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Toric real loci

Master project in Mathematics

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Abstract

In this master thesis, we study the fixed point sets (real loci) of certain anti-symplectic involutions of toric symplectic manifolds. Based on a theorem of Duistermaat [13], we show that they are branched coverings over the moment polytope, and exploit this fact to get a simple description of their topology.

Additionally, we carefully present the proof of a theorem of Oda [33], which shows that the moment polygon of any 4-dimensional toric symplectic manifold can be obtained from a triangle or a quadrilateral by a finite sequence of corner-choppings. Using this, we are able to explicitly determine the full list of possibilities for the real locus in dimension 4, up to diffeomorphism.

Résumé

Dans cette thèse de master, on étudie les ensembles de points fixes (lieux réels) de certaines involutions anti-symplectiques de variétés symplectiques toriques. Sur la base d'un théorème de Duistermaat [13], on montre qu'ils sont des revêtements ramifiés sur le polytope moment, et on exploite ce résultat pour obtenir une description simple de leur topologie.

En outre, on présente soigneusement la preuve d'un théorème d'Oda [33], qui montre que le polygone moment de toute variété symplectique torique à dimension 4 peut être obtenu à partir d'un triangle ou d'un quadrilatère par une suite finie de "coupes de coin". Grâce à cela, on parvient à déterminer explicitement la liste complète de possibilités pour le lieu réel en dimension 4, à difféomorphisme près.

Contents

Introduction

A toric symplectic manifold is a symplectic manifold equipped with an effective Hamiltonian action of a torus with half the dimension of the manifold. Objects of this kind were first studied by algebraic geometers as *toric varieties* (see e.g. [33, 14]), but since the foundational work of Delzant [12] they have too become mainstays of symplectic geometry. He proved that these manifolds are classified by their moment map image, which is a convex polytope, according to an earlier result obtained independently by Guillemin and Sternberg [19] and by Atiyah [3]. This allows many geometric questions to be translated into more amenable combinatorial statements. In this way, toric symplectic manifolds have served as a testing ground for new theories and developments in the field, while still forming a rich and extensive theory of their own.

Given a toric symplectic manifold, we consider the notion of a *real locus*, which is the fixed point set of an anti-symplectic involution compatible with the Hamiltonian torus action. The motivating example is that of complex projective space $\mathbb{CP}^n$ under the complex conjugation map, or more generally any complex submanifold thereof. These real loci were considered by Duistermaat in [13], albeit in a slightly more general setting. Building up on the work of Meyer [29], who had proven that they are compact embedded Lagrangians, Duistermaat proved that their moment map image is the whole moment polytope. His results were later extended by Goldin and Holm in [15], and by Biss, Guillemin and Holm in [6]. Additionally, O'Shea and Sjamaar have generalised these results to Hamiltonian actions of non-abelian groups [32] (see also [36]).

In the toric setting, it can moreover be shown that the restriction of the moment map to the real locus is a branched covering with 2^n sheets, where $2n$ is the dimension of the symplectic manifold. This fact has been known for a long time, and we are able to trace it at least to the work of Guillemin (see [18]), although the very closely related concept of a *small cover* had already been studied by Davis and Januszkiewicz in [11]. In Theorem 3.11, we present a proof of this fact, establishing as well that the real locus can be obtained as a quotient of the disjoint union of 2^n copies of the moment polytope, glued along their boundaries in a manner which is fully determined by the geometry of the polytope. This has already been explored in the work of Abreu and Macarini in [2], and of Abreu and Gadbled in [1].

In the 4-dimensional case ($n = 2$), this result can be further exploited. In fact, it exhibits the real locus in a very explicit way as the compact connected surface obtained from a polygonal region by gluing its edges in pairs. Furthermore, a theorem of Oda [33] shows that the moment polygon of a toric symplectic manifold of dimension 4 can be obtained from a certain family of triangles and quadrilaterals by iterated *corner-choppings*, a procedure which is defined in Section 2.5. Combining these ingredients, we are able to arrive at a complete understanding of the topology of 2-dimensional real loci.

Note that Oda, being an algebraic geometer, formulated this result in the setting of (complete non-singular) toric varieties, although it can be ported to the symplectic setting without substantial difference. This result is often quoted in works in symplectic geometry, but seldom proved, usually leaving the proof to algebraic geometry references (for instance, the later account by Fulton in [14]). This situation served as the motivation to include in the present thesis a self-contained account of this result, and more generally of 4-dimensional toric symplectic manifolds. In the process, we discovered another account of this proof by Audin in [4].

We must also mention the very closely related work of Brendel, Joontae Kim and Moon [8]. They study *real Lagrangians*, in a more general setting than the real loci considered here, and explore their topology, including a complete description of the monotone 4-dimensional case (toric symplectic del Pezzo surfaces). Moreover, Brendel [7] and Jin Hong Kim [24], among others, have explored the realization problem of a given Lagrangian submanifold of a symplectic manifold as a real Lagrangian.

Overview of this work The present work is an EPFL master's thesis, which was written by the author as a project exchange student at ETH Zurich, during the fall semester of 2023/24.

The main body of this thesis is divided into three chapters.

In Chapter 1, we present some background material which is relevant to the remaining of the thesis. This allows us to fix notations and conventions, as well as to easily reference this content when needed.

In Chapter 2, we look in detail at 4-dimensional toric symplectic manifolds. We classify the associated moment polygons in the triangle and quadrilateral case, and present a proof of Oda's theorem that the general case is obtained from these by a finite sequence of corner-choppings.

Finally, in Chapter 3, we consider the notion of a real structure and the associated real locus. After establishing its basic properties, we prove, as promised, that the real locus is a branched covering of the moment polytope. We combine this result and Oda's theorem to analyse the 4-dimensional case, in which the real locus is a compact connected surface.

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Chapter 1

Background

In this chapter, we include some background material which will be relevant later, for ease of reference.

Whenever it is not explicitly stated otherwise, all manifolds, diffeomorphisms, vector fields, differential forms, Lie group actions, etc. are assumed to be smooth, i.e. infinitely differentiable. By default, manifolds do not have boundary or corners, and submanifolds are smoothly embedded.

1.1 Linear algebra and lattices

Notation 1.1. If V is a real vector space and V^* is its dual, there is a natural pairing between them, which we denote

$$\langle \cdot, \cdot \rangle \colon V^* \times V \to \mathbb{R}, \quad \langle \ell, v \rangle := \ell(v).$$

Definition 1.2. Let V be a finite-dimensional real vector space. A *lattice* Λ in V is a discrete additive subgroup of V which spans V.

Proposition 1.3. *Let V be a finite-dimensional real vector space. An additive subgroup $\Lambda \subseteq V$ is a lattice if and only if there exists an $\mathbb{R}$-basis of V which is a $\mathbb{Z}$-basis of Λ.*

Definition 1.4. Let V be a finite-dimensional real vector space and $\Lambda \subseteq V$ a lattice. A lattice vector $v \in \Lambda$ is called *primitive* if it cannot be written as $v = ku$ with $u \in \Lambda$, $k \in \mathbb{Z}$ and $|k| > 1$.

Proposition 1.5 ([5, (11.4)]). *Let V be a finite-dimensional real vector space and $\Lambda \subseteq V$ a lattice. Then*

$$\Lambda^* := \{\, \ell \in V^* : \langle \ell, v \rangle \in \mathbb{Z} \textit{ for all } v \in \Lambda \,\} \subseteq V^*$$

is a lattice in V^*. It is called the dual lattice *of Λ.*

Proposition 1.6. *Let V be a finite-dimensional real vector space and $\Lambda \subseteq V$ a lattice. Then, under the canonical isomorphism $V^{**} \cong V$, the double dual lattice Λ^{**} is identified with Λ.*

1.2 Lie theory

Definition 1.7. Let G be a Lie group acting smoothly on a manifold M. We say that the action is *effective*, or *faithful*, if no element of the group acts trivially, apart from the identity. Equivalently, the kernel of the action is trivial.

Definition 1.8. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and M a manifold equipped with a smooth action of G. For each $X \in \mathfrak{g}$, we define the *fundamental vector field* $\widetilde{X} \in \Gamma(TM)$[1] associated to X by

$$\widetilde{X}_p := \frac{d}{dt}(\exp tX \cdot p)\Big|_{t=0}$$

for each $p \in M$.

Proposition 1.9 ([26, Theorem 20.18(a)])**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and M a manifold equipped with a smooth action of G. Then, the map*

$$\mathfrak{g} \to \Gamma(TM), \quad X \mapsto \widetilde{X}$$

is a Lie algebra anti-homomorphism.

Definition 1.10. A *torus* T is a compact connected abelian Lie group.

Theorem 1.11 ([9, Theorem 3.6])**.** *Every torus of dimension n is isomorphic (as a Lie group) to the* standard torus $\mathbb{T}^n = (S^1)^n$.

Proposition 1.12. *Let T be a torus with Lie algebra $\mathfrak{t}$, and exponential map denoted by* $\exp\colon \mathfrak{t} \to T$. *Then*

$$\mathfrak{t}_{\mathbb{Z}} := \ker \exp \subseteq \mathfrak{t}$$

is a lattice in $\mathfrak{t}$, which we call the integral lattice *of T. Moreover,*

$$\mathfrak{t}^*_{\mathbb{Z}} := \{\, \ell \in \mathfrak{t}^* : \langle \ell, v\rangle \in 2\pi\mathbb{Z} \textit{ for all } v \in \mathfrak{t}_{\mathbb{Z}} \,\} \subseteq \mathfrak{t}^*$$

is a lattice in $\mathfrak{t}^*$, which we call the weight lattice *of T.*

Proof. In light of Theorem 1.11, it is enough to consider the case where $T = \mathbb{T}^n$ is a standard torus. This is done below in Example 1.13. □

[1]The smoothness of $\widetilde{X}$ follows from an application of [26, Proposition 9.7].

Example 1.13. Consider the standard 1-dimensional torus $\mathbb{T}^1 = S^1$, and regard it as the unit circle in the complex plane $\mathbb{C} \cong \mathbb{R}^2$.

If we pick the vector $(0,1)$ as a basis of the tangent space to S^1 at the identity, we obtain an isomorphism $\operatorname{Lie} S^1 \cong \mathbb{R}$, under which the exponential map of S^1 takes the form

$$\exp\colon \mathbb{R} \to S^1, \quad \theta \mapsto e^{i\theta}.$$

The integral lattice of S^1 is then $\ker \exp = 2\pi\mathbb{Z} \subset \mathbb{R}$.

By dualising, we also get an isomorphism $(\operatorname{Lie} S^1)^* \cong \mathbb{R}$, under which the weight lattice of S^1 is identified with

$$\{\, x \in \mathbb{R} : xy \in 2\pi\mathbb{Z} \text{ for all } y \in 2\pi\mathbb{Z} \,\} = \mathbb{Z} \subset \mathbb{R}.$$

More generally, we can consider the standard n-dimensional torus $\mathbb{T}^n = (S^1)^n$. If we proceed analogously, we obtain an isomorphism $\operatorname{Lie} \mathbb{T}^n \cong \mathbb{R}^n$ under which the exponential map of $\mathbb{T}^n$ takes the form

$$\exp\colon \mathbb{R}^n \to \mathbb{T}^n, \quad (\theta_1, \dots, \theta_n) \mapsto (e^{i\theta_1}, \dots, e^{i\theta_n}),$$

and the integral lattice of $\mathbb{T}^n$ is identified with $(2\pi\mathbb{Z})^n \subset \mathbb{R}^n$. By dualising, we get an isomorphism $(\operatorname{Lie} \mathbb{T}^n)^* \cong \mathbb{R}^n$, which identifies the weight lattice of $\mathbb{T}^n$ with $\mathbb{Z}^n \subset \mathbb{R}^n$. $\triangle$

Definition 1.14. We define the group $\mathrm{AGL}(n, \mathbb{Z})$ to be the group of affine linear maps $T\colon \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$T(x) = Ax + b,$$

for some $A \in \mathrm{GL}(n, \mathbb{Z})$ and $b \in \mathbb{R}^n$.

More generally, if V is a finite-dimensional real vector space and Λ is a lattice in V, we define the group $\mathrm{AGL}(\Lambda)$ to be the group of affine linear maps $T\colon V \to V$ of the form

$$T(v) = Av + b,$$

for some $A \in \mathrm{GL}(\Lambda)$ and $b \in V$.

Theorem 1.15 ([26, Theorem 19.26])**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there exists a unique connected subgroup of G whose Lie algebra is $\mathfrak{h}$.*

1.3 Convex polytopes

Definition 1.16. Let V be a finite-dimensional real vector space. A *(convex) polytope* Δ in V is the convex hull of a finite number of points in V.

Definition 1.17. Let V be a finite-dimensional real vector space. A *(convex) polyhedron* P in V is a subset of V which is the intersection of finitely many half-spaces, i.e.

$$P = \bigcap_{i=1}^{d} \{ x \in V : \langle \ell_i, x \rangle \geq \lambda_i \},$$

where $\lambda_1, \dots, \lambda_d \in \mathbb{R}$ are scalars, and $\ell_1, \dots, \ell_d \in V^* \setminus \{0\}$ are non-zero linear functions on V.

Theorem 1.18 (Weyl–Minkowski, [5, Theorem 4.4 and Theorem 4.7])**.** *Let V be a finite-dimensional real vector space. Then $P \subseteq V$ is a polytope if and only if it is a bounded polyhedron.*

Definition 1.19. Let V be a finite-dimensional real vector space and $P \subseteq V$ a polyhedron. A subset $F \subseteq P$ is a *face* of P if it is of the form

$$F = P \cap \{ x \in V : \langle \ell, x \rangle = \lambda \},$$

where $\ell \in V^*$ is a linear function on V and $\lambda \in \mathbb{R}$ is a scalar such that for all $x \in P$ we have $\langle \ell, x \rangle \geq \lambda$. We also say that $\{ x \in V : \langle \ell, x \rangle = \lambda \}$ is a *supporting hyperplane* for P.

Definition 1.20. Let V be a finite-dimensional real vector space and S a subset of V. The *(affine) dimension* of S, denoted by $\dim S$, is the dimension of the affine subspace of V generated by S.[2]

Definition 1.21. Let V be a finite-dimensional real vector space and $P \subseteq V$ a polyhedron. A face of dimension 0 is necessarily a singleton, whose unique element is called a *vertex* of P. A face of dimension 1 is called an *edge*, and a face of dimension $\dim P - 1$ is called a *facet*.

If F is a face of P, the corresponding *open face* F° is the relative interior of F.

Proposition 1.22 ([5, Theorem 4.15(2)])**.** *Let V be a finite-dimensional real vector space and $P \subseteq V$ a polyhedron. Then every point $p \in P$ is contained in exactly one open face of P.*

Remark 1.23 ([17, Propositions 2.6.2 and 2.6.3]). Let V be a finite-dimensional real vector space and $P \subseteq V$ a polyhedron. Write P as an *irredundant* intersection of half-spaces,

$$P = \bigcap_{i=1}^{d} \{ x \in V : \langle \ell_i, x \rangle \geq \lambda_i \},$$

for some covectors $\ell_1, \dots, \ell_d \in V^* \setminus \{0\}$ and scalars $\lambda_1, \dots, \lambda_d \in \mathbb{R}$. More precisely, by the irredundancy condition we mean that no proper subset of this family of half-spaces has P as its intersection.

[2]Note that the dimension of the empty set as an affine subspace of V is -1.

If the polyhedron P is full-dimensional ($\dim P = \dim V$), then it has exactly d facets, which are given by

$$F_i = P \cap \{\, x \in V : \langle \ell_i, x \rangle = \lambda_i \,\}, \quad i = 1, \ldots, d.$$

Definition 1.24. Let V be a real vector space of dimension n, Λ a lattice in V, and $\Delta \subseteq V$ a convex polytope. We say that Δ is a *unimodular polytope* (with respect to Λ) if it is such that

- each vertex of Δ belongs to exactly n edges (we say that these n edges *meet* at the vertex);
- at each vertex $p \in \Delta$, the n edges meeting at p are contained in rays of the form $\{\, p + tu_i : t \geq 0 \,\}$, where $(u_1, \ldots, u_n)$ is a $\mathbb{Z}$-basis of Λ.

It is worth stressing that the vertices of a unimodular polytope are not required to belong to the lattice. Additionally, it follows from the definition that unimodular polytopes are necessarily full-dimensional.

Remark 1.25. Let V be a finite-dimensional real vector space, Λ a lattice in V, and $\Delta \subseteq V^*$ a unimodular polytope.

It is immediate that, for any non-zero constant $\lambda \in \mathbb{R} \setminus \{0\}$, the scaled image $\lambda\Delta$ is also unimodular.

Similarly, for any transformation $T \in \mathrm{AGL}(\Lambda)$, the image $T(\Delta)$ is again a unimodular polytope.

