Morse theory and symplectic toric manifolds

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semester project supervised by Prof. Ana Cannas Da Silva

Contents

1 Introduction

The following semester project covers some aspects of Morse theory. The work is divided in three sections:

Section 1 introduces the basics of Morse theory. Following Milnor's book we introduce Morse functions, show their basic properties and prove that critical points of Morse functions determine the homotopy type of closed manifolds. We also prove Morse inequalities and conclude the section by showing that not only Morse functions exist on any closed manifold, but they are actually dense.

Section 2 introduces a class of manifolds often considered in symplectic geometry due to the nice properties of its objects: toric symplectic manifolds. We will briefly explain how toric symplectic manifolds can be completely described in terms of so-called Delzant polytopes, which are constructed via the moment map. The polytope associated to a toric symplectic manifold M is a considerably easier object to study and this will become apparent in section 3.

In the final section we combine Morse theory and toric symplectic manifolds. We use the moment map coming from the Hamiltonian action to produce interesting Morse functions and use the results from section 1 to obtain information about the structure of the homology ring.

The idea of using the moment map as a Morse function originally appeared in T. Frankel's work "Fixed points and torsion on Kähler manifolds", where a broader class of functions (admitting critical, non-degenerate submanifolds) was considered.

Acknowledgments

I am very thankful to Prof. Cannas Da Silva for introducing me to Morse theory, symplectic toric manifolds as well as for supervising this project. Her support throughout the semester was very valuable and the conversations we had extremely helpful.

2 Morse theory

In this chapter we introduce Morse functions and use Morse theory to prove that manifolds can be obtained as CW complexes. We then show standard Morse inequalities and finally prove that Morse functions are generic. We closely follow Milnor's book [**Mil**].

2.1 Basics and main results

Definition 2.1. For a smooth function $f : M \to \mathbb{R}$ where M is a smooth manifold, a point $x \in M$ is said to be a **critical point** if $Df(x) : T_xM \to T_{f(x)}\mathbb{R} = 0$. The value $f(x)$ is called a **critical value**.

Definition 2.2. Associated to a critical point x of f we define the **Hessian** of f at x by

$$H_f(v,w)(x) = X(Y(f))(x)$$

where $v, w \in T_xM$ and X, Y are arbitrary vector fields on M satisfying $X(x) = v$ and $Y(x) = w$.

The Hessian is symmetric since

$$X(Y(f))(x) - Y(X(f))(x) = [X,Y](f)(x) = df(x)[X,Y] = 0$$

(using that x is a critical point) and since $X(f)(x)$ is independent of the choice of X, it is a well-defined symmetric bilinear form. Notice that with a choice of basis for T_xM the Hessian can be represented by the matrix $(\frac{\partial^2 f}{\partial_i \partial_j})_{ij}$.

Definition 2.3. A critical point x of $f : M \to \mathbb{R}$ is called **non-degenerate** if the Hessian $H_f(x)$ is non-degenerate as a bilinear form.

Definition 2.4. The **index** of a bilinear form H on a vector space V is the maximal n such that there exists a subspace $W \subset V$ of dimension n with $H|_V$ being negative definite. The **nullity** of H is the dimension of the subspace $\{v \in V \mid H(v,w) = 0 \text{ for all } w \in V\}$.

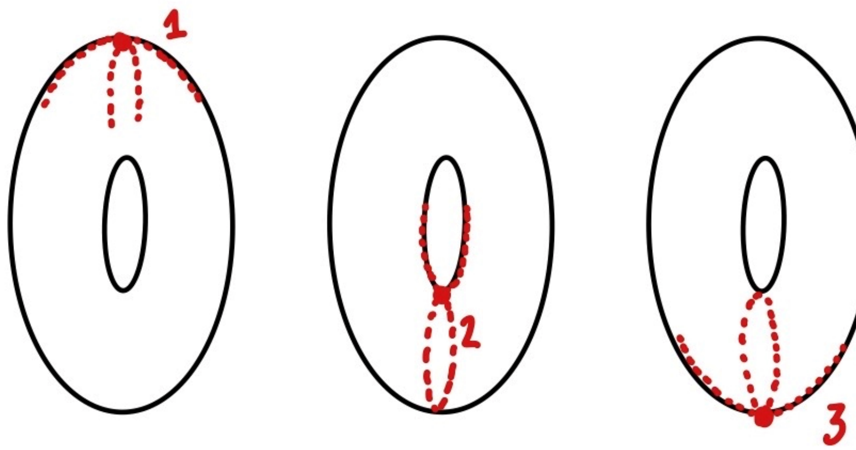
The nullity of $H_f(x)$ in the setting above will be simply denoted by the nullity of f at x. We start by showing that the function around non-degenerate critical points can be described in terms of a special coordinate system:

Lemma 2.1 (Lemma of Morse)**.** *Let x be a non-degenerate critical point for f. There exists a chart φ on a neighborhood U of x, $\varphi(x) = 0$ and*

$$f(y) = f(x) - (y^1)^2 - ... - (y^\lambda)^2 + (y^{\lambda+1})^2 + ... + (y^n)^2$$

where λ denotes the index of f at x.

Remark. Consider the following picture to see an example of critical points with different indices. In case 1, the index is 2, in case 2 the index in 1 and in the last image, the index is 0.



Proof. As a first step, notice that if such φ exists, then in the basis of T_xM associated to the chart, the Hessian is represented by

$$\begin{pmatrix} -2 \cdot 1_\lambda & 0 \\ 0 & 2 \cdot 1_{n-\lambda} \end{pmatrix}.$$

It follows that the index of f at x must be at least λ and since the Hessian is positive definite on the complementary subspace, the index has to be equal to λ. Here we use that the index of a quadratic form is preserved by pull-backs. In order to show existence of such chart, let us start by picking an arbitrary coordinate system with $(x) = 0$. Consider

$$f(x^1, ..., x^n) = f(0) + \sum_i x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx^1, ..., tx^n)$$

$$= f(0) + \sum_{i=1}^n x_i \sum_{j=1}^n \int_0^1 \int_0^1 x_j \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i}(tsx^1, ..., tsx^n)$$

$$= f(0) + \sum_{i,j=i}^n x_i x_j h_{ij}(x^1, ..., x^n)$$

for appropriate smooth functions h_{ij} which we can assume w.l.o.g. to be symmetric ($h_{ij} = h_{ji}$). Notice moreover that by definition of h_{ij},

$$h_{ij}(0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0, ..., 0)$$

which is non-degenerate by assumption.
To conclude it remains to diagonalize the quadratic form in a neighborhood of 0. By the Principal Axes Theorem there exists an orthogonal transformation $Q \in \mathrm{Mat}_n(\mathbb{R})$ such that for coordinates $(z) = Q^T(y)$,

$$f(y^1, ..., y^n) = f(0) + \lambda_1 (z^1)^2 + ... + \lambda_n (z^n)^2$$

for the eigenvalues λ of the quadratic form. Notice that in this notation Q depends on the point (y). Re-scaling the coordinates (z) by these eigenvalues, i.e.

