J-holomorphic Curves in Symplectic Topology

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Abstract

This paper provides a quick overview of some of the most important results concerning J-holomorphic curves and their structure together with some applications to symplectic geometry and topology. The purpose of this exposition is to provide a summary of the general ideas involved in the theory of J-holomorphic curves like elliptic regularity and transversality arguments used in the context of moduli spaces as well as hinting at the importance of these methods in different aspects of symplectic topology like Floer homology. This paper closely follows and summarizes the books by McDuff and Salamon [8] and by Oh [10]. Most proofs, if presented at all, are taken from these sources. Whenever proofs are omitted, we refer to [8] for a complete argument.

1 Introduction

The study of J-holomorphic curves and their Moduli spaces for symplectic manifolds provides an extremely useful tool in symplectic topology introduced by Gromov in the 80's of the last century, see [6]. Many interesting results like the Non-Squeezing Theorem can be proven by using the methods surrounding J- holomorphic curves and thus make it worth to study the ideas behind J-holomorphic curves. Furthermore, similar ideas as used for example in the study of Moduli spaces of J-holomorphic curves appear in different circumstances like Morse Theory and Floer homology, see [1]. Especially the ideas involving Banach manifolds and the Implicit Function Theorem are used both in Morse Theory and Floer Homology to define appropriate homological structures.

The study of J-holomorphic curves is naturally inspired by the corresponding considerations for holomorphic curves in complex manifolds. J-holomorphic curves are defined on almost complex manifolds which is a far bigger class of manifolds than the complex manifolds are. However, it turns out that many local properties of holomorphic curves carry over to this more general setting (see Carleman's Similarity Principle in Section 2 or [8]). A Moduli spaces is the "manifold" of J-holomorphic curves representing some homology class, and the notion of Gromov compactness is key in the understanding of properties of J-holomorphic curves and their structure. The applications range from considerations of uniqueness of symplectic forms on certain manifolds like the complex projective plane to invariants of manifolds such as symplectic capacities which are intimately related to the celebrated non-squeezing theorem as well as the Gromov-Witten invariants. The latter are a family of invariants defined on symplectic manifolds by "counting" J-holomorphic curves. These invariants are interesting not only for themselves but they also play a major role in the study of quantum cohomology where these invariants naturally appear in the quantum cup product. (This brief overview of the importance of J-holomorphic curves is due to [4], for further informations we refer to said source.)

Our main source is the book by McDuff and Salamon [8] of which this paper presents a summary of and the paper is divided into several sections in the following way:

In Section 2, we highlight some properties of J-holomorphic curves which are reminiscent of holomorphic maps. One of the main ideas is that several good properties from holomorphic maps carry over to this more general setup due to a result called *Carleman Similarity principle*. Moreover, we introduce the notion of simple J-holomorphic curves which will play an important role in the study of Moduli spaces as simplicity will be used in order to deduce that the universal Moduli space forms a Banach manifold which is a key ingredient in the proof that generic almost complex structures are regular. We will clarify the notion of regular almost complex structures later.

In Section 3, we introduce some of the basic vocabulary surrounding differentiability in Banach spaces as well as Banach manifolds with some examples. The main results are the generalized Implicit Function Theorem which is later used to prove that Moduli spaces of simple J-holomorphic curves with respect to a regular almost complex structure J are indeed finite-dimensional manifolds. Moreover, we introduce the Sard-Smale Theorem which generalizes Sard's Theorem and enables us to more easily prove genericity of certain properties like regularity of almost complex structures. Again, similar ideas involving the Sard-Smale Theorem appear in Morse Theory and Floer Homology in order to deduce that generic structures lead to equivalent homologies on a given manifold.

In Section 4, we finally introduce Moduli spaces as well as the main results about their manifold structure and cobordism class. The main idea here is to proceed as we would do in the finite-dimensional case and try to invoke the Implicit Function Theorem for a special section in a infinite-dimensional vector bundle. We already note that the infinite dimensionality of the spaces involved will pose a major difficulty when establishing this result. This is mainly due to the fact that we will have to abandon smoothness of the J-holomorphic curves (because smooth maps do not form a Banach manifold) in order to apply the Implicit Function Theorem. It will turn out that an appropriate generalisation of Sobolev spaces for maps between manifolds yields the necessary Banach manifold structure in order to invoke the results in Section 3. Finally, by using local coordinate expression for appropriate Banach manifold structures, we can prove the desired results concerning the manifold structure of the Moduli spaces this way.

In Section 5, we study some compactness properties of J-holomorphic curves and especially the so-called bubbling process. By elliptic regularity, one can see that a sequence of J-holomorphic curves might have unbounded differentials which leads to the formation of a bubble which is nothing else than an adjoined J-holomorphic sphere. Under appropriate assumptions on the homology class which the J-holomorphic curves represent, we can see that no bubbling can occur and thus under certain conditions the Moduli spaces modulo reparametrisation are compact. Finally, we introduce stable maps as a generalisation of J-holomorphic spheres which naturally appear in the convergence considerations of sequences of J-holomorphic spheres. An important result for stable maps is that they satisfy Gromov compactness, i.e. a sequence of stable maps with bounded energy has a Gromov convergent subsequence. One can see that Gromov convergence introduces a topology on the space of stable maps and this topology is even metrizable.

In Section 6, we give two applications of the methods of J-holomorphic curves described up to now, namely the Non-Squeezing Theorem, which enables us to introduce a symplectic capacity and thus an invariant for symplectomorphisms, and C^0-closedness of the space of sypmlectomorphisms of a symplectic manifold to itself. Before we give these statements in general, we describe the linear case.

As previously mentioned, this paper intends to provide a summary and overview of some of the main results on J-holomorphic curves as found in [8]. This summary closely follows the presentation in this book in structure and almost all results and proofs, if given, are taken from this reference.

Finally, I would like to thank Dr. Ana Cannas da Silva for providing me with the opportunity to write this semester thesis as well as her advise throughout my work on it.

2 J-holomorphic Curves

J-holomorphic curves are the main object of interest in this paper and will be used to derive some results concerning symplectic manifolds and symplectomorphisms between them. In this chapter, we lay the foundation for the upcoming results concerning J-holomorphic curves and therefore define what almost complex structures and symplectic manifolds are. An important consequence for the applicability of the methods here is that every symplectic manifold admits a compatible almost complex structure. Afterwards, we start investigating pseudoholomorphic curves and their basic properties which are mostly reminiscent of properties of holomorphic curves. For example, we derive a unique continuation result for J-holomorphic curves as well as some local reparametrisation properties. The relation between the results in the J-holomorphic and the holomorphic case are due to *Carleman's Similarity principle* which we will present. Finally, we mention the most important results derived from the ellipticity of the Cauchy-Riemann-equations for J-holomorphic curves. These will be important in the study of moduli spaces and their compactness properties. Considering the defining partial differential equation for J-holomorphic curves, we can locally apply elliptic techniques by trying to uncover a relation to the Laplace operator and derive important estimates allowing us to deduce these results.

2.1 Almost Complex Structures and Symplectic Geometry

Before entering the study of almost complex structures and J-holomorphic curves, we remind ourselves of the definition of a symplectic manifold and almost complex structure:

Definition 2.1. *A* symplectic manifold *is a pair (M, ω) consisting of a smooth, even dimensional manifold M equipped with a non-degenerate, closed 2-form ω.*

Non-degeneracy simply means that for all $p \in M$, the alternating form on T_pM induced by ω is non-degenerate, i.e. ω induces and isomorphism between T_pM and its dual space T_pM^*. We note that many manifolds do not admit a symplectic structure, for example the non-degeneracy of ω implies that all symplectic manifolds are orientable. But there are also orientable manifolds which do not possess a symplectic structure, see for example S^4 due to its second cohomology group $H^2_{dR}(S^4) = \{0\}$.
On the other hand, manifolds admitting a symplectic structure can have several different ones, hence there is generally no uniqueness concerning the symplectic structure on a given manifold. Consider for example any orientable 2-manifold, then every volumeform induces a symplectic structure.
One key feature which distinguishes symplectic geometry from Riemannian geometry is that symplectic manifolds are all locally symplectomorphic by Darboux' theorem. This implies for example that there cannot be any local invariants on symplectic manifolds allowing us to distinguish them. This sharphly contrasts the situation in Riemannian geometry where curvature already defines local invariants for isometries. We formally state Darboux's theorem here:

Theorem 2.1. *Let (M, ω) be a symplectic $2n$-manifold and $p \in M$ any point. Then there exists a coordinate chart (U, ϕ), $\phi : U \subset \mathbb{R}^{2n} \to \phi(U)$ centered at p such that:*

$$\phi^*\omega = \omega_0,$$

where ω_0 denotes the standard symplectic structure on $\mathbb{R}^{2n}$.

A proof of this result can be found in [3, p.46] or [7, p.110]. Instead of writing it down, we show the following result which states that even local neighbourhoods of symplectic submanifolds allow Darboux charts:

Theorem 2.2. *Let (M,ω) be a symplectic $2n$-manifold, $N \subset M$ a symplectic $2k$-submanifold and $p \in N$ any point. Then there exists a coordinate chart (U,ϕ) as above with the additional property that:*

$$x \in N \Leftrightarrow \phi^{-1}(x) \in U \cap \mathbb{R}^{2k} \times \{0\}^{2(n-k)}.$$

This result is an exercise in [7, p.124]. It states that there exist adapted Darboux charts to symplectic submanifolds. The proof can be adapted such that the statement also holds for Lagrangian, isotropic and coisotropic submanifolds.

Proof. We will first construct a chart which is adapted to the submanifold *and* satisfies a similar property to Darboux charts on the submanifold. Let $p \in N$ be any point and (U,ψ) a chart adapted to the submanifold with $\psi(0) = p$. Note that the pullback under ψ of the symplectic form ω is still symplectic and that $V := U \cap \mathbb{R}^{2k} \times \{0\}^{2(n-k)}$ is a symplectic submanifold of U. Thus it suffices to work locally. By using a linear transformation, we can moreover assume:

$$\psi^*\omega = \sum_{i=1} dx_i \wedge dy_i, \tag{1}$$

coincides with the usual symplectic form on $\mathbb{R}^{2n}$ in 0. Note that we use coordinates $(x_1, y_1, \ldots, x_n, y_n)$.
Now, because V is a symplectic submanifold of U, there is a Darboux chart on V around 0. By possibly shrinking U, we can extend this map to a diffeomorphism on U by mapping the last $2(n-k)$ coordinates to themselves by the identity and the first $2k$ according to a Darboux chart on V. Note that by pulling-back $\psi^*\omega$ under this map, we see that the pull back of $\psi^*\omega$ under this diffeomorphism and then again pulling-back under the inclusion map of the image of V, we get the standard symplectic form due to the choice of a Darboux chart. Moreover, in 0, we still see that equation (1) holds. In order to keep the notation simple, we will denote the composition of ψ and the constructed diffeomorphism again by ψ and the set on which this map is defined again by U. V will gets adapted analogously. Note that this chart has now our desired properties.

Next, we remark that the usual symplectic form on U and our pullback symplectic form agree in 0 and their pullbacks to V agree. Therefore, we note that due to closedness and Proposition 6.8 in [3, p.39], we know, if we possibly take a slightly smaller U to get a tubular neighbourhood:

$$\sum_{i=1}^{n} dx_i \wedge dy_i - \psi^*\omega = d\mu, \tag{2}$$

where $\mu_x = 0$, for all $x \in V$. We define:

$$\forall t \in [0,1] : \omega_t := \psi^*\omega + td\mu. \tag{3}$$

Additionally, by again taking a smaller U and by recalling (1), we can assume that ω_t is non-degenerate for all $x \in U$ and all $t \in [0,1]$. Due to this, we are now able to argue by using Moser's trick. Namely, we want to find a time dependent vectorfield X_t with flow ρ_t, such that:

$$\frac{d}{dt}(\rho_t^*\omega_t) = 0, \tag{4}$$

thus by Fisherman's formula:

$$\begin{aligned} \frac{d}{dt}(\rho_t^*\omega_t) &= \rho_t^*(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}) \\ &= \rho_t^*(di_{X_t}\omega_t + d\mu) \\ &= 0. \end{aligned} \tag{5}$$

Noting that ρ_t are diffeomorphisms, this holds therefore, if we have:

$$i_{X_t}\omega_t = -\mu, \tag{6}$$

for all t and $x \in U$. Note that we can define therefore an appropriate smooth, time-dependent vectorfield due to non-degeneracy. Furthermore, due to $\mu_x = 0$, for $x \in V$, we see:

$$X_t = 0, \tag{7}$$

on V. Now, by further restricting U, we can assume that the flow is defined for $0 \leq t \leq 1$ and by (7), the flow does not move points in V. Taking thus ρ_1, we get the desired diffeomorphism which provides the desired Darboux chart due to our previous calculations. □

Even though local invariants do not exist, we are still interested in defining global invariants in order to be able to distinguish different symplectic manifolds from one another. One way to define such an invariant takes the route of J-holomorphic curves and might lead eventually to Gromov-Witten invariants, for example.

Now we introduce one of the most important notions in this paper:

Definition 2.2. *An* almost complex manifold (M, J) *consists of a smooth manifold M together with an almost complex structure J, i.e. a vectorbundle isomorphism $J : TM \to TM$, such that $J^2 = -Id$.*

The easiest examples of almost complex manifolds are the complex ones equipped with the almost complex structure defined by the complex structure of the manifold, i.e. defined using the Cauchy-Riemann equations in complex charts. But there are many other examples which are not complex manifolds, for example the connected sum of three copies of $\mathbb{CP}^2$ as proven by Taubes, see [3, p.103].
An important family of almost complex manifolds are the symplectic manifolds. But before stating the important existence result on almost complex structures, let us define:

Definition 2.3. *Let (M, ω) be a symplectic manifold and J an almost complex structure on M. Then J is called:*

- ω-tame*, if $\forall p \in M, \forall v \neq 0 \in T_pM : \omega_p(v, J_pv) > 0$.*
- ω-compatible*, if it is ω-tame and moreover:*

$$\forall p \in M, \forall v, w \in T_pM : \omega_p(v, w) = \omega_p(J_pv, J_pw).$$

Note that an ω-tame almost complex structure can be used to define a Riemannian metric g on M by defining:

$$\forall p \in M, \forall v, w \in T_pM : g_p(v, w) := \frac{1}{2}\big(\omega_p(v, J_pw) + \omega_p(J_pv, w)\big). \tag{8}$$

We sometimes denote g by $|\cdot|_J$ or g_J. The following existence result is of interest in our study of J-holomorphic curves as it guarantees the existence of almost complex structures on symplectic manifolds:

Theorem 2.3. *Let (M, ω) be a symplectic manifold. Then there exists an ω-compatible almost complex structure J on M.*

A proof of this theorem can be found in [3, p.70] or [7, p.153]. For the rest of this paper, we denote by $\mathcal{J}_\tau(M, \omega)$ the set of all ω-tame almost complex structures and by $\mathcal{J}(M, \omega)$

the set of all ω-compatible almost complex structures on the symplectic manifold $(M\omega)$. Clearly:

$$\mathcal{J}(M,\omega) \subset \mathcal{J}_\tau(M,\omega),$$

and by [7, p.153], we even know that these spaces are contractible. Moreover, we let $\mathcal{J}^l(M)$ and similarily for the other spaces denote the almost complex structures on M which are of class C^l, for any $l \in \mathbb{N}$. Note that by taking $l = \infty$, we just uncover the space introduced before.

2.2 Definition and Examples

With the notions of the previous section in mind, we can now define what J-holomorphic curves are. From now on, M will always denote an almost complex manifold.

Definition 2.4. *Let (M, J) be a almost complex manifold and (Σ, j) a Riemannian surface. A* J-holomorphic curve *is a smooth map $u : \Sigma \to M$, such that the differential du is complex linear with respect to the induced complex structures on the tangent bundles, i.e.*

$$du(x) \circ j_x = J_{u(x)} \circ du(x), \forall x \in \Sigma \tag{9}$$

We note that one can define:

$$\overline{\partial}_J u := \frac{1}{2}\big(du + J \circ du \circ j\big) \in \Omega^{0,1}(\Sigma, u^*TM), \tag{10}$$

where $\Omega^{0,1}(\Sigma, u^*TM)$ denotes the vectorbundle of complex-antilinear 1-forms on Σ with values in the pullback bundle as above. Note that u is J-holomorphic if and only if $\overline{\partial}_J u = 0$. Before entering the more abstract part of our discussion involving J-holomorphic curves, we give some easy examples:

Example 2.1.

