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Geometric Quantization

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Introduction

Quantization is the process of transitioning from classical physical systems and phenomena to their quantum mechanical counterparts, in particular with the focus on keeping intact certain analogies between both descriptions, such as the symmetries of a system under consideration. While many different strategies have been developed during the 20th century to address this problem, it is now known that on the one hand not all quantum mechanical systems are represented by a correpsonding classical one and on the other hand it is impossible in a lot of cases to define a quantization procedure without giving up on at least some desirable properties. To this day the theory of Geometric Quantization remains one of the most sucessful ones in unifying various different quantization techniques that have proven to be successful when considering particular physical systems and in presenting a generalized procedure that can be applied in a vast multitude of cases.

This thesis is concerned with presenting the most fundamental aspects of Geometric Quantization, originally developed in its modern form by Betram Kostant and Jean-Marie Souriau, in a mostly self-contained way. We will begin by setting up the framework of quantization and motivating the desirable aspects of a quantization procedure by contrasting the formalisms of classical and quantum mechanics. The second section contains a description of the fundamental mathematical theories that lie at the heart of geometric quantization. In the third section we will discuss the process of prequantization in detail which will lead to a preliminary version of geometric quantization. Finally, the last section discusses how this can be extended through geometric quantization, in particular when applied to Kähler manifolds which will eventually lead to the description of holomorphic quantization.

At this point I would like to thank Prof. Dr. Cannas da Silva for suggesting the topic of this paper and for her patient supervision throughout the semester.

Contents

1. Canonical Quantization

1.1. **Classical Mechanics.** To understand the concepts and goals of geometric quantization, we quickly review the basic formulation of classical mechanics in the language of symplectic geometry. In order to establish some comparability, we list the different aspects of the formalism in the same way as the postulates for the quantum mechanical formalism in subsection 1.2.

Classical Mechanical Systems

1.) **Phase Space**
An arbitrary classical physical system is described by a symplectic manifold (M, ω) (or more generally a Poisson Manifold $(M, \{\})$). The state of the system is given by a point $x \in M$.

2.) **Time Evolution**
The evolution of a classical physical system is determined by a Hamiltonian $H \in \mathcal{C}^\infty(M)$ and the corresponding Hamiltonian equations. In particular, the evolution of an observable in time is determined by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = X_H(f) = \{f, H\}.$$

3.) **Composite Systems**
The phase space of a composite physical system is the Cartesian product of the individual phase spaces

$$M_{AB} = M_A \times M_B.$$

4.) **Observables**
An observable f is a smooth (possibly complex-valued) function on M

$$f \in \mathcal{C}^\infty(M).$$

The value of an observable for a system state $x \in M$ is simply given by evaluating the function at the corresponding point $f(x)$.

1.2. **Quantum Mechanics.** Before we discuss the formalism of quantum mechanics, let us review the basic definition of a Hilbert space which will play a central role.

Definition 1.1 (Hilbert space)**.** A complex Hilbert space $\mathcal{H}$ is a complex vector space equipped with an inner product $\langle \cdot, \cdot \rangle$, that is complete with respect to the metric induced by $\langle \cdot, \cdot \rangle$.

If $\mathcal{H}$ is a Hilbert space, the inner product induces an isomorphism

$$\Phi \colon \mathcal{H} \to \mathcal{H}^*, v \mapsto \langle v, \cdot \rangle.$$

We write elements of $\mathcal{H}$ as so-called "kets" $|\psi\rangle \in \mathcal{H}$ and the corresponding elements of the dual space under the isomorphism above as bras $\Phi(|\psi\rangle) = \langle\psi| \in \mathcal{H}^*$. Note that the formalism is chosen in a way, such that

$$\langle \psi_1 | \psi_2 \rangle := \langle \psi_1 | \left(|\psi_2\rangle \right) = \langle \psi_1, \psi_2 \rangle.$$

This is the popular Bra-ket or Dirac notation introduced by Dirac in 1939. For a more detailed account, see for example [12].

The most important aspects of the formalism of quantum mechanics can be listed as a number of postulates (see [12],[4]). Note that this list is not exhaustive and one can take different axiomatic approaches to derive some of the postulates from others. However, here we simply try to summarize the most fundamental aspects in the mathematical formulation of quantum mechanics. The listed postulates can partly be derived from the original description of Quantum Mechanics by Dirac in his work "The Principles of Quantum Mechanics" [2].

The Postulates of Quantum Mechanics

1.) **State Space**
Any physical system is represented by a Hilbert space $\mathcal{H}$. The state of the system is described by an element $|\psi\rangle \in \mathcal{H}$, such that $\langle\psi|\psi\rangle = 1$. $|\psi\rangle$ is called the state vector of the system.

2.) **Time Evolution**
The evolution of a closed system in time is described by a Hermitian operator H called the Hamiltonian operator and the Schrödinger equation

$$i\hbar \frac{\mathrm{d}\,|\psi\rangle}{\mathrm{d}t} = H\,|\psi\rangle\,.$$

It follows, that the evolution of a closed system in time is described by some unitary operation U, i.e. if we denote by $|\psi(t)\rangle$ the state of the system at time t, we have

$$|\psi(t_2)\rangle = U(t_2, t_1)\,|\psi(t_2)\rangle$$

where $U(t_2, t_1) = \exp\left(\frac{-iH(t_2-t_1)}{\hbar}\right)$ is a unitary operator called the time evolution operator for t_1, t_2.

3.) **Composite Systems**
The state space of a composite physical system is the tensor product of the individual systems under consideration

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

4.) **Observables**
Observables are linear self-adjoint operators A on $\mathcal{H}$. The expectation value of an observable A for a system in the state $|\psi\rangle$ is given by

$$\langle A\rangle = \langle\psi|\,A\,|\psi\rangle\,.$$

5.) **Quantum Measurements**
Measurements are described by a set of measurement operators $\{M_m\}$ that satisfy

$$\sum_m M_m^\dagger M_m = \mathbb{1},$$

where the index m denotes the possible outcomes of the measurement. The probability of measuring outcome m for a state $|\psi\rangle$ is then given by

$$p(m) = \langle\psi|\,M_m^\dagger M_m\,|\psi\rangle$$

and after the measurement of an outcome m, the state collapses to

$$|\psi'\rangle = \frac{M_m\,|\psi\rangle}{p(m)}.$$

Remark 1.2 (The Heisenberg Picture). The postulate about time evolution as formulated above refers to the so-called Schrödinger picture of quantum mechanics where operators are fixed and states evolve in time. Another formalism is provided by the Heisenberg picture. Here the state of the system is fixed and doesn't change but the operators corresponding to observables undergo a transformation in time, which then allows for changes in measurements or expectation values of the system. The evolution of quantum operators in the Heisenberg picture is determined by the Heisenberg equation

$$i\hbar \frac{\mathrm{d}A(t)}{\mathrm{d}t} = [A(t), H(t)]$$

where H again denotes the Hamiltonian operator of the system. Note the clear resemblance to the time evolution of classical observables in the language of symplectic geometry as discussed in subsection 1.1.

1.3. **Symmetry and Irreducibility.** Two other important concepts we would like to capture with our quantization procedure are the related concepts of symmetry and irreducibility. In particular, if a classical system carries some kind of symmetry, our quantized system should carry an analogous symmetry of some form. We will largely follow [3] and begin by comparing the concepts for the classical and quantum mechanical case. First, consider the notion of a complete set of observables, which intuitively describes a set of observables that fully determine the state of the system.

Definition 1.3 (Complete Set of Classical Observables)**.** Let (M, ω) be a symplectic manifold. A set of classical observables $f_1, \ldots, f_n \in \mathcal{C}^\infty(M)$ is called complete if for an arbitrary observable $g \in \mathcal{C}^\infty(M)$, it holds that

$$\{g, f_i\} = 0 \text{ for all } i \in \{1, \cdots, n\} \implies g = \text{const.}$$

Definition 1.4 (Complete Set of Quantum Observables)**.** Let $\mathcal{H}$ be a Hilbert space. A set of self-adjoint operators (quantum observables) $\hat{f}_1, \ldots, \hat{f}_n$ is called complete if for all operators $\hat{g} \in \mathcal{O}(\mathcal{H})$

$$[\hat{g}, \hat{f}_i] = 0 \text{ for all } i \in \{1, \ldots, n\} \implies \hat{g} = z\mathbb{1} \quad \text{for some } z \in \mathbb{C}.$$

When discussing symmetries and the definition of irreducible spaces, we will need the concept of a group representation

Definition 1.5 (Representation)**.** A representation of a group G on a vector space V is a group homomorphism

$$\rho\colon G \to GL(V).$$

Every element of G thus induces a linear isomorphism $\rho(g)\colon V \to V$ and it holds that $\rho(g_1) \circ \rho(g_2) = \rho(g_1 g_2)$.

Definition 1.6 (Irreducibility)**.** Let V be a vector space and $\rho\colon G \to GL(V)$ some representation of a group G on V. A linear subspace $W \subseteq V$ is called invariant under ρ, if $\rho(W) \subseteq W$. The representation $\rho\colon G \to V$ is called irreducible, if the only invariant subspaces of V with respect to ρ are the trivial space $\{0\}$ and V itself. In a slight abuse of notation, one sometimes simply calls the vector space V irreducible.

The concepts of completeness and irreducibility are related in the following way (see [3]).

Lemma 1.7. *Let $\mathcal{H}$ be a Hilbert space and $\hat{f}_i$ a complete set of self-adjoint operators on $\mathcal{H}$. Then the only proper closed subspace of $\mathcal{H}$ that is invariant under all operators $\hat{f}_i$ is the trivial space $\{0\}$.*

Proof. Let $W \subseteq \mathcal{H}$ be a closed linear subspace that is invariant under the operators $\hat{f}_i$. We can decompose any element $\psi \in \mathcal{H}$ uniquely as

$$|\psi\rangle = |\psi\rangle_W + |\psi\rangle_{W^\perp} \qquad |\psi\rangle_W \in W,\ |\psi\rangle_{W^\perp} \in W^\perp.$$

Note that if W is invariant, then so is $W^\perp$, because the operators $\hat{f}_i$ are self-adjoint. In particular for $|\psi\rangle \in W^\perp$ and arbitrary $|\psi'\rangle \in W$

$$\left\langle \hat{f}_i(\psi) \middle| \psi' \right\rangle = \left\langle \psi \middle| \hat{f}_i(\psi') \right\rangle = 0.$$

Denote by $\pi_W : \mathcal{H} \to W$ the projection operator of W. Then for all $i \in \{1, 2, \cdots, n\}$

$$\pi_W \circ \hat{f}_i(|\psi\rangle) = \pi_W(\hat{f}_i(|\psi\rangle_W) + \hat{f}_i(|\psi\rangle_{W^\perp})) = \hat{f}_i(|\psi\rangle_W) = \hat{f}_i \circ \pi_W(|\psi\rangle).$$

Hence π_W commutes with all operators $\hat{f}_i$. However, then $\pi_W = z\mathbb{1}$ for some $z \in \mathbb{C}$ and it follows that $W = \mathcal{H}$ or $W = \{0\}$. □

Definition 1.8 (Classical Symmetry)**.** In classical mechanics a group of symmetries on a symplectic manifold (M, ω) is a Lie group G with a symplectic action ψ on M, i.e. a group homomorphism

$$\psi \colon G \to \mathrm{Symp}(M).$$

In order to describe quantum symmetries we consider the formulation of quantum mechanics in terms of ray spaces. Recall that the state of a quantum system was described by a normalized vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. Equivalently one can also describe the system with the projective line space of $\mathcal{H}$

$$\mathcal{PH} := \{z\,|\psi\rangle\,| z \in \mathbb{C}\} = \mathcal{H}/\sim$$
$$|\psi_1\rangle \sim |\psi_2\rangle \Leftrightarrow |\psi_1\rangle = z\,|\psi_2\rangle \text{ for some } z \in \mathbb{C}$$

The state of the quantum system is then described by some element in projective space $[|\psi\rangle]_\sim \in \mathcal{PH}$.
Note that according to the Measurement Postulate, the transition probability from one quantum state $|\varphi\rangle$ to another quantum state $|\psi\rangle$, i.e. the probability to measure a state $|\varphi\rangle$ as another state $|\psi\rangle$ is given by

$$\frac{|\langle\psi|\varphi\rangle|^2}{\langle\psi|\psi\rangle\,\langle\varphi|\varphi\rangle}.$$

This motivates the following definition.

Definition 1.9 (Quantum Symmetry)**.** A quantum symmetry is a bijective map

$$g \colon \mathcal{PH} \to \mathcal{PH},$$

such that the ray product (and therefore the transition probability)

$$\kappa \colon \mathcal{PH} \times \mathcal{PH} \to \mathbb{R}^+, \quad ([|\psi\rangle]_\sim, [|\varphi\rangle]_\sim) \mapsto \frac{|\langle\psi|\varphi\rangle|^2}{\langle\psi|\psi\rangle\,\langle\varphi|\varphi\rangle}$$

is conserved under g.

An elementary theorem in the mathematical formulation of quantum mechanics proved by Eugene Wigner in 1931 ([16]) relates quantum symmetries in the sense of Definition 1.9 to unitary and anti-unitary operators on the underlying Hilbert space. An accessible proof of Wigner's Theorem 1.10 can be found in [14].

