The Duistermaat-Heckman Theorem with an Application to Harish-Chandra-Itzykson-Zuber

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Abstract

We give a proof of two related theorems in symplectic geometry, both referred to as the Duistermaat-Heckman theorems. The setting is always that of a hamiltonian torus space (M, ω) with moment map μ. Considering symplectic reduction, the first theorem relates the reduced space at levels $t \in \mathfrak{t}^*$ in a neighbourhood of any regular value of the moment map to the reduced space at the regular value itself; the reduced symplectic form at level t is found to vary linearly in t. Pushing the Liouville measure on M forward by μ gives the **Duistermaat-Heckman measure** on $\mathfrak{t}^*$; The second version of the theorem computes the Fourier transform of a one-dimensional projection of this measure as a sum over the fixed points of the action. This theorem is then used in conjunction with the method of coadjoint orbits to prove the Harish-Chandra-Itzykson-Zuber integral formula.

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Chapter 1

The Coisotropic Embedding Theorem

We start off this report by presenting the key theorem in the proof of the first Duistermaat-Heckman theorem. Let us give the relevant definitions.

Definition 1.1. *Let (V, ω) be a symplectic vector space, and $W \subset V$ a subset. Then the **symplectic complement** of W is the set*

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \, \forall w \in W\}.$$

In case W is a linear subspace, the complement satisfies $(W^\omega)^\omega = W$, but not necessarily $W \cap W^\omega = \{0\}$.

Definition 1.2. *We say the subspace W is **coisotropic** if $W^\omega \subset W$.*

If $i : Z \hookrightarrow M$ is an embedding for a manifold Z and a symplectic manifold (M, ω), the embedding is called coisotropic if $T_i(z)Z^{\omega_{i(z)}}$ is a coisotropic subspace of $T_{i(z)}M$.

Our first goal is to prove the coisotropic embedding theorem outside the setting of hamiltonian spaces:

Theorem 1.3 (Coisotropic Embedding Theorem)**.** *Let Z^k a differentiable manifold and $i_j : Z \hookrightarrow M_j$ embeddings into symplectic manifolds (M_j, ω_j), for $j = 1, 2$, such that*

1. *i_1 and i_2 are coisotropic.*

2. *$i_1^*\omega_1 = i_2^*\omega_2 = \tau$ for τ a closed 2-form of constant rank on Z.*

Then there exist neighbourhoods U_j of $i_j(Z) \subset M_j$ and a symplectomorphism $f : U_1 \to U_2$ such that $i_2 = f \circ i_1$.

1.1 Main Ingredients

Proving the theorem in this form makes use of the tubular neighbourhood theorem and the Darboux-Weinstein theorem. Let us give the necessary definitions to state them. The statements about the tubular neighbourhood theorem are from [1], section 6.2, whereas the Darboux-Weinstein theorem is as in [2], theorem 22.1.

Let M^n a manifold and $i : X^k \hookrightarrow M$ an embedded submanifold. Write $x = i(x)$ in M, as well. At each point $x \in M$, view the tangent space to X at x as a linear subspace of $T_i(x)M$ via the inclusion $Di(x)$. Define the **normal space to** x by $N_x := TxM/TxX$, which is an $n-k$ dimensional vector space. Define the **normal bundle** as

$$NX := \{(x,v) \in TM \mid x \in X, v \in N_xX\}.$$

This has the structure of a vector bundle of rank $n-k$ over X, and hence as a manifold, NX is n-dimensional.

Regard X as an embedded submanifold of NX via the zero section

$$i_0 : X \hookrightarrow NX, \qquad x \mapsto (x,0).$$

Theorem 1.4 (Tubular Neighbourhood Theorem)**.** *There exists a neighbourhood U_0 of X in NX, a neighbourhood U_1 of X in M, and a diffeomorphism $\varphi : U_0 \to U_1$ such that the following diagram commutes:*

$$\begin{array}{ccc} NX \supseteq U_0 & \xrightarrow[\cong]{\varphi} & U \subseteq M \\ & {\scriptstyle i_0}\nwarrow \quad \nearrow{\scriptstyle i} & \\ & X & \end{array}$$

Theorem 1.5. *[Darboux-Weinstein] Let X a submanifold of M and ω_0 and ω_1 nonsingular, closed 2-forms on M such that $i^*\omega_0 = i^*\omega_1$, for $i : X \hookrightarrow M$ the inclusion. Then there exists a neighbourhood U of X and a diffeomorphism f from U into M such that*

1. *$f(x) = x$ for all $x \in X$, and*
2. *$f^*\omega_1 = \omega_0$.*

1.2 Proof of the Coisotropic Embedding Theorem

This proof follows that of the uniqueness part of theorem 39.2 in [2], combined with some observations in 9.1 of [1]. First, we will show how the normal bundles corresponding to $i_1(Z)$ and $i_2(Z)$ can be identified. We start by linear algebra. Suppose (V^n, Ω) is a symplectic vector space, and U^k is a coisotropic subspace, that is, one such that $U^\Omega \subset U$. The quotient space V/U is then a vector space of dimension $n-k$.

Claim: The pairing defined by

$$\Omega' : V/U \times U^\Omega \to \mathbb{R}, \qquad ([v], u) \mapsto \Omega(v, u)$$

is well-defined and nondegenerate.

Indeed, if $v \sim v'$, then $v - v' \in U^\Omega$, so that for $u \in U^\Omega$, we have $0 = \Omega(v - v', u) \iff \Omega(v, u) = \Omega(v', u)$, hence Ω' is well-defined. For nondegeneracy, suppose $\Omega'([v], u) = 0$ for all $u \in U^\Omega$. Then also $\Omega(v, u) = 0$ for all $u \in U^\Omega$, whence $v \in (U^\Omega)^\Omega = U$, so that $[v] = 0$.

If $\Omega'([v], u) = 0$ for all $[v]$, then also $\Omega(v, u) = 0$ for all $v \in V$, which by nondegeneracy of Ω implies $v = 0 \implies [v] = 0$.

From this we obtain an isomorphism

$$\tilde{\Omega} : V/U \longrightarrow (U^\Omega)^*, \qquad [v] \mapsto \Omega'([v], \cdot).$$

Hence as both our embeddings are coisotropic, $T_{i(z)}i(Z) \subset T_{i(z)}M$ is a coisotropic subspace for both embeddings, and thus the above considerations yield an identification between the normal bundle $N_j = TM|_{i_j(Z)}/Ti_j(Z) \to Z$ and the bundle $(Ti_j(Z)^{\omega_j})^* \to Z$.

Since we can view T_zZ as a linear subspace of $T_{i_j(z)}M_j$ via $Di_j(z)$, we may write a generic element of $Ti_j(Z)$ as $(z, Di_j(z)[v])$ for some $v \in T_zZ$. If this is an element of the symplectic complement, we have that

$$\begin{aligned} 0 &= (\omega_j)_{i_j(z)}\,(Di_j(z)[v], \cdot) \\ &= (i_j^*\omega_j)_z(v, \cdot) \\ &= \tau_z(v, \cdot), \end{aligned}$$

which tells us that $v \in T_zZ^\tau$, yielding an identification of N_j with $(TZ^\tau)^*$, and hence between N_1 and N_2 themselves (recall that τ was not required to be nondegenerate). Let $A : N_1 \to N_2$ the corresponding vector bundle isomorphism.

Next, the tubular neighbourhood theorem applied to i_1 and i_2 guarantees the existence of diffeomorphisms $\varphi_j : U_1^j \to U_2^j$ for $U_1^j \subset N_j$ a neighbourhood of the zero section $0_j(Z)$ and $U_2^j \subset M_j$ a neighbourhood of $i_j(Z)$. These diffeomorphisms satisfy $\varphi_j \circ 0_j = i_j$.

Set $\Phi := \varphi_2 \circ A \circ \varphi_1^{-1}$ and $\bar{\omega}_1 := \Phi^*\omega_2$. We then have $\Phi \circ i_1 = i_2$:

$$\begin{aligned} \Phi \circ i_1 &= \varphi_2 \circ A \circ \varphi_1^{-1} \circ i_1 \\ &\overset{(1)}{=} \varphi_2 \circ A \circ 0_1 \\ &\overset{(2)}{=} \varphi_2 \circ 0_2 \\ &\overset{(1)}{=} i_2, \end{aligned}$$

where (1) uses $\varphi_j \circ 0_j = i_j$ and (2) follows from the fact that A, as a vector bundle isomorphism, is linear on fibres.

This implies that

$$\begin{aligned} i_1^* \bar{\omega}_1 &= i_1^* \Phi^* \omega_2 \\ &= (\Phi \circ i_1)^* \omega_2 \\ &= i_2^* \omega_2 \\ &= i_1^* \omega_1, \end{aligned}$$

so ω_1 and $\bar{\omega}_1$ agree on $i_1(Z)$ and we are in position to use Weinstein's theorem. We obtain a neighbourhood U of $i_1(Z) \subset M_1$ and a diffeomorphism g from U into a neighbourhood of $i_1(Z)$ such that $g^* \bar{\omega}_1 = \omega_1$ and $g \circ i_1 = i_1$.

Setting $f := \Phi \circ g$ hence defines a diffeomorphism from $U \supset i_1(Z)$ to a neighbourhood of $i_2(Z)$, satisfying

$$f^* \omega_2 = g^* \Phi^* \omega_2 = g^* \bar{\omega}_1 = \omega_1,$$

that is, f is a symplectomorphism from a neighbourhood of $i_1(Z)$ to a neighbourhood of $i_2(Z)$. Also, as $g \circ i_1 = i_1$, we have

$$f \circ i_1 = \Phi \circ g \circ i_1 = \Phi \circ i_1 = i_2,$$

and so f is the symplectomorphism we seek.

1.3 Introducing Group Actions

Let us now assume that, in addition to the hypotheses of the coisotropic embedding theorem, a compact Lie group G acts on M_j, preserving the symplectic forms on M_j. We claim

Proposition 1.6. *The symplectomorphism from theorem 1.3 can be chosen to be G-equivariant.*

Proving this boils down to showing that the diffeomorphisms obtained by Darboux' and the tubular neighbourhood theorem, respectively, can be chosen to be G-equivariant. Indeed, recalling

$$f = \varphi_2 \circ A \circ \varphi_1^{-1} \circ g,$$

if g and the φ_j are equivariant, then the action on the tangent bundle, and hence the normal bundle, induced by differentiating the action is linear and commutes with A, so that by equivariance of φ_2, we obtain equivariance of f.

To see that it is possible to choose these diffeomorphisms as equivariant, we must consult the proofs of the tubular neighbourhood theorem and Darboux' theorem. These will rely heavily on properties of the exponential map, so let us recall its definition and a few basic properties, as stated in lectures 43 and 52 of [3].

1.3.1 The Exponential Map

Definition 1.7. *Let (M, g) a riemannian manifold and $\mathbb{S} \in \mathfrak{X}(TM)$ the geodesic spray associated to the Levi-Civita connection induced by g. Denote by $\Theta : \mathcal{D} \to TM$ its flow, for $\mathcal{D} \subset \mathbb{R} \times TM$ the maximal domain of definition. Let $\mathcal{S}_x \subset TM$ the set of all tangent vectors such that $(1, (x, v)) \in \mathcal{D}$, and set $\mathcal{S} = \bigcup_{x \in M} \mathcal{S}_x$. Let $\pi : TM \to M$ the projection.*

*The **exponential map** associated to g is defined as*

$$\exp : \mathcal{S} \to M, \quad \exp(x, v) = \pi(\Theta_1(x, v)),$$

and we write

$$\exp_x := \exp|_{\mathcal{S}_x} : \mathcal{S}_x \to M.$$

Proposition 1.8. *Let $x \in M$ and recall the canonical identification between T_xM and $T_{0_x}TM$ given by*

$$\mathcal{J}_{0_x} : T_xM \to T_{0_x}TM, \qquad v \mapsto \left.\frac{d}{dt}\right|_{t=0} 0_x + tv.$$

The following hold:

(i) For each $x \in M$, $\exp_x$ satisfies

$$D\exp_x(0_x) \circ \mathcal{J}_{0_x} = \mathrm{id}_{T_xM},$$

and as $\mathcal{J}_{0_x}$ is a diffeomorphism, this means that $\exp_x$ has maximal rank near 0_x and is hence a local diffeomorphism from a neighbourhood of $0_x \in T_xM$ to a neighbourhood of x in M.

(ii) For each $x \in M$, $(\pi, \exp) : \mathcal{S} \to M \times M$ has rank $2n$ at 0_x and thus maps a neighbourhood of 0_x in T_xM diffeomorphically onto a neighbourhood of $(x, x) \in M \times M$. Moreover, if $0(M) \subset TM$ denotes the zero section, there exists a neighbourhood $\mathcal{U}$ of $0(M)$ such that $(\pi, \exp)$ maps $\mathcal{U}$ diffeomorphically onto a neighbourhood of the diagonal $\Delta = \{(x, x) \mid x \in M\}$.

Proposition 1.9. *Let (M, g) a riemannian manifold and d the distance metric on M induced by g. Set*

$$O(x, \varepsilon) = \{v \in T_xM \mid \sqrt{g_x(v, v)} < \varepsilon\}, \qquad U(x, \varepsilon) = \{y \in M \mid d(x, y) < \varepsilon\}.$$

Further set

$$\mathrm{inj}_g(x) = \sup\{r > 0 \mid \exp_x|_{O(x,r)} \text{ is a diffeomorphism onto its image}\}.$$

Then for $\varepsilon \in (0, \mathrm{inj}_g(x))$, $\exp_x|_{O(x,\varepsilon)}$ maps $O(x, \varepsilon)$ diffeomorphically onto $U(x, \varepsilon)$. Hence for every $y \in U(x, \varepsilon)$, there exists a unique length-minimizing geodesic joining x to y.

With this machinery in hand, we examine the proof of the tubular neighbourhood theorem in order to identify the diffeomorphisms φ_j from the proof of the coisotropic embedding theorem.

1.3.2 Identifying the Morphisms from the Tubular Neighbourhood Theorem

We will do this by sketching a proof of the theorem in case the submanifold X is compact, which is an adaptation from the outline given in 6.2 of [1].

Endow M with a riemannian metric g, and let $d(p,q)$ the corresponding distance metric on M. Define for $\varepsilon > 0$ the ε-neighbourhood of X by

$$\mathcal{U}^\varepsilon := \{p \in M \mid d(p,q) < \varepsilon \text{ for some } q \in X\}.$$

Our strategy is to find a neighbourhood of the zero section in NX to identify with a neighbourhood of the zero section in TM which is mapped diffeomorphically onto $\mathcal{U}^\varepsilon$ by the exponential map. The main points of this proof thus consist in defining the appropriate neighbourhood and choosing ε small enough so that we may make use of the various properties from section 1.3.1.

For the neighbourhood, note that we can identify the normal space at x with the following subspace of T_xM:

$$N_xM \cong \{v \in T_xM \mid g_x(v,w) = 0 \; \forall w \in T_xX\} =: T_xX^\perp.$$

To see this, consider the pairing

$$T_xM|_X/T_xX \times T_xX^\perp \to \mathbb{R}, \qquad ([v],u) \mapsto g'_x([v],u) := g_x(v,u).$$

If $v \sim v'$, then $v - v' \in T_xX$, so that $0 = g_x(v-v',u)$ for any $u \in T_xX^\perp$, and by linearity of g_x, this proves the pairing is well-defined. It is also non-degenerate, since if there is $[v] \in T_xM|_M/T_xX$ such that $g'_x([v],u) = 0$ for all $u \in T_xX^\perp$, then also $g_x(v,u) = 0$ for any representative v and all $u \in T_xX^\perp$, and thus $v \in (T_xX^\perp)^\perp = T_xX$, meaning $[v] = 0$. If $g'_x([v],u) = 0$ for all $[v]$, then $u = 0$ follows simply from nondegeneracy of g_x.

This now induces an isomorphism

$$N_xX \to (T_xX^\perp)^*, \qquad [v] \mapsto g_x(v,\cdot),$$

which we can identify with $T_xX^\perp$ by $g_x(v,\cdot) \mapsto v$ since g_x is an inner product.

Now define

$$NX^\varepsilon = \{(x,v) \in NX \mid \sqrt{g_x(v,v)} < \varepsilon\}.$$

This is the neighbourhood of the zero section we want to map onto $\mathcal{U}^\varepsilon$.

For this, choose $0 < \varepsilon < \mathrm{inj}_g(X) := \inf\{\mathrm{inj}_g(y) \mid y \in X\}$.

Claim: For X compact, $\mathrm{inj}_g(X) > 0$.

This is proposition 52.16 in [3]. Using the second claim in proposition 1.8, we obtain a neighbourhood V of $0(X)$ such that $(\pi, \exp)|_V$ is an embedding. By compactness, there are finitely many x_i, ε_i such that

$$X \subset \bigcup_{i=1}^k U(x_i, \varepsilon_i),$$

with the property that

$$U(x_i, 3\varepsilon_i) \times U(x_i, 3\varepsilon_i) \subset (\pi, \exp)(V), \quad i = 1, \dots, k.$$

Hence if $y \in X$, there is an index i such that $y \in U(x_i, \varepsilon_i)$. As $\exp_y$ is an embedding on $O(x_i, 3\varepsilon_i)$, this gives $\mathrm{inj}_g(y) \geq 2\varepsilon_i$, and hence

$$\mathrm{inj}_g(X) \geq \min_{i=1,\dots,k} 2\varepsilon_i > 0.$$

With this in hand, note that $N_x X^\varepsilon = O(x, \varepsilon)$ is mapped diffeomorphically onto $U(x, \varepsilon)$ by the exponential map by proposition 1.9, so that we obtain a family of diffeomorphisms $(\exp_x)_{x \in X}$ onto $U(x, \varepsilon)$. The following lemma, taken from [4], is the final step to prove the theorem:

Lemma 1.10. *Let T a metric space, V, W and D subspaces with $W \subset V$ and $W \subset D$. Let $f : D \to V$ a continuous map such that $f|_W = \mathrm{id}_W$ and assume there exists for each $y \in W$ some $\varepsilon_y > 0$ such that the restriction of f to $B_{\varepsilon_y}(y) \cap D$ is a homeomorphism onto an open subset of V. Then there exists a neighbourhood $Y \subset D' \subset D$, open in D, on which f is injective.*

Taking $T = TM$, V the zero section $0(M) \subset TM$, W the zero section of the submanifold X and $D = NX^\varepsilon$ gives that the restriction of exp to NX^ε considered as a map $NX^\varepsilon \to 0(M) \cong M$ satisfies the requirements on f above, so that we can find a neighbourhood of the zero section $0(X)$ in NX^ε on which exp is injective. This neighbourhood is open in NX^ε, so that we can just choose ϵ small enough to be contained in it, and then take $U_0 = NX^\varepsilon$. Since exp is smooth and maps $0(X)$ to X, we conclude that it maps U_0 diffeomorphically onto a neighbourhood $U \subset M$.

