^{-1,*} \left(\frac{(j_\mu^*\sigma)^{n-k}}{(n-k)!} \wedge \overline{\alpha} \right).\end{aligned}$$

On one hand we have $j_\mu^*\sigma = \pi_0^*\overline{\sigma}_\mu$ and $\pi_0 \circ \varphi_i^{-1} = \pi_{\overline{W}_i}$, so that

$$\varphi_i^{-1,*}(j_\mu^*\sigma)^{n-k} = \varphi_i^{-1,*}(\pi_0^*\overline{\sigma}_\mu)^{n-k} = \pi_{\overline{W}_i}^* \overline{\sigma}_\mu^{n-k},$$

where $\overline{\sigma}_\mu$ is the reduced form at μ. On the other hand we have that the trivializations φ_i send the infinitesimal vectors $\xi^\#$ to the right invariant vectors $\xi^R \in \mathfrak{X}(T)$ (i.e., $\xi^R_g = r_{g,*}\xi$), as can be checked using the fact that φ_i is T-equivariant and that $\varphi_i^{-1}|_{G\times\{[p]\}}$ is the orbit map at the point $p = \varphi_i^{-1}(e, [p])$.

Thus, $\varphi_i^{-1,*}\overline{\alpha}$ is a k-form on $T \times \overline{W}_i$, invariant under the T-action, and such that

$$\varphi_i^{-1,*}\overline{\alpha}(\xi_1^R, \ldots, \xi_k^R) = 1.$$

Since we are in a torus, this means that $\varphi_i^{-1,*}\overline{\alpha} = \pi_T^* d\mathrm{vol}_T$, i.e., is the pullback via $\pi_T : T \times \overline{W}_i \to T$ of the standard volume form $d\mathrm{vol}_T$ of T. Again by virtue of Fubini's Theorem we finally obtain:

$$\begin{aligned}
\lambda(\mu) &= \sum_i \int_{\overline{W}_i} \theta_i \circ \pi_{\overline{W}_i} \frac{\overline{\sigma}_\mu^{n-k}}{(n-k)!} \int_T d\mathrm{vol}_T \\
&= (2\pi)^k \int_{\overline{W}_i} \sum_i \theta_i \circ \pi_{\overline{W}_i} \frac{\overline{\sigma}_\mu^{n-k}}{(n-k)!} \\
&= (2\pi)^k \int_{\overline{S}_0} \frac{\overline{\sigma}_\mu^{n-k}}{(n-k)!} \\
&= (2\pi)^k \mathrm{vol}_{\overline{\sigma}_\mu}(\overline{S}_0) \\
&= (2\pi)^k \mathrm{vol}_{\overline{\omega}_\mu}(\overline{S}_\mu).
\end{aligned}$$

The last identity follows from the fact that $(\overline{S}_\mu, \overline{\omega}_\mu)$ is symplectomorphic to $(\overline{S}_0, \overline{\sigma}_\mu)$. □

As an immediate application we get:

Theorem 5.20 (Duistermaat–Heckman, [12]) *Consider a Hamiltonian T-space (M, ω, Φ) with $T = \mathbb{T}^k$ a torus and a proper moment map Φ. Then, the Duistermaat–Heckman measure is a piecewise polynomial of at most degree $n-k$ multiple of the Lebesgue measure on $\mathfrak{t}^* \cong \mathbb{R}^k$. Specifically, its Radon–Nikodym derivative with respect to the Lebesgue measure is given by a fixed polynomial of at most degree $n-k$ in every connected region of regular values of Φ.*

The Radon–Nikodym derivative $\lambda : \mathfrak{t}^* \to \mathbb{R}$ is called the *Duistermaat–Heckman polynomial*, and the theorem is generally referred to as the *Duistermaat–Heckman formula.*

Proof Following the previous computation, we note that

$$\begin{aligned}
\mathrm{vol}_{\overline{\omega}_\mu}(\overline{S}_\mu) &= \mathrm{vol}_{\overline{\sigma}_\mu}(\overline{S}_{\mu_0}) \\
&= \int_{\overline{S}_{\mu_0}} \frac{\overline{\sigma}_\mu^{n-k}}{(n-k)!} \\
&= \frac{1}{(n-k)!} \int_{\overline{S}_{\mu_0}} (\overline{\omega}_{\mu_0} - \langle \mu - \mu_0, C \rangle)^{n-k},
\end{aligned}$$

where μ_0 is a fixed regular value that is chosen for each component. Since the integral only depends on the de Rham cohomology class of a form, the integration of products of the representative C of the first characteristic class do not depend on μ nor on the diffeomorphisms chosen in the previous section, and we thus obtain a polynomial on μ of degree at most $n-k$. □

If M has dimension $2n$ and the torus has half-dimension $k=n$, in particular in the case of a symplectic toric manifold, we obtain that the polynomials of each region have degree 0, that is, the Radon–Nikodym derivative λ is constant on such regions. We get:

Corollary 5.21 *The Duistermaat–Heckman polynomial of a symplectic toric manifold is constant and equal to $(2\pi)^n$, that is, its Duistermaat–Heckman measure is $(2\pi)^n$ times the Lebesgue measure. In particular, its symplectic volume is*

$$\mathrm{vol}_\omega(M) = (2\pi)^n \mathrm{vol}(\Delta_M)$$

where Δ_M is its Delzant polytope.

Proof The reduced space at a free value consists of 1 point, and thus it has symplectic volume 1, so that $\lambda(\mu) = (2\pi)^k$. We note that by Delzant's Theorem, the interior of the moment polytope of a symplectic toric manifold is dense in its image and thus the region of free values of Φ is dense. □

Example 5.22 (Volume of $\mathbb{S}^2$) A result known already to Archimedes is the fact that the area bounded by two parallels of a 2-sphere depends only on the height Δh between the two parallels, measured orthogonally. In fact, this area is simply $2\pi\Delta h$. This is precisely the above corollary applied to the case of the Hamiltonian $\mathbb{S}^1$-action on $\mathbb{S}^2$ (i.e., Example 4.3). Since the moment map $H = x_3$ is the height function, denoting dh the Lebesgue measure on $\mathbb{R}$, we see that

$$\rho = 2\pi dh.$$

For any $-1 < h_0 < h_1 < 1$, with $\Delta h = h_1 - h_0$, we have

$$\mathrm{Area}(\{x \in \mathbb{S}^2 : h_0 < x_3 < h_1\}) = \int_{H^{-1}(h_0,h_1)} \omega = \rho([h_0,h_1]) = 2\pi\Delta h. \quad \square$$

Example 5.23 (Volume of $\mathbb{CP}^n$) In Example 4.46 we saw that $(\mathbb{CP}^n, \omega_{FS}, \Phi)$ is a Hamiltonian T/H-space with moment map

$$\Phi : \mathbb{CP}^n \to \mathfrak{h}^0 : z \mapsto -\frac{(|z_i|^2)_i}{2||z||^2} - c,$$

for $c \in \mathbb{R}^{n+1}$ such that $\sum_i c_i = 1/2$ and $\mathfrak{h}^0 = \ker k$, $k : \mathbb{R}^{n+1} \to \mathbb{R} : x \mapsto \sum_i x_i$. We recall that $T = \mathbb{T}^{n+1}$ and $H \cong \mathbb{S}^1$ is the diagonal subgroup. In order to apply the previous corollary, we first obtain the Delzant polytope of $\mathbb{CP}^n$ in $\mathfrak{t}^* \cong \mathbb{R}^n$, identified in the way described in this section. To do so, we compose Φ with the linear projection $p : \mathbb{R}^{n+1} \to \mathbb{R}^n : (x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$. If we consider the basis $\{e_i^*\}_{i=0}^n$ dual to the standard basis $\{e_i\}_{i=0}^n$ of $\mathfrak{t} \cong \mathbb{R}^{n+1}$, the last n elements $\{e_i^*\}_{i=1}^n$ are a basis of $\mathfrak{h}^0$ dual to the basis $\{e_i\}_{i=1}^n$ of the Lie subalgebra $\mathfrak{g} = \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$ of the subgroup $G = \{1\} \times \mathbb{T}^n \subset \mathbb{T}^{n+1}$. This subgroup $G \cong T^n$ is such that $\pi|_G : G \to T/H$ is a Lie group isomorphism and we choose this description of T/H as $\mathbb{T}^n$. Since the dual of the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{t}$ is precisely p, we see that $p \circ \Phi$ is the desired moment map and its image is

$$\Delta_{\mathbb{CP}^n} = \operatorname{im} p \circ \Phi = -\frac{1}{2}\Delta_n - p(c)$$

where Δ_n denotes the standard n-simplex in $\mathbb{R}^n$ generated by 0 and the standard basis, i.e.