Theorem 1.10 (Wigner's Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and $g\colon \mathcal{PH} \to \mathcal{PH}$ a quantum symmetry. Denote by $\pi\colon \mathcal{H} \to \mathcal{PH}$ the projection under complex lines. Then there exists a unitary or an anti-unitary operator U_g, such that for all $|\psi\rangle \in \mathcal{H}$*

$$\pi \circ U_g(|\psi\rangle) = g \circ \pi(|\psi\rangle).$$

If U_g, U'_g are two such operators inducing g, it further holds that

$$U_g = e^{i\alpha} U'_g.$$

1.4. **Canonical Quantization.** The previous discussion of the mathematical formulation of classical and quantum mechanics now allows us to define certain conditions that should hold when quantizing a classical system. A detailed introduction to quantization and a presentation of different elementary methods can be found in [1]. To begin with, we consider canonical quantization of a classical mechanical system with n degrees of freedom in Euclidean space. Here the phase space is simply given by $M = \mathbb{R}^n \times \mathbb{R}^n$ with canonical coordinates p_i, q_i and the corresponding Hilbert space for the quantum system is the space of square-integrable functions $\mathcal{H} = L^2(R^n)$.
The goal of canonical quantization is to assign to every classical observable $f \in \mathcal{C}^\infty(M)$ a quantum observable, i.e. some self-adjoint operator $Q_f \in \mathcal{O}(\mathcal{H})$, such that the following quantization conditions are satisfied (cf. [1]). Note that any classical observable is a function in the coordinates of M, hence a function of q_i, p_i.

Quantization Conditions

1.) **Linearity**
The assignment $f \mapsto Q_f$ is linear

$$Q_{f+g} = Q_f + Q_g \qquad Q_{\lambda f} = \lambda Q_f, \quad \lambda \in \mathbb{C}.$$

2.) **Identity Operator**
If we denote by 1 the constant function of value 1 on M, then 1 induces the identity operator $\mathbb{1}$ on $\mathcal{H}$

$$Q_1 = \mathbb{1}.$$

3.) **Commutation Relations**
The assignment $f \mapsto Q_f$ preserves the structure of the Poisson bracket $\{\cdot,\cdot\}$ on M and the commutator $[\cdot,\cdot]$ on $\mathcal{H}$ up to a constant $i\hbar$

$$[Q_f, Q_g] = i\hbar Q_{\{f,g\}}.$$

4.) **Position and Momentum Operators**
For the canonical coordinates q_i, p_i, we recover the standard position and momentum operators defined Q_{q_i}, Q_{p_i}, defined by

$$Q_{q_i}(|\psi\rangle) = q_i\,|\psi\rangle \qquad Q_{p_i}(|\psi\rangle) = -i\hbar \frac{\partial\,|\psi\rangle}{\partial q_i}.$$

5.) **von Neumann Rule**
For any function $g\colon \mathbb{R} \to \mathbb{R}$ and any observable $f \in \mathcal{C}^\infty(M)$ for which the operator expression $g(Q_f)$ is defined, it holds that

$$Q_{g\circ f}(|\psi\rangle) = g(Q_f)(|\psi\rangle).$$

(Note that for the operator expression we identify multiplication in $\mathbb{R}$ with composition in $\mathcal{O}(\mathcal{H})$, e.g. if $g(x) = x^2$, then $g(Q_f) = Q_f \circ Q_f$).

Unfortunately the conditions as stated above are inconsistent with each other and one can show that there exists no mapping $f \mapsto O_f$ defined on $\mathcal{C}^\infty(M)$ satisfying 1.) - 5.). In fact, any choice of three out of the listed conditions 1.), 3.), 4.) and 5.) already leads to a contradiction ([1]). As one example we consider

Theorem 1.11 (Groenewold's Theorem)**.** *There is no quantization operator $f \mapsto Q_f$ satisfying quantum conditions 1.), 3.) and 4.) that is defined for all $f \in \mathcal{C}^\infty(M)$ which can be expressed as polynomials in q_i, p_i of degree up to 4.*

This result is due to Hilbrand J. Groenewold [5].

There have been different approaches in altering the requirements for a quantization procedure. The method of deformation quantization for example only considers conditions 1.)-4.) and only requires the commutation relations 3.) to hold in the limit of $\hbar \to 0$.
The approach of geometric quantization explained herein is based on restricting the space of quantizable observables and not taking the von Neumann Rule into account. In doing so, we obtain a theory that can be extended to arbitrary classical mechanical systems modeled on symplectic manifolds (M, ω). We will review the quantization conditions when discussing prequantization in earnest in section 3.

2. Mathematical Preliminaries

In this section, we will describe most necessary mathematical concepts that will be needed in the precise formulation of prequantization and holomorphic geometric quantization in sections 3 and 4. We assume that the reader is familiar with differential geometry and basic concepts of symplectic geometry, however, we will quickly review the concepts used most prominently in the following discussions.
The most relevant topics are covered in the subsections about Polarizations 2.5 and Complex Line Bundles 2.7. Readers familiar with both symplectic and complex geometry should be able to read only these parts or even jump straight ahead to the description of Prequantization in section 3 and still follow the main procedure.

2.1. **Symplectic Manifolds.**

A detailed introduction to symplectic geometry including proofs of the statements listed here can be found in [13].

Definition 2.1 (Symplectic Manifold)**.** Let M be a smooth manifold. A symplectic form ω on M is a closed non-degenerate differential 2-form. The pair (M, ω) is called a symplectic manifold.

Remark 2.2. Note that any symplectic manifold necessarily has even dimension.

Definition 2.3 (Local Symplectic Potential)**.** Let (M, ω) be a symplectic manifold. A one-form θ defined on some subset $U \subseteq M$, such that on U it holds that

$$d\theta = \omega$$

is called a local symplectic potential of (M, ω).

Definition 2.4 (Hamiltonian Vector Field)**.** Let (M, ω) be a symplectic manifold and let $f \in \mathcal{C}^\infty(M)$ be a smooth function on M. Then there exists a unique vector field $X_f \in \mathcal{X}(M)$, such that

$$\omega(X_f, \cdot) = -df.$$

X_f is called the Hamiltonian vector field of f.

Definition 2.5 (Poisson Manifold)**.** Let M be a smooth manifold. A Poisson structure on M is a Lie bracket $\{\cdot,\cdot\}$ on $\mathcal{C}^\infty(M)$ that satisfies the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

The pair $(M, \{\cdot,\cdot\})$ is called a Poisson manifold.

Lemma 2.6 (Poisson Structure of a Symplectic Manifold)**.** *Let (M, ω) be a symplectic manifold. Then the following defines a Poisson structure on M*

$$\{f, g\} := \omega(X_f, X_g) = X_f(g).$$

Lemma 2.7. *Let $(M.\omega)$ be a symplectic manifold. The assignment*

$$\mathcal{C}^\infty(M) \to \mathcal{X}(M), \qquad f \mapsto X_f$$

is a Lie algebra homomorphism with respect to the Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{C}^\infty(M)$ and the Lie bracket of vector fields $[\cdot,\cdot]$, i.e. $X_{\{f,g\}} = [X_f, X_g]$.

Definition 2.8 (Symplectomorphism)**.** Let $(M, \omega), (M', \omega')$ be two symplectic manifolds. A diffeomorphism $\varphi \colon M_1 \to M_2$ is called a symplectomorphism if

$$\varphi^* \omega' = \omega.$$

We denote the set of symplectomorphisms on M by $\mathrm{Symp}(M)$.

2.2. **Complex Vector Spaces.** In the procedure of prequantization we will eventually work with complexified tangent bundles and complex line bundles. In the following subsections we will discuss the most important concepts involved when discussing complex as opposed to real manifolds.
To begin with, we consider different constructions that allow us to turn real vector spaces into complex ones, i.e. vector spaces over $\mathbb{C}$. A detailed introduction to complex finite vector spaces can be found in [6].

Definition 2.9 (Complex Structure)**.** Let V be a real vector space. A complex structure on V is a linear operator

$$J \colon V \to V, \quad \text{such that } \; J^2 = -1.$$

Lemma 2.10. *Let V be a real vector space equipped with a complex structure J. Then J induces a complex vector space structure on V where scalar multiplication is given by*

$$\cdot \colon \mathbb{C} \times V \to V, \quad (a + bi, v) \mapsto av + bJ(v).$$

Proof. The scalar multiplication defined by J as above is clearly distributive with respect to addition in $\mathbb{C}$ and V and trivially $1 \cdot v = v$. It remains to show that $\cdot$ is associative. Let therefore $a+bi, c+di \in \mathbb{C}$ and consider

$$\begin{aligned}((a+bi)(c+di)) \cdot v &= ((ac-bd)+(bc+ad)i) \cdot v \\ &= (ac-bd)v + (bc+ad)J(v) \\ &= acv + adJ(v) + bcJ(v) - bd(v) \\ &= (a+bi) \cdot (cv + dJ(v)) \\ &= (a+bi) \cdot ((c+di) \cdot v).\end{aligned}$$

□

In the literature, complex structures as defined in 2.9 are often referred to as almost complex structures. The distinction will only become important once we consider complex structures on manifolds.

Definition 2.11 (Complexification)**.** Let V be a real vector space. We define the complexification of V as the complex vector space given by the tensor product of V and $\mathbb{C}$ over the real numbers $V_{\mathbb{C}} := V \times \mathbb{C}$, where scalar multiplication is defined via

$$\cdot \colon \mathbb{C} \times V_{\mathbb{C}}, \quad (z(v, z')) \mapsto (v, zz').$$

Lemma 2.12. *Let V be a real finite-dimensional vector space. Then we can identify $V_{\mathbb{C}} \cong V \oplus V$ as complex vector spaces, where scalar multiplication on $V \oplus V$ is defined via*

$$\cdot \colon \mathbb{C} \times (V \oplus V), \quad (a+bi, (v_1, v_2)) \mapsto (av_1 - bv_2, bv_1 + av_2)$$

Proof. Any decomposable element $(v, z) \in V_{\mathbb{C}}$ can be uniquely written as a sum

$$(v, z) = (v, a+bi) = (av, 1) + (bv, i).$$

This already sets up the isomorphism $\Psi \colon V_{\mathbb{C}} \to V \oplus V$ defined on decomposable elements as $(v, a+bi) \mapsto (av, bv)$ and extended linearly to $V_{\mathbb{C}}$.
Ψ is linear by definition. Choose some basis $v_1, \ldots, v_n$ of V and let $v, v' \in V$. Then we can write $v = \sum_j \alpha^j v_j, v' = \sum_j \beta^j v_j$ and the inverse of Ψ is given by

$$\Psi^{-1}(v, v') = \Psi^{-1}\left(\sum_j (\alpha j v_j, \beta^j v_j)\right) = \sum_j (v_j, \alpha^j + \beta^j i).$$

Hence Ψ is bijective. Furthermore, scalar multiplication is respected, since

$$\begin{aligned}\Psi((a+bi)(v, c+di)) &= \Psi(v, (ac-bd)+(ad+bc)i) \\ &= ((ac-bd)v), ((ad+bc)v) \\ &= (a+bi) \cdot (cv, dv) \\ &= (a+bi) \cdot \Psi(v, c+di).\end{aligned}$$

□

Definition 2.13. Under the above identification, we will often write vectors in $V_{\mathbb{C}}$ as $v = x + iy$, where $x, y \in V$ denote the components of v as an element of $V \oplus V$. We then define the complex conjugate

$$\overline{v} := x - iy$$

and denote elements v with $v = \bar{v}$ as real elements.

Remark 2.14. Let V be a real vector space of dimension n equipped with a complex complex structure as defined in 2.9. Then

$$\dim_{\mathbb{C}}(V) = \dim_{\mathbb{R}}(V) = n \quad \dim_{\mathbb{C}}(V_{\mathbb{C}}) = 2n$$

Consider now a vector space V equipped with an almost complex structure J. We can extend J linearly over $\mathbb{C}$ to define an automorphism on $V_{\mathbb{C}}$, i.e.

$$J(v, z) = (J(v), z).$$

Since $J^2 = -1$, the eigenvalues of J are then given by $i, -i$, which leads to the following definition.

Definition 2.15 (r-s-Forms)**.** Let V be a real vector space and denote by J the extension of an almost complex structure on V to $V_{\mathbb{C}}$. A vector $v \in V_{\mathbb{C}}$ is called a $(1,0)$-vector if $J(v) = iv$ and a $(0,1)$-vector if $J(v) = -iv$.
A 1-form $\omega \in V_{\mathbb{C}}^*$ is called a $(1,0)$-form if it vanishes on all $(0,1)$-vectors and a $(0,1)$-form if it vanishes on all $(1,0)$-vectors.
An (r,s)-form is an element of $\Lambda^{(r+s)}(V^*)$ that can be written as a linear combintation of exterior products of exactly r $(1,0)$-forms and s $(0,1)$-forms. Any p-form $\alpha \in \Lambda^p(V^*)$ can be uniquely written as a linear combination of $(p,0),(p-1,1),\ldots,(0,p)-$forms.