$$(z') = D \cdot Q^T(y)$$

$$D = \mathrm{diag}(|\lambda_1|^{-1}, ..., |\lambda_n|^{-1})$$

we obtain

$$f(z') = f(x) - (z'^1)^2 - ... - (z'^\lambda)^2 + (z'^{\lambda+1})^2 + ... + (z'^n)^2.$$

The matrix D is well-defined in a neighborhood of 0 by the non-degeneracy assumption. This concludes the proof.

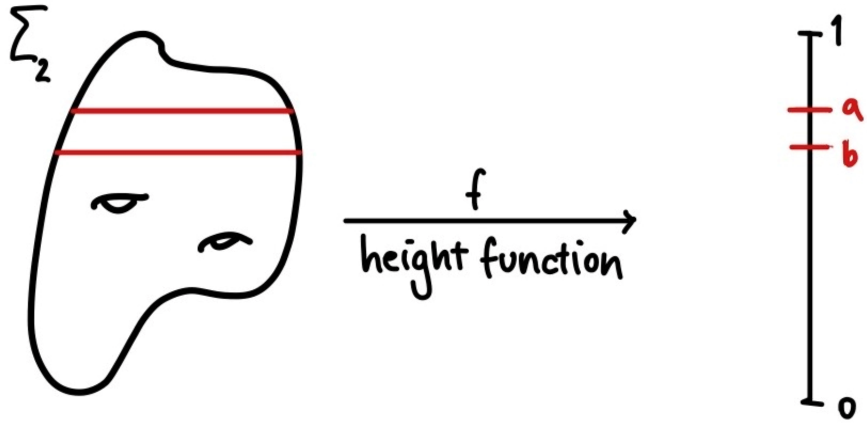
$\square$

Corollary 2.1.1. *Non-degenerate critical points are isolated and Morse functions on compact manifolds have at most finitely many critical points.*

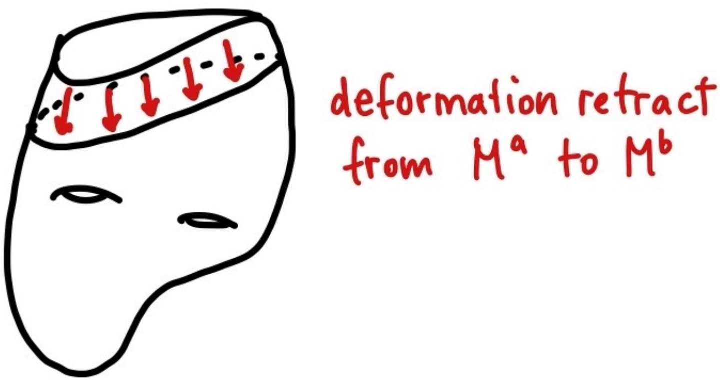
Definition 2.5. Let M be a smooth manifold and $f : M \to \mathbb{R}$ be a smooth function. We denote by M^a the subset $\{f \leq a\}$.

Critical points of Morse functions $f : M \to \mathbb{R}$ can be used to understand the topology of the manifold. Roughly speaking, at critical values of f the homotopy type of M^a (where a is a critical value) changes by attaching a cell of dimension equal to the index of the critical point.

Theorem 2.2. *Let $f \in C^\infty(M, \mathbb{R})$ and $a < b \in \mathbb{R}$. If the set $f^{-1}([a, b])$ is compact and contains no critical points of f, then M^a is diffeomorphic to M^b and the inclusion $M^a \hookrightarrow M^b$ is a deformation retract.*



Remark. One could consider the height function on a surface of genus two as depicted above. The function is Morse and, as long as we stay away from the critical points, sublevel sets are deformation retracts of each other.



The main idea in the proof will be to let the points flow along the gradient of f, as shown by the red arrows.

Proof. Endow M with an arbitrary Riemannian metric g and define

$$p(x) = \rho(x) \frac{1}{g(\nabla f(x), \nabla f(x))}$$

where ρ is a smooth function taking values 1 on $f^{-1}([a, b])$ and vanishing outside a compact set (which we assume small enough for the expression to be well-defined).
Define the vector field

$$X(x) = p(x) \nabla f(x),$$

and denote by $\varphi : \mathbb{R} \times M \to M$ its flow which is complete since X has compact support. Consider the values of the function along the flow:

$$\left.\frac{d}{dt}\right|_{t=t_0} f(\varphi_t(x)) = Df(\varphi_{t_0}(x))[X(\varphi_{t_0}(x))] = \rho(\varphi_{t_0}(x)).$$

It follows that $\varphi_{b-a}(M^a) = M^b$, thus proving that M^a and M^b are diffeomorphic. It remains to show that M^a is a deformation retract of M^b. Consider the map

$$F : M^b \times I \to M$$

$$F(x,t) = \begin{cases} \varphi_{t(a-f(x))}(x) \text{ if } f(x) \geq a \\ x \text{ otherwise .} \end{cases}$$

Then F is smooth, $F(x,0) = x$ and $F(x,1) \in M^a$ and $F(y,1) = y$ for all $y \in M^a$. $\square$

Remark. The compactness assumption cannot be omitted. Indeed, removing single points from the example above provides a counterexample.