Proposition 1.26. *Let V be a finite-dimensional real vector space, Λ a lattice in V, and $\Delta \subseteq V^*$ a full-dimensional convex polytope in the dual space V^*. Consider the dual lattice $\Lambda^* \subseteq V^*$.*

Following Remark 1.23, describe Δ as an irredundant intersection of half-spaces (we are using the canonical identification $V^{**} \cong V$):

$$\Delta = \bigcap_{i=1}^{d} \{\, \xi \in V^* : \langle \xi, v_i \rangle \geq \lambda_i \,\},$$

where $v_1, \ldots, v_d \in V \setminus \{0\}$ are vectors and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ are scalars.

If Δ is a unimodular polytope with respect to the dual lattice Λ^*, then each pair (v_i, λ_i) can be rescaled, without changing the associated half-space, so that v_i is a primitive element of the lattice Λ. In that case, we have moreover that

(*) *for each vertex $p \in \Delta$, the vectors v_i associated to the facets of Δ containing p form a $\mathbb{Z}$-basis of Λ.*

Conversely, if the v_i are primitive elements of Λ and condition **(*)** *holds, then Δ is unimodular (with respect to Λ^*).*

1.4 Toric symplectic manifolds

Definition 1.27. Let (M, ω) be a symplectic manifold and G a Lie group acting smoothly on M by symplectomorphisms. Denote by $\mathfrak{g}$ the Lie algebra of G. We say that the action is *Hamiltonian* if there exists a map $\mu\colon M \to \mathfrak{g}^*$ such that:

- for each $X \in \mathfrak{g}$, the function $\mu^X := \langle \mu(\cdot), X \rangle \colon M \to \mathbb{R}$ is a Hamiltonian function for the fundamental vector field $\widetilde{X}$, i.e. $\omega(\widetilde{X}, \cdot) = d\mu^X$;
- μ is equivariant with respect to the action $\psi\colon G \to \mathrm{Symp}(M, \omega)$, i.e. for all $g \in G$ we have $\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu$, where Ad^* denotes the coadjoint representation of G on $\mathfrak{g}^*$.

The map μ is said to be a *moment map* for the action. We say that (M, ω, G, μ) is a *Hamiltonian G-space*.

If the group G is abelian, the coadjoint representation is trivial, and hence the second condition for a moment map reduces to invariance with respect to the action of G on M. This applies in particular when G is a torus, which will be the main case of interest in the following.

Theorem 1.28 (Atiyah–Guillemin–Sternberg, [3, Theorem 1] or [19, Theorem 4])**.** *Let (M, ω) be a compact connected symplectic manifold equipped with a Hamiltonian action of a torus T. Let $\mu\colon M \to \mathfrak{t}^*$ be a choice of moment map for the action, where $\mathfrak{t}$ denotes the Lie algebra of T. Then $\Delta := \mu(M) \subseteq \mathfrak{t}^*$ is a convex polytope, which is called the* moment polytope *associated to the action. The vertices of this polytope are images under μ of fixed points of the torus action.*

Definition 1.29. A *toric symplectic manifold* of dimension $2n$ is a compact connected symplectic manifold (M, ω) of dimension $2n$ equipped with an effective Hamiltonian action of a torus T of dimension n, together with a choice of corresponding moment map μ.

The following terminology is based on [34, 35].

Definition 1.30. Let T be a torus, and $(M_1, \omega_1, T, \mu_1)$, $(M_2, \omega_2, T, \mu_2)$ toric symplectic manifolds. We say that they are

- *isomorphic* if there exists a T-equivariant symplectomorphism
$$\varphi\colon (M_1, \omega_1, T) \to (M_2, \omega_2, T)$$
such that $\mu_2 \circ \varphi = \mu_1$;
- *weakly isomorphic* if there exists an automorphism $h\colon T \to T$ and a symplectomorphism $\varphi\colon (M_1, \omega_1) \to (M_2, \omega_2)$ such that, for any $g \in T$ and $p \in M_1$, we have
$$\varphi(g \cdot p) = h(g) \cdot \varphi(p).$$

We already know that the image of the moment map is a polytope. It turns out that, in the toric case, this map has a close connection to the action of the torus, and is amenable to a quite concrete description.

Lemma 1.31 ([12, Lemma 2.2]). *Let (M, ω, T, μ) be a toric symplectic manifold, with moment polytope $\Delta = \mu(M) \subseteq \mathfrak{t}^*$, where $\mathfrak{t}$ denotes the Lie algebra of T. Then:*

- *the fibres of $\mu\colon M \to \Delta$ are the orbits of the T-action on M;*
- *for each $\xi \in \Delta$, the fibre $\mu^{-1}(\xi)$ is a torus of dimension equal to that of the open face of Δ containing ξ (cf. Proposition 1.22);*
- *for each $p \in M$, the isotropy group $T_p = \{ g \in T : g \cdot p = p \}$ is the connected subgroup of T whose Lie algebra is the annihilator in $\mathfrak{t}$ of the open face of Δ containing $\mu(p)$ (cf. Theorem 1.15).*

However, something much stronger is true: the moment polytope completely characterises its toric symplectic manifold, up to isomorphism. Moreover, there is a simple characterisation of the convex polytopes which are the moment polytope of some toric symplectic manifold.

Theorem 1.32 (Delzant, [12, Theorem 2.1 and Section 3]). *Fix a torus T.*

For any toric symplectic manifold (M, ω, T, μ), its moment polytope $\Delta \subseteq \mathfrak{t}^*$ is a unimodular polytope with respect to the weight lattice $\mathfrak{t}^*_{\mathbb{Z}}$ of T.

Additionally, any unimodular polytope $\Delta \subseteq \mathfrak{t}^*$ can be realised as the moment polytope of some toric symplectic manifold $(M_\Delta, \omega_\Delta, T, \mu_\Delta)$.

Moreover, this induces a one-to-one correspondence between toric symplectic manifolds (M, ω, T, μ) up to isomorphism, and unimodular polytopes Δ in $\mathfrak{t}^*$.

This descends to a one-to-one correspondence between toric symplectic manifolds (M, ω, T, μ) up to weak isomorphism and unimodular polytopes $\Delta \subseteq \mathfrak{t}^*$ up to a transformation in $\mathrm{AGL}(\mathfrak{t}^*_{\mathbb{Z}})$.

Remark 1.33. It is important to note that the weight lattice $\mathfrak{t}^*_{\mathbb{Z}}$ of a torus T is *not* the dual lattice $(\mathfrak{t}_{\mathbb{Z}})^*$ of the integral lattice $\mathfrak{t}_{\mathbb{Z}}$, in the sense of Proposition 1.5, due to the extra factor of 2π. However, by inspecting Definition 1.24, one sees that the notion of unimodular polytope is the same with respect to both of these lattices, since they differ only by a positive real scaling factor.

Hence, Proposition 1.26 applies, giving an alternate characterisation of the moment polytope Δ of a toric symplectic manifold.

Example 1.34. Let $M = \mathbb{CP}^n$ be the n-dimensional complex projective space, with symplectic structure given by the Fubini–Study form ω_{FS} (for more details, see for instance [10, Homework 12] or [28, Example 4.3.3]). Recall that, on each open set

$$U_j = \{ [z_0 : \cdots : z_n] \in \mathbb{CP}^n : z_j \neq 0 \} \subseteq \mathbb{CP}^n,$$

the Fubini–Study form is given by

$$\omega_{\mathrm{FS}} = \frac{i}{2}\partial\bar{\partial}\log\left(\frac{z_0\overline{z_0} + \cdots + z_n\overline{z_n}}{z_j\overline{z_j}}\right).$$

The standard n-torus $\mathbb{T}^n$ acts on $\mathbb{CP}^n$ by

$$(t_1, \ldots, t_n) \cdot [z_0 : z_1 : \cdots : z_n] = [z_0 : t_1 z_1 : \cdots : t_n z_n].$$

It is straightforward to check that $\mathbb{T}^n$ acts by symplectomorphisms, since $t\bar{t} = 1$ for any $t \in S^1$, and that this action is effective.

Moreover, it can be seen that this action is Hamiltonian. Under the identifications of Example 1.13, we claim that this $\mathbb{T}^n$-action of $\mathbb{CP}^n$ is Hamiltonian with moment map

$$\mu \colon \mathbb{CP}^n \to \mathbb{R}^n,$$
$$\mu([z_0 : \cdots : z_n]) = -\frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + \cdots + |z_n|^2}, \ldots, \frac{|z_n|^2}{|z_0|^2 + \cdots + |z_n|^2}\right).$$

It is clear that μ is invariant with respect to the action. Therefore, it remains to show, for each $X \in \mathfrak{t} \cong \mathbb{R}^n$, that μ^X is a Hamiltonian function for the fundamental vector field $\widetilde{X}$ on $\mathbb{CP}^n$.

By Proposition 1.9, it is enough to show this property for a basis of $\mathfrak{t}$. In other words, it is enough to show that the components functions $\mu^1, \ldots, \mu^n$ of the moment map are Hamiltonian functions for the fundamental vector fields associated to $e_1, \ldots, e_n \in \mathbb{R}^n \cong \mathfrak{t}$. This can be verified through computations in local coordinates.

We conclude that $(\mathbb{CP}^n, \omega_{\mathrm{FS}}, \mathbb{T}^n, \mu)$ is a toric symplectic manifold. Note that its moment polytope $\Delta = \mu(\mathbb{CP}^n)$ is

$$\Delta = \left\{ x \in \mathbb{R}^n : x_1 + \cdots + x_n \geq -\frac{1}{2},\ x_1, \ldots, x_n \leq 0 \right\},$$

the n-dimensional simplex in $\mathbb{R}^n$ with vertices $0, -\frac{1}{2}e_1, \ldots, -\frac{1}{2}e_n$. △

Theorem 1.32 states that the weak isomorphism type of a toric symplectic manifold is determined by the equivalence class of its moment polytope $\Delta \subseteq \mathfrak{t}^*$ under the action of $\mathrm{AGL}(\mathfrak{t}^*_{\mathbb{Z}})$. Given a moment polytope Δ, we will now see how to modify its toric symplectic manifold in order to exhibit the other elements of its equivalence class as moment polytopes.

Remark 1.35. Let (M, ω, T, μ) be a toric symplectic manifold with moment polytope $\Delta = \mu(M) \subset \mathfrak{t}^*$.

If $c \in \mathfrak{t}^*$ is any constant, then $\mu + c$ is also a moment map for this Hamiltonian T-action of M, and the associated moment polytope is $\Delta + c$, a translation of Δ.

Remark 1.36. Let (M, ω, T, μ) be a toric symplectic manifold with moment polytope $\Delta = \mu(M) \subset \mathfrak{t}^*$. Denote the action by $\rho\colon T \to \mathrm{Symp}(M)$. For convenience, we will use Theorem 1.11 to identify $T \cong \mathbb{T}^n$, which induces isomorphisms $\mathfrak{t} \cong \mathfrak{t}^* \cong \mathbb{R}^n$ as in Example 1.13.

Consider a transformation

$$A \in \mathrm{GL}(\mathfrak{t}^*_{\mathbb{Z}}) \cong \mathrm{GL}(n, \mathbb{Z}) \subset \mathrm{GL}(n, \mathbb{R}) \cong \mathrm{GL}(\mathfrak{t}^*).$$

On $\mathfrak{t} \cong \mathbb{R}^n$, we have the transpose map $A^\mathsf{T}\colon \mathbb{R}^n \to \mathbb{R}^n$. This map descends to an automorphism $h\colon \mathbb{T}^n \to \mathbb{T}^n$ such that the following diagram commutes,

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "A^\mathsf{T}"] \arrow[d, "\exp"] & \mathbb{R}^n \arrow[d, "\exp"] \\ \mathbb{T}^n \arrow[r, "h"] & \mathbb{T}^n. \end{tikzcd}$$

This is because $A^\mathsf{T}(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$. The invertibility of $A^\mathsf{T} \in \mathrm{GL}(n, \mathbb{Z})$ implies the invertibility of h.

More concretely, if $A = \left(a_{ij}\right)_{i,j=1}^n$, then h can be defined by

$$h(t_1, t_2, \ldots, t_n) = (t_1^{a_{11}} t_2^{a_{21}} \cdots t_n^{a_{n1}}, t_1^{a_{12}} t_2^{a_{22}} \cdots t_n^{a_{n2}}, \ldots, t_1^{a_{1n}} t_2^{a_{2n}} \cdots t_n^{a_{nn}}).$$

We consider a new $\mathbb{T}^n$-action $\widetilde{\rho}\colon \mathbb{T}^n \to \mathrm{Symp}(M)$ defined by $\widetilde{\rho} := \rho \circ h$. It is clear that this action is still symplectic and effective. We claim moreover that this action is Hamiltonian with moment map $\widetilde{\mu} := A \circ \mu$.

Indeed, let $X \in \mathfrak{t} \cong \mathbb{R}^n$. If

$$\widetilde{X} \in \Gamma(TM), \quad \widetilde{X}_p := \frac{d}{dt}(\exp tX \cdot p)\Big|_{t=0}$$

is the fundamental vector field associated to X with respect to the action ρ, then its fundamental vector field with respect to the action $\widetilde{\rho} = \rho \circ h$ is

$$\frac{d}{dt}(h(\exp tX) \cdot p)\Big|_{t=0} = \frac{d}{dt}(\exp A^\mathsf{T}(tX) \cdot p)\Big|_{t=0} = \widetilde{A^\mathsf{T} X}.$$

Then we can see that,

$$\omega(\widetilde{A^\mathsf{T} X}, \cdot) = d\mu^{A^\mathsf{T} X} = d\langle \mu(\cdot), A^\mathsf{T} X \rangle = d\langle A\mu(\cdot), X \rangle = d\widetilde{\mu}^X,$$

showing that $\widetilde{\mu}$ is a moment map for this modified action.

The associated moment polytope $\widetilde{\Delta}$ is then the transformed image of Δ under A.

It is also true that if we scale a unimodular polytope by a non-zero constant, we obtain again a unimodular polytope. We will now see that we can modify a given toric symplectic manifold to change the moment polytope by such a scaling. Note however that this transformation does not preserve the weak isomorphism class.

Remark 1.37. Let $(M, \omega, \mathbb{T}, \mu)$ be a toric symplectic manifold with moment polytope $\Delta = \mu(M) \subset \mathfrak{t}^*$.

For any non-zero constant $\lambda \in \mathbb{R} \setminus \{0\}$, we can consider the same action of T on the symplectic manifold $M, \lambda\omega$. It is clear that this action is still effective and symplectic. Moreover, it is Hamiltonian, with moment map $\lambda\mu : M \to \mathfrak{t}^*$ scaled by the same factor: for any $X \in \mathfrak{t}$ we have

$$(\lambda\omega)(\widetilde{X}, \cdot) = \omega(\lambda\widetilde{X}, \cdot) = d\mu^{\lambda X} = d\langle\mu(\cdot), \lambda X\rangle = d\langle\lambda\mu(\cdot), X\rangle = d(\lambda\mu)^X.$$

We finish with yet another result from Delzant's paper [12]. It tells us that every toric moment map locally looks identical.

Lemma 1.38 ([12, Lemma 2.5], or [4, Proposition IV.4.21]). *Let (M, ω, T, μ) be a toric symplectic manifold of dimension $2n$, with moment polytope $\Delta = \mu(M) \subseteq \mathfrak{t}^*$. Let F be a k-dimensional face of Δ, and V an open ball in F such that its closure $\overline{V} \subset F^{\circ}$ is compact and contained in the interior of F.*

Identify $\mathfrak{t}^* \cong \mathbb{R}^n$, by choosing an integral basis of the k-dimensional subspace of $\mathfrak{t}^*$ parallel to F, and extending it to an integral basis of $\mathfrak{t}^*$. Integrality is meant with respect to the weight lattice, so that this isomorphism identifies $\mathfrak{t}^*_{\mathbb{Z}} \cong \mathbb{Z}^n$. Under this identification, V is an open subset of a standard $\mathbb{R}^k \subseteq \mathbb{R}^n$.

Let $B(\varepsilon)$ denote the open ball in $\mathbb{C}$ centered at the origin and with radius ε.

Then, there exists a neighbourhood U of $\mu^{-1}(V)$ in M, a number $\varepsilon > 0$, and a diffeomorphism

$$\Phi \colon U \to \mathbb{T}^k \times V \times B(\varepsilon)^{n-k} \subset \mathbb{T}^k \times V \times \mathbb{C}^{n-k}$$

such that:

- *Φ pulls back the symplectic form*

$$\sum_{j=1}^{k} d\theta_j \wedge d\mu_j + \sum_{j=1}^{n-k} dx_j \wedge dy_j$$

  to the symplectic form ω;

- *it transforms the T-action on U into the $\mathbb{T}^n$-action on $\mathbb{T}^k \times V \times B(\varepsilon)^{n-k}$ defined by*

$$(t_1, \dots, t_n) \cdot (e^{i\theta_1}, \dots, e^{i\theta_k}, \mu_1, \dots, \mu_k, z_1, \dots, z_{n-k}) = \\ = (t_1 e^{i\theta_1}, \dots, t_k e^{i\theta_k}, \mu_1, \dots, \mu_k, t_{k+1} z_1, \dots, t_n z_{n-k}).$$

The moment map $\mu\colon \mathbb{T}^k \times V \times B(\varepsilon)^{n-k} \to \mathbb{R}^n$ is given by

$$\mu(s_1,\dots,s_k,\mu_1,\dots,\mu_k,z_1,\dots,z_{n-k}) = \\ = c + \left(\mu_1,\dots,\mu_k,-\frac{1}{2}|z_1|^2,\dots,-\frac{1}{2}|z_{n-k}|^2\right).$$

Note that the expression of the moment map differs slightly from the ones in the references given, due to differences in the background conventions.