The final statement left to show is commutativity of the diagram in 1.4, but this is evident as $\exp \circ i_0(x) = \exp(x, 0) = i(x) \in M$.

Proof of lemma 1.10. For each $y \in W$, $f(B_{\varepsilon_y/2}(y) \cap D)$ is open in V. Hence as $f(y) = y$, it contains $B_{\varepsilon'_y}(y)$ for some $\varepsilon'_y < \varepsilon_y/4$. As f is a homeomorphism when restricted to this ball, its preimage $Z_y = f^{-1}(B_{\varepsilon'_y}(y))$ is open in D. Set $D' = \bigcup_{y \in Y} Z_y$. Then D' contains Y as $y \in Z_y$ for all y. Take $z_1 \in Z_{y_1}$ and $z_2 \in Z_{y_2}$ are such that $f(z_1) = f(z_2) = y_0$, and without loss of generality, $\varepsilon_{y_1} \geq \varepsilon_{y_2}$. Note further that $y_0 = f(z_i) \in f(Z_{y_i}) \subset B_{\varepsilon'_{y_i}}(y_i)$ for both i. Then

$$\begin{aligned} d(z_2, y_1) &\leq d(z_2, y_2) + d(y_2, y_0) + d(y_0, y_1) \\ &< \varepsilon_{y_2}/2 + \varepsilon'_{y_2} + \varepsilon'_{y_1} \\ &< \varepsilon_{y_2}/2 + \varepsilon_{y_2}/4 + \varepsilon_{y_1}/4 \\ &< \varepsilon_{y_1}, \end{aligned}$$

so that $z_1, z_2 \in B_{\varepsilon_{y_1}}(y_1) \cap D$. f restricted to this set is a homeomorphism, however, so that $z_1 = z_2$. □

In the general case, the strategy is to replace ε by a continuous function $\varepsilon : X \to \mathbb{R}_+$ which tends to zero quickly enough. In any case, the diffeomorphism we obtain is the exponential map restricted to an appropriate domain.

1.3.3 Achieving Equivariance of the Exponential Map

While we will not state the proof here, the diffeomorphism obtained from Darboux' theorem is also an exponential map, as seen in the proof of theorem 22.1 in [2]. Hence we must show that it is possible to arrange that the exponential maps commute with the G-action, that is, if we denote the action by $\psi : g \mapsto \psi_g \in \mathrm{Diff}(M)$, then

$$\exp(\psi_g(x), D\psi_g(x)[v]) = \psi_g \circ \exp(x, v). \qquad (*)$$

Proposition 1.11. *Let (M, g) a riemannian manifold and let G a Lie group acting on M. If g is invariant with respect to the action, that is, if for all $x \in M, u, v \in T_xM, g \in G$, we have*

$$g_{\psi_g(x)}(D\psi_g(x)[u], D\psi_g(x)[v]) = g_x(u, v),$$

then $(*)$ holds.

Proof. We prove that if $\gamma : (t^-, t^+) \to M$ is a non-constant geodesic, then $\psi_g \circ \gamma$ is, too. Assume $\gamma(0) = x$, $\gamma'(0) = v$, and let $T = \frac{\partial}{\partial t} \in \mathfrak{X}(\mathbb{R})$. Then we need to show that $\nabla_T((\psi_g \circ \gamma)') = 0$ for the Levi-Civita connection. Consider

$$\begin{aligned} T\langle(\psi_g \circ \gamma)', (\psi_g \circ \gamma)'\rangle &= T\langle D\psi_g(\gamma)[\gamma'], D\psi_g(\gamma)[\gamma']\rangle \\ &= T\langle\gamma', \gamma'\rangle \\ &= 2\langle\nabla_T(\gamma'), \gamma'\rangle \\ &= 0 \end{aligned}$$

as γ is a geodesic. However, we also have

$$T\langle(\psi_g \circ \gamma)', (\psi_g \circ \gamma)'\rangle = 2\langle\nabla_T((\psi_g \circ \gamma)'), (\psi_g \circ \gamma)'\rangle,$$

and $(\psi_g \circ \gamma)' \neq 0$ as we assume γ to be non-constant implies $\nabla_T((\psi_g \circ \gamma)') = 0$. Note that $(\psi_g \circ \gamma)(0) = \psi_g(x)$ and $(\psi_g \circ \gamma)'(0) = D\psi_g(x)[v]$.

Now recall the defining property of the geodesic spray $\mathbb{S}$, namely that a curve $\delta : (-\varepsilon, \varepsilon) \to TM$ is an integral curve of $\mathbb{S}$ if and only if $\pi \circ \delta$ is a geodesic. Thus for Θ the flow of $\mathbb{S}$, we have shown that

$$\psi_g \circ \pi(\Theta_t(x, v)) = \pi(\Theta_t(\psi_g(x), D\psi_g(x)[v])).$$

In particular for $t = 1$, we obtain

$$\psi_g \exp(x, v) = \exp(\psi_g(x), D\psi_g(x)[v]).$$

□

Hence in order to complete the proof, what is left to show is that we can always find such an invariant riemannian metric. This is according to [5].

Proposition 1.12. *Let M a manifold and G a compact Lie group acting on M. Then there exists a G-invariant riemannian metric on M.*

Proof. Let g' any riemannian metric on M and define

$$g = \int_G \psi_a^* g' da,$$

where the integral is taken with respect to the Haar measure on G. This metric is invariant:

$$\begin{aligned} g_{\psi_b(x)}(D\psi_b(x)[u], D\psi_b(x)[v]) &= \int_G (\psi_a^* g')_{\psi_b(x)}(D\psi_b(x)[u], D\psi_b(x)[v]) da \\ &= \int_G g'_{\psi_{ab}(x)}(D\psi_{ab}(x)[u], D\psi_{ab}(x)[v]) da \\ &= \int_G g'_{\psi_a(x)}(D\psi_a(x)[u], D\psi_a(x)[v]) da \\ &= g_x(u, v), \end{aligned}$$

where the second to last equality is due to translation invariance of the Haar measure. □

1.4 Hamiltonian Group Actions

As we may always choose an invariant riemannian metric, and in this case the corresponding exponential map commutes with the G-action, we have proven the following version of the coisotropic embedding theorem:

Theorem 1.13 (Equivariant Coisotropic Embedding Theorem)**.** *Let (M_j, ω_j) two symplectic manifolds of dimension $2n$, and Z a differentiable manifold of dimension k. Assume we have two embeddings $i_j : Z \hookrightarrow M_j$ and an action of a compact Lie group G on both M_j preserving ω_j. Assume the embeddings satisfy*

1. *$i_1^*\omega_1^* = i_2^*\omega_2 = \tau$ for τ a closed 2-form of constant rank on Z;*
2. *The embeddings i_j are coisotropic and G-equivariant.*

Then there exist G-invariant neighbourhoods U_j of $i_j(Z)$ in M_j and a G-equivariant symplectomorphism $f : U_1 \to U_2$ such that $f \circ i_1 = i_2$.

Taking this one step further and letting the actions be hamiltonian with G-equivariant moment maps, we can also prove

Proposition 1.14. *Suppose in addition to the above hypotheses that the action by G on M_j is hamiltonian with moment maps $\mu_j : M_j \to \mathfrak{g}^*$ satisfying $\mu_1 \circ i_1 =$*

$\mu_2 \circ i_2$. *Then f satisfies $\mu_2 \circ f = \mu_1$, that is, the following diagram commutes:*

$$\begin{array}{ccccccccc} & & i_1(Z) & \hookrightarrow & U_1 & \hookrightarrow & M_1 & & \\ & \overset{i_1}{\nearrow} & & & \Big\downarrow f & & & \overset{\mu_1}{\searrow} & \\ Z & & & & & & & & \mathfrak{g}^* \\ & \underset{i_2}{\searrow} & & & & & & \underset{\mu_2}{\nearrow} & \\ & & i_2(Z) & \hookrightarrow & U_2 & \hookrightarrow & M1 & & \end{array}$$

Proof. We claim that $\mu_2 \circ f$ is a moment map for the action on M_1. We can take f equivariant thanks to the preceding discussion, so it remains to verify that $\langle \mu_2 \circ f, X \rangle$ is a hamiltonian function for all $X \in \mathfrak{g}$. Indeed, for $p \in M_1$ and $v \in T_p M_1$,

$$\begin{aligned} d\langle \mu_2 \circ f, X \rangle|_p(v) &= \langle D(\mu_2 \circ f)(p)[v], X \rangle \\ &= \langle D\mu_2(f(p))Df(p)[v], X \rangle \\ &= d\langle \mu_2, X \rangle|_{f(p)}(Df(p)[v]) \\ &= -\omega^2_{f(p)}(\xi^2_X(f(p)), Df(p)[v]), \end{aligned}$$

where we used that μ_2 is a moment map. Next, note that

$$Df(p)[\xi^1_X(p)] = \frac{d}{dt}\bigg|_{t=0} f(\exp(tX) \cdot p) = \frac{d}{dt}\bigg|_{t=0} \exp(tX) \cdot f(p) = \xi^2_X(f(p))$$

by equivariance of f. Hence the last term above becomes

$$(-f^*\omega_2)_p(\xi^1_X(p), v) = -(\omega_1)_p(\xi^1_X(p), v),$$

using that f is a symplectomorphism. This establishes that $\mu_2 \circ f$ is a moment map, and thus $\mu_2 \circ f$ and μ_1 differ by a constant. However, as $\mu_1 \circ i_1 = \mu_2 \circ i_2$ and $f \circ i_1 = i_2$, we also have

$$\mu_2 \circ f \circ i_1 = \mu_2 \circ i_2 = \mu_1 \circ i_1,$$

whence we see that $\mu_2 \circ f = \mu_1$ on the nose on $i_1(Z)$. Hence the constant must be zero. □

1.5 The Equivariant Darboux Theorem

Using quite similar techniques as above together with the Weinstein theorem, we can prove an equivariant version of the Darboux theorem. Although we will only use it in chapter 3, we shall state and prove it here due to this similarity. This formulation of the theorem is analogous to 1.4.7 in [6], the proof an adaptation of an outline given in [2] after theorem 22.1.

Theorem 1.15 (Equivariant Darboux). *Let $(M^{2N}, \omega, \mathbb{T}^n, \mu)$ a hamiltonian torus space, and p a fixed point of the action. Then there exists a $\mathbb{T}^n$-invariant neighbourhood U of p, coordinate functions $(x_1, \dots, x_N, y_1, \dots, y_N)$ centered at p and constants $\lambda^{(1)}, \dots, \lambda^{(N)} \in \mathbb{Z}^n$ such that on U*

1. $\omega_U = \omega_0 = \sum_{j=1}^N dx_j \wedge dy_j = \frac{i}{2} \sum_{j=1}^N dz_j \wedge \overline{dz_j}$*, where* $z_j = x_j + iy_j$.
2. *The action becomes the action of $\mathbb{T}^n$ by multiplication with weights $\lambda^{(j)}$:*
$$(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot (z_1, \dots z_N) = \left(e^{i\langle \lambda^{(1)}, \theta \rangle} z_1, \dots, e^{i\langle \lambda^{(N)}, \theta \rangle} z_N \right).$$
3. *The moment map becomes*
$$\mu_U = \mu(p) + \frac{1}{2} \sum_{j=1}^{n} \lambda^{(j)} |x^2 + y^2| = \mu(p) + \frac{1}{2} \sum_{j=1}^{N} \lambda^{(j)} |z_j|^2.$$

The proof will make use of the Darboux-Weinstein theorem we just used to prove the coisotropic embedding theorem, together with the fact that riemannian metrics invariant under the action of a compact Lie group always exist, and have the property that their associated exponential maps commute with the action.

Proof. First consider the setting of a hamiltonian G-space for G compact. Choose an invariant riemannian metric and consider the associated exponential map. Then $\exp_p$ is a local diffeomorphism between neighbourhoods of $0_p \in T_pM$ and $p \in M$, and T_pM is a symplectic vector space under the form ω_p. G acts on T_pM by
$$g \cdot u := D\psi_g(p)[u]$$
because p is a fixed point. This action is linear, so it is a representation of G on T_pM, called the isotropy representation.

At the point $0_p \in T_pM$, we have
$$(\exp_p^* \omega)_{0_p}[\mathcal{J}_{0_p}(u), \mathcal{J}_{0_p}(v)] = \omega_p(D \exp_p(0_p)[\mathcal{J}_{0_p}(u)], D \exp_p(0_p)[\mathcal{J}_{0_p}(v)]) = \omega_p(u, v),$$
so that the two symplectic forms ω_p and $\exp_p^* \omega$ agree at this point. Using the Darboux-Weinstein theorem, this guarantees the existence of a diffeomorphism $g : T_pM \to T_pM$ defined on a neighbourhood of 0_p such that $g(0_p) = 0_p$ and $g^* \exp_p^* \omega = \omega_p$. Thus if we set
$$\phi = (\exp_p \circ g)^{-1},$$
we see that ϕ is a local symplectomorphism from a neighbourhood of $p \in M$ to a neighbourhood of $0_p \in T_pM$. As we took the metric to be G-invariant, the exponential map, and hence also g and ϕ, are equivariant. Let U be the domain of ϕ.

View T_pM as a complex vector space. Then by invariance of the riemannian metric g we chose, we have that for T_pM endowed with the inner product g_p, the isotropy representation is unitary.

Assuming further that G is not only compact, but also abelian, we may decompose T_pM into its weight spaces (see, for example, section 3.5.2 in [7])

$$T_pM = \bigoplus_{\chi} V_\chi,$$

where the sum is over the characters, that is, continuous group homomorphisms $\chi : G \to S^1$, and

$$V_\chi = \{v \in T_pM \mid g \cdot v = \chi(g)v\}.$$

As T_pM has complex dimension N, at most N of the χ may be nontrivial; denote these by $(\chi_1, \ldots, \chi_N)$.

Now construct a (real) symplectic basis $(u_1, \ldots, u_N, v_1, \ldots, v_N)$ of T_pM such that each u_i and v_i is contained in some V_χ, and let $(e_1, \ldots, e_N, f_1, \ldots, f_N)$ the standard symplectic basis of $\mathbb{C}^N$. The map β sending $u_i \mapsto e_i$ and $v_i \mapsto f_i$, extended linearly, is then a symplectomorphism. Concerning the action, we obtain for $v = a^i u_i + b^j v_j \in T_pM$ and $g \in G$ that

$$\begin{aligned} g \cdot v &= a^i(g \cdot u_i) + b^j(g \cdot v_j) \\ &= a^i \chi_i(g) u_i + b^j \chi_j(g) v_j. \end{aligned}$$

Thus applying β,

$$\begin{aligned} \beta(g \cdot v) &= a^i \chi_i(g) \beta(u_i) + b^j \chi_j(g) \beta(v_j) \\ &= a^i \chi_i(g) e_i + b^j \chi_j(g) f_j \\ &=: g \cdot (a^i e_i + b^j f_j) \\ &= g \cdot \beta(v). \end{aligned}$$

Hence under this definition of the action, $\mathbb{C}^N$ has the same weights in its weight space decomposition. What we have shown so far is that for a compact abelian Lie group G acting on a symplectic manifold M^{2N}, we have a local equivariant symplectomorphism

$$(M, \omega) \cong (\mathbb{C}^N, \omega_0),$$

where the action on $\mathbb{C}^N$ is as above and ω_0 is the standard symplectic form. The first statement is thus proved.

Now suppose $G = \mathbb{T}^n$. It can be shown (corollary 3.67 in [7]) that the characters of G are of the form

$$\chi(\theta) = e^{i\langle \lambda, \theta \rangle}$$

for $\theta \in \mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ and some $\lambda \in \mathbb{Z}^n$. In this case, this means we may decompose $\mathbb{C}^N$ into

$$\mathbb{C}^N = \bigoplus_{i=1}^{N} W_{\lambda^{(i)}},$$

where $\mathbb{T}^n$ acts on $W_{\lambda^{(i)}}$ by

$$\theta \cdot z = e^{i\langle \lambda^{(i)}, \theta \rangle} z.$$

Choosing an appropriate basis of $\mathbb{C}^N$, we may thus write

$$z = (z_1, \ldots, z_N) \in \bigoplus_{j=1}^{N} W_{\lambda^{(j)}}$$

for $z_j \in W_{\lambda^{(j)}}$, so that the action is given by

$$\theta \cdot z = \left(e^{i\langle \lambda^{(1)}, \theta \rangle} z_1, \ldots, e^{i\langle \lambda^{(N)}} z_N \right),$$

which proves the second claim.