2.3. **Complex Vector Bundles.** In the following subsection we will generalize the concepts discussed above to complex vector bundles. Most ideas discussed here are completely analogous to the real case and are covered in most introductory texts to (complex) differential geometry (such as [7],[9]).

Definition 2.16 (Complex Vector Bundle)**.** Let M be a smooth manifold. A complex vector bundle on M is a real vector bundle

$$\pi \colon E \to M,$$

such that the fibres of E are complex vector spaces and there exist vector bundle charts that are linear over $\mathbb{C}$.

We would like a way to classify what it means for two vector bundles to be equivalent. This leads us to the definition of isomorphies of vector bundles.

Definition 2.17 (Complex Vector Bundle Isomorphism)**.** Let E, E' be two complex vector bundles over the same manifold M. A vector bundle isomorphism is a bijective map $\Psi : E \to E'$, such that Ψ restricts to a complex isomorphism $\Psi_x \colon E_x \to E'_x$ on all fibres E_x of E. We write $E \cong E'$ for two isomorphic vector bundles.

Definition 2.18 (Transition Functions)**.** Let E be a complex vector bundle of dimension n and $\{U_i, \varphi_i\}$ a vector bundle atlas of E. The corresponding transition functions are the unique functions

$$c_{ij} \colon U_i \cap U_j \to \mathrm{GL}(\mathbb{C}, n),$$

such that for all $z \in \mathbb{C}^n, x \in U_i \cap U_j$

$$(\varphi_i \circ \varphi_j^{-1})(x, z) = (x, c_{ij}(x)z).$$

We can use transition functions to construct new vector bundles from scratch.

Lemma 2.19. *Let E be a complex vector bundle and $\{U_i \varphi_i\}$ a trivialization of E. Then the corresponding transition functions c_{ij} satisfy the following cocycle conditions*

i) $c_{ij} = c_{ji}^{-1}$ *on* $U_{ij} := U_i \cap U_j$

ii) $c_{ij}c_{jk} = c_{ik}$ *on* $U_{ik} := U_i \cap U_j \cap U_k$.

Conversely, if M is a smooth manifold and $\{U_i\}$ an open covering of M with a family of functions $c_{ij} \colon U_{ij} \to GL(\mathbb{C}, n)$, such that the conditions above are satisfied, we can construct a vector bundle E on M of dimension n, such that the c_{ij} are the transition functions of E and E is uniquely defined up to isomorphy of vector bundles.

While two vector bundles with the same transition functions are always isomorphic, the converse does not hold necessarily. The following Lemma classifies isomorphy in terms of the transition functions.

Lemma 2.20. *Let E_1, E_2 be two complex vector bundles of dimension n over M that trivialize over an open covering $\{U_i\}_{i \in I}$ of M with corresponding transition functions $c_{ij}, c'_{ij} \colon U_i \cap U_j \to GL(\mathbb{C}, n)$. E_1 and E_2 are isomorphic if and only if there exist smooth functions*

$$h_i \colon U_i \to GL(\mathbb{C}, n),$$

such that for all $x \in U_i \cap U_j$

$$c'_{ij}(x) = h_i^{-1}(x) c_{ij}(x) h_j(x).$$

Remark 2.21. As in the real case, we can construct new complex vector bundles from existing ones using ordinary vector space constructions. In particular if E, E_1, E_2 are complex vector bundles on M, we can consider the fibrewise defined bundles

$$E_1 \otimes_{\mathbb{C}} E_2 := \bigcup_{x \in M} (E_1)_x \otimes_{\mathbb{C}} (E_2)_x$$

$$\mathrm{Hom}_{\mathbb{C}}(E_1, E_2) := \bigcup_{x \in M} \mathrm{Hom}_{\mathbb{C}}((E_1)_x, (E_2)_x).$$

$$E^* := \bigcup_{x \in M} E_x^*$$

Furthermore, using the concept of complexification discussed in subsection 2.2, we can obtain a complex vector bundle $E_{\mathbb{C}}$ from a real vector bundle E through

$$E_{\mathbb{C}} := \bigcup_{x \in M} (E_x)_{\mathbb{C}}$$

called the complexification of E.

In the following, when dealing with complex vector bundles we will simply write $\mathrm{Hom}(E_1, E_2)$ and $E_1 \otimes E_2$ without specifying that we're dealing with $\mathbb{C}$ as the underlying field.

By making use of the complexification of the tangent space we can now define complex vector fields and differential forms in a completely analogous way to their real counterparts.

Definition 2.22 (Complex Vector Fields and Differential Forms)**.** Let M be a smooth manifold. A complex vector field X is a smooth section of the complexified

tangent bundle $TM_{\mathbb{C}} \cong TM \otimes \mathbb{C}$.
A complex tensor field of type (r,s) over M is a section of

$$(TM_{\mathbb{C}})^r \otimes (TM_{\mathbb{C}}^*)^s.$$

A complex differential form of degree p is an element of $\Lambda^p(TM_{\mathbb{C}}^*)$.

Remark 2.23. Note that when we consider the dual space V^* of a complex vector space V, the elements of V^* are complex- not real-valued functions on V. In particular if we apply a complex 1-form $\omega \in \Lambda_{\mathbb{C}}(M)$ to a complex vector field $X \in \mathcal{X}_{\mathbb{C}}(M)$, we generally obtain a complex-valued function on M

$$\omega(X) \in \mathcal{C}^{\infty}_{\mathbb{C}}(M)$$

In the context of prequantization, we will discuss line bundles with a hermitian structure. Recall the definition of a Riemannian metric on smooth manifolds.

Definition 2.24 (Riemannian Metric)**.** Let $\pi\colon E \to M$ be a smooth vector bundle. A Riemannian metric g on E is a section $g \in \Gamma(E^* \otimes E^*)$, such that g_x is an inner product on E_x for all $x \in M$, i.e. linear, symmetric and positive definite.
A Riemannian metric on M is a Riemannian metric on the tangent bundle TM.

Analogously we can define hermitian structures when considering complex vector bundles.

Definition 2.25 (Hermitian Structure)**.** Let $\pi\colon E \to M$ be a complex vector bundle. A hermitian structure on E is a Hermitian inner product on the fibres E_x of E that varies smoothly with $x \in M$.

2.4. **Complex Manifolds.** Up to this point, we have only considered complex vector bundles over real manifolds. We will now discuss the notion of a complex manifold.

Definition 2.26 (Complex Manifold)**.** A complex manifold of dimension n is a smooth manifold M with an atlas of complex charts $\{U_i, \phi_i\colon U_i \to \phi_i(U_i) \subset \mathbb{C}^n\}_{i\in I}$, such that the corresponding transition functions $\phi_i \circ \phi_j^{-1}$ are holomorphic.
Note that any such atlas can be considered as a smooth atlas under the identification $\mathbb{C} \cong \mathbb{R}^2$ and M necessarily has even real dimension $2n$.

Based on the definition of an almost complex structure on vector spaces 2.9, we can define the weaker notion of an almost complex structure on a smooth manifold.

Definition 2.27 (Almost Complex Manifold)**.** Let M be a smooth manifold. An almost complex structure on M is a vector bundle homomorphism on the tangent bundle of M

$$J\colon TM \to TM,$$

such that for all $x \in M$, the fibre homomorphism J_x is an almost complex structure on the vector space T_xM, i.e.

$$J_x^2 = -1.$$

A manifold M equipped with an almost complex structure is called an almost complex manifold.
If (M,ω) is a symplectic manifold equipped with a symplectic form ω, we say that J is compatible with ω if for all $X, Y \in \mathcal{X}(M)$

$$\omega(J(X), J(Y)) = \omega(X, Y).$$

As it turns out, the two concepts are closely related and an almost complex structure gives rise to a complex structure if it's integrable.

Definition 2.28. Let M be a smooth manifold equipped with an almost complex structure J. We can extend J linearly to define a vector bundle homomorphism $J_{\mathbb{C}}: TM_{\mathbb{C}} \to TM_{\mathbb{C}}$. Then there is a decomposition

$$TM_{\mathbb{C}}(M) \cong T^{(1,0)}(M) \oplus T^{(0,1)}(M),$$

such that $J_{\mathbb{C}}$ acts as multiplication by i on $T^{(1,0)}(M)$ and as multiplication by $-i$ on $T^{(0,1)}(M)$ (cf. Definition 2.15). $T^{(1,0)}(M)$ and $T^{(0,1)}(M)$ are called the holomorphic and anti-holomorphic tangent bundles of the almost complex manifold (ω, J).

Definition 2.29 (Integrability)**.** Let M be a smooth manifold. An almost complex structure J on M is called integrable if the Lie bracket is closed on the holomorphic tangent bundle $T^{(1,0)}(M)$, i.e.

$$X, Y \subseteq T^{(1,0)}(M) \implies [X, Y] \subseteq T^{(1,0)}(M).$$

One can in fact show that integrable almost complex structures are equivalent to complex structures in the sense of definition 2.26.

Proposition 2.30. *Every integrable almost complex structure is induced by a unique complex structure.*

Another way to characterize integrability of an almost complex structure is by considering the corresponding Nijenhius tensor.

Definition 2.31 (Nijenhius-Tensor)**.** Let M be a smooth manifold and $A \in \mathcal{T}^{(1,1)}(M)$ be a $(1,1)$-tensor. The Nijenhius-Tensor N_A of A is the $(1,2)$-tensor defined by

$$N_A(X, Y) := -A^2[X, Y] + A([AX, Y] + [X, AY]) - [AX, AY].$$

Theorem 2.32 (Newlander–Nirenberg Theorem)**.** *An almost complex structure J is integrable if and only if the corresponding Nijenhius-Tensor N_J vanishes.*

A partial proof for Proposition 2.30 and Theorem 2.32 can be found in [8].

The connection betwen symplectic and (almost-)complex manifolds lies in the definition of a Kähler manifold. These will provide an important subcase for the geometric quantization procedure and lead to the formalism of holomorphic quantization.

Definition 2.33 (Kähler Manifold)**.** An almost-Kähler Manifold is a triple (M, ω, J), such that

i) (M, ω) is a symplectic manifold, i.e. ω is a symplectic form on M,
ii) (M, J) is an almost complex manifold, i.e. J is an almost complex structure on M
iii) J is compatible with ω, i.e. for all $X, Y \in \mathcal{X}(M)$

$$\omega(J(X), J(Y)) = \omega(X, Y).$$

If J is additionally integrable, we say that (M, ω, J) is a Kähler manifold.

As it turns out, we can consider several different view points of Kähler manifolds, depending on whether one would like to emphasize the symplectic, complex or Riemannian structure. This is captured in the following definition.

Definition 2.34 (Compatible Triple)**.** Let M be a smooth manifold equipped with a Riemannian metric g, a symplectic form ω and an almost complex structure J. For ω and g define corresponding isomorphisms

$$\Psi_g \colon TM \to TM^*, \quad v \mapsto g(v, \cdot) \qquad \Psi_\omega \colon TM \to TM^*, \quad v \mapsto \omega(v, \cdot).$$

The triple (g, ω, J) is called compatible if for all $u, v \in TM$

i) $g(u, v) = \omega(u, Jv)$,
ii) $\omega(u, v) = g(Ju, v)$,
iii) $J(u) = \Psi_g^{-1} \circ \Psi_\omega(u)$.

Note that this implies, that each structure is completely determined by the other two.

2.5. **Polarizations.** When passing from prequantization to the formalism of geometric quantization, the main concept one needs is that of a complex polarization. In the following let (M, ω) denote a symplectic manifold.

Definition 2.35 (Real Polarization)**.** A real polarization of M is a smooth distribution P which is integrable and Lagrangian.

Definition 2.36 (Complex Polarization)**.** A complex polarization of M is a complex distribution P on M, such that

i) P is Lagrangian with respect to $TM_{\mathbb{C}}$, i.e. for all $x \in M$, P_x is a Lagrangian subspace of $(TM_{\mathbb{C}})_x$,
ii) P is integrable,
iii) the dimension of $P \cap \bar{P} \cap TM$ is constant.

The second condition can be replaced by the following.

Lemma 2.37. *Let P be a complex distribution on M. P is integrable, if and only if for all $x \in M$ there exists a neighborhood U and complex-valued functions $f_1, \dots, f_n$ defined on U, such that $\bar{P}$ is spanned by the corresponding Hamiltonian vector fields $X_{f_1}, \dots, X_{f_n}$.*

We can construct two kinds of real polarizations from a complex one.

Definition 2.38 ((Co-)isotropic Distributions)**.** Let P be a complex polarization of M. Then

$$D \coloneqq P \cap \bar{P} \cap TM$$

defines a real isotropic distribution and

$$E \coloneqq (P + \bar{P}) \cap TM = D^\perp$$

defines a real co-isotropic distribution. We call D and E the corresponding isotropic and co-isotropic distributions of P, respectively.

The following definitions will be needed when we introduce polarized sections for the description of geometric quantization in section 4.

Definition 2.39 (Strongly integrable)**.** Let P be a complex polarization of M and denote by D and E the corresponding isotropic and coisotropic distributions. P is strongly integrable or reducible if

i) E is integrable,

ii) The spaces of integral manifolds of D and E

$$\mathcal{D} = M/D \quad \mathcal{E} = M/E$$

are smooth manifolds,

iii) The projection $\pi\colon \mathcal{D} \to \mathcal{E}$ is a submersion.