1. *Let M be a complex manifold. Then it is clear, that any holomorphic map $u : \Sigma \to M$ is also J-holomorphic if J denotes the almost complex structure induced on the manifold M via complex charts as described earlier.*
2. *Let $\Sigma = M = S^2$, our favourite example from now on. Then, S^2 is a complex manifold. For instance, one can define a complex structure on S^2 by using the usual homeomorphism $S^2 \cong \mathbb{CP}^1$ to the complex projective space. Therefore, we can now deduce that Möbius-transformations lead to an important family of examples of J-hoilomorphic spheres, as these maps are clearly holomorphic. Furthermore, even though this might seem a bit less clear, this implies that rotations of the sphere define J-holomorphic curves as well.*

We note that (10) provides an important interpretation of J-holomorphic curves as the zeros of a section in an appropriate vectorbundle which in turn inspires the arguments leading to moduli spaces. For this, we informally introduce as in [8, p.19] an infinite-dimensional vectorbundle $\mathcal{E} \to C^\infty(\Sigma, M)$ with fibers:

$$\mathcal{E}_u := \Omega^{0,1}(\Sigma, u^*TM), \forall u \in C^\infty(\Sigma, M). \tag{11}$$

Note that now, the map:

$$C^\infty(\Sigma, M) \to \mathcal{E}, u \mapsto (u, \overline{\partial}_J u), \tag{12}$$

defines a section and by our earlier observation, the zeros correspond bijectively to J-holomorphic curves over Σ. This insight inspires considerations related to transversality of this section as one would study in the finite-dimensional framework in order to see that the preimage of regular values lead to submanifolds. Note that in this context, if we were able

to use this type of reasoning, we might be able to conclude that the space of J-holomorphic curves modelled over Σ has a manifold structure. However, it will turn out to be more difficult to deduce these results due to the missing Banach manifold structure on the space of smooth maps. We will later return to such considerations and see how to overcome these difficulties.
Before investigating some properties of J-holomorphic curves, we want to introduce two more things: local coordinate conditions for J-holomorphic curves and the notion of energy: Let $u : \Sigma \to M$ be a smooth map and (U, ϕ) a complex coordinate chart on Σ, $\phi : U \subset M \to \mathbb{C}$. We then set $u_0 = u \circ \phi^{-1}$ and by direct calculation, we see due to ϕ being compatible with the complex structure and using the identification $z = s + it$ in the coordinate chart:

$$\begin{aligned}\overline{\partial}_J(u_0)\partial_s &= \frac{1}{2}\big(du_0 + J(u_0) \circ du_0 \circ j\big)\partial_s \\ &= \frac{1}{2}\big(du_0(\partial_s) + J(u_0) \circ du_0 \circ j(\partial_s)\big) \\ &= \frac{1}{2}\big(du_0(\partial_s) + J(u_0) \circ du_0(\partial_t)\big) \\ &= \frac{1}{2}\big(\partial_s u_0 + J(u_0) \circ \partial_t u_0\big), \end{aligned} \tag{13}$$

where we used the usual identification of tangent vectors and partial derivatives. Similarily, one obtains:

$$\overline{\partial}_J(u_0)\partial_t = \frac{1}{2}\big(\partial_t u_0 - J(u_0) \circ \partial_s u_0\big). \tag{14}$$

Note that the change of sign is due to:

$$j(\partial_t) = -\partial_s. \tag{15}$$

From this we can now deduce that:

$$\overline{\partial}_J(u_0) = \frac{1}{2}\big(\partial_s u_0 + J(u_0) \circ \partial_t u_0\big)ds + \frac{1}{2}\big(\partial_t u_0 - J(u_0) \circ \partial_s u_0\big)dt, \tag{16}$$

because we know the behaviour of this one form on the basis ∂_s, ∂_t. Therefore, by (16), we see that u being J-holomorphic is equivalent to:

$$\partial_s u_0 + J(u_0)\partial_t u_0 = 0, \tag{17}$$

for all complex coordinate charts on Σ. Note the similarity of (17) to the usual Cauchy-Riemann-equations encountered in complex analysis. Indeed, if both Σ, M are complex manifolds, one sees easily that (17) is nothing else than the usual Cauchy-Riemann-equations in this situation and thus the J-holomorphic curves in a complex manifold with the induced almost complex structure are necessarily holomorphic.
Finally, we introduce the notion of energy of a J-holomorphic curve which will become essential in our study of sequences of J-holomorphic spheres:

Definition 2.5. *Let (M, ω) be an almost symplectic manifold with a ω-tame almost comlex structure J. Moreover, let $(\Sigma, j, dvol_\Sigma)$ be a compact Riemann surface with volume form $dvol_\Sigma$ and $u : \Sigma \to M$ be a smooth map. The* energy *of the map u is then defined by:*

$$E(u) := \frac{1}{2}\int_\Sigma |du|_{J,j} dvol_\Sigma \tag{18}$$

In this definition, we have used the following norm of the differential for every $x \in \Sigma$:

$$|du(x)|_{J,j} := |v|_j^{-1}\sqrt{|du(x)v|_J^2 + |du(x)j(x)v|_J^2}, \forall v \neq 0 \in T_xM. \tag{19}$$

Here, we use the definition of $|\cdot|_J$ as in (8) and note that one can similarily define a Riemannian metric on the Riemann surface leading to $|\cdot|_j$. The following argument shows that this norm is independent of the choice of vector $v \neq 0 \in T_xM$:
Note that if we take some $e \in T_x\Sigma$, then $e, j(x)e$ form a basis and by representing $v = \lambda e + \mu j(x)e$ as a linear combination, we see:

$$\begin{aligned} |du(x)v|_J^2 + |du(x)j(x)v|_J^2 &= |du(x)(\lambda e + \mu j(x)e)|_J^2 + |du(x)j(x)(\lambda e + \mu j(x)e)|_J^2 \\ &= |du(x)(\lambda e + \mu j(x)e)|_J^2 + |du(x)(\lambda j(x)e - \mu e)|_J^2 \\ &= \lambda^2(|du(x)e|_J^2 + |du(x)j(x)e|_J^2) + \mu^2(|du(x)e|_J^2 + |du(x)j(x)e|_J^2) \\ &= (\lambda^2 + \mu^2)(|du(x)e|_J^2 + |du(x)j(x)e|_J^2). \end{aligned} \tag{20}$$

Similarily, using orthogonality of j:

$$|v|_j^2 = (\lambda^2 + \mu^2)|e|_j^2. \tag{21}$$

Combining (20) and (21), we conclude that the expression is independent of the chosen vector v in the expression above.

The most important property of the energy is that it can be described by purely topological invariants of the manifolds as long as u is J-holomorphic and $J \in \mathcal{J}_\tau(M\omega)$:

Lemma 2.1. *Let ω be a symplectic form and J ω-tame. Then for every J-holomorphic curve $u : \Sigma \to M$, we find that the following equation holds:*

$$E(u) = \int_\Sigma u^*\omega = [u^*\omega] \cap [\Sigma] = [\omega] \cap u_*[\Sigma], \tag{22}$$

If J is even ω-compatible, then for every smooth map $u : \Sigma \to M$, one sees:

$$E(u) = \int_\Sigma |\overline{\partial}_J u|_{J,j}^2 dvol_\Sigma + \int_\Sigma u^*\omega \geq \int_\Sigma u^*\omega. \tag{23}$$

Note that this lemma implies that if J is ω-compatible, then J-holomorphic curves minimize energy among all smooth maps u representing the same homology class $A \in H_2(M, \mathbb{Z})$, i.e. among all maps u, such that $u_*[\Sigma] = A$. This is an immediate consequence of the equation (23).

Additionally, note that the energy is invariant with respect to confomal transformations on the base space. Namely, if we have a conformal, holomorphic map $\phi : \Sigma' \to \Sigma$, then for $v \neq 0$ and j' the complex structure on Σ':

$$\begin{aligned} |d(u\circ\phi)(x)|_{J,j'} &= |v|_{j'}^{-1}\sqrt{|d(u\circ\phi)(x)v|_J^2 + |d(u\circ\phi)(x)j'(x)v|_J^2} \\ &= |v|_{j'}^{-1}|d\phi(x)v|_j|d\phi(x)|_j^{-1}\sqrt{|du(\phi(x))d\phi(x)v|_J^2 + |du(\phi(x))d\phi(x)j'(x)v|_J^2} \\ &= |v|_{j'}^{-1}|d\phi(x)v|_j|d\phi(x)|_j^{-1}\sqrt{|du(\phi(x))d\phi(x)v|_J^2 + |du(\phi(x))j(x)d\phi(x)v|_J^2} \\ &= |du(\phi(x))|_{J,j}|d\phi(x)|_{j,j'}. \end{aligned} \tag{24}$$

Moreover, calculating the pull-back volume form gives:

$$\begin{aligned} \phi^* dvol_\Sigma &= dvol_\Sigma(d\phi\partial_s, d\phi\partial_t)ds \wedge dt \\ &= dvol_\Sigma(d\phi\partial_s, d\phi j'\partial_s)ds \wedge dt \\ &= dvol_\Sigma(d\phi\partial_s, jd\phi\partial_s)ds \wedge dt \\ &= |d\phi|_{j,j'}^2 dvol_{\Sigma'}. \end{aligned} \tag{25}$$

Now, calculating the energy of the map $u \circ \phi$, we see:

$$\begin{aligned} E(u \circ \phi) &= \frac{1}{2} \int_{\Sigma'} |d(u \circ \phi)(x)|_{J,j'} dvol_{\Sigma'} \\ &= \frac{1}{2} \int_{\Sigma'} |du(\phi(x))|_{J,j} |d\phi(x)|_{j,j'} |d\phi|^{-2}_{j,j'} \phi^* dvol_{\Sigma} \\ &= \frac{1}{2} \int_{\Sigma'} |du(\phi(x))|_{J,j} |d\phi(x)|^{-1}_{j,j'} \phi^* dvol_{\Sigma} \\ &= \frac{1}{2} \int_{\Sigma} |du(x)|_{J,j} dvol_{\Sigma} \\ &= E(u), \end{aligned} \tag{26}$$

where we used a change of variables in order to change domains. Thus we deduce the desired invariance under conformal transformations. Note that this also easily follows from the formula given in the previous Lemma.

Lastly, we want to point out how energy will become important in our considerations. Later in the paper, we will study convergence of sequences of J-holomorphic curves and this leads naturally to stable maps. More precisely, using elliptic regularity, one can derive conditions (i.e boundedness of derivatives) under which sequences of J-holomorphic curves possess a converging subsequence in the C^{∞}-topology. If this condition is violated however, we can still say something about a more general convergence behaviour and thus about a "limiting curve". At any point where the derivatives are unbounded, bubbling occurs. Thus new J-holomorphic curves are formed leading to stable maps and the energy spreads among the bubbles and the limiting curve which exists on the complement of the points with unbounded derivatives by the convergence result. The energy will be a key ingredient in proving features of the bubbles such as connections of bubbles to the original curve which is due to energy bounds. Moreover, we will use energy and bubbling arguments to prove that certain Moduli spaces are compact. These topics will come up later in this paper.

2.3 Main properties

Even though almost complex manifolds are generally not complex, J-holomorphic curves share many features with holomorphic maps. Among these features is a unique continuation properties as well as a local representation lemma reminiscent of holomorphic curves. Thus some of the intuition from complex manifolds and complex analysis carries over to the more general setting of almost complex manifolds and J-holomorphic curves.
Before entering the discussion, we want to point out as in [8], that most of the results in this subsection are stated in greater generality than is actually needed in our applications and all statements together with their proofs can be found in [8, Chapter 2]. Nevertheless, we will often use the corresponding statements in situations where the almost complex structure is not smooth, hence we will need these more general statements instead of the smooth results. This again is due to the fact that when studying Moduli spaces, one abandons the smoothness assumption to work over Sobolev spaces and thus we generally have much less regularity guaranteed.
As suggested, there is a kind of link between J-holomorphic curves and holomorphic ones which allows us in certain situations to generalize results for holomorphic maps. This is essentially due to the *Caleman Similarity principle* which ensures that J-holomorphic curves can be locally represented by holomorphic maps modulo some modifications:

Theorem 2.4 (Carleman Similarity Principle)**.** *Let $p > 2$, $C(z) \in L^p(B_\epsilon(0), End(\mathbb{R}^{2n}))$ and $J \in W^{1,p}(B_\epsilon(0), GL_{2n}(\mathbb{R}))$ such that $J^2 = -Id$. Moreover, assume that $u \in W^{1,p}(B_\epsilon(0)\mathbb{R}^{2n})$*

with $u(0) = 0$ solves the first order elliptic equation, under the identification $z = s + it$:

$$\partial_s u(z) + J(z)\partial_t u(z) + C(z)u(z) = 0. \tag{27}$$

Then there exist $0 < \delta < \epsilon$, $\Phi \in W^{1,p}(B_\delta(0), Hom_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R}^{2n}))$ and a holomorphic map $\sigma : B_\delta(0) \to \mathbb{C}^n$ with $\sigma(0) = 0$, such that $\Phi(z)$ is invertible for all $z \in B_\delta(0)$ and:

$$u(z) = \Phi(z)\sigma(z), \quad \Phi(z)^{-1}J(z)\Phi(z) = i, \tag{28}$$

where i just denotes the usual endomorphism induced by the complex-linear structure on $\mathbb{C}^n$.

This theorem can be applied to J-holomorphic curves by defining $\tilde{J}(z) := J(u(z))$ for some given J-holomorphic curve u. Note that by taking local coordinates the conditions of the theorem are satisfied if J is sufficiently smooth.

For the next theorem, the notion of *vanishing to infinite order* is important:

Definition 2.6. *A integrable function $u : B_\epsilon(0) \to \mathbb{C}^n$ is said to* vanish to order infinity at 0*, if:*

$$\int_{|z|\leq r} |u(z)| = O(r^k), \forall k \in \mathbb{N}. \tag{29}$$

We note that vanishing to order infinity at 0 for some smooth function u is equivalent to saying that all its derivatives up to order infinity vanish at 0. This is simply due to Taylor expansion Namely, assume that $u^{(k)}(0) = 0, \forall k \leq n$, but $u^{(n+1)}(0) \neq 0$. By Taylor expansion, we therefore find:

$$u(z) = O(|z|^{n+1}), \tag{30}$$

which in turn implies for $k = n+2$ that the desired asymptotic behaviour cannot hold. Thus we reach a contradiction.
This integral formulation of vanishing becomes important in proving the following property in full generality:

Theorem 2.5. *Let Σ be a connected Riemann surface and J be a C^1-almost complex structure. Moreover, assume that $u, v : \Sigma \to M$ are J-holomorphic curves, such that $u-v$ vanishes to order infinity at some point in local coordinates, i.e. u, v agree up to order infinity at some point in Σ. Then $u = v$.*

Note that if we would replace almost complex with complex, the statement would be trivial by just considering local representations of the curves and applying the corresponding well-known theorem from complex analysis. In this case, the proof requires a bit more care, especially to deal with the possible lack of regularity due to J just being C^1.
There are essentially two ways to prove the result: Either by using an inequality and invoking Aronszajn's Theorem or by applying the Carleman Similarity principle as stated in Theorem 2.4. Note that due to connectedness of Σ, the result reduces to the usual kind of connectedness argument using local representations making the proof accessible by Theorem 2.4:

Proof. Our proof here follows the one given in [8] closely. We first show a local version for maps $u, v : B_\epsilon(0) \to \mathbb{R}^{2n}$ which are J-holomorphic with respect to an C^1-almost complex structure on $\mathbb{R}^{2n}$. Note that by expressing everything locally, we reach this situation in the Theorem we want to prove.

Assume that $w = u - v$ vanishes to order infinity at 0. We want to apply the Carleman Similarity principle. Therefore, we need to find a almost complex structure defined on $B_\epsilon(0)$ and we choose $\hat{J}(z) := J(u(z))$ as suggested before. Note that $\hat{J}$ satisfies the regularity conditions as well as the almost complex-condition. We thus need to determine $C(z)$ and for this, we calculate:

$$\begin{aligned}\partial_s w + \hat{J}(z)\partial_t w &= \partial_s u + J(u(z))\partial_t u - (\partial_s v + J(u(z))\partial_t v)\\ &= \partial_s u + J(u(z))\partial_t u - (\partial_s v + J(v(z))\partial_t v) + (J(v(z)) - J(u(z)))\partial_t v\\ &= (J(v(z)) - J(u(z)))\partial_t v, \end{aligned} \tag{31}$$

where we used that both u, v are J-holomorphic. Note that the remaining expression should be $C(z)w(z)$. By using the mean value theorem, we see:

$$\begin{aligned}(J(v(z)) - J(u(z)))\partial_t v &= \Big(\int_0^1 dJ(u(z) + \tau(v(z) - u(z)))(v(z) - u(z))d\tau\Big)\partial_t v(z)\\ &= -\Big(\int_0^1 dJ(u(z) - \tau w(z))w(z)d\tau\Big)\partial_t v(z), \end{aligned} \tag{32}$$

which now has the desired form. One sees that by defining $C(z)$ accordingly, the required integrability holds due to continuity of $C(z)$.
Therefore, we can apply the Carleman-Similarity principle for any point inside the ball and hence see $u(z) = \Phi(z)\sigma(z)$ for some σ holomorphic. Note that due to the Sobolev embeddings and $p > 2$, Φ is continuous.
The result follows now by the usual connectedness argument. Let $z_0 \in B_\epsilon(0)$ be a point where u vanishes to order infinity, Applying Carleman's similarity principle around this point, we see that the corresponding holomorphic map σ vanishes to order infinity at 0. Thus it is constant in the neighbourhood, leading to openness of the set of points in the ball where u vanishes to order infinity. Closedness follows by considering a converging subsequence, taking a Carleman representation of u in a neighbourhood of the limit point and noticing that thus σ vanishes on a converging subsequence. Thus by complex analysis, σ is constant in a neighbourhood and thus u vanishes to order infinity in the limit point. Therefore, by connectedness, we conclude that the result holds. □

In a similar manner, other Theorems reminiscent of results from complex analysis carry over to almost complex situations. For example, we see that critical values are isolated:

Theorem 2.6. *Assume that Σ is a compact Riemann surface and $u : \Sigma \to M$ is a non-constant J-holomorphic curve of class C^1 for the C^1-almost complex structure J. Then the set:*

$$u^{-1}\big(\{u(z)|z \in \Sigma, du(z) = 0\}\big),$$

is finite and thus a discrete subset of Σ. Moreover, $u^{-1}\big(\{u(z)\}\big)$ is finite for all $z \in \Sigma$.

There are two ways to prove this result, either by Carleman for the general statement or by using Taylor polynomial expansions at critical points and deducing that the Taylor polynomial is holomorphic if one just uses Taylor expansion to the order of vanishing. This suffices due to asymptotic properties of Taylor expansion and hence we can conclude that the critical points are isolated, similar reasoning shows that the preimage of any point is finite. We note however, that the second approach involving Taylor series requires smoothness of the objects involved and thus does not prove Theorem 2.6 in full generality.

We now show that there are "preferred" coordinate charts for a given J-holomorphic curve around non-critical points:

Theorem 2.7. *Assume that J is a C^l-almost complex structure on M and $u : \Omega \to M$ is a J-holomorphic curve defined on some neighbourhood $0 \in \Omega \subset \mathbb{C}$. Assume that $du(0) \neq 0$,* *then there exists a C^{l-1}-coordinate chart $\psi : U \subset M \to \mathbb{C}^n$ defined in a neighbourhood of $u(0)$, such that:*

$$\psi(u(z)) = (z, 0, \dots, 0), \quad d\psi(u(z)) \circ J(u(z)) = J_0 d\psi(u(z)), \quad \forall z \in \Omega \cap u^{-1}(U), \tag{33}$$

where J_0 denotes the standard almost complex structure on $\mathbb{C}^n$.