The third claim now simply follows by computing the moment map for this action on $\mathbb{C}^N$. Indeed, the fundamental vector field associated to $v \in \mathfrak{t}^n \cong \mathbb{R}^n$

$$\begin{aligned}
\xi_v(z) &= \left.\frac{d}{dt}\right|_{t=0} \exp(tv) \cdot z \\
&= \left.\frac{d}{dt}\right|_{t=0} (e^{itv_1}, \ldots, e^{itv_n}) \cdot (z_1, \ldots, z_N) \\
&= \left.\frac{d}{dt}\right|_{t=0} (e^{i\langle \lambda^{(1)}, tv \rangle} z_1, \ldots, e^{i\langle \lambda^{(N)}, tv \rangle} z_N) \\
&= \left.\frac{d}{dt}\right|_{t=0} (r_1 e^{i(\langle \lambda^{(1)}, tv \rangle + \phi_1)}, \ldots, r_N e^{i(\langle \lambda^{(N)}, tv \rangle + \phi_N)}) \\
&= \sum_j \left.\frac{d}{dt}\right|_{t=0} (\langle \lambda^{(j)}, tv \rangle + \phi_j) \partial_{\phi_j} \\
&= \sum_j \langle \lambda^{(j)}, v \rangle \partial_{\phi_j}.
\end{aligned}$$

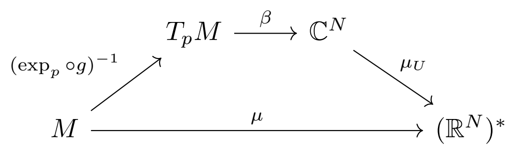
Hence $\xi_{e_k} = \sum_l \langle \lambda^{(l)}, e_k \rangle \partial_{\phi_l}$. Thus if $Y \in \mathfrak{X}(\mathbb{C}^N)$ is arbitrary,

$$\begin{aligned}
\omega_0(\xi_{e_k}, Y) &= \sum_j r_j dr_j \wedge d\phi_j(\xi_{e_k}, Y) \\
&= -\sum_j r_j dr_j(Y) d\phi_j \left(\sum_l \langle \lambda^{(l)}, e_k \rangle \partial_{\phi_l} \right) \\
&= -\sum_l \langle \lambda^{(l)}, e_k \rangle r_l dr_l(Y) \\
&= -\left(\sum_l \langle \lambda^{(l)}, e_k \rangle \right) \frac{1}{2} dr_l^2(Y).
\end{aligned}$$

Thus the k-th component of the moment map is $\frac{1}{2} \left(\sum_l \langle \lambda^{(l)}, e_k \rangle \right) r_l^2$, which means that, up to a constant,

$$\mu_U = \frac{1}{2} \sum_k \lambda^{(k)} r_k^2.$$

Hence our moment map agrees with the claimed one up to the constant $\mu(p)$. But since following diagram must commute and the composition of all the upper maps takes the value 0 at p, we see that the constant must be $\mu(p)$, finishing the proof.



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Chapter 2

The Duistermaat-Heckman Theorem

2.1 Reduction near the Zero Level

In Marsden-Meyer-Weinstein's reduction theorem, one considers a hamiltonian G-space (M, ω, G, μ) for G compact and the restriction of the action to $Z := \mu^{-1}(0)$ being free. Let us further assume that μ be proper, so that $\mu^{-1}(t)$ is compact for all $t \in \mathfrak{g}^*$. But if G acts freely on Z, it also acts freely on $\mu^{-1}(t)$ for $t \in \mathfrak{g}^*$ in a neighbourhood of zero, so that we may consider the reduced spaces

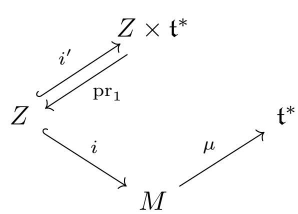
$$M_0 = \mu^{-1}(0)/G, \qquad M_t = \mu^{-1}(t)/G,$$

with respective symplectic forms ω_0 and ω_t. We will use the coisotropic embedding theorem to compare the symplectic structure of these spaces for $G = \mathbb{T}^n$.

Let $\dim M = 2N$. We do this by considering the mainfold $M' = Z \times \mathfrak{t}^*$, endowed with the action induced by $\mathbb{T}^n$ by

$$\theta \cdot (z, v) = (\theta \cdot z, v).$$

Introducing the projection and inclusion map $i' : z \mapsto (z, 0)$, we are in the following situation:



We would like to expand this diagram to obtain one analogous to the one from proposition 1.14. Hence, we need to endow $Z \times \mathfrak{t}^*$ with a symplectic form and a hamiltonian $\mathbb{T}^n$-action such that the conditions of the equivariant coisotropic embedding theorem are satisfied, to obtain a local symplectomorphism between neighbourhoods of $i'(Z)$ and $i(Z)$. We proceed along the lines of chapter 30 of [1], or section 2.1 of [8].

Recall first the principal bundle obtained by reduction

$$\begin{array}{ccc} Z & \overset{i}{\hookrightarrow} & M \\ \downarrow{\scriptstyle \pi} & & \\ M_0 & & \end{array}$$

and the definition of a connection form on principal bundles:

Definition 2.1. *Let $\pi : P \to M$ a principal G-bundle. Then $\alpha \in \Omega^1(P, \mathfrak{g})$ is called a* ***connection form*** *if*

- *$\alpha(\xi_v) = v$ for ξ_v the fundamental vector field and all $v \in \mathfrak{g}$;*
- *α is equivariant with respect to the adjoint action, that is,*

$$\alpha(g \cdot p) = \mathrm{Ad}_g(\alpha(p)).$$

Now choose a connection form $\alpha \in \Omega^1(Z, \mathfrak{t})$ with regard to this bundle for $\mathfrak{t}$ the Lie algebra of $\mathbb{T}^n$. $\mathbb{T}^n$ being abelian means the adjoint action is trivial, so α has to be $\mathbb{T}^n$-invariant and satisfy $\alpha(\xi_v) = v$ for all $v \in \mathfrak{g}$.

Let $(t_i)_i$ an orthonormal basis of $\mathfrak{t}$ and $(t^i)_i$ the corresponding dual basis. This induces a function

$$t_i : Z \times \mathfrak{t}^* \longrightarrow \mathbb{R}, \qquad (z, p) \mapsto p(t_i).$$

With regard to this basis, we may also write the connection form as $\alpha = \sum_{i=1}^n \alpha_i t_i$ for some $\alpha_i \in \Omega^1(Z)$, which satisfy $\alpha_i(\xi_v) = t^i(v)$. Let us use this to define a symplectic form on $Z \times \mathfrak{t}^*$ by

$$\omega' := \mathrm{pr}_1^* \pi^* \omega_0 + \sum_{i=1}^n d(t_i \mathrm{pr}_1^*(\alpha_i)).$$

Note that the t_i may be regarded as global coordinates on $\mathfrak{t}^*$, so that ∂_{t_i} is a well-defined vector field on $\mathfrak{t}^*$.

For $t \in \mathfrak{t}^*$, the last term in the definition of ω' can be rewritten:

$$\begin{aligned} \langle t, \mathrm{pr}_1^*(\alpha) \rangle &= \langle t_i(t) t^i, \mathrm{pr}_1^*(\alpha_j t_j) \rangle \\ &= t_i(t) \mathrm{pr}_1^*(\alpha_j) \langle t^i, t_j \rangle \\ &= t_i(t) \mathrm{pr}_1^*(\alpha_i), \end{aligned}$$

where we used the Einstein summation convention. Hence sometimes, we will also supress the pullback by the first projection from notation and write ω' as

$$\omega' = \pi^*(\omega_0) + d\langle t, \alpha \rangle.$$

We claim this is the symplectic form we are looking for.

2.1.1 Conditions on the Form

This section is dedicated to proving that ω' satisfies the conditions of the coisotropic embedding theorem. We will prove ω' is locally symplectic and $\mathbb{T}^n$-invariant, that i and i' are coisotropic embeddings, and that $i^*\omega = (i')^*\omega'$.

Proposition 2.2. *ω' is symplectic in a neighbourhood of $Z \times \{0\}$.*

Proof. As the exterior differential commutes with pullbacks, ω' is clearly closed. As for nondegeneracy, note that on $Z \times \{0\}$, we have

$$\omega' = \mathrm{pr}_1^* \pi^* \omega_0 + \sum_{i=1}^n dt_i \wedge \mathrm{pr}_1^*(\alpha_i).$$

Then we have for the second term in the definition of ω' evaluated at the vector field $((0, \partial_{t_i}), (\xi_{t_i}, 0))$:

$$\begin{aligned}\sum_{k=1}^n dt_k \wedge \mathrm{pr}_1^*(\alpha_k)((0, \partial_{t_i}), (\xi_{t_i}, 0)) &= \sum_{k=1}^n \underbrace{dt_k(0, \partial_{t_i})}_{=\delta_{ki}} \underbrace{\mathrm{pr}_1^*(\alpha_k)(\xi_{t_i}, 0)}_{=\alpha_k(\xi_{t_i})} \\ &\quad - \underbrace{dt_k(\xi_{t_i}, 0)}_{=0} \mathrm{pr}_1^*(\alpha_k)(0, \partial_{t_i}) \\ &= \alpha_i(\xi_{t_i}) \\ &= t^i(t_i) \\ &= 1.\end{aligned}$$

As $\mathrm{pr}_1^* \pi^* \omega_0((0, \partial_{t_i}), (\xi_{t_i}, 0)) = \pi^* \omega_0(0, \xi_{t_i}) = 0$, we obtain nondegeneracy of ω' on $Z \times \{0\}$. Hence as nondegeneracy is an open condition, ω' is nondegenerate in a neighbourhood of $Z \times \{0\}$. □

Proposition 2.3. *ω' is $\mathbb{T}^n$-invariant.*

Proof. Take $(z, p) \in Z \times \mathfrak{t}^*$, $\theta \in \mathbb{T}^n$ and $(u, v), (u', v') \in T_{(z,p)}(Z \times \mathfrak{t}^*)$. Denote the action by $\psi_\theta : Z \times \mathfrak{t}^* \to Z \times \mathfrak{t}^*$ as defined above, that is, ψ_θ is the identity on the $\mathfrak{t}^*$-factor, and compute

$$\begin{aligned}&(\mathrm{pr}_1^* \pi^* \omega_0)_{\psi_\theta(z,p)}(D\psi_\theta(z, p)[(u, v)], D\psi_\theta(z, p)[(u', v')]) \\ &= (\mathrm{pr}_1^* \pi^* \omega_0)_{(\psi_\theta(z),p)}((D\psi_\theta(z)[u], v), (D\psi_\theta(z)[u'], v')) \\ &= (\omega_0)_{[z]}(D(\pi \circ \psi_\theta)(z)[u], D(\pi \circ \psi_\theta)(z)[u']) \\ &= (\omega_0)_{[z]}(D\pi(z)[u], D\pi(z)[u']) \\ &= (\mathrm{pr}_1^* \pi^* \omega_0)_{(z,p)}((u, v), (u', v')).\end{aligned}$$

Similarly for the other term: expanding $d(t_i \mathrm{pr}_1^*(\alpha_i))$, it is sufficient to check that dt_i, $\mathrm{pr}_1^*(\alpha_i)$, and $\mathrm{pr}_1^*(d\alpha_i)$ are $\mathbb{T}^n$-invariant. The latter two are evident since the α_i are $\mathbb{T}^n$-invariant by invariance of α, and we have

$$\begin{aligned}(dt_i)_{(\psi_\theta(z),p)}(D\psi_\theta(z)[u], v) &= v(t_i)\\ &= (dt_i)_{(z,p)}(u, v)\end{aligned}$$

under the identification of $T_p\mathfrak{t}^* \cong \mathfrak{t}^*$. □

Verifying that the embedding $Z = \mu^{-1}(0) \hookrightarrow M$ is coisotropic will require a preliminary lemma, as discussed in [6], or in section 23.2 of [1]:

Lemma 2.4. *Let* (M, ω, G, μ) *a hamiltonian* G*-space, and* $p \in M$*. Denote by* $\mathcal{O} \subset M$ *the orbit of* p*. Identifying the tangent space to* $\mathfrak{g}^*$ *with* $\mathfrak{g}^*$*, consider the differential of the moment map as*

$$D\mu(p) : T_pM \to \mathfrak{g}^*.$$

Then we have $\ker D\mu(p) = T_p\mathcal{O}^{\omega_p}$*.*

Proof. Recall that μ is a moment map and hence satisfies $d\langle\mu, X\rangle = -\imath_{\xi_X}(\omega)$ for $X \in \mathfrak{g}$. Recall that for functions, we have $df_x = Df(x)$ under the identification of $T_{f(x)}\mathbb{R} \cong \mathbb{R}$, so that if $v \in T_pM$ and $p \in M$, we have

$$\begin{aligned}d\langle\mu, X\rangle_p(v) &= D\langle\mu, X\rangle(p)[v]\\ &= \langle D\mu(p)[v], X\rangle\\ &= -\omega_p(\xi_X(p), v).\end{aligned}$$

Note that the vector fields $\xi_X(p)$ at p span $T_p\mathcal{O}$ for $X \in \mathfrak{g}$. Thus by nondegeneracy of ω_p, it follows that

$$\begin{aligned}D\mu(p)[v] = 0 &\iff \langle D\mu(p)[v], X\rangle = 0 \quad \forall X \in \mathfrak{g}\\ &\iff -\omega_p(\xi_X(p), v) = 0 \quad \forall X \in \mathfrak{g}\\ &\iff v \in T_p\mathcal{O}^{\omega_p}.\end{aligned}$$

□

Corollary 2.5. $i : Z = \mu^{-1}(0) \hookrightarrow M$ *is a coisotropic embedding.*

Proof. By the implicit function theorem, we have $Di(z)[T_zZ] = T_zZ = \ker D\mu(z)$, which is equal to $T_z\mathcal{O}^{\omega_z}$ for $\mathcal{O}$ the orbit through z by the previous lemma. Hence note that $T_zZ^{\omega_z} = \ker D\mu(z)^{\omega_z} = (T_z\mathcal{O}^{\omega_z})^{\omega_z} = T_z\mathcal{O}$. μ being invariant with respect to the action means it is constant on orbits, so that $\mathcal{O}$ is contained in Z, and hence also

$$T_zZ^{\omega_z} = T_z\mathcal{O} \subset T_zZ.$$

□

Proposition 2.6. $i'^*\omega' = i^*\omega$ *in* $\Omega^2(Z)$*, and the inclusion* i' *is also coisotropic.*

Proof. A simple computation shows

$$\begin{aligned}
i'^*\omega' &= i'^*\mathrm{pr}_1^*\pi^*\omega_0 + \sum_{i=1}^n d(i^*(t_i\mathrm{pr}_1(\alpha_i))) \\
&= (\pi\circ \mathrm{pr}_1 \circ i')^*\omega_0 + \sum_{i=1}^n d((t_i\circ i')\alpha_i) \\
&= \pi^*\omega_0 + 0 \\
&= i^*\omega.
\end{aligned}$$

We used that $i'\circ \mathrm{pr}_1 = \mathrm{id}_Z$, $t_i \circ i' = 0$ and the identity $\pi^*\omega_0 = i^*\omega$ from symplectic reduction.

Let us now verify that i' is coisotropic. Note that $i'(Z) = Z\times\{0\}$, and that we have for any $z\in Z$ that $Di'(z)[Z] = T_zZ\times\{0\}$. We must show that this subspace is coisotropic, which amounts to showing that if $(u,v)\in T_{(z,0)}(Z\times\{0\})^{\omega'}$, then $u\in T_zZ$ and $v=0$. For all $(u',0)\in T_{(z,0)}(Z\times\{0\})$, we have

$$\begin{aligned}
0 &= \omega'_{(z,0)}\left((u,v),(u',0)\right) \\
&= (\mathrm{pr}_1^*\pi^*\omega_0)_{(z,0)}((u,v),(u',0)) + \sum_{k=1}^n dt_k((u,v))\mathrm{pr}_1^*(\alpha_k)((u',0)) \\
&\qquad\qquad\qquad\qquad - dt_k((u',0))\mathrm{pr}_1^*(\alpha_k)_{(z,0)}((u,v)) \\
&= (\pi^*\omega_0)_z(u,u') + \sum_{k=1}^n v(t_k)(\alpha_k)_z(u') - dt_k(0)(\alpha_k)_z(u) \\
&\implies (i^*\omega)_z(u,u') = -\sum_{k=1}^n v(t_k)(\alpha_k)_z(u').
\end{aligned}$$

Hence as the left hand side is independent of v, equality implies that neither does the right side; since the t_i form a basis of $\mathfrak{t}$, this implies $v=0$ and thus

$$\omega_{i(z)}(u,u') = 0 \quad \forall u'\in T_{i(z)}Z,$$

where we view $T_{i(z)}Z$ as a subspace of $T_{i(z)}M$. This means, of course, that $u\in T_{i(z)}Z^{\omega}$, which is a subspace of $T_{i(z)}Z$ by the last corollary, finishing the proof. ∎

Proposition 2.7. *The action of $\mathbb{T}^n$ on (M',ω') defined above is hamiltonian with moment map*

$$\mathrm{pr}_2 : Z\times\mathfrak{t}^*\to\mathfrak{t}^*, \qquad (z,p)\mapsto p.$$

Proof. The fundamental vector field associated to this action is just

$$\zeta_v(z,p) = \left.\frac{d}{dt}\right|_{t=0}\exp(tv)\cdot(z,p) = \left.\frac{d}{dt}\right|_{t=0}(\exp(tv)\cdot z,p) = (\xi_v(z),p),$$

and thus its flow is

$$\phi_t : Z \times \mathfrak{t}^* \to Z \times \mathfrak{t}^*, \qquad (z,p) \mapsto (\exp(tv)\cdot z, p) = (\varphi_t(z), p)$$

for φ the flow of ξ_v. Hence we compute

$$\begin{aligned}\imath_{\zeta_{t_j}}\omega' &= \mathrm{pr}_1^*\pi^*\omega_0(\zeta_{t_j},\cdot) + \sum_{i=1}^n \imath_{\zeta_{t_j}}(d(t_i\mathrm{pr}_1^*(\alpha_i)) \\ &= \omega_0(D\pi[\xi_{t_j}],\cdot) + \sum_{i=1}^n \mathcal{L}_{\zeta_{t_j}}(t_i\mathrm{pr}_1^*(\alpha_i)) - d\imath_{\zeta_{t_j}}(t_i\mathrm{pr}_1^*(\alpha_i)).\end{aligned}$$

The first term vanishes since $\pi(\exp(tt_j)\cdot z)$ is constant. For $(z,p)\in Z\times\mathfrak{t}^*$ and (u,v) a tangent vector, we have

$$\begin{aligned}\mathcal{L}_{\zeta_{t_j}}(t_i\mathrm{pr}_1^*(\alpha_i))_{(z,p)}(u,v) &= \left.\frac{d}{dt}\right|_{t=0} \phi_t^* t_i\mathrm{pr}_1^*(\alpha_i)_{(z,p)}(u,v) \\ &= \left.\frac{d}{dt}\right|_{t=0} (t_i\circ\phi_t)(z,p)(\alpha_i)_{\exp(tt_j)\cdot z}(D\varphi_t(z)[u]) \\ &= \left.\frac{d}{dt}\right|_{t=0} p(t_i)(\alpha_i)_z(u) \\ &= 0\end{aligned}$$

by $\mathbb{T}^n$-invariance of the α_i. For the other term, we have

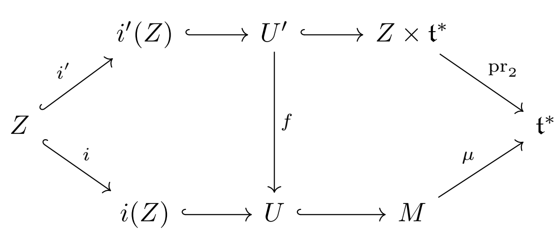
$$\begin{aligned}\imath_{\zeta_{t_j}}(t_i\mathrm{pr}_1^*(\alpha_i))_{(z,p)} &= t_i(z,p)(\alpha_i)_z(\xi_{t_j}(z)) \\ &= p(t_i)t^i(t_j).\end{aligned}$$

Hence summing over i, we obtain

$$\imath_{\zeta_{t_j}}\omega' = 0 - d\,(p(t_j))\,.$$

□

Hence the coisotropic embedding theorem guarantees the existence of neighbourhoods $i(Z)\subset U\subset M$ and $i'(Z)\subset U'\subset M'$, along with an equivariant symplectomorphism $f : U'\to U$, such that the following diagram commutes:



2.1.2 Comparison of the Reduced Spaces

In order to compare the reduced spaces M_0 and M_t, this allows us to work in our newly constructed manifold M' with moment map given by pr_2, and thus instead consider the reduced spaces

$$\mathrm{pr}_2^{-1}(t)/\mathbb{T}^n = Z \times \{t\}/\mathbb{T}^n$$

which are hence symplectomorphic to M_t for $t \in \mathfrak{t}^*$ in a neighbourhood of zero.