1.5 Holomorphic fibre bundles

Definition 1.39. Let M and F be complex manifolds. A *holomorphic fibre bundle over M with fibre F*, or simply an *F-bundle over M*, is a complex manifold E together with a surjective holomorphic map $\pi\colon E \to M$ such that, for each $p \in M$, there exists a neighbourhood U of p in M and a biholomorphism $\Phi\colon \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd},$$

where $\pi_1\colon U \times F \to U$ denotes the projection onto the first factor. The map Φ is called a *local trivialization of E over U*.

Example 1.40. The simplest example of a holomorphic fibre bundle is the *trivial bundle $M \times F$*, with $\pi\colon M \times F \to M$ the projection onto the first factor. △

When the fibre F is a vector space, we can impose an additional linearity requirement.

Definition 1.41. Let M be a complex manifold. A *holomorphic vector bundle of rank k over M* is a holomorphic fibre bundle $\pi\colon E \to M$ with fibre $F = \mathbb{C}^k$, such that:

- for each $p \in M$, the fibre $E_p = \pi^{-1}(p)$ is endowed with the structure of a k-dimensional complex vector space;
- the local trivializations $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^k$ can be chosen such that, for every $p \in U$, the restriction $\Phi|_{E_p}\colon E_p \to \{p\} \times \mathbb{C}^k$ is a linear isomorphism.

When $k = 1$, we say that E is a *holomorphic line bundle*.

Example 1.42. As before, an immediate example is the trivial holomorphic vector bundle $\pi\colon M \times \mathbb{C}^k \to M$. △

Example 1.43. The trivial holomorphic line bundle $\mathbb{CP}^n \times \mathbb{C} \to \mathbb{CP}^n$ is denoted by $\mathcal{O}_{\mathbb{CP}^n}$, or simply $\mathcal{O}$ when n is clear from the context. △

Definition 1.44. Let $\pi\colon E \to M$ be a holomorphic vector bundle. A *(holomorphic) sub-bundle* of E is a holomorphic vector bundle $\pi'\colon D \to M$, where D is an embedded complex submanifold of E and $\pi' = \pi|_D$, such that for each $p \in M$ the fibre $D_p = D \cap E_p$ is a linear subspace of E_p.

Example 1.45. Consider the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ over $\mathbb{CP}^n$. Recall that elements of $\mathbb{CP}^n$ are (complex) lines in $\mathbb{C}^{n+1}$. Hence, it is natural to define

$$\mathcal{O}_{\mathbb{CP}^n}(-1) := \{\, (\ell, x) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : x \in \ell \,\}. \tag{1.1}$$

Together with the associated projection $\pi\colon \mathcal{O}_{\mathbb{CP}^n}(-1) \to \mathbb{CP}^n$, it can be checked that this is a sub-bundle of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, and hence a holomorphic line bundle over $\mathbb{CP}^n$. It is called the *tautological line bundle*, and denoted simply by $\mathcal{O}(-1)$ when there is no risk of confusion. △

Proposition 1.46 ([21, Example 2.2.4.iv])**.** *Let $E \to M$ be a holomorphic vector bundle. Then there exists the* dual bundle *$E^* \to M$, with fibres $(E^*)_p = (E_p)^*$, for each $p \in M$.*

Example 1.47. Recall the tautological line bundle $\mathcal{O}_{\mathbb{CP}^n}(-1) \to \mathbb{CP}^n$. Its dual is called the *hyperplane line bundle* on $\mathbb{CP}^n$ and is denoted by $\mathcal{O}_{\mathbb{CP}^n}(1)$, or simply $\mathcal{O}(1)$. △

Proposition 1.48 ([21, Example 2.2.4.ii])**.** *Let $E \to M$, $F \to M$ be holomorphic vector bundles over the manifold M. Then there exists the* tensor product bundle *$E \otimes F \to M$, with fibres $(E \otimes F)_p = E_p \otimes F_p$, for each $p \in M$.*

Example 1.49. Recall the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{CP}^n$. For each positive integer k, we define the line bundle $\mathcal{O}(k) \to \mathbb{CP}^n$ by

$$\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k} = \underbrace{\mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1)}_{k \text{ times}}.$$

We define moreover the line bundle $\mathcal{O}(-k) \to \mathbb{CP}^n$ by

$$\mathcal{O}(-k) = \mathcal{O}(-1)^{\otimes k} = \underbrace{\mathcal{O}(-1) \otimes \cdots \otimes \mathcal{O}(-1)}_{k \text{ times}}.$$

There is an embedding of $\mathcal{O}(-k)$ as a sub-bundle of a trivial bundle that generalises Eq. (1.1). More precisely, $\mathcal{O}(-k)$ can be identified with a sub-bundle of $\mathbb{CP}^n \times (\mathbb{C}^{n+1})^{\otimes k} \to \mathbb{CP}^n$ as follows:

$$\mathcal{O}(-k) = \{\, (\ell, x) \in \mathbb{CP}^n \times (\mathbb{C}^{n+1})^{\otimes k} : x \in \ell^{\otimes k} \,\}.$$ △

Proposition 1.50 ([21, Example 2.2.4.i]). *Let $E \to M$, $F \to M$ be holomorphic vector bundles over the manifold M. Then there exists the* direct sum bundle *$E \oplus F \to M$, with fibres $(E \oplus F)_p = E_p \oplus F_p$, for each $p \in M$.*

Proposition 1.51 ([21, Example 2.2.4.vi]). *Let $E \to M$ be a holomorphic vector bundle of rank $k+1$ over the manifold M. Then there exists the* projective bundle associated to *E, or the* projectivisation of *E, which is a holomorphic $\mathbb{CP}^k$-bundle over M denoted by $\mathbb{P}(E) \to M$, with fibres $(\mathbb{P}(E))_p = \mathbb{P}(E_p)$, for each $p \in M$.*

1.6 Principal bundles

It will also be useful to consider principal G-bundles, where G is a Lie group. Essentially, these are fibre bundles together with a group whose fibres are copies of G, but without having a preferred choice of identity element.

In this section, we will be working in the smooth category, as opposed to the holomorphic category as in the previous section. For that reason, we begin with the definition of a fibre bundle in this setting.

Definition 1.52. Let M and F be smooth manifolds. A *smooth fibre bundle over M with fibre F*, or simply an *F-bundle over M*, is a smooth manifold E together with a surjective smooth map $\pi\colon E \to M$ such that, for each $p \in M$, there exists a neighbourhood U of p in M and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd},$$

where $\pi_1\colon U \times F \to U$ denotes the projection onto the first factor. The map Φ is called a *local trivialization of E over U*.

Example 1.53. The simplest example of a smooth fibre bundle is the *trivial bundle $M \times F$*, with $\pi\colon M \times F \to M$ the projection onto the first factor.

We say that a fibre bundle $\pi\colon E \to M$ is *trivializable* if there exists a local trivialization $\Phi\colon E \to M \times F$ defined over the whole base. △

Definition 1.54. Let G be a Lie group. A *(smooth) principal G-bundle* is a smooth fibre bundle $\pi\colon P \to M$ with fibre G, together with a smooth free right action $P \times G \to P$, such that

- the action of G preserves the fibers of P, i.e. $\pi(p \cdot g) = \pi(p)$ for all $p \in P$ and $g \in G$;
- G acts transitively on each fiber of P;

- the local trivializations $\Phi\colon \pi^{-1}(U) \to U \times G$ can be chosen to be G-equivariant.

Example 1.55. Again, the simplest example of a principal G-bundle is the *trivial bundle* $M \times G$, with $\pi\colon M \times G \to M$ the projection onto the first factor. Here, G acts on $M \times G$ by multiplication on the right in the second factor.

We say that a principal bundle $\pi\colon P \to M$ is *trivializable* if there exists a local trivialization $\Phi\colon P \to M \times G$ defined over the whole base. △

Example 1.56. Let (M, ω, T, μ) be a toric symplectic manifold with moment polytope $\Delta = \mu(M)$. We claim that the restriction

$$\mu|_{\mu^{-1}(\Delta^\circ)}\colon \mu^{-1}(\Delta^\circ) \to \Delta^\circ$$

of the moment map is a principal T-bundle over the interior Δ° of the moment polytope.

In fact, most of this follows from Lemma 1.31:

- T acts freely on $\mu^{-1}(\Delta^\circ)$ (since the isotropy groups of these points are trivial);
- the fibres of μ are precisely the orbits of the T-action on M, thus T preserves them and acts transitively on them;
- the fibres over Δ° are tori of the same dimension as T, and thus they are diffeomorphic to T by Theorem 1.11.

It remains to show the existence of equivariant local trivializations. For this, it is enough to apply Lemma 1.38 with $k = n$. △

We finish this section by stating a very important result about principal bundles.

Theorem 1.57. *Let G be a Lie group and M a contractible smooth manifold. Then, any principal G-bundle over M is trivializable.*

Proof. This follows from [25, Proposition A.8], since the identity map on M is homotopic to a constant map. Note that this reference treats principal bundles in the topological setting. However, in light of Remark A.1 in the same reference, the result should translate to the smooth setting. □

Since convex subsets of a finite-dimensional vector space are contractible, this yields the following consequence.

Corollary 1.58. *Let (M, ω, T, μ) be a toric symplectic manifold. Then the principal T-bundle*

$$\mu|_{\mu^{-1}(\Delta^\circ)}\colon \mu^{-1}(\Delta^\circ) \to \Delta^\circ$$

of Example 1.56 is trivializable.

Chapter 2

Toric symplectic 4-manifolds

2.1 The 2-dimensional case

We begin with a brief look at the simpler case of toric symplectic manifolds of dimension 2. By Delzant's theorem, they are classified by their moment polytopes, which are line segments.

We assume that the manifolds have an action of the standard 1-torus $\mathbb{T}^1 = S^1$, and we work under the identifications $\mathfrak{t} \cong \mathfrak{t}^* \cong \mathbb{R}$ of Example 1.13. References to unimodularity should be understood with respect to the weight lattice $\mathbb{Z} \subset \mathbb{R}$.

We have already seen one example.

Example 2.1. We specialise Example 1.34 to the case $n = 1$. Consider $\mathbb{CP}^1$ with the Fubini–Study symplectic form ω_{FS}, and the action of S^1 on $\mathbb{CP}^1$ given by

$$t \cdot [z_0 : z_1] = [z_0 : tz_1].$$

This is an effective Hamiltonian action with moment map

$$\mu_0 \colon \mathbb{CP}^1 \to \mathbb{R}, \quad \mu_0([z_0 : z_1]) = -\frac{1}{2}\frac{|z_1|^2}{|z_0|^2 + |z_1|^2}.$$

Its moment polytope is the line segment $[-\frac{1}{2}, 0]$.

Of course, any translated version of μ_0 is also a moment map for this action, whose associated moment polytope is a translated image of the above line segment. For instance, if we choose instead

$$\widetilde{\mu}_0 \colon \mathbb{CP}^1 \to \mathbb{R}, \quad \widetilde{\mu}_0([z_0 : z_1]) = \frac{1}{2} - \frac{1}{2}\frac{|z_1|^2}{|z_0|^2 + |z_1|^2} = \frac{1}{2}\frac{|z_0|^2}{|z_0|^2 + |z_1|^2},$$

the moment polytope is the line segment $[0, \frac{1}{2}]$.

In this way, we can exhibit any line segment in $\mathbb{R}$ of length $\frac{1}{2}$ as a moment polytope of a toric symplectic manifold. △

Example 2.2. Alternatively, we can consider $\mathbb{CP}^1$ with a scaled symplectic form $\lambda\omega_{\mathrm{FS}}$, for some non-zero real number λ. Geometrically, this corresponds to a copy of $\mathbb{CP}^1$ with different surface area and/or orientation. The same action of S^1 is still Hamiltonian with respect to this new symplectic form, but the moment map, and hence also the moment polytope, are scaled by λ.

For instance, this action of S^1 on $(\mathbb{CP}^1, -2\omega_{\mathrm{FS}})$ has a moment map given by

$$-2\,\mu_0([z_0 : z_1]) = \frac{|z_1|^2}{|z_0|^2 + |z_1|^2},$$

and the associated moment polytope is the line segment $[0,1]$. △

In the examples above we have identified, for any line segment $I \subset \mathbb{R}$, a toric symplectic 2-manifold whose moment polytope is I. By Delzant's theorem (Theorem 1.32), we have identified all toric symplectic 2-manifolds, up to isomorphism.

Proposition 2.3. *Up to the action of* $\mathrm{AGL}(1,\mathbb{Z})$*, any unimodular polytope in* $\mathbb{R}$ *is equivalent to exactly one interval of the form* $[0,L]$*, with* $L > 0$*.*

Equivalently, any toric symplectic manifold (M, ω, S^1, μ) *of dimension* 2 *is weakly isomorphic to* $(\mathbb{CP}^1, \lambda\omega_{FS}, S^1, \lambda\mu_0)$*, for some unique* $\lambda > 0$*.*

Proof. An element of $\mathrm{AGL}(1,\mathbb{Z})$ is an affine linear map $L\colon \mathbb{R} \to \mathbb{R}$ of the form $L(x) = \pm x + c$, for some $c \in \mathbb{R}$.

Thus, it is clear that, up to the action of this group, line segments are classified by their length. □

In particular, we see that any toric symplectic manifold of dimension 2 is diffeomorphic to the sphere S^2.

Example 2.4. For the sake of concreteness, we may identify $\mathbb{CP}^1$ and S^2 via stereographic projection from the south pole,

$$\sigma\colon S^2 \to \mathbb{CP}^1,\ (x_1, x_2, x_3) \mapsto \begin{cases} [1 + x_3 : x_1 + ix_2] & \text{if } (x_1, x_2, x_3) \neq (0,0,-1) \\ [0:1] & \text{if } (x_1, x_2, x_3) = (0,0,-1) \end{cases},$$

$$\sigma^{-1}\colon \mathbb{CP}^1 \to S^2, \quad [z_0 : z_1] \mapsto \left(\frac{\mathrm{Re}(\overline{z_0} z_1)}{|z_0|^2 + |z_1|^2}, \frac{\mathrm{Im}(\overline{z_0} z_1)}{|z_0|^2 + |z_1|^2}, \frac{|z_0|^2 - |z_1|^2}{|z_0|^2 + |z_1|^2} \right).$$

It can be seen that this map satisfies $\sigma^*\omega_{\mathrm{FS}} = \frac{1}{4}\omega_{\mathrm{std}}$, where ω_{std} denotes the standard area form on S^2, seen as the unit sphere in $\mathbb{R}^3$.

Thus, let us consider the toric symplectic manifold $(\mathbb{CP}^1, 4\omega_{FS}, S^1, 4\mu + 1)$, and transfer this structure to S^2 through the above identification.

It turns out that the resulting action of S^1 on S^2 is by rotations around the vertical axis, and the moment map is the height function (see Fig. 2.1):

$$h := \sigma \circ (4\mu + 1) \colon S^2 \to \mathbb{R}, \quad (x_1, x_2, x_3) \mapsto x_3. \qquad \triangle$$

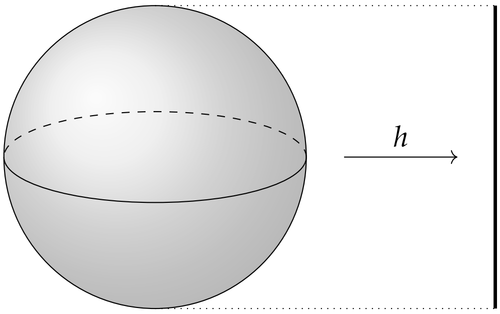

Figure 2.1: The moment map of the Hamiltonian S^1-action on S^2 by rotations around the vertical axis, as in Example 2.4.

2.2 General facts regarding unimodular polygons

Having fully understood the toric symplectic manifolds in dimension 2, we move to the next simplest case of dimension 4. By Delzant's theorem, such manifolds are classified by 2-dimensional unimodular polytopes, i.e. unimodular *polygons*.

For the rest of this chapter, we assume that the manifolds have an action of the standard 2-torus $\mathbb{T}^2 = S^1 \times S^1$, and we work under the identifications $\mathfrak{t} \cong \mathfrak{t}^* \cong \mathbb{R}^2$ of Example 1.13. References to unimodularity should be understood with respect to the weight lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. By *(convex) polygon*, we mean a convex polytope of dimension 2.

We begin by establishing a useful piece of terminology, to which we will refer repeatedly later.

Definition 2.5. Let $P \subset \mathbb{R}^2$ be a unimodular polygon. We say that P is in *standard position* if it has a vertex at the origin, and the two edges incident at that vertex lie along the positive coordinate axes.