The proof relies on choosing a complex frame and applying a standard argument involving the exponential map. Using Theorem 2.7, we are able to deduce further properties. For example, if two J-holomorphic curves assume "similar" values in a neighbourhood of some point and one of the curves has non-vanishing derivative at this point, then they agree up to composition with a holomorphic map. More precisely, it holds:

Theorem 2.8. *Let J be a C^2-almost complex structure on M and $u, v : \Omega \subset \mathbb{C} \to M$ J-holomorphic curves, $0 \in \Omega$, such that $u(0) = v(0), du(0) \neq 0$. Furthermore, let $(z_n), (w_n) \subset \Omega$ be sequences converging to 0, such that:*

$$u(z_n) = v(w_n), w_n \neq 0, \forall n \in \mathbb{N}.$$

Then there is a holomorphic function $\phi : B_\epsilon(0) \subset \Omega \to \Omega$, such that $\phi(0) = 0$ and:

$$v = u \circ \phi.$$

The proof of this relies on choosing coordinate charts as described in Theorem 2.7 and apply an estimate on the function described by the local expressions. This will then imply, that the local expression of v vanishes in the last $2n-2$ coordinates, thus lying in the image of u. Therefore, the result follows in the case of sufficient regularity. The general case follows again by using Carleman's Similarity principle.

Next, we want to consider the set of common values of two J-holomorphic curves in even more detail and therefore deduce the following result:

Theorem 2.9. *Let J be a C^2-almost complex structure on M and Σ_0, Σ_1 compact and connected Riemann surfaces without boundary. Furthermore, let $u_j : \Sigma_j \to M$, $j \in \{0, 1\}$, be J-holomorphic curves, such that u_0 is non-constant and $u_0(\Sigma_0) \neq u_1(\Sigma_1)$. Then $u_0^{-1}\big(u_1(\Sigma_1)\big)$ is at most countable and can only accumulate at critical points of u_0.*

The main idea of the proof is to invoke a connectedness argument and use contradiction, i.e. assuming that the preimage accumulates at a non-critical point. The key ingerdient is that a neighbourhood of some point in Σ_0 gets mapped to $u_1(\Sigma_1)$ exactly if there is a sequence $z_n \neq z_0$ converging to z_0, such that $u_0(z_n) \in u_1(\Sigma_1), \forall n \in \mathbb{N}$. The reason this holds is due to Theorem 2.8 which guarantees this under the assumption of non-vanishing differential. Thus by removing the points which have vanishing differential (note that there are only finitely many such due to Theorem 2.6), we still have a connected Riemann surface and can thus apply the outlined connectedness argument leading to $u_0(\Sigma_0) \subset u_1(\Sigma_1)$. By exchanging roles of Σ_0 and Σ_1, we get the result as the converse inclusion follows similarily.

To finish this section, we introduce the notion of simple J-holomorphic curves. This kind of J-holomorphic curves will be the space of curves we build our moduli spaces upon. Therefore, it will be of interest to be able to recognize simple curves. For this we will introduce a necessary condition after stating the basic definition:

Definition 2.7. *Let (M, J) be almost complex and (Σ, j) a compact Riemann surface. A J-holomorphic curve $u : \Sigma \to M$ is called* multiply covered*, if there exists another compact*

Riemann surface (Σ', j') *and a J-holomorphic curve* $u' : \Sigma' \to M$ *as well as a holomorphic branched covering* $\phi : \Sigma \to \Sigma'$ *with* $deg(\phi) > 1$*, such that:*

$$u = u' \circ \phi.$$

If u is not multiply covered, it is called simple.
Finally, u is called somewhere injective*, if* $\exists z \in \Sigma$*:*

$$du(z) \neq 0, u^{-1}\big(\{u(z)\}\big) = \{z\}.$$

Such points are called injective*.*

Note that a *holomorphic branched covering* $\phi : \Sigma \to \Sigma'$ is a holomorphic map ϕ which is a covering map after removing a nowhere dense subset of Σ.

At this point, we want to give two easy examples to illustrate the notions introduced. First, consider the identity map $u : S^2 \to S^2$. It is clear that this map is a simple J-holomorphic curve due to its degree being 1. We remind ourselves that the almost complex structure stems from the complex one on $\mathbb{CP}^1$.
On the other hand, if we consider the map $v : S^2 \to S^2$ which maps a point

$$(\sqrt{1-h^2}\cos\psi, \sqrt{1-h^2}\sin\psi, h)$$

to another the one on S^2 given in cylindical coordinates by

$$(\sqrt{1-h^2}\cos 2\psi, \sqrt{1-h^2}\sin 2\psi, h).$$

We note, due to expressing everything in local coordinates, this map is holomorphic and thus a J-holomorphic curve. Moreover, the map is clearly multiply cover by letting $v' = id$ and $\phi = v$ and noting that $deg(v) = 2$. Thus we have a multiply covered J-holomorphic curve. Finally, note that the map v is no covering but a branched covering by ignoring the poles of the sphere.

As promised, we turn now to a criterion for simplicity of J-holomorphic curves:

Theorem 2.10. *Let J be a C^2-almost complex structure on M and Σ a compact Riemann surface without boundary. If a J-holomorphic curve $u : \Sigma \to M$ is simple, then u is somewhere injective. Moreover, the set of non-injective points is at most countable and can only accumulate at critical points of u.*

We briefly return to our two examples above. For the identity map on the sphere, we note that all points are in fact injective, whereas for the rotation map only the poles could be injective. Thus by the result above we could also conclude that the rotation map is not simple.

Finally, we can deduce some uniqueness statements for simple J-holomorphic curves:

Corollary 2.1. *Let J be a C^2-almost complex structure on M and Σ_0, Σ_1 compact, connected Riemann surfaces without boundary and $u_j : \Sigma_j \to M$ be simple J-holomorphic curves for $j \in \{0, 1\}$, such that:*

$$u_0\big(\Sigma_0\big) = u_1\big(\Sigma_1\big).$$

Then there exists a holomorphic diffeomorphism $\phi : \Sigma_0 \to \Sigma_1$, such that:

$$u_1 \circ \phi = u_0.$$

Note that this corollary states that simple J-holomorphic maps on compact, connected Riemannian surfaces are characterised by their image. Thus two such maps with the same image are just holomorphic reparametrisations from one another. In the context of $\Sigma_0 = \Sigma_1 = S^2$, this means that if we have two J-holomorphic spheres in a manifold M with the same image, then they just differ by precomposition with a Möbius transformation. This will be used to identify the space of simple, unparametrised J-holomorphic spheres with the quotient of all simple J-holomorphic spheres under the natural identification provided by the action of the group of Möbius transformations on these maps.

The proof of this corollary relies on defining the biholomorphic map directly on the set of common injective points by using that the images coincide. The resulting map is holomorphic due to non-injective points only accumulating at critical points, thus injective points having open neighbourhoods containing only injective points. From the inverse function theorem, we conclude that we have a holomorphic diffeomorphism and by the removal of singularities, we can extend the map. The result follows.

Corollary 2.2. *Let J be a C^2-almost complex structure on M and $\Sigma_0, \dots, \Sigma_N$ compact, connected Riemann surfaces without boundary and $u_j : \Sigma_j \to M$ be J-holomorphic curves for $j \in \{0, \dots, N\}$, such that u_0 is simple and:*

$$u_0(\Sigma_0) \neq u_j(\Sigma_j), \forall j \in \{1, \dots, N\}.$$

Then, for every $z_0 \in \Sigma_0$ and every open neighbourhood $z_0 \in U_0 \subset \Sigma_0$, there is an annulus $A_0 \subset U_0$ centered around z_0, such that $u_0 : A_0 \to M$ is an embedding and:

$$u_0^{-1}\big(u_0(A)\big), u_0(A_0) \cap u_j(\Sigma_j) = \emptyset, \forall j \in \{1, \dots, N\}.$$

Instead of proving all the results mentioned up to now, we will outline a proof of the following statement. The reason for this is that the actual proof requires the same ideas as used for example in the proof of the existence of adapted local coordinates:

Proposition 2.1. *Let Q be a compact codimension 2 submanifold of an almost complex manifold (M, J), such that $J(TQ) = TQ$. Moreover, let $u : B_1(0) \to M$ be a J-holomorphic curve such that $u(0) \in Q$. Then the point of intersection is either isolated or $u(B_1(0)) \subset Q$.*

This result is Exercise 2.6.1.i) in [8, p.35]. The idea is that the proof of this theorem uses a coordinate representation, which can be adapted to prove Theorem 2.7, and an argument which is used in a similar manner to show Theorem 2.8. Thus by giving the proof of this exercise instead, we basically provide a proof of these two Theorems as well.

Proof. We just give the main ideas involved: First, we want to find coordinate charts adapted to the almost complex submanifold Q. In order to do this, choose locally around the point $u(0)$ a complex frame on said submanifold and extend it to a local family of bases by adding one more complex section. Using the exponential map appropriately in a neighbourhood around the point $u(0)$. The remainder of the proof consists of invoking a Theorem by Aronszajn, see [8], in order to deduce the statement. □

2.4 Elliptic Regularity

To end our preliminary investigation of J-holomorphic curves, we state the fundamental regularity and compactness results for J-holomorphic curves as presented in [8, Chapter 4]. These results are based on elliptic regularity estimates as described in [8] in Appendix B. We assume that M is a smooth $2n$-manifold with an almost complex structure $J \in \mathcal{J}^l(M)$ and Σ a complex Riemann surface with boundary and complex structure $j \in \mathcal{J}(\Sigma)$. Moreover, we take a n-submanifold $L \subset M$ which is totally real with respect to J, i.e. $TL \cap J(TL) = \{0\}$. The main results used later on are the following two:

Theorem 2.11 (Regularity)**.** *Let $l \geq 2, p > 2$ be integers, $J \in \mathcal{J}^l(M)$ and $u : \Sigma \to M$ a $W^{1,p}$-function such that:*

$$du \circ j = J \circ du, \tag{34}$$

and $u(\partial\Sigma) \subset L$. Then $u \in W^{l,p}$ and if $l = \infty$, then u is smooth.

Theorem 2.12 (Compactness)**.** *Let $l \geq 2, p > 2$ be integers, $J_k \in \mathcal{J}^l(M)$ a sequence of almost complex structures converging in the C^l-topology to some $J \in \mathcal{J}^l(M)$. Similarily let $j_k \in \mathcal{J}(\Sigma)$ be a sequence of complex structures on Σ converging in the C^∞-topology to the complex structure $j \in \mathcal{J}(\Sigma)$. Furthermore, assume that $U_k \subset \Sigma$ is an increasing sequence of open sets with covering Σ and $u_k : U_k \subset \Sigma \to M$ (J_k, j_k)-holomorphic curves in $W^{1,p}$ such that $\forall k \in \mathbb{N}$:*

$$u_k(\partial\Sigma \cap U_k) \subset L. \tag{35}$$

If for every compect subset $Q \subset \Sigma$ with smooth boundary, there exists a compact subset $K \subset M$ and a constant $c > 0$, such that for all $k \in \mathbb{N}$ sufficiently big:

$$\|du_k\|_{L^p(Q)} \leq c, \quad u_k(Q) \subset K, \tag{36}$$

then there is a subsequence of $(u_k)_{k\in\mathbb{N}}$ which converges in the C^{l-1}-topology on compact subsets of Σ to a J-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (M, L)$.

These results rely on methods of elliptic regularity and Sobolev spaces applied in the context of J-holomorphic curves.

The considerations split into three parts:

1. Treat local J-holomorphic curves defined on open sets in $\Omega \subset \mathbb{C}$ with image in $\mathbb{R}^{2n}$ with an almost complex structure. An important idea is to consider J-holomorphic curves there as solutions of a linear partial differential equation. Note that if $u : \Omega \to \mathbb{R}^{2n}$ is a weak J-holomorphic for the standard almost complex structure, then:

$$\partial_s u + J(u)\partial_t u = 0. \tag{37}$$

   Note that this PDE is non-linear due to J depending on u. But if we define $J'(z) := J(u(z))$, then u also solves the linear PDE:

$$\partial_s u + J'\partial_t u = 0, \tag{38}$$

   but note that the regularity of u now restricts regularity of J'. In order to apply some iterative arguments, it is more convenient to consider the inhomogenous PDE:

$$\partial_s v + J'\partial_t v = \nu, \tag{39}$$

   because by direct calculation, if u is J-holomorphic then $\partial_s u$ solves the inhomogenous equation with $\nu = -(\partial_s J')\partial_t u$, enabling us to use elliptic bootstrapping methods to increase regularity.

2. One now proceeds to consider the inhomogenous linear PDE as given before and look for weak solutions of it. By appropriately choosing test functions, one can derive regularity from corresponding results for the Laplace operator. This gives then a first basic estimate. By an iterative argument one can thus obtain higher regularity using Sobolev embeddings in order to reach better integrability properties.

3. Finally, we apply the previous considerations to J-holomorphic curves in a symplectic manifold by using appropriate coordinate charts and use the Sobolev embedding theorems to deduce our main results. Note that increasing regularity of u by considering J' as defined before, we can also improve regularity of J' which again enables us to improve regularity of u.

For more details, we refer to [8, p.572-577].

We want to point out that similar arguments involving elliptic regularity can be invoked in the study of further objects like Floer homology. The main idea there is to consider the Floer equation in a similar manner as we consider the generalized Cauchy-Riemann equations and apply Sobolev theory in order to get regularity and compactness results which are of interest in this situation. For further informations on this, we refer to [1].

3 Banach Manifolds

In this chapter we will introduce the necessary vocabulary surrounding Banach manifolds. Banach manifolds will play an important role when analyzing Moduli spaces of J-holomorphic curves as generalisations of well-known theorems from finite-dimensional calculus like the Implicit Function Theorem, which will be useful in our studies, only hold in the context of Banach manifolds. The main idea is that we can show that the Moduli spaces are finite dimensional manifolds by using the Implicit Function Theorem for Banach manifolds which in turn, as in the finite dimensional case, is just a local application of the Implicit Function Theorem for Banach spaces.
We start by defining differentiability on Banach spaces and stating some of its properties reminiscent of the usual properties of differentiability encountered in the finite-dimensional context. Similar to the finite-dimensional setting, differentiability can be derived from sufficiently regular directional derivatives as seen in the next section. Next, we introduce the notion of a Banach manifold and give the two most important examples in our situation: generalized Sobolev spaces for sections in vectorbundles and maps between manifolds. These spaces will naturally include the J-holomorphic curves and thus enable us to apply the theorems from this section. Afterwards, we state the main results which mostly carry over from theorems from calculus. It is important that we can generalize Sard's theorem as it provides us with a tool to prove genericity of certain properties. In our applications, we will use this to deduce that regularity of almost complex structures is a generic property, but more on these notions later.

3.1 Differentiability on Banach spaces

First, let us state the formal definition of differentiability in analogy to the well-known definition for finite-dimensional vector spaces.

Definition 3.1 (Differentiability)**.** *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed vector spaces, $U \subset X$ an open subset of X in the norm topology and $f : U \to Y$ a map. Moreover, let $x_0 \in U \subset X$ be any point. Then f is called* differentiable at x_0 with differential $T \in L(X, Y)$ *if:*

$$\lim_{\|h\|_X \to 0} \frac{1}{\|h\|_X} \big(f(x_0 + h) - f(x_0) - Th \big) = 0. \tag{40}$$

We write $df(x) = T$.
The map f is called differentiable*, if f is differentiable at each point $x \in U \subset X$ and we then call the map:*

$$df : U \to L(X, Y), x \mapsto df(x), \tag{41}$$

mapping each point $x \in U$ to the derivative $df(x)$ at the point x the differential *of f. Moreover, the map f is called* continuously differentiable *if the map df is continuous with respect to the operator norm.*

We note that in the case of X, Y being finite-dimensional, normed vector spaces, this definition coincides with the usual notion of differentiability. Moreover, higher orders of differentiability are defined in the usual, iterative way and so is smoothness of maps. Note for this that the space of continuous linear maps forms a Banach space when equipped with the operator norm.

We remark that many of the previously known results from finite-dimensional calculus carry over to this situation such as the chain rule. Namely, we want to mention the following results from [11]:

Theorem 3.1. *Let X, Y, Z be Banach spaces and $U \subset X$, $V \subset Y$ open subsets. Then the following results hold:*

- *If $f, g : U \to Y$ are differentiable in $x_0 \in U$, then so are $f + g$ and λf, for $\lambda \in \mathbb{R}$ and we have:*
$$d(f+g)(x_0) = df(x_0) + dg(x_0), \quad d(\lambda f)(x_0) = \lambda df(x_0), \tag{42}$$
i.e. linearity generalizes to our more general notion of differentiability.

- *If $f : u \to Y$, $g : V \to Z$ with $f(U) \subset V$ are differentiable in x_0 and $f(x_0)$ respectively, then $g \circ f$ is differentiable in x_0 with differential:*
$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0), \tag{43}$$
i.e. the chain rule generalises to this setting.

The proofs of these two properties are straight-forward generalisations of the proofs in the finite-dimensional setting. For more details consult [11], chapter 3.

Next, we give a simple example of a differentiable map to illustrate this concept, namely let X be a Hilbert space and $\langle \cdot, \cdot \rangle$ denote the scalar product of X. We show that the following map:
$$f : X \to \mathbb{R}, x \mapsto \langle x, x \rangle = \|x\|_X^2, \tag{44}$$
is differentiable. For this, we first let $h \in X$ and calculate:
$$\begin{aligned} f(x+h) - f(x) &= \|x+h\|_X^2 - \|x\|_X^2 \\ &= \langle x+h, x+h \rangle - \langle x, x \rangle \\ &= 2\langle x, h \rangle + \|h\|_X^2, \end{aligned} \tag{45}$$
where we used symmetry and bilinearity. Note that $h \mapsto 2\langle x, h \rangle$ is a continuous linear operator for any $x \in X$. Thus we want to prove that this is actually the differential $df(x)$, i.e., we need to see:
$$\lim_{\|h\|_X \to 0} \frac{f(x+h) - f(x) - df(x)h}{\|h\|_X} = \lim_{\|h\|_X \to 0} \frac{\|h\|_X^2}{\|h\|_X} = \lim_{\|h\|_X \to 0} \|h\|_X = 0. \tag{46}$$
By definition, we thus see that f is differentiable everywhere with the differential defined above. Moreover, one immediately sees that the map is even smooth, as the differential defines a linear map, hence smooth.