Proposition 2.8. *The reduced space* (M_t, ω_t) *is symplectomorphic to*

$$(M_0, \omega_0 + \langle t, \beta \rangle),$$

where β *is the unique 2-form on* M_0 *such that* $\pi^*\beta = A$ *for* $A = d\alpha$ *the curvature form of the connection* α *on* Z*.*

Proof. As explained above, the space (M_t, ω_t) may be identified with $(Z \times \{t\}/\mathbb{T}^n, \omega_t')$, where ω_t' is the form obtained by symplectic reduction, which means it satisfies $i_t^*\omega' = \pi^*\omega_t'$ for $i_t : Z \times \{t\} \hookrightarrow Z \times \mathfrak{t}^*$ the inclusion.

Restricting ω' to $Z \times \{t\}$ is

$$\begin{aligned} i_t^*\omega' &= \pi^*\omega_0 + \sum_{i=1}^n d((t_i \circ i_t)\alpha_i) \\ &= \pi^*\omega_0 + \sum_{i=1}^n t(t_i) d\alpha_i \\ &= \pi^*\omega_0 + \langle t, d\alpha \rangle \\ &= \pi^*\omega_0 + \langle t, \pi^*\beta \rangle \\ &= \pi^*(\omega_0 + \langle t, \beta \rangle). \end{aligned}$$

This proves the claim by uniqueness of the reduced form. □

By homotopy invariance of the de Rham cohomology classes, this implies our first main result, which is theorem 2.7 in [8]:

Theorem 2.9 (Duistermaat-Heckman)**.** *The cohomology class of the reduced symplectic form* $[\omega_t]$ *varies in* t *according to*

$$[\omega_t] = [\omega_0] + \langle t, [\beta] \rangle.$$

Note that by the Chern-Weil isomorphism, $[\beta]$ *is in fact independent of the choice of connection* α *on* Z*.*

2.2 The Duistermaat-Heckman Measure

The following definitions are taken from 30.1 in [1]. For any symplectic manifold (M^{2N}, ω), we have that $\frac{\omega^N}{N!}$ is a volume form. This gives rise to a measure on M:

Definition 2.10. *For $U \subset M$ a Borel set, that is, a set in the σ-ring generated by compact subsets of M, define the* ***Liouville*** *or* ***symplectic measure*** *m_ω on M by*

$$m_\omega(U) := \int_U \frac{\omega^N}{N!}.$$

If (M, ω, G, μ) is a hamiltonian G-space with μ a *proper* moment map, we may push this measure forward by μ:

Definition 2.11. *The* ***Duistermaat-Heckman measure*** *m_{DH} on $\mathfrak{g}^*$ is defined to be the pushforward measure of m_ω by μ, that is, for U a Borel set of $\mathfrak{g}^*$, define*

$$m_{DH}(U) := \int_{\mu^{-1}(U)} \frac{\omega^N}{N!}.$$

As $\mathfrak{g}^*$ is a vector space, we may identify it with $\mathbb{R}^n$, where we also have the Lebesgue measure λ. We will see that m_{DH} is absolutely continuous with respect to λ, and investigate the corresponding Radon-Nikodym derivative, that is, a measurable function f on $\mathfrak{g}^*$ such that

$$m_{DH}(U) = \int_U f \, d\lambda.$$

Note that in the case of $G = \mathbb{T}^n$, the Lebesgue measure on $\mathfrak{t}$ comes from $dt_1 \wedge \ldots \wedge dt_n$, for the t_i an orthonormal basis of $\mathfrak{t}$.

2.2.1 Integration on Fibres

In the computations for the Duistermaat-Heckman measure, we will need to integrate on the principal fibre bundle $\pi : Z \to M_0$. Here we introduce the notion of how to do this, following section 3.4.5 of [9].

As in the case of a single manifold, we require the bundle to be *orientable*, which for fibre bundles is defined as follows:

Definition 2.12. *Let $\pi : E \to M$ a fibre bundle with fibre F. The bundle is said to be* ***orientable*** *if*

1. *The fibre F is orientable;*
2. *There exists an open cover (U_α) of M and trivialisations $\Psi_\alpha : \pi^{-1}(U_\alpha) \to F \times U_\alpha$ such that the transition maps*

$$\rho_{\beta\alpha} = \Psi_\beta \circ \Psi_\alpha^{-1} : F \times (U_\alpha \cap U_\beta) \to F \times (U_\alpha \cap U_\beta)$$

are fibrewise orientation preserving, that is, for each $y \in U_\alpha \cap U_\beta$, the diffeomorphism

$$F \to F, \qquad f \mapsto \rho_{\beta\alpha}(f, y)$$

is orientation preserving.

Let $\pi : E \to M$ an orientable fibre bundle with fibre F.

Proposition 2.13. *There exists a linear operator*

$$\pi_* : \Omega_c(E) \to \Omega_c(M)$$

whose degree is $-r$ for $r = \dim F$, locally defined as follows on coordinates over a local trivialization of E of the form (x, y) for $x = (x^i)$ coordinates on F and $y = (y^j)$ coordinates on M: If $\omega = f dx^I \wedge dy^J \in \Omega_c^k(E) \cong \Omega_c^k(F \times M)$ locally for some $f \in C_c^\infty(F \times M)$ and $k = |I| + |J|$, set

$$\pi_*(\omega) = \left(\int_F f(x, \cdot) dx^I \right) dy^J \in \Omega^{k-r}(M),$$

where the notation is meant to indicate that we integrate over x. Note that whenever $|I| \neq r$, we have $\pi_*(\omega) = 0$. Whenever $k - r < 0$, we also define $\pi_*(\omega)$ to be zero.

We call π_* the ***integration-along-fibres*** *operator.*

The main property of this operator we will use is the so-called projection formula:

Proposition 2.14. *Let $\pi : E \to M$ an oriented fibre bundle with fibre F, $\omega \in \Omega_c(E)$ and $\eta \in \Omega_c(M)$. Then we have*

$$\int_E \omega \wedge \pi^*(\eta) = \int_M \pi_* \omega \wedge \eta.$$

Proof. Consider local coordinates over a trivialisation (U_α) of the bundle, along with the projection maps $\mathrm{pr}_1 : F \times M \to F$ and $\mathrm{pr}_2 : F \times M \to M$. Then we can view pullbacks of forms on M by π as forms on $F \times U_\alpha$ and write

$$\begin{aligned} \omega &= f dx^I \wedge dy^J, \quad f \in C_c^\infty(F \times M), \\ \eta &= g dy^K, \quad g \in C_c^\infty(M) \\ \pi^*(\eta) &= \mathrm{pr}_2^*(\eta) = g(\mathrm{pr}_2) d(y \circ \mathrm{pr}_2)^K, \\ \pi_*(\omega) &= \left(\int_F f(x, \cdot) dx^I \right) dy^J. \end{aligned}$$

Then

$$\begin{aligned} \int_{U_\alpha} \pi_* \omega \wedge \eta &= \int_{U_\alpha} \left(\int_F f(x, \cdot) dx^I \right) dy^J \wedge g dy^K \\ &= \int_{U_\alpha \times F} f d(x \circ \mathrm{pr}_1)^I \wedge d(y \circ \mathrm{pr}_2)^J \wedge g(\mathrm{pr}_2) d(y \circ \mathrm{pr}_2)^K \\ &= \int_{U_\alpha \times F} \omega \wedge \pi^*(\eta). \end{aligned}$$

Hence as U_α is an open cover, the result follows. □

2.2.2 Computing the Measure

To start, we consider the symplectic volume of the reduced spaces (M_t, ω_t)

Proposition 2.15. $\mathrm{vol}(M_t)$ *is a polynomial in* $t \in \mathfrak{t}^* \cong \mathbb{R}^n$ *of degree at most* $N - n$.

Proof. By the symplectomorphism of proposition 2.8

$$\mathrm{vol}(M_t) = \int_{M_t} \frac{\omega_t^{N-n}}{(N-n)!} = \int_{M_0} \frac{(\omega_0 + \langle t, \beta \rangle)^{N-n}}{(N-n)!}.$$

Let us write $\beta = \beta^i t_i$ for the t_i the basis of $\mathfrak{t}$ and $\beta^i \in \Omega^2(M_0)$, and $t = a_j t^j$ for $a_j = t(t_j)$. Then we can write

$$\begin{aligned}\langle t, \beta \rangle &= a_j \beta^i \langle t^j, t_i \rangle \\ &= a_i \beta^i.\end{aligned}$$

We expand using the multinomial theorem:

$$\mathrm{vol}(M_t) = \frac{1}{(N-n)!} \int_{M_0} \sum_{k=0}^{N-n} \binom{N-n}{k} \omega_0^{N-n-k} \sum_{|\alpha|=k} \binom{k}{\alpha} a^\alpha \beta_1^{\alpha_1} \cdots \beta_n^{\alpha_n},$$

or equivalently,

$$\mathrm{vol}(M_t) = \sum_{|\alpha| \le N-n} \frac{a^\alpha}{(N-n-|\alpha|)! a_1! \cdots a_n!} \int_{M_0} \beta_1^{\alpha_1} \cdots \beta_n^{\alpha_n} \omega_0^{N-n-|\alpha|},$$

so under the identification of $t \in \mathfrak{t}^*$ with the vector $a \in \mathbb{R}^n$, the symplectic volume of the reduced space M_t for t in a neighbourhood of zero is indeed a polynomial of degree at most $N - n$. □

Now let $U \subset \mathfrak{g}^*$ a Borel subset contained in the neighbourhood of 0 which is such that Z times this neighbourhood is symplectomorphic to a neighbourhood of $i(Z) \subset M$. Then $(\mu^{-1}(U), \omega)$ is symplectomorphic to $(Z \times U, \omega')$, and thus

$$m_{DH}(U) = \int_{\mu^{-1}(U)} \frac{\omega^N}{N!} = \int_{Z \times U} \frac{(\omega')^N}{N!}.$$

Next, we evaluate $(\omega')^N = (\pi^*(\omega_0) + d\langle t, \alpha \rangle)^N$. First, we expand the differential, so that we get

$$(\pi^*(\omega_0) + \langle dt, \alpha \rangle + \langle t, d\alpha \rangle)^N.$$

As the above summands are all two-forms, their wedge product is symmetric (that is, $\eta_1 \wedge \eta_2 = \eta_2 \wedge \eta_1$ for any two-forms η_1, η_2), so that we can use the binomial theorem to obtain

$$\sum_{k=0}^{N} \binom{N}{k} (\pi^*(\omega_0 + \langle t, \beta \rangle))^{N-k} \langle dt, \alpha \rangle^k.$$

Here we used that $d\alpha = \pi^*\beta$. Since $\omega_0 + \langle t, \beta\rangle$ is a two-form on M_0, which has dimension $2(N-n)$, its powers (and hence powers of $\pi^*(\omega_0 + \langle t, \beta\rangle)$) vanish whenever $N - k > N - n \iff k < n$. On the other hand, by antisymmetry of the wedge product on one-forms, we have that $\alpha_i \wedge \alpha_i = 0$ for all i, so that only factors with distinct α_i survive in the expansion of $\langle dt, \alpha\rangle^k$. As i ranges in $1, \ldots, n$, this means that $\langle dt, \alpha\rangle^k = 0$ whenever $k > n$. Hence only the $k = n$-term survives, which is

$$\frac{N!}{n!(N-n)!}(\pi^*(\omega_0 + \langle t, \beta\rangle))^{N-n}(\langle dt, \alpha\rangle)^n.$$

Expanding the last term, we have again due to anti-symmetry that only terms of the following form survive:

$$dt_{\sigma(1)} \wedge \alpha_{\sigma(1)} \wedge \ldots \wedge dt_{\sigma(n)} \wedge \alpha_{\sigma(j)},$$

for any $\sigma \in S_n$. Using symmetry of the wedge product on two-forms, we may rearrange the sum over S_n to give

$$n! dt_1 \wedge \alpha_1 \wedge \ldots \wedge dt_n \wedge \alpha_n,$$

which, up to sign, is equal to

$$n! \alpha_1 \wedge \ldots \wedge \alpha_n \wedge dt_1 \wedge dt_n.$$

Hence the form we integrate over $Z \times U$, remembering the factor $\frac{1}{N!}$, takes the form

$$\frac{1}{(N-n)!}(\pi^*(\omega_0 + \langle t, \beta\rangle))^{N-n}\alpha_1 \wedge \ldots \wedge \alpha_n \wedge dt_1 \wedge \ldots \wedge dt_n,$$

so that by Fubini,

$$m_{DH}(U) = \int_U \left[\int_Z \frac{(\pi^*(\omega_0 + \langle t, \beta\rangle))^{N-n}}{(N-n)!} \wedge \alpha_1 \wedge \ldots \wedge \alpha_n\right] dt_1 \wedge \ldots \wedge dt_n.$$

Using the projection formula from proposition 2.14, the integral over Z is equal to

$$\int_{M_0} \frac{(\omega_0 + \langle t, \beta\rangle)^{N-n}}{(N-n)!}\pi_*(\alpha_1 \wedge \ldots \wedge \alpha_n), \qquad (**)$$

where $\pi_*(\alpha_1 \wedge \ldots \wedge \alpha_n) = 1$ because we may choose a chart such that the coordinate vector fields are the ξ_{t_i}. Using that $\alpha_1 \wedge \ldots \wedge \alpha_n(\xi_{t_1}, \ldots, \xi_{t_n}) = 1$, the integration-along-fibres operator becomes

$$\pi_*(\alpha_1 \wedge \ldots \wedge \alpha_n) = \int \alpha_1 \wedge \ldots \wedge \alpha_n(\xi_{t_1}, \ldots, \xi_{t_n}) = 1.$$

With this, $(**)$ becomes precisely the expression we deduced for $\mathrm{vol}(M_t)$ in proposition 2.15, so that

$$m_{DH}(U) = \int_U \mathrm{vol}(M_t) dt.$$

The following theorem summarizes what we have proven in this discussion.

Theorem 2.16. *Let $(M, \omega, \mathbb{T}^n, \mu)$ a hamiltonian $\mathbb{T}^n$-space. Then*

1. *m_{DH} is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{t}^*$.*
2. *The Radon-Nikodym derivative $f(t) = \mathrm{vol}(M_t)$ is a piecewise polynomial of degree at most $N - n$.*

Proof. Both claims are immediate from the above arguments if $t \approx 0$, in which case the Radon-Nikodym derivative $f(t) = \mathrm{vol}(M_t)$ is in fact a polynomial by proposition 2.15. But the same holds for a neighbourhood of any other regular value of μ, since we may change the moment map by a constant. In this neighbourhood, it may occur that f is a different polynomial, which is why f is only piecewise polynomial. □

2.2.3 Application to Delzant Spaces

This section is an application of theorem 2.16; we show that the symplectic volume of a Delzant space is equal to the Lebesgue measure of its corresponding Delzant polytope. In the following, let $\Delta \subset (\mathbb{R}^n)^*$ a Delzant polytope and $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu)$ the corresponding symplectic toric manifold. We briefly recall Delzant's construction and refer to [10] or [1], chapter XI, for more details.

Recall that M_Δ is constructed in the following manner: If Δ has d facets, start by writing

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \geq \lambda_i,\ i = 1, \ldots, d\}$$

for some constants λ_i and $v_i \in \mathbb{R}^n$ the primitive inward pointing normal vectors. The map

$$\tilde{\pi} : \mathbb{R}^d \to \mathbb{R}^n, \qquad e_i \mapsto v_i$$

induces a map $\pi : \mathbb{T}^d \to \mathbb{T}^n$ with kernel N. Let $i : N \hookrightarrow \mathbb{T}^d$ the inclusion.