$$\Delta_n := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i \leq 1, x_i \geq 0\}.$$

Since $\mathrm{vol}_{\mathbb{R}^n}(\Delta_n) = 1/n!$, we get

$$\mathrm{vol}_{\omega_{FS}}(\mathbb{CP}^n) = (2\pi)^n \mathrm{vol}_{\mathbb{R}^n}(\operatorname{im} p \circ \Phi) = (2\pi)^n \frac{1}{2^n} \mathrm{vol}_{\mathbb{R}^n}(\Delta_n) = \frac{\pi^n}{n!}. \qquad \square$$

Appendix A

Appendix

A.1 Fubini-Study structure on the complex projective space

The complex projective space $\mathbb{CP}^n$ for a positive integer $n > 0$ is defined as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ via the equivalence relation given by

$$(z_0, \dots, z_n) \sim (z'_0, \dots, z'_n) \iff \exists \lambda \in \mathbb{C}^* : (z_0, \dots, z_n) = \lambda(z'_0, \dots, z'_n).$$

We define its topology as the quotient topology given by the projection

$$\mathrm{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n : z = (z_0, \dots, z_n) \mapsto [z] := [z_0 : \dots : z_n].$$

It has the structure of a complex manifold of complex dimension n with the complex charts given by the domains

$$U_i := \{[z] \in \mathbb{CP}^n : z_i \neq 0\},\ i = 0, 1, \dots, n,$$

and the coordinates

$$\psi_i : U_i \to \mathbb{C}^n : [z] \mapsto (\frac{z_0}{z_i}, \frac{z_1}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i}),$$
$$\psi_i^{-1} : \mathbb{C}^n \to U_i : (w_1, \dots, w_n) \mapsto [w_1 : \dots : 1 : \dots : w_n],$$

where the hat means that the i-th term is not present. The change of coordinate maps $\psi_j \circ \psi_i^{-1}$ defined on $W_j := \psi_i(U_i \cap U_j) = \{(w_1, \dots, w_n) \in \mathbb{C}^n : w_j \neq 0\}$ are homolomorphic (for e.g. $i < j$),

$$\psi_j \circ \psi_i^{-1} : W_j \to W_i : (w_1, \dots, w_n) \mapsto (\frac{w_1}{w_j}, \dots, \frac{1}{w_j}, \dots, \frac{\widehat{w_j}}{w_j}, \dots, \frac{w_n}{w_j}).$$

The canonical complex structure J on $\mathbb{CP}^n$ coincides with multiplication by the complex unit $i = \sqrt{-1}$ in any of the coordinates ψ_i. Furthermore, if we consider in $\mathbb{C}^{n+1} \setminus \{0\}$ its standard complex structure, then the projection pr is holomorphic.

This coordinates can also be used to see that $\mathrm{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a $\mathbb{C}^*$-bundle, via the holomorphic local trivializations

$$\mathrm{pr}^{-1}(U_i) \to \mathbb{C}^* \times U_i : (z_0, \dots, z_n) \mapsto (z_i, [z]).$$

We now define the Fubini-Study form ω_{FS} on $\mathbb{CP}^n$. Define in $\mathbb{C}^{n+1} \setminus \{0\}$ the form

$$\omega_1 = \frac{i}{2}\partial\overline{\partial} f(z), \quad f(z) = \log(|z|^2).$$

We compute

$$\begin{aligned}
\omega_1 &= \frac{i}{2}\partial\overline{\partial}\log(|z|^2) \\
&= \frac{i}{2}\partial\left(\frac{1}{|z|^2}\overline{\partial}(\overline{z}\cdot z)\right) \\
&= \frac{i}{2}\partial\left(\frac{1}{|z|^2} z \cdot d\overline{z}\right) \\
&= \frac{i}{2}\left(-\frac{1}{|z|^4}\partial(\overline{z}\cdot z)\wedge(z\cdot d\overline{z}) + \frac{1}{|z|^2}(dz\wedge d\overline{z})\right) \\
&= \frac{i}{2|z|^4}\left(|z|^2(dz\wedge d\overline{z}) - (\overline{z}\cdot dz)\wedge(z\cdot d\overline{z})\right) \\
&= \frac{i}{2|z|^4}\sum_{i,j}\left(|z_i|^2 dz_j\wedge d\overline{z}_j - z_j\overline{z}_i dz_i\wedge d\overline{z}_j\right).
\end{aligned}$$

Either looking at this coordinate expression or since $f(z)$ is a $\mathbb{C}^*$-homogeneous function (i.e., $f(\lambda z) = \lambda f(z)$ for $\lambda \in \mathbb{C}^*$), it is clear that ω_1 is constant on the fibers of $\mathrm{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Furthermore, at a fixed $z \in \mathbb{C}^{n+1} \setminus \{0\}$ we have

$$a_{ij} := \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}(z) = \frac{1}{|z|^4}\left(|z|^2\delta_{ij} - z_j\overline{z}_i\right).$$

Hence, $A = (a_{ij})_{ij}$ is Hermitian and gives a sesquilinear 2-form

$$(u, v) \mapsto A(u, v) := \sum_{ij} a_{ij} u_i \overline{v}_j = \frac{1}{|z|^2}\left(\langle u, v\rangle - \langle u, \hat{z}\rangle\langle \hat{z}, v\rangle\right),$$

for the unit vector $\hat{z} = z/|z|$ and the standard hermitian inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{C}^{n+1}$. Denoting the projection $\langle\cdot,\cdot\rangle$-orthogonal to $\hat{z}$ of $u \in \mathbb{C}^{n+1}$ by $u_\perp$, we get

$$A(u, v) = \frac{1}{|z|^2}\langle u_\perp, v_\perp\rangle.$$

Thus, $\ker A$ is given by the $\mathbb{C}$-span of z, precisely the kernel of the differential at z of the projection $\mathrm{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. All in all, ω_1 descends to a non-degenerate 2-form in $\mathbb{CP}^n$. We define this form to be the *Fubini-Study form*, i.e., the unique form such that

$$\mathrm{pr}^*\omega_1 = \omega_{FS}.$$

The Fubini-Study form is a Kähler form on $\mathbb{CP}^n$, since it is a symplectic, real form and the complex structure is compatible. It is also $U(n+1)$-invariant, since f is.

Alternatively, it can also be defined using the local Kähler potentials given by $\psi_i^{-1,*}h_i(w) = \log(|w|^2+1)$ in each U_i. Clearly, $\frac{i}{2}\partial\overline{\partial}h_i$ all provide the same form, as can also be checked using that $\partial\overline{\partial}(\log(z_i) + \log(\overline{z}_i)) = 0$ for a local complex logarithm. In ψ_i coordinates we get an expression for ω_{FS} at $w = \psi_i(p)$:

$$\omega_{FS} = \frac{i}{2(1+|w|^2)^2}\left((1+|w|^2)\sum_i dw_i \wedge d\overline{w}_i - \sum_{ij} w_j\overline{w}_i dw_i \wedge d\overline{w}_j\right).$$

We note that $\mathrm{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ can be factored through $\mathbb{S}^{2n+1}$:

$$\mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{\mathrm{pr}_1} \mathbb{S}^{2n+1} \xrightarrow{\mathrm{pr}_2} \mathbb{CP}^n,$$

by first taking the partial quotient

$$\mathrm{pr}_1 : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{S}^{2n+1} : z \mapsto \frac{z}{|z|},$$

followed by

$$\mathrm{pr}_2 : \mathbb{S}^{2n+1} \to \mathbb{CP}^n : z \mapsto [z].$$

Proposition A.1 *The pullback of the Fubini Study form via* pr_2 *coincides with the restriction of the standard symplectic form* ω_0 *of* $\mathbb{R}^{2n+2}$ *to the sphere* $\mathbb{S}^{2n+1}$.