Definition 2.40 (Polarized Function)**.** Let P be a complex polarization on M. A complex function $f \in \mathcal{C}^\infty(M)$ is called polarized if

$$P(f) = 0.$$

Definition 2.41 (Adapted Symplectic Potential)**.** Let P a complex polarization on M. A symplectic potential θ is called adapted to P, if

$$\theta(X) = 0 \qquad \forall X \subseteq P.$$

Definition 2.42 (Admissible Polarization)**.** Let (M, ω) be a symplectic manifold. A complex polarization P on M is admissible if for all $x \in M$ there exists a neighborhood U_x and a local symplectic potential adapted to P.

Lemma 2.43. *Let (M, ω) be a symplectic manifold and P a complex polarization on M. If P is strongly integrable, then P is admissible.*

Let now P be a complex polarization on (M, ω) and define on P

$$\tilde{h}(X, Y) := i\omega(X, \overline{Y}).$$

Then $\ker(\tilde{h}) = P \cap \overline{P}$, since clearly $P \cap \overline{P} \subseteq \ker(\tilde{h})$ and if $X \in \ker(\tilde{h})$, then necessarily $X \in P$ (since $\tilde{h}$ was defined on P) and at the same time

$$\omega(X, \overline{Y}) = -ih(X, Y) = 0$$

for all $Y \in P$, hence $X \in P \cap \overline{P}$ since $\overline{P}$ is Lagrangian. Thus $\tilde{h}$ projects to define a non-degenerate two-form on $P/(P \cap \overline{P})$. If wen denote this form by h, then we can define the type of a polarization.

Definition 2.44 (Type of a Polarization)**.** Let P be a complex polarization and h the non-degenerate 2-form constructed as above. Then the type (r, s) of P is exactly the type of h.

We will further discuss two extreme cases

Definition 2.45. Let P be a complex polarization of M with type (r, s). P is called

i) Kähler if $P \cap \bar{P} = 0 \Leftrightarrow r + s = n$,

ii) real if $P = \bar{P} \Leftrightarrow r = s = 0$,

iii) positive if $r = n$.

The following proposition relates the concept of a Kähler polarization to the previously introduced definition of Kähler manifolds 2.33 and hence motivates the name.

Proposition 2.46. *Let (M, ω) be a symplectic manifold. The existence of Kähler polarizations is equivalent to the existence of a compatible complex structure, i.e. a Kähler manifold structure on M.*
In particular,

i) if (M, ω, J) is a Kähler manifold, the corresponding holomorphic and anti-holomorphic polarizations are Kähler polarizations.

ii) if (M,ω) is a symplectic manifold equipped with a Kähler polarization P, we can construct a complex structure J on M that is compatible with ω, such that P and $\overline{P}$ are the corresponding holomorphic and anti-holomorphic tangent bundles.

Proof. $\Rightarrow$:
Let (M,ω,J) be a Kähler manifold and consider the holomorphic and anti-holomorphic tangent bundles $P,\overline{P}$. From Definition 2.28, it immediately follows that $P\cap\overline{P}=0$. Furthermore, if the dimension of M is $2n$, then the dimensions of P and $\overline{P}$ are both n. It remains to be shown that P is integrable, but that follows from the integrability of J.

$\Leftarrow$:
For the converse, let P be a Kähler polarization on (M,ω), then $TM_{\mathbb{C}} = P\oplus\overline{P}$, since both have dimension n and $P\cap\overline{P}=0$. We can write any $X\in TM_{\mathbb{C}}$ in a unique way as $X = X_P + X_{\overline{P}}$, where $X_P\in P$ and $X_{\overline{P}}\in\overline{P}$. Then define a complex structure J in the following way

$$J\colon TM_{\mathbb{C}}\to TM_{\mathbb{C}}, \quad X\mapsto iX_P - iX_{\overline{P}}.$$

It holds that

$$J^2(X) = J(iX_P - iX_{\overline{P}}) = i^2X_P + i^2X_{\overline{P}} = -X$$

and clearly

$$X\in P \implies J(X) = iX \qquad X\in\overline{P} \implies J(X) = -iX,$$

hence $T^{(1,0)}(M) = P$ and $T^{(0,1)}(M) = \overline{P}$. Since P and $\overline{P}$ are involutive, J is integrable by definition 2.29. Furthermore J is compatible with ω, since

$$\begin{aligned}\omega(JX,JY) &= \omega(iX_P - iX_{\overline{P}}, iY_P - iY_{\overline{P}})\\ &= \omega(iX_P, -iY_{\overline{P}}) + \omega(-iX_{\overline{P}}, iY_P)\\ &= \omega(X_P, Y_{\overline{P}}) + \omega(X_{\overline{P}}, Y_P)\\ &= \omega(X_P + X_{\overline{P}}, Y_P + Y_{\overline{P}})\\ &= \omega(X,Y),\end{aligned}$$

since P and $\overline{P}$ are Lagrangian and thus (M,ω,J) is a Kähler manifold. □

Proposition 2.47. *Let (M,ω) be a Kähler manifold and denote by $P,\overline{P}$ the Kähler polarizations of the holomorphic and anti-holomorphic tangent bundles. Denote by z_i the complex coordinates, such that*

$$P = span\left\{\frac{\partial}{\partial z_j}\right\} \qquad \overline{P} = span\left\{\frac{\partial}{\partial \overline{z}_j}\right\}.$$

Then there exists a function K on M called a Kähler scalar, such that the symplectic form ω can be expressed as

$$\omega = i\sum_{i,j}\frac{\partial^2 K}{\partial z_i\partial\overline{z}_j}dz_i\wedge d\overline{z}_j.$$

Thus there exist local symplectic potentials

$$\theta = i\sum_j\frac{\partial K}{\partial z_j}dz_j \qquad \overline{\theta} = i\sum_j\frac{\partial K}{\partial\overline{z}_j}d\overline{z}_j$$

that are adapted to P and $\overline{P}$, respectively.

For a proof of this statement, see [10].

2.6. **Complex Connection Form and Curvature.** This subsection deals with the definition and properties of connections on complex vector bundles, which lie at the heart of the prequantization procedure. While we discuss the most fundamental aspects for arbitrary vector bundles, we are mainly interested in the often much simpler case of a line bundle. The latter is discussed in more detail in the following subsection 2.7.
Recall the definition of a real connection:

Definition 2.48 (Connection)**.** Let $\pi\colon E \to M$ be a smooth vector bundle. A connection ∇ (or covariant derivative operator) on E is an operator

$$\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$$

with the following properties

i) $\nabla_{X+Y}(s) = \nabla_X(s) + \nabla_Y(s)$
ii) $\nabla_{fX}(s) = f\nabla_X(s)$
iii) $\nabla_X(s_1 + s_2) = \nabla_X(s_1) + \nabla_X(s_2)$
iv) $\nabla_X(fs) = f\nabla_X(s) + X(f)s$

for all $X, Y \in \mathcal{X}(M), f \in \mathcal{C}^\infty(M), s, s_1, s_2 \in \Gamma(E)$. In other words, ∇ has to be $\mathcal{C}^\infty$-linear in the first argument (i), ii)), linear in the second argument (iii)) and satisfy the so-called Leibniz condition (iv)).
A connection on M is a connection on the tangent bundle TM. In this case, ∇ is an operator

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M).$$

We can define connections on complex vector bundles in an equivalent way.

Definition 2.49. Let $\pi\colon E \to M$ be a complex vector bundle. A connection ∇ (or covariant derivative operator) on E is an operator

$$\nabla : \mathcal{X}^{\mathbb{C}}(M) \times \Gamma(E) \to \Gamma(E)$$

with the following properties

i) $\nabla_{X+Y}(s) = \nabla_X(s) + \nabla_Y(s)$
ii) $\nabla_{fX}(s) = f\nabla_X(s)$
iii) $\nabla_X(s_1 + s_2) = \nabla_X(s_1) + \nabla_X(s_2)$
iv) $\nabla_X(fs) = f\nabla_X(s) + X(f)s$

for all $X, Y \in \mathcal{X}^{\mathbb{C}}(M), f \in \mathcal{C}^\infty_{\mathbb{C}}(M), s, s_1, s_2 \in \Gamma(E)$. In other words, ∇ has to be $\mathcal{C}^\infty_{\mathbb{C}}$-linear in the first argument (i), ii)), linear (over the complex numbers) in the second argument (iii)) and satisfy the Leibniz condition (iv)).
A complex connection on M is a connection on the complexified tangent bundle $TM_{\mathbb{C}}$. In this case, ∇ is an operator

$$\nabla\colon \mathcal{X}^{\mathbb{C}}(M) \times \mathcal{X}^{\mathbb{C}}(M) \to \mathcal{X}^{\mathbb{C}}(M).$$

In the following we will consider only complex connections on complex manifolds M and will therefore refer to them simply as connections on M.

Remark 2.50. Consider again the definition of a (complex) connection 2.49. Properties i) and ii) guarantee that for a fixed section $s \in \Gamma(E)$), the map $X \to \nabla_X(s)$ is $\mathcal{C}^\infty$-linear. Hence we can view ∇ as an operator

$$\nabla\colon \Gamma(E) \to \mathrm{Hom}_{\mathcal{C}^\infty}(\mathcal{X}(M), \Gamma(E)) \cong \Gamma(E) \otimes \Omega^1(M).$$

If we denote by $\Omega^p(M, E)$ the E-valued p-forms on M, we thus have

$$\nabla\colon \Gamma(E) \cong \Omega^0(M, E) \to \Omega^1(M, E).$$

Proposition 2.51 (Exterior covariant derivative)**.** *Let $\pi\colon E \to M$ be a vector bundle with connection ∇. We can extend ∇ to a family of $\mathbb{C}$-linear operators*

$$\nabla^p\colon \Omega^p(M, E) \to \Omega^{p+1}(M, E)$$

by recursively defining ∇^p on decomposable elements via

$$\nabla^p(\omega \otimes s) \coloneqq d\omega \otimes s + (-1)^p \omega \wedge \nabla s$$

for all $\omega \in \Omega^p(M), s \in \Gamma(E)$.

Definition 2.52 (Curvature Form)**.** Let $\pi\colon E \to M$ be a (complex) vector bundle with connection ∇. The curvature form of ∇ is the $\mathrm{Hom}(E, E)$-valued 2-form

$$\Omega \coloneqq \nabla^1 \circ \nabla\colon \Gamma(E) \cong \Omega^0(M, E) \to \Omega^2(M, E).$$

In particular, for $X, Y \in \mathcal{X}(M)$ and $s \in \Gamma(E)$

$$\Omega(X, Y)(s) = \nabla^1(\nabla(s))(X, Y)$$

One way to characterize a connection locally is the notion of a connection form, which is dependent on the choice of a local frame.

Definition 2.53 (Connection Form)**.** Let $\pi\colon E \to M$ be a (complex) vector bundle of fibre dimension n and let ∇ be a connection on E. Choose a local frame $e \coloneqq \{e_i\}_{i=1\dots n}$ of E over some $U \subseteq M$. The connection form associated with (U, e) is defined as the $n \times n$-matrix of one-forms ω_{ij}, such that for all $i \in \{1, 2, \dots n\}$

$$\nabla e_i = \sum_{j=1}^{k} \omega_{ij} \otimes e_j.$$

Remark 2.54. While the connection form associated with a local frame is not necessarily defined globally (on all of M), we can always choose a covering $\{U_k\}_{k\in K}$ of M and corresponding local frames $e^k = \{e_i^k\}_{i=1\dots n}$, such that we obtain a matrix of one forms for each $k \in K$.

Because any arbitrary section can locally be expressed via a local frame, we can express the covariant derivative of an arbitrary section, using a local frame and the corresponding connection form. In particular if $s = f^i e_i$ is an arbitrary local section (with the Einstein summation convention)

$$\nabla s = \nabla(f^i e_i) = df^i \otimes e_i + f^i \nabla e_i = df^i \otimes e_i + f^i w_i^j \otimes e_j.$$

Remark 2.55. Note that the definition of the connection 1-form in the literature sometimes includes a prefactor of i or $2\pi i$, particularly in discussions of Geometric Quantization. This leads to slightly differing statements in some cases, however, we will try to make apparent whenever this is the case.

Let us now consider how the connection form changes when we choose a different local frame.

Lemma 2.56 (Change of Frame)**.** *Let ∇ be a connection on a complex vector bundle E of fibre dimension k and let $e \coloneqq \{e_i\}_{i=1\ldots k}$ be a local frame of E with corresponding conection form w_{ij}. The connection form w'_{ij} of a second local frame defined via*

$$e'_i = \sum_j f_{ij} e_j$$

where $f_{ij} \in \mathcal{C}^\infty_{\mathbb{C}}(M)$, is then given by

$$w' = df f^{-1} + f w f^{-1}$$

where w, w' denote the $k \times k$-matrices of the corresponding connection one-forms and f denotes the invertible $k \times k$ matrix with entries f_{ij}.