Define the map

$$J : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^*, \qquad (z_1, \ldots, z_d) \longmapsto \frac{1}{2}(|z_1|^2, \ldots, |z_d|^2),$$

and consider the standard action of $\mathbb{T}^d$ on $\mathbb{C}^d$ by multiplication with the moment map

$$\mu = J + (\lambda_1, \ldots, \lambda_d).$$

Restricting the action to N, the moment map becomes $i^* \circ \mu$.

Set $Z = (i^* \circ \mu)^{-1}(0)$ and let $j : Z \hookrightarrow \mathbb{C}^d$ the inclusion. Set $M_\Delta = Z/N$.

Letting $\tilde{\pi}^* : (\mathbb{R}^n)^* \to (\mathbb{R}^d)^*$ the induced dual map, one can show

$$\tilde{\pi}^*(\Delta) = \mu(Z).$$

Hence for a vertex τ of Δ, we can find $z \in Z$ such that $\mu(z) = \tilde{\pi}^*(\tau)$, and show that the restriction of $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$ is a bijection. From this we obtain a right inverse σ_τ for $\pi : \mathbb{T}^d \to \mathbb{T}^n$ and thereby an isomorphism

$$\mathbb{T}^d = N \times \mathbb{T}^n.$$

Letting the $\mathbb{T}^n$-factor act on M_Δ by

$$\theta \cdot p(z) = p(\sigma_\tau(\theta) \cdot z)$$

is hamiltonian with a moment map μ_Δ such that the following diagram commutes:

$$\begin{array}{ccccccc} Z & \xhookrightarrow{j} & \mathbb{C}^d & \xrightarrow{\mu} & (\mathbb{R}^d)^* & \xrightarrow{\sim} & \mathfrak{n}^* \oplus (\mathbb{R}^n)^* \\ \Big\downarrow{\scriptstyle p} & & & & & & \Big\downarrow{\scriptstyle \mathrm{pr}_2} \\ M_\Delta & & & \xrightarrow{\qquad\quad \mu_\Delta \quad\qquad} & & & (\mathbb{R}^n)^* \end{array}$$

That is,

$$\mu_\Delta \circ p = \mathrm{pr}_2 \circ \mu \circ j.$$

The action is independent of the choice of vertex τ: Suppose ν is another vertex of Δ and σ_ν the corresponding right inverse. Then we have

$$\begin{aligned} \pi(\sigma_\tau(\theta)\sigma_\nu(\theta)^{-1}) &= \pi(\sigma_\tau(\theta))\pi(\sigma_\nu(\theta^{-1})) \\ &= \theta\theta^{-1}, \end{aligned}$$

so that $\sigma_\tau(\theta)\sigma_\nu(\theta)^{-1} \in N$, which means

$$\begin{aligned} p(\sigma_\tau(\theta)\cdot) &= p(n \cdot (\sigma_\nu(\theta) \cdot z)) \\ &= p(\sigma_\nu(\theta) \cdot z) \end{aligned}$$

for some $n \in N$.

The proof will follow exercises 2.20-2.23 in [8].

Lemma 2.17. *If $a \in \Delta$, then $\mu^{-1}(a)$ is a single $\mathbb{T}^n$-orbit.*

Proof. Consider first the map J from above. Evidently, the image of J is the set

$$\{(x_1, \dots, x_d) \in \mathbb{R}^n \mid x_i \geq 0\}.$$

We start by noting that if a lies in this set, then $J^{-1}(a) \subset \mathbb{C}^d$ is a single $\mathbb{T}^d$-orbit.

The $\mathbb{T}^d$-action on $\mathbb{C}^d$ is given by

$$(e^{i\theta_1}, \dots, e^{i\theta_d}) \cdot (r_1 e^{i\alpha_1}, \dots, r_d e^{i\alpha_d}) := (r_1 e^{i(\alpha_1+\theta_1)}, \dots, r_d e^{i(\alpha_d+\theta_d)}),$$

from which we see that the orbits are of the form

$$\mathbb{T}^d \cdot (r_1 e^{i\alpha_1}, \dots, r_d e^{i\alpha_d}) = \{(z_1, \dots, z_d) \in \mathbb{C}^d \mid |z_j| = r_j\}.$$

From the definition of J, writing $a = (x_1, \dots, x_d)$,

$$\begin{aligned} J^{-1}(x_1, \dots, x_d) &= \{(z_1, \dots, z_d) \mid \frac{1}{2}(|z_1|^2, \dots, |z_d|^2) = (x_1, \dots, x_d)\} \\ &= \left\{(z_1, \dots, z_d) \mid |z_j| = \sqrt{2x_j}\right\}, \end{aligned}$$

which is precisely a single $\mathbb{T}^d$-orbit. Evidently, if we shift J by the constant λ, the condition for $(J+\lambda)^{-1}(a)$ being a single orbit is that a be in

$$\{(x_1, \dots, x_d) \mid x_i \geq \lambda_i\}.$$

Using the decomposition $\mathbb{T}^d = N \times \mathbb{T}^n$ and comparing with the commutative diagram for μ_Δ above, it follows that orbits by $\mathbb{T}^n$ on M_Δ are projections of $\mathbb{T}^d$-orbits on Z, and that $a = \mathrm{pr}_2(x)$ for some x in the image of $J+\lambda$, so that indeed, the preimage by $(J+\lambda)\circ j$ is the intersection of a $\mathbb{T}^d$-orbit with Z. Hence its projection, which is $\mu_\Delta(a)^{-1}$, is a $\mathbb{T}^n$-orbit. $\square$

Lemma 2.18. *The action of* $\mathbb{T}^n$ *on* $\mu_\Delta^{-1}(a)$ *is free if and only if* a *is in the interior of* Δ.

Proof. Suppose θ fixes $p(z) \in M_\Delta$, which means that

$$p(\sigma(\theta)\cdot z) = p(z),$$

so that either $\sigma(\theta) \in N$, or $\sigma(\theta) \in \mathrm{Stab}_{\mathbb{T}^d}(z)$. As σ is a right inverse of π and $N = \ker(\pi)$, we may already conclude that $\sigma(\theta) \in N$ implies $\theta = 1$. We next compute the stabilizer of z.

Take $\tau \in \Delta$. For all $z \in Z$ with $\mu(z) = \tilde{\pi}^*(\tau)$, we have on one hand that $\mu_\Delta(p(z)) = \tau$, which means of course $z \in \mu_\Delta^{-1}(\tau)$, and on the other, comparing to the proof of 3.11 in [10], we have that for all $i = 1, \dots, d$

$$\begin{aligned}
\tau \in \mathrm{int}(\Delta) &\iff \langle \tau, v_i\rangle > \lambda_i \\
&\iff \langle \tau, \tilde{\pi}(e_i)\rangle > \lambda_i \\
&\iff \langle \tilde{\pi}^*(\tau), e_i\rangle > \lambda_i \\
&\iff \langle \mu(z), e_i\rangle > \lambda_i \\
&\iff \frac{1}{2}|z_i|^2 + \lambda_i > \lambda_i \\
&\iff z_i \neq 0.
\end{aligned}$$

However, the multiplicative action on $\mathbb{C}^d$ by $\mathbb{T}^d$ fixes precisely the zero coordinates in z, so that we conclude that the stabilizer satisfies

$$\tau \in \mathrm{int}(\Delta) \iff \mathrm{Stab}_{\mathbb{T}^d}(z) = \{e\}.$$

Hence taking $\tau = a$ concludes the proof. $\square$

From this, we conclude that if $a \in \mathrm{int}\Delta$, as the action by $\mathbb{T}^n$ is free, we can consider the reduced space

$$M_a = \mu_\Delta^{-1}(a)/\mathbb{T}^n$$

and deduce that it is a single point as $\mu_\Delta^{-1}(a)$ consists of a single $\mathbb{T}^n$-orbit.

We are now ready to compute the symplectic volume of symplectic toric manifolds.

Theorem 2.19. *The symplectic volume of M_Δ is equal to the Lebesgue measure of Δ.*

Proof. As M_a is a 0-dimensional manifold, the exponential of the symplectic form on M_a is just 1. Letting $\sigma : M_a = \{p\} \to \{0\}$, we have that

$$\mathrm{vol}(M_a) = \int_{M_a} 1 = \int_{\{0\}} \sigma_*(1) = 1$$

whenever $a \in \mathrm{int}(\Delta)$. If $a \notin \Delta$, the preimage $\mu_\Delta^{-1}(a)$ is empty, and hence the reduced space is, too, so that the integral vanishes. Thus we conclude that

$$\mathrm{vol}(M_a) = \begin{cases} 1, & a \in \mathrm{int}(\Delta), \\ 0, & a \notin \Delta. \end{cases}$$

Now as the Duistermaat-Heckman measure is the pushforward of the Liouville measure by the moment map, we have that

$$\mathrm{vol}(M_\Delta) = \int_{M_\Delta} \frac{\omega_\Delta^n}{n!} = m_{DH}(\mathfrak{g}^*).$$

As $\mu_\Delta(M_\Delta) = \Delta$, we could take any $\Delta \subset U \subset \mathfrak{g}^*$ instead of $\mathfrak{g}^*$. Recalling now that we showed $dm_{DH} = \mathrm{vol}(M_t)dt$, we conclude

$$\mathrm{vol}(M_\Delta) = \int_{\mathfrak{g}^*} \mathrm{vol}(M_t)\, dt = \int_\Delta 1\, dt = \mathrm{vol}(\Delta).$$

We remark at this point that it is unsubstantial that we did not define $\mathrm{vol}(M_a)$ for $a \in \partial\Delta$ since it is a null set with respect to the Lebesgue measure. $\square$

Chapter 3

The Stationary Phase Lemma

This section presents the Stationary Phase formulation of the Duistermaat-Heckman theorem. We begin by quoting the general lemma of stationary phase, as in [2]:

Theorem 3.1 (Lemma of Stationary Phase)**.** *Let X^n a compact orientable manifold, λ a volume form on X, and $f : X \to \mathbb{R}$ a Morse function. Then*

$$\left(\frac{t}{2\pi}\right)^{n/2} \int_X e^{it\,f}\,\lambda = \sum_{p \in \mathrm{Crit}(f)} c(p) e^{it\,f(p)} + R(t),$$

where the remainder term $R(t)$ is of order $O(\frac{1}{t})$ and the constants $c(p)$ are given by

$$c(p) = \exp\left(\frac{\pi i}{4} \operatorname{sgn} \operatorname{Hess}(f)(p)\right) |\det(\operatorname{Hess}(f)(p)[e_i, e_j])_{ij}|^{-\frac{1}{2}},$$

the (e_i) being a basis of T_pM such that $\lambda(e_1, \ldots, e_n) = 1$.

In some cases, the remainder term is in fact identically zero, for which the Gaussian integral serves as an example we will get back to later:

Lemma 3.2. *Let Q a diagonal matrix and $\xi \in \mathbb{R}^n$. Then*

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i}{2}x^T Q x} e^{-i\xi \cdot x} dx = |\det Q|^{-\frac{1}{2}} \exp\left(\frac{\pi i}{4} \operatorname{sgn} Q\right) \exp\left(-\frac{i}{2}\xi^T Q^{-1} \xi\right).$$

See [2], section 8, for a proof. If we take f to be a quadratic function, that is,

$$f(x) = f(0) + \sum_{i=1}^{n} \frac{\alpha_i}{2} x_i^2 = f(0) + \frac{1}{2} x^T \underbrace{\begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_r \end{bmatrix}}_{=:Q} x,$$

and evaluate the Gaussian integral at $\xi = 0$, we see that for any t,

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{it\,(f(x)-f(0))} dx = |t \det Q|^{-\frac{1}{2}} \exp\left(\frac{\pi i}{4} \operatorname{sgn} Q\right),$$

which is equivalent to

$$\left(\frac{t}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{it\,f(x)}\, dx = |\det Q|^{-\frac{1}{2}} \exp\left(\frac{\pi i}{4} \operatorname{sgn} Q\right) \exp\left(itf(0)\right).$$

We have

$$\nabla f(x) = \begin{pmatrix} \alpha_1 x_1 \\ \vdots \\ \alpha_n x_n \end{pmatrix}, \qquad \operatorname{Hess}(f) = Q,$$

so that the only critical point of f is 0. Hence

$$c(0) = \exp\left(\frac{\pi i}{4} \operatorname{sgn} Q\right) |\det Q|^{-\frac{1}{2}}$$

as desired. Hence if we set $X = \mathbb{R}^n$, λ the Lebesgue measure and f a quadratic function, there is indeed no error term. Notice though that in this example, X is not compact.

In this section, we would like to prove that for compact hamiltonian $\mathbb{T}^n$-spaces and the right Morse function, the error term also vanishes.

Theorem 3.3 (Duistermaat-Heckman Stationary Phase Lemma). *Let $(M^{2N}, \omega, \mathbb{T}^n, \mu)$ a compact hamiltonian $\mathbb{T}^n$-space and $v \in \mathfrak{t}$ such that $\xi_v(p) = 0 \iff p$ is a fixed point. Call such v* ***nondegenerate****.*

Letting $X = M$, $f = \mu^v = \langle \mu, v \rangle$, and $\lambda = \frac{\omega^n}{n!}$ in theorem 3.1, the error term vanishes:

$$\left(\frac{t}{2\pi}\right)^N \int_M e^{it\,\mu^v} \frac{\omega^N}{N!} = \sum_{p \in \operatorname{Crit}(\mu^v)} c(p) e^{it\,\mu^v(p)}. \tag{3.1}$$

Note that in the case where v is nondegenerate, p is a critical point of μ^v if and only if it is a fixed point of the action.

We prove this in a series of steps, following section 33 in [2], or exercises 2.11 to 2.15 in [8]. In the following, we always consider the hamiltonian space $(M^{2N}, \omega, \mathbb{T}^n, \mu)$ for M compact, and $v \in \mathfrak{t}$ nondegenerate. Denote the Liouville form by $\sigma = \frac{\omega^N}{N!}$, and let $\xi = \xi_v$ the fundamental vector field associated to v.

Lemma 3.4. *Let $\Omega(M) = \bigoplus_{i=1}^{2N} \Omega^i(M)$ be the deRham complex on M and let Ω_{inv} denote the subcomplex of $\mathbb{T}^n$-invariant forms. Define the operator*

$$\delta_\xi = d + \imath_\xi.$$

δ maps Ω_{inv} into Ω_{inv} and satisfies $\delta_\xi^2 = 0$ on Ω_{inv}. Note also that δ_ξ carries the the space of even forms into the space of odd forms and vice versa.

Proof. We have for $\omega \in \Omega_{\text{inv}}(M)$ that $\psi_\theta^*(d\omega) = d(\psi_\theta^*\omega) = d\omega$. Furthermore, note that for $x \in M$

$$D\psi_\theta(x)[\xi(x)] = \left.\frac{d}{dt}\right|_{t=0} \theta \cdot (\exp(tv) \cdot x) = \left.\frac{d}{dt}\right|_{t=0} \exp(tv) \cdot (\theta \cdot x) = \xi(\theta \cdot x).$$

Consider $\omega \in \Omega^2_{\text{inv}}(M)$ as the general case only introduces more notational complexity. Then for $v \in T_xM$,

$$\begin{aligned}
\psi_\theta^*(\imath_\xi\omega)_x(v) &= \imath_\xi\omega_{\theta\cdot x}(D\psi_\theta(x)[v]) \\
&= \omega_{\theta\cdot x}(\xi(\theta \cdot x), D\psi_\theta(x)[v]) \\
&= \omega_{\theta\cdot x}(D\psi_\theta(x)[\xi(x)], D\psi_\theta(x)[v]) \\
&= \omega_x(\xi(x), v) \\
&= \imath_\xi\omega_x(v).
\end{aligned}$$

This proves the first claim. For the second, take $\omega \in \Omega_{\text{inv}}(M)$ and compute

$$\delta_\xi^2\omega = \underbrace{d^2\omega}_{=0} + \underbrace{d\imath_\xi\omega + \imath_\xi d\omega}_{=\mathcal{L}_\xi(\omega)} + \underbrace{\imath_\xi^2\omega}_{=0}$$

by Cartan's magic formula. By invariance, we have

$$\begin{aligned}
\mathcal{L}_\xi(\omega) &= \left.\frac{d}{dt}\right|_{t=0} \psi^*_{\exp(tv)}\omega \\
&= \left.\frac{d}{dt}\right|_{t=0} \omega \\
&= 0.
\end{aligned}$$

□

Lemma 3.5. *Suppose that the set M_{fix} of points fixed by $\mathbb{T}^n$ is finite, and let $M_0 = M \setminus M_{fix}$. Suppose $\mu = \mu_0 + \mu_2 + \ldots + \mu_{2m} \in \bigoplus_{i=1}^N \Omega^{2i}_{\text{inv}}$ for some $m < N \in \mathbb{N}$ is δ_ξ-closed. Then μ_{2m} is d-exact on M_0.*

Proof. First, equip M with a $\mathbb{T}^n$-invariant riemannian metric and define

$$\alpha = \frac{\langle \xi, \cdot \rangle}{\|\xi\|^2}.$$

α is linear and smooth as a composition of smooth functions, so it defines a 1-form on M_0. It is not well defined on all of M since the denominator is zero at the fixed points; Note that it may not become zero for any other point as we took $v \in \mathfrak{t}$ to be nondegenerate.

α is invariant, as for $x \in M_0$ and $u \in T_xM_0$,

$$\begin{aligned}
\langle \xi, \cdot \rangle_{\theta\cdot x}(D\psi_\theta(x)[u]) &= \langle \xi(\theta \cdot x), D\psi_\theta(x)[u] \rangle \\
&= \langle D\psi_\theta(x)[\xi(x)], D\psi_\theta(x)[u] \rangle \\
&= \langle \xi(x), u \rangle.
\end{aligned}$$

Thus α satisfies the following properties:

$$\mathcal{L}_\xi(\alpha) = 0, \qquad \alpha(\xi) = 1, \tag{3.2}$$

and also

$$\imath_\xi(d\alpha) = 0.$$

This can be seen by Cartan's magic formula:

$$\imath_\xi(d\alpha) = \mathcal{L}_\xi(\alpha) - d\imath_\xi\alpha,$$

where both terms on the right hand side are zero due to (3.2).