For completeness, we want to note that continuous multilinear maps define smooth maps. This can be proven in a similar way as in the finite dimensional case and also similar to our proof in the case of the scalar product above. Thus especially continuous linear operators define smooth maps as well as continuous bilinear maps.

Lastly, we want to present a generalisation of an important theorem form finite dimensional calculus which enables us to reduce differentiability considerations to directional derivatives. For this, we say that a map $f : U \subset Y$ between Banach spaces is *Gateaux-differentiable* in x_0, if there is a continuous linear operator $T : X \to Y$, such that:
$$\lim_{h \to 0} \frac{f(x+hv) - f(x)}{h} = T(v), \tag{47}$$
for any $v \in X$ and $h \in \mathbb{R}$ small enough. Thus directional derivatives in any direction exist and they satisfy some linear compatibility relation described by the continuous linear operator T. We then call T the *Gateaux-differential* of f in x_0 and also denote it by $df(x_0)$. If f is Gateaux-differentiable in any point of $U \subset X$ it defines a map:
$$df : U \to L(X, Y). \tag{48}$$

Note that if f is differentiable in x_0, then it is Gateaux-differentiable in x_0. The converse is generally false, as the convergence need not be uniform as required in the definition of differentiability. But similar to the finite-dimensional setting, we can get a partial converse by requiring continuity of the Gateaux-differential mapping:

Theorem 3.2. *If $f : U \subset X \to Y$ is Gateaux-differentiable in U and $df : U \to L(X,Y)$ is continuous, then f is differentiable in U with differential $df(x_0)$ in $x_0 \in U$ and thus f is even continuously differentiable.*

The main idea of the proof is to use the norm characterisation in normed vector spaces using duality (Hahn-Banach Theorem, see [11]) and to apply the mean value theorem in finite dimensions. The result is taken from [11] as the previous results in this section. Moreover, the proof presented below is again taken from [11, p.121]:

Proof. Let $x_0 \in U$ be given and take $h \in X$ small enough. We want to estimate:

$$\|f(x_0 + h) - f(x_0) - df(x_0)h\|. \tag{49}$$

First, note that there exists a $y' : Y \to \mathbb{R}$, such that the functional has operatornorm $\|y'\| = 1$ and:

$$\|f(x_0 + h) - f(x_0) - df(x_0)h\| = y'(f(x_0 + h) - f(x_0) - df(x_0)h), \tag{50}$$

due to the Hahn-Banach theorem. We now want to consider the following map:

$$g : [0,1] \to \mathbb{R}, t \mapsto y'(f(x_0 + th) - f(x_0)). \tag{51}$$

Note that we can now apply the usual mean value theorem as well as the chain rule in order to see:

$$\exists c \in (0,1) : \frac{g(1) - g(0)}{1 - 0} = y'(f(x_0 + h) - f(x_0)) = g'(c), \tag{52}$$

where:

$$\forall t \in (0,1) : \frac{d}{dt} g(t) = y'(df(x_0 + th)h). \tag{53}$$

which follows by smoothness of linear maps as well as the chain rule.

Therefore, we find:

$$\begin{aligned} \|f(x_0 + h) - f(x_0) - df(x_0)h\| &= y'(f(x_0 + h) - f(x_0) - df(x_0)h) \\ &= y'(df(x_0 + ch)h - df(x_0)h) \\ &\leq |y'(df(x_0 + ch)h - df(x_0)h)| \\ &\leq \|y'\|\|df(x_0 + ch) - df(x_0)\|\|h\| \\ &= \|df(x_0 + ch) - df(x_0)\|\|h\|. \end{aligned} \tag{54}$$

The proof follows now by taking for a given $\epsilon > 0$ some $\delta > 0$ small enough, such that for all $x \in X$ with $\|x - x_0\| < \delta$, we have:

$$\|df(x) - df(x_0)\| < \epsilon, \tag{55}$$

which is possible due to continuity of the Gateaux-differential. Taking $\delta \leq 1$ and h from before with $\|h\| < \delta$, we get the desired result by the inequality above:

$$\begin{aligned} \frac{\|f(x_0 + h) - f(x_0) - df(x_0)h\|}{\|h\|} &\leq \frac{\|df(x_0 + ch) - df(x_0)\|\|h\|}{\|h\|} \\ &= \|df(x_0 + ch) - df(x_0)\| \\ &< \epsilon \end{aligned} \tag{56}$$

if $\|h\| < \delta$. □

3.2 Banach Manifolds

In order to deal with Moduli spaces of J-holomorphic curves, we will need to study Banach manifolds in order to be able to understand the Sobolev spaces involved in the arguments. Some important results for Banach manifolds which will play an important role in our situation will be presented in the next subsection. But first we define:

Definition 3.2. *A* Banach manifold *is a topological space M with an atlas $(U_\alpha, \phi_\alpha, E_\alpha)_{\alpha \in A}$ of coordinate maps with $U_\alpha \subset M$ open, E_α Banach spaces and:*

$$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset E_\alpha, \tag{57}$$

*where $\phi_\alpha(U_\alpha) \subset E_\alpha$ is open and such that all coordinate changes $\phi_\alpha \circ \phi_\beta^{-1}$ are differentiable, $\forall \alpha, \beta \in A$. It is a C^l-*Banach manifold *if all coordinate changes are of class C^l and* smooth *if all coordinate changes are smooth.*

We use the same notion of atlas as in the finite dimensional case to denote an open covering of coordinate maps with sufficient smoothness.

We note that the notions of differentiable and smooth maps carry over to this setting by using local coordinates as in the case of finite-dimensional manifolds. Moreover, we immediately get quite a few examples by just considering any Banach space X together with the norm topology and the trivial atlas given by the single coordinate chart (X, id, X). But there are also further examples and we want to provide some which will be of interest in the rest of this paper:

Example 3.1. *Let M be a compact, smooth n-manifold and $p : E \to M$ a smooth vector bundle over M. We can then define a generalised notion of Sobolev space under these circumstances. Namely, we call a section:*

$$s : M \to E \tag{58}$$

to be of class $W^{k,p}$ for $k \in \mathbb{N}, p \in [1, \infty]$, if all its local coordinate representations are in $W^{k,p}$. Note that due to the change of variables formula for Sobolev spaces, this is well-defined and this space naturally inherits a vectorspace structure by using pointwise operations. We call this space $W^{k,p}(M, E)$.
We define now a norm on this space by using the Sobolev norms in coordinate charts. Therefore, take finitely many coordinate charts with bounded coordinate domains, due to compactness of M, which cover M and define the norm by taking the sum of the finitely many local Sobolev norms. It is immediate that the resulting function is a norm. Moreover, we see that if we choose different coordinate charts, we end up with an equivalent norm. This follows by considering coordinate changes between both coverings and using the change of variables formula for Sobolev space. Finally, we can show that this space is complete, hence even a Banach space and thus a Banach manifold. For this, choose a Cauchy sequence and observe that due to the definition of the norm, the sequence converges in local coordinates to functions. We need to show that we can patch them together to get a global limit. But note that the limit is well-defined globally by using changes of variables between the charts and convergence of the restricted Sobolev maps. Thus we really have a converging sequence.

Example 3.2. *We can further extend the previous considerations to introduce Sobolev spaces of maps between closed, smooth manifolds M, X. However, this requires a bit more care. Namely, we first need to assume that $kp > \dim X$ in order to be able to get a well-defined notion. More precisely, we define the space $W^{k,p}(X, M)$ to be the completion of the set of smooth maps $C^\infty(X, M)$ with respect to a Sobolev-distance function obtained by using an embedding $M \hookrightarrow \mathbb{R}^N$. This implies that all Sobolev-maps in local coordinates belong to the*

respective Sobolev spaces on their coordinate domains. In order for the coordinate changes to preserve weak derivatives, we need to assume $kp > dim\ X$*.*
Our main interest lies in defining a Banach manifold structure on this space and we do so by using the Sobolev space $W^{k,p}(X, u^*TM)$ *for some smooth* $u : X \to M$*. We then define coordinates by taking the map:*

$$W^{k,p}(X, u^*TM) \to W^{k,p}(X, M), \quad \xi \mapsto exp_u(\xi), \tag{59}$$

which indeed yields a continuous, injective map between these spaces. One can check that it even is a homeomorphism. Additionally, changes of coordiantes yield smooth maps and thus the resulting atlas provides a smooth Banach manifold structure on the space of Sobolev maps between the manifolds X *and* M*.*
A bit more precisely, we first note that the coordinate charts are surely well-defined and injective. The first observation is due to passing to local coordinates and using the smoothness of the exponential. The second one follows from the injectivity of the exponential. Next, we want to show that it is continuous, thus we use an embedding of M *into* $\mathbb{R}^N$ *as described earlier and just need to show continuity with respect to the Sobolev distance function there. Again, using finitely many coordinate charts on X, we can check this claim.*
The most important and difficult part is to check that the maps are homeomorphisms onto some open subset. For this, we check that the image of any open set is a neighbourhood to all its points. The argument then uses local diffeomorphism properties of the exponential to conclude that it is a homeomorphism on some neighbourhood of the zero section. By density of the smooth maps in the Sobolev space, we thus have an atlas and smoothness of the coordinate changes is a result from the theory of Sobolev spaces.

For further details, we refer to [10].

We finally note that similar to the finite-dimensional case, we are able to define a notion of tangent space to a Banach manifold by using curves in said Banach manifold. This yields a well-defined notion and allows us to investigate differentials as usual on the tangent space.

3.3 Inverse Function Theorem

We now start to head into direction of the most important results concerning Banach manifolds for our applications by introducing some Theorems which will enable us to identify submanifolds more easily. The main theorem in this regard is a generalization of the finite dimensional Inverse Function Theorem:

Theorem 3.3. *Let* X, Y *be Banach spaces,* $U \subset X$ *open and* $f : U \to Y$ *continuously differentiable. Moreover, let* $x_0 \in U$ *be a point such that the differential* $df(x_0)$ *is invertible. Then there exist open neighbourhoods* $U_0 \subset U$ *of* x_0 *and* $V_0 \subset Y$ *of* $f(x_0)$*, such that the restricted map:*

$$f : U_0 \to V_0 \tag{60}$$

is a diffeomorphism, i.e. the inverse map is also continuously differentiable and satisfies:

$$df^{-1}(y) = df(f^{-1}(y))^{-1}. \tag{61}$$

The proof of this theorem follows along the same lines as in the finite-dimensional case by using the Banach fixed point theorem to construct a local inverse. We refer to [8, p.539] for a complete proof.

An important consequence of Theorem 3.3 is the *Implicit Function Theorem* which enables us to find Banach submanifolds in Banach spaces. The key notion for this is a generalisation of regular values for differentiable maps between infinite-dimensional Banach spaces:

Definition 3.3. *For a differentiable map $f : U \subset X \to Y$, an element $y \in Y$ is called* regular*, if $df(x)$ is surjective and admits a right-inverse for all $x \in f^{-1}(y)$.*

Note that in contrast to the finite-dimensional setting, it does not suffice to just assume surjectivity of the differential. We also need existence of a right-inverse. An important class of surjective linear operators with continuous right-inverse are the surjective Fredholm operators. These also play an important role in our considerations, as maps whose differentials are Fredholm maps satisfy the Sard-Smale-theorem which we will introduce later.

The result is now a straight-forward generalisation of the finite-dimensional Implicit Function Theorem:

Theorem 3.4. *Let X, Y be Banach spaces, $U \subset X$ open, $l \in \mathbb{N}$ and $f : U \to Y$ be of class C^l. Then, if $y \in Y$ is a regular value of f, then:*

$$\mathcal{M} := f^{-1}(y) \subset Y, \tag{62}$$

is a C^l-Banach manifold and:

$$T_x\mathcal{M} = ker\ df(x), \tag{63}$$

for all $x \in \mathcal{M}$. Moreover, if f is a Fredholm map, i.e. all differentials are Fredholm operators, then $\mathcal{M}$ is a finite-dimensional manifold with:

$$dim\ \mathcal{M} = index(f). \tag{64}$$

This theorem will be of particular interest in the context of Moduli spaces, as it will enable us to find submanifolds in coordinate charts of appropriate Banach manifold which will be translated to finite-dimensional Banach submanifolds by using local coordinate charts. Again, we refer to [8, p.541] for a complete proof of this statement.

Finally, another important result concerns a more general Sard-Smale-Theorem which will enable us to prove genericity of regular almost complex structures in the next section by using a Banach manifold called the universal Moduli space and a projection map defined on it. More details will be provided in the next section.

Theorem 3.5. *Let X, Y be separable Banach spaces and $U \subset X$ an open subset. Suppose that $f : U \subset X \to Y$ is a Fredholm map of class C^l, $l \in \mathbb{N}$ with:*

$$l \geq max\{1, index(f) + 1\}. \tag{65}$$

Then we define:

$$Y_{reg}(f) := \{y \in Y | x \in U, f(x) = y \Rightarrow\ im\ df(x) = Y\} \tag{66}$$

to be the set of regular values of f. Then $Y_{reg}(f)$ is residual in Y.

These are the main analytic tools in the study of Moduli spaces. Applying these, we will be able to deduce that under appropriate assumptions the space of simple J-holomorphic curves is a finite-dimensional smooth manifold equipped with the C^∞-topology. This will be the main result of the upcoming section.

4 Moduli Spaces

4.1 Main Results and Some Examples

We now begin our investigations of the space of simple J-holomorphic curves as well as its structure. We will see that, under certain conditions, these maps form finite-dimensional manifolds and these manifolds are cobordant for appropriate, but generic choices of almost complex structures. This will become important in applications, where we will use such considerations to see that evaluation maps are cobordant which is an important observation in the proof of the non-squeezing theorem.
We start the section by introducing the main results for J-holomorphic curves. For the remainder of this section, we will always assume that (M, ω) is a finite-dimesional, compact and smooth manifold and $(\Sigma, j, dvol)$ is a compact Riemann surface with a given complex structure j as well as a volume form $dvol$. Moreover, we let $J \in \mathcal{J}_\tau(M, \omega)$ and $A \in H_2(M; \mathbb{Z})$ a given homology class on M. We then denote by:

$$\mathcal{M}(A, \Sigma; J) := \{u \in C^\infty(\Sigma, M) | J \circ du = du \circ j, [u] = u_*[\Sigma] = A\}, \tag{67}$$

the space of all J-holomorphic curves on Σ representing the homology class A, where $[\Sigma]$ denotes the fundamental class of the Riemannian surface. Furthermore, we denote by:

$$\mathcal{M}^*(A, \Sigma; J) := \{u \in \mathcal{M}(A, \Sigma; J) | u \text{ is simple}\}, \tag{68}$$

the space of all J-holomorphic curves in $\mathcal{M}(A, \Sigma; J)$ which are simple. We call the space from (67) the *Moduli space of J-holomorphic curves.* Finally, in the case $\Sigma = S^2 \cong \mathbb{CP}^1$, we abbreviate:

$$\mathcal{M}(A; J) := \mathcal{M}(A, \mathbb{CP}^1; J), \quad \mathcal{M}^*(A; J) := \mathcal{M}^*(A, \mathbb{CP}^1; J). \tag{69}$$

To state the key results of this section, we have to introduce the long-awaited notion of a *regular almost complex structure* as well as the definition of a *regular homotopy of almost complex structures.* These will provide the set of almost complex structures for which the results of this section will hold. An important observation is that these sets are residual and thus the property of being regular is generic by definition. However, to apply our results later on we will be interested in criteria which enable us to check whether a given almost complex structure, for example one induced by a complex structure, is regular. For example, we can think of the standard complex structure on the sphere and wonder whether it is regular as an almost complex structure. Otherwise, we might accidentally look at a almost complex structure for which the results do not hold.

Before introducing regular almost complex structures, we have to take a look at the condition we want to be satisfied by J-holomorphic curves in order to get some intuition. We remind ourselves that, as we have seen in Section 2.1 in equation (12), we can think of J-holomorphic curves as the zeros of a section in an appropriate vector bundle. Intuitively, this section should be transverse to the zero section in order for the Moduli space of simple curves to be a manifold, because this would encourage the use of the Implicit Function Theorem in this situation. By using (12) and the splitting of the tangent space due to base space and fibers, we see that we would like the projected differential:

$$D_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM), \tag{70}$$

to be surjective for all $u \in \mathcal{M}^*(A, \Sigma; J)$. We note at this point that due to the vector bundle just being a Frechet space, this does not yet suffice to conclude but we can use this operator nevertheless in an appropriate manner to find a submanifold of some Sobolev space. We mention that we sometimes denote $D_u = D_{J,u}$ in order to stress dependence on the almost complex structure.

We can calculate D_u by using local coordinates as in (17) to find:

$$D_u\xi = \overline{\partial}_J\xi - \frac{1}{2}(J\partial_\xi J)(u)\partial_J(u), \tag{71}$$

where $\xi : \mathbb{C} \to \mathbb{R}^{2n}$ is a smooth map representing a vector field along u. Note that ∂_J is the complex linear part of the differential and defined by:

$$\partial_J(u) = \frac{1}{2}(du - J \circ du \circ j). \tag{72}$$

From here, there are two comments we need to make before finally defining what a regular almost comlex structure is: Firstly, by (71) we can show that D_u is a real Cauchy-Riemann operator because if f is smooth and real-valued, we see by direct computation:

$$\begin{aligned} D_u(f\xi) &= \overline{\partial}_J(f\xi) - \frac{1}{2}(J\partial_{f\xi}J)(u)\partial_J(u) \\ &= f\overline{\partial}_J\xi + \overline{\partial}(f)\xi - f\frac{1}{2}(J\partial_\xi J)(u)\partial_J(u) \\ &= fD_u\xi + \overline{\partial}(f)\xi. \end{aligned} \tag{73}$$

By the definition of real Cauchy-Riemann operator, we can therefore conclude that D_u is such an operator for all simple J-holomorphic curves. This has some important consequences. For example, due to the Riemann-Roch theorem, this implies that D_u is a Fredholm operator with index $n\chi(\Sigma) + \mu(u^*TM, F)$, where n is the complex rank of u^*TM, F is the totally real-subbundle of u^*TM along the boundary and μ denotes the so-called boundary Maslov index. For more details we refer to [8] in Appendix C, where one can also find a complete proof of the necessary results involving Cauchy-Riemann operators. These properties will be important to deduce manifold properties of the Moduli spaces by using arguments involving the Implicit Function Theorem. Moreover, it will also be useful to find a Banach manifold structure on the so-called *universal Moduli space* which will be used to check genericity of regular almost complex structures.