In particular, by convexity, a unimodular polygon in standard position is contained in the principal quadrant $(\mathbb{R}_0^+)^2$. See Fig. 2.2 for an example.

Lemma 2.6. *Let $P \subset \mathbb{R}^2$ be a unimodular polygon and $p \in P$ a vertex of P. Then, there exists a transformation $T \in \mathrm{AGL}(2, \mathbb{Z})$ such that $T(p) = 0$ and $T(P)$ is a unimodular polygon in standard position.*

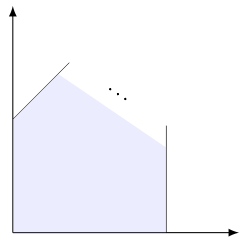

Figure 2.2: A unimodular polygon in standard position.

Proof. If necessary, begin by applying a translation sending p to the origin.

Since P is unimodular, there are now two edges meeting at the origin, lying along rays generated by vectors u_1, u_2, for some basis (u_1, u_2) of $\mathbb{Z}^2$.

To transform P into standard position, it is enough to apply a change of basis to the lattice $\mathbb{Z}^2$, which corresponds to a transformation in $\mathrm{GL}(2, \mathbb{Z})$. $\square$

Before proceeding the study of unimodular polygons, we stop to explicitly formulate some basic fact about general convex polygons which will be important. They will allow us to rigorously define what it means to list the edges of a convex polygon in *anticlockwise* order.

Lemma 2.7. *Let $P \subset \mathbb{R}^2$ be a convex polygon and $p \in P$ a vertex. Then, p belongs to exactly two edges of P.*

Proof. By [17, Proposition 2.6.4], p is the intersection of two edges of P. The polygon P must be contained in the convex cone C generated by these edges. Moreover, if a straight line L through p contains P in one of the closed half-planes it defines, then the whole cone C must be contained in this half-plane. Thus, the intersection $L \cap P$ is either the vertex p, or one of the two edges already in consideration; there are no other edges containing p. $\square$

Proposition 2.8. *Let $P \subset \mathbb{R}^2$ be a convex polygon. Let E be an edge of P and $v \in \mathbb{R}^2$ an inward-pointing normal vector to E.*

The edge E has two vertices. By Lemma 2.7, it is adjacent to exactly two other edges, say F_1 and F_2. Let $u_1, u_2 \in \mathbb{R}^2$ be inward-pointing normal vectors to F_1 and F_2, respectively.

Then the vectors u_1, u_2 are not parallel to v, and the ordered bases (v, u_1) and (v, u_2) of $\mathbb{R}^2$ have opposite orientations.

In the above conditions, if (v, u_1) is negatively oriented and (v, u_2) is positively oriented, we say that F_1 is the *previous* edge (before E), and F_2 is the *next* edge (after E). This corresponds to ordering the edges in anticlockwise order. We also say that $E \cap F_1$ is the *first* vertex of E, and $E \cap F_2$ is the *second* vertex of E. See Fig. 2.3 for an illustration.

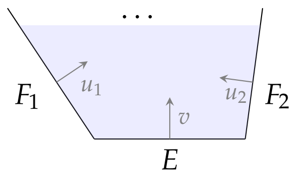

Figure 2.3: An illustration of Proposition 2.8.

Proof. First of all, if v and u_1 were parallel, then the edges E and F_1 would be parallel. Since they intersect at a vertex, this is impossible.

Write $v = (v_1, v_2)$, $u_1 = (u_{11}, u_{12})$, $u_2 = (u_{21}, u_{22})$. Note that the vector $(v_2, -v_1)$ is parallel to the edge E. This choice of vector allows us to distinguish the two vertices of E. Let $(x_0, y_0) \in \mathbb{R}^2$ be the vertex of E such that

$$E = \{ (x_0, y_0) + t(v_2, -v_1) : t \in [0, T] \},$$

for some $T > 0$.

This vertex is also a vertex of one of the F_i; without loss of generality, let us say us that it is a vertex of F_1. Choose a vector $w = (w_1, w_2) \in \mathbb{R}^2$ such that

$$F_1 = \{ (x_0, y_0) + t(w_1, w_2) : t \in [0, T'] \},$$

for some $T' > 0$.

By convexity of P, we have that $\varepsilon(v_2 + w_1, -v_1 + w_2) \in P$, for some small enough $\varepsilon > 0$. In particular, this vector must be in the open half-plane determined by an inward normal to the edge F_1:

$$(u_{11}, u_{12}) \cdot \varepsilon(v_2 + w_1, -v_1 + w_2) = \varepsilon(u_{11}v_2 - u_{12}v_1) > 0,$$

where we used the fact that u_1 and w are orthogonal vectors. This shows that

$$\det(v, u_1) = v_1 u_{12} - v_2 u_{11} < 0,$$

and hence the ordered basis (v, u_1) of $\mathbb{R}^2$ is negatively oriented.

An analogous argument at the other vertex of E shows that the ordered basis (v, u_2) of $\mathbb{R}^2$ is positively oriented, establishing the claim. □

Proposition 2.9. *Let $P \subset \mathbb{R}^2$ be a convex polygon. Make a list of edges of P in the following way:*

- *choose an edge E_1 of P to begin with;*
- *at each step, given a list $E_1, \ldots, E_k$, add at the end the next edge, in the sense of Proposition 2.8;*
- *stop when you find that the next edge is again E_1.*

Then, this process terminates, and results in a list of every edge of P, without repetition. Such a list is an ordering of the edges of P in anticlockwise order.

Proof. The polygon P finitely many edges. Hence, if this process were to continue indefinitely, we would necessarily have repeated entries in the list. Let E_i be the first edge to appear a second time in the list.

If $E_i \neq E_1$, then the previous edge E_{i-1} and the next edge E_{i+1} have also already appeared in the list. These are the only two other edges with intersect E_i. Hence, E_i can only appear in the list if it is immediately preceded by E_{i-1}. This contradicts the minimality hypothesis on E_i.

We conclude that $E_i = E_1$ is the first edge that would appear repeated on the list, if the described process were allowed to continue indefinitely. This shows that this process terminates, and that the result is a list without repetitions.

It remains to show that the resulting list contains every edge of P. Equivalently, we must show that every two vertices of P are connected by a finite sequence of edges. Our argument is based on the proof of [17, Proposition 11.3.2].

Let p be a vertex of P, and $L \subset \mathbb{R}^2$ a supporting line for P (in the sense of Definition 1.19) such that $L \cap P = \{p\}$. For the sake of contradiction, suppose that not all vertices are connected to p by a finite sequence of edges. Over all such vertices, let q be one with minimum distance to L. Note that this distance is necessarily positive.

The vertex q is adjacent to two other vertices. If both of them were at least as distant from L as q is, then the convex cone generated by the edges meeting at q, which contains the polygon P, would not intersect the line L, which is absurd. Hence, one of the vertices adjacent to q must be closer to L than q is.

By our minimality hypothesis on q, this latter vertex must be connected to p by a finite sequence of edges. However, this sequence can be enlarged to connect p to q, which is a contradiction. □

We give now a description of unimodular polygons, as a specialisation of Proposition 1.26. It allows us to decide whether a given convex polygon is unimodular, given inward normal vectors to its edges.

Proposition 2.10. *Let $P \subset \mathbb{R}^2$ be a convex polygon with d edges.*

If P is unimodular, then each of its edges has a unique inward normal vector which is a primitive element of $\mathbb{Z}^2$. Moreover, if $v_1, \dots, v_d \in \mathbb{Z}^2$ are the primitive inward normal vectors to the edges of P, in anticlockwise order, then they must satisfy

(∗) $\det(v_{i-1}, v_i) = 1$, *for all $i = 1, \dots, d$,* *where we take $v_0 = v_d$.*

Conversely, suppose that $v_1, \dots, v_d \in \mathbb{Z}^2$ are primitive inward normal vectors to the edges of P, in anticlockwise order. If condition (*) holds, then P is unimodular.

Proof. Suppose first that P is unimodular. This means that, at each vertex of P, the two edges meeting there are generated by a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^2$.

Since a quarter-turn rotation is a lattice automorphism of $\mathbb{Z}^2$, each edge of P has a normal vector in $\mathbb{Z}^2$. Dividing by the greatest common divisor of its (integer) components, we obtain a primitive normal vector, which is uniquely determined up to sign: one inward-pointing and one outward-pointing.

Let p be a vertex of P, at which two edges E_1 and E_2 meet. By unimodularity, there exist primitive vectors $u_1, u_2 \in \mathbb{Z}^2$ which generate these edges and form a basis of $\mathbb{Z}^2$, i.e. $\det(u_1, u_2) = \pm 1$. If $\tilde{u}_1, \tilde{u}_2$ denote the images of u_1, u_2 under a quarter-turn rotation, they are primitive (not necessarily inward) normal vectors to E_1 and E_2, respectively, and still satisfy $\det(\tilde{u}_1, \tilde{u}_2) = \pm 1$.

If v_1, v_2 are the primitive inward normal vectors to E_1 and E_2, respectively, then

$$\det(v_1, v_2) = \pm \det(\tilde{u}_1, \tilde{u}_2) \in \{-1, 1\}.$$

However, by Proposition 2.8, we see that this determinant must be positive, if we consider the edges in anticlockwise order.

Conversely, we can reverse the argument: if the normals to a consecutive pair of edges are a basis of $\mathbb{Z}^2$, then the edges are themselves generated by a basis of $\mathbb{Z}^2$ obtained from the first one, up to sign, by a quarter-turn rotation. □

2.3 Unimodular triangles

In this section, we begin by looking at the simplest kind of polygons: triangles.

Example 2.11. We specialise Example 1.34 to the case $n = 2$. Consider $\mathbb{CP}^2$ with the Fubini–Study symplectic form ω_{FS}, and the action of $\mathbb{T}^2$ on $\mathbb{CP}^2$ given by

$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2].$$

This is an effective Hamiltonian action with moment map

$$\mu\colon \mathbb{CP}^2 \to \mathbb{R}^2$$
$$\mu([z_0 : z_1 : z_2]) = -\frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right).$$

The associated moment polygon is an isosceles right triangle with vertices $(0,0)$, $\left(-\frac{1}{2}, 0\right)$ and $\left(0, -\frac{1}{2}\right)$, as shown in Fig. 2.4.

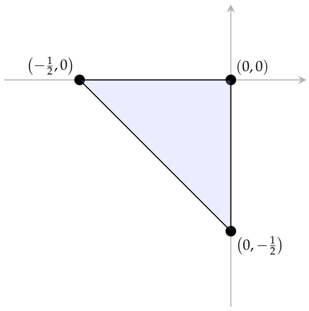

Figure 2.4: The triangle of Example 2.11, which is the moment polygon of a $\mathbb{CP}^2$.

Following Remarks 1.35, 1.36 and 1.37, one can modify this example to obtain as moment polytope any image of this triangle under scaling and the action of $\mathrm{AGL}(2,\mathbb{Z})$. △

It turns out that the previous example already covers all possible unimodular triangles in $\mathbb{R}^2$.

Proposition 2.12. *Up to the action of* $\mathrm{AGL}(2,\mathbb{Z})$*, any unimodular triangle in* $\mathbb{R}^2$ *is equivalent to the isosceles right triangle with vertices* $(0,0)$*,* $(\lambda,0)$ *and* $(0,\lambda)$*, for a unique value of* $\lambda > 0$*.*

Proof. By Lemma 2.6, we may assume, up to the action of $\mathrm{AGL}(2,\mathbb{Z})$, that our triangle is in standard position. Hence, it is enough to consider triangles with vertices at $(0,0)$, $(a,0)$ and $(0,b)$, for some positive real numbers a,b.

Unimodularity implies that the slope b/a must be rational. Let m,n be coprime positive integers such that $b/a = m/n$.

Then, the primitive inward normals vectors are $(1,0)$, $(0,1)$ and $(-m,-n)$.

According to Proposition 2.10, these vectors must satisfy

$$\det\begin{pmatrix} 0 & -m \\ 1 & -n \end{pmatrix} = \det\begin{pmatrix} -m & 1 \\ -n & 0 \end{pmatrix} = 1.$$

Therefore, we have $m = n = 1$ and $a = b$, which establishes the proposition.

Uniqueness holds because the transformations in $\mathrm{AGL}(2,\mathbb{Z})$ have determinant 1 or -1, and thus they preserve unsigned area. □

2.4 Unimodular quadrilaterals

After exploring the case of unimodular triangles, the natural next step is to consider quadrilaterals. We start with some examples.

Example 2.13. Recall Example 2.1. We can consider the symplectic form $p_1^*\omega_{\mathrm{FS}} + p_2^*\omega_{\mathrm{FS}}$ on the product $\mathbb{CP}^1 \times \mathbb{CP}^1$, where $p_1, p_2 \colon \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ denote the projections to each factor.

Consider the action of $\mathbb{T}^2$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by

$$(t_1, t_2) \cdot ([z_0 : z_1], [w_0 : w_1]) = ([z_0 : t_1 z_1], [w_0 : t_2 w_1]).$$

It is straightforward to check that this is an effective Hamiltonian action with moment map

$$\mu \colon \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{R}^2$$
$$\mu([z_0 : z_1], [w_0 : w_1]) = -\frac{1}{2}\left(\frac{|z_0|^2}{|z_0|^2 + |z_1|^2}, \frac{|w_0|^2}{|w_0|^2 + |w_1|^2}\right).$$

The moment polytope is the square $\left[-\frac{1}{2}, 0\right] \times \left[-\frac{1}{2}, 0\right] \subset \mathbb{R}^2$, which is depicted in Fig. 2.5. △

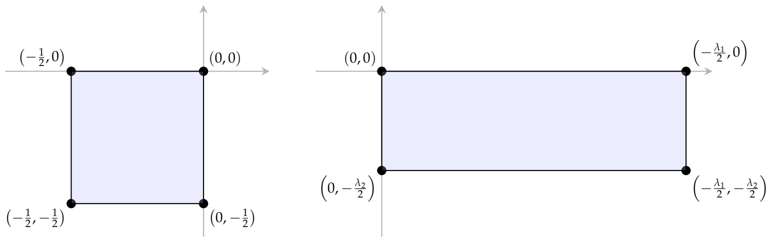

Figure 2.5: The square of Example 2.13 and one rectangle as in Example 2.14. Each of them is the moment polygon of a $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Example 2.14. More generally, for any constants $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, one can consider the symplectic form

$$\omega = \lambda_1 p_1^*\omega_{\mathrm{FS}} + \lambda_2 p_2^*\omega_{\mathrm{FS}}$$

on $\mathbb{CP}^1 \times \mathbb{CP}^1$ (cf. [10, Section 3.4]).

Similarly, the above $\mathbb{T}^2$-action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is effective and Hamiltonian with respect to this symplectic form, with moment map

$$\mu \colon \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{R}^2$$
$$\mu([z_0 : z_1], [w_0 : w_1]) = -\frac{1}{2}\left(\frac{\lambda_1|z_0|^2}{|z_0|^2 + |z_1|^2}, \frac{\lambda_2|w_0|^2}{|w_0|^2 + |w_1|^2}\right).$$

The moment polytope is a rectangle. For instance, if $\lambda_1, \lambda_2 > 0$, it is the rectangle $[-\lambda_1/2, 0] \times [-\lambda_2/2, 0] \subset \mathbb{R}^2$. Another possibility is depicted in Fig. 2.5. △

One might hope that every unimodular quadrilateral in $\mathbb{R}^2$ is $\mathrm{AGL}(2,\mathbb{Z})$-equivalent to a rectangle. However, this is not true: one must consider a more general class of trapezoids, as in the following result.

Proposition 2.15. *Up to the action of* $\mathrm{AGL}(2,\mathbb{Z})$*, any unimodular quadrilateral in* $\mathbb{R}^2$ *is equivalent to a trapezoid with vertices* $(0,0)$*,* $(0,\beta)$*,* (α,β) *and* $(\alpha+k\beta,0)$*, for some positive real numbers* α,β *and some non-negative integer* k*.*

Proof. Let $P \subset \mathbb{R}^2$ be a unimodular quadrilateral. Again by Lemma 2.6, we may assume, up to the action of $\mathrm{AGL}(2,\mathbb{Z})$, that P is in standard position.

Following Proposition 2.10, let $(1,0)$, $(0,1)$, (a,b), (c,d) be the primitive inward normal vectors to the edges of the unimodular quadrilateral, in anticlockwise order. These must satisfy

$$\det\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} = \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \det\begin{pmatrix} c & 1 \\ d & 0 \end{pmatrix} = 1.$$

The first and last determinants imply that $a = d = -1$, and then the middle one simplifies to

$$1 - bc = 1 \iff b = 0 \text{ or } c = 0.$$

Geometrically, this shows that P has at least two consecutive right angles. Note that up to a translation and a quarter-turn rotation (which are elements of $\mathrm{AGL}(2,\mathbb{Z})$), the two alternatives are equivalent. Thus, we may assume without loss of generality that $c = 0$, i.e. that we have right angles at the vertices which lie on the vertical coordinate axis.