Next, define

$$\nu = \alpha \wedge (1 + d\alpha)^{-1} \wedge \mu,$$

where we let $(1 + d\alpha)^{-1}$ be defined as the von Neumann series

$$(1 + d\alpha)^{-1} = \sum_{i=0}^{2N} (-d\alpha)^i.$$

We now want to show that

$$\imath_\xi(d\nu) = \imath_\xi(\mu).$$

As $d(1 + d\alpha)^{-1} = 0$, we obtain

$$d\nu = d\alpha \wedge (1 + d\alpha)^{-1} \wedge \mu - \alpha \wedge (1 + d\alpha)^{-1} \wedge d\mu.$$

Using that $\delta_\xi(\mu) = 0$, we have $d\mu = -\imath_\xi(\mu)$, and so

$$d\nu = d\alpha \wedge (1 + d\alpha)^{-1} \wedge \mu + \alpha \wedge (1 + d\alpha)^{-1} \wedge \imath_\xi(\mu).$$

Applying $\imath_\xi$ to both sides, we obtain, using (3.2):

$$\begin{aligned}\imath_\xi(d\nu) &= d\alpha \wedge (1 + d\alpha)^{-1} \wedge \imath_\xi(\mu) + \underbrace{\imath_\xi(\alpha)}_{=1} \wedge (1 + d\alpha)^{-1} \wedge \imath_\xi(\mu) \\ &= (d\alpha \wedge (1 + d\alpha)^{-1} + (1 + d\alpha)^{-1}) \wedge \imath_\xi(\mu) \\ &= \underbrace{(\sum_{i=0}^{2N} (-1)^i ((d\alpha)^{i+1} + (d\alpha)^i)}_{=1} \imath_\xi(\mu).\end{aligned}$$

Finally, on M_0, $\imath_\xi : \Omega^N(M) \to \Omega^{m-1}(M)$ is injective, which implies that $\mu_m = d\nu_{m-1}$. □

We want to apply this lemma to the form $\beta = \mu^v - \omega \in \Omega^0 \oplus \Omega^2$. Indeed, β is δ_ξ-closed:

$$\delta_\xi(\beta) = d\mu^v + \underbrace{\imath_\xi(\mu^v)}_{=0} + \underbrace{d\omega}_{=0} + \imath_\xi(\omega) = 0$$

by definition of the moment map. Hence also $\exp(it\beta)$ is δ_ξ-closed, and its component of highest degree 2^N is then, up to constant multiple, $e^{it\mu^v}\sigma$. Hence by the last lemma, there exists $\nu_{2N-1} \in \Omega^{2N-1}(M \setminus M_0)$ such that $d\nu_{2N-1} = e^{it\mu^v}\sigma$.

Proposition 3.6. *Let B_p denote a small open neighbourhood of the critical point p. Then we have by Stokes' Theorem*

$$\int_M \exp(it\mu^v)\sigma = \sum_{p\in \mathrm{Crit}(\mu^v)} \left(\int_{B_p} \exp(it\mu^v)\sigma - \int_{\partial B_p} \nu_{2N-1} \right) \tag{3.3}$$

Proof. As we assume the fixed points to be isolated, we may take the B_p to be disjoint. Note that $e^{it\,\mu^v}$ is exact outside U, where U is the union of the B_p, so that

$$\int_M \exp(it\mu^v)\sigma = \int_{M\setminus U} \exp(it\mu^v)\sigma + \sum_{p\in \mathrm{Crit}(\mu^v)} \int_{B_p} \exp(it\mu^v)\sigma.$$

Now as $\int_{M\setminus U} \exp(it\mu^v)\sigma = -\sum_{p\in \mathrm{Crit}(\mu^v)} \int_{\partial B_p} \nu_{2N-1}$, we are done. □

We may evaluate the right hand side of 3.3 by using the equivariant Darboux theorem 1.15, which we proved in chapter 1:

Theorem 3.7 (Equivariant Darboux)**.** *Let $(M^{2N}, \omega, \mathbb{T}^n, \mu)$ a hamiltonian torus space, and p a fixed point of the action. Then there exists a $\mathbb{T}^n$-invariant neighbourhood U of p, coordinate functions $(x_1, \ldots, x_N, y_1, \ldots, y_N)$ centered at p and constants $\lambda^{(1)}, \ldots, \lambda^{(N)} \in \mathbb{Z}^n$ such that on U*

1. *$\omega_U = \omega_0 = \sum_{j=1}^N dx_j \wedge dy_j = \frac{i}{2}\sum_{j=1}^N dz_j \wedge \overline{dz_j}$, where $z_j = x_j + iy_j$.*
2. *The action becomes multiplication with weights $\lambda^{(j)}$:*

$$(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot (z_1, \ldots z_N) = \left(e^{i\langle \lambda^{(1)}, \theta\rangle} z_1, \ldots, e^{i\langle \lambda^{(N)}, \theta\rangle} z_N \right).$$

3. *The moment map becomes*

$$\mu_U = \mu(p) + \frac{1}{2}\sum_{j=1}^{n} \lambda^{(j)} |x^2 + y^2| = \mu(p) + \frac{1}{2}\sum_{j=1}^{N} \lambda^{(j)} |z_j|^2.$$

If we evaluate the right hand side of 3.3 in these coordinates, we may use Stokes' theorem again, which yields

$$\int_{B_p} \exp(it\mu_U^v)\omega_0 - \int_{\partial B_p} \tilde{\nu}_{2N-1} = \int_{\mathbb{R}^{2N}} \exp(it\mu_U^v)\omega_0.$$

Lemma 3.2 has already established that this integral has no error term in its stationary phase expansion, and so the same holds true for $\exp(it\mu^v)\sigma$, concluding the proof of theorem 3.3.

It is also simple to compute the constants $c(p)$ in these coordinates. The Hessian of μ_U^v is then

$$\mathrm{Hess}(\mu_U^v)(p) = \begin{bmatrix} \langle \lambda^{(1)}, v \rangle & & & & \\ & \langle \lambda^{(1)}, v \rangle & & & \\ & & \ddots & & \\ & & & \langle \lambda^{(N)}, v \rangle & \\ & & & & \langle \lambda^{(N)}, v \rangle \end{bmatrix},$$

so its determinant is $\prod_{j=1}^N \alpha_j(p,v)^2$ for $\alpha_j(p,v) = \langle \lambda^{(j)}, v \rangle$. Recall that the $\lambda^{(i)}$ depend on p, and note that this also shows that μ^v is Morse.

Denoting the number of negative eigenvalues of the Hessian by l, its signature is $2N - 4l$. In the formula for the $c(p)$, this yields for the first factor

$$\exp\left(\frac{\pi i}{4} \mathrm{sgn}\, \mathrm{Hess}(f)(p)\right) = i^N (-1)^l.$$

The second factor becomes

$$\left(\prod_{j=1}^N |\alpha(p,v)|\right)^{-1} = (-1)^l \left(\prod_{j=1}^N \alpha(p,v)\right)^{-1},$$

so that the signs cancel and we obtain

$$c(p) = (i)^N \prod_{j=1}^N (\alpha(p,v))^{-1}.$$

Inserting this into the expression 3.1 gives our final result:

Theorem 3.8. *Under the assumptions of theorem 3.3, we have*

$$\left(\frac{t}{2\pi i}\right)^N \int_M e^{it\,\langle \mu, v \rangle} \frac{\omega^N}{N!} = \sum_p \frac{e^{it\,\langle \mu(p), v \rangle}}{\prod_{j=1}^N \alpha(p,v)},$$

the sum being over the fixed points of the action.

Chapter 4

The Harish-Chandra-Itzykson-Zuber Integral

4.1 Outline

In this section we compute explicitly a certain integral, as an application of the Duistermaat-Heckman theorem. Let $A, B \in \mathrm{Gl}_n(\mathbb{C})$ be hermitian matrices with eigenvalues $\lambda_1(A) \leq \ldots \leq \lambda_n(A)$ and $\lambda_1(B) \leq \ldots \leq \lambda_n(B)$. Consider the integral

$$\mathcal{Z}(A, B) = \int_{U(n)} e^{t\mathrm{Tr}(AUBU^*)}\, dU$$

for $U(n)$ the unitary group and $t \in \mathbb{C}^\times$, the integral being with respect to the Haar probability measure of the unitary group. Then the Harish-Chandra-Itzykson-Zuber formula asserts that

Theorem 4.1 (Harish-Chandra-Itzykson-Zuber)**.**

$$\mathcal{Z}(A, B) = c_n \frac{\det((e^{t\lambda_i(A)\lambda_j(B)})_{1\leq i,j\leq n})}{t^{\frac{n^2-n}{2}} \Delta(\lambda(A)\Delta(\lambda(B))},$$

whenever there are no multiple eigenvalues. In this expression, $\Delta(\lambda(M))$ denotes the Vandermonde determinant

$$\Delta(\lambda(M)) := \prod_{1\leq i<j\leq n} (\lambda_j(M) - \lambda_i(M)),$$

and c_n is the constant

$$c_n := \prod_{i=1}^{n-1} i!.$$

We proceed along the lines of section 3 of [11] in proving the theorem.

Proposition 4.2. *To prove the Integral Formula, it suffices to show that*

$$\int_{U(n)} e^{t\mathrm{Tr}(AUBU^*)}\, dU = \frac{C_{A,B}}{t^{\frac{n^2-n}{2}}} \det\left((e^{t\lambda_i(A)\lambda_i(B)})_{i,j} \right) \tag{4.1}$$

for some constant $C_{A,B}$ which only depends on n, A, and B.

Proof. Set $N := \frac{n^2-n}{2}$ and $X(t) = (e^{t\lambda_i(A)\lambda_j(B)})_{i,j}$. We consider the Taylor expansion of $\det(X(t))$ near zero. If the proposed equality holds, then

$$\det(X(t)) = \frac{t^N}{C_{A,B}} \int_{U(n)} e^{t\mathrm{Tr}(AUBU^*)}\, dU,$$

and thus we obtain for the derivative

$$\frac{d^j}{dt^j} t^N \int_{U(n)} e^{t\mathrm{Tr}(AUBU^*)}\, dU$$
$$= \sum_{k=0}^{j} \binom{j}{k} N(N-1)\cdots(N-k+1)t^{N-k} \int_{U(n)} \mathrm{Tr}(AUBU^*)^{j-k} e^{t\mathrm{Tr}(AUBU^*)}\, dU.$$

Evidently, this is only nonzero at $t=0$ for $k=N$, so that the N-th Taylor coefficient becomes

$$\frac{1}{C_{A,B}} \frac{1}{N!} \binom{N}{N} N! \int_{U(n)} 1\, dU = \frac{1}{C_{A,B}}.$$

However, we may also express the determinant by the Leibniz formula

$$\det(X(t)) = \sum_{\sigma \in S_n} \mathrm{sgn}\,(\sigma) \prod_{i=1}^{n} e^{t\lambda_i(A)\lambda_{\sigma(i)}(B)}$$

and compute the N-th coefficient of its Taylor expansion. For σ fixed, the derivative is

$$\begin{aligned} \frac{d^N}{dt^N}\Big|_{t=0} \prod_{i=1}^{n} e^{t\lambda_i(A)\lambda_{\sigma(i)}(B)} &= \sum_{|\alpha|=N} \binom{N}{\alpha} \prod_{i=1}^{n} (\lambda_i(A)\lambda_{\sigma(i)}(B))^{\alpha_i} \\ &= \sum_{|\alpha|=N} N! \prod_{i=1}^{n} \frac{1}{\alpha_i!} (\lambda_i(A)\lambda_{\sigma(i)}(B))^{\alpha_i} \end{aligned}$$

for $\alpha \in \mathbb{N}^n$ a multiindex. Hence the N-th Taylor coefficient must be

$$\sum_{\sigma \in S_n} \mathrm{sgn}\,(\sigma) \sum_{|\alpha|=N} \prod_{i=1}^{n} \frac{1}{\alpha_i!} (\lambda_i(A)\lambda_{\sigma(i)}(B))^{\alpha_i}.$$

Note that for a summand with $\alpha_k = \alpha_l$ for some indices, this summand will cancel out with the one arising from the permutation exchanging $\sigma(k)$ and $\sigma(l)$; hence it suffices to consider only multiindices with all components distinct. However, as $N = \frac{1}{2}(n^2 - n) = 0+1+\ldots+n-1$, we conclude that the components of α are a permutation of $\{0, 1, 2, \ldots, n-1\}$, that is, $\alpha_i = \alpha(i) - 1$ for some $\alpha \in S_n$. Hence we may rearrange this sum to give

$$\sum_{\sigma \in S_n} \mathrm{sgn}\,(\sigma) \sum_{\alpha \in S_n} \prod_{i=1}^{n} \frac{1}{(i-1)!} (\lambda_{\alpha^{-1}(i)}(A) \lambda_{\alpha^{-1} \circ \sigma(i)}(B))^{i-1}.$$

Setting $\beta = \alpha^{-1} \circ \sigma$ gives that $\mathrm{sgn}\,(\sigma) = \mathrm{sgn}\,(\alpha)\mathrm{sgn}\,(\beta)$, and noting that we may as well sum over α^{-1} lets us rewrite this as

$$\sum_{\alpha, \beta \in S_n} \mathrm{sgn}\,(\alpha)\mathrm{sgn}\,(\beta) \prod_{i=1}^{n} \frac{1}{(i-1)!} (\lambda_{\alpha(i)}(A) \lambda_{\beta(i)}(B))^{i-1}.$$

As the Vandermonde determinant can also be expressed by the Leibniz formula as

$$\Delta(\lambda(A)) = \sum_{\sigma \in S_n} \mathrm{sgn}\,(\sigma) \prod_{i=1}^{n} \lambda_i(A)^{\sigma(i)-1},$$

we see that the expression for the N-th Taylor coefficient is just

$$\Delta(\lambda(A))\Delta(\lambda(B)) \prod_{i=1}^{n} \frac{1}{(i-1)!}.$$

Comparing this with our first computation, where we obtained that the Taylor coefficient must be equal to $\frac{1}{C_{A,B}}$, we substitute in equation 4.1 the expression for $C_{A,B}$ we just obtained to see that

$$\int_{U(n)} e^{t\mathrm{Tr}(AUBU^*)}\, dU = \frac{(n-1)! \cdots 0!}{\Delta(\lambda(A))\Delta(\lambda(B)) t^{\frac{n^2-n}{2}}} \det\left((e^{t\lambda_i(A)\lambda_j(B)})_{i,j} \right),$$

which is what we want to prove. □

It remains to prove that 4.1 holds. As the trace is invariant under conjugation, it suffices to take A and B diagonal matrices. In the case where the integral formula is supposed to hold, we may assume that A and B have no multiple eigenvalues, so that we can write

$$A = \mathrm{diag}(a) = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}, \qquad B = \mathrm{diag}(b) = \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{bmatrix}$$

for $a = (a_i)_i, b = (b_i)_i \in \mathbb{R}^n$ both with pairwise distinct components. By analytic continuation, we may take t to be imaginary. Furthermore, by subtracting a constant from A, we may take A to have trace zero, which is a condition we will need later. With these simplifications in place, we must show

$$\int_{U(n)} e^{it\mathrm{Tr}(AUBU^*)}\, dU = C_{a,b} \det((e^{ita_j b_k})_{j,k}) t^{-(n^2-n)/2}. \tag{4.2}$$

Expand the right hand side as

$$C_{a,b} \sum_{\sigma \in S_n} \mathrm{sgn}\,(\sigma) e^{ita\cdot\sigma(b)} t^{-(n^2-n)/2}.$$

This already bears some resemblance to the formula from the Duistermaat-Heckman theorem in its stationary phase formulation. To see that we can in fact apply it, we will seek to interpret the integral on the left hand side as over a hamiltonian torus space with an action whose fixed points are parameterized by S_n. We will introduce the notion of coadjoint orbits for this purpose.

4.2 Coadjoint Orbits

The goal of this section is to show how for a matrix Lie group G, the orbits of the coadjoint action on the dual of the Lie algebra can be endowed with a symplectic structure. We will focus on matrix Lie groups as this will make the proofs easier, and is the primary case we will be concerned with in the proof of the Harish-Chandra-Itzykson-Zuber integral formula. This section follows [12], section 8.3.3.

4.2.1 Adjoint and Coadjoint Representations

We start by recalling the basic definitions. Let us consider a Lie group G with Lie algebra $\mathfrak{g}$. Conjugation defines a diffeomorphism on G:

$$\begin{aligned} \Psi_g : G &\longrightarrow G, \\ h &\longmapsto hgh^{-1}. \end{aligned}$$

Hence $g \mapsto \Psi_g$ defines a smooth action on G.

We define the **adjoint** at an element $g \in G$ to be the derivative of Ψ_g at the identity, which is a linear isomorphism since Ψ_g is a diffeomorphism:

$$\mathrm{Ad}_g = D\Psi_g(e) : \mathfrak{g} \to \mathfrak{g},$$

and call

$$\begin{aligned} \mathrm{Ad} : G &\longmapsto \mathrm{Gl}(\mathfrak{g}) \\ g &\longmapsto \mathrm{Ad}_g \end{aligned}$$

the adjoint representation.