Secondly, we want to mention that we can extend the definition of D_u to more general maps by introducing the map:

$$\mathcal{F}_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM), \tag{74}$$

for a general smooth map $u : \Sigma \to M$ by setting:

$$\mathcal{F}_u(\xi) := \Phi_u(\xi)^{-1}\big(\overline{\partial}_J(\exp_u(\xi))\big), \tag{75}$$

where exp denotes the exponential map with respect to the Riemannian metric defined as in (8) and:

$$\Phi_u(\xi) : u^*TM \to \exp_u(\xi)^*TM, \tag{76}$$

denotes the parallel transport with respect to the connection $\tilde{\nabla}$ derived from the Levi-Civita connection ∇ by setting:

$$\tilde{\nabla}_v X := \nabla_v X - \frac{1}{2}J(\nabla_v J)X. \tag{77}$$

The choice of connection is due to the fact that we want the parallel transport to preserve the almost complex structure in order to get complex antilinear 1-forms by parallel transporting complex antilinear 1-forms. Thus this requirement ensures well-definition of $\mathcal{F}_u$. We quickly check that $\tilde{\nabla}$ really preserves J:

$$\begin{aligned}\tilde{\nabla}_v(JX) &= \nabla_v(JX) - \frac{1}{2}J(\nabla_v J)(JX)\\ &= (\nabla_v J)X + J\nabla_v X + \frac{1}{2}J^2(\nabla_v J)(X)\\ &= (\nabla_v J)X + J\nabla_v X - \frac{1}{2}(\nabla_v J)(X)\\ &= J\nabla_v X + \frac{1}{2}(\nabla_v J)(X)\\ &= J(\nabla_v X - \frac{1}{2}J(\nabla_v J)(X))\\ &= J\tilde{\nabla}_v X. \end{aligned} \tag{78}$$

Note that we used that J is almost complex in the following way:

$$0 = \nabla_v(-Id) = \nabla_v(J^2) = J(\nabla_v J) + (\nabla_v J)J. \tag{79}$$

Moreover, note that for J-holomorphic curves u, the differential of $\mathcal{F}_u$ in 0 gives exactly D_u, for more general maps we use this as the definition and see:

Proposition 4.1. *For any smooth map $u : \Sigma \to M$, we define:*

$$D_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM), \tag{80}$$

by setting:

$$D_u(\xi) := d\mathcal{F}_u(0)\xi. \tag{81}$$

We then have the following representation:

$$D_u\xi = \frac{1}{2}\Big(\nabla\xi + J(u)\nabla\xi \circ j\Big) - \frac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J(u). \tag{82}$$

As usual, we refer to [8] for a proof of this proposition.

Additionally, note that we can extend the map $\mathcal{F}_u$ for u of class $W^{k,p}$, with $k \geq 1, p > 2$. The condition on the parameters ensures continuity of the Sobolev maps. Therefore, we get a map:

$$\mathcal{F}_u : W^{k,p}(\Sigma, u^*TM) \to W^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM). \tag{83}$$

This will be important, as we will show that we can apply the Implicit Function Theorem in this context for regular J and thus find a submanifold in this space. By using appropriate coordinate charts, this manifold structure carries over to the Moduli space of simple J-holomorphic curves which is the result we are after. We note here that what we computed in the case of general Sobolev spaces and their Banach manifold structures used the notion of tangent vector briefly mentioned in the previous Section.

Finally, we introduce the definition announced earlier:

Definition 4.1. *An almost complex structure J on M is called* regular *for some given $A \in H_2(M;\mathbb{Z})$ and Σ, if for all $u \in \mathcal{M}^*(A, \Sigma; J)$, the map D_u is surjective. We denote the set of all regular almost complex structures by $\mathcal{J}_{reg}(A, \Sigma)$.*

Moreover, a smooth homotopy of almost complex structures is a smooth map:

$$[0, 1] \to \mathcal{J}_\tau(M, \omega), \quad \lambda \mapsto J_\lambda. \tag{84}$$

For any two $J_0, J_1 \in \mathcal{J}_{\text{reg}}(A, \Sigma)$, we denote by $\mathcal{J}(J_0, J_1)$ the set of all smooth homotopies between J_0 and J_1. We define:

Definition 4.2. *A smooth homotopy $J_\lambda \in \mathcal{J}(J_0, J_1)$ is called* regular *for $A \in H_2(M;\mathbb{Z})$ and Σ, if:*

$$\Omega^{0,1}(\Sigma, u^*TM) = im\ D_{J_\lambda,u} + \mathbb{R}v_\lambda, \tag{85}$$

for all $\in [0,1], u \in \mathcal{M}^*(A,\Sigma;J_\lambda)$, where:

$$v_\lambda := (\partial_\lambda J_\lambda) du \circ j, \tag{86}$$

which is the image in $\Omega^{0,1}(\Sigma, u^*TM)$ of the tangent vector to the homotopy. We denote the set of all regular homotopies between J_0, J_1 by $\mathcal{J}_{reg}(A,\Sigma;J_0,J_1)$.

We are now able to state the central results of this section:

Theorem 4.1. *If $J \in \mathcal{J}_{reg}(A,\Sigma)$, then $\mathcal{M}^*(A,\Sigma;J)$ is a smooth, finite-dimensional manifold of dimension:*

$$dim\ \mathcal{M}^*(A,\Sigma;J) = n(2-2g) + 2c_1(A), \tag{87}$$

which carries a natural orientation. Here g denotes the genus of the surface Σ. Additionally, the set of all regular almost complex structures $\mathcal{J}_{reg}(A,\Sigma) \subset \mathcal{J}_\tau(M,\omega)$ is residual.

Note that the dimension of the manifold follows from the index of D_u for a simple J-holomorphic curve u. As noted earlier, this index equals $n\chi(\Sigma) + \mu(u^*TM, F)$ and in our case, we see:

$$\chi(\Sigma) = 2 - 2g, \tag{88}$$

as well as:

$$\mu(u^*TM, F) = \mu(u^*TM) = 2c_1(A), \tag{89}$$

because we assume the Riemann surface to not have a boundary. Thus the desired dimension formula follows.

This theorem ensures a manifold structure on the Moduli space of simple J-holomorphic curves. But we can do even better and relate the manifold structures for different regular almost complex structures due to the following result:

Theorem 4.2. *We define $\mathcal{W}^*(A,\Sigma;(J_\lambda)_{\lambda\in[0,1]})$ to be the following set:*

$$\mathcal{W}^*(A,\Sigma;(J_\lambda)_{\lambda\in[0,1]}) := \{(\lambda,u) | \lambda \in [0,1], u \in \mathcal{M}^*(A,\Sigma;J_\lambda)\}, \tag{90}$$

for any homotopy $(J_\lambda)_{\lambda\in[0,1]}$. Then if the homotopy is regular, i.e. $(J_\lambda)_\lambda \in \mathcal{J}_{reg}(A,\Sigma;J_0,J_1)$, the space $\mathcal{W}^*(A,\Sigma;(J_\lambda)_{\lambda\in[0,1]})$ is a smooth, oriented manifold with boundary such that:

$$\partial\mathcal{W}^*(A,\Sigma;(J_\lambda)_{\lambda\in[0,1]}) = \mathcal{M}^*(A,\Sigma;J_0) \cup \mathcal{M}^*(A,\Sigma;J_1), \tag{91}$$

and the boundary orientation agrees with the one on $\mathcal{M}^*(A,\Sigma;J_1)$ and is opposite to the one on $\mathcal{M}^*(A,\Sigma;J_0)$. Additionally, the set of regular homotopies $\mathcal{J}_{reg}(A,\Sigma;J_0,J_1)$ between two regular almost complex structures J_0, J_1 is residual in the set of all smooth homotopies between those two.

Thus, we know that different regular almost complex structures give rise to cobordant Moduli manifolds of simple J-holomorphic curves. As an easy example, we will now try to see how these results work in the case $\Sigma = M = S^2$ with the usual symplectic and complex structure as well as $A = [S^2]$ being the fundamental class.

4.1.1 First Examples

In this case, note that due to our earlier considerations, any J-holomorphic curve is actually a holomorphic map from the Riemannian sphere to itself. We want to know which of these maps are simple. Note that simplicity means that there is an injective point by a previous result in Section 2. Due to this, there is a point on the Riemannian sphere which has exactly one preimage point and, additionally, this point has non-vanishing and thus invertible differential. By the usual way to calculate degree of maps between spheres, this implies that a simple J-homolomorphic curve $u : S^2 \to S^2$ satisfies:

$$\deg\, u = \pm 1. \tag{92}$$

Note that as holomorphic maps preserve the orientation induced by the complex sturcture, we can even conclude that deg $u = 1$.

Now we have enough information to determine all simple J-holomorphic curves in this case: Note that as the degree is non-vanishing, the map has to be surjective because the sphere with a single point removed is contractible. Next, we show that it is also injective. Note that every value which is assumed by at least two points needs to be singular due to the usual method for the computation of degrees as for example found in [2]. Therefore, we just have to check whether there are any points with vanishing differential.
Now assume that there is a point where the derivative vanishes. By going over to local holomorphic coordinates, we can assume that in local coordinates $\tilde{u} : B_\epsilon(0) \to \mathbb{C}$ with $\tilde{u}(0) = 0$ and $\tilde{u}'(0) = 0$. We assume moreover that $2 \leq n \in \mathbb{N}$ is the smallest number, such that:

$$\tilde{u}^{(k)}(0) = 0, \forall k \leq n-1 \tag{93}$$

and $\tilde{u}^{(n)}(0) \neq 0$. Then by a result from complex analysis which is presented in [5, p.158], we can represent:

$$\tilde{u}(z) = g(z)^n, \forall z \in B_\delta(0), \tag{94}$$

for some $0 < \delta \leq \epsilon$ and some holomorphic map g such that $g(0) = 0, g'(0) \neq 0$. Note that thus g maps an open neighbourhood of 0 to an open set containing 0 and thus we see that $\tilde{u}$ assumes all values in a neighbourhood of 0 at least n times. Note that if we stay close enough to 0, these $\tilde{u}'$ at these points is invertible, hence we reach a contradiction to our degree, as any point with non-vanishing derivative contributes positively to the degree for all elements in the preimage of any regular value. We remark, that since non-injective points are countable, we are guaranteed to get the desired contradiction.

Therefore, we know now that the map u is bijective and all differentials are invertible. Thus it is a biholomorphic map of the Riemannian sphere to itself. From complex analysis as in [5], we recall that any such map is a Möbius transformation. Thus we know in this case (with J denoting the complex structure):

$$\mathcal{M}^*([S^2], S^2; J) = PGL(2, \mathbb{C}). \tag{95}$$

where $PGL(2, \mathbb{C})$ denotes the group of Möbius transformations. We note here that we do not know that J is a regular almost complex structure or not. Nevertheless, we can see that:

$$\dim\, PGL(2, \mathbb{C}) = 6, \tag{96}$$

which coincides with the desired dimension suggested by our main result for Modulispaces, since for the sphere we have $n = 1, g = 0$ and:

$$c_1([S^2]) = 2, \tag{97}$$

and therefore:

$$n(2-2g)+c_1([S^2])=6. \tag{98}$$

Additionally, we note that if we take the Moduli space of simple J-holomorphic curves and quotient by reparametrisation, i.e. by the action of the group of Möbius transformations on the J-holomorphic spheres, we end up with a one-point, zero dimensional manifold. Note that this space is compact, whereas the Moduli space itself is not compact at all. We will investigate this phenomenon further in the next section as it is an instance of a more general situation.

Similarily, these considerations can know be applied to investigate some further simple Moduli spaces which will be important in the context of the non-squeezing theorem. Namely, if we take V a compact, symplectic $2n-2$ manifold with trivial second homotopy group $\pi_2(V)=0$ and choose:

$$[S^2\times\{x_0\}]\in H_2(S^2\times V;\mathbb{Z}), \tag{99}$$

for any point $x_0\in V$, we can try to calculate the space $\mathcal{M}^*([S^2\times\{x_0\}],S^2;J)$ for the symplectic product manifold with the product almost complex structure as before. For the almost complex structure on this product space, we take, as usual, the one induced by the complex structures on the sphere and for V any given tame almost complex structure. This gives us a so-called *product almost complex structure*.

We first note that any simple J-holomorphic curve $u:S^2\to S^2\times V$ induces J-holomorphic curves $u_1:S^2\to S^2$ and $u_2:S^2\to V$ by projecting on both factors separately. By using the induced map in homology of the projection onto the first factor, we discover:

$$u_1\in\mathcal{M}^*([S^2],S^2;J_{S^2})=PGL(2,\mathbb{C}), \tag{100}$$

as calculated before. Thus it remains to determine u_2.
Note that by $\pi_2(V)=0$, we already know that u_2 is homotopic to a constant map. We want to show that it is even constant. Therefore, the map induces the trivial map in homology and cohomology. Thus, we get:

$$\int u_2^*\omega=0, \tag{101}$$

where ω denotes the symplectic form on V. We note that due to j_2 respecting the almost complex structure on the Riemann surface, this implies that the differential of u_2 vanishes and thus it is constant, i.e. note that:

$$\begin{aligned}u_2^*\omega(v,j_{S^2}(v))&=\omega(du_2(v),du_2(j_{S^2}v))\\&=\omega(du_2(v),J_Vdu_2(v))\\&>0,\end{aligned} \tag{102}$$

if $v\neq 0, du_2\neq 0$ due to tameness of the almost complex structure. Expressing everything in local coordinates and integrating the differential form, we see that our claim holds.

Thus we now know the Moduli space in this case:

$$\mathcal{M}^*([S^2\times\{x_0\}],S^2;J)=\mathcal{M}^*([S^2],S^2;J_{S^2})\times V, \tag{103}$$

and if we take the quotient with respect to reparametrisation of the J-holomorphic curves, i.e. by the group action of the Möbius transformations, we once again uncover a compact manifold, this time V. We note once more that the dimension formula follows in this case by similar considerations as before and remind ourselves that we do not know yet whether the almost complex structure considered here is actually regular. We will return to this example later on and use our results from here to deduce the degree of the evaluation map.

4.2 "Proofs"

In this section, we give a more detailed outline of the technical proof of Theorems 4.1 and 4.2. The proof relies heavily on several different analytic results concerning Sobolev spaces and elliptic regularity on one hand as well as the Riemann-Roch Theorem in order to apply the Implicit Function Theorem on the other hand. Some parts have been dealt with before in Section 3 and for others we refer to [8] and the appendices therein for a more complete exposition.

The first important step is to remind ourselves that we have considered J-holomorphic curves as the points of intersection of a section with the zero section in an appropriate vectorbundle. Now, in order to apply Banach manifold methods as introduced in the previous Section, we have to generalize the section in (12) to a section in Banach manifolds. For this, we will need to work in the corresponding Sobolev space completions. What we get this way is a section:

$$W^{k,p}(\Sigma, M) \to \mathcal{E}_J^{k-1,p}, \tag{104}$$

where $\mathcal{E}_J^{k-1,p}$ is a fiber bundle over $W^{k,p}(\Sigma, M)$ with fiber $W^{k-1,p}(\Sigma, \Lambda \otimes_J u^*TM)$. Similarily, in order to get genericity, one considers a slightly more general fiber bundle, namely:

$$\mathcal{E}^{k-1,p} \to W^{k,p}(\Sigma, M) \times \mathcal{J}^l, \tag{105}$$

where $\mathcal{J}^l$ denotes the set of all C^l-almost complex structures. The Banach manifold structure is defined using parallel transport in the fibers to get charts to identify the different fibers with one another. By using these fiber bundles, one obtains differentials corresponding to the Cauchy-Riemann operator we have already mentioned before. Afterwards, we are in a position where we are able to apply the Implicit Function Theorem in this situation. For the first vectorbundle, we see that locally, we have a Fredholm map around every J-holomorphic curve which, by using local coordinates, gives us the desired manifold properties of the moduli space. For the second bundle, one has to work a bit more as the differential is not necessarily Fredholm, thus one has to check whether 0 is a regular value. But it turns out that 0 is a regular value and thus the preimage is therefore again a Banach manifold which is called the *universal Moduli space*.

The rest of the argument now follows by checking transversality of the projection of the universal Moduli space onto the space of almost complex structures. By applying Sard-Smale, on sees then that almost all values are regular and thus almost all J are regular. This outline skips quite a few steps in the proof as we are required to proof genericity of smooth almost complex structures which requires some more work.

Therefore, a general recipe of the proof summarizing the main ingredients is outlined below:

1. Consider the section:

$$\mathcal{F} : \mathcal{B}^{k,p}(\Sigma, M) \times \mathcal{J}^l \to \mathcal{E}^{k-1,p}, \mathcal{F}(u, J) := (u, J, \overline{\partial}_J(u)), \tag{106}$$

   in the Banach vector bundle, where $\mathcal{B}^{k,p}(\Sigma, M)$ denotes the subset of $W^{k,p}(\Sigma, M)$ of maps representing the desired homology class A given in the theorem. Here, we use that $p > 2$ in order to assume continuity of the maps by Sobolev embedding. Using local coordinate charts on both manifolds, we see that the differential at any (u, J) with $\mathcal{F}(u, J) = 0$ is given by:

$$d\mathcal{F}(u, J)(\eta, Y) := D_u\eta + \frac{1}{2}Y(u) \circ du \circ j. \tag{107}$$

Now, by reminding ourselves that D_u is a real Cauchy-Riemann operator, we see that it is Fredholm and thus has closed image. This in turn implies that $d\mathcal{F}(u,J)$ has closed image and thus if one shows density of the image, the operator is surjective. Density of the image is shown by using formal adjoints of Cauchy-Riemann operators as seen in [8, p.51]. In order to conclude that 0 is a regular value of $\mathcal{F}$ and thus to invoke the Implicit Function Theorem, one needs to prove existence of a right-inverse operator but this follows from properties of the Fredholm operator. Thus we can deduce that:

$$\mathcal{F}^{-1}(0) = \mathcal{M}^*(A,\Sigma;\mathcal{J}^l), \tag{108}$$

is a Banach manifold. We note at this point that it is actually essential to keep track of the regularity of the Banach manifolds as this will be used in the Sard-Smale theorem. Moreover, we want to point out that we skipped the explicit definition of appropriate Banach manifold structures on the spaces involved which uses parallel transport. For more details, we refer to [8].