Theorem 2.3. *Let $f : M \to \mathbb{R}$ be a smooth function and let x be a non-degenerate critical point of index λ. If for some $\epsilon > 0$ the set $f^{-1}([f(x) - \epsilon, f(x) + \epsilon])$ is compact and only contains the critical point x, then $M^{f(x)+\delta}$ has the homotopy type of $M^{f(x)-\delta}$ with a λ-cell attached for all sufficiently small δ.*

Proof. Let us start by choosing a coordinate system as in Lemma 2.1. In these coordinates the critical point $x \in M$ corresponds to $(0, ..., 0)$ and the function f takes the form

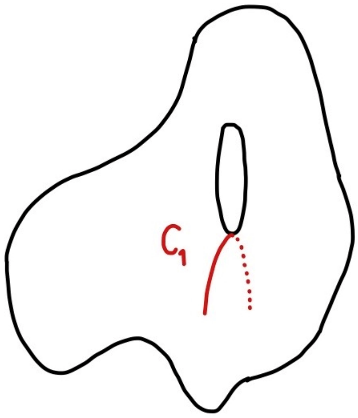
$$f(y) = f(x) - (y^1)^2 - ... - (y^\lambda)^2 + (y^{\lambda+1})^2 + ... + (y^n)^2. \tag{1}$$

Assume that for some $\epsilon > 0$ the set $f^{-1}([f(x) - \epsilon, f(x) + \epsilon])$ is compact and contains no other critical points. Up to choosing a smaller ϵ, we can assume that the chart coming from Lemma 2.1, $\varphi : U \subset M \to V \subset \mathbb{R}^n$ satisfies[1] $B_{2\epsilon}(0) \subset \varphi(U)$. We define the embedded λ-cell

$$C_\lambda = \{y \in U \mid (y^1)^2 + ... + (y^\lambda)^2 \leq \epsilon \text{ and } y^{\lambda+1} = ... = y^n = 0\}.$$

One could think of this cell as the "downward-pointing piece" around the critical point x, as sketched in the picture below.

[1] $B_r(x)$ denotes the closed ball of radius r around x.



Combining this with (1), we obtain $\partial C_\lambda^{f(x)-\epsilon} = \partial C_\lambda$ so that C_λ is attached to $M^{f(x)-\epsilon}$.
Claim 1: $M^{f(x)-\epsilon} \cup C_\lambda$ is a deformation retract of $M^{f(x)+\epsilon}$. To prove the claim we want to apply Theorem 2.2 to a suitably chosen function g.
We start by defining a smooth function

$$\mu : \mathbb{R} \to \mathbb{R}_{\geq 0},$$

satisfying

$$\begin{cases} \mu(0) > \epsilon, \\ \mu(x) = 0 \text{ for all } x \geq 2\epsilon, \\ -1 < \mu'(x) \leq 0 \text{ for all } x \in \mathbb{R}. \end{cases}$$

To define g, set

$$g = \begin{cases} g(y) = f(y) \text{ for all } y \notin U \\ f(y) - \mu\big((y^1)^2 + ... + (y^\lambda)^2 + 2(y^{\lambda+1})^2 + ... + 2(y^n)^2\big) \text{ for } y \in U \end{cases}$$

In order to make the notation more transparent, we also define

$$\xi, \eta : U \to \mathbb{R}$$

$$\xi(y) = (y^1)^2 + ... + (y^\lambda)^2$$

$$\eta(y) = (y^{\lambda+1})^2 + ... + (y^n)^2$$

and rewrite the functions f, g as

$$f(y) = f(x) - \xi(y) + \eta(y)$$

$$g(y) = f(x) - \xi(y) + \eta(y) - \mu(\xi(y) + 2\eta(y)).$$

It follows immediately from the definition of μ that the two functions f and g agree outside a neighborhood O of the critical point x corresponding to a 2ϵ ball in the chart φ.

Claim 2: $\{g^{-1}((-\infty, f(x)+\epsilon])\} = \{f^{-1}((-\infty, f(x)+\epsilon])\} = M^{f(x)+\epsilon}$.
In order to see this, we start by noticing that $g = f$ outside $\{\xi + 2\eta \leq 2\epsilon\}$. On the complement of this region, $g \leq f \leq c + \epsilon$.
Claim 3: g and f have the same critical points.
In the region $\{\xi + 2\eta \leq 2\epsilon\}$ we can compute the differential

$$dg = \frac{\partial g}{\partial \xi} d\xi + \frac{\partial g}{\partial \eta} d\eta = (-1 - \mu'(\xi + 2\eta))d\xi + (1 - 2\mu'(\xi - 2\eta))d\eta$$

by the assumption on μ', this can only be zero if $d\eta = 0$, i.e. the only critical point in this region is the origin. It follows from Claim 2 and $g \leq f$ that $g^{-1}([f(x)-\epsilon, f(x)+\epsilon])$ can only contain x as a critical point, but $g(x) < f(x) - \epsilon$ so that $g^{-1}([f(x)-\epsilon, f(x)+\epsilon])$ does not contain critical points. This enables us to invoke Theorem 2.2 and obtain:
Claim 4: $g^{-1}((-\infty, f(x)-\epsilon])$ is a deformation retract of $g^{-1}((-\infty, f(x)+\epsilon]) = M^{f(x)+\epsilon}$.
We introduce the notation

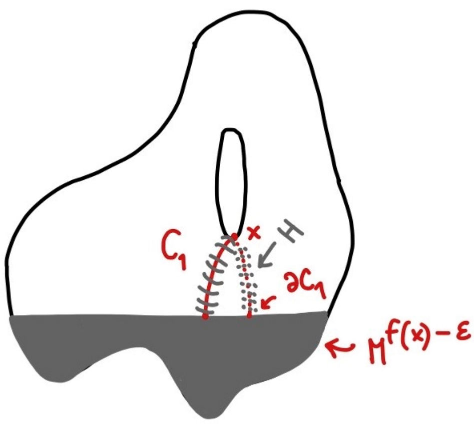
$$H = \overline{g^{-1}((-\infty, f(x)-\epsilon]) - M^{f(x)-\epsilon}}$$

so that we can write

$$g^{-1}((-\infty, f(x)-\epsilon]) = M^{f(x)-\epsilon} \cup H$$

(H is sometimes referred to as a handle). Notice that the λ-cell C_λ introduced above can now be described by the more concise equations $C_\lambda = \{y \in U \mid \xi(y) \leq \epsilon \text{ and } \eta(y) = 0\}$.

Claim 5: $C_\lambda \subset H$. Indeed, $g(y) \leq g(x) < f(x) - \epsilon$ for all $y \in U$ but $f(y) \geq f(x) - \epsilon$ for all $y \in C_\lambda$ by definition of the cell combined with equation (1).