Proof. We prove the statement with the defining property of a connection one-form from Definition 2.53. Consider therefore

$$\begin{aligned}
\sum_{j=1}^k w'_{ij} \otimes e'_j = \nabla e'_i &= \sum_{j=1}^k \nabla(f_{ij} e_j) \\
&= \sum_{j=1}^k (df_{ij} \otimes e_j + f_{ij} \nabla e_j) \\
&= \sum_{j=1}^k \left(df_{ij} \otimes \sum_{l=1}^k f^{-1}_{jl} e'_l + f_{ij} \sum_{m=1}^k w_{jm} \otimes e_m \right) \\
&= \sum_{j=1}^k \left(df_{ij} \otimes \sum_{l=1}^k f^{-1}_{jl} e'_l + f_{ij} \sum_{m=1}^k w_{jm} \otimes \sum_{l=1}^k f^{-1}_{ml} e'_l \right) \\
&= \sum_{l=1}^k \left(\sum_{j=1}^k df_{ij} f^{-1}_{jl} + \sum_{j=1}^k \sum_{m=1}^k f_{ij} w_{jm} f^{-1}_{ml} \otimes e'_l \right) \\
&= \sum_{j=1}^k \left((df f^{-1})_{ij} + (f w f^{-1})_{ij} \otimes e'_j \right),
\end{aligned}$$

where in the last step we swapped the indices j and l. □

We can now relate the connection one-form to the curvature form from Definition 2.52. Recall that the curvature form Ω is a $Hom(E, E)$-valued 2-form. In particular, if we choose a local frame $e_1, \ldots, e_n$, we can express Ω as a matrix of 2-forms.

Proposition 2.57 (Cartan's Structure Identity)**.** *Let E be a vector bundle with connection ∇ and denote by Ω the corresponding curvature form. If we choose a local frame $e_1, \ldots, e_n$ of E and denote by w the connection one-form of ∇, it holds that*

$$\Omega = dw - [w, w].$$

In particular if we consider Ω as a matrix of bundle-valued 2 forms, then

$$\Omega_{ij} = dw_{ij} - \sum_{k=1}^n w_{ik} \wedge w_{kj}.$$

Proof. It suffices to show the result for any section e_i. Consider therefore

$$\begin{aligned}
\Omega(e_i) &= \nabla^1 \circ \nabla e_i \\
&= \nabla^1 \left(\sum_j w_{ij} \otimes e_j \right) \\
&= \sum_j dw_{ij} \otimes e_j - \sum_j w_{ij} \wedge \nabla e_j \\
&= \sum_j dw_{ij} \otimes e_j - \sum_{j,k} w_{ij} \wedge w_{jk} e_k .
\end{aligned}$$

Thus the individual entries of Ω are given by

$$\Omega_{ij} = e_j^*(\Omega(e_i)) = dw_{ij} - \sum_{k=1}^{n} w_{ik} \wedge w_{kj}.$$

□

2.7. **Complex Line Bundles.** As we will see, in the formalism of prequantization, quantum states will eventually be described by sections in a complex vector bundle of dimension 1. Such a bundle is called a line bundle.

Definition 2.58 (Line Bundle)**.** A (complex) line bundle is a one-dimensional (complex) vector bundle.

As we will only consider complex line bundles in the following, we will simply refer to them as line bundles.
We will see that most of the concepts discussed so far become more trivial if we consider line bundles.

Remark 2.59 (Transition Functions). The transition functions of a line bundle $\pi \colon L \to M$ over some trivialization U_i are now functions

$$c_{ij} \colon U_i \cap U_j \to \mathbb{C}^*$$

and two line bundles L, L' that trivialize over the same open covering U_i with corresponding transition functions c_{ij}, c'_{ij} are isomorphic if and only if there exist functions $h_i \colon U_i \to \mathbb{C}^*$, such that for all $x \in M$

$$c'_{ij}(x) = h_i^{-1}(x) c_{ij}(x) h_j(x).$$

Remark 2.60 (Connection and Curvature Form). On line bundles the connection 1-form as defined in 2.53 is now simply a one-form. Note that any non-vanishing section s in a line bundle already defines a local frame and hence has an associated connection 1-form w defined by

$$\nabla s = w \otimes s.$$

Cartan's Structure Identity 2.57 simplifies to

$$\Omega = dw.$$

Lemma 2.61. *Let L be a line bundle with a connection ∇ and denote by Ω the corresponding curvature form. Then for all $X, Y \in \mathcal{X}(M), s \in \Gamma(L)$*

$$\Omega(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

Proof. We prove the statement locally. Consider a local non-vanishing section s and denote by w the corresponding connection 1-form. Then

$$\begin{aligned}
&\nabla_X\nabla_Y s = \nabla_X w(Y)s = X(w(Y))s + w(Y)\nabla_X s = X(w(Y))s + w(Y)w(X)s\\
&\nabla_Y\nabla_X s = \nabla_Y w(X)s = Y(w(X))s + w(X)\nabla_Y s = Y(w(X))s + w(Y)w(X)s\\
&\nabla_{[X,Y]}s = w([X,Y])s
\end{aligned}$$

and thus

$$\begin{aligned}
\nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s &= X(w(Y))s - Y(w(X))s - w([X,Y])s\\
&= dw(X,Y)s.
\end{aligned}$$

□

Note that since line bundles are one-dimensional, any non-vanishing section determines a local frame and by choosing a particular section, one can thus relate arbitrary sections with functions on the base manifold. Given some trivialization, we can always pick out so-called unit sections.

Definition 2.62 (Unit Section)**.** Let $\pi\colon E \to M$ be a line bundle and let (U,φ) be a vector bundle chart

$$\varphi\colon \pi^{-1}(U) \to U \times \mathbb{C}.$$

Then the unit section of (U,φ) is the section

$$s\colon M \to E, \quad x \mapsto \varphi^{-1}(x,1).$$

Lemma 2.63. *Any trivialization of a line bundle is completely determined by its unit sections.*

Proof. Let (U_i,φ_i) be a trivialization of a line bundle $\pi\colon L \to M$ and denote by s_i the corresponding unit sections. Then each s_i is in particular non-vanishing and thus defines a local frame for L. We can thus recover the original trivialization

$$\varphi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}, \quad zs_i(x) \mapsto (x,z).$$

□

The following will play a central role in deriving the prequantization procedure.

Remark 2.64. Let $\pi\colon L \to M$ be an arbitrary line bundle with connection ∇ and let $s\colon M \to L$ be a section in M. Let $\{U_i,\varphi_i\}$ be a trivialization of E and denote by s_i the corresponding unit sections and by w_i the corresponding connection 1-form. Then locally we can write $s|_{U_i} = \psi_i s_i$ for some functions $\psi_i\colon U_i \to \mathbb{C}$ and hence for any $X \in \mathcal{X}(M)$

$$\begin{aligned}
\nabla_X(s) &= \nabla_X(\psi_i s_i)\\
&= X(\psi_i)s_i + \psi_i\nabla_X(s_i)\\
&= X(\psi_i)s_i + \psi_i w_i(X)s_i\\
&= (X(\psi) + w_i(X)\psi_i)s_i,
\end{aligned}$$

hence if we identify a section s with its corresponding function ψ_i for some unit section s_i, the covariant derivative acts on these functions via

$$\nabla_X : \psi \mapsto X(\psi) + w_i(X) \tag{2.65}$$

One might wonder how the functions ψ_i and the corresponding connection one forms of unit sections transform under a change in trivialization.

Lemma 2.66 (Change of Trivialization)**.** *Let $\pi\colon L \to M$ be an arbitrary line bundle with connection ∇ and let $\{U_i, \varphi_i\}$ be a trivialization with corresponding transition functions $c_{ij}\colon U_{ij} \to \mathbb{C}^*$ and unit sections $s_i\colon U_i \to L$.*

i) For any section $s = \psi_i s_i = \psi_j s_j$ of L defined on U_{ij}, the corresponding functions transform via

$$\psi_i = c_{ij}\psi_j.$$

ii) The corresponding connection 1-forms w_i, w_j (defined on U_{ij}) transform via

$$w_i = w_j + \frac{dc_{ji}}{c_{ji}}.$$

Proof. i) This follows immediately from the transformation of the unit sections

$$\begin{aligned} & s_i(x) = \varphi_i^{-1}(x,1) = \varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1}(x,1) = c_{ji}(x)s_j(x) \\ \Longrightarrow\ & \psi_i s_i = c_{ij}\psi_j s_i = \psi_j s_j \end{aligned}$$

ii) Note that this is just a special case of Lemma 2.56. In particular, it follows from the definition of the connection one-form and

$$\nabla s_i = \nabla c_{ji}s_j = dc_{ji} \otimes s_j + c_{ji}\nabla s_j = (dc_{ji} + c_{ji}w_j) \otimes s_j = \left(\frac{dc_{ji}}{c_{ji}} + w_j\right) \otimes s_i.$$

□

In fact, every family of one-forms that satisfy the conditions of Lemma 2.66 can be used to construct a connection.

Lemma 2.67. *Let $\pi\colon L \to M$ be a line bundle with a trivialization $\{U_i, \varphi_i\}$ and corresponding transition functions c_{ij}. If there exists a family of one-forms w_i defined on U_i, such that*

$$w_i = w_j + \frac{dc_{ji}}{c_{ji}} \qquad \text{in } U_{ij} := U_i \cap U_j,$$

then there exists a connection ∇ on L, such that the connection one-forms of the unit sections s_i are given by w_i.

Proof. We use the defining property of a connection one-form to define a covariant derivative operator. In particular, if s is any section in L we can locally write $s = f_i s_i$ for some $f_i \in \mathcal{C}^\infty_{\mathbb{C}}(U_i)$. Then for any $X \in \mathcal{X}(M)$, we define locally

$$\nabla_X s := (X(f_i) + f_i w_i(X))s_i.$$

To see that the expression is well-defined consider another unit section s_j. Then it holds that (compare Lemma 2.66)

$$s_i = c_{ji}s_j \quad f_i = c_{ij}f_j$$

and hence

$$\begin{aligned}(X(f_i)+f_iw_i(X))s_i &= \left(X(c_{ij}f_j)+c_{ij}f_j\left(w_j(X)+\frac{dc_{ji}(X)}{c_{ji}}\right)\right)c_{ji}s_j\\ &= X(f_j)s_j+X(c_{ij})c_{ji}f_js_j+w_j(X)f_js_j+c_{ij}dc_{ji}(X)f_js_j\\ &= \left(X(f_j)+f_jw_j(X)\right)s_j+\left(X(c_{ij})c_{ji}+X(c_{ji})c_{ij}\right)f_js_j\\ &= \left(X(f_j)+f_jw_j(X)\right)s_j+\left(X(c_{ij}c_{ji})\right)f_js_j\\ &= \left(X(f_j)+f_jw_j(X)\right)s_j\end{aligned}$$

where in the last line the last term vanished since $c_{ij}c_{ji}=1$.
It's clear that the operator ∇ defined in this way is linear in the argument of s and $\mathcal{C}^\infty$-linear in the argument of X. It remains to check the Leibniz property. Let $s=f_is_i$ be any section and let $s' := gs$ for some function $g\in\mathcal{C}^\infty(M)$. Then locally

$$\begin{aligned}\nabla_X(s') &= (X(gf_i)+gf_iw_i(X))s_i\\ &= X(g)f_is_i+g(X(f_i)+f_iw_i(X))s_i\\ &= X(g)s'+g\nabla_X(s)\end{aligned}$$

and thus ∇ defined as above is in fact a valid connection. □

Let us consider now line bundles with a hermitian structure.

Definition 2.68 (Hermitian Connection)**.** Let $\pi\colon E\to M$ be a complex line bundle with a hermitian structure h. A connection ∇ on E is called hermitian with respect to h if for all $X\in\mathcal{X}(M)$ and all $s,s'\in\Gamma(E)$

$$X(h(s,s'))=h(\nabla_Xs,s')+h(s,\nabla_Xs').$$

Lemma 2.69. *Let L be a line bundle with hermitian structure h and connection ∇. If ∇ is hermitian with respect to h, the corresponding curvature form Ω is purely imaginary.*

Proof. Consider a local frame (U_i,s_i) of L and denote by w_i the corresponding connection one-forms. For any vector field X defined in U_i

$$\begin{aligned}&X(h(s_i,s_i))=h(\nabla_Xs_i,s_i)+h(s_i,\nabla_Xs_i)=(w_i(X)+\overline{w_i(X)})h(s_i,s_i)\\ &\implies w_i+\overline{w_i}=\frac{dh(s_i,s_i)}{h(s_i,s_i)}\\ &\implies \Omega+\overline{\Omega}=d(w_i+\overline{w_i})=0.\end{aligned}$$

This proves the statement locally, but since U_i was arbitrary, it holds globally as well. □

A hermitian structure h on L further allows us to choose a specific trivialization that is normalized with respect to h.

Definition 2.70 (Normalized Trivialization)**.** Let L be a line bundle with hermitian structure h and a compatible connection ∇. A trivialization $\{U_i,\varphi_i\}$ of L is normalized if the corresponding unit sections s_i satisfy

$$h(s_i,s_i)=1.$$

We can always go from an arbitrary trivialization $\{U_i,\varphi_i\}$ to a normalized one $\{U_i',\varphi_i'\}$ by normalizing the corresponding bundle charts

$$\varphi_i\to\varphi' := \sqrt{h(s_i,s_i)}\varphi_i.$$

Lemma 2.71. *For a normalized trivialization of a line bundle L equipped with some hermitian structure h, the corresponding transition functions c_{ij} fulfil $|c_{ij}| = 1$. If L is further equipped with a compatible connection ∇, the corresponding connection 1-forms are purely imaginary.*
Conversely if there exists a trivialization of L with transition functions c_{ij}, such that $|c_{ij}| = 1$ and unit sections s_i, such that the corresponding connection 1-forms are purely imaginary, then there exists a hermitian structure on L that is compatible with ∇.