We use this to define the **coadjoint representation**

$$\begin{aligned}\mathrm{Ad}^* : G &\longrightarrow \mathrm{Gl}(\mathfrak{g}^*)\\ g &\longmapsto \mathrm{Ad}^*_g\end{aligned}$$

as follows. For $\phi \in \mathfrak{g}^*$, we let $\mathrm{Ad}^*_g(\phi)$ the element which acts on $v \in \mathfrak{g}$ by

$$\langle \mathrm{Ad}^*_g(\phi), v\rangle = \langle \phi, \mathrm{Ad}_{g^{-1}}(v)\rangle.$$

We will use the notation common for group actions: Letting $g \in G$, $v \in \mathfrak{g}$ and $\phi \in \mathfrak{g}^*$, write

$$\begin{aligned}g \cdot v &= \mathrm{Ad}_g(v),\\ g \cdot \phi &= \mathrm{Ad}^*_g(\phi).\end{aligned}$$

Of course, we should check these are indeed group actions. In doing so, we will see why taking the inverse of g in the definition above is necessary to obtain a left group action:

Proposition 4.3. *For $g, h \in G$, we have $\mathrm{Ad}_g \circ \mathrm{Ad}_h = \mathrm{Ad}_{gh}$, as well as $\mathrm{Ad}^*_g \circ \mathrm{Ad}^*_h = \mathrm{Ad}^*_{gh}$.*

Proof. The first statement is a simple application of the chain rule:

$$\mathrm{Ad}_g \circ \mathrm{Ad}_h = D\Psi_g(e) \circ D\Psi_h(e) = D\Psi_g\Psi_h(e) = D\Psi_{gh}(e) = \mathrm{Ad}_{gh}.$$

For the second, take $\phi \in \mathfrak{g}^*$ and $v \in \mathfrak{g}$ and compute

$$\begin{aligned}\langle \mathrm{Ad}^*_g \circ \mathrm{Ad}^*_h(\phi), v\rangle &= \langle \phi, \mathrm{Ad}_{h^{-1}} \circ \mathrm{Ad}_{g^{-1}}(v)\rangle\\ &= \langle \phi, \mathrm{Ad}_{(gh)^{-1}}(v)\rangle\\ &= \langle \mathrm{Ad}^*_{gh}(\phi), v\rangle.\end{aligned}$$

□

We can take this one step further by taking the derivative of Ad:

$$\mathrm{ad} := D\mathrm{Ad}(e) : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

where $\mathfrak{gl}(\mathfrak{g})$ are all linear maps $\mathfrak{g} \to \mathfrak{g}$. The map ad satisfies for $v, w \in \mathfrak{g}$

$$\mathrm{ad}_v(w) = [v, w].$$

See for example [3], proposition 10.23. Define the coadjoint version similarly by

$$\langle \mathrm{ad}^*_v(\phi), w\rangle := \langle \phi, \mathrm{ad}_w(v)\rangle = \langle \phi, [w, v]\rangle.$$

We now lay the foundations which will help us prove that the symplectic structure constructed in the nexts section is well-defined. Same as with a Lie group acting on any manifold, we obtain fundamental vector fields associated to $v \in \mathfrak{g}$ for each of the representations, defined by

$$\xi_v \in \mathfrak{X}(\mathfrak{g}) \text{ by } \xi_v(u) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(tv)}(u),$$
$$\xi_v^* \in \mathfrak{X}(\mathfrak{g}^*) \text{ by } \xi_v^*(\phi) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*_{\exp(tv)}(\phi).$$

We will also need the following lemma from problem M.5 in [3].

Lemma 4.4. *For any action, the map $v \mapsto \xi_v$ is a Lie algebra homomorphism, that is, it respects the Lie bracket.*

Proof. For any action μ, denote by μ^p the map $\mu^p(g) = g \cdot p$. Then we may rewrite the fundamental vector field as

$$\xi_v(p) = \frac{d}{dt}\bigg|_{t=0} \mu^p(\exp(tv)) = D\mu^p(e)[v].$$

Hence we have

$$\begin{aligned}
[\xi_v, \xi_w](p) &= [D\mu^p(e)[v], D\mu^p(e)[w]] \\
&= D\mu^p(e)[[v,w](e)] \\
&= \xi_{[v,w]}(p).
\end{aligned}$$

□

In the case where G is a matrix Lie group, we have for $v \in G$, $w \in \mathfrak{g}$ that

$$\mathrm{Ad}_v(w) = vwv^{-1}.$$

This gives that $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is an algebra homomorphism:

$$\mathrm{Ad}_g([v,w]) = g(vw - wv)g^{-1} = gvg^{-1}gwg^{-1} - gwg^{-1}gvg^{-1} = [\mathrm{Ad}_g(v), \mathrm{Ad}_g(w)].$$

Implicitly identifying $T_v\mathfrak{g} \cong \mathfrak{g}$, this also simplifies the expressions we obtain for the fundamental vector fields, in that

$$\xi_v(w) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(tv)}(w) = \frac{d}{dt}\bigg|_{t=0} \exp(tv)w\exp(-tv) = vw - wv = [v,w].$$

Similarly for ξ_v^*:

$$\begin{aligned}
\langle \xi_v^*(\phi), w \rangle &= \langle \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*_{\exp(tv)}(\phi), w \rangle \\
&= \langle \phi, \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(-tv)}(w) \rangle \\
&= \langle \phi, [-v, w] \rangle \\
&= \langle \phi, [w, v] \rangle \\
&= \langle \mathrm{ad}_v^*, w \rangle.
\end{aligned}$$

Let us quickly summarize these results:

Proposition 4.5. *Let G a matrix Lie group, $v, w \in \mathfrak{g}$, and $\phi \in \mathfrak{g}^*$. Then we have*

$$\xi_v(w) = \mathrm{ad}_v(w) = [v, w], \qquad \langle \xi_v^*(\phi), w\rangle = \langle \mathrm{ad}_v^*(\phi), w\rangle = \langle \phi, [w, v]\rangle.$$

Let us also give a proposition relating the two vector fields:

Lemma 4.6. *Let G a matrix Lie group, $g \in G$, $v \in \mathfrak{g}$, $\phi \in \mathfrak{g}^*$. Then we have*

$$\xi^*_{\mathrm{Ad}_g(v)}(g \cdot \phi) = \mathrm{Ad}_g^*(\xi_v^*(\phi)).$$

Proof. Take another element $w \in \mathfrak{g}$ and compute

$$\begin{aligned}
\langle \mathrm{Ad}_g^*(\xi_v^*(\phi)), w\rangle &= \langle \xi_v^*(\phi), \mathrm{Ad}_{g^{-1}}(w)\rangle \\
&= \langle \phi, [\mathrm{Ad}_{g^{-1}}(w), v]\rangle \\
&= \langle \phi, \mathrm{Ad}_{g^{-1}}[w, \mathrm{Ad}_g(V)]\rangle \\
&= \langle \mathrm{Ad}_g^*(\phi), [w, \mathrm{Ad}_g(v)]\rangle \\
&= \langle \mathrm{Ad}_g^*(\phi), \mathrm{ad}_{\mathrm{Ad}_g(v)}(w)\rangle \\
&= \langle \mathrm{ad}^*_{\mathrm{Ad}_g(v)}(\mathrm{Ad}_g^*(\phi)), w\rangle \\
&= \langle \xi^*_{\mathrm{Ad}_g(v)}(g \cdot \phi), w\rangle.
\end{aligned}$$

□

4.2.2 Symplectic Structure on Coadjoint Orbits

Here we define a symplectic form on the **coadjoint orbit**

$$\mathcal{O}_\phi = \{\mathrm{Ad}_g^*(\phi) \mid g \in G\}$$

for $\phi \in \mathfrak{g}^*$.

Proposition 4.7. *Let $\phi \in \mathfrak{g}^*$ and consider $\mathfrak{g}_\phi \subset \mathfrak{g}$, the Lie algebra of the stabilizer of ϕ in G, that is, of*

$$\mathrm{Stab}_G(\phi) = \{g \in G \mid \mathrm{Ad}_g^*(\phi) = \phi\}.$$

We claim that

$$\mathfrak{g}_\phi = \{v \in \mathfrak{g} \mid \langle \phi, [w, v]\rangle = 0, \ \textit{for all } w \in \mathfrak{g}\}.$$

Moreover, $v \in \mathfrak{g}_\phi$ if and only if $\xi_v^*(\phi) = 0$.

Proof. Let us denote the set in question by X. Note that $\langle \phi, [w, v]\rangle = \langle \mathrm{ad}_v^*(\phi), w\rangle = \langle \xi_v^*(\phi), w\rangle$, so $v \in X \iff \xi_v^*(\phi) = 0$.

As the exponential map associated to $\mathrm{Stab}_G(\phi)$ and $\mathfrak{g}_\phi$ is just the restriction of the exponential map $\mathfrak{g} \to G$, we have that if $v \in \mathfrak{g}_\phi$, then $\exp(tv) \in \mathrm{Stab}_G(\phi)$ for all t. Hence $\mathrm{Ad}^*_{\exp(tv)}(\phi) = \phi$, so that $\xi_v^*(\phi) = 0$ and thus $v \in X$.

If $v \in X$, it suffices to show that $\exp(tv) \in \mathrm{Stab}_G(\phi)$ for all t in a neighbourhood of 0 to conclude that $v = \frac{d}{dt}\big|_{t=0} \exp(tv) \in \mathfrak{g}_\phi$. This is however clear from $\xi_v^*(\phi) = 0$. □

We can now begin the construction of the symplectic form.

Lemma 4.8. *For $\phi \in \mathfrak{g}^*$, define an alternating form on $\mathfrak{g}$ by setting for $v, w \in \mathfrak{g}$*

$$\tilde{\omega}_\phi(v, w) = \langle \phi, [v, w] \rangle.$$

$\tilde{\omega}_\phi$ vanishes precisely on $\mathfrak{g}_\phi$.

Proof. $\tilde{\omega}_\phi$ is evidently alternating and bilinear due to bilinearity and antisymmetry of the Lie bracket. If $v \in \mathfrak{g}_\phi$, then by the previous proposition, we have that $\langle \phi, [v, w] \rangle = \tilde{\omega}_\phi(v, w) = 0$ for all $w \in \mathfrak{g}$, and vice versa. □

To define a symplectic form on $\mathcal{O}_\phi$, we give a description of its tangent space.

Lemma 4.9. *We can identify the tangent space $T_\phi \mathcal{O}_\phi$ with the quotient $\mathfrak{g}/\mathfrak{g}_\phi$.*

Proof. The map

$$\begin{aligned} \mathfrak{g} &\longrightarrow T_\phi \mathcal{O}_\phi \\ v &\longmapsto \xi_v^*(\phi) \end{aligned}$$

is linear in v:

$$\begin{aligned} \xi_{v+w}^*(\phi) &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(tv)\exp(tw)}(\phi) \\ &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(tv)\exp(0)}(\phi) + \left.\frac{d}{ds}\right|_{s=0} \mathrm{Ad}^*_{\exp(0)\exp(sw)}(\phi) \\ &= \xi_v^*(\phi) + \xi_w^*(\phi). \end{aligned}$$

It is furthermore surjective and has kernel $\mathfrak{g}_\phi$ by proposition 4.7, whence the claim follows. □

Theorem 4.10. *$\tilde{\omega}_\phi$ induces a nondegenerate two-form on the tangent bundle to the coadjoint orbit $\mathcal{O}_\phi$ passing through ϕ. Moreover, the induced form is closed on the tangent bundle to $\mathcal{O}_\phi$ through ϕ, making it a symplectic form.*

Proof. Every element of $T_\phi(\mathcal{O}_\phi)$ can be written as $\xi_v^*(\phi)$ for some $v \in \mathfrak{g}$. Set

$$\omega_\phi(\xi_v^*(\phi), \xi_w^*(\phi)) := -\tilde{\omega}_\phi(v, w).$$

This is well-defined, since if $\xi_v^*(\phi) = \xi_w^*(\phi)$, then $0 = \xi_{v-w}^*(\phi)$, so that $v - w \in \mathfrak{g}_\phi$. Hence $\tilde{\omega}_\phi(v, \cdot) = \tilde{\omega}_\phi(w, \cdot)$.

If $\omega_\phi(\xi_v^*(\phi), \cdot) = -\tilde{\omega}_\phi(v, \cdot) = 0$, then $v \in \mathfrak{g}_\phi$ by the last lemma, so that $\xi_v^*(\phi) = 0_{T_\phi \mathcal{O}_\phi}$, implying nondegeneracy.

Lastly, we show ω is closed. Take $u, v, w \in \mathfrak{g}$ to compute

$$\begin{aligned} d\omega(\xi_u^*, \xi_v^*, \xi_w^*) &= -\omega([\xi_u^*, \xi_v^*], \xi_w^*) + \omega([\xi_u^*, \xi_w^*], \xi_v^*) - \omega([\xi_v^*, \xi_w^*], \xi_u^*) \\ &\quad + \xi_u^* \omega(\xi_v^*, \xi_w^*) - \xi_v^* \omega(\xi_u^*, \xi_w^*) + \xi_w^* \omega(\xi_u^*, \xi_v^*) \\ &\overset{(1)}{=} + \langle \cdot, [[u, v], w] \rangle - \langle \cdot, [[u, w], v] \rangle + \langle \cdot, [[v, w], u] \rangle \\ &\quad - \xi_u^* \langle \cdot, [v, w] \rangle + \xi_v^* \langle \cdot, [u, w] \rangle - \xi_w^* \langle \cdot, [u, v] \rangle \\ &\overset{(2)}{=} -\xi_u^* \langle \cdot, [v, w] \rangle + \xi_v^* \langle \cdot, [u, w] \rangle - \xi_w^* \langle \cdot, [u, v] \rangle. \end{aligned}$$

(1) used that $v \mapsto \xi_v$ is a Lie algebra homomorphism and (2) used antisymmetry together with the Jacobi identity. To see that the last line equals 0, note that

$$\xi_u^*(\langle \cdot, [v,w] \rangle)(\phi) = \frac{d}{dt}\bigg|_{t=0} \langle \mathrm{Ad}^*_{\exp(tu)}(\phi), [v,w] \rangle = \frac{d}{dt}\bigg|_{t=0} \langle \phi, \mathrm{Ad}_{\exp(-tu)}([v,w]) \rangle.$$

The last expression is equal to $\langle \phi, -\xi_u([v,w]) \rangle = \langle \phi, [[v,w],u] \rangle$ by proposition 4.5. Hence we may rewrite the last line as

$$\langle \cdot, [[v,w],u] \rangle + \langle \cdot, [[w,u],v] \rangle + \langle \cdot, [[u,v],w] \rangle = 0$$

by the Jacobi identity. □

The symplectic form defined in theorem 4.10 is called the **Kostant-Kirillov-Souriau** form. This makes any coadjoint orbit $\mathcal{O}_\phi$ into a symplectic manifold, and hence in particular of even dimension.

Theorem 4.11. *For any $\phi \in \mathfrak{g}^*$ and $\mathcal{O}_\phi \subset \mathfrak{g}^*$ the coadjoint orbit through ϕ, the inclusion map*

$$i : \mathcal{O}_\phi \hookrightarrow \mathfrak{g}^*$$

is a moment map for the coadjoint action. Hence $(\mathcal{O}_\phi, \omega_K, G, i)$ is a hamiltonian G-space.

Proof. We have $i^v(\phi) = \langle i(\phi), v \rangle = \langle \phi, v \rangle$, and

$$\begin{aligned}
(di^v)_\phi(\xi_w^*(\phi)) &= \xi_w^*(\phi)(i^v) \\
&= \frac{d}{dt}\bigg|_{t=0} i^v(\mathrm{Ad}^*_{\exp(tw)}(\phi)) \\
&= \frac{d}{dt}\bigg|_{t=0} \langle \mathrm{Ad}^*_{\exp(tw)}(\phi), v \rangle \\
&= \langle \xi_w^*(\phi), v \rangle \\
&= \langle \mathrm{ad}_w^*(\phi), v \rangle \\
&= \langle \phi, [v,w] \rangle \\
&= -(\omega_K)_\phi(\xi_v^*(\phi), \xi_w^*(\phi)) \\
&= -(\imath_{\xi_v^*}\omega_K)_\phi(\xi_w^*(\phi)).
\end{aligned}$$

Lastly, we also have

$$\langle i(\mathrm{Ad}_g^*(\phi)), w \rangle = \langle \mathrm{Ad}_g^*(\phi), w \rangle = \langle \phi, \mathrm{Ad}_{g^{-1}}(w) \rangle = \langle i(\phi), \mathrm{Ad}_{g^{-1}}(w) \rangle = \langle \mathrm{Ad}_g^*(i(\phi)), w \rangle,$$

proving equivariance. □

4.3 Finding the Proper Hamiltonian Space

We want to take for our symplectic manifold a coadjoint orbit, so let us make sense of the adjoint and coadjoint representations in this case. Here we follow again [11], section 3.

4.3.1 Adjoint and Coadjoint on $U(n)$

Recall $U(n) < \mathrm{Gl}_n(\mathbb{C})$ is a Lie group of real dimension n^2. Let us compute its Lie algebra, given by

$$\mathfrak{u} = \{\dot\gamma(0) \mid \gamma \text{ path in } U(n) \text{ with } \gamma(0) = \mathrm{id}\}.$$

As γ must be a path in $U(n)$, it satisfies $\gamma(t)\gamma(t)^* = \mathrm{id}$. Differentiating this identity yields

$$\dot\gamma(0)\gamma(0)^* + \gamma(0)\dot\gamma^*(0) = 0,$$

so if $\dot\gamma(0) = A \in M_{n\times n}$, we have that

$$\mathfrak{u} = \{A \in M_{n\times n} \mid A^* = -A\}.$$

We now seek to give a description of $\mathfrak{gl}(n)^*$. Define on $M_{n\times n}$ the symmetric bilinear form

$$\langle A, B\rangle := \mathrm{Tr}(AB).$$

Recall that Tr is invariant under cyclic permutations, which also implies it is invariant under conjugation:

$$\mathrm{Tr}(UAU^{-1}) = \mathrm{Tr}(AU^{-1}U) = \mathrm{Tr}(A).$$

Then each $\phi_A \in \mathfrak{gl}(n)^*$ is of the form $\mathrm{Tr}(A\cdot)$, and in particular, every $\psi_B \in \mathfrak{u}^*$ is of the form $\mathrm{Tr}(B\cdot)$ for some B anti-hermitian. Let us also write $\phi_A = A$, and thus $\langle \phi_A, B\rangle = \langle A, B\rangle$.