Thus far, we reduced to the case where the primitive inward normal vectors to the edges of P are $(1,0)$, $(0,1)$, $(-1,k)$ and $(0,-1)$, for some integer n. In this case, the vertices of P are $(0,0)$, $(0,\beta)$, (α,β) and $(\alpha-n\beta,0)$, for some positive real numbers α,β.

Finally, up to a translation and a vertical reflection (which are elements of $\mathrm{AGL}(2,\mathbb{Z})$), we may assume that $n \leq 0$. Geometrically, this means that the lower horizontal edge is not shorter than the upper one. Taking $k = -n$, we arrive at the desired result. □

The following example introduces the *Hirzebruch surfaces*, which are a family of toric symplectic manifolds of dimension 4 whose moment polygons are the above-described trapezoids. For more details, check [4, Section IV.5.a, Exercise IV.4, Exercise IV.17] and [21, Exercise 2.4.5].

Example 2.16. Recall the material in Section 1.5. For each non-negative integer k, we define the *k-th Hirzebruch surface* $\mathcal{H}_k$ to be the total space of the projective

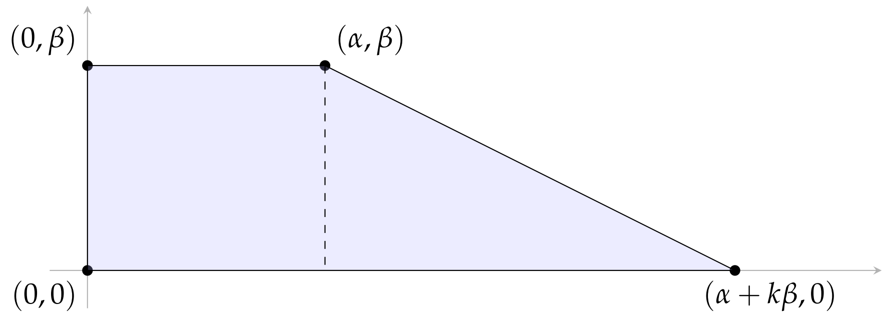

Figure 2.6: A trapezoid among the family described in Proposition 2.15. This example has $\alpha = 3.14$, $\beta = 2.71$ and $k = 2$.

bundle[1]

$$\mathcal{H}_k = \mathbb{P}(\mathcal{O}(-k) \oplus \mathcal{O}) \to \mathbb{CP}^1.$$

These are complex surfaces, and thus real 4-manifolds.

In Example 1.49, we saw that $\mathcal{O}(-k)$ can be identified with a holomorphic sub-bundle of the trivial bundle $\mathbb{CP}^1 \times (\mathbb{C}^2)^{\otimes k}$. In fact, that embedding extends to

$$i_k \colon \mathcal{H}_k \hookrightarrow \mathbb{CP}^1 \times \mathbb{P}((\mathbb{C}^2)^{\otimes k} \oplus \mathbb{C}) \cong \mathbb{CP}^1 \times \mathbb{CP}^{2^k},$$

which exhibits the Hirzebruch surface $\mathcal{H}_k$ as a complex submanifold of a product of complex projective spaces. Hence, we get symplectic forms on $\mathcal{H}_k$ by pulling back any Kähler form on $\mathbb{CP}^1 \times \mathbb{CP}^{2^k}$.

Alternatively, it can be seen that, as a holomorphic $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$, $\mathcal{H}_k$ is isomorphic to

$$\{ ([a:b],[x:y:z]) \in \mathbb{CP}^1 \times \mathbb{CP}^2 : a^k y = b^k x \} \subset \mathbb{CP}^1 \times \mathbb{CP}^2.$$

From now on, we consider this presentation of $\mathcal{H}_k$, since it is more convenient for our purposes.

Let $p_1 \colon \mathbb{CP}^1 \times \mathbb{CP}^2 \to \mathbb{CP}^1$ and $p_2 \colon \mathbb{CP}^1 \times \mathbb{CP}^2 \to \mathbb{CP}^2$ be the natural projections to each factor. We consider on $\mathbb{CP}^1 \times \mathbb{CP}^2$ the Kähler form

$$\omega = \alpha\, p_1^* \omega_{\mathbb{CP}^1} + \beta\, p_2^* \omega_{\mathbb{CP}^2},$$

for some constants $\alpha, \beta > 0$. If $i \colon \mathcal{H}_k \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^2$ denotes the above embedding, then $i^*\omega$ is a symplectic form on $\mathcal{H}_k$.

[1] We could also have defined $\mathcal{H}_k = \mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O})$, since these two projective bundles are actually isomorphic as complex manifolds. This is a consequence of [37, Chapter 3, Exercise 2].

We consider an action of the standard 2-torus $\mathbb{T}^2$ on $\mathcal{H}_k$, defined by

$$(u,v)\cdot([a:b],[x:y:z]) = ([ua:b],[u^k x:y:vz]).$$

The action is effective, since for instance the element $([1:1],[1:1:1]) \in \mathcal{H}_k$ has trivial isotropy.

We can verify that the action is Hamiltonian. Indeed, the fundamental vector field associated to the first coordinate of $\mathbb{T}^2$ is

$$X = \frac{d}{dt}([e^{it}a:b],[e^{ikt}x:y:z])\Big|_{t=0} = (X_1, X_2).$$

Then we can compute that

$$\begin{aligned}
\omega(X,\cdot) &= \alpha\,\omega_{\mathbb{CP}^1}(X_1,(p_1)_*\cdot) + \beta\,\omega_{\mathbb{CP}^2}(X_2,(p_2)_*\cdot)\\
&= \alpha\, d\left(-\frac{1}{2}\frac{|a|^2}{|a|^2+|b|^2}\right) + \beta\, d\left(-\frac{k}{2}\frac{|x|^2}{|x|^2+|y|^2+|z|^2}\right)\\
&= d\left(-\frac{1}{2}\left(\frac{\alpha|a|^2}{|a|^2+|b|^2} + \frac{k\beta|x|^2}{|x|^2+|y|^2+|z|^2}\right)\right).
\end{aligned}$$

Here, we used that we already know the moment map of the usual toric actions on $\mathbb{CP}^1$ and $\mathbb{CP}^2$.

For the second coordinate, the computations are analogous, and slightly simpler. We get that this action is Hamiltonian with moment map

$$\mu\colon \mathcal{H}_k \to \mathbb{R}^2,$$
$$\mu([a:b],[x:y:z]) = -\frac{1}{2}\left(\frac{\alpha|a|^2}{|a|^2+|b|^2} + \frac{k\beta|x|^2}{|x|^2+|y|^2+|z|^2}, \frac{\beta|z|^2}{|x|^2+|y|^2+|z|^2}\right).$$

It is not easy at first sight to identify the image of this map. However, recall from Lemma 1.31 that the vertices of the moment polygon are the images of the fixed points of the toric action.

By looking at the explicit expression of this action, we see that there are exactly four fixed points:

$$([0:1],[0:0:1]),\ ([0:1],[0:1:0]),\ ([1:0],[1:0:0]),\ ([1:0],[0:0:1]).$$

Hence, we conclude that the image of the map -2μ is the convex polygon with vertices $(0,\beta)$, $(0,0)$, $(\alpha+k\beta,0)$ and (α,β), corresponding to the family of trapezoids described in Proposition 2.15. $\triangle$

We remark that the Hirzebruch surfaces are an instance of the more general notion of a *Bott tower*, which are iterated $\mathbb{CP}^1$-bundles. For more on this, see e.g. [16].

2.5 Chopping corners

We have explored the simplest unimodular polygons: triangles and quadrilaterals. We will now introduce the operation of *corner-chopping*, which allows us to build more complex unimodular polygons out of a given one.

Definition 2.17. Let $P \subset \mathbb{R}^2$ be a unimodular polygon and $p \in P$ a vertex of P. Let $u_1, u_2 \in \mathbb{Z}^2$ be the primitive vectors along the edges meeting at p. Let $\varepsilon > 0$ be small enough, such that $p + \varepsilon u_i$ still lies in the relative interior of the corresponding edge, for $i = 1, 2$.

We define the *ε-corner-chopping* of P at p to be the convex polygon $\tilde{P}$ with the same vertices as P, except that the vertex p is replaced by $p + \varepsilon u_1$ and $p + \varepsilon u_2$ (cf. Fig. 2.7).

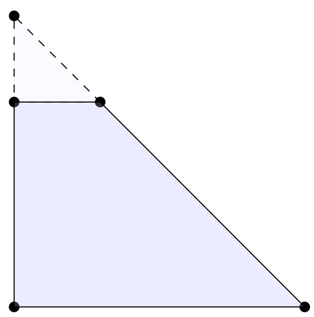

Figure 2.7: An illustration of the corner-chopping procedure.

Proposition 2.18. *Let $P \subset \mathbb{R}^2$ be a unimodular polygon, and consider some ε-corner-chopping of P at a vertex p. If v_1, v_2 are the primitive inward normal vectors to the edges meeting at p, then the primitive inward normal vector to the new edge is $v_1 + v_2$.*

In particular, any corner-chopping of P is again unimodular.

Proof. Let E_1, E_2 be the two edges meeting at p, in anticlockwise order, and let $u_1, u_2 \in \mathbb{Z}^2$ be primitive vectors which generate these edges and form a basis of $\mathbb{Z}^2$.

Denote by $J : \mathbb{R}^2 \to \mathbb{R}^2$ a quarter-turn rotation in the anticlockwise direction. Then, one can see, as in the proof of Proposition 2.8, that $v_1 = Ju_1$ and $v_2 = -Ju_2$. It follows that

$$v_1 + v_2 = J(u_1 - u_2)$$

is the primitive inward normal vector to the new edge created by the chopping. □

Note that the definition of corner-chopping generalises naturally to higher-dimensional polytopes, even though we consider it here only in the two-dimensional case.

Remark 2.19. There is a geometric construction on toric symplectic manifolds which corresponds to corner-chopping on polygons. It is a particular case of the general procedure known as *blowing-up*. Essentially, it involves removing a small open ball from the manifold, and collapsing its boundary according to the Hopf fibration $S^{2n-1} \to \mathbb{CP}^{n-1}$. We will not cover in detail, but we refer to [23, Remark 2.12], [22, Sections 2 and 3] and [28, Section 7.1] for more details.

The precise statement is as follows: if M is a toric symplectic manifold with moment polytope Δ, and $\widetilde{\Delta}$ is a unimodular polytope that can be obtained from Δ by a corner-chopping, then $\widetilde{\Delta}$ is the moment polytope of a toric symplectic manifold obtained from M by an equivariant symplectic blow-up at a T-fixed point.

Notably, if (M, ω) is a 4-dimensional symplectic manifold and $(\widetilde{M}, \widetilde{\omega})$ is a blow-up of the former, then we have a diffeomorphism $\widetilde{M} \cong M \# \overline{\mathbb{CP}^2}$ with the connected sum of M and a copy of $\mathbb{CP}^2$ with reversed orientation.

Example 2.20. As illustrated in Fig. 2.7, the corner-chopping of a unimodular triangle is a unimodular quadrilateral with $k = 1$, in the notation of Proposition 2.15.

In particular, in light of the previous remark, the blow-up of $\mathbb{CP}^2$ at a $\mathbb{T}^2$-fixed point is isomorphic to a first Hirzebruch surface $\mathcal{H}_1$, as a toric symplectic manifold. △

2.6 Classification of unimodular polygons

We are now ready to state and prove a classification result for unimodular polygons. This result is due to Oda [33, Theorem 8.2], and was originally considered in the context of algebraic toric varieties. For other later accounts of this proof, check [14, Section 2.5 and Notes to Chapter 2] and [4, Theorem VII.4.1].

Theorem 2.21. *Up to the action of* $\mathrm{AGL}(2, \mathbb{Z})$, *all unimodular polygons in* $\mathbb{R}^2$ *can be obtained by a finite sequence of corner-choppings from one of the following polygons:*

- *the isosceles right triangle with vertices* $(0, 0)$, $(\lambda, 0)$ *and* $(0, \lambda)$, *for some positive real number* λ;
- *the trapezoid with vertices* $(0, 0)$, $(0, \beta)$, (α, β) *and* $(\alpha + n\beta, 0)$, *for some positive real numbers* α, β *and some non-negative integer* n.

Proof. Let $P \subset \mathbb{R}^2$ be a unimodular polygon, and let $v_1, \ldots, v_d$ be the sequence of primitive inward normal vectors to its edges, in anticlockwise order. The indices are to be interpreted modulo d, e.g. we consider $v_{d+1} = v_1$.

We begin with the following lemma (cf. Fig. 2.8).

Lemma 2.22. *Let $\{v, v'\}$ and $\{w, w'\}$ be $\mathbb{Z}$-bases of $\mathbb{Z}^2$. If w lies in the interior of the sector $\mathbb{R}_0^+ v + \mathbb{R}_0^+ v'$, and w' is in the sector $\mathbb{R}_0^+ v' + \mathbb{R}_0^+(-v)$, then it must be that $w' = v'$ or $w' = -v$.*

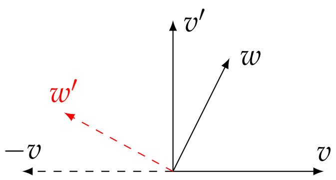

Figure 2.8: An impossible arrangement of bases of $\mathbb{Z}^2$, according to the conclusion of Lemma 2.22.

Proof of lemma. Since $\{v, v'\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, we can write $w = av + bv'$ and $w' = cv + dv'$, for some integers a, b, c, d. The assumptions on w and w' translate to $a > 0, b > 0, c \leq 0, d \geq 0$.

If $c \neq 0$ and $d \neq 0$, we see that

$$\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc \geq 2,$$

which contradicts the fact that $\{w, w'\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. Hence, we conclude that $c = 0$ or $d = 0$. □

Lemma 2.23. *If $d \geq 4$, there must be two opposite vectors in the sequence, i.e. $v_j = -v_i$ for some i, j. Equivalently, this means that P has at least one pair of parallel edges.*

Proof of lemma. Suppose, for the sake of contradiction, that this does not hold. Possibly after a cyclic relabelling of the edges, we may assume that, if we start at v_1 and list the normal vectors in anticlockwise order, more than half of them appear before $-v_1$ (note that, by assumption, $-v_1$ is not in the sequence).

Let v_j be the last of these vectors. Since $d \geq 4$, we have by the previous assumption that $j > 2$.

By Lemma 2.22 applied to $\{v_2, -v_1\}$ and $\{v_j, v_{j+1}\}$, we see that v_{j+1} must lie strictly between $-v_2$ and v_1. Moreover, since (v_j, v_{j+1}) is a positively oriented basis, the oriented angle from v_j to v_{j+1} must be less than π, and thus v_{j+1}

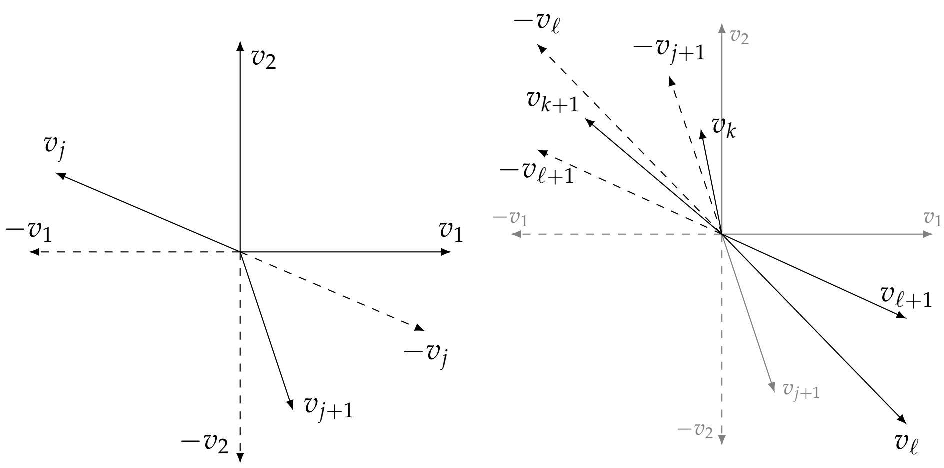

Figure 2.9: An illustration of the proof of Lemma 2.23.

must lie strictly between $-v_2$ and $-v_j$. This implies that $-v_{j+1}$ is strictly between some consecutive pair v_k and v_{k+1}, for some $k \in \{2, \dots, j-1\}$.

Now, there exists some index $\ell \in \{j+1, \dots, d\}$ such that $-v_\ell$ is strictly between v_k and v_{k+1}, but $-v_{\ell+1}$ is strictly between v_{k+1} and $-v_1$. Loosely speaking, if we start at v_{j+1} and continue through the sequence, v_ℓ is the "last" one such that $-v_\ell$ is "still" strictly between v_k and v_{k+1}. Applying Lemma 2.22 to the bases $\{v_k, v_{k+1}\}$ and $\{-v_\ell, -v_{\ell+1}\}$, we get a contradiction. □

Lemma 2.24. *For every index k, there is some integer a_k such that*

$$a_k v_k = v_{k-1} + v_{k+1}.$$

In fact, this integer a_k has a geometric interpretation: it is the Euler class of the normal bundle of the $\mathbb{CP}^1$ corresponding to the edge E_k [4, Section VII.4].