To conclude the proof of the theorem we show the following,
Claim 6: $M^{f(x)-\epsilon} \cup C_\lambda$ is a deformation retract of $M^{f(x)-\epsilon} \cup H$ which, combined with

Claim 4 yields that the homotopy type of $M^{f(x)+\epsilon}$ is the one obtained from $M^{f(x)-\epsilon}$ by attaching a λ-cell. We need to distinguish between three different cases:

$$F(y,t) = \begin{cases} \text{id outside of } U \\ (y^1, ..., y^\lambda, ty^{\lambda+1}, ..., ty^n) \text{ if } \xi(y) \leq \epsilon \\ (y^1, ..., y^\lambda, c_t y^{\lambda+1}, ..., c_t y^n) \text{ if } \epsilon \leq \xi \leq \eta + \epsilon \text{ where } c_t = t + (1-t)\sqrt{\frac{\xi-\epsilon}{\eta}} \\ \text{id if } \eta + \epsilon \leq \xi \end{cases}.$$

This concludes the proof.

□

Theorem 2.3 can be used to prove the following result:

Theorem 2.4. *Let $f : M \to \mathbb{R}$ be a smooth function with no degenerate critical points and assume that M^a is compact for all $a \in \mathbb{R}$. Then M is homotopy equivalent to a CW-complex with one cell of dimension λ for each critical point of dimension λ.*

Another related result relating the critical points of Morse functions to the topology of the domain is the following result by Reeb.

Theorem 2.5 (Reeb)**.** *Let $f : M \to \mathbb{R}$ be a smooth function on a compact manifold such that f has exactly two critical points, both of them non-degenerate. Then M is homeomorphic to a sphere.*

Proof. By compactness of M the two critical points must be the minimum and the maximum which can be assumed to satisfy $f(x_{max}) = 1$ and $f(x_{min}) = 0$ by re-scaling and translating. By Lemma 2.1 the indices must be 0 and n respectively and by Theorem 2.3

$$f^{-1}([0,\epsilon]) = \{y \mid ((y^1)^2 + ... + (y^n)^2) \in [0,\epsilon]\}$$

$$f^{-1}([1-\epsilon,\epsilon]) = \{y \mid 1 - ((y^1)^2 + ... + (y^n)^2) \in [1-\epsilon, 1]\}$$

are both homeomorphic to closed n-cells.It follows from Theorem 2.3 that M is the union of two n-cells attached along their boundaries and therefore homeomorphic to the sphere.

□

2.2 Morse inequalities

Morse inequalities allow us to understand the homology rings of a given space M using the critical points of Morse functions $f : M \to \mathbb{R}$.

Definition 2.6. A function S is called **subadditive** if

$$S(X, Z) \leq S(X, Y) + S(Y, Z)$$

for all $\subset Y \subset Z$. The function is called **additive** if equality holds, i.e.

$$S(X, Z) = S(X, Y) + S(Y, Z).$$

Remark. Subadditive functions we are interested include

$$R_n(X, Y) = \operatorname{rank}(H_n(X, Y; \mathbb{F}))$$

for any dimension $n \in \mathbb{N}$ and field $\mathbb{F}$. The subadditivity can be seen using the long exact sequence of the tuple (X, Y, Z)

$$\ldots \to H_n(Y, Z) \to H_n(X, Z) \to H_n(X, Y) \to \ldots$$

The Euler characteristic is additive.

Lemma 2.6. *Let S be subadditive and $X_0 \subset X_1 \subset \ldots \subset X_n$. Then $S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_{i-1})$. Equality holds for additive functions.*

Proof. We proceed by induction, $n = 2$ being true by definition. Assume the result holds for $n-1$. Then $S(X_n, X_0) \leq S(X_{n-1}, X_0) + S(X_n, X_{n-1}) \leq \sum_{i=1}^{n} S(X_i, X_{i-1})$. □

Definition 2.7. A Morse function f is said to be **perfect** if the inequalities in Theorem 2.7 are all equalities.

Let $f : M \to \mathbb{R}$ be a smooth function with isolated non-degenerate critical points. Let $a_1 < \ldots < a_k$ be such that $M = M^{a_i}$ contains i critical points and M^{a_k}. Then

$$H_*(M^{a_i}, M^{a_{i-1}}) = H_*(M^{a_{i-1}} \cup C_{\lambda_i}, M^{a_{i-1}}) \overset{\text{excision}}{=} H_*(C_{\lambda_i}, \partial C_{\lambda_i}) = \begin{cases} \mathbb{F} \text{ if } \lambda_i \neq 0 \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

where λ_i denotes the index of the critical point contained in $M^{a_i} - M^{a_{i-1}}$.

Theorem 2.7 (Weak Morse inequalities)**.** *Let N_λ denote the number of critical points of index λ of any smooth function $f : M \to \mathbb{R}$, where M is a compact manifold and the critical points of f are non-degenerate. Then*

$$R_\lambda(M) \leq N_\lambda \tag{3}$$

$$\sum(-1)^\lambda R_\lambda(M) = \sum(-1)^\lambda N_\lambda. \tag{4}$$

Proof. Using equation (2) together with Lemma 2.6 and the increasing sequence $\varnothing = M^{a_0} \subset M^{a_1} \subset ... \subset M^{a_k} = M$,

$$R_\lambda(M) \leq \sum_{i=1}^{n} R_\lambda(M^{a_i}, M^{a_{i-1}}) \overset{(2)}{=} N_\lambda.$$

To prove equality (4) we use the Euler characteristic together with Lemma 2.6,

$$\chi(M) = \sum_{i=1}^{n} \chi(M^{a_i}, M^{a_{i-1}}) = \sum_{i=1}^{n} (-1)^\lambda N_\lambda.$$

□

In the setting of Theorem 2.7, we can prove stronger inequalities. We define

$$S_\lambda(X,Y) = R_\lambda(X,Y) - R_{\lambda-1}(X,Y) + R_{\lambda-2}(X,Y) - ... \pm R_0(X,Y).$$

Theorem 2.8 (Morse inequality)**.**

$$S_\lambda(M) \leq \sum_{i=1}^{k} S_\lambda(M) \tag{5}$$

$$R_\lambda(M) - R_{\lambda-1}(M) + ... \pm R_0(M) \leq N_\lambda - N_{\lambda-1} + ... \pm N_0 \tag{6}$$