Proof. Let (U_i, φ_i) be a normalized trivialization. Then

$$\begin{aligned} &h(s_i, s_i) = 1 = h(s_j, s_j) = h(c_{ij}s_i, c_{ij}s_i) = |c_{ij}|^2 h(s_i, s_i) \\ \Longrightarrow\ &|c_{ij}| = 1. \end{aligned}$$

Denote now by w_i the corresponding connection 1-forms. For any vector field X on M we have

$$\begin{aligned} 0 &= X(h(s_i, s_i)) \\ &= h(\nabla_X s_i, s_i) + h(s_i, \nabla_X s_i) \\ &= h(w_i(X)s_i, s_i) + h(s_i, w_i(X)s_i) \\ &= w_i(X) + \overline{w_i(X)} \end{aligned}$$

and hence $w_i(X)$ is purely imaginary.
For the converse, let (U_i, φ_i) be a trivialization, such that the connection 1-forms w_i of the corresponding unit sections s_i are purely imaginary. Let $v_1, v_2 \in L_x$ be two arbitrary elements in a fibre of L. Then we can always write $v_1 = z_1 s_i(x), v_2 = z_2 s_i(x)$ for some i and uniquely defined $z_1, z_2 \in \mathbb{C}$. We can use this to define a hermitian structure on L

$$h_x(z_1 s_i(x), z_2, s_i(x)) \coloneqq z_1 \overline{z_2}.$$

To see that h is well defined, consider another unit section s_j. Then

$$\begin{aligned} h_x(v_1, v_2) &= h_x(z_1 s_i(x), z_2 s_i(x)) \\ &= h_x(z_1 c_{ji}(x) s_j(x), z_2 c_{ji}(x) s_j(x)) \\ &= z_1 \overline{z_2} |c_{ji}(x)|^2 \\ &= z_1 \overline{z_2}. \end{aligned}$$

The smoothness of h follows from the smoothness of the unit sections and h is clearly symmetric sesquilinear and positive definite on each fibre of L. □

Remark 2.72. Note that different conventions in the prefactor of the connection form, as described in Remark 2.55 can lead to a similar statement, where the connection 1-forms are real forms.

Remark 2.73. With Lemma 2.71 we can deduce that the transition functions for normalized trivializations can be written as $c_{ij}(x) = e^{if_{ij}}$ for some real functions $f_{ij} \in \mathcal{C}^\infty(M)$. Then the transformations in Lemma 2.66 become

$$\psi_i = e^{if_{ij}} \psi_j \qquad w_i = w_j + \frac{d \exp\{if_{ji}\}}{\exp\{if_{ji}\}} = w_j + i df_{ji}. \tag{2.74}$$

In the following we will try to classify the different connections a line bundle can exhibit and how they are related when they carry the same curvature or are compatible with a hermitian structure.

Proposition 2.75 (Connections on a Line Bundle)**.** *Let $\pi\colon L \to M$ be a line bundle with hermitian structure h and let ∇, ∇' be two connections on L. Then the following hold*

i) *∇ and ∇' differ by an element in $\Omega^1_{\mathbb{C}}(M)$*

$$\nabla - \nabla' = \lambda \quad \textit{for some } \lambda \in \Omega^1_{\mathbb{C}}(M)$$

ii) *If ∇ and ∇' are further both compatible with h, they differ by a purely imaginary one-form on M*

$$\nabla - \nabla' = \lambda \quad \textit{for some } \lambda \in \Omega^1_{i}(M)$$

iii) *If ∇ and ∇' further have the same curvature form Ω, they differ by a closed purely imaginary one-form on M*

$$\nabla - \nabla' = d\mu \quad \textit{for some } \lambda \in \Omega^1_{i}(M) \textit{ where } d\lambda = 0.$$

Proof. i) To begin with consider two arbitrary connections ∇, ∇'. Let s be a local frame of L defined on some $U \subseteq M$ and denote by w, w' the corresponding connection 1-forms. Then for any $X \in \mathcal{X}(U)$ and any local section fs, where $f \in \mathcal{C}^\infty(U)$

$$\begin{aligned}\nabla_X(fs) - \nabla'_X(fs) &= f\nabla_X s - f\nabla'_X s \\ &= fw(X)s - fw'(X)s \\ &= (w - w')(X)fs\end{aligned}$$

$$\implies \nabla - \nabla' = w - w' \in \Omega^1_{\mathbb{C}}(U).$$

Since U was arbitrary, the statements holds globally as well.
Consider now an arbitrary one-form $\lambda \in \Omega^1_{\mathbb{C}}(M)$. Then for any connection ∇ on L

$$\nabla'_X s \coloneqq \nabla_X s + \lambda(X)s$$

is clearly linear in the arguments of X and s and $\mathcal{C}^\infty$-linear in the argument of X. Furthermore

$$\begin{aligned}\nabla'_X(fs) &= \nabla_X(fs) + \lambda(X)fs \\ &= X(f)s + f\left(\nabla_X s + \lambda(X)s\right) \\ &= X(f)s + f\nabla'_X s\end{aligned}$$

and hence ∇' is another connection on L.

ii) Consider now two connections ∇, ∇' that are both compatible with some hermitian structure h on L and let $\lambda \in \Omega^1_{\mathbb{C}}(M)$, such that $\nabla' = \nabla + \lambda$. Then for any vector field $X \in \mathcal{X}(M)$ and any arbitray section $s \in \Gamma(L)$

$$\begin{aligned}0 &= X(h(s,s)) - X(h(s,s)) \\ &= h(\nabla'_X s, s) + h(s, \nabla'_X s) - h(\nabla_X s, s) - h(s, \nabla_X s) \\ &= h(\lambda(X), s) + h(s, \lambda(X)s) \\ &= h(s,s)(\lambda + \overline{\lambda})(X)\end{aligned}$$

and hence λ is purely imaginary.

iii) Let ∇, ∇' be two connections on L with the same curvature Ω that are both compatible with respect to some hermitian structure h and let $\lambda \in \Omega^1_i(M)$, such that $\nabla' = \nabla + \lambda$. Then for any local frame s of L and the corresponding connection 1-forms w, w', it holds that $\lambda = w' - w$ (see proof of i)) and thus

$$0 = \Omega - \Omega = dw' - dw = d\lambda.$$

□

3. Prequantization

3.1. **Dirac's Quantum Conditions.** The goal of geometric quantization is to assign to every classical system described by a symplectic manifold (M, ω) a separable Hilbert space $\mathcal{H}$ and to every classical observable f in a subalgebra $\mathcal{Q}(M)$ of $\mathcal{C}^\infty(M)$, a quantum observable $O_f \in \mathcal{O}(\mathcal{H})$.
To begin with, we review the quantization conditions based on the different aspects of classical and quantum mechanical systems discussed in section 1 as listed in [1].

Quantization Conditions

1.) **Linearity**
The assignment $f \mapsto O_f$ is linear

$$O_{f+g} = O_f + O_g \qquad O_{\lambda f} = \lambda O_f, \quad \lambda \in \mathbb{C}.$$

2.) **Identity Operator**
If we denote by 1 the constant function of value 1 on M, then 1 induces the identity operator $\mathbb{1}$ on $\mathcal{H}$

$$O_1 = \mathbb{1}.$$

3.) **Commutation Relations**
The assignment $f \mapsto O_f$ is a Lie algebra homomorphism with respect to the Poisson bracket $\{\cdot, \cdot\}$ on M and the commutator $[\cdot, \cdot]$ on $\mathcal{H}$ up to a constant $i\hbar$

$$[O_f, O_g] = i\hbar O_{\{f,g\}}.$$

4.) **Position and Momentum Operators**
For Euclidean space $M = \mathbb{R}^n$ equipped with the standard symplectic form ω and canonical coordinates q_i, p_i, we recover the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ with standard position and momentum operators O_{q_i}, O_{p_i} defined by

$$O_{q_i}(|\psi\rangle) = q_i |\psi\rangle \qquad O_{p_i}(|\psi\rangle) = -i\hbar \frac{\partial |\psi\rangle}{\partial q_i}.$$

5.) **Functoriality**
For two symplectic manifolds $(M_1, \omega_1), (M_2, \omega_2)$ with a symplectomorphism $\varphi \colon M_1 \to M_2$, there should exist a unitary operator U_φ between the corresponding Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, such that

$$f \in \mathcal{Q}(M_2) \implies f \circ \varphi \in \mathcal{Q}(M_1)$$
$$O_{f \circ \varphi} = U_\varphi^\dagger O_f U_\varphi$$

Definition 3.1 (Quantization)**.** Let (M, ω) be a classical mechanical system and $\mathcal{Q}(M)$ a subalgebra of $\mathcal{C}^\infty(M)$. A separable Hilbert space $\mathcal{H}$ together with an assignment

$$Q \colon \mathcal{Q}(M) \to \mathcal{O}(\mathcal{H})$$

satisfying the conditions stated above is called a quantization of the system represented by M.

Instead of Postulates 4 and 5 one often finds an irreducibility postulate which is based on the preservation of symmetries when quantizing a classical system.

6.) **Irreducibility Postulate**
For every complete set of observables $\{f_i\}$ of M, the corresponding quantum operators Q_{f_i} form a complete set of quantum observables for $\mathcal{H}$.

In the description of prequantization, however, we will only be concerned with the conditions 1.)-3.). These are the conditions for quantization originally posed by Dirac and we will refer to them from now on as Q1)-Q3).

3.2. **Prequantum Bundles.** In the description of prequantization we largely follow the procedure described in [17] and [3], [15] in more detail.
Step 1:
For any classical observable $f \in \mathcal{C}^\infty(M)$, the symplectic structure of M already allows us to assign to f an operator, namely the Hamiltonian vector field X_f acting itself on smooth functions on M. We obtain a Hilbert space by only considering the square integrable functions on M, where M is equipped with the natural volume form induced by ω, the Liouville volume form

$$\Lambda_\omega \coloneqq (-1)^{\frac{1}{2}n(n-1)} \frac{1}{n!}\omega^n.$$

Hence, a first approach for prequantization would be to map

$$(M,\omega) \to \mathcal{H}_M \coloneqq L^2(M,\Lambda_\omega) \qquad f \in \mathcal{C}^\infty(M) \mapsto -i\hbar X_f.$$

The assignment is clearly linear and satisfies the commutation relations, since $f \mapsto X_f$ is a Lie algebra homomorphism and thus

$$[Q_f, Q_g] = [-i\hbar X_f, -i\hbar X_g] = (-i\hbar)^2 X_{\{f,g\}} = -i\hbar Q_{\{f,g\}}.$$

However, one can easily see, that quantum condition Q2) is violated. In particular, if f is constant, the corresponding Hamiltonian vector field X_f vanishes everywhere and hence $Q_1 = 0$.