Proof of lemma. Consider a triple v_{k-1}, v_k, v_{k+1} of consecutive vectors. We can write

$$v_{k+1} = Av_{k-1} + Bv_k,$$

for some integers A, B. The pair (v_k, v_{k+1}) is a basis of $\mathbb{Z}^2$, so their coordinates in the basis (v_{k-1}, v_k) must satisfy the unimodularity condition

$$\det \begin{pmatrix} 0 & A \\ 1 & B \end{pmatrix} = 1 \iff A = -1.$$

Hence, we see that $Bv_k = v_{k-1} + v_{k+1}$. □

Lemma 2.25. *If $d \geq 5$, there exists some index j such that (v_{j-1}, v_{j+1}) is a positively oriented $\mathbb{Z}$-basis of $\mathbb{Z}^2$, and $v_j = v_{j-1} + v_{j+1}$.*

Proof of lemma. By Lemma 2.23, possibly after a cyclic relabelling of the edges, there is some index i such that $v_i = -v_1$. Since $d \geq 5$, we can moreover make this choice such that $i \geq 4$ (if needed, relabel the sequence starting at v_i).

According to Lemma 2.24, our goal is to show that there must exist some index j such that $a_j = 1$.

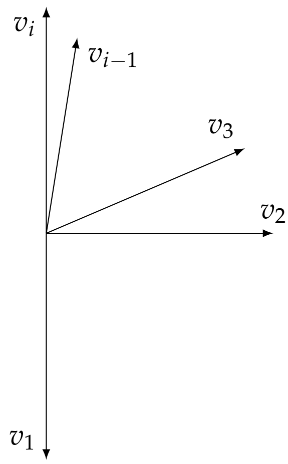

Figure 2.10: An illustration of the proof of Lemma 2.25.

Note that $(v_2, v_i = -v_1)$ is a positively oriented basis of $\mathbb{Z}^2$. Relative to it, all the vectors $v_3, \ldots, v_{i-1}$ are contained in the first quadrant. Hence, for each index k we can write

$$v_k = b_k v_2 + b'_k v_i,$$

for some integers b_k, b'_k, which are both positive when $k \in \{3, \ldots, i-1\}$. By Lemma 2.24, we also have

$$a_k v_k = v_{k-1} + v_{k+1},$$

and this integer a_k is positive when $k \in \{2, \ldots, i-1\}$ (since in this case the oriented angle from v_{k-1} to v_{k+1} is less than π).

Combining both expressions, we can see that

$$a_k b_k = b_{k-1} + b_{k+1}, \quad a_k b'_k = b'_{k-1} + b'_{k+1}.$$

We can also consider the quantity $c_k = b_k + b'_k$, which satisfies the analogous identity

$$a_k c_k = c_{k-1} + c_{k+1}.$$

Since $c_2 = c_i = 1$ and $c_3 \geq 2$, there must exist some $j \in \{2, \ldots, i-1\}$ such that $c_{j-1} \leq c_j$ and $c_{j+1} < c_j$. For such a j, we have from the above identity that

$$a_j c_j = c_{j-1} + c_{j+1} < 2c_j,$$

which implies that $a_j = 1$.

Since (v_{j-1}, v_j) is a positively oriented basis of $\mathbb{Z}^2$, it follows immediately that the same is true for $(v_{j-1}, v_{j+1} = v_j - v_{j-1})$. □

Since the pair (v_{j-1}, v_{j+1}) is positively oriented, this means that the lines spanned by the edges E_{j-1} and E_{j+1} of P intersect at a point separated from P by the line spanned by E_j.

Hence, the figure obtained from P by removing the edge E_j is still bounded, and hence a convex polygon. In fact, it is a unimodular polygon with $d-1$ edges, and P is obtained from it by a corner-chopping.

We conclude that any unimodular polygon can be obtained by a finite sequence of corner-choppings from a unimodular polygon with at most 4 edges. By Propositions 2.12 and 2.15, we are done. □

We have now concluded the proof of Theorem 2.21. In light of Delzant's theorem (Theorem 1.32), this yields a classification result for toric symplectic manifolds of dimension 4 (cf. Remark 2.19).

Theorem 2.26. *Fix the standard* 2*-torus* $\mathbb{T}^2$*. Up to isomorphism, every toric symplectic* 4*-manifold is obtained from a* $\mathbb{CP}^2$ *or a Hirzebruch surface* $\mathcal{H}_k$ *by a finite sequence of equivariant symplectic blow-ups at* $\mathbb{T}^2$*-fixed points.*

Finally, we note that there are some partial results by Oda and Miyake in the case $n = 3$ [33, Section 9, and particularly Theorem 9.6]. Nevertheless, it is hopeless to expect a result analogous to Theorem 2.26 to hold in higher dimensions (cf. [35], namely Theorems 2.24 and 2.25).

As a follow-up to the contents of this chapter, we should mention the work of Pelayo, Pires, Ratiu and Sabatini [34] on the moduli space of toric symplectic 4-manifolds. Pelayo and Santos have further extended these results to higher dimensions in [35].

Additionally, Karshon, Kessler and Pinsonnault have explored in [23] the question of uniqueness of Hamiltonian toric actions on a given compact connected symplectic 4-manifold.

Chapter 3

Real loci

3.1 Real structures

Even though our main setting of interest is toric symplectic manifolds, the contents of this section hold just as well for more general Hamiltonian torus actions. For that reason, we state them in this level of generality. We are essentially following the work of Duistermaat in [13].

Definition 3.1. Let T be a torus and (M, ω, T, μ) be a compact connected Hamiltonian T-space. A *real structure*[1] on (M, ω, T, μ) is a smooth map $\tau \colon M \to M$ such that:

- τ is an anti-symplectic involution, i.e. $\tau^2 = \mathrm{id}_M$ and $\tau^*\omega = -\omega$;
- the moment map μ is invariant under τ, i.e. $\mu \circ \tau = \mu$.

Moreover, the *real locus* of $(M, \omega, T, \mu, \tau)$ is defined as

$$M^\tau := \{\, p \in M : \tau(p) = p \,\}.$$

Example 3.2. Recall from Example 1.34 that $(\mathbb{CP}^n, \omega_{\mathrm{FS}}, \mathbb{T}^n, \mu)$ is a toric symplectic manifold. We define a smooth involution $\tau \colon \mathbb{CP}^n \to \mathbb{CP}^n$ given by complex conjugation,

$$\tau([z_0 : \cdots : z_n]) = [\overline{z_0} : \cdots : \overline{z_n}].$$

From the local expression of the Fubini–Study form given in Example 1.34, it is clear that τ is anti-symplectic. Similarly, we can see that the formula for the moment map μ is invariant under complex conjugation. The real locus is then $(\mathbb{CP}^n)^\tau = \mathbb{RP}^n$. △

[1]For some authors, a real structure on (M, ω, T, μ), or perhaps just on (M, ω), is simply an anti-symplectic involution. Our stricter notion can in that case be called, for instance, a *toric real structure*.

In the above definition of a real structure, we included a compatibility condition between the anti-symplectic involution and the Hamiltonian torus action. However, this is not the only possible natural condition that one might think of imposing. We see now two other equivalent such conditions.

We remark that (a) and (c) are the conditions we are most interested in and which we will apply in the sequel; condition (b) serves essentially as a connecting point between them, in the proof.

Proposition 3.3. *Let T be a torus with Lie algebra $\mathfrak{t}$, (M, ω, T, μ) a compact connected Hamiltonian T-space, and $\tau\colon M \to M$ an anti-symplectic smooth involution. Then, the following are equivalent:*

(a) $\mu \circ \tau = \mu$;

(b) *for all $X \in \mathfrak{t}$ we have $\tau_* \widetilde{X} = -\widetilde{X}$, where as before $\widetilde{X}$ denotes the fundamental vector field on M associated to X;*

(c) *for all $g \in T$ and $p \in M$ we have $\tau(g \cdot p) = g^{-1} \cdot \tau(p)$.*

Proof. We will prove the implications a $\Rightarrow$ b $\Rightarrow$ c and c $\Rightarrow$ b $\Rightarrow$ a.

a $\Rightarrow$ b Let $X \in \mathfrak{t}$. We know that $\mu^X \circ \tau = \mu^X$. Recall from the definition of a moment map that $d\mu^X = \omega(\widetilde{X}, \cdot)$. At each point $p \in M$, the differential of the left-hand side is

$$d(\mu^X \circ \tau)_p = (d\mu^X)_{\tau(p)} \circ d\tau_p = \omega_{\tau(p)}(\widetilde{X}_{\tau(p)}, d\tau_p(\cdot)).$$

Using that τ is an anti-symplectic involution, the above equals

$$\begin{aligned} \omega_{\tau(p)}(d\tau_p \, d\tau_{\tau(p)} \widetilde{X}_{\tau(p)}, d\tau_p(\cdot)) &= (\tau^*\omega)_p(d\tau_{\tau(p)} \widetilde{X}_{\tau(p)}, \cdot) \\ &= \omega_p(-(\tau_* \widetilde{X})_p, \cdot). \end{aligned}$$

Hence, by differentiating both sides of (a) we see that

$$\omega(-\tau_* \widetilde{X}, \cdot) = \omega(\widetilde{X}, \cdot).$$

By non-degeneracy of the symplectic form ω, it follows that $\tau_* \widetilde{X} = -\widetilde{X}$.

b $\Rightarrow$ c Fix an arbitrary $p \in M$. What we wish to show is then equivalent to that, for every $g \in T$, we have

$$g \cdot \tau(g \cdot p) = \tau(p).$$

Define $F\colon T \to M$ by $F(g) = g \cdot \tau(g \cdot p)$. Since T is connected and $F(e) = \tau(p)$, it is enough to show that the differential of F is identically zero. Note that, for each $g \in T$, the tangent space to T at g is $\operatorname{im}(dL_g)_e$, where $L_g\colon T \to T$ denotes left multiplication by g.

For each $g \in T$ and $X \in \mathfrak{t}$, we can compute

$$\begin{aligned}
dF_g((dL_g)_e X) &= \frac{d}{dt} F(g \exp tX)\Big|_{t=0} \\
&= \frac{d}{dt}[(g \exp tX) \cdot \tau(g \exp tX \cdot p)]\Big|_{t=0} \\
&= \frac{d}{dt}[(g \exp tX) \cdot \tau(g \cdot p)]\Big|_{t=0} + \frac{d}{dt}[g \cdot \tau(g \exp tX \cdot p)]\Big|_{t=0}, \qquad (3.1)
\end{aligned}$$

where in the last equality we have applied the chain rule. Recall that T is an abelian group, and thus we can see that

$$\begin{aligned}
\frac{d}{dt}[(g \exp tX) \cdot \tau(g \cdot p)]\Big|_{t=0} &= \frac{d}{dt}[\exp tX \cdot (g \cdot \tau(g \cdot p))]\Big|_{t=0} \\
&= \widetilde{X}_{g \cdot \tau(g \cdot p)}. \qquad (3.2)
\end{aligned}$$

On the other hand, if we denote by $\psi \colon T \to \mathrm{Symp}(M, \omega)$ the T-action on M, we have

$$\begin{aligned}
\frac{d}{dt}[g \cdot \tau(g \exp tX \cdot p)]\Big|_{t=0} &= (d\psi_g)_{\tau(g \cdot p)} (d\tau)_{g \cdot p} \frac{d}{dt}[\exp tX \cdot (g \cdot p)]\Big|_{t=0} \\
&= (d\psi_g)_{\tau(g \cdot p)} (d\tau)_{g \cdot p} \widetilde{X}_{g \cdot p} \\
&= -(d\psi_g)_{\tau(g \cdot p)} \widetilde{X}_{\tau(g \cdot p)} \\
&= -\frac{d}{dt}[g \exp tX \cdot \tau(g \cdot p)]\Big|_{t=0} \\
&= -\frac{d}{dt}[\exp tX \cdot (g \cdot \tau(g \cdot p))]\Big|_{t=0} \\
&= -\widetilde{X}_{g \cdot \tau(g \cdot p)}. \qquad (3.3)
\end{aligned}$$

Note that in the third equality we used that (b) holds.

Combining Eqs. (3.1) to (3.3), we conclude that the differential of F does indeed vanish identically.

c ⇒ b Let $X \in \mathfrak{t}$. We wish to show that for all $p \in M$ we have

$$d\tau_p \widetilde{X}_p = -\widetilde{X}_{\tau(p)}.$$

By definition of $\widetilde{X}$ and of the differential of a smooth map, we see that

$$d\tau_p \widetilde{X}_p = \frac{d}{dt} \tau(\exp tX \cdot p)\Big|_{t=0} = \frac{d}{dt}[\exp(-tX) \cdot \tau(p)]\Big|_{t=0},$$

where the last equality is an application of (c). By the chain rule, the above equals

$$-\frac{d}{dt}[\exp tX \cdot \tau(p)]\Big|_{t=0} = -\widetilde{X}_{\tau(p)}.$$

b ⇒ a It is clearly equivalent to showing that $\mu^X \circ \tau = \mu^X$, for every $X \in \mathfrak{t}$.

As in the proof of a ⇒ b, we can see that $d(\mu^X \circ \tau) = \omega(-\tau_* \widetilde{X}, \cdot)$. By (b), this equals $\omega(\widetilde{X}, \cdot) = d\mu^X$.

Hence, $\mu^X \circ \tau$ and μ^X are real-valued smooth functions on the connected manifold M whose differentials are identical. It follows that these functions differ by an additive constant $c^X \in \mathbb{R}$. However, if we precompose both sides of this identity with the involution τ, we see that

$$\mu^X \circ \tau = \mu^X + c^x \implies \mu^X = \mu^X + 2c^X.$$

It follows that $c^X = 0$ and $\mu^X \circ \tau = \mu^X$, as we wished to show. □

We establish now some basic properties of the real locus.

Proposition 3.4 ([13, Section 2])**.** *Let T be a torus and $(M, \omega, T, \mu, \tau)$ a compact connected Hamiltonian T-space equipped with a real structure. Then the real locus M^τ is a compact Lagrangian submanifold of (M, ω).*

Proof. M^τ is compact, since it is a closed subset of the compact manifold M:

$$M^\tau = (\mathrm{id}_M, \tau)^{-1}(\{\, (p, p) \in M \times M : p \in M \,\}).$$

We will now show that M^τ is a Lagrangian submanifold of M. The main ingredient for this is the following fact: every point $p \in M^\tau$ has a τ-invariant neighbourhood U in M, together with local coordinates $x \colon U \to \mathbb{R}^{2n}$, such that the local representation of τ in these coordinates is *linear*. This is shown separately as Lemma 3.5.

Thus, let $p \in M^\tau$ and apply this fact to get an appropriate coordinate chart (U, x). Denote $V := x(U) \subseteq \mathbb{R}^{2n}$, and let $L_p \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear map such that the local representation of τ in this chart is a restriction of L_p. Then, note that

$$x(M^\tau \cap U) = \{\, x \in V : L_p(x) = x \,\} = V \cap \ker(L_p - \mathrm{id}), \tag{3.4}$$

which is an open subset of an eigenspace of L_p. Hence, $M^\tau \cap U$ is an embedded submanifold of M of dimension $\dim\ker(L_p - \mathrm{id}) = \dim\ker(d\tau_p - \mathrm{id})$, and for all $p \in U$

$$T_p(M^\tau \cap U) = \ker(d\tau_p - \mathrm{id}) \subseteq T_p M.$$

Note that the linear map $d\tau_p$ is also an involution, and thus it can only have 1 or -1 as eigenvalues. We can moreover see that $d\tau_p$ is diagonalisable: its eigenvectors span $T_p M$, since every vector $v \in T_p M$ can be written in the form

$$v = \frac{v + d\tau_p v}{2} + \frac{v - d\tau_p v}{2}.$$

This implies that we have a decomposition

$$T_pM = \ker(d\tau_p - \mathrm{id}) \oplus \ker(d\tau_p + \mathrm{id}).$$

We claim that both summands are isotropic subspaces of the symplectic vector space (T_pM, ω_p), from which it follows that they are in fact Lagrangian subspaces. Indeed, if $v, w \in \ker(d\tau_p - \mathrm{id})$, we have that

$$\omega_p(v,w) = \omega_p(d\tau_p v, d\tau_p w) = (\tau^*\omega)_p(v,w) = -\omega_p(v,w),$$

showing that $\omega_p(v,w) = 0$. The same argument applies to $\ker(d\tau_p + \mathrm{id})$.

This shows that, for every $p \in M^\tau$, we have $\dim\ker(d\tau_p - \mathrm{id}) = n$. Together with the above, we can conclude that the real locus M^τ is a Lagrangian submanifold of M. □

Lemma 3.5. *Let M be a smooth manifold of dimension d and $\tau\colon M \to M$ a smooth involution. Then, for every point $p \in M$ with $\tau(p) = p$, there exists a neighbourhood U of p in M satisfying $\tau(U) = U$, together with local coordinates $x\colon U \to \mathbb{R}^d$ on U, such that the local representation of τ in these coordinates is the restriction to $x(U) \subseteq \mathbb{R}^d$ of a linear map $L\colon \mathbb{R}^d \to \mathbb{R}^d$.*

Proof. Let $p \in M$ be such that $\tau(p) = p$. Consider some local coordinates $x\colon U_1 \to \mathbb{R}^d$ for M around p. Restricting the coordinates to

$$U_2 := U_1 \cap \tau^{-1}(U_1) = U_1 \cap \tau(U_1),$$

and using the fact that τ is a continuous involution, we find a coordinate neighbourhood U_2 of p satisfying $\tau(U_2) = U_2$.