Proof. Apply Lemma 2.9 to the increasing sequence

$$\varnothing \subset M^{a_1} \subset ... \subset M^{a_k}$$

□

Lemma 2.9. *The function S_λ is subadditive.*

Proof. Consider an exact sequence

$$... \xrightarrow{\phi_1} A \xrightarrow{\phi_2} B \xrightarrow{\phi_3} C \to ...$$

and notice that $\operatorname{rank}(\phi_1) + \operatorname{rank}(\phi_2) = \operatorname{rank}(A)$. We can inductively use this equality to obtain

$$\operatorname{rank}(\phi_1) = \operatorname{rank}(A) - \operatorname{rank}(\phi_2) = ... = \operatorname{rank}(A) - \operatorname{rank}(B) + \operatorname{rank}(C) - \operatorname{rank}(D) \geq 0.$$

Applying this to $\partial : H_{\lambda+1}(X,Y) \to H_\lambda(Y,Z)$,

$$\operatorname{rank}(\partial) = R_\lambda(Y,Z) - R_\lambda(X,Z) + R_\lambda(X,Y) - R_{\lambda-1}(Y,Z) + ... \geq 0.$$

In other words $S_\lambda(Y,Z) - S_\lambda(X,Z) + S_\lambda(X,Y) \geq 0$.

$\square$

Algebraic manipulations of these inequalities allow us to prove the following result:

Corollary 2.9.1. *If* $N_{\lambda+1} = N_{\lambda-1} = 0$ *then* $R_\lambda = N_\lambda$ *and* $R_{\lambda+1} = R_{\lambda-1}$.

2.3 Critical points of Morse functions and orientability

Let S be a closed surface. If S is non-orientable, it can be decomposed into a connected sum of real projective planes $R\mathbb{P}^2$. As a concrete example, one can show that the Klein bottle is homeomorphic to $R\mathbb{P}^2 \# R\mathbb{P}^2$. Following exercise 3.3.6a in [**Hat**], we know that the first homology of the connected sum can be computed as follows:

$$H_1(M_1 \# M_2; \mathbb{Z}) \cong (H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z}))^*$$

where we denote by * the operation of replacing one of the torsion factors $\mathbb{Z}_2$ by the free abelian group $\mathbb{Z}$. In particular, for any surface S containing at least two projective planes in their direct sum decomposition, the rank of $H_1(S; \mathbb{Z})$ is at least one. Combining this information with Theorem 2.7, more precisely with the inequality

$$R_1(S) \leq N_1,$$

we conclude that any Morse function on S must have at least one critical point of index 1 (moreover since we assume S to be compact we also have a maximum and a minimum, i.e. critical points of index 2 and 0 respectively).

2.4 Existence of Morse functions

So far, the main examples of Morse functions have been so called height functions. As we will see, there is an abundance of functions whose critical points are non-degenerate. The first step consists of finding Morse functions for submanifolds of the Euclidean space. We follow section 1.2 of [**AD**].

Proposition 2.10. *Let $M \subset \mathbb{R}^n$ be a submanifold. For almost every point $p \in \mathbb{R}^n$ the function*

$$f_p : M \to \mathbb{R}$$
$$x \mapsto ||x-p||^2$$

is a Morse function.

Proof. The differential of f_p at the point x is the map

$$v \in T_xM \mapsto \langle 2(x-p), v\rangle,$$

which vanishes if and only if the tangent space at x is orthogonal to the vector $x-p$. Consider the Hessian (in local normal coordinates)

$$\frac{\partial^2 f_p}{\partial u_i \partial u_j} = 2\big(\frac{\partial x}{\partial u_i}\cdot\frac{\partial x}{\partial u_j} + (x-p)\cdot\frac{\partial^2 x}{\partial u_i \partial u_j}\big). \tag{7}$$

Non-degenerate critical points correspond to points x having tangent space orthogonal to $p-x$ as well as non-degenerate Hessian.
In order to show that f_p is Morse (the critical points are non-degenerate), for almost every p, we express the points p for which this is not the case in terms of critical values of a smooth enough function. Once we have this characterization, we can conclude by applying Sard's Theorem.
Consider the normal bundle

$$N \subset M \times \mathbb{R}^n$$

and the map

$$E : N \to N$$
$$(x, v) \mapsto x + v.$$

We claim that critical values of E coincide with values $p \in \mathbb{R}^n$ for which the matrix (7) is not invertible.
TO prove the claim, pick local coordinates for N, such that the coordinates induced on TN satisfy the following condition: if We denote by u_i coordinates on M and by t_i coordinates on $\mathbb{R}^\ltimes$, then

$$\forall i, j \langle \frac{\partial}{\partial u_i}, v_j\rangle = 0$$

where $v_j = \frac{\partial}{\partial t_j}$.
The matrix representing the differential of E in the basis induced on the tangent bundle takes the form

$$(\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j}) - \sum_k v_k \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}.$$

This shows that the non-degeneracy of the Hessian for the critical points of f_p is equivalent to p being a critical value for E, which concludes the proof. $\square$

We can now approximate arbitrary smooth functions by Morse function.

Theorem 2.11. *Let M be a compact manifold. Then Morse functions are dense in the space of smooth functions $C^\infty(M)$ (endowed with the compact-open topology).*

Proof. Let f be a smooth function on M. Let $\iota : M \to \mathbb{R}^N$ be an embedding of M into some Euclidean space. Define a new embedding h into a space of higher dimension by

$$h(x) = (f(x), \iota(x)) \in \mathbb{R}^{N+1}.$$

By Proposition 2.10 for almost every $p = (-c + \epsilon_1, ..., \epsilon_{N+1})$ the function f_p is Morse. If this is the case, the function

$$g(x) = \frac{1}{2c}(f_p(x) - c^2)$$

is Morse too. Moreover,

$$g(x) = f(x) + \frac{1}{2c}(f(x)^2 + \sum_{i=1}^{N} \iota_i(x)^2) - \frac{1}{c}(\epsilon_1 f(x) + \sum_{i=1}^{N+1} \epsilon_{i+1}\iota_i) + \sum_{i=2}^{N+1} \epsilon_i^2 - \epsilon_1,$$

which shows that we can choose g arbitrarily closed to f in the compact-open topology. $\square$

3 Toric symplectic manifolds

In this section we introduce Hamiltonian actions, prove some of their properties and then concentrate on Hamiltonian toric actions. These enable us to define toric symplectic manifolds and explain how they can be classified by so-called Delzant polytopes. The main reference for this section is Part B of [**ACL**].