Step 2: To account for the vanishing operator of the constant function, we slightly adapt the previous assignment to

$$f \in \mathcal{C}^\infty(M) \mapsto -i\hbar X_f + f$$

where in a slight abuse of notation f acts on elements of $\mathcal{H}$ by multiplication, i.e.

$$Q_f(\psi) = -i\hbar X_f(\psi) + f\psi \quad \text{ for all } \psi \in \mathcal{H}.$$

While the assignment is still linear and now satisfies Q2), we lose the Lie algebra homomorphism property. To see this, consider for $f, g \in \mathcal{C}^\infty(M), \psi \in \mathcal{H}$

$$[X_f, g](\psi) = X_f(g\psi) - gX_f(\psi) = X_f(g)\psi$$

and therefore

$$\begin{aligned}[Q_f, Q_g] &= [-i\hbar X_f + f, -i\hbar X_g + g] \\ &= (-i\hbar)^2[X_f, X_g] - i\hbar([X_f, g] + [f, X_g]) + [f, g] \\ &= -i\hbar(-i\hbar X_{\{f,g\}} + X_f(g) - X_g(f)) \\ &= -i\hbar(-i\hbar X_{\{f,g\}} + 2\{f, g\}) \\ &= -i\hbar Q_{\{f,g\}} - i\hbar\{f, g\}.\end{aligned}$$

Step 3:
Finally, to correct for the unwanted term in Step 2 of the construction, we introduce a symplectic potential θ of ω and define the assignment as

$$f \in \mathcal{C}^\infty(M) \mapsto Q_f = -i\hbar X_f - \theta(X_f) + f = -i\hbar(X_f - \frac{i}{\hbar}\theta(X_f)) + f \tag{3.2}$$

Linearity is preserved and since X_f vanishes for $f = 1$, Q2) is satisfied as well. We can easily check the homomorphism property

$$\begin{aligned}[Q_f, Q_g] &= [-i\hbar X_f - \theta(X_f) + f, -i\hbar X_g - \theta(X_g) + g] \\ &= [-i\hbar X_f + f, -i\hbar X_g + g] - [\theta(X_f), -i\hbar X_g] - [-i\hbar X_f, \theta(X_g)] \\ &= -i\hbar\left(-i\hbar X_{\{f,g\}} + 2\{f, g\} - [\theta(X_f), X_g] - [X_f, \theta(X_g)]\right) \\ &= -i\hbar\left(-i\hbar X_{\{f,g\}} + 2\{f, g\} + X_g(\theta(X_f)) - X_f(\theta(X_g))\right) \\ &= -i\hbar\left(-i\hbar X_{\{f,g\}} + 2\{f, g\} + d(i_{X_f}\theta)(X_g) - d(i_{X_g}\theta)(X_f)\right) \\ &= -i\hbar\left(-i\hbar X_{\{f,g\}} + \{f, g\} - \theta(X_{\{f,g\}})\right) \\ &= -i\hbar Q_{\{f,g\}}.\end{aligned}$$

Introduction of Line Bundles
While the above construction satisfies all the original quantum conditions, the introduction of some symplectic potential θ means that the assignment is dependent on the choice of θ. Furthermore for a general symplectic manifold (M, ω) there might not exist a global symplectic potential and hence the above construction is only valid locally. To further investigate this dependence we consider how a change in symplectic potential affects the induced quantum operator.
Let θ and θ' be two symplectic potentials of ω that are both defined on some open $U \subseteq M$. Without loss of generalization, we can assume that U is contractible. Due to the Lemma of Poincaré, θ and θ' then differ by an exact one-form on U

$$\theta - \theta' = d\lambda \quad \text{for some } \lambda \in \mathcal{C}^\infty(U).$$

Thus, the difference in the corresponding quantum operators is given by

$$Q_f - Q'_f = -d\lambda(X_f) = -X_f(\lambda).$$

We can account for this term when allowing for changes in the functions ψ by some complex phase

$$\theta \to \theta' = \theta + d\lambda \qquad \psi \to \psi' = e^{\frac{i}{\hbar}\lambda}\psi. \tag{3.3}$$

$$\begin{aligned}
\implies Q'_f(\psi') &= -i\hbar X_f(e^{\frac{i}{\hbar}\lambda}\psi) - (\theta(X_f) + X_f(\lambda) - f)e^{\frac{i}{\hbar}\lambda}\psi \\
&= -i\hbar(e^{-\frac{i}{\hbar}\lambda}X_f(\psi) + \psi X_f(e^{\frac{i}{\hbar}\lambda})) - (\theta(X_f) + X_f(\lambda) - f)e^{\frac{i}{\hbar}\lambda}\psi \\
&= e^{\frac{i}{\hbar}\lambda}\left(-i\hbar\left(X_f(\psi) + \frac{i}{\hbar}\psi X_f(\lambda)\right) - (\theta(X_f) + X_f(\lambda) - f)\psi\right) \\
&= e^{\frac{i}{\hbar}\lambda}\left(-i\hbar X_f(\psi) - (\theta(X_f) - f)\psi\right) \\
&= e^{\frac{i}{\hbar}\lambda} Q_f(\psi)
\end{aligned}$$

By comparing 3.2 with 2.65 and 3.3 with 2.74, one can see that a possible way to model prequantization is by considering sections of a hermitian line bundle and the action of a covariant derivative, such that for corresponding connection one-forms w_i and symplectic potentials θ_i, we have

$$w_i = -\frac{i}{\hbar}\theta_i.$$

The curvature form and the symplectic form allow us to make this precise in a global way, which motivates the following definition.

Definition 3.4 (Prequantum Bundle)**.** A symplectic manifold is called quantizable if there exists a line bundle $\pi\colon L \to M$ equipped with a hermitian structure h and a compatible connection ∇, such that the corresponding curvature Ω is given by

$$\Omega = -\frac{i}{\hbar}\omega.$$

In this case, we call (L, π, M, h, ∇) or in a slight abuse notation just L a prequantum bundle.

Remark 3.5. Note that as in Remark 2.55 different conventions lead to different prefactors in Definition 3.4. The resulting integrality conditions discussed in the next subsection, however, do not change.

Lemma 3.6. *Let L be a line bundle with hermitian structure h and denote by $\Gamma(L)_C$ the set of sections in L with compact support. Then h defines an inner product on $\Gamma(L)_C$ via*

$$\langle\psi_1|\psi_2\rangle := \int_M h(\psi_1, \psi_2)\Lambda_\omega.$$

We are now in a position to describe the full prequantization procedure of a prequantizable symplectic manifold.

Proposition 3.7. *Let (L, π, M, h, ∇) be a prequantum bundle. Let $\mathcal{H}$ be the completion of the inner product space of compactly supported sections $\Gamma_C(L)$ in L, equipped with the inner product defined in Lemma 3.6 and define an assignment*

$$Q\colon \mathcal{C}^\infty(M) \to \mathcal{O}(\mathcal{H}), \quad f \mapsto Q_f = -i\hbar\nabla_{X_f} + f.$$

Then the tuple $(\mathcal{H}, Q)$ satisfies all of Dirac's Quantum Conditions Q1)-Q3) specified in 3.1 and we call it a prequantization of the classical system (M, ω).

Proof. Linearity of the given map is obvious. For the identity function $f = 1$, the corresponding Hamiltonian vector field X_f vanishes and hence we recover the identity operator $Q_1 = \mathbb{1}$. Lastly, the commutation relations are satisfied

$$\begin{aligned}
[Q_f.Q_g] &= [-i\hbar\nabla_{X_f} + f, -i\hbar\nabla_{X_g} + g] \\
&= (-i\hbar)^2(\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_g}) - i\hbar(\nabla X_f g + f\nabla X_g - \nabla_{X_g} f - g\nabla_{X_f}) \\
&= (-i\hbar)^2(\Omega(X_f, X_g) + \nabla_{[X_f,X_g]}) - i\hbar(X_f(g) + g\nabla X_f + f\nabla_{X_g} - X_g(f) - f\nabla_{X_g} - g\nabla_{X_f}) \\
&= -i\hbar(-i\hbar\nabla_{X_{\{f,g\}}} + \{f,g\}) - i\hbar(-i\hbar\Omega(X_f, X_g) - \{f,g\}) \\
&= -i\hbar(-i\hbar\nabla_{X_{\{f,g\}}} + \{f,g\}) \\
&= -i\hbar Q_{\{f,g\}}.
\end{aligned}$$

□

3.3. **Integrality Conditions.** In this subsection, we will discuss under which circumstances a symplectic manifold (M, ω) is prequantizable. We present two different integrality conditions (cf. [17]).

Proposition 3.8 (Weil's Integrality Condition)**.** *Let (M, ω) be a symplectic manifold. If ω is prequantizable, the integral of ω over a closed (i.e. compact and without boundary) oriented 2-surface in M is an integral multiple of $2\pi\hbar$.*

Before we begin the proof, we will need the following Lemma, the proof of which was adapted from [11].

Lemma 3.9. *Let $\pi\colon L \to M$ be a line bundle with connection ∇ and corresponding curvature Ω. Let Σ be a compact submanifold of M, such that its boundary $\delta\Sigma = \gamma$ is a loop in M. Then*

$$hol(\gamma, \nabla) = exp\left(-\int_\Sigma \Omega\right).$$

Proof. Consider first the case where Σ is contained in some $U \subseteq M$, such that there exists a local non-vanishing section s in U and denote by w the corresponding connection 1-form. Let δ be a parallel section along γ, i.e.

$$\nabla_{\dot\gamma}\delta = 0.$$

Assume that γ is parametrized by the unit interval $[0, 1]$. Then we can write $\delta(t) = f(\gamma(t))s(\gamma(t))$ for some complex-valued function $f\colon \gamma([0, 1]) \to \mathbb{C}$. The above then yields

$$\begin{aligned}
0 = \nabla_{\dot\gamma} fs = \dot\gamma(f)s + f\nabla_{\dot\gamma}s = \dot\gamma(f)s + fw(\dot\gamma)s = (\dot\gamma(f) + fw(\dot\gamma))\, s \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t)) = -f(\gamma(t))w(\dot\gamma(t)) \qquad \forall t \in [0, 1] \\
\implies f(\gamma(t)) = \exp\left(-\int_0^t dtw(\dot\gamma(t))\right) f(\gamma(0)) = \exp\left(-\int_\gamma w\right) f(\gamma(0)).
\end{aligned}$$

Using Stoke's Theorem, we can then deduce that

$$\mathrm{hol}(\nabla, \gamma) = \exp\left(-\int_\gamma w\right) = \exp\left(-\int_\Sigma \Omega\right).$$

For the general case, due to the compactness of Σ, we can find a triangulation of Σ, such that each triangle is contained in some $U \subseteq M$ with a local non-vanishing section defined on U. Then we obtain the general result by composition of the holonomies of all triangles, since the paths along all inner edges of the triangulation cancel.

□

We can now prove the original statement from Proposition 3.8.

Proof. If ω is prequantizable, there exists some line bundle $\pi\colon L \to M$ with connection ∇, such that the corresponding curvature is given by $\Omega = \frac{i}{\hbar}\omega$. Consider any closed oriented 2-surface D in M. There exists a loop $\gamma \subseteq D$ that splits D into two separate surfaces Σ_1 and Σ_2, such that the boundaries are given by γ and the reversed curve γ^-, respectively (keeping in mind the orientations of Σ_1, Σ_2), so

$$\partial\Sigma_1 = \gamma \quad \partial\Sigma_2 = \gamma^- .$$

Then due to Lemma 3.9, we can deduce that

$$1 = \mathrm{hol}(\nabla, \gamma)\mathrm{hol}(\nabla, \gamma^-) = \exp\left(-\int_{\Sigma_1} \Omega\right) \exp\left(-\int_{\Sigma_2} \Omega\right) = \exp\left(-\int_D \Omega\right)$$

$$\Longrightarrow \frac{1}{2\pi i}\int_D \Omega \in \mathbb{Z} \implies \frac{1}{2\pi\hbar}\int_D \omega \in \mathbb{Z}.$$

□

If M is simply connected, Weil's Integrality Condition is also sufficient.

Proposition 3.10. *Let (M, ω) be a symplectic manifold and simply connected. If the integral of ω over any closed oriented 2-surface in M is an integral multiple of $2\pi\hbar$, ω is prequantizable.*

For a proof of Proposition 3.10, see [17].
In the general case, we obtain a different integrality condition that is necessary and sufficient.

Proposition 3.11. *Let (M, ω) be a symplectic manifold. (M, ω) is prequantizable if and only if the class of $\frac{\omega}{2\pi\hbar}$ in $\hat{H}^2(M, \mathbb{R})$ under the identification of the deRham isomorphy lies in the image of $\hat{H}^2(M, \mathbb{Z})$.*

Proof. To begin with, consider what it means for $[\frac{\omega}{2\pi\hbar}]$ to lie in the image of $\hat{H}^2(M, \mathbb{Z})$. We can identify the deRham cohomology class of an arbitrary closed 2-form Θ with an element in the Čech cohomology via the following construction. Choose a contractible cover U_i, i.e. a cover U_i where arbitrary intersections of the sets U_i are again contractible and denote these intersections by

$$U_{ij} \coloneqq U_i \cap U_j, \qquad U_{ijk} \coloneqq U_i \cap U_j \cap U_k,$$

and so on. Since Θ is closed, according to the Poincaré Lemma there exist local one-forms θ_i defined on each U_i, such that

$$d\theta_i = \Theta \text{ on } U_i.$$

On each U_{ij}, θ_i and θ_j differ by a closed one-form $\theta_i - \theta_j$. Since we are dealing with a contractible cover, there further exist functions f_{ji} defined on each U_{ij}, such that

$$df_{ij} = \theta_i - \theta_j \text{ on } U_{ij}.$$

The functions $g_{ijk} \coloneqq f_{ij} + f_{ik} - f_{jk}$ are constant on each U_{ijk}, since

$$d(f_{ji} + f_{ik} - f_{jk}) = \theta_i - \theta_j + \theta_k - \theta_i - \theta_k + \theta_j = 0.$$

Thus the family $\{U_{ijk}, g_{ijk}\}$ defines a 2-cochain and an element in $\check{H}^2(M, \mathbb{R})$. One can show that this assignment in fact defines an isomorphism between the deRham cohomology $H^2_{dR}(M, \mathbb{R})$ and the Čech cohomology $\check{H}^2(M, \mathbb{R})$.
Taking into account this construction of the deRham isomorphism (as described

also in [3]), the statement is equivalent to the existence of a contractible cover $\{U_i\}$ with corresponding symplectic potentials θ_i and a family of functions f_{ij} defined on $U_{ij} := U_i \cap U_j$, such that

i) $\theta_i - \theta_j = df_{ji}$ on U_{ij}
ii) $\frac{1}{2\pi\hbar}(f_{ji} + f_{ik} - f_{jk}) \in \mathbb{Z}$ on $U_{ijk} := U_i \cap U_j \cap U_k$.