2. Note that similar considerations can be applied to the section:

$$\mathcal{F}_J : \mathcal{B}^{k,p}(\Sigma, M) \to \mathcal{E}_J^{k-1,p}, \mathcal{F}(u) = (u, \overline{\partial}_J(u)), \tag{109}$$

in order to prove that the Moduli spaces are finite-dimensional manifolds. By going over to local coordinates and calculating differentials, the operator D_u appears for u a J-holomorphic curve and hence if J is regular, we get the desired result due to the Implicit Function Theorem and the fact that the operator D_u is Fredholm. This also provides the desired dimensional statement for the Moduli space.

3. The theorem also guarantees an orientation. This is given by considering determinant bundles as in [8, p.52] using the Fredholm operator D_u.

4. Finally, to conclude the proof of our first theorem, we have to check genericity of regular almost complex structures. For this, one considers the map:

$$\pi : \mathcal{M}^*(A,\Sigma;\mathcal{J}^l) \to \mathcal{J}^l, (u,J) \mapsto J, \tag{110}$$

which projects elements in the universal Moduli space to its almost complex structure. Again by going over to local coordinates and calculating differentials, one sees that the differential is surjective if and only if D_u is surjective due to using tangent space identification of the universal Moduli space as a subvectorspace of the tangent space of $\mathcal{B}^{k,p}(\Sigma, M) \times \mathcal{J}^l$. One can moreover show that $d\pi$ is Fredholm everywhere, thus given sufficient regularity of the Banach manifolds involved, we can apply the Sard-Smale theorem from section 3 to deduce that genericity holds. However, we are not yet done, as we have only shown genericity of regular almost complex structures of class C^l, some more work is required to get the same statement for smooth almost complex structures, see [8, p.54].

5. To prove the remaining theorem concerning cobordisms, one follows a similar procedure by defining appropriate Banach manifold structures, then invoking the Implicit Function Theorem and Sard-Smale to get the desired results.

To conclude our discussion of Moduli spaces, we would like to remark that the ideas appearing here are used again in Morse Theory and Floer Homology.

5 Bubbling and Compactness

Next, we want to take a look at convergence properties of J-holomorphic curves. This will be important in applications, because compactness of the Moduli spaces will allow us to define invariants on them. We will consider the process of *Bubbling* occurring in general sequences of J-holomorphic curves with bounded energy. Note that for example J-holomorphic maps within a Moduli space with fixed homology class satisfy this boundedness property due to the topological characterisation of the energy in Section 2. Moreover, we will then use the impossibility of bubbling for certain minimal homology classes to deduce that the space of unparametrized J-holomorphic spheres is sometimes compact. Moreover, more careful investigations of the bubbling process lead to results of general sequences of J-holomorphic maps converging modulo bubbling at finitely many points and due to the energy bound we will naturally get maps which we will later identify as *Stable maps*. For these maps, more general considerations and an appropriate notion of convergence lead to a topology on stable maps which is metrizable and we can derive an important compactness result in this case called *Gromov compactness*.

5.1 Energy, Bubbling and Examples

5.1.1 Properties of the Energy

As we have already seen in the examples of Moduli spaces, we cannot expect the Moduli spaces themselves to be compact. But sometimes, we can make the space compact by using quotients with respect to reparametrisations as seen in the previous section. In this section, we try to find a more general result telling us when a Moduli space admits a compact quotient. Moreover, we try to discuss what can go wrong and introduce the notion of *Bubbling*. The main idea is that we cannot expect compactness but there is more structure than one might guess at first glance. These considerations together with the notion of *Stable Maps* allow for a compactness statement called *Gromov Compactness* for an appropriately defined notion of convergence which is useful in the study of J-holomorphic curves.
Most of this section relies on the compactness result from Theorem 2.12. In fact, bubbling occurs when the conditions of said theorem are violated, i.e. the differentials are unbounded. This will allow us to deduce several properties. But let us start by introducing the following result, which will become important when dealing with Bubbling as it enables us to extend the resulting maps to J-holomorphic curves on a slightly larger set under certain boundedness assumptions:

Theorem 5.1 (Removal of Singularities)**.** *Let (M, ω) be a compact symplectic manifold, $L \subset M$ a compact Lagrangian submanifold as well as J an ω-tame almost complex structure on M. As before, we denote by g_J the Riemannian metric as in* (8)*. Then:*

(i.) *If $u : B_1(0) \setminus \{0\} \subset \mathbb{C} \to M$ is a J-holomorphic curve with energy $E(u) < \infty$. Then u extends to a smooth map on $B_1(0)$.*

(ii.) *If $u : (B_1(0) \cap \mathbb{H} \setminus \{0\}, B_1(0) \cap \mathbb{R} \setminus \{0\} \to (M, L)$ is a J-holomorphic curve with $E(u) < \infty$, where $\mathbb{H}$ denotes the upper half plane. Then u extends to a smooth map on $B_1(0) \cap \mathbb{H}$.*

In our applications we will mostly use (*i.*) for spheres. By inversion, we can apply the result to extend J-holomorphic curves at ∞ in the Riemannian sphere.

It will be an important feature of J-holomorphic curves to have energy bounded from below. This will be used to show that only finitely many J-holomorphic spheres can "bubble off" for a sequence of J-holomorphic spheres with bounded energy. This, in turn, will naturally lead to the introduction of stable maps to generalize J-holomorphic curves. The result reads as follows:

Proposition 5.1. *Let (M, J) be a compact, almost complex manifold and L a totally real submanifold and assume we are given a Riemannian metric g on M. Then there exists a $\hbar > 0$, such that:*

$$E(u) \geq \hbar, \tag{111}$$

for all non-constant J-holomorphic spheres $u : S^2 \to M$ and more generally for all non-constant J-holomorphic discs $u : (B_1(0), S^1) \to (M, L)$.

The proof of this proposition uses an estimate for J-holomorphic curves which is also of interest in the proof of the Removal of Singularities-Theorem. Moreover, we remark that the constant $\hbar$ depends on M, ω, L and J.

5.1.2 Bubbling

Now we want to introduce the process of bubbling in the same way as seen in [8, Chapter 4]. The main idea is that given a sequence $(u_n)_{n\in\mathbb{N}}$ of J-holomorphic spheres, such that:

$$\sup_{n\in\mathbb{N}} E(u_n) < \infty, \tag{112}$$

hence with bounded energy. We then recall Theorem 2.12: In the case of spheres, there is no boundary and thus no restrictions by boundary conditions. Moreover, we can choose the sequence of almost complex structures to be the constant sequence which obviously converges in the C^∞-topology. Similarily, take the constant sequence of complex structures on S^2. Moreover, we recall that we assume M to be compact and symplectic and J to be tame with respect to the symplectic form on M. Then there are two cases:

- Either for all compact $Q \subset S^2$, there is an upper bound for $\|du_n\|_{L^\infty}$ for n sufficiently large. Then we can conclude that the sequence of J-holomorphic spheres converges on compact subsets in the C^∞-topology to another J-holomorphic sphere. In this case, we have everything we can hope for. We note that the condition of the images being contained in a compact subset is automatically satisfied due to compactness of the manifold M.

- On the other hand, it could happen that for some compact subset $Q \subset S^2$, we have:

  $$\limsup_n \|du_n\|_{L^\infty(Q)} = \infty. \tag{113}$$

  Note that then:

  $$\limsup_n \|du_n\|_{L^\infty(S^2)} = \infty. \tag{114}$$

  In this case, the compactness result in Theorem 2.12 cannot be applied. But we can still try to deduce some convergence properties for some subsequence. We will follow [8, p.80] closely in our discussion.

In the second case, the idea is now to take a sequence $z_n \in S^2$, such that:

$$|du_n(z_n)| = \|du_n\|_{L^\infty(S^2)} = c_n. \tag{115}$$

We take a subsequence such that c_n converges monotone to ∞. Moreover, note that due to compactness, z_n has a converging subsequence. Thus passing to another subsequence, we can assume that the z_n converge in S^2 to some z_0. We will still denote the subsequences the same way as the original sequences in order to avoid unnecessary subscript complications.

Next, we want to construct another sequence of maps from the one we have which will satisfy the condition needed in Theorem 2.12. This will lead to the "bubble" emerging in this case. The construction presented here is taken from [8], chapter 4, thus for further details we refer to said source.

The key idea is that we can reduce to local coordinates around z_0 by using a holomorphic chart $\phi : U \subset \mathbb{C} \to S^2$ such that $0 \in U$ and $\phi(0) = z_0$. If we pull back the volume form $dvol$, we get using as usual the identification $z = s + it$:

$$\phi^* dvol = \lambda^2 ds \wedge dt, \tag{116}$$

for some function $\lambda : U \to]0; \infty[$. Note that we use here that the orientation induced by the volume form coincides with the one given by the complex structure. By rescaling and possibly shrinking U, we can assume wlog:

$$\lambda(0) = 1, \quad \frac{1}{2} \leq \lambda(z) \leq 2, \tag{117}$$

for all $z \in U$. We now define an auxilliary sequence $(\hat{u}_n)_{n\in\mathbb{N}}$ by:

$$\hat{u}_n(z) := u_n(\phi(z)), \forall n \in \mathbb{N}, \forall z \in U. \tag{118}$$

Similarily, we express the sequence (z_n) in these coordinates and set $\hat{z}_n := \phi^{-1}(z_n)$. Then, using the definition of the numbers c_n and the norm of the differentials, we see:

$$\begin{aligned} c_n := \|du_n\|_{L^\infty(S^2)} &= |du_n(z_n)| \\ &= |d(\hat{u}_n \circ \phi^{-1})(z_n)| \\ &= |d\hat{u}_n(\hat{z}_n) d\phi^{-1}(z_n)| \\ &= |d\hat{u}_n(\hat{z}_n)||d\phi^{-1}(z_n)| \\ &= |d\hat{u}_n(\hat{z}_n)|\lambda(\hat{z}_n)^{-1} \\ &= \frac{|d\hat{u}_n(\hat{z}_n)|}{\lambda(\hat{z}_n)} \\ &= \sup_{z\in U} \frac{|d\hat{u}_n(z)|}{\lambda(z)}. \end{aligned} \tag{119}$$

Moreover, it is clear by the established convergence of the sequence $(z_n)_{n\in\mathbb{N}}$ that:

$$\lim_{n\to\infty} \hat{z}_n = 0. \tag{120}$$

Finally, we define for all $n \in \mathbb{N}$ a map $v_n : B_{c_n\epsilon}(0) \to M$ with $\epsilon > 0$ small enough by:

$$v_n(z) := \hat{u}_n\big(\hat{z}_n + \frac{z}{c_n}\big), \forall z \in B_{c_n\epsilon}(0). \tag{121}$$

We want to check that this new sequence satisfies the conditions of Theorem 2.12 and has bounded energy. Therefore, we need to calculate norms of differentials and see:

$$dv_n(z) = d\hat{u}_n\big(\hat{z}_n + \frac{z}{c_n}\big)c_n^{-1}. \tag{122}$$

Thus by (119) and (117), we see:

$$\|dv_n\|_{L^\infty(B_{c_n\epsilon}(0))} \leq 2. \tag{123}$$

Moreover, we also know that $|dv_n(0)| \geq \frac{1}{2}$. As for the energy, we note that we only used a conformal transformation to define v_n from $\hat{u}_n$, thus by our earlier considerations on conformal invariance in section 2:

$$\begin{aligned} E(v_n; B_{c_n\epsilon}(0)) &= E(\hat{u}_n; B_\epsilon(\hat{z}_n)) \\ &\leq E(u_n), \end{aligned} \tag{124}$$

by using conformality of coordinate charts and positivity of the integrand. Therefore:

$$\sup_{n\in\mathbb{N}} E(v_n) \leq \sup_{n\in\mathbb{N}} E(u_n) < \infty. \tag{125}$$

Due to boundedness, we can invoke Theorem 2.12 (by relabeling, we once more assume that already v_n itself is the desired sequence) to get a J-holomorphic map $v : \mathbb{C} \to M$ and we notice:

$$0 < E(v) \leq \sup_{n\in\mathbb{N}} E(u_n), \tag{126}$$

due to convergence by the compactness result and $|dv(0)| \geq \frac{1}{2}$ by convergence. But finiteness of the energy enables us to remove the singularity at ∞ of v by invoking the previously introduced Removal of Singularity-Theorem. Namely, as inversion is a conformal map, it preserves energy and thus $\hat{v}(z) = v(1/z), \forall z \neq 0$ has again finite energy and is J-holomorphic. Invoking Theorem 5.1, we see that $\hat{v}$ extends to a J-holomorphic map on $\mathbb{C}$. Therefore, v extends to a J-holomorphic sphere $v : S^2 \to M$ with non-vanishing energy.
The map v is a so-called *bubble* formed by the sequence. It is an important feature that the bubble is nothing more than the limit of "magnifications" of the maps u_n of neighbourhoods where the differential become unbounded. The magnification process spreads the differential out, allowing us to bound it and in the process, we get a sequence for which the compactness result is then applicable. It is important to see that for small neighbourhoods of z_0, the maps u_n have positive energy at least $E(v)$, thus the bubble corresponds to concentration of the energy at a certain point. More precisely, we have:

$$\liminf_{n\infty} E(u_n; B_\epsilon(z_0) \geq E(v), \tag{127}$$

for any $\epsilon > 0$. This is due to the following estimate for any R:

$$\begin{aligned} E(v; B_R(0)) &= \lim_{n\to\infty} E(v_n; B_R(0)) \\ &= \lim_{n\to\infty} E(\hat{u}_n; B_{R/c_n}(\hat{z}_n)) \\ &\leq \liminf_{n\to\infty} E(u_n; B_\epsilon(z_0)), \end{aligned} \tag{128}$$

by noticing $\lim_{n\to\infty} R/c_n = 0$ and using conformal invariance. Taking $R \to \infty$, we get the desired concentration of energy.

5.1.3 A few Examples

Next, we want to have a more concrete example to get a feeling for the process of Bubbling and therefore work out some of the examples presented in [8] in chapter 4. We will be working on S^2 with the Fubini-Study form as symplectic form which is compatible with the complex structure or on the unit disc in $\mathbb{C}$. If we consider a sequence (ϕ_n) of Möbius transformations on S^2 and note that in this case, we have for $z \in \mathbb{C}$:

$$|d\phi_n(z)| := \sqrt{2}|\phi_n'(z)|\frac{1+|z|^2}{1+|\phi_n(z)|^2}, \tag{129}$$

we see that if the sequence of derivative is bounded, we can employ the compactness result Theorem 2.12. Note that we could have also deduced existence of a convergent subsequence directly using Arzela-Ascoli and Weierstrass' convergence result for holomorphic maps. But what happens when the transformations are unbounded? For example, we take:

$$\phi_n(z) := \frac{z - z_n}{1 - z_n z}, \forall z \in \mathbb{C}, \tag{130}$$

where $(z_n) \subset]0, 1[$ is a sequence converging to 1. Considering these maps on the unit disc, we see:

$$\phi_n'(z) = \frac{1 - z_n^2}{(1 - z_n z)^2}. \tag{131}$$

Note that this converges for $n \to \infty$ to 0, if $z \neq 1$. But if $z = 1$, then the sequence diverges with generalized limit ∞. Also, we observe that on compact subsets $B_1(0) \setminus \{1\}$ the sequence converges uniformly to 0. Here is an example where one observes bubbling. Note that $\phi_n(1) = 1, \forall n \in \mathbb{N}$, thus due to uniform convergence, arbitrarily small neighbourhoods of 1 get "stretched out" to cover the whole disc. This leads to unbounded derivatives and thus to a bubble, here a "bubble-disc".

Another example can be found by considering maps:

$$u_n(z) := \frac{zp(z)}{(z - a_n)q(z)}, \tag{132}$$

for all n and all $z \in S^2$ as our sequence of J-holomorphic curves. Moreover, we assume that p, q are coprime polynomials with $p(0) = q(0) = 1$ and (a_n) is a sequence in S^2 converges to 0, i.e. the "south pole". By direct calculation using the Fubini-Study structure, we see that a similar formula to (129) holds. Once again, we note that the sequence converges uniformly on compact subsets of $S^2 \setminus \{0\}$ to $u(z) = p(z)/q(z)$ by direct investigation. We want to show that there is a bubble forming. Calculating the derivative of the map, we see:

$$u_n'(z) = \frac{(p(z) + zp'(z))(z - a_n)q(z) - zp(z)(q(z) + (z - a_n)q'(z))}{(z - a_n)^2 q(z)^2}. \tag{133}$$

Note that the derivatives are unbounded in 0, thus there will emerge a bubble in the limit. Therefore, we. again encounter an example of bubbling on the sphere.

For further examples, we refer as usual to [8], chapter 4.

5.1.4 Compactness of Moduli Spaces

One important consequence of the bubbling process is that sometimes the Moduli spaces discussed earlier are compact after identifying reparametrisations of the same J-holomorphic sphere. In order to see this, we take A to be a homology class which minimizes energy, i.e.:

$$[\omega] \cap [A] = \hbar, \tag{134}$$

where $\hbar$ is defined as before to be the infimum of possible values attained by the energy of non-constant J-holomorphic spheres. The main idea is now that if we consider the space $\mathcal{M}^*(A; J)$ of simple J-holomorphic spheres with values in the compact symplectic manifold M with ω-tame almost complex structure, then we can say more about its structure than before by using considerations involving bubbling. Namely, having a sequence of maps in this Moduli space and possibly reparametrising it using Möbius transformations, we see that either there is a converging subsequence or bubbling occurs. But note that the resulting bubble would have energy $\geq \hbar$. Thus all energy of the maps concentrates completely close to some point. But by reparametrising the sequence, we can show that this leads to a contradiction, thus we will have the following result:

Theorem 5.2. *Let* (M, ω) *be a compact symplectic manifold with* J ω*-tame and* $A \in H_2(M; \mathbb{Z})$ *be a spherical homology class with:*

$$[\omega] \cap [A] = \hbar, \tag{135}$$

i.e. a homology class in the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M; \mathbb{Z})$ *satisfying this condition. Then the Moduli space* $\mathcal{M}(A; J)/PSL_2(\mathbb{C})$ *is compact, i.e. for any sequence* $u_n : S^2 \to M$ *of J-holomorphic spheres in* $\mathcal{M}(A; J)$*, there exists a sequence of* $\phi_n \in PSL_2(\mathbb{C})$*, such that* $u_n \circ \phi_n$ *has a* C^∞*-convergent subsequence.*

We note that in the case of the theorem above, any J-holomorphic sphere representing A is necessarily simple and that the quotient can be identified with the space of all unparametrised J-holomorphic spheres representing A.