Definition 3.1. An action $\psi : G \times M \to M$ on a symplectic manifold (M, ω) is called a **Hamiltonian action** if there exists a **moment map** $\mu : M \to \mathfrak{g}^*$ satisfying the following conditions:

1. for each $X \in \mathfrak{g}$, $d\mu^X = \omega(X^\#, \cdot)$, where $\mu^X(p) := \mu(p)(X)$ and $X^\#$ is the vector field on M associated to the one-parameter subgroup $\{\exp(tX) \in G\}$,
2. μ is equivariant with respect to the action ψ and the coadjoint action Ad^* of G on $\mathfrak{g}^*$, i.e.
$$\mu \circ \psi_g = Ad_g^* \circ \mu \text{ for all } g \in G.$$

Remark. Let X_H be a Hamiltonian vector field, $dH = \omega(X_H, \cdot)$. Let $\psi : \mathbb{R} \to \mathrm{Diff}(M)$ be the action generated by the flow of the vector field, i.e. $t \mapsto \theta_t$. Let e be the canonical generator of the lie algebra of $\mathbb{R}$. Then

$$e^\#(f)(p) = \frac{d}{dt}\bigg|_{t=0} f(\exp(te) \cdot p) = \frac{d}{dt}\bigg|_{t=0} f(\theta_t(p)) = X_H(p)(f),$$

$$i_{e^\#}\omega = i_{X_H}\omega = dH \overset{!}{=} d\mu.$$

To show that H is a moment map for the action it remains to check equivariance. Notice that the adjoint action is trivial on any abelian Lie group and it therefore suffices to check that the moment map is invariant with respect to the action, i.e. $\mu \circ \psi_g = \mu$. This is clear in our case, since

$$\frac{d}{dt}\bigg|_{t=0} (H(\theta_t(p))) = dH(X_H)(p) = \omega(X_H, X_H)(p) = 0.$$

Remark. The moment map is uniquely determined up to a constant.

It is important to point out that in the higher dimensional case, the definition of Hamiltonian action is not merely requiring to have Hamiltonian vector fields in every direction, but additionally requiring these actions to be compatible with each other.
Consider the following example:
Let $\mathbb{R}^2$ act on $(\mathbb{R}^2, \omega_{\mathrm{std}})$ by translations, i.e.

$$(g, x) \mapsto x + g.$$

The action of every one-dimensional group of $\mathbb{R}^2$ is Hamiltonian, but the action is transitive and if a moment map existed, it would have to be constant.
We call this type of action a **weakly Hamiltonian action**.

Theorem 3.1. *Let (M, ω) be a compact connected symplectic manifold and let $\mathbb{T}^m$ be an m-torus. Suppose $\psi : \mathbb{T}^m \to \mathrm{Sympl}(M, \omega)$ is a Hamiltonian action with moment map μ. Then:*

1. *the levels of μ are connected,*
2. *the image of μ is convex,*
3. *the image of μ is the closed convex hull of the images of the fixed points of the action.*

Definition 3.2. The image $\mu(M)$ is called the **moment polytope**.

Definition 3.3. An action of a group G on a manifold M is said to be **effective** if the map $\psi : G \to \mathrm{Diff}(M)$ is injective.

Remark. For a smooth effective action of a k-dimensional group on a smooth manifold M there exist at least one orbit of dimension k.

Theorem 3.2. *Let $(M, \omega, \mathbb{T}^m, \mu)$ be an effective Hamiltonian action on a symplectic manifold. If the action is effective, then $\dim(M) \geq 2m$.*

Proof. By definition of the moment map, for any $p \in M$

$$\ker(d\mu(p)) = (T_pO)^{\perp},$$

where $^{\perp}$ denotes the symplectic orthogonal complement and O denotes the orbit of the action containing p.
Since the action is assumed to be effective, there exists at least one m-dimensional orbit, which is also an isotropic submanifold. It follows that $\dim(M) \geq 2m$. □

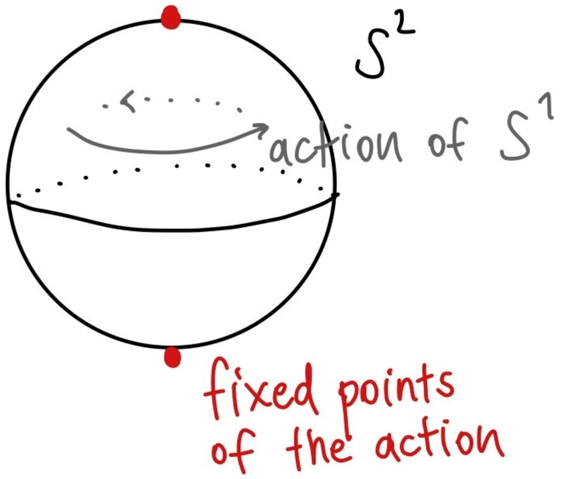
Definition 3.4. A **symplectic toric manifold** is a compact symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian action of a torus $\mathbb{T}^n$ with a choice of moment map μ.

Definition 3.5. Two symplectic toric manifolds are **equivalent** if there exists an isomorphism $\lambda : \mathbb{T}_1 \to \mathbb{T}_2$ and a λ-equivariant symplectomorphism $\varphi : M_1 \to M_2$ such that $\mu_1 = \mu_2 \circ \varphi$.

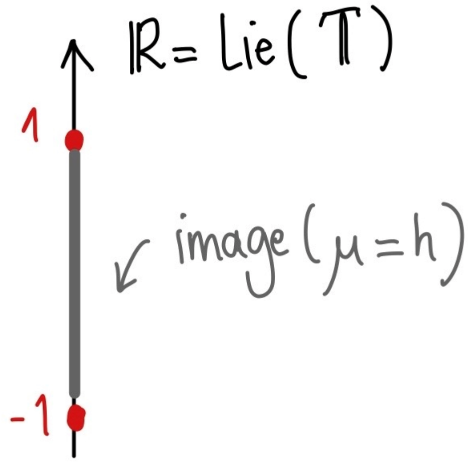
3.1 Example

A possible example of a symplectic toric manifold is the 2-sphere $(S^2, \omega_{\text{std}} = d\theta \wedge dh)$ endowed with the Hamiltonian action of S^1

$$e^{it} \cdot (\theta, h) = (\theta + t, h).$$



The 1-torus (circle) acts by rotations and the two poles are fixed points of the action. The moment maps is given by the height function (up to a constant) and it is clear that the image of the moment map is the closed convex hull of the images of the fixed points (marked in red).