The adjoint acts on $\mathfrak{u}$ by

$$\mathrm{Ad}_U(A) = UAU^{-1} = UAU^*$$

for $U \in U(n)$. Hence for the coadjoint, we have

$$\begin{aligned}\langle \mathrm{Ad}_U^*(B), A\rangle &= \langle B, \mathrm{Ad}_{U^{-1}}(A)\rangle \\ &= \langle B, U^*AU\rangle \\ &= \langle UBU^*, A\rangle,\end{aligned}$$

so that also $\mathrm{Ad}_U^*(B) = UBU^*$. Hence the coadjoint orbit of $B \in \mathfrak{u}^*$ is just

$$\mathcal{O}_B = \{UBU^* \mid U \in U(n)\}.$$

Recall that we may write the tangent vectors in $T_P\mathcal{O}_P$ as $\xi_A^*(\phi_P) = \mathrm{ad}_A^*(\phi_P)$. Compute

$$\begin{aligned}\langle \xi_A^*(\phi_P), C\rangle &= \langle P, [C, A]\rangle \\ &= \langle P, CA - AC\rangle \\ &= \langle PA^* - A^*P, C\rangle \\ &= \langle [P, A^*], C\rangle \\ &= \langle [A, P], C\rangle,\end{aligned}$$

where the last step used that $A \in \mathfrak{u}$ is anti-hermitian. Hence let us write tangent vectors as $\xi_A^*(P) = [A, P] \in T_P\mathcal{O}_P$. With this identification in mind, the Kirillov-Kostant-Souriau symplectic form on $\mathcal{O}_B$ becomes

$$\omega_P([S,P],[T,P]) = -\langle P,[S,T]\rangle = -\mathrm{Tr}(P[S,T]).$$

4.3.2 Rotation of the Orbit

Let us come back to the left hand side of (4.2). As we consider A and B to be hermitian rather than anti-hermitian, as are the elements of $\mathfrak{u}$, we define

$$M = i\mathcal{O}_B,$$

and also the diagonal map

$$\Phi : M \to \mathbb{R}^n, \quad (c_{i,j})_{1\le i,j\le n} \mapsto \begin{bmatrix} c_{11} \\ \vdots \\ c_{nn} \end{bmatrix}.$$

Then recalling we may take $A = \mathrm{diag}(a)$ for $a \in \mathbb{R}^n$, we have for any matrix B that $\mathrm{Tr}(AB) = a \cdot \Phi(B)$. Hence we can rewrite the integral as

$$\int_{U(n)} e^{it\mathrm{Tr}(AUBU^*)}\, dU = \int_M e^{ita\cdot\Phi(x)}\, d\mu(x)$$

for μ some Haar measure on M. The discussion about coadjoint orbits lets us interpret M as a symplectic manifold; Letting $iP \in i\mathcal{O}_P$, we have that tangent vectors in $T_{iP}i\mathcal{O}_P$ are of the form iv for $v \in T_P\mathcal{O}_P$.

Indeed, letting γ a path in $\mathcal{O}_P$, we have that $i\gamma$ is a path in M, and thus $i\dot{\gamma}(0)$ is a tangent vector to M.

Recall that v is of the form $[S, P]$ for some $S \in \mathfrak{u}$, that is, S an anti-hermitian matrix. Hence define the symplectic form on M such that multiplication by i becomes a symplectomorphism:

$$\omega_{iP}(i[S,P], i[T,P]) := \omega_P([S,P],[T,P]) = -\mathrm{Tr}(P[S,T]).$$

Thus if we just write $B = iP \iff P = -iB$ for a hermitian matrix B, we may write tangent vectors as $i[S,P] = [S,iP] = [S,B]$, and the symplectic form becomes

$$\omega_B([S,B],[T,B]) = \mathrm{Tr}(iB[S,T]).$$

The next step is to interpret Φ as a moment map for a suitable action.

Proposition 4.12. *Let $a \in \mathbb{R}^n$ and $X_a \in \mathfrak{X}(M)$ defined by $X_a(P) = [i\mathrm{diag}(a), P]$. Then $d(a\cdot\Phi) = -\imath_{X_a}\omega$.*

Proof. Differentiate the map $\Phi^a = a \cdot \Phi$ at an arbitrary vector field $Y \in \mathfrak{X}(M)$, which we can take to be of the form $Y(P) = [S(P), P]$ for $S : M \to \mathfrak{u}$ smooth.

Notice first that $a \cdot \Phi(P) = \mathrm{Tr}(\mathrm{diag}(a)P)$, so that $a \cdot \Phi = \mathrm{Tr} \circ \ell_{\mathrm{diag}(a)}$, where $\ell_{\mathrm{diag}(a)}$ denotes left multiplication by $\mathrm{diag}(a)$. Thus

$$\begin{aligned} D(a \cdot \Phi)(P)[Y(P)] &= D\mathrm{Tr}(\mathrm{diag}(a)P) D\ell_{\mathrm{diag}(a)}(P)[[S(P), P]] \\ &= \mathrm{Tr}(\ell_{\mathrm{diag}(a)}[S(P), P]) \\ &= \mathrm{Tr}(\mathrm{diag}(a)[S(P), P]) \end{aligned}$$

since both Tr and $\ell_{\mathrm{diag}(a)}$ are linear. Using the cyclical symmetries of the trace, we obtain the identities

$$\mathrm{Tr}(A[P, S]) = \mathrm{Tr}(P[S, A]) = \mathrm{Tr}(S[A, P]),$$

which holds for arbitrary square matrices. Thus

$$\begin{aligned} \mathrm{Tr}(\mathrm{diag}(a)[S(P), P]) &= \mathrm{Tr}(P[\mathrm{diag}(a), S(P)]) \\ &= -\mathrm{Tr}(iP[i\mathrm{diag}(a), S(P)]) \\ &= -\omega_P([i\mathrm{diag}(a), P], [P, S(P)]) \\ &= -\omega_P(X_a(P), Y(P)). \end{aligned}$$

As Y was arbitrary, we conclude

$$d\Phi^a = -\imath_{X_a}\omega. \tag{4.3}$$

□

We now restrict the conjugation action by $U(n)$ to an appropriate subgroup which we can identify with a torus, such that its moment map is Φ. Consider

$$T := \{D \in U(n) \mid D \text{ diagonal, and } \det(D) = 1\}.$$

As $T \subset U(n)$, its diagonal entries must have absolute value 1. The map

$$\begin{aligned} \tau : \mathbb{T}^{n-1} &\longrightarrow T \\ (\theta_1, \dots, \theta_{n-1}) &\longmapsto \mathrm{diag}(\theta_1, \dots, \theta_{n-1}, \theta) \end{aligned}$$

for $\theta = (\prod_{j=1}^{n-1} \theta_j)^{-1}$ is a group homomorphism and evidently bijective.

Its Lie algebra $\mathfrak{t}$ must be a subset of the anti-hermitian diagonal matrices in $\mathfrak{u}$; an arbitrary curve in T can be written as $\gamma(t) = \tau(e^{i\theta_1(t)}, \dots, e^{i\theta_{n-1}(t)})$ for some arbitrary functions $\theta_j : \mathbb{R} \to \mathbb{R}$ with $\theta_j(0) = 0$. If $\dot{\theta}_j(0) = v_i$, then for $e^{i\theta(t)} = (\prod_{j=1}^{n-1} e^{i\theta_j(t)})^{-1} = e^{-i\sum_{j=1}^{n-1}\theta_j(t)}$, we must have

$$\left.\frac{d}{dt}\right|_{t=0} e^{i\theta(t)} = -i\sum_{j=1}^{n-1} v_i.$$

From this we see that the Lie algebra of T consists of all anti-hermitian diagonal matrices of trace zero. A diagonal matrix being anti-hermitian is equivalent to its entries being purely imaginary, so that we can identify $\mathfrak{t}$ with a subset of $\mathbb{R}^n$ by dividing by i and taking the diagonal. Let us state this as a lemma for clarity.

Lemma 4.13. *The Lie algebra of T is the set*

$$\mathfrak{t} = \{A \in \mathfrak{u} \mid \mathrm{Tr}(A) = 0, A \text{ diagonal}\},$$

which we may identify with $\mathbb{R}^{n-1}$ via the map

$$\nu : \mathbb{R}^{n-1} \to \mathfrak{t}, \qquad (v_1, \ldots, v_{n-1}) \mapsto i\mathrm{diag}(v_1, \ldots, v_{n-1}, v)$$

for $v = \sum_{j=1}^{n-1} v_j$.

Verifying that X_a is the fundamental vector field for the action by this subgroup will establish Φ as a moment map.

Lemma 4.14. *X_a is the vector field generated by the action of T on M by conjugation.*

Proof. Recall that for the action by $U(n)$ and $S \in \mathfrak{u}$, the fundamental vector field is just

$$\xi^*_S(P) = [S, P].$$

Let $v \in \mathfrak{t} \cong \mathbb{R}^{n-1}$ and $a = \nu(v) = (v, -\sum_i v_i) \in \mathbb{R}^n$. Then the vector field generated by the action restricted to T is

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(t\nu(v))}(P) = [\nu(v), P] = [i\mathrm{diag}(a), P] = X_a(P).$$

□

It is now quick to verify the conditions to use the Stationary Phase Duistermaat-Heckman theorem; we need to verify M is compact and to give a description of the nondegenerate elements of $\mathfrak{t}$.

Proposition 4.15. *The manifold M has dimension $n^2 - n$ and is compact.*

Proof. As the map $U \mapsto UU^*$ is continuous and has kernel $U(n)$, we have that $U(n)$ is closed. Thus any coadjoint orbit $\mathcal{O}_P$ is also closed, since any path in $\mathcal{O}_P$ is of the form $U(t)PU(t)^*$ for $U(t)$ a path in $U(n)$. Hence as the limit of $U(t)$ lies in $U(n)$, so does the limit of $U(t)PU(t)^*$ lie in $\mathcal{O}_P$.

Any coadjoint orbit by $U(n)$ is bounded since $\|UPU^*\|^2 = \mathrm{Tr}(UP^*U^*UPU^*) = \mathrm{Tr}(UP^*PU^*) = \mathrm{Tr}(P^*P) = \|P\|^2$.

Hence any coadjoint orbit by $U(n)$ is compact, and so $M = i\mathcal{O}_B$ is also compact.

As for the dimension of M, recall by proposition 4.7 and the proof of theorem 4.10, that the tangent space of the orbit admits the identification $T_B\mathcal{O}_B \cong \mathfrak{u}/\mathfrak{u}_B$, and so the dimension of the tangent space, which coincides with the dimension of $\mathcal{O}_B$ and M, is given by $\dim(\mathfrak{u}) - \dim(\mathfrak{u}_B)$. We have that $\dim(\mathfrak{u}) = n^2$. The stabilizer is given by

$$\mathrm{Stab}_{U(n)}(B) = \{U \in U(n) \mid \mathrm{Ad}^*_U(B) = UBU^* = B\}.$$

This condition is equivalent to $UB = BU$. As we took B to be diagonal and without multiple eigenvalues, U commutes with B if and only if U is also diagonal. The unitary diagonal matrices can be identified with $\mathbb{T}^n$, so that $\dim(\mathrm{Stab}_{U(n)}(B)) = \dim(\mathfrak{u}_B) = n$, and hence

$$\dim(M) = \dim(T_B\mathcal{O}_B) = n^2 - n.$$

□

4.3.3 Critical Values of the Moment Map

Suppose $P \in M$ is a critical value of Φ^a for $a \in \mathfrak{t}$, that is,

$$0 = d\Phi^a|_P = -\omega_P([i\mathrm{diag}(a), P], \cdot).$$

Thus by nondegeneracy, $d\Phi^a|_P = 0 \iff [i\mathrm{diag}(a), P] = 0 \iff P$ commutes with $i\mathrm{diag}(a)$. This is the case if and only if P is block diagonal (diagonal if a has no multiple entries).

On the other hand, the fixed points of the action are those for which $DPD^* = P \iff DP = PD$ for all $D \in T$. Hence as T contains diagonal matrices without multiple entries, P is a fixed point if and only if P is diagonal.

As we also took $B = \mathrm{diag}(b)$ to be diagonal, and as unitary diagonalization matrices are unique up to permutations of their columns, we have that the diagonal matrices in M consist of

$$\mathrm{Fix}(M) = \left\{ \begin{pmatrix} b_{\sigma(1)} & & \\ & \ddots & \\ & & b_{\sigma(n)} \end{pmatrix} \mid \sigma \in S_n \right\}.$$

Evidently, this is a subset of the block diagonal matrices in M, and coincides with the critical points of Φ^a if a has no multiple entries. This proves

Proposition 4.16. *The nondegenerate elements $a \in \mathfrak{t}$, that is, those such that $X_a(P) = 0 \iff P$ is fixed by T, are those elements with pairwise distinct entries.*

We are now in place to use the Duistermaat-Heckman theorem. It tells us that, for $a \in \mathfrak{t}$ nondegenerate,

$$\left(\frac{t}{2\pi i}\right)^{\frac{n^2-n}{2}} \int_M e^{it\langle\Phi,\mathrm{diag}(a)\rangle} \underbrace{\frac{\omega^{(n^2-n)/2}}{((n^2-n)/2)!}}_{=:\mu} = \sum_{\sigma\in S_n} \frac{e^{it\langle\Phi(\mathrm{diag}(\sigma(b))),\mathrm{diag}(a)\rangle}}{\prod_j \alpha_j(\sigma(b), a)},$$

which we rearrange to obtain

Theorem 4.17. *Let $A = \mathrm{diag}(a)$ without multiple eigenvalues and trace zero, $B = \mathrm{diag}(b)$ without multiple eigenvalues, and $t \in \mathbb{R}$. Then*

$$\int_M e^{it\, a\cdot\Phi}\mu = C_{A,B} \sum_{\sigma\in S_n} e^{ita\cdot\sigma(b)} t^{-\frac{n^2-n}{2}}$$

for $M = \{iUBU^* \mid U \in U(n)\}$, $\Phi : M \to \mathbb{R}^n$ the diagonal map, and μ the Liouville-measure associated to the Kostant-Kirillov-Souriau symplectic form on M, and

$$C_{A,B} := \left((2\pi i)^{\frac{n^2-n}{2}} \sum_{\sigma \in S_n} \frac{1}{\prod_{j=1}^{n-1} \alpha_j(\sigma(b), a)} \right).$$

Corollary 4.18. *The Harish-Chandra-Itzykson-Zuber integral formula holds.*

Proof. Using the discussion just after the proof of proposition 4.2, all that is left to show is that

$$\int_{U(n)} e^{it\mathrm{Tr}(AUBU^*)} dU = C \int_M e^{ita\cdot\Phi(x)} d\mu(x)$$

for some constant C, possibly modifying the constant $C_{A,B}$ from above. For A and B diagonal and $x = UBU^*$, $a \cdot \Phi(x) = \mathrm{Tr}(AUBU^*)$, so we may as well consider the integral on the right hand side to be over $U(n)$ instead. We show that μ is a Haar measure on M, giving that it must be a multiple of the Haar probability measure, which implies the claim.

This means we must show μ is $U(n)$-invariant. This follows quickly from the conjugation invariance of the trace:

$$\begin{aligned}
((\mathrm{Ad}_U^*)\omega)_B([S,B],[T,B]) &= \omega_{UBU^*}(U[S,B]U^*, U[T,B]U^*) \\
&= \omega_{UBU^*}([USU^*, UBU^*], [UTU^*, UBU^*]) \\
&= \mathrm{Tr}(iUBU^*[USU^*, UTU^*]) \\
&= \mathrm{Tr}(iUBU^*U[S,T]U^*) \\
&= \mathrm{Tr}(iUB[S,T]U^*) \\
&= \mathrm{Tr}(iB[S,T]) \\
&= \omega_B([S,B],[T,B]).
\end{aligned}$$

Hence ω, and by extension μ, are $U(n)$-invariant. $\square$

Bibliography

[1] Ana Cannas da Silva. *Lecture Notes on Symplectic Geometry.* Springer-Verlag Berlin Heidelberg, 2001. ISBN: 9783540453307.

[2] Victor Guillemin and Shlomo Sternberg. *Symplectic Techniques in Physics.* Cambridge University Press, 1990. ISBN: 978-0521389907.

[3] Will J. Merry. *Lecture Notes on Differential Geometry.* URL: `https://drive.google.com/file/d/1DE0X9YS3RlfW_FpBt-HGH8mwtY_w2Vhj/view` (visited on 05/10/2019).

[4] Dan Ramras (https://mathoverflow.net/users/4042/dan-ramras). *On a proof of the existence of tubular neighborhoods.* MathOverflow. URL: `https://mathoverflow.net/q/58124%20(version:%202017-04-13)` (visited on 25/11/2019).

[5] Encyclopedia of Mathematics. *Invariant Metric.* URL: `https://www.encyclopediaofmath.org/index.php/Invariant_metric` (visited on 05/10/2019).

[6] Ana Cannas da Silva. *Seminar on Symplectic Toric Manifolds.* URL: `http://www.vvz.ethz.ch/Vorlesungsverzeichnis/lerneinheit.view?semkez=2019S&ansicht=ALLE&lerneinheitId=130781&lang=en` (visited on 28/05/2019).

[7] Manfred Einsiedler and Thomas Ward. *Functional Analysis, Spectral Theory, and Applications.* Springer International Publishing, 2017.

[8] Victor Guillemin. *Moment Maps and Combinatorial Invariants of Hamiltonian $\mathbb{T}^n$-Spaces.* Birkhäuser, 1994. ISBN: 978-0-8176-3770-5.

[9] Liviu Nicolaescu. *Lectures on the Geometry of Manifolds.* 2018. URL: `https://www3.nd.edu/~lnicolae/Lectures.pdf` (visited on 25/11/2019).

[10] Joël Beimler. "Classification of Surfaces Equipped with Area Forms". ETH Zürich, 2019.

[11] Terence Tao. *The Harish-Chandra-Itzykson-Zuber integral formula.* URL: `https://terrytao.wordpress.com/2013/02/08/the-harish-chandra-itzykson-zuber-integral-formula/` (visited on 18/11/2019).

[12] Yasushi Honma. シンプレクテイック幾何入門. URL: `http://www.f.waseda.jp/homma_yasushi/homma2/download/symplectic-intro.pdf` (visited on 18/11/2019).