"$\Leftarrow$

Assume that $\{U_i, \theta_i\}$ is a family of symplectic potentials with functions f_{ij} defined on U_{ij} as above. Define

$$c_{ij} := \exp(\frac{i}{\hbar} f_{ij}).$$

Choosing $i = j = k$ yields $c_{ii} = 1$ and thus

i) $c_{ij}c_{jk} = \exp(\frac{i}{\hbar}(f_{ij} + f_{jk})) = \exp(\frac{i}{\hbar}(f_{ik} + 2\pi\hbar n)) = c_{ik}$ (for some $n \in \mathbb{Z}$)
ii) $c_{ij}c_{ji} = c_{ii} = 1$.

The c_{ij} fulfil the cocycle conditions from Lemma 2.19 and can be used to define a line bundle L on M. Furthermore

$$\frac{dc_{ji}}{c_{ji}} = \frac{i}{\hbar} df_{ji} = \frac{i}{\hbar}(\theta_i - \theta_j)$$

and using 2.67, we can thus equip L with a connection ∇, such that the corresponding connection forms on U_i are given by $w_i = \frac{i}{\hbar}\theta_i$. But then in particular the curvature of ∇ is given (locally) by

$$\Omega = dw_i = \frac{i}{\hbar} d\theta_i = \frac{i}{\hbar}\omega,$$

as required. Lastly, the symplectic potentials and the functions f_{ij} are real, which means the connection one-forms are purely imaginary and $|c_{ij}| = 1$. Using Lemma 2.71, L can then be equipped with a hermitian structure compatible with ∇.

"$\Rightarrow$"

Let (M, ω) be a prequantizable manifold. Then there exists a line bundle $\pi \colon L \to M$ on M equipped with a hermitian structure h and a compatible connection ∇, such that the corresponding curvature is given by $\Omega = \frac{i}{\hbar}\omega$. Let (U_i, φ_i) be some trivialization of L on a contractible cover U_i and denote by w_i the connection one-forms of the corresponding unit sections s_i. Since $dw_i = dw_j$ and all $U_{ij} = U_i \cap U_j$ are contractible, we can find functions f_{ji} defined on U_{ij}, such that

$$df_{ji} = w_i - w_j = \frac{dc_{ji}}{c_{ji}} \implies f_{ji} = \log c_{ji}$$

using the Lemma of Poincaré. Then, due to the cocycle condition

$$\begin{aligned} c_{ji}c_{ik} = c_{jk} &\implies \log c_{ji} + \log c_{ik} - \log c_{jk} \in 2\pi i\mathbb{Z} \\ &\implies f_{ji} + f_{ik} - f_{jk} \in 2\pi i\mathbb{Z} \\ &\implies \frac{1}{2\pi i}(f_{ji} + f_{ik} - f_{jk}) \in \mathbb{Z}. \end{aligned}$$

If we define the symplectic potentials $\theta_i := -i\hbar w_i$ and $\hat{f}_{ji} := -i\hbar f_{ji}$, then it follows that

$$\theta_i - \theta_j = d\hat{f}_{ji}, \qquad \frac{1}{2\pi\hbar}\left(\hat{f}_{ji} + \hat{f}_{ik} - \hat{f}_{jk}\right) \in \mathbb{Z}$$

and hence $\left[\frac{1}{2\pi\hbar}\omega\right] \in \mathbb{Z}$. □

3.4. **Examples.** Before we consider the more complete procedure of geometric quantization, we will illustrate the shortcomings of prequantization at the fundamental example of an arbitrary cotangent bundle.

Example 3.12 (Free Particle). Consider a configuration space Q and the symplectic manifold given by the cotangent bundle T^*Q of Q equipped with the canonical symplectic form ω. Denote by θ the canonical one-form. Then θ is a global symplectic potential and T^*Q is prequantizable with the trivial bundle $L = M \times \mathbb{C}$. Sections in L can be identified with complex-valued functions on M and we can equip L with the connection

$$\nabla_X f := X(f) - \frac{i}{\hbar}\theta(X)f.$$

Equivalently, we can define the constant function $s = 1$ as a global frame of L and define ∇ as the connection such that s carries the connection one-form $w_i = \frac{i}{\hbar}\theta$. Either way, the curvature of the corresponding connection satisfies $\Omega = \frac{i}{\hbar}\omega$.
We can now extend an arbitrary coordinate system q_i of Q to a canonical coordinate system q_i, p^j on T^*Q. Using the prequantization assignment from proposition 3.7, we obtain the quantized operators

$$Q_{q_i} = -i\hbar\nabla_{X_{q_i}} + q_i = i\hbar\frac{\partial}{\partial p^i} + q_i$$
$$Q_{p^j} = -i\hbar\nabla_{X_{p^j}} + p^j = -i\hbar\frac{\partial}{\partial q_j}.$$

Clearly, we don't recover the standard quantum operators for position and momentum as specified in quantization conditions of subsection 1.4. Note that in particular our quantum operators Q_{q_i} depend on the momenta p^j, which should not be the case.

4. Quantization

As we'ven seen in the last subsection 3.4, the prequantization procedure fails to produce the correct quantizations for elementary systems, such as the harmonic oscillator. Intuitively, the problem that arises lies in the fact, that the prequantum states at a certain point in time depend on too many variables. The idea of geometric quantization is therefore to limit the Hilbert space constructed in the method of prequantization by introducing polarizations and only considering polarized sections. In doing so, one also has to limit the set of prequantum operators and therefore also the set of quantizable classical operators to be consistent with the restriction of the original Hilbert space. We will see that in the general case new problems arise, however, when dealing exclusively with Kähler polarizations, the described method aleady yields correctly quantized systems for a number of elementary examples. This section offer a short overview of the basics for holomorphic quantization.

4.1. **Polarized Sections.** We begin with the definition of polarized sections.

Definition 4.1 (Polarized Section)**.** Let (L, π, M, h, ∇) be a prequantum bundle and let P be a strongly integrable polarization of M. A section s in L is called polarized with respect to P if

$$\nabla_X s = 0 \quad \forall X \subseteq P.$$

We denote the set of sections in L that are polarized with respect to P by $\Gamma_P(L)$.

The quantization procedure follows immediately from this definition.

Definition 4.2 (Quantum States)**.** Let (M, ω) be a symplectic manifold with a strongly integrable polarization P and let (L, π, M, h, ∇) be a prequantum bundle on M. Then we define the space of quantum states under the polarization P as the set of sections $\Gamma_P(L)$ that are polarized with respect to P.

Before we consider the problems that arise with this new definition, let us try to relate it to the idea that the states should be dependent on less variables than before. To do so, we need the following Lemma as can be found in [17].

Lemma 4.3. *Let P be a strongly integrable polarization on a symplectic manifold (M, ω). For any $x \in M$, there exists a neighborhood U and real coordinates*

$$\{x_1, \dots x_{n-l}, y_1, \dots, y_{n-l}, u_1, \dots, u_l, v_1, \dots, v_l\}$$

with complex coordinates $\{z_1, \dots z_l\}$ defined as $z_k := u_k + iv_k$, such that P is spanned in U by

$$\frac{\partial}{\partial x_j}, j \in \{1, \dots, n-l\} \quad \frac{\partial}{\partial \overline{z}_k}, k \in \{1, \dots, l\}$$

and the isotropic distribution $D = P \cap \overline{P} \cap TM$ is spanned by

$$\frac{\partial}{\partial x_j}, j \in \{1, \dots, n-l\}.$$

Consider now a section in a prequantum bundle (L, π, M, h, ∇) that is polarized with respect to a strongly integrable polarization P. We know from 2.43, that P is admissible, hence locally there exists an adapted symplectic potential θ. If we define $w := \frac{i}{\hbar}\theta$, then w_i is the connection one-form of a non-vanishing section s and we can again represent any section $s' = \psi s$ by a complex function ψ. Let us now consider what condition Definition 4.1 poses on these functions. For all $X \subseteq P$

$$0 = \nabla_X \psi s = (X(\psi) + \psi w(X))s = X(\psi)s$$

since θ is adapted to P and hence $w(X) = \frac{i}{\hbar}\theta(X) = 0$. From the previous Lemma 4.3, it follows that in these local coordinates, the functions ψ can only depend on the variables $\{y_1, \dots, y_{n-l}, \bar{z}_1, \dots \bar{z}_l\}$ which make up exactly half of the defining coordinates for M.

So while this construction theoretically resolves our problem of dependencies on too many variables, new issues arise. The main problem lies in the fact that in general the polarized sections are not square-integrable and thus do not form a Hilbert space. At the same time we cannot just restrict $\Gamma_P(L)$ to the intersection with the original Hilbert space $\mathcal{H}$ of square-integrable section, since in a lot of cases $\Gamma_P(L) \cap \mathcal{H} = \emptyset$. To see this, consider for any real vector field $X \subseteq D$ and $s_1, s_2 \in \Gamma_P(L)$

$$\nabla_X(h(s_1, s_2)) = h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2) = 0.$$

Thus, the function $h(s_1, s_2)$ is constant on the leaves of the foliation of D and if possible we would like to integrate it in the quotient manifold $\mathcal{D} := M/D$. However, to do so we need to define a measure in $\mathcal{D}$ and thus in the general case, quantization requires another extra step. There is one special case however, where this problem resolves itself, namely when we are dealing with Kähler polarizations. In that case $P \cap \overline{P} = \emptyset$ and hence we obtain $\mathcal{D} = M$. The formalism obtained in this way is called holomorphic quantization and we will make it more precise in subsection 4.3. Before discussing it in more detail, however, we will consider the restrictions Definition 4.2 puts on the set of classical observables that are quantizable.

4.2. **Quantum Operators.** Under the new definition of the quantum Hilbert space $\mathcal{H}$ as described in subsection 4.1, the assignment of quantum operators to classical observables from the procedure of prequantization becomes invalid, because we need to make sure that $\Gamma_P(L)$ is invariant under the action of the resulting operators, i.e. that polarized sections are mapped to polarized sections. As a result, we have to restrict the set of classical observables $f \in \mathcal{C}^\infty(M)$ that are quantizable. In the following let (L, π, M, h, ∇) be a prequantum bundle on a symplectic manifold and let P be a strongly integrable polarization of M.

Definition 4.4 (Quantizable Observable)**.** A classical observable $f \in \mathcal{C}^\infty(M)$ is quantizable with respect to P if the corresponding quantum operator Q_f maps polarized sections to polarized sections

$$s \in \Gamma_P(L) \implies Q_f(s) = -i\hbar\nabla_{X_f} s + fs \in \Gamma_P(L).$$

One might pose the question under which circumstances a classical observable f is quantizable.

Proposition 4.5. *A classical observable f is quantizable if and only if*

(1) the flow of X_f leaves P invariant, hence $[X_f, X] \subseteq P \quad \forall X \subseteq P$.
(2) the corresponding quantum operator Q_f satisfies

$$Q_f(\nabla_X s) = \nabla_X Q_f(s) \quad \forall X \subseteq P, s \in \Gamma_P(L).$$

A proof can be found in [3].

4.3. **Holomorphic Quantization.** Let us now consider the case where we are dealing with a Kähler manifold, and specifically with P a Kähler polarization. We know by Proposition 2.46, that M can be equipped with a complex structure, such that the Kähler polarizations P, $\overline{P}$ are exactly the holomorphic and anti-holomorphic bundles on M. Furthermore, Proposition 2.47 tells us, that there exist local symplectic potentials θ and $\overline{\theta}$ that are adapted to P and $\overline{P}$, respectively, given in local complex coordinates by

$$\theta = i \sum_j \frac{\partial K}{\partial z_j} \qquad \overline{\theta} = -i \sum_j \frac{\partial K}{\partial \overline{z}_j}$$

where K denotes a Kähler scalar, i.e. some function on M. If we define a connection one-form $w := \frac{i}{\hbar}\theta$ and denote by s the corresponding local frame of L, then following the same logic as the discussion after Lemma 4.3, we can deduce that polarized sections can be represented as $s' = \psi s$, where ψ is a function on M, such that

$$\frac{\partial \psi}{\partial z_j} = 0 \quad \forall j \qquad \text{or} \qquad \frac{\partial \psi}{\partial \overline{z}_j} = 0 \quad \forall j,$$

depending on whether we are considering the polarization P or $\overline{P}$, since they are spanned by $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$, respectively. However, these are exactly the holomorphic and anti-holomorphic functions on M, which also.motivates the name Holomorphic Quantization.
The main result then, which sets apart the case of Kähler manifolds from the more general cases is captured in the following proposition.

Proposition 4.6. *Let (L, π, M, h, ∇) be a prequantum bundle and P a Kähler polarization on M. Denote by $\mathcal{H}$ the Hilbert space of square-integrable sections that arises from the prequantization procedure. Then the set of holomorphic square-integrable sections $\Gamma_P(L) \cap \mathcal{H}$ is a closed subspace of $\mathcal{H}$. In particular $\Gamma_P(L)$ is a Hilbert space.*

A proof of 4.6 can be found in [3].

4.4. **Outlook.** While we have seen in the previous subsection that the procedure of geometric quantization as described up to this point can successfully applied when dealing with Kähler manifolds, some work remains to be done in the most general case. In particular we need to define a notion of integration in the quotient manifold $\mathcal{D}$ and this leads to the introduction of half-forms and the procedure of metaplectic correction (see [17][3]). Unfortunately a full description of these methods would exceed the scope of this thesis.

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