As an example, we remind ourselves of the example in Section 4. There, we have seen that $\mathcal{M}^*([S^2 \times \{x_0\}], S^2; J) = \mathcal{M}^*([S^2], S^2; J_{S^2}) \times V$ for V compact, symplectic with trivial second homology group. By taking quotient with respect to reparametrisation and noticing that the homology class $[S^2 \times \{x_0\}]$ satisfies the required condition, we note that:

$$\begin{aligned} \mathcal{M}^*([S^2 \times \{x_0\}], S^2; J)/PSL_2(\mathbb{C}) &= (\mathcal{M}^*([S^2], S^2; J_{S^2}) \times V)/PSL_2(\mathbb{C}) \\ &\cong V, \end{aligned} \tag{136}$$

thus we see that the expected compactness is reflected by our calculations in this case. Note that we can interpret this by noting that through any point in V, there passes a J-holomorphic curve representing the class $[S^2 \times \{x_0\}]$. This will be useful in Section 6 when dealing with evaluation maps.

5.2 Convergence of J-holomorphic Maps

In the last subsection, we have started to consider convergence and bubbling at a single point. If we have a general sequence of J-holomorphic spheres with values in some compact symplectic manifold M with tame almost complex structure J, then we can derive a stronger statement. Namely, up to finitely many points, where bubbling will occur, the sequence converges uniformly on compact subspaces in the C^∞-topology and we can characterise the energy of the limit curve by the remaining energy which is not concentrated at the bubbling points. The following theorem states this more systematically:

Theorem 5.3. *Let* J_n *be a sequence of* ω*-tame almost complex structures on* (M, ω) *converging to* J *in the* C^∞*-topology and* U_n *be a sequence of open sets in a Riemann surface* Σ*, such that* $\bigcup_n U_n = \Sigma$*. Moreover, let* $u_n : (U_n, U_n \cap \partial\Sigma) \to (M, L)$ *be a sequence of* J_n*-holomorphic curves with:*

$$\sup_n E(u_n) < \infty, \tag{137}$$

where L *is a compact Lagrangian submanifold of* M*. Then there exists a subsequence* (u_{n_k})*, a J-holomorphic curve* $u : (\Sigma, \partial\Sigma) \to (M, L)$ *and a finite set* $Z := \{z_1, \dots z_l\} \subset \Sigma$*, such that the following holds:*

(i.) u_{n_k} *converges uniformly with all derivatives to* u *on compact subsets of* $\Sigma \setminus Z$*.*

(ii.) *For every* $j \in \{1, \dots, l\}$ *and every* $\epsilon > 0$*, such that* $B_\epsilon(z_j) \cap Z = \{z_j\}$*, the limit:*

$$m_\epsilon(z_j) := \lim_{n_k \to \infty} E(u_{n_k}; B_\epsilon(z_j)), \tag{138}$$

exists and is a continuous function of ϵ *as well as satisfying:*

$$m(z_j) := \lim_{\epsilon \to 0^+} m_\epsilon(z_j) \geq \hbar. \tag{139}$$

(iii.) *For every compact subset $K \subset \Sigma$ with $Z \subset int\, K$, we have:*

$$E(u;K) + \sum_{j=1}^{l} m(z_j) = \lim_{n_k \to \infty} E(u_{n_k};K). \tag{140}$$

For our application, the most important case is $\Sigma = S^2$ which does not have a boundary. Thus the statement guarantees convergence uniformly with all derivatives up to some finite set of bubbling points with some continuity properties of the energy.
We note that the theorem stated here neither requires Σ to be compact nor to have no boundary.

The main idea in the proof is to first remove points until we can apply Theorem 2.12 to the remaining Riemann surface. This process will terminate, as only finitely many bubbles can emerge due to the energy bound. Note that due to the removal of singularities in Theorem 5.1, the limit map extends to the whole space. Thus statement (i.) is pretty easy to deduce.
The continuity of energy is deduced by using convergence on annuli between ϵ-balls. The lower bound is due to similar considerations as we have made for our first description of bubbling. Note that here it becomes harder to prove the statement, if the Riemann surface has non-trivial boundary and we then need to prove a further lemma which mainly examines the convergence property of bubbling points which lie on the boundary.
The last part follows by removing small neighbourhoods around each bubbling point z_j in K and considering the limit as these balls let their radius go to 0. Note that on the remaining set, we have uniform convergence.

Note that for holomorphic functions, we know Weierstrass' Theorem which guarantees local uniform convergence of derivatives of a sequence of functions, if the sequence itself converges uniformly by using the standard representation formulae for the derivatives, see for example [5]. The following Lemma allows us to extend this statement to J-holomorphic curves in a similar way:

Theorem 5.4. *Let J_n, J and the other objects involved as in the previous theorem. Let $U \subset \mathbb{C}$ be an open subset and $u_n : U \to M$ be a sequence of J_n-holomorphic curves which converge uniformly to some continuous map $u : U \to M$. Then u is even J-holomorphic and the u_n even converge uniformly with all derivatives on compact subsets of U to u.*

Note that any J-holomorphic curve can locally be seen as a J-holomorphic curve defined on an open subset of $\mathbb{C}$ due to Riemann surfaces being complex manifolds. Thus we can apply the statement locally if we have uniformly convergent sequences of pseudoholomorphic curves in our more general setting.

The proof is straight-forward, one just has to distinguish two cases as usual: Either we can apply the compactness result in Theorem 2.12, which immediately implies the statement given above, or there is a compact subset where the sequence of derivatives are unbounded. Thus we just have to consider the second case. Note that in this case, there will emerge a bubble at some point of the compact set where the sequence of derivatives is unbounded. Note that due to uniform convergence, the image of the bubble must lie in an arbitrarily small neighbourhood of the bubbling point. Thus the bubble is constant, which leads to a contradiction, as the energy will vanish.

Before finishing this subsection, we want refine the previous convergence result a bit. Namely, we want to get some connectedness property of the emerging bubbles to the limit curve which will become useful afterwards in our discussion of stable maps which basically are the limits of sequences of J-holomorphic curves with bounded energy. The result below heavily builds upon some energy estimates on cylinders and thus becomes more technical than the previous result.

Theorem 5.5. *Let J_n be a sequence of tame almost complex structures on (M,ω) converging in the C^∞-topology to J and let $z_0 \in \mathbb{C}$ as well as $r > 0$. Assume that $u_n : B_r(z_0) \to M$ is a sequence of J_n-holomorphic curves and $u : B_r(z_0) \to M$ is a J-holomorphic curve, such that:*

- *u_n converges uniformly with all derivatives to u on compact subsets of $B_r(z_0) \setminus \{z_0\}$.*
- *The following limit exists and is positive:*

$$m_0 := \lim_{\epsilon\to 0^+} \lim_{n\to\infty} E(u_n; B_\epsilon(z_0)). \tag{141}$$

Then there exists a subsequence (u_{n_k}) and a sequence $(\psi_{n_k}) \subset PSL_2(\mathbb{C})$ of Möbius transformations as well as a J-holomorphic sphere $v : S^2 \to M$ and finitely many distinct points $z_1, \ldots, z_l, z_\infty \in S^2$, such that:

(i.) *ψ_{n_k} C^∞-converges to z_0 on every compact subset of $S^2 \setminus \{z_\infty\}$.*

(ii.) *The sequence $v_{n_k} := u_{n_k} \circ \psi_{n_k}$ coverges to v uniformly with all derivatives on any compact subset of $S^2 \setminus \{z_1, \ldots, z_l, z_\infty\}$ and the following limits exist and are positive:*

$$m_j := \lim_{\epsilon\to 0^+} \lim_{n\to\infty} E(v_n; B_\epsilon(z_j)), \forall j \in \{1, \ldots, l\}. \tag{142}$$

(iii.)

$$E(v) + \sum_{j=1}^{l} m_j = m_0. \tag{143}$$

(iv.) *If v is constant, then $l \geq 2$.*

Note that if we are given a sequence as in Theorem 5.3, then we can surely apply Theorem 5.5 on neighbourhoods of the bubbling points as the conditions are satisfied. The statement then tells us that a reparametrised sequence with reparametrisations from $PSL_2(\mathbb{C})$ converges up to possibly bubbling at finitely many points.
The importance of conclusion (iv.) will become apparent in the context of stable maps where where we will see that it relates to stability conditions put on stable maps.

The next proposition will also beome important in the study of stable maps. It basically tells us that v and u are connected at the point $u(z_0) = v(z_\infty)$.

Proposition 5.2. *Let $J_n \in \mathcal{J}_\tau(M,\omega)$, $z_0 \in \mathbb{C}$ and u_n, u as in Theorem 5.5 and let ψ_{n_k}, v and $z_1, \ldots, z_l, z_\infty \in S^2$ as in the conclusion of said theorem satisfying* (i.) *to* (iii.). *Then we have $u(z_0) = v(z_\infty)$ and additionally, for every $\epsilon > 0$, there exists a $\delta > 0$ and $n_0 \in \mathbb{N}$, such that:*

$$d(z, z_0) + d(\psi_{n_k}^{-1}(z), z_\infty) < \delta \quad \Rightarrow \quad d(u_{n_k}(z), u(z_0)) < \epsilon, \tag{144}$$

for every $n_k \geq n_0$ and every $z \in S^2$.

For the proofs of these results, we refer to [8, p.105] due to their technical nature. These results mainly rely on estimates of energies for cylinders/annuli.

5.3 Stable Maps and Gromov Compactness

The upcoming section provides an overview of the main results concerning stable maps summarizing the exposition provided in [8], chapter 5.

5.3.1 Definitions

Finally, we introduce the announced notion of stable maps as a generalisation of J-holomorphic curves in the spirit of our convergence considerations in the last subsection. Basically, a stable map is a collection of J-holomorphic spheres modeled on a tree in the graph-theoretical sense. We recall that a n-labelled tree (T, E, Λ), given a non-negative integer n, is a tree (T, E) equipped with a function:

$$\Lambda : \{1, \ldots, n\} \to T, i \mapsto \alpha_i. \tag{145}$$

We moreover set for any $\alpha \in T$:

$$\Lambda_\alpha := \{i \in \{1, \ldots, n\} | \Lambda(i) = \alpha\}. \tag{146}$$

More precisely, we have the following definition:

Definition 5.1. *Given (M, ω) a compact, symplectic manifold, $j \in \mathcal{J}_\tau(M, \omega)$, $n \geq 0$ and (T, E, Λ) an n-labelled tree. A* stable J-holomorphic map in M with n marked points *(modelled over (T, E, Λ)) is a tupel:*

$$(\boldsymbol{u}, \boldsymbol{z}) := (\{u_\alpha\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{(\alpha,\beta) \in E}, \{\alpha_i, z_i\}_{i \in \{1,\ldots,n\}}), \tag{147}$$

consisting of a collection of J-holomorphic spheres $u_\alpha : S^2 \to M$ for each vertex α of the tree T, a collection of nodal points *$z_{\alpha\beta}$ modelled on the set of (oriented) edges $(\alpha, \beta) \in E$ of the tree and lastly a sequence of* marked points *$z_1, \ldots, z_n \in S^2$, such that the following conditions hold:*

- *If $\alpha, \beta \in T$ with $(\alpha, \beta) \in E$, then $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$, i.e. J-holomorphic spheres corresponding to adjacent vertices of the tree map the nodal points corresponding to the edge to the same point in M. We denote the set of nodal points on the sphere corresponding to $\alpha \in T$ by:*

$$Z_\alpha := \{z_{\alpha\beta} | (\alpha, \beta) \in E\}. \tag{148}$$

- *For every $\alpha \in T$ the nodal points in the α-sphere Z_α and the marked points z_i with $i \in \Lambda_\alpha$ are pairwise distinct and called the* special points*. We denote these points by:*

$$Y_\alpha := Z_\alpha \cup \{z_i | i \in \Lambda_\alpha\}. \tag{149}$$

- *If u_α is a constant map, then $|Y_\alpha| \geq 2$, this is called the* stability condition*.*

Thus stable maps modelled on a given tree correspond to certain assignments of J-holomorphic spheres to each vertex. We note that in the bubbling process, we naturally encountered stable maps as limiting curves and by Proposition 5.2, these limiting curves have surely the first property. For the second one, we note that in the case of limits of J-holomorphic spheres, we do not need marked points, by Theorem 5.5 condition 3 is automatically satisfied without specifying further marked points.
Naturally, one can model stable maps on $T \times S^2$ modulo the natural identifiction $(\alpha, z) \sim (\beta, w) \Leftrightarrow (\alpha = \beta, z = w) \vee ((\alpha, \beta) \in E, z = z_{\alpha\beta}, w = z_{\beta\alpha})$.

Next, we note that the notion of energy naturally extends to stable maps by summing the energies of all individual J-holomorphic spheres:

$$E(\mathbf{u}) := \sum_{\alpha \in T} E(u_\alpha), \tag{150}$$

and we define the energy of the subtree $T_{\alpha\beta}$ resulting from taking the connected component of T containing β after removing the edge $(\alpha, \beta) \in E$ by summing only over said subtree:

$$m_{\alpha\beta}(\mathbf{u}) := \sum_{\gamma \in T_{\alpha\beta}} E(u_\gamma). \tag{151}$$

This is inspired by the notion introduced in Theorems 5.3 and 5.5 which do relate to the energies of subtrees emerging from bubbling at points on the sphere. Similarily, we can define the homology class represented by a stable map by adding the homology classes represented by the individual J-holomorphic spheres.

We define an appropriate notion of equivalence of stable maps and thus a reparametrisation group. This will be important when defining more general Moduli spaces of stable maps:

Definition 5.2. *Two stable maps* $(\boldsymbol{u}, \boldsymbol{z})$ *and* $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{z}})$ *modelled over trees* (T, E, Λ) *and* $(\tilde{T}, \tilde{E}, \tilde{\Lambda})$ *are called* equivalent*, if there exists a tree isomorphism* $f : T \to \tilde{T}$ *and a map* $\phi : T \to PSL_2(\mathbb{C}), \alpha \mapsto \phi_\alpha$*, such that:*

$$\tilde{u}_{(\alpha)} \circ \phi_\alpha = u_\alpha, \quad \tilde{z}_{f(\alpha)f(\beta)} = \phi_\alpha(z_{\alpha\beta}), \quad \tilde{z}_i = \phi_{\alpha_i}(z_i). \tag{152}$$

Note that thus $\tilde{\Lambda}_{(\alpha)} = \Lambda_\alpha, \forall \alpha \in T$*.*

We define the reparametrisation group G_T of stable map modelled on a tree (T, E, Λ) to be given by such collections of maps, where f is a tree automorphism of T. The group structure is defined by composition of the automorphisms and the fractional linear maps separately. Note that the group naturally acts on the set of all stable maps modelled on T. We remark that the stability condition ensures that the isotropy subgroup of G_T of any stable map is finite as fractional linear maps are uniquely determined by fixing the values at three distinct points.

Similar to the generalisation of energy to include stable maps, we can say that a stable map represents a homology class $A \in H_2(M; \mathbb{Z})$, if:

$$\sum_{\alpha \in T} u_{\alpha *}[S^2] = A. \tag{153}$$

We then define $\overline{\mathcal{M}}_{0,n}(M, A; J)$ to be the set of equivalence classes of stable maps modelled on n-labelled trees representing the homology class A. One important observation due to the Gromov Compactness result will yield that this space is actually a compact metrizable space. Moreover, we denote for any n-labelled tree T the set of all stable maps modelled on T representing A by $\tilde{\mathcal{M}}_{0,T}(M, A; J)$ and we set $\mathcal{M}_{0,T}(M, A; J) := \tilde{\mathcal{M}}_{0,T}(M, A; J)/G_T$ and call this space a *stratum* of the set $\overline{\mathcal{M}}_{0,n}(M, A; J)$. Note that hence $\overline{\mathcal{M}}_{0,n}(M, A; J)$ is the union over all equivalence classes of trees T of the spaces $\mathcal{M}_{0,T}(M, A; J)$. We call the *top stratum* to be the space $\mathcal{M}_{0,T}(M, A; J)$ where T is the unique tree on one vertex with an n-labelling. We abbreviate this space by $\mathcal{M}_{0,n}(M, A; J)$ and similarily $\tilde{\mathcal{M}}_{0,n}(M, A; J)$. Notice that if $A \neq 0$ and $n = 0$, then we have:

$$\tilde{\mathcal{M}}_{0,0}(M, A; J) = \mathcal{M}(A; J), \quad \mathcal{M}_{0,n}(M, A; J) = \mathcal{M}(A; J)/PSL_2(\mathbb{C}), \tag{154}$$

the usual Moduli space we have already encountered in Section 4. Moreover, if $n = 1$, we see:

$$\tilde{\mathcal{M}}_{0,1}(M, A; J) = \mathcal{M}(A; J) \times S^2, \tag{155}$$

and:

$$\mathcal{M}_{0,1}(M, A; J) = \mathcal{M}(A; J) \times S^2/PSL_2(\mathbb{C}) =: \mathcal{M}(A; J) \times_{PSL_2(\mathbb{C})} S^2. \tag{156}$$

These spaces naturally appear in the study of evaluation maps related to the non-squeezing theorem.