3.2 Delzant's theorem

In this final chapter we combine Morse theory and symplectic toric manifolds: we show that it is possible to use the moment map coming from the Hamiltonian action as a Morse function (or rather an appropriately chosen component of the moment map) to obtain a complete descripion of the homology ring. As we will see, the Morse function in question is perfect The polytope $[0, 1]$ spanned by the red fixed points in Example 3.1 can be generalized to arbitrary symplectic toric manifolds. The associated polytopes can be used to classify symplectic toric manifolds.

Definition 3.6. A **Delzant polytope** Δ in $\mathbb{R}^n$ is a polytope satisfying:

1. **simplicity**: there are n edges meeting at each vertex
2. **rationality**: the edges meeting at vertex p are rational, i.e. each edge is of the form $p + tu_i$ for some $u_i \in \mathbb{Z}^n$
3. **smoothness** for each vertex p the corresponding u_i can be chosen to be a $\mathbb{Z}$ basis for $\mathbb{Z}^n$.

Theorem 3.3 (Delzant)**.** *Toric manifolds are classified by Delzant polytopes. There is a bijective correspondence*

$$\{\textit{toric manifolds}\} \leftrightarrow \{\textit{Delzant polytopes}\}$$

$$(M, \omega, \mathbb{T}^n, \mu) \mapsto \mu(M).$$

4 Homology of symplectic toric manifolds

In this section we use the moment map μ to obtain Morse functions on the toric symplectic manifold M that we are considering. This can be done by choosing a suitable direction of the moment map. As a result, it is possible to apply Morse inequalities as well as other results from section 1 to understand the homology ring of M.

4.1 Equivariant Darboux theorem

As a first step we prove the equivariant version of Darboux's Theorem, which will allow us to locally describe Hamiltonian actions around fixed points.

Theorem 4.1 (Equivariant Darboux). *Let (M^{2n}, ω) be a symplectic manifold with a symplectic action of a compact Lie group G and let $q \in M$ be a fixed point. Then there exists a G-invariant chart $(U, x_1, ..., x_n, y_i, ..., _y{}_n)$ centered at q and G-equivariant with respect to a linear action of G on $\mathbb{R}^{2n}$ such that*

$$\omega|_U = \sum_{k=1}^{n} dx_k \wedge dy_k.$$

The result above can be obtained as a Corollary of the following Theorem:

Theorem 4.2 (Theorem 3.2, [**DHJH**]). *Let $N \subset M$ be a submanifold of a symplectic manifold (M^{2n}, ω). Let $\omega_0, \omega_1 \in \Omega^2(M)$ be two closed 2-forms on M satisfying $\omega_0|_N = \omega_1|_N$. Then there exists a neighborhood U of N and a diffeomorphism $f : U \to U$ such that*

1. $f|_N = \mathrm{id}$
2. $f^*\omega_1 = \omega_0$

Moreover, if $\psi : G \times M \to M$ is an action on M by a compact Lie group G preserving N, ω_0 and ω_1, then f can be chosen to be equivariant, i.e.

$$f \circ \psi_g = \psi_g \circ f \textit{ for all } g \in G.$$

Proof. Consider the closed forms

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

Claim 1: there exists a 1-form β on a neighborhood U of N such that $d\beta = \omega_0 - \omega_1$.
Proof of claim: If N is contained in a contractible neighborhood, the claim is true (for example when N consists of a single point). Otherwise, choose a family of equivariant maps $\varphi_T : U \to U$ satisfying

1. $\varphi_t|_N = \mathrm{id}$
2. $\varphi_0 : U \to N$
3. $\varphi_1 = \mathrm{id}$

Pick a tubular neighborhood for N, $V \subset \nu(N)$, where $\nu(N)$ denotes the normal bundle of N in TM. Then

$$\phi_t : X \to N$$
$$(p, v) \mapsto tv$$

is a deformation retract of N. For any differential form η on M we have

$$\phi_1^*\eta - \phi_0^*\eta = \int_0^1 \frac{d}{dt}(\phi_t^*\eta)dt$$
$$= \int_0^1 \phi_t^*(L_{\xi_t\eta})$$
$$= \int_0^1 \phi_t^*(d\iota_{\xi_t}\eta + \iota_{\xi_t}d\eta)dt$$

Setting $\eta = \omega_0 - \omega_1$, $d\eta = 0$ implies that

$$\eta = d\int_0^1 \phi_t^*(\iota_{\xi_t}\eta)dt.$$

Notice that $\eta|_N = 0$ so that we can assume w.l.o.g.

$$\beta_Y = 0$$

for some neighborhood $Y \subset N$. In particular, $\omega_t|_Y$ is non-degenerate for all $t \in [0, 1]$ and therefore this family of closed forms is non-degenerate on a small neighborhood of N.

We can therefore find a well-defined time-dependent vector field

$$\iota_{X_t}\omega_t = \beta$$

and since β can be chosen to be G- invariant, so does X_t. Te isotopy generated by the vector fields X_t,

$$\frac{d}{dt}f_t = X_t \circ f_t$$

$$f_0 = \mathrm{id}$$

are G-equivariant by equivariance of the X_t's. Moreover,

$$\begin{aligned} f_1^*\omega_1 - \omega_0 &= \int_0^1 \frac{d}{dt}(f_t^*\omega_t)dt \\ &= \int_0^1 f_t^*(d\beta + \omega_1 - \omega_0)dt \\ &= 0. \end{aligned}$$

This concludes the proof. $\square$

Proof. In order to prove Theorem 4.1 consider the special case of a fixed point $q \in M$ and let $V \subset T_qM$ be diffeomorphic to a neighborhood U of q. Endow the tangent space with the canonical symplectic form ω_0. Since q is a fixed point of the symplectic action, G also acts linearly on T_qM via

$$g \mapsto D\psi_g(q).$$

Applying Theorem 4.2 to the symplectic manifold $U \subset M$, the submanifold q and the symplectic forms $\omega|_U$ and $\sigma^*\omega_0$, we obtain a neighborhood U' of q with the desired properties. $\square$

For the action of a torus, we have a weight-space decomposition around every fixed point q:

Theorem 4.3. *Let $(M^{2n}, \omega, \mathbb{T}^m, \mu)$ be a Hamiltonian action on a symplectic space and let $q \in M$ be a fixed point. Then there exists a chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ centered at q and weights $\lambda^1, ..., \lambda^n \in \mathbb{Z}^m$ such that*

1. $\omega|_U = \sum_{k=1}^n dx_k \wedge dy_k$, *and*
2. $\mu|_U = \mu(q) - \frac{1}{2}\sum_{k=1}^n \lambda^k(x_k^2 + y_k^2)$.