Proof Since $\mathrm{pr} = \mathrm{pr}_2 \circ \mathrm{pr}_1$, both pr_1 and pr_2 submersions, and we have defined $\mathrm{pr}^*\omega_{FS} = \omega_1$, it is enough to check that $\mathrm{pr}_1^*\omega_0|_{\mathbb{S}^{2n+1}} = \omega_1$. Hence we compute, denoting $\varphi := \mathrm{pr}_1$ for convenience:

$$\begin{aligned}
\mathrm{pr}_1^*\omega_0 &= \varphi^*\left(\frac{i}{2}\sum_i dz_i \wedge d\overline{z}_i\right)\\
&= \frac{i}{2}\sum_i d\varphi_i \wedge d\overline{\varphi}_i\\
&= \frac{i}{2}\sum_i (\partial\varphi_i + \overline{\partial}\varphi_i) \wedge (\partial\overline{\varphi}_i + \overline{\partial}\overline{\varphi}_i)
\end{aligned}$$

$$= \frac{i}{2}\sum_i \left(\frac{1}{|z|}dz_i - \frac{z_i}{2|z|^3}(z \cdot d\overline{z} + dz \cdot \overline{z})\right)$$
$$\wedge \left(\frac{1}{|z|}d\overline{z}_i - \frac{\overline{z}_i}{2|z|^3}(z \cdot d\overline{z} + dz \cdot \overline{z})\right)$$
$$= \frac{i}{2|z|^4}\sum_{i,j}\left(|z_i|^2 dz_j \wedge d\overline{z}_j - z_j\overline{z}_i dz_i \wedge d\overline{z}_j\right)$$
$$= w_{FS}. \qquad \square$$

A.2 Principal bundles, connections and characteristic classes

In this section we survey the main definitions and results concerning connections, curvature forms, characteristic classes, and their application to principal bundles, following [27].

A.2.1 Vector valued forms, connections and curvature

Consider a finite-dimensional real vector space V.

Definition A.2 (V-valued k-form) A *V-valued k-form* on a smooth manifold M is a smooth function assigning to each point $p \in M$ a V-valued k-covector on the tangent space T_pM, i.e., an element of $(\bigwedge^k T_p^*M) \otimes V$, where $\bigwedge^k$ denotes the k-th alternate product. Alternatively, a V-valued k-form is a smooth section of the vector bundle $(\bigwedge^k T^*M) \otimes V$ over M. We denote them by

$$\Omega^k(M;V) := \Gamma\left(\left(\bigwedge^k T^*M\right) \otimes V\right).$$

Given a basis $\{v_i\}_i^n$ of V and $\alpha \in \Omega^k(M;V)$, then one can write

$$\alpha = \sum_i^n \alpha^i v_i,$$

for the components $\alpha^i \in \Omega^k(M;\mathbb{R})$ of α in the basis $\{v_i\}_i^n$.

Let V, W, Z be finite-dimensional vector spaces and $\mu : V \times W \to Z$ a bilinear map. We define the *exterior product with respect to* μ of $\alpha \in \Omega^k(M;V)$ and $\beta \in \Omega^l(M;W)$ by

$$(\alpha \cdot \beta)(t_1, \ldots, t_{k+l}) :=$$
$$\frac{1}{k!l!}\sum_{\sigma \in S_{k+l}} (-1)^\sigma \mu(\alpha(t_{\sigma(1)}, \ldots, t_{\sigma(k)}), \beta(t_{\sigma(k+1)}, \ldots, t_{\sigma(k+l)})),$$

where S_{k+l} is the permutation group of $k+l$ elements. If we fix basis of V, W, and Z, $\{v_i\}_i$, $\{w_j\}_j$ and $\{z_k\}_k$ respectively, we have

$$\alpha \cdot \beta = \sum_{i,j,k} \alpha^i \wedge \beta^j c_{ij}^k z^k \in \Omega^{k+l}(M; Z),$$

where c_{ij}^k is the k-th coordinate of $\mu(v_i, w_j)$ and α^i and β^j are the coordinates of α and β. The *exterior differential* of $\alpha \in \Omega^k(M; V)$ is

$$d\alpha := \sum_i^n d\alpha^i v_i.$$

Similarly the *Lie derivative* is

$$\mathfrak{L}_X \alpha = \sum_i^n \mathfrak{L}_X \alpha^i v_i.$$

If $X \in \mathfrak{X}(M)$ has flow φ_t^X, it coincides with the definition by the standard formula:

$$\mathfrak{L}_X \alpha := \left.\frac{d}{dt}\right|_{t=0} \varphi_{-t,*}^X \alpha,$$

where one defines the notion of the pullback of V-valued forms in the exact same manner as for usual forms. Clearly, one still has

$$d(\alpha \cdot \beta) = (d\alpha) \cdot \beta + (-1)^{\deg \alpha} \alpha \cdot (d\beta).$$

Consider now a Lie group with Lie algebra $\mathfrak{g}$. Then, the $\mathfrak{g}$*-valued k-forms* are given by $\Omega^k(M; \mathfrak{g})$, i.e., we take $V := \mathfrak{g}$. Furthermore we have a canonical bilinear form, the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, and we define the *Lie bracket of* $\mathfrak{g}$*-valued forms* $\alpha \in \Omega^k(M; \mathfrak{g})$ and $\beta \in \Omega^l(M; \mathfrak{g})$ by

$$[\alpha, \beta](t_1, \ldots, t_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma [\alpha(t_{\sigma(1)}, \ldots, t_{\sigma(k)}), \beta(t_{\sigma(k+1)}, \ldots, t_{\sigma(k+l)})],$$

i.e.,

$$[\alpha, \beta] = \sum_{i,j} \alpha^i \wedge \beta^j [\xi_i, \xi_j] \in \Omega^{k+l}(M; \mathfrak{g}),$$

for a basis $\{\xi_i\}_i$ of $\mathfrak{g}$. The skew-symmetry of the Lie bracket allows to show that, for $\alpha \in \Omega^k(M; \mathfrak{g})$ and $\beta \in \Omega^l(M; \mathfrak{g})$,

$$[\alpha, \beta] = (-1)^{kl+1} [\beta, \alpha].$$

Similarly, one has

$$d[\alpha, \beta] = [d\alpha, \beta] + (-1)^k [\alpha, d\beta].$$

Finally, all definitions and properties of V-valued k-forms generalise to the case of forms with values in a smooth vector bundle $\pi : E \to M$, defined as the sections of the vector bundle $(\bigwedge^k T^*M) \otimes E$, that is:

$$\Omega^k(M; E) := \Gamma\left(\left(\bigwedge^k T^*M\right) \otimes E\right).$$

That is, at each point $p \in M$ we have a skew-symmetric multilinear map

$$\alpha_p : \bigwedge^k T_p^*M \to E_p.$$

Let $\pi : E \to M$ be a smooth vector bundle over M.

Definition A.3 (Connection) A *connection* on E is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) : (X, s) \mapsto \nabla_X s,$$

such that for any $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$,

1. $\nabla_X s$ is $\mathcal{C}^\infty(M, \mathbb{R})$-linear in X and $\mathbb{R}$-linear in s;
2. (Leibniz rule) given $f \in \mathcal{C}^\infty(M, \mathbb{R})$, then

$$\nabla_X(fs) = (Xf)s + f\nabla_X s.$$

It can be checked that connections are local in s in the sense that the value of $\nabla_X s$ at a point $p \in M$ depends only on the values of s in any neighborhood of p. Moreover, the dependence of $\nabla_X s$ on X is not only local but pointwise, i.e., it only depends on X_p.