5.3.2 Gromov Convergence

Let us now introduce the notion of *Gromov convergence* of a sequence of J-holomorphic spheres with n marked points each, i.e. an appropriate notion of convergence in $\tilde{\mathcal{M}}_{0,n}(M, A; J)$:

Definition 5.3. *Let J_k be a sequence of ω-tame almost complex structures on M converging in the C^∞-topology to $J \in \mathcal{J}_\tau(M, \omega)$ and let:*

$$(\boldsymbol{u}, \boldsymbol{z}) := \left(\{u_\alpha\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{(\alpha,\beta)\in E}, \{\alpha_i, z_i\}_{1\le i\le n}\right) \tag{157}$$

be a stable map and $u_k : S^2 \to M$ be a sequence of J_k-holomorphic spheres with n distinct marked points $z_1^k, \dots, z_n^k \in S^2$. The sequence $(u_k, \{z_1^k, \dots, z_n^k\}) = (u_k, \boldsymbol{z}_k)$ Gromov converges *to $(\boldsymbol{u}, \boldsymbol{z})$, if there exists a family of Möbius transformations $\{\psi_k^\alpha\}_{\alpha\in T, k\in\mathbb{N}}$ such that the following holds:*

(i.) *For all $\alpha \in T$, the sequence $u_k^\alpha := u_k \circ \psi_k^\alpha$ converges in the C^∞-topology to u_α on compact subsets of $S^2 \setminus Z_\alpha$. This is also called the* Map property.

(ii.) *If $(\alpha, \beta) \in E$, then:*

$$m_{\alpha\beta}(u) = \lim_{\epsilon\to 0}\lim_{k\to\infty} E(u_k^\alpha; B_\epsilon(z_{\alpha\beta}). \tag{158}$$

This is also called the Energy property.

(iii.) *If $(\alpha, \beta) \in E$, then the sequence $\psi_k^{\alpha\beta} := (\psi_k^\alpha)^{-1} \circ \psi_k^\beta$ converges to $z_{\alpha\beta}$ in the C^∞-topology on compact subsets of $S^2 \setminus \{z_{\beta\alpha}\}$. This is also called the* Rescaling property.

(iv.) *$z_i = \lim_{k\to\infty}(\psi_k^{\alpha_i})^{-1}(z_i^k), \forall i \in \{1, \dots, n\}$. This is also called the* Marked Points property.

We note that the Map property basically tells us that the sequence of J-spheres in appropriate parametrisations converges up to bubbling at finitely many points. The Energy property guarantees that energy is preserved by the convergence process, i.e. the limit of the energy concentrated for the sequence at some point is the same energy as the one of the limiting stable map in the corresponding subtree. Rescaling describes the bubbling properties of the compositions of the rescalings. Moreover, as these compositions can be used to link u_k^α and u_k^β, one can interpret the bubbling process in the Gromov convergent sequence from this point of view. Lastly, the Marked Points property just states pointwise convergence at the marked point.

Gromov convergent sequences have many interesting properties, among which we have the following special case of Gromov compactness:

Theorem 5.6. *Let (M, ω) be a compact symplectic manifold and J_k be a sequence of tame almost complex structures on M converging to J in the C^∞-topology.Moreover, assume that we have a sequence of J-spheres $u_k : S^2 \to M$ with $\sup_{k\in\mathbb{N}} E(u_k) < \infty$ and n-tuples of pairwise distinct marked points $\boldsymbol{z}^k := (z_1^k, \dots, z_n^k)$ in S^2 for all $k \in \mathbb{N}$. Then there is a stable map $(\boldsymbol{u}, \boldsymbol{z})$ and a subsequence $(u_{k_j}, \boldsymbol{z}^{k_j})$, such that $(u_{k_j}, \boldsymbol{z}^{k_j})$ Gromov converges to $(\boldsymbol{u}, \boldsymbol{z})$.*

Additionally, the limit of a Gromov convergent sequence is generally not unique due to the ambiguity in the choice of Möbius transformations. However, one can at least see that the limit is unique up to equivalence and thus Gromov convergence is well-defined on equivalence classes:

Theorem 5.7. *Let J_k be a sequence of tame almost complex structures on M which converges in the C^∞-topology to J and assume that the sequence $(u_k, \boldsymbol{z}^k)$ of J_k-holomorphic spheres with n distinct marked points Gromov converges to $(\boldsymbol{u}, \boldsymbol{z})$ and $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{z}})$. Then these two stable maps are equivalent. Thus, limits of Gromov convergent sequences are well-defined up to equivalence.*

Therefore, Gromov convergence gives us a welldefined notion of convergence on the set of equivalence classes of stable maps.

For the proof of these two important results, we refer to [8].

5.4 Compactness and Metrization

We want to generalize the notion of Gromov convergence to general sequences of stable maps and then to generalize Gromov compactness to this new situation. Note that, due to the possibly different trees on which the stable maps are modelled, this is not completely straightforward. For this, we define:

Definition 5.4. *Let J_k be a sequence of tame almost complex structures on (M, ω) converging to J in the C^∞-topology. A sequence*

$$(\boldsymbol{u}_k, \boldsymbol{z}_k) = (\{u_k^\alpha\}_{\alpha \in T_k}, \{z_{\alpha\beta}^k\}_{\alpha E \beta}, \{\alpha_i^k, z_i^k\}_{1 \le i \le n}), \tag{159}$$

of stable maps with n marked points each modelled on trees T_k is said to Gromov converge *to a stable map:*

$$(\boldsymbol{u}, \boldsymbol{z}) = (\{u^\alpha\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le n}), \tag{160}$$

if and only if for sufficiently large $k \in \mathbb{N}$, there exists a surjective tree homomorphism:

$$f_k : T \to T_k, \tag{161}$$

as well as a collection of Möbius transformations $\{\psi_k^\alpha\}_{\alpha \in T}$, such that the following hold:

(MAP) *For every $\alpha \in T$, the sequence $u_k^{f_k(\alpha)} \circ \psi_k^\alpha$ converges to u^α uniformly on compact subsets of $S^2 \setminus Z_\alpha$ with all derivatives.*

(ENERGY)*If $\alpha E \beta$, then:*

$$m_{\alpha\beta}(\boldsymbol{u}) = \lim_{\epsilon \to 0} \lim_{k \to \infty} E_{f_k(\alpha)}\Big(\boldsymbol{u}_k; \psi_k^\alpha\big(B_\epsilon(z_{\alpha\beta})\big)\Big). \tag{162}$$

(RESCALING) *If $\alpha, \beta \in T$ are adjacent and k_j is a subsequence such that $f_{k_j}(\alpha) = f_{k_j}(\beta)$, then $\psi_{k_j}^{\alpha\beta} := \big(\psi_{k_j}^\alpha\big)^{-1} \circ \psi_{k_j}^\beta$ converges to $z_{\alpha\beta}$ uniformly with all derivatives on compact subsets of $S^2 \setminus \{z_{\beta\alpha}\}$.*

(NODAL POINT) *If $\alpha, \beta \in T$ are adjacent and again k_j a subsequence such that $f_{k_j}(\alpha) \neq f_{k_j}(\beta)$, then:*

$$z_{\alpha\beta} = \lim_{j \to \infty} (\psi_{k_j}^\alpha)^{-1}\Big(z_{f_{k_j}(\alpha) f_{k_j}(\beta)}^{k_j}\Big). \tag{163}$$

(MARKED POINT) *For all $1 \le i \le n$, we have $\alpha_i^k = f_k(\alpha_i)$ as well as:*

$$z_i = \lim_{k \to \infty} \big(\psi_k^{\alpha_i}\big)^{-1}(z_i^k). \tag{164}$$

We note that the definition shares many features with our previous definition of Gromov convergence. The main differences are the necessity to include surjective tree homomorphisms to account for different modelling trees as well as including one futher condition, the *(NODAL POINT)* condition, as in our previous setup no such condition was necessary due to the sequence just consisting of J-holomorphic spheres together with some marked points.

The interpretation as previously discussed carry over to this more general notion.

The main interest in this complicated notion of convergence arises from the fact that we can easily generalize the Gromov convergence theorem to this setup. This is simply due to the fact that the energy bound provides us with an upper bound on the number of vertices in the tree and hence the stable maps of the sequence have are modelled on only finitely many possible trees. By choosing a subsequence, one might assume that they are all modelled on the same tree. Now apply our previous Gromov compactness result to stable maps which we get by considering just all J-holomorphic curves corresponding to a vertex in the common tree together with their marked points. To conclude, one just has to check that the resulting limiting stable maps form together a stable map and that all conditions for Gromov convergence are satisfied, which is now rather easy. For more details, we refer to [8, p.140 - 141].

Thus we have derived:

Theorem 5.8 (Gromov Compactness)**.** *Let J_k be a sequence of tame almost complex structures on a compact symplectic manifold M which converges in the C^∞-topology to J. Moreover, assume that $(\boldsymbol{u}_k, \boldsymbol{z}_k)$ is a sequence of stable maps with bounded energy, i.e.:*

$$\sup_{k\in\mathbb{N}} E(\boldsymbol{u}_k) < \infty, \tag{165}$$

then the sequence contains a Gromov convergent subsequence.

Additionally, one can also generalize the uniqueness result for Gromov convergence to this situation.

Lastly for this section, we want to note that one can use the notion of Gromov convergence to introduce a topology on the space of equivalence classes of stable maps $\overline{\mathcal{M}}_{0,n}(M, A; J)$ representing a certain homology class A. For this, define a set to be closed, if it is closed under Gromov convergent sequences. Due to certain properties of Gromov convergence, see [8, p.147], this yields a topology on said space. Moreover, this topology is even metrizable and $\overline{\mathcal{M}}_{0,n}(M, A; J)$ is compact in this topology due to Gromov compactness and the fixed homology class providing a common bound on the energy. This in turn can be used in order to note that the evaluation maps which can be defined as in [8], are continuous with respect to the Gromov topology induced this way and such considerations will be useful in the context of Gromov-Witten invariants, where one proves that the evaluation map is a pseudocycle under certain conditions which leads to the definition of the Gromov-Witten invariants. For more details, we refer to [8, p.150] as well as [8, Chapter 6].

6 Applications

Finally, after introducing several different notions of convergence of J-holomorphic maps, we are able to give some applications. We will be working on a proof of the non-squeezing theorem by using some considerations form Section 3 and evaluation maps. A key observation will be that in our situation, there are J-holomorphic spheres passing through any given point.

Using the non-squeezing theorem, we will be able to prove an interesting property of the group of symplectomorphisms, namely that it is closed under C^0-limits. This follows from certain observations related to symplectic capacities and their invariance under symplectomorphism, which gives a useful characterisation of symplectomorphisms for our argument.

The presentation given in the upcoming section gives an overview over some of the ideas involved in the proof of the important results given. The treatment follows [8] and [7] and all results as well as the structure and presentation are taken from these sources.

6.1 Linear Symplectic Results

We follow the presentation in [7, p.55 - 63]: The easiest examples of symplectomorphisms are the affine linear symplectic maps on $\mathbb{R}^{2n}$, where we equip $\mathbb{R}^{2n}$ with the usual symplectic structure. These are maps $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, where:

$$\psi(x) = Ax + b, \tag{166}$$

with A being a symplectic matrix, i.e. $A^T J_0 A = J_0$. As usual, J_0 denotes the standard almost complex structure endomorphism on this space, and b is any vector. As in [7, p.55], we will now outline how to deduce some properties of these maps. For this, we follow the notation in [7] and denote by:

$$Z^{2n}(R) := B^2_R(0) \times \mathbb{R}^{2n-2}, \forall R \in \mathbb{R}_{>0}, \tag{167}$$

the cylinder of radius R. Let moreover $e_1, \dots, e_n, f_1, \dots, f_n$ be the standard basis of $\mathbb{R}^{2n}$ labelled in a way that they form a symplectic basis with $e_i^T J_0 f_j = \delta_{ij}$. We have the following theorem:

Theorem 6.1 (Linear Non-Squeezing Theorem)**.** *If ψ is an affine linear symplectic map such that for some $r, R > 0$:*

$$\psi\big(B^{2n}_r(0)\big) \subset Z^{2n}(R), \tag{168}$$

then $r \leq R$.

The main idea of the proof as given in [7, p.55] is to first reduce to the case $r = 1$ for simplicity and then express the condition $\psi\big(B^{2n}_1(0)\big) \subset Z^{2n}(R)$ by the following inequality:

$$\sup_{\|x\|=1} \Big((\langle e_1, \psi(x)\rangle)^2 + (\langle f_1, \psi(x)\rangle)^2 \Big) \leq R^2. \tag{169}$$

This follows immediately by using the definition of our basis. Then by inserting the definition of ψ as an affine linear map and cleverly choosing x, we get the desired result.

Theorem 6.1 basically tells us that one cannot embed balls of large radius into smaller cylinders. Therefore, from a symplectic viewpoint, there must be a difference between these two spaces which the symplectic structure is able to detect. This naturally leads to the notion of symplectic width which forms an invariant for subsets and is invariant under symplectic

linear maps.
It is remarkable, that having the non-squeezing property, i.e. that a map satisfies the property of Theorme 6.1, characterises symplectic and anti-symplectic maps, i.e. we have the following result:

Theorem 6.2. *Let A be a non-singular matrix, such that A, A^{-1} both have the non-squeezing property, i.e.:*

$$\forall r, R > 0 : A\big(B_r^{2n}(0)\big) \subset Z^{2n}(R) \Rightarrow r \leq R, \tag{170}$$

and similarily for A^{-1}. Then A is either symplectic or anti-symplectic.

For a proof, we refer to [7, p.56].

6.2 Non-Squeezing Theorem

After having seen the linear version of the Non-Squeezing Theorem, we will now state the result for more general symplectomorphisms as in [8]:

Theorem 6.3 (Non-Squeezing Theorem)**.** *Let $i : B_r^{2n}(0) \to \mathbb{R}^{2n}$ be a symplectic embedding with respect to the usual symplectic structure, such that:*

$$i\big(B_r^{2n}(0)\big) \subset Z^{2n}(R). \tag{171}$$

Then $r \leq R$.

The main idea of the proof which is presented in [8] is to reduce our considerations to symplectic embeddings of the ball into $B_R^2 \times V$, where V is a closed symplectic manifold of dimension $2n-2$ which is aspherical. One does this by embedding $\mathbb{R}^{2n-2}$ symplectically into a symplectic torus. Moreover, we next embed the ball $B_r^2(0)$ symplectically into $\mathbb{CP}^1 \approx S^2$. Then, we recall from Section 3, where we have seen that the Moduli space for the product space $S^2 \times V$ is $2n+4$ dimensional. By considering the evaluation map:

$$\text{ev} : \mathcal{M}^*(A, J) \times S^2 \to S^2 \times V, \tag{172}$$

we note that the map is invariant under the action of the Möbius group $G = PSL_2(\mathbb{C})$, thus we get a map:

$$\text{ev} : \mathcal{M}^*(A, J) \times_G S^2 \to S^2 \times V. \tag{173}$$

Note that now the RHS is a compact manifold of dimension $2n$. Moreover, due to our calculations in Section 4, we know that the map has degree 1. By cobordisms, this thus holds for all regular almost complex structures.

This will now become our main ingredient to prove the Non-Squeezing Theorem. Namely, we have that through every point in $S^2 \times V$, there passes a J-holomorphic curve, thus especially through the image of the center of the ball $B_r^{2n}(0)$ under the embedding. Pulling this J-holomorphic sphere back to a J-holomorphic curve in $B_r^{2n}(0)$ results in a minimal surface in $B_r^{2n}(0)$ through the origin due to the almost complex structure on the ball being the standard one and therefore, the result now follows by seeing that the area of the J-curve must be bigger or equal than πr^2 and the area can be calculated directly by using integrals of the symplectic structures. For more details, we refer to [8, p.8,p.323].
Additionally, we would like to mention that in [8], another proof of this theorem is outlined in chapter 9. The proof there uses complex blow-up techniques rather than minimal surface theory to arrive at the desired conclusion.

6.3 Group of Symplectomorphisms

Again, our treatment, definitions and results follow closely the one given in [7, p.457]: The Non-Squeezing Theorem has some interesting applications. For example, we can use it to deduce that symplectomorphisms are stable under C^0-limits. But before we are able to deduce this statement, we have to derive a characterisation of symplectomorphisms which uses symplectic capacities:

Theorem 6.4. *Let $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a diffeomorphism and let c be a symplectic capacity on $\mathbb{R}^{2n}$ with $c(B_1^{2n}(0)) = c(Z^{2n}(1)) = \pi$, then the following are equivalent:*

- *ψ preserves the symplectic capacity of ellipsoids*
- *ψ is either a symplectomorphism or an anti-symplectomorphism*

We note that a *symplectic capacity* c on $\mathbb{R}^{2n}$ is nothing else than a map assigning to each $A \subset \mathbb{R}^{2n}$ a number $c(A) \in [0, \infty]$, such that the following properties hold for this map:

- If there is a symplectomorphism ψ on $\mathbb{R}^{2n}$, such that $\psi(A) \subset B$ for some subsets $A, B \subset \mathbb{R}^{2n}$, then $c(A) \leq c(B)$. This is called the *monotonicity property.*
- For all $\lambda > 0$, we have $c(\lambda A) = \lambda^2 c(A)$. This is called the *conformality property.*
- $\mathrm{c}(\mathrm{B}_1^{2n}(0)) > 0$ and $c(Z^{2n}(1)) < \infty$. This is called the *non-triviality* condition.

Note that one can use the Non-Squeezing Theorem to construct a symplectic capacity by defining:

$$\bar{c}(A) := \inf\{\pi r^2 | A \text{ embeds symplectically into } Z^{2n}(r)\}, \tag{174}$$

for all $A \subset \mathbb{R}^{2n}$. That it fullfills the properties of a symlectic capacity follows directly from the definition. Moreover, one sees that $\bar{c}(B_1^{2n}(0)) = \bar{c}(Z^{2n}(1)) = \pi$.
Using the theorem above, we can now deduce:

Theorem 6.5. *For every symplectic manifold (M, ω), the group of symplectomorphisms of the manifold M to itself is closed under C^0-limits in the space of all diffeomorphisms.*

The idea of the proof is to consider everything locally, as being a symplectomorphism is a local property of a diffeomorphism. Thus one can reduce to the case $M = \mathbb{R}^{2n}$ via Darboux-charts. Then one uses symplectic capacities to deduce that the limiting map preserves symplectic capacity of ellipsoids which in turn implies that the map needs to be symplectic or anti-symplectic. One concludes that it is symplectic by considering product maps which again are symplectic and by similar reasoning lead to a symplectic or anti-symplectic limit. But this leads to the observation that the limit map must have been a symplectomorphism and hence we are done.

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