Let $V_2 := x(U_2) \subseteq \mathbb{R}^d$ and denote by $\widehat{\tau}\colon V_2 \to V_2$ the local representation of τ in the (U_2, x) coordinate chart. Let

$$L = d\widehat{\tau}_{x(p)}\colon \mathbb{R}^d \to \mathbb{R}^d$$

be the derivative of $\widehat{\tau}$ at the point $x(p)$. More concretely, we have that

$$L = dx_p \circ d\tau_p \circ (dx_p)^{-1}. \tag{3.5}$$

Note that the linear map L is also an involution.

We want to find new local coordinates $\widetilde{x}$ on a neighbourhood of p such that $\widetilde{x} \circ \tau \circ \widetilde{x}^{-1}$ is a restriction of the linear map L. In other words, we are searching for a map $\widetilde{x}\colon U_2 \to \mathbb{R}^d$ such that $\widetilde{x} \circ \tau = L \circ \widetilde{x}$. If one looks at maps of the form $\widetilde{x}(p) = x(p) + F(p)$, one might arrive at the following candidate:

$$\widetilde{x}\colon U_2 \to \mathbb{R}^d, \quad \widetilde{x} = x + L \circ x \circ \tau.$$

Indeed, this is a smooth map, and we can compute

$$\widetilde{x} \circ \tau = x \circ \tau + L \circ x \circ \tau^2 = L^2 \circ x \circ \tau + L \circ x = L \circ \widetilde{x}.$$

Moreover, using Eq. (3.5) we see that

$$d\widetilde{x}_p = dx_p + L \circ dx_p \circ d\tau_p = dx_p + dx_p \circ (d\tau_p)^2 = 2\, dx_p,$$

which is an invertible linear map. By the inverse function theorem, we can find a neighbourhood $U_3 \subseteq U_2$ of p such that $\widetilde{x} : U_3 \to \mathbb{R}^d$ is a diffeomorphism into its open image, i.e. a local coordinate map.

As before, to ensure τ-invariance, we take $U := U_3 \cap \tau(U_3)$. It is now possible to consider the local representation of τ in the $(U, \widetilde{x})$ chart, and check that it is the restriction to $\widetilde{x}(U)$ of the linear map L previously defined. □

In [13], Duistermaat proved a convexity theorem for the real locus. We will not cover its proof, but it is closely related to the Atiyah–Guillemin–Sternberg convexity theorem (Theorem 1.28).

Theorem 3.6 (Duistermaat, [13, Theorem 2.5])**.** *Let T be a torus and $(M, \omega, T, \mu, \tau)$ a compact connected Hamiltonian T-space equipped with a real structure. Suppose that the real locus M^τ is not empty. Then M^τ has full image under the moment map, i.e. $\mu(M^\tau) = \mu(M) = \Delta$. In fact, the same is true if we replace M^τ with any one of its connected components.*

3.2 The real locus as a branched covering

From now on, we focus again on the toric case. In this context, one can obtain stronger results describing the real locus. This goes back at least to the work of Guillemin in [18]. Moreover, the very closely related concept of a *small cover* had already been studied by Davis and Januszkiewicz in [11].

First of all, in the toric case, the real locus is always non-empty.

Lemma 3.7. *Let $(M, \omega, T, \mu, \tau)$ be a toric symplectic manifold equipped with a real structure. Then the real locus M^τ is non-empty, and contains all the fixed points of the T-action on M.*

Proof. Let $p \in M$ be a fixed point of the T-action on M. Then $q := \tau(p)$ is also a fixed point of the T-action: for any $g \in T$ we have that

$$g \cdot \tau(p) = \tau(g^{-1} \cdot p) = \tau(p).$$

Moreover, we have that $\mu(q) = \mu(\tau(p)) = \mu(p)$. By Lemma 1.31, it follows that p and q belong to the same T-orbit. Since p is a fixed point of the T-action, we conclude that $q = \tau(p) = p$, and thus $p \in M^\tau$.

Finally, Lemma 1.31 also implies that the preimage of each vertex of the moment polytope Δ is one fixed point of the T-action. It follows that these fixed points do exist, and hence $M^\tau \neq \varnothing$. $\square$

Moreover, one can show that, in the toric case, the real locus is *connected*.

Proposition 3.8. *Let $(M, \omega, T, \mu, \tau)$ be a toric symplectic manifold equipped with a real structure. Then the real locus M^τ is a compact connected Lagrangian submanifold of (M, ω).*

Proof. Recall from Proposition 3.4 and Lemma 3.7 that M^τ is non-empty, and a compact embedded Lagrangian submanifold of (M, ω).

Let $\Delta = \mu(M)$ be the moment polytope. By Theorem 3.6, we know that $\mu(C) = \Delta$, for any connected component C of M^τ. However, by Lemma 1.31, we know that the pre-images of the vertices of the moment polytope Δ are singletons. This implies that any two connected components must intersect, and hence that there is at most one connected component. $\square$

We will now see that the order 2 elements in T generate a finite subgroup which acts on M^τ.

Notation 3.9. Let T be a torus. Then we define the subgroup

$$T^{\mathbb{R}} := \{ g \in T : g^2 = e \}.$$

For instance, if $T = \mathbb{T}^n$ is the standard n-torus, then $T^{\mathbb{R}} = \{-1, 1\}^n$ is a finite subgroup of order 2^n.

Lemma 3.10. *Let $(M, \omega, T, \mu, \tau)$ be a toric symplectic manifold equipped with a real structure. Then, for each T-orbit $\mathcal{O} \subseteq M$, the intersection $M^\tau \cap \mathcal{O}$ is exactly one $T^{\mathbb{R}}$-orbit.*

Proof. We begin by showing that, for each $g \in T^{\mathbb{R}}$ and $p \in M^\tau$, we have $g \cdot p \in M^\tau$. This shows that each $M^\tau \cap \mathcal{O}$ is a union of $T^{\mathbb{R}}$-orbits. Indeed, we have that

$$\tau(g \cdot p) = g^{-1} \cdot \tau(p) = g^{-1} \cdot p = g \cdot p.$$

Now, suppose that $p, q \in M^\tau$ and $g \in T$ are such that $q = g \cdot p$. We will show that there exists $g' \in T^{\mathbb{R}}$ such that $q = g' \cdot p$. Indeed, we can see that

$$q = \tau(q) = \tau(g \cdot p) = g^{-1} \cdot \tau(p) = g^{-1} \cdot p.$$

This implies that $g^2 \cdot p = p$, i.e. $g^2 \in T_p$, the isotropy group of p. Note that, by Lemma 1.31, T_p is a subtorus of T. Hence, by Theorem 1.11 we can find

$h \in T_p$ such that $h^2 = g^2$. Now, take $g' = gh^{-1}$. It is immediate that $q = g' \cdot p$. Moreover,

$$(g')^2 = g^2 h^{-2} = e,$$

and thus $g' \in T^{\mathbb{R}}$.

Thus far, we can conclude that each intersection $M^\tau \cap \mathcal{O}$ is either empty or exactly one $T^{\mathbb{R}}$-orbit. However, combining Lemma 1.31 and Theorem 3.6 we see that M^τ intersects every T-orbit. □

Theorem 3.11. *Let $(M, \omega, T, \mu, \tau)$ be a toric symplectic manifold of dimension $2n$ equipped with a real structure, with real locus M^τ and moment polytope $\Delta = \mu(M) = \mu(M^\tau)$. Then, the following holds.*

(a) *The T-action on M restricts to a $T^{\mathbb{R}}$-action on M^τ.*

(b) *There exists a $T^{\mathbb{R}}$-equivariant (topological) quotient map*

$$q \colon \Delta \times T^{\mathbb{R}} \to M^\tau,$$

which makes the following diagram commute

$$\begin{tikzcd} \Delta \times T^{\mathbb{R}} \arrow[rr, "q"] \arrow[dr, "p"'] & & M^\tau \arrow[dl, "\mu"] \\ & \Delta & \end{tikzcd}.$$

Here, $T^{\mathbb{R}}$ acts on $\Delta \times T^{\mathbb{R}}$ by multiplication in the second factor, and we denote by $p \colon \Delta \times T^{\mathbb{R}} \to \Delta$ the natural projection onto the first factor.

Note that we are not claiming that the map q is smooth[2]*; in fact, it follows from the proof that this is not true, if $n > 0$.*

(c) *Over the interior Δ^{o}, this quotient map restricts to a ($T^{\mathbb{R}}$-equivariant) diffeomorphism*

$$\Psi = q|_{\Delta^{\mathrm{o}} \times T^{\mathbb{R}}} \colon \Delta^{\mathrm{o}} \times T^{\mathbb{R}} \xrightarrow{\sim} M^\tau \cap \mu^{-1}(\Delta^{\mathrm{o}}).$$

This result implies that $\mu|_{M^\tau} \colon M^\tau \to \Delta$ is a *branched covering map*, which breaks into 2^n connected components over the interior Δ^{o}. More precisely, $M^\tau \cap \mu^{-1}(\Delta^{\mathrm{o}})$ is the disjoint union of 2^n open subsets of M^τ, each of which is mapped diffeomorphically onto Δ^{o} by μ.

[2]We should remark that, for $n > 0$, Δ is not a smooth manifold, and if $n > 1$ it is not even a smooth manifold with boundary. Thus, it might not be clear what smoothness of q would mean in the first place. There are two equivalent ways to solve this issue: we may regard Δ as a smooth manifold with corners; or we may regard it as a subset of the finite-dimensional vector space $\mathfrak{t}^*$. In both cases, a map from Δ to a manifold is smooth if and only if it can be locally extended to a smooth map on an open subset of $\mathfrak{t}^* \cong \mathbb{R}^2$.

Moreover, by part (b), since $T^{\mathbb{R}}$ has the discrete topology, this means that M^τ is (homeomorphic to) a quotient of the disjoint union of 2^n copies of the moment polytope Δ.

Finally, the restriction $\mu|_{M^\tau \cap \mu^{-1}(\Delta^\circ)} \colon M^\tau \cap \mu^{-1}(\Delta^\circ) \to \Delta^\circ$ of the moment map is a trivializable principal $T^{\mathbb{R}}$-bundle.

Proof. The fact that $T^{\mathbb{R}}$ acts on M^τ is a direct consequence of (half of) Lemma 3.10. This shows part (a).

(b) Choose be a connected component P_0 of $M^\tau \cap \mu^{-1}(\Delta^\circ)$. We claim that the restriction $\mu|_{\overline{P_0}} \colon \overline{P_0} \to \Delta$ is a homeomorphism.

If we show that $\mu|_{\overline{P_0}} \colon \overline{P_0} \to \Delta$ is a proper local homeomorphism, it follows by [27, Exercise 11-9] that it is a covering map. Since Δ is simply connected, this must in fact be a homeomorphism. Note that $\overline{P_0}$ is compact and Δ is Hausdorff, ensuring that $\mu|_{\overline{P_0}}$ is a proper map.

We are thus left to show that $\mu|_{\overline{P_0}} \colon \overline{P_0} \to \Delta$ is a local homeomorphism.

To do this, we use Lemma 1.38, which gives a local normal form for the moment map. More precisely, let $p \in M^\tau$. Then, in a neighbourhood of p in M, the moment map looks like $\mu \colon \mathbb{T}^k \times V \times B(\varepsilon)^{n-k} \to \mathbb{R}^n$, given by

$$\begin{aligned} &\mu(s_1, \dots, s_k, \mu_1, \dots, \mu_k, z_1, \dots, z_{n-k}) = \\ &\qquad = c + \left(\mu_1, \dots, \mu_k, -\frac{1}{2}|z_1|^2, \dots, -\frac{1}{2}|z_{n-k}|^2 \right). \end{aligned}$$

Adapting the argument of Duistermaat's lemma [13, Proposition 2.2], we may also assume that the involution τ is locally given by

$$\begin{aligned} &\tau \colon \mathbb{T}^k \times V \times B(\varepsilon)^{n-k} \to \mathbb{T}^k \times V \times B(\varepsilon)^{n-k}, \\ &\tau(s_1, \dots, s_k, \mu_1, \dots, \mu_k, z_1, \dots, z_{n-k}) = \\ &\qquad = (s_1^{-1}, \dots, s_k^{-1}, \mu_1, \dots, \mu_k, \overline{z_1}, \dots, \overline{z_{n-k}}). \end{aligned}$$

Hence, the real locus locally looks like

$$\begin{aligned} (\mathbb{T}^k \times V \times B(\varepsilon)^{n-k}) \cap M^\tau &= \{-1, 1\}^k \times V \times (-\varepsilon, \varepsilon)^{n-k} \\ &\subset \{-1, 1\}^k \times V \times \mathbb{R}^{n-k}. \end{aligned}$$

Note that the points lying over the interior Δ° are those whose coordinates $z_1, \dots, z_{n-k}$ are all non-zero, since these are the points with a trivial isotropy group. When intersected with the real locus, this gives locally

$$\{-1, 1\}^k \times V \times ((-\varepsilon, 0) \cup (0, \varepsilon))^{n-k}.$$

This corresponds to 2^n disjoint copies of $V \times (0,\varepsilon)^{n-k}$, which are open neighbourhoods in each of the 2^n connected components of $M^\tau \cap \mu^{-1}(\Delta^{\mathrm{o}})$. We can conclude that $\overline{P_0}$ locally looks like $V \times [0,\varepsilon)^{n-k}$, with the moment map locally given by

$$\mu\colon V \times [0,\varepsilon)^{n-k} \to \mathbb{R}^n,$$
$$(\mu_1,\dots,\mu_k,r_1,\dots,r_{n-k}) \mapsto \left(\mu_1,\dots,\mu_k,-\frac{r_1^2}{2},\dots,-\frac{r_{n-k}^2}{2}\right).$$

This is in fact a homeomorphism to its image. (Note, though, that for $k < n$ it is not a diffeomorphism!)

We have now shown that, for each component P of $M^\tau \cap \mu^{-1}(\Delta^{\mathrm{o}})$, its closure $\overline{P}$ is a homeomorphic copy of Δ; more precisely, it is mapped homeomorphically onto Δ by μ. It also follows that these 2^n copies of Δ are permuted under the action of $T^{\mathbb{R}}$.

We must also show that these copies cover the whole real locus M^τ, i.e.

$$M^\tau = \bigcup_{g\in T^{\mathbb{R}}} g\cdot \overline{P} = \overline{\bigcup_{g\in T^{\mathbb{R}}} g\cdot P} = \overline{M^\tau \cap \mu^{-1}(\Delta^{\mathrm{o}})}.$$

Note that $\mu\colon M^\tau \to \Delta$ is a quotient relative to a group action (of $T^{\mathbb{R}}$), and hence is an open map [see 26, Lemma 21.1].

Then, $\mu\left(M^\tau \setminus \bigcup_{g\in T^{\mathbb{R}}} g\cdot\overline{P}\right)$ is an open subset of Δ which does not intersect the interior Δ^{o}. It follows that this subset must be empty, and thus we get the desired equality.

At this point we are able to define a surjective map $q\colon \Delta \times T^{\mathbb{R}} \to M^\tau$, such that

$$q|_{\Delta\times\{g\}} = (\mu|_{g\cdot\overline{P_0}})^{-1},$$

for each $g \in T^{\mathbb{R}}$. This maps each $\Delta \times \{g\}$ to $g \cdot \overline{P_0} \subseteq M^\tau$.

(c) Equivalently, we must show that, for all $g \in T^{\mathbb{R}}$, the restriction

$$q|_{\Delta^{\mathrm{o}}\times\{g\}}\colon \Delta^{\mathrm{o}} \times \{g\} \to g\cdot P_0$$

is a diffeomorphism. By symmetry and definition of the map q, this reduces to showing that

$$\mu|_{P_0}\colon P_0 \to \Delta^{\mathrm{o}}$$

is a diffeomorphism.

We already know, from the proof of the previous point, that this map is a homeomorphism. Moreover, that proof also shows that this map is a local diffeomorphism (we are in the case $k = n$). It follows that it is a diffeomorphism, as we wanted to show. □

3.3 The case $n = 2$

Let $(M, \omega, T, \mu, \tau)$ be a toric symplectic manifold of dimension 4 equipped with a real structure. In this case, the real locus M^τ is a compact connected surface embedded in M.

In light of the classification of surfaces, this raises the following question: for any given example, is it possible to easily identify the diffeomorphism type of M^τ? Moreover, can every diffeomorphism type of compact connected surfaces arise in this way, or is the topology of M^τ constrained?