Example A.4 In a trivial bundle $\pi_M : E = M \times V \to M$ for some finite-dimensional vector space V, one always has the trivial connection such that, given $X \in \mathfrak{X}(M)$ and $s = \sum_i h^i e_i \in \Gamma(M \times V)$, for a basis $\{e_i\}_i$ of V and smooth $h^i \in \mathcal{C}^\infty(M, \mathbb{R})$,

$$\nabla_X s := \sum_i (Xh^i)e_i. \qquad \square$$

Clearly, the space of connections on a vector bundle $\pi : E \to M$ is affine, since given ∇, ∇' and $t \in \mathbb{R}$ then $\nabla^t := t\nabla + (1-t)\nabla'$ is also a connection:

$$\begin{aligned}
\nabla_X^t(fs) &= t((Xf)s + f\nabla_X s) + (1-t)((Xf)s + f\nabla'_X s) \\
&= (Xf)s + f(t\nabla + (1-t)\nabla')_X s \\
&= (Xf)s + f\nabla_X^t s.
\end{aligned}$$

For this reason, one can always obtain a connection on E by covering U with local trivializations, choosing the trivial connection of the previous example on the domain of each such local trivialization, and finally combining these connections by use of a partition of unity subordinate to the covering.

Definition A.5 (Curvature of a connection) The *curvature* of a connection ∇ is the map

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$
$$(X, Y, s) \mapsto R(X, Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

Clearly, $R(X, Y)s$ is skew-symmetric in X, Y, and a computation shows that it is $\mathcal{C}^\infty(M, \mathbb{R})$-linear in each of its three arguments, and thus it defines a pointwise linear map $R_p : T_pM \times T_pM \times E_p \to E_p$.

Let ∇ be a connection on a vector bundle $\pi : E \to M$ of rank r, and suppose we have a local frame $\{e_i\}_i^r$ of E over a trivializing open set U (i.e., $e_i \in \Gamma(E|_U)$ form a basis of E_p at every $p \in U$). Then, we can express the connection ∇ over the elements of the local frame as

$$\nabla_X e_j = \sum_i \alpha^i_j(X) e_i$$

for some $\alpha^i_j \in \Omega^1(U; \mathbb{R})$, and similarly

$$R(X, Y)e_j = \sum_i \Omega^i_j(X, Y) e_i,$$

for $\Omega^i_j \in \Omega^2(U; \mathbb{R})$.

Proposition A.6 *Let ∇ be a connection on a vector bundle $\pi : E \to M$ of rank r, and suppose we have a local frame $\{e_i\}_i^r$ of E over a trivializing open set U. Then, the coefficients Ω^i_j of the curvature of ∇ are given by*

$$\Omega^i_j = d\alpha^i_j + \sum_k \alpha^i_k \wedge \alpha^k_j.$$

The proof is a computation; see also Theorem 11.1 in [27].

This can be abbreviated by writing

$$\Omega = d\alpha + \alpha \wedge \alpha,$$

where the differential of a matrix of forms is defined component-wise, and where the exterior product of two matrices of forms is defined by taking as the bilinear form μ (as in the beginning of this subsection) the standard matrix multiplication.

A.2.2 Characteristic classes and the Chern–Weil homomorphism

As we have seen, a connection ∇ on a vector bundle E of rank r over a manifold M can be represented locally by a matrix $\alpha \equiv (\alpha^i_j)_{ij}$ of 1-forms relative to a frame for E over an open set U, and similarly for its curvature R by a matrix $\Omega := (\Omega^i_j)_{ij}$ of 2-forms. If we change the local frame on U, i.e., a change of basis, from $e \equiv (e_i)^r_i$ (as a row matrix) to $\overline{e} \equiv (\overline{e}_i)^r_i = ea$, for some smooth matrix section $a : U \to GL(r, \mathbb{R})$, then one has for the matrices $\overline{\alpha}$ and $\overline{\Omega}$ expressing ∇ and R in terms of $\overline{e}$:

$$\overline{\alpha} = a^{-1}\alpha a + a^{-1}da$$

and

$$\overline{\Omega} = a^{-1}\Omega a,$$

as one checks using the properties of the connection ∇ (for $\nabla e = e\alpha$ and $Re = e\Omega$ versus $\nabla\overline{e} = \overline{e\alpha}$ and $R\overline{e} = \overline{e}\overline{\Omega}$).

Thus, given a polynomial $P(X)$ in r^2 variables invariant under conjugation of the argument, regarded as a square matrix, by elements of $GL(r, \mathbb{R})$, we can define the form $P(\Omega)$ independently of the frame as a global form of M (one defines it using the matrix expressions Ω given by local frames of a covering of M by local trivializations; since $P(\Omega)$ does not depend on the frame, this is well defined globally). It turns out that this form is closed and furthermore independent of the original connection:

Theorem A.7 *Let E be a smooth vector bundle of rank r over M, ∇ a connection on E, and $P(X)$ an invariant homogeneous polynomial of degree k on $\mathfrak{gl}(r, \mathbb{R})$; then:*

(i) the global $2k$-form $P(\Omega)$ is closed;

(ii) the cohomology class $[P(\Omega)] \in H^{2k}(M; \mathbb{R})$ is independent of the connection ∇.

For a proof, we refer to e.g. Theorem 23.3 of [27].

This gives rise to an algebra homomorphism

$$c_E : \mathrm{Inv}(\mathfrak{gl}(r, \mathbb{R})) \to H^*(M; \mathbb{R}) : P(X) \mapsto [P(\Omega)],$$

called the *Chern–Weil homomorphism*, where the algebra structure (over $\mathbb{R}$) on $H^*(M; \mathbb{R})$ is the de Rham structure. Here, $\mathrm{Inv}(\mathfrak{gl}(r, \mathbb{R}))$ is the algebra of polynomials on $\mathfrak{gl}(r, \mathbb{R})$ that are invariant under the $GL(r, \mathbb{R})$-action on $\mathfrak{gl}(r, \mathbb{R})$ by conjugation. For each polynomial $P(X) \in \mathrm{Inv}(\mathfrak{gl}(r, \mathbb{R}))$ homogeneous of degree k, $[P(\Omega)] \in H^{2k}(M; \mathbb{R})$ is an isomorphism-invariant of the vector bundle E. In this sense, $[P(\Omega)]$ is characteristic of the vector bundle E and is called a *characteristic class* of E; concretely, we define:

Definition A.8 (Characteristic class) A *characteristic class* on real smooth vector bundles associates to each smooth manifold M a map

$$c_M : \left\{ \begin{array}{c} \text{isomorphism classes of real} \\ \text{vector bundles over } M \end{array} \right\} \to H^*(M;\mathbb{R}),$$

such that if $f : N \to M$ is a smooth map and E is a vector bundle over M, then

$$c_N(f^*E) = f^*c_M(E),$$

(for the pullback vector bundle f^*E).

In category theory language, we have three categories: the category of smooth manifolds with smooth maps, the category of (sets of isomorphism classes of) real vector bundles of rank k over smooth manifolds with vector bundle homomorphisms, and the category of (graded) rings with (graded) ring homomorphisms. We then have two functors: Vect_k assigns to each smooth manifold M the set of isomorphism classes of real vector bundles of rank k over M, and H^* assigns $H^*(M;\mathbb{R})$ to M. A characteristic class is a *natural transformation* between the functors Vect_k and H^*. Thus, each invariant polynomial $P(X) \in \mathrm{Inv}(\mathfrak{gl}(r,\mathbb{R}))$ gives rise to a natural transformation $\mathrm{Vect}_k \to H^*$ via the characteristic class c such that $c_M : E \mapsto [P(\Omega)]$ for the curvature matrix Ω of any connection on E.

Example A.9 (Invariant polynomials) There are two fundamental groups of invariant polynomials in $\mathrm{Inv}(\mathfrak{gl}(r,\mathbb{R}))$:

- The *coefficients of the characteristic polynomial*: let $X = (x^i_j)_{ij}$ be an $r \times r$ matrix of indeterminates x^i_j, and λ another indeterminate. The coefficients $f_k(X)$ of the monomial of order $r-k$ of the polynomial in λ given by the determinant

$$\det(\lambda \mathrm{id} + X) = \lambda^r + f_1(X)\lambda^{r-1} + \cdots + f_{r-1}(X)\lambda + f_r(X),$$

  are polynomials on $\mathfrak{gl}(r,\mathbb{R})$. They are $GL(r,\mathbb{R})$-invariant since det is and hence $\det(\lambda\mathrm{id} + aXa^{-1}) = \det(\lambda\mathrm{id} + X)$.

- The *trace polynomials*, defined as

$$\Sigma_k(X) := \mathrm{tr}(X^k)$$

  are $GL(r,\mathbb{R})$-invariant since the trace is and $(aXa^{-1})^k = aX^ka^{-1}$. □

They are related by Newton's identity:

$$\Sigma_k - f_1\Sigma_{k-1} + f_2\Sigma_{k-2} - \cdots + (-1)^{k-1}f_{k-1}\Sigma_1 + (-1)^k k f_k = 0.$$

A crucial fact regarding this relation is the following:

Theorem A.10 *The ring* $\mathrm{Inv}(\mathfrak{gl}(r,\mathbb{R}))$ *of invariant polynomials on* $\mathfrak{gl}(r,\mathbb{R})$ *is generated as a ring either by the coefficients* $f_k(X)$ *of the characteristic polynomial or by the trace polynomials* $\Sigma_k(X)$*:*

$$\mathrm{Inv}(\mathfrak{gl}(r,\mathbb{R})) = \mathbb{R}[f_1,\ldots,f_r] = \mathbb{R}[\Sigma_1,\ldots,\Sigma_r].$$

A proof can be seen in Appendix B of [27].

Using the invariance of a class $[P(\Omega)]$ from the connection, one can always choose a Riemannian metric on E and then choose a connection compatible with the metric. This will imply that Ω is skew-symmetric, and hence also all its odd powers Ω^{2k+1}. Since the trace of a skew-symmetric matrix is 0 and since the trace polynomials generate $\mathrm{Inv}(\mathfrak{gl}(r,\mathbb{R}))$, we obtain:

Theorem A.11 *If a homogeneous invariant polynomial* $P(X) \in \mathrm{Inv}(\mathfrak{gl}(r,\mathbb{R}))$ *has odd degree* k*, then for any connection* ∇ *on any vector bundle* E *over* M *with curvature matrix* Ω*, the cohomology class* $[P(\Omega)]$ *is zero in* $H^{2k}(M;\mathbb{R})$*.*

For a proof we refer to Theorem 24.3 of [27].

The theorem states that the polynomials $\mathrm{tr}(\Omega^k)$ and $f_k(\Omega^k)$ are all zero for odd k. Thus, the ring of characteristic classes of E has two sets of generators,

(i) the trace polynomials of even degrees:

$$[\mathrm{tr}(\Omega^2)], [\mathrm{tr}(\Omega^4)], [\mathrm{tr}(\Omega^6)], \ldots;$$

(ii) the coefficients of even degrees of the characteristic polynomial $\det(\lambda \mathrm{id} + X)$:

$$[f_2(\Omega)], [f_4(\Omega)], [f_6(\Omega)], \ldots.$$

Definition A.12 (Pontrjagin class) The k-th *Pontrjagin class* $p_k(E)$ of a real vector bundle E over M is

$$p_k(E) := \left[f_{2k}\left(\frac{i}{2\pi}\Omega\right)\right] \in H^{4k}(M;\mathbb{R}).$$

The factor $i = \sqrt{-1}$ ensures that other formulas are sign-free (since f_{2k} is homogeneous of degree $2k$, the purely imaginary number i disappears); on the other hand, the factor $1/2\pi$ ensures that $p_k(E)$ integrates an integer over any compact oriented submanifold of M of dimension $4k$.

Chern class of a complex vector bundle A *smooth complex vector bundle* of complex rank r is a defined as a smooth map $\pi : E \to M$ of smooth manifolds that is locally of the form to $U \times \mathbb{C}^r \to U$. All the theory of real vector bundles generalises easily to that of complex vector bundles. For example, one can define Hermitian metrics on complex bundles in the same way as standard metrics are defined on real bundles. Connections and connections compatible with a Hermitian metric generalise similarly. In the complex case, a change of basis at a local trivializing set U will be given by a matrix section $a : U \to GL(r, \mathbb{C})$ such that $\bar{e} = ea$ for some moving basis $e, \bar{e}$ of $E|_U$. The same transformation equations hold for the matrix of the curvature Ω, and one considers the $GL(r, \mathbb{C})$-invariant polynomials over $\mathfrak{gl}(R, \mathbb{C})$, $\mathrm{Inv}(\mathfrak{gl}(r, \mathbb{C}))$. Characteristic classes of complex vectors bundles are thus obtained similarly.

Chern classes are the complex analogue of the Pontrjagin classes:

Definition A.13 (Chern class) The k-th *Chern class* $c_k(E)$ of a complex vector bundle E over M is

$$c_k(E) := \left[f_k \left(\frac{i}{2\pi} \Omega \right) \right] \in H^{2k}(M; \mathbb{R}).$$

One should notice however that Chern classes associated to odd degree polynomials need not be zero as was the case for Pontrjagin classes, since the trace of the curvature of a connection compatible with a Hermitian metric need not be zero (it will be however purely imaginary, thus Chern classes have real coefficients).

A.2.3 Principal bundles

A principal G-bundle is a locally trivial family of Lie groups, factoring over some smooth base manifold. One of their points of interest is that the theory of connections on a vector bundle can be generalised to the theory of connections on a principal bundle, with the advantage of having basis-free connection forms.

Definition A.14 (Principal G-bundle) A smooth fiber-bundle $\pi : P \to M$ is a *smooth principal G-bundle* if the fiber is a Lie group G, and G acts smoothly and freely on M such that the local trivializations

$$\psi_U : \pi^{-1}(U) \to U \times G$$

are G-equivariant for the G-self-action on the G-factor.

We recall that we are considering exclusively left actions. In particular, it follows that π is G-invariant and that G acts transitively on each fiber.

Example A.15

- Product G-bundles: The product of a smooth manifold M and a Lie group G, endowed with the G-self-action on the G factor, is the simplest example of principal G-bundle (with a global trivialization given by the identity).
- According to Theorem 2.12, an homogeneous space G/H yields a natural principal bundle $\pi_G \to G/H$. □

Given principal G-bundles $\pi_P : P \to M$ and $\pi_Q : Q \to N$, a *morphism* of principal G-bundles is a pair of maps $(\overline{f} : Q \to P, f : N \to M)$ such that $\overline{f}$ is G-equivariant and we have a commuting diagram

$$\begin{array}{ccc} Q & \xrightarrow{\overline{f}} & P \\ \downarrow{\scriptstyle \pi_Q} & & \downarrow{\scriptstyle \pi_P} \\ N & \xrightarrow{f} & M. \end{array}$$

In particular, as in any fiber-bundle, π is a submersion and one defines the *vertical tangent space* (consisting of *vertical vectors*) as $V_p := \ker d_p\pi \subset T_pP$, completing the short exact sequence

$$0 \longrightarrow V_p \longrightarrow T_pP \xrightarrow{\pi_{*,p}} T_{\pi(p)}M \longrightarrow 0.$$

For a principal G-bundle, vertical vectors coincide with the space generated by the infinitesimal generators of the action, i.e., with the tangent space at the orbit. Since the action is free, we obtain an isomorphism

$$d_e j_p : \mathfrak{g} \overset{\cong}{\to} V_p = \operatorname{im} d_e j_p.$$

A *horizontal distribution* is a smooth distribution $p \in P \mapsto H_p \subset T_pP$ that is complementary to V_p at each p, that is, such that $V_p \oplus H_p = T_pP$. In other words, a horizontal distribution H is a choice of a splitting of the short exact sequence of vector bundles

$$0 \longrightarrow V \longrightarrow TP \xrightarrow{\pi_*} TM \longrightarrow 0,$$

such that we have a right-inverse of π_* given by the inverse of the restriction

$$\pi_*| : H \overset{\cong}{\to} TM.$$

Since π is G-invariant, so is $V \subset TP$, and we ask $H \subset TP$ to be as well. Given such a G-invariant horizontal distribution, this allows to define a G-invariant projection $\nu : TP \to V$, given by the *vertical component*:

$$\nu_p : T_pP = V_p \oplus H_p \to V_p.$$

Similarly, we denote the *horizontal component* $h : TP \to H$ as

$$h_p : T_pP = V_p \oplus H_p \to H_p.$$

Both projections are G-invariant, in the sense that $\psi_{g,*} \circ \nu_p = \nu_{gp} \circ \psi_{g,*}$ and similarly for h.

Thus, one can consider the $\mathfrak{g}$-valued 1-form α on P that chooses the generator in $\mathfrak{g}$ associated to the vertical component of a given tangent vector:

$$\alpha_p := (d_e j_p)^{-1} \circ \nu_p : T_p P \xrightarrow{\nu_p} V_p \xrightarrow{(d_e j_p)^{-1}} \mathfrak{g}.$$

This construction is equivalent to the choice of the horizontal distribution H, that is, the choice of an Ehresmann connection:

Definition A.16 (Ehresmann connection) An *Ehresmann connection* or simply a *connection* on a principal G-bundle $P \to M$ is a $\mathfrak{g}$-valued 1-form $\alpha \in \Omega^1(P; \mathfrak{g})$ satisfying:

(i) for any $\xi \in \mathfrak{g}$, $p \in P$, we have $\alpha_p(\xi_p^{\#}) = \xi$;

(ii) (G-equivariance) for any $g \in G$ we have $\psi_g^* \alpha = \mathrm{Ad}_g \circ \alpha$.

Reciprocally, one obtains a horizontal distribution from a connection α by defining $H_p := \ker \alpha_p$ for $p \in P$.

Example A.17 In a product G-bundle $U \times G$, one can choose the distribution $H := TU \times \{0\} \subset T(U \times G)$. This is equivalent to choosing as connection

$$\alpha_{(p,g)}(v, \xi) := \xi g^{-1}, \qquad (p, g) \in U \times G,\ (v, \xi) \in T_p U \times T_g G. \qquad \square$$

Lemma A.18 *Given a principal G-bundle, there exists a connection.*

Proof There are several ways to prove it, the simplest using partitions of unity and local trivializations together with the fact that connections form an affine space, as for connections on vector bundles.

Alternatively, for a compact group G, one considers the embedding $\varphi : P \times \mathfrak{g} \to TP : (p, \xi) \mapsto (p, \xi_p^{\#})$ as a vector subbundle and chooses a G-invariant Riemannian metric on P by averaging over G. Then, we let $TP \to \operatorname{im} \varphi : (p, v) \mapsto (p, \mathrm{pr}_p^{\perp}(v))$ be the orthogonal projection to that subbundle with respect to the metric. The 1-form α is defined by $\alpha_p(v) := (d_e j_p)^{-1}(\mathrm{pr}_p^{\perp}(v))$ for $v \in T_p P$. The fact that $\alpha_p(\xi_p^{\#}) = \xi$ follows immediately. The G-equivariance follows from the fact that the metric is G-invariant, so that $\mathrm{pr}_p^{\perp} = d_e j_p \circ \alpha_p$ also is and hence $g(d_e j_p \circ \alpha_p)(v) = d_e j_{gp} \circ \alpha_{gp}(gv)$. Since $g d_e j_p = d_e j_{gp} \circ \mathrm{Ad}_g$, then $\alpha_{gp}(gv) = (d_e j_{gp})^{-1} g (d_e j_p \circ \alpha_p)(v) = \mathrm{Ad}_g \circ \alpha_p(v)$ and we are done. $\square$

In Proposition A.6 we saw that given a aconnection ∇ on a vector bundle E over M, then its connection and curvature matrices α_e and Ω_e for a frame e on an open set U are related by

$$\Omega_e = d\alpha_e + \alpha_e \wedge \alpha_e.$$

In a principal bundle we do not have a matrix representation since the group G need not be linear; however we notice that the commutation of matrices is the Lie bracket in the Lie algebra of any matrix group. In fact, one can write the previous expression for vector bundles as

$$\Omega_e = d\alpha_e + \frac{1}{2}[\alpha_e, \alpha_e],$$

for $[\cdot,\cdot]$ given by matrix commutation in $\mathfrak{gl}(r,\mathbb{R})$. Thus, the natural generalization to principal bundles is to define:

Definition A.19 (Curvature of an Ehresmann connection) Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\alpha \in \Omega^1(P;\mathfrak{g})$ be a connection on a principal G-bundle $\pi : P \to M$. The *curvature* of α is the $\mathfrak{g}$-valued 2-form given by

$$\Omega := d\alpha + \frac{1}{2}[\alpha, \alpha].$$

Theorem A.20 *Consider a principal G-bundle $\pi : P \to M$, α a connection on P, and Ω the curvature form of α; then:*

(i) (Horizontality) Given $p \in P$, $X_p, Y_p \in T_pP$,

$$\Omega_p(X_p, Y_p) = (d\alpha)_p(h_p(X_p), h_p(Y_p)).$$

(ii) (G-equivariance) For $g \in G$, we have $\psi_g^*\Omega = \mathrm{Ad}_g \circ \Omega$.

(iii) (Second Bianchi identity) $d\Omega = [\Omega, \alpha]$.

For a proof, see Theorem 30.4 of [27].

Consider now a principal G-bundle $\pi : P \to M$ and a finite-dimensional representation of G (i.e., a linear left G-action on V) $\rho : G \to GL(V)$, for some finite-dimensional vector space V.

An important construction related to principal bundles is the *associated bundle* $P \times_\rho V$, defined as the quotient of the product space $P \times V$ over the relation

$$(p, v) \sim (gp, gv), \qquad g \in G,\ (p, v) \in P \times V,$$

where $gv := \rho(g)v$. In other words, it is the orbit space of the product $P \times V$ under the diagonal G-action, with the quotient topology. In the following we will show that it is in fact a smooth V-vector bundle over M. We denote the equivalence class of (p, v) by $[p, v]$. The associated bundle comes with a natural projection

$$p : P \times_\rho V \to M : [p, v] \mapsto \pi(p).$$

Clearly p is well-defined, since $p([pg, gv]) = \pi(gp) = \pi(p) = p([p, v])$.

Example A.21 Consider a trivial product bundle $U \times G$ for some smooth manifold U and a Lie group G, and consider a finite-dimensional representation $\rho : G \to GL(V)$. Then, the map

$$\begin{aligned} \varphi_U : (U \times G) \times_\rho V &\xrightarrow{\cong} U \times V, \\ [(x, g), v] &\mapsto (x, g^{-1}v), \end{aligned}$$

is a fiber-preserving homeomorphism. It is clearly surjective, and it is well-defined and injective since

$$\begin{aligned}((x,g),v) \sim ((x',g'),v') &\iff \exists h \in G : ((x,hg),hv) = ((x',g'),v') \\ &\iff x' = x,\ v' = g'g^{-1}v \\ &\iff \varphi_U([(x,g),v]) = \varphi_U([(x',g'),v']).\end{aligned}$$

It is continuous and so is its inverse

$$\begin{aligned}\varphi_U^{-1} : U \times V &\xrightarrow{\cong} (U \times G) \times_\rho V, \\ (x,v) &\mapsto [(x,e),v].\end{aligned}$$ □

In fact, for a general principal G-bundle $\pi : P \to M$, using local trivializations (ψ_U, U) we can work locally on trivial product bundles. This allows us to define a unique smooth structure on $E := P \times_\rho V$ such that the induced maps φ (under localization) are diffeomorphisms. Furthermore, it turns $p : E \to M$ into a V-vector bundle. We have a diagram, making explicit the induced trivialization for E, $\varphi_U \circ \overline{\psi_U}$:

$$\begin{tikzcd} p^{-1}(U) = \pi^{-1}(U) \times_\rho V \arrow[r, "p"] \arrow[d, "\overline{\psi_U}", "\cong"'] \arrow[dr, "\varphi_U \circ \overline{\psi_U}", "\cong"'] & U \\ (U \times G) \times_\rho V \arrow[r, "\varphi_U", "\cong"'] & U \times V, \arrow[u, "\mathrm{pr}_U"'] \end{tikzcd}$$

where $\overline{\psi_U}$ is induced by $\psi_U \times \mathrm{id}_V : \pi^{-1}(U) \times V \to (U \times G) \times V$ after quotienting both sides over the G-action (note that $\psi_U \times \mathrm{id}_V$ is G-equivariant for the diagonal actions).

This local trivialization provides a canonical way of identifying the fiber E_x with the vector space V for each $x \in M$. We define the map, for $p \in P$ such that $\pi(p) = x$:

$$f_p : V \to E_x : v \mapsto [p, v].$$

This map is a linear isomorphism and satisfies $f_{gp} = f_p \circ \rho(g^{-1})$.

Example A.22 1. If the representation $\rho : G \to GL(V)$ is trivial, then $E = P \times_\rho V$ is just the trivial bundle $M \times V$, since $(P \times V)/G = (P/G) \times V \cong M \times V$.

2. If $V = \mathfrak{g}$ and the representation is the adjoint representation of the Lie group G on its Lie algebra $\mathfrak{g}$, $\rho \equiv \mathrm{Ad}$, the associated vector bundle $\mathrm{Ad}P := P \times_{\mathrm{Ad}} \mathfrak{g}$ is called *the adjoint bundle of P*. □

Definition A.23 (Left-equivariant and tensorial form) A V-valued k-form α on P is *left-equivariant of type ρ* if for every $g \in G$,

$$\psi_g^* \alpha = \rho_g \circ \alpha.$$

It is *tensorial of type* ρ if it is left-equivariant of type ρ and horizontal. We denote the set of all smooth tensorial V-valued k-forms of type ρ by $\Omega^k_\rho(P;V)$.

Example A.24 We have already seen the case of such a form: the connection of a principal G-bundle is horizontal and left-equivariant of type Ad, for $V := \mathfrak{g}$ and the G-representation given by the adjoint action. $\square$

Fixed a connection α and its associated horizontal distribution H, we say that a form $\omega \in \Omega^k(P;V)$ is *horizontal* if $i_Y\omega = 0$ for any vertical vector field $Y_p \in V_p$. Furthermore, we define the *horizontal component* $\omega^h \in \Omega^k(P;V)$ of $\omega \in \Omega^k(P;V)$ as

$$\omega^h_p(v_1, \dots, v_k) := \omega_p(h_p(v_1), \dots, h_p(v_k)).$$

Theorem A.25 *There exists a linear isomorphism between the set $\Omega^k_\rho(P;V)$ of tensorial k-forms of type ρ and the set $\Omega^k(M;E)$ of k-forms of M with values on E given by*

$$\Omega^k_\rho(P;V) \to \Omega^k(M;E) : \varphi \mapsto \varphi^b,$$

defined as follows: for $x \in M$ and $u_1, \dots, u_k \in T_xM$, choose $p \in P$ such that $\pi(p) = x$ and lifts $v_1, \dots, v_k \in T_pP$ such that $\pi_{*,p}(v_i) = u_i$, we put

$$\varphi^b_x(u_1, \dots, u_k) := f_p \circ \varphi_p(v_1, \dots, v_k).$$

The inverse map

$$\Omega^k(M;E) \to \Omega^k_\rho(P;V) : \psi \mapsto \psi^\#,$$

is defined, for $p \in P$ and $v_1, \dots, v_k \in T_pP$, as

$$\psi^\#_p(v_1, \dots, v_k) := (f_p)^{-1} \circ \psi_{\pi(p)}(\pi_{*,p}(v_1), \dots, \pi_{*,p}(v_k)).$$

This result is an easy verification; see also Theorem 31.9 in [27].

Example A.26 The curvature Ω of a connection on a principal G-bundle P is tensorial of type Ad and thus can always be viewed as an element Ω^b of $\Omega^2(M; \mathrm{Ad}P)$. $\square$

Definition A.27 (Basic form) A differential form $\omega \in \Omega^k(P;V)$ is *basic* if it is the pullback $\pi^*\overline{\omega}$ of a form $\overline{\omega} \in \Omega^k(M;V)$.

Theorem A.28 *Let $\pi : P \to M$ be a principal G-bundle. A form $\omega \in \Omega^k(P;V)$ is basic if and only if it is horizontal and G-invariant.*

Proof It follows directly from the previous result choosing a trivial representation; one can also argue directly as follows.

First, a basic form $\omega = \pi^*\overline{\omega}$ is obviously horizontal, and G-invariant, since $\psi_g^*\pi^* = (\pi \circ \psi_g)^* = \pi^*$. Reciprocally, given a horizontal and G-equivariant form we can define, for $p \in P$ and $u_i \in T_{\pi(p)}M$,

$$\overline{\omega}_{\pi(p)}(u_1, \ldots, u_k) := \omega_p(v_1, \ldots, v_k),$$

for any vectors $v_i \in T_pP$ such that $\pi_{*,p}(v_i) = u_i$. Since ω is horizontal this definition does not depend on the choice of v_i, and since it is G-equivariant it does not depend on the choice of p (note that since $\pi \circ \psi_g = \pi$, we can choose $v_i' := \psi_{g*}(v_i)$ if we had chosen $p' := gp$). $\square$

Example A.29 Consider the curvature Ω of a connection on a principal G-bundle P for an abelian group G. Then, Ad is trivial so that Ω is horizontal and G-invariant, and thus basic. There exists a unique $C \in \Omega^2(M; \mathfrak{g})$ such that

$$\Omega = \pi^*C. \qquad \square$$

Definition A.30 (Covariant derivative) Let $\pi : P \to M$ be a principal G-bundle with a connection α, and let V be a real vector space. The *covariant derivative* of a V-valued k-form $\alpha \in \Omega^k(P; V)$ is

$$D\alpha := (d\alpha)^h.$$

Example A.31 The curvature of a connection α on a principal G-bundle is, by Theorem A.20, precisely the covariant derivative of α (for $V = \mathfrak{g}$). It should be noted that α is not horizontal and hence not in $\Omega^1_{\mathrm{Ad}}(P; \mathfrak{g})$. $\square$

We conclude this Appendix section by explaining how to obtain characteristic classes of principal G-bundles, similarly to how we associate the Pontrjagin classes to a real bundle, and the Chern classes to a complex bundle.

For that, we have to generalize the notion of invariant polynomials on matrices. We let V be again a finite-dimensional vector space with dual V^*. We can understand polynomials of degree k on V as elements of $\mathrm{Sym}^k(V^*)$, that is, the symmetrized direct sum of k copies of V^*. Relative to a basis $\{e_i\}_i$ of V with corresponding dual basis $\{e_i^*\}_i$, a polynomial Q of degree k is expressible as a sum of monomials of degree k in $\{e_i^*\}_i$,

$$Q = \sum_I a_I e_{i_1}^* \cdots e_{i_k}^*,$$

(using multi-index notation $I \equiv (i_1, \ldots, i_k)$, I ranging over all subgroups with repetition of $\{1, \ldots, k\}$). For the case of $V := \mathfrak{g}$, a polynomial $Q : \mathfrak{g} \to \mathbb{R}$ is $\mathrm{Ad}(G)$-invariant if for all $g \in G$ and $\xi \in \mathfrak{g}$ one has

$$Q(\mathrm{Ad}_g(\xi)) = Q(\xi).$$

We denote them by $\mathrm{Inv}(\mathfrak{g})$; for $G = GL(r, \mathbb{R})$ this reduces to $\mathrm{Inv}(\mathfrak{gl}(r, \mathbb{R}))$.

Consider the curvature Ω of a connection α on the principal G-bundle $\pi : P \to M$ and a basis $\{\xi_i\}_i$ of $\mathfrak{g}$ with corresponding dual basis $\{\mu_i\}_i$ of $\mathfrak{g}^*$. We have that

$$\Omega = \sum_i \Omega^i \xi_i$$

for the components $\Omega^i \in \Omega^2(P; \mathfrak{g})$. For an invariant polynomial in $\mathrm{Inv}(\mathfrak{g})$, $Q = \sum_I a_I \mu_{i_1} \cdots \mu_{i_k}$, we define $Q(\Omega)$ to be the $2k$-form

$$Q(\Omega) := \sum_I a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}.$$

Clearly, this definition in basis-independent, since the transformation of the coefficients a_I compensates that of the coordinates Ω^i.

Theorem A.32 *Let Ω be the curvature of a connection α on a principal G-bundle $\pi : P \to M$, and let Q be an $\mathrm{Ad}(G)$-invariant polynomial of degree k on $\mathfrak{g}$; then:*

(i) $Q(\Omega)$ is a basic form on P, i.e., there is $\Lambda \in \Omega^{2k}(M; \mathbb{R})$ such that $Q(\Omega) = \pi^*\Lambda$.

(ii) Λ is closed.

(iii) The cohomology class $[\Lambda] \in H^{2k}(M; \mathbb{R})$ is independent of the connection α.

A proof can be found in Theorem 32.2 in [27].

Thus, just as in the vector bundle case, one can associate to every $\mathrm{Ad}(G)$-invariant polynomial Q on $\mathfrak{g}$ a *characteristic* cohomology class $[\Lambda] \in H^*(M; \mathbb{R})$ such that $Q(\Omega) = \Lambda$. This homomorphism, i.e.,

$$c_P : \mathrm{Inv}(\mathfrak{g}) \to H^*(M; \mathbb{R}) : Q \mapsto [\Lambda],$$

for the unique $\Lambda \in \Omega^{2k}(M; \mathbb{R})$ such that $Q(\Omega) = \pi^*\Lambda$, is called the *Chern–Weil homomorphism*.

As we already mentioned, in the case of an abelian Lie group G, particularly that of a torus $\mathbb{T}^k$, the adjoint action is trivial and every polynomial on $\mathfrak{g}$ induces a characteristic class. In particular, Ω is $\mathrm{Ad}(\mathbb{T}^k)$-invariant and thus basic and equal to π^*C for $C \in \Omega^2(M; \mathfrak{g})$. Most importantly, the theorem states that the class $c = [C] \in H^2(M; \mathfrak{g})$ is independent of the connection. This is the crucial ingredient used in the corollaries of the Duistermaat–Heckman Theorem, like in Theorem 5.20, and in general justifies the interest of the Duistermaat–Heckman statement, since it makes the 'linear' coefficient relating the forms of reduced spaces at different free values be constant in cohomology and independent of any choices, namely, the choices made to identify different fibers.

Definition A.33 (First characteristic class of a torus-bundle) The *first characteristic class of a* $\mathbb{T}^k$*-bundle* $\pi : E \to M$ is the unique class $c = [C] \in H^2(M; \mathfrak{g})$ such that

$$\Omega = \pi^* C,$$

where Ω is the curvature of any connection on E.

In particular, the components of $[C]$ in terms of any basis of $\mathfrak{g}$ are characteristic classes of E, in the sense introduced above.

Bibliography

[1] M. F. Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14:1–15, 1982.

[2] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23:1–28, 1984.

[3] M. Audin. *Torus Actions on Symplectic Manifolds*. Progress in Mathematics. Birkhäuser Basel, second edition, 2012.

[4] N. Berline and M. Vergne. Zéros d'un champ de vecteurs et classes caractéristiques équivariantes. *Duke Math. J.*, 50:539–549, 1983.

[5] G. Bredon. *Introduction to Compact Transformation Groups*. Pure and Applied Mathematics. Academic Press, New York-London, first edition, 1972.

[6] A. Cannas da Silva. Symplectic toric manifolds. https://people.math.ethz.ch/~acannas/Papers/toric.pdf.

[7] A. Cannas da Silva. *Lectures on Symplectic Geometry*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, first edition, 2008.

[8] F. C. Caramello. Introduction to orbifolds. https://arxiv.org/abs/1909.08699.

[9] P. Crooks and J. Weitsman. Abelianization and the Duistermaat–Heckman Theorem. https://arxiv.org/abs/2302.02421v1.

[10] G. Darboux. Sur le problème de Pfaff. *Bull. Sci. Math.*, 6:49–68, 1882.

[11] T. Delzant. Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. Math. France*, 116:315–339, 1988.

[12] J. J. Duistermaat and G. J. Heckman. On the variation in the cohomology of the symplectic form of the reduced phase space. *Invent. Math.*, 69:259–268, 1982.

[13] M. J. Gotay. On coisotropic imbeddings of presymplectic manifolds. *Advances in Math*, 84(1):111–114, 1982.

[14] V. Guillemin. *Moment Maps and Combinatorial Invariants of Hamiltonian Tn-spaces.* Progress in Mathematics. Birkhäuser Boston, MA, first edition, 1994.

[15] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, 67:491–513, 1982.

[16] G. Heckman. In memoriam Hans Duistermaat (1942-2010): Recollections of a godsend talent. *Nieuw Archief voor Wiskunde*, 11:240–241, 2010.

[17] N. Jacobson. *Lie Algebras.* Dover Publications, Inc., New York, republication of the 1962 original edition, 1979.

[18] J. M. Lee. *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics. Springer New York, NY, second edition, 2013.

[19] J. M. Lee. *Introduction to Riemannian Manifolds.* Graduate Texts in Mathematics. Springer Nature Switzerland AG, second edition, 2018.

[20] E. Lerman and S. Tolman. Hamiltonian torus actions on symplectic orbifolds and toric varieties. *Trans. Amer. Math. Soc.*, 349:4201–4230, 1997.

[21] C.-M. Marle. Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique. *Rendiconti del Seminario Matematico*, 43(2):227–251, 1985.

[22] D. Mcduff and D. Salamon. *J-holomorphic Curves and Symplectic Topology*, volume 52 of *Colloquium Publications.* American Mathematical Society, second edition, 2012.

[23] D. Mcduff and D. Salamon. *Introduction to Symplectic Topology.* Oxford Graduate Texts in Mathematics. Oxford University Press, third edition, 2017.

[24] E. Meinrenken. Symplectic geometry lecture notes. https://www.math.toronto.edu/mein/teaching/LectureNotes/sympl.pdf.

[25] J. K. Moser. On the volume elements on manifolds. *Trans. Amer. Math. Soc.*, 120:286–294, 1965.

[26] I. Satake. On a generalization of the notion of manifold. *Sci. U.S.A. Proc. Nat. Acad.*, 42:359–363, 1956.

[27] W. L. Tu. *Differential Geometry.* Graduate Texts in Mathematics. Springer, first edition, 2017.

[28] A. Weinstein. Symplectic manifolds and their Lagrangian submanifolds. *Advances in Math*, 6:329–346, 1971.