Proof. By Theorem 4.1 it suffices to understand the moment map for a linear Hamiltonian action on a standard symplectic vector space. W.l.o.g. identify T_qM with $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ and let ψ be a linear Hamiltonian action of $\mathbb{T}^m$ on $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$.
We have a decomposition in 2-dimensional weight spaces

$$\mathbb{R}^{2n} = \oplus V_k,$$

$$V_k = \{v \in \mathbb{R}^{2n} \mid \psi_g(v) = \chi_k(g)v = e^{2\pi\lambda_k g}v\}$$

for some $\lambda_k \in \mathbb{Z}^m$. Choose a basis

$$\{x_1, ..., x_n, y_1, ..., y_n\}$$

such that

$$\{x_k, y_k\}$$

is a real basis for V_k. To conclude the proof we show that the moment map $\mu : \mathbb{R}^{2n} \to \mathbb{R}^m$ restricted to the weight space V_k is given by

$$\mu : V_k \to \mathbb{R}^m$$

$$v \mapsto \lambda_k \cdot (x_k^2 + y_k^2) \in \mathbb{R}^m.$$

Indeed, for the standard basis $\{e_1, ..., e_m\}$ of $\mathbb{R}^m$, the vector field associated to the action and the vector e_i is

$$e_i^{\#} = \lambda_i(\frac{\partial}{\partial x_k} + \frac{\partial}{\partial y_k}).$$

Finally, we use the definition of the moment map to obtain

$$d\mu^{e_i} = -\sum_{k=1}^{n} \lambda_k(dx_k + dy_k)$$

for an appropriate basis $\{x_1, ..., x_n, y_1, ..., y_n\}$ of $\mathbb{R}^{2n}$. By Theorem 4.1 this concludes the proof. $\square$

4.2 Moment map as a Morse function

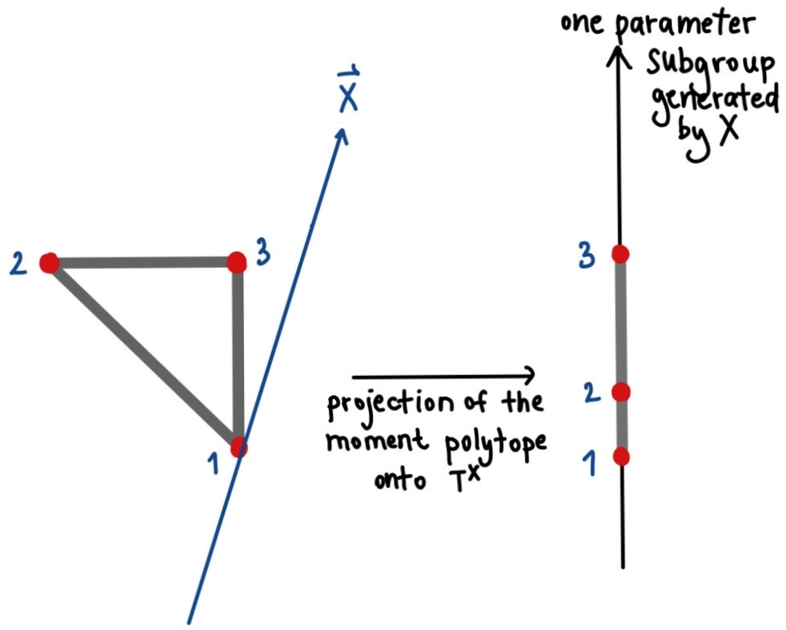
The moment map associated to a symplectic toric manifold M can be used to define a variety of Morse functions on M, which in turn allow us to study the homology ring of M via Morse inequalities.
Let $(M, \omega, \mathbb{T}^n, \mu)$ be a $2n$-dimensional symplectic toric manifold. Choose a generic direction X in $\mathbb{R}^n$ whose components are independent over $\mathbb{Q}$. Then

1. the subgroup generated by X is dense in $\mathbb{T}^n$
2. X is not parallel to the facets of the moment polytope
3. the vertices of the moment polytope have different projections along X.

Define

$$\mu^X = \langle \mu, X \rangle : M \to \mathbb{R},$$

the component of μ along X.



Theorem 4.4. *Let $(M, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold and let $X \in \mathbb{R}^n$ have components independent over $\mathbb{Q}$. Then the degree $2k$ homology group of M has dimension equal to the number of vertices of the moment polytope where there are exactly k edge vectors pointing up relative to the projection along X. All odd-degree homology groups vanish.*

Proof. By Theorem 4.3 for any fixed point of the Hamiltonian action $q \in M$ there exists a chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ and weights $\lambda^1, ..., \lambda^n \in \mathbb{Z}^n$ such that

$$\mu^X|_U = \mu^X(q) - \frac{1}{2}\sum_{k=1}^{n}\langle\lambda^k, X\rangle(x_k^2 + y_k^2).$$

Notice that since the components of X are independent over $\mathbb{Q}$ and the action is effective, all coefficients $\langle\lambda^k, X\rangle$ must be non-zero.
The critical point q of μ^X is therefore non-degenerate and the index is equal to

$$2 \cdot \#\{k \text{ such that } -\langle\lambda^k, X\rangle < 0\}.$$

Geometrically speaking, these $-\lambda^k$'s are exactly the edges of the polytope that point upwards when projected onto the subgroup generated by X (the scalar product with X is positive).
Notice that each critical point has even index and the function μ^X is therefore a perfect Morse function. The Theorem follows. $\square$