Using the description of the real locus as a branched covering over the moment polygon $\Delta = \mu(M) = \mu(M^\tau)$, given in the previous section, we will be able to use Δ to recover the diffeomorphism type of M^τ, and answer the questions raised above. In particular, we will see that the diffeomorphism type of M^τ is in fact independent of τ, and depends only on the weak isomorphism type of (M, ω, T, μ), or equivalently on the $\mathrm{AGL}(2, \mathbb{Z})$-orbit of Δ.

We should note that the methods in the previous section only allow us to identify the real locus M^τ up to homeomorphism. However, it is a well-known fact that, in dimensions at most 3, two manifolds are diffeomorphic if and only if they are homeomorphic (see [31] or [20] for dimension 2 and [30, Theorem 6.3] for dimension 3).

Proposition 3.12. *Let $(M, \omega, T, \mu, \tau)$ be a toric symplectic manifold of dimension 4 equipped with a real structure, with real locus M^τ and moment polygon $\Delta = \mu(M) = \mu(M^\tau)$. Suppose that μ is chosen such that Δ has a vertex at the origin.*

Fix an isomorphism $\mathfrak{t}^* \cong \mathbb{R}^2$ which identifies the lattices $\mathfrak{t}^*_{\mathbb{Z}} \cong \mathbb{Z}^2$ and sends Δ to standard position. By exponentiating, this induces an associated isomorphism $T \cong \mathbb{T}^2$.

Let $E_1, \dots, E_d$ be the edges of Δ. Following Proposition 1.26, write Δ as an irredundant intersection of half-planes,

$$\Delta = \{ x \geq 0 \} \cap \{ y \geq 0 \} \cap \bigcap_{j=3}^{d} \{ (x, y) \in \mathbb{R}^2 : a_j x + b_j y \geq \lambda_j \},$$

where, for each $j = 3, \dots, d$, the vector $(a_j, b_j) \in \mathbb{Z}^2$ is the inward-pointing primitive normal vector to the edge E_j, and λ_j is a real constant.

Denote $\Delta_{(+1,+1)} := \Delta$ and let $\Delta_{(-1,+1)}$ be the image of $\Delta_{(+1,+1)}$ under reflection with respect to the coordinate x-axis. Similarly, we define $\Delta_{(+1,-1)}$ and $\Delta_{(-1,-1)}$, respectively, as the images of $\Delta_{(+1,+1)}$ and $\Delta_{(-1,+1)}$ under reflection with respect to the coordinate y-axis.

The union of these four copies of Δ can be seen as a polygonal region $P \subset \mathbb{R}^2$ with $4(d-2)$ edges.

Then, M^τ is the compact connected surface obtained by identifying the edges of P in pairs, as follows: for each $j = 3, \dots, d$, and $(\varepsilon_1, \varepsilon_2) \in \{-1, +1\}^2$, the edge E_j of $\Delta_{(\varepsilon_1,\varepsilon_2)}$ is identified with its copy in $\Delta_{((-1)^{a_j}\varepsilon_1,(-1)^{b_j}\varepsilon_2)}$.

Notably, the way in which the edges of P are identified depends only on the parity of the components a_j, b_j of the primitive normal vectors to the edges of P.

Proof. According to Theorem 3.11, we have a $T^{\mathbb{R}}$-equivariant quotient map

$$q\colon \Delta \times T^{\mathbb{R}} \to M^\tau,$$

which is such that the composition $\mu \circ q$ equals the canonical projection $\Delta \times T^{\mathbb{R}} \to \Delta$. Moreover, when restricted to $\Delta^{o} \times T^{\mathbb{R}}$, the map q is a diffeomorphism to its image.

We begin by understanding precisely which is the equivalence relation on $\Delta \times T^{\mathbb{R}}$ induced by q. Note that if two points in $\Delta \times T^{\mathbb{R}}$ are identified by q, they must also have the same image under p, i.e. they correspond to the same point in Δ in different "slices".

First, let $v \in \Delta^{o}$ be an interior point. Since q is injective on $\Delta^{o} \times T^{\mathbb{R}}$, each point in $\{v\} \times T^{\mathbb{R}}$ is only identified with itself.

Now, let $v \in \Delta$ be a vertex. By Lemma 1.31 and Lemma 3.7, we know that $\mu^{-1}(v) \subseteq M^\tau$ is a singleton. This implies that all the four points in $\{v\} \times T^{\mathbb{R}}$ are identified together.

Finally, let $v \in E_j^{o} \subset \Delta$ lie in the interior of an edge. Recall that we have fixed an isomorphism $\mathfrak{t}^* \cong \mathbb{R}^2$, along with the induced isomorphisms $\mathfrak{t} \cong \mathbb{R}^2$ and $T \cong \mathbb{T}^2$. Moreover, this last one restricts to an isomorphism $T^{\mathbb{R}} \cong \{-1, +1\}^2$.

According to these identifications, the annihilator in $\mathfrak{t} \cong \mathbb{R}^2$ of the edge E_j is the line spanned by the normal vector (a_j, b_j). By Lemma 1.31, we can conclude that, for any $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \{-1, +1\}^2$, the isotropy group of $q(v, \varepsilon) \in M^\tau$ relative to the $\mathbb{T}^2$-action on M is the 1-dimensional subtorus

$$T_j := \{\, (e^{ia_jt}, e^{ib_jt}) : t \in \mathbb{R} \,\} \subset \mathbb{T}^2.$$

Recall that, by assumption, a_j and b_j are coprime integers. Then, the isotropy group of $q(v, \varepsilon)$ relative to the $\{-1, +1\}^2$-action on M^τ is

$$T_j \cap \{-1, +1\}^2 = \{(1, 1), ((-1)^{a_j}, (-1)^{b_j})\}.$$

By equivariance of q, we have then that $q(v, \varepsilon) = \varepsilon \cdot q(v, 1)$, which coincides with $q(v, 1)$ if and only if $\varepsilon \in T_j \cap \{-1, +1\}^2$.

It follows that the four points in $\{v\} \times \{-1, +1\}^2$ are identified in pairs: each $(v, \varepsilon) \in \Delta \times \{-1, +1\}^2$ is identified with the point $(v, ((-1)^{a_j}, (-1)^{b_j})\varepsilon)$.

Let us look in particular at points in the edges

$$E_1 = \Delta \cap \{ (0, y) : y \in \mathbb{R} \} \text{ and } E_2 = \Delta \cap \{ (x, 0) : x \in \mathbb{R} \},$$

for which $(a_1, b_1) = (1, 0)$ and $(a_2, b_2) = (0, 1)$.

Note that the polygonal region P can be obtained as a quotient of $\Delta \times \{-1, +1\}^2$, under the following identifications:

- $(v, (1, 1)) \sim (v, (-1, 1))$ and $(v, (1, -1)) \sim (v, (-1, -1))$, for $v \in E_1$;
- $(v, (1, 1)) \sim (v, (1, -1))$ and $(v, (-1, 1)) \sim (v, (-1, -1))$, for $v \in E_2$.

In other words, P is obtained from $\Delta \times \{-1, +1\}^2$ by gluing the copies of Δ along the edges E_1 and E_2. By identifying the remaining edges, we obtain that M^τ is a quotient of P.

Moreover, we can check that the quotient $\Delta \times \{-1, 1\}^2$ respects our labelling of the copies of Δ. More precisely, for any $\varepsilon \in \{-1, 1\}^2$, $\Delta \times \{\varepsilon\}$ is mapped to $\Delta_\varepsilon \subset P$. This shows that identifications of the edges of P are as claimed in the statement of this result. $\square$

This theorem gives us a very explicit description of the real locus. Let us look at some examples.

Example 3.13. Consider $M = \mathbb{CP}^2$ as in Example 2.11. The moment polytope is a triangle (cf. Proposition 2.12). If we follow the construction in Proposition 3.12, we get that the real locus M^τ is diffeomorphic to the compact connected surface obtained from the polygon in Fig. 3.1.

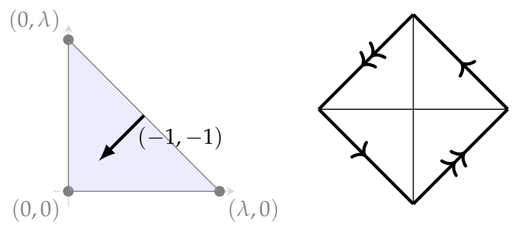

Figure 3.1: The real locus of a $\mathbb{CP}^2$ is a $\mathbb{RP}^2$.

It is well-known that the compact connected surface obtained from this polygon identification scheme is the real projective plane $\mathbb{RP}^2$.

Note that this agrees with Example 3.2. $\triangle$

Example 3.14. Consider now the family of trapezoids from Proposition 2.15, corresponding to Hirzebruch surfaces $\mathcal{H}_k$. The primitive normal vectors to the edges are $(1, 0), (0, 1), (-1, -k), (0, -1)$.

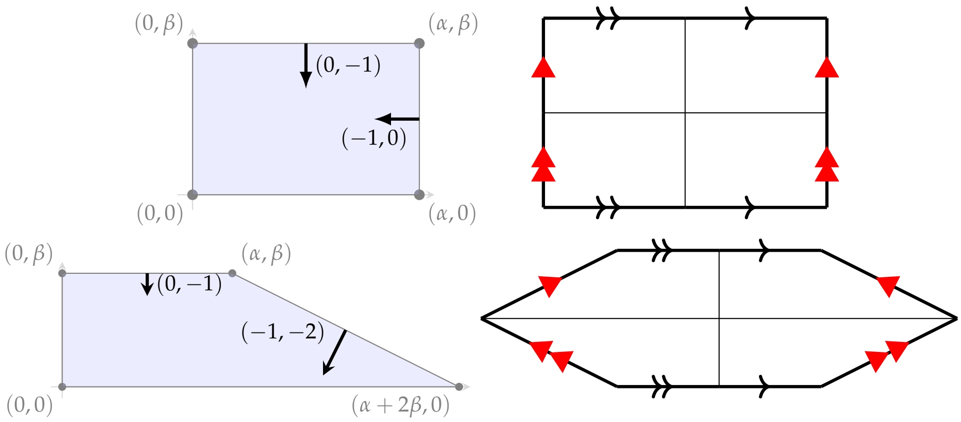

Figure 3.2: The construction of Proposition 3.12 applied to Hirzebruch surfaces $\mathcal{H}_k$ with $k = 0$ and $k = 2$. Their real loci are 2-tori $\mathbb{T}^2$.

According to the construction of Proposition 3.12, the diffeomorphism type of the real locus of $\mathcal{H}_k$ will depend only on the parity of k.

If k is even, the real locus of $\mathcal{H}_k$ is diffeomorphic to a 2-torus, as exemplified in Fig. 3.2.

On the other hand, if k is odd, we see that the real locus of $\mathcal{H}_k$ is diffeomorphic to a Klein bottle, as in Fig. 3.3. Recall that a Klein bottle is diffeomorphic to the connected sum of two real projective planes, $K \cong \mathbb{RP}^2 \# \mathbb{RP}^2$. △

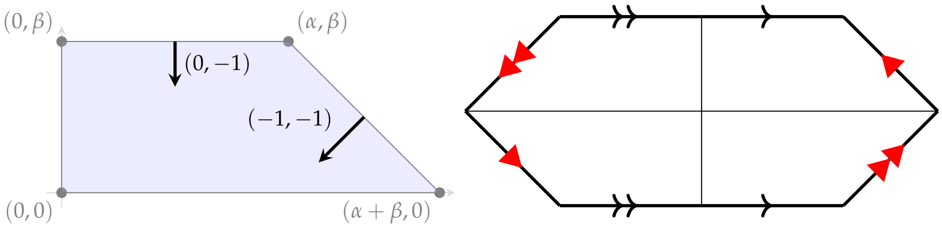

Figure 3.3: The construction of Proposition 3.12 applied to a Hirzebruch surface $\mathcal{H}_k$ with $k = 1$. Its real locus is a Klein bottle.

If one wants to understand the topology of the real locus of an arbitrary 4-dimensional toric symplectic manifold, Theorems 2.21 and 2.26 are highly useful. In fact, it is enough to consider, as we already have, the cases of a $\mathbb{CP}^2$ or a Hirzebruch surface, and then examine what happens after a blow-up. At the level of moment polygons, this corresponds to a corner-chopping, as we have seen in Remark 2.19.

Example 3.15. Let $\Delta \subset \mathbb{R}^2$ be a unimodular polygon, and $\widetilde{\Delta}$ the unimodular polygon obtained from Δ after a corner-chopping.

Up to the action of $\mathrm{AGL}(2, \mathbb{Z})$, we may assume that Δ is in standard position.

Moreover, we suppose that this is done such that the corner-chopping is performed at the vertex at the origin.

Applying the construction of Proposition 3.12 to Δ, we obtain a polygonal region $P \subset \mathbb{R}^2$, together with a labelling scheme for identifying its edges. If the same construction is applied to the subpolygon $\widetilde{\Delta} \subset \Delta$, we obtain a polygonal subregion $\widetilde{P} \subset P$, which differs from P by the removal of a small square centered at the origin and with diagonals aligned along the coordinates axes (see Fig. 3.4). Following Proposition 3.12, the edges of this square are identified in the same way as in Fig. 3.1.[3]

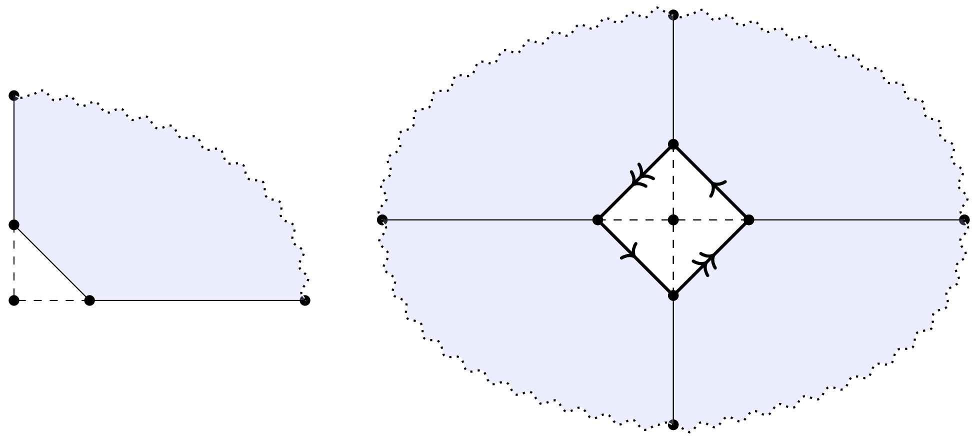

Figure 3.4: The construction of Proposition 3.12 performed after a corner-chopping.

We know that the real locus with respect to any real structure on the toric symplectic manifold (M, ω, T, μ) associated to P is homeomorphic to a certain compact connected surface S. The picture shows that, after blow-up, the real locus is transformed by connected sum with a real projective plane, i.e. it becomes $\widetilde{S} \cong S \# \mathbb{RP}^2$.

Note the similarity with the identity $\widetilde{M} \cong M \# \overline{\mathbb{CP}^2}$, mentioned in Remark 2.19. $\triangle$

In this way, we have fully determined the possible homeomorphism and diffeomorphism types of the real locus of a 4-dimensional toric symplectic manifold. Note that, with the exception of the even Hirzebruch surfaces, M^τ is never orientable.

Corollary 3.16. *Let $(M, \omega, T, \mu, \tau)$ be a 4-dimensional toric symplectic manifold equipped with a real structure, and M^τ be the real locus.*

[3] Note that, strictly speaking, Proposition 3.12 does not apply directly to $\widetilde{\Delta}$ and $\widetilde{P}$ (e.g., $\widetilde{\Delta}$ does not have a vertex at the origin, even though it has edges along the coordinate axes). However, it is easily seen that the result extends to this case.

Then M^τ is a compact connected surface which, up to diffeomorphism, is independent of the particular choice of τ.

If M^τ is orientable, then (M, ω, T, μ) is weakly isomorphic to a Hirzebruch surface $\mathcal{H}_k$ for some even k, and M^τ is a 2-torus.

In particular, M^τ is never diffeomorphic to a sphere, or to an compact connected orientable surface of genus higher than 1.

By contrast, any compact connected non-orientable surface can be realised as such a M^τ:

- *if $M^\tau \cong \mathbb{RP}^2$, then (M, ω, T, μ) is weakly isomorphic to the standard $\mathbb{CP}^2$;*
- *if $M^\tau \cong \mathbb{RP}^2 \# \mathbb{RP}^2 = K$ is a Klein bottle, then (M, ω, T, μ) is weakly isomorphic to a Hirzebruch surface $\mathcal{H}_k$, for some odd k;*
- *if $M^\tau \cong \underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{g+1 \text{ times}}$, for some $g > 1$, then (M, ω, T, μ) is weakly isomorphic to a $(g-1)$-fold blow-up of a Hirzebruch surface $\mathcal{H}_k$, for some odd k;*

Proof. Recall from Example 2.20 that a blow-up of $\mathbb{CP}^2$ is isomorphic to a Hirzebruch surface $\mathcal{H}_1$. For the last point, we recall the identity

$$T \# \mathbb{RP}^2 \cong K \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2. \qquad \square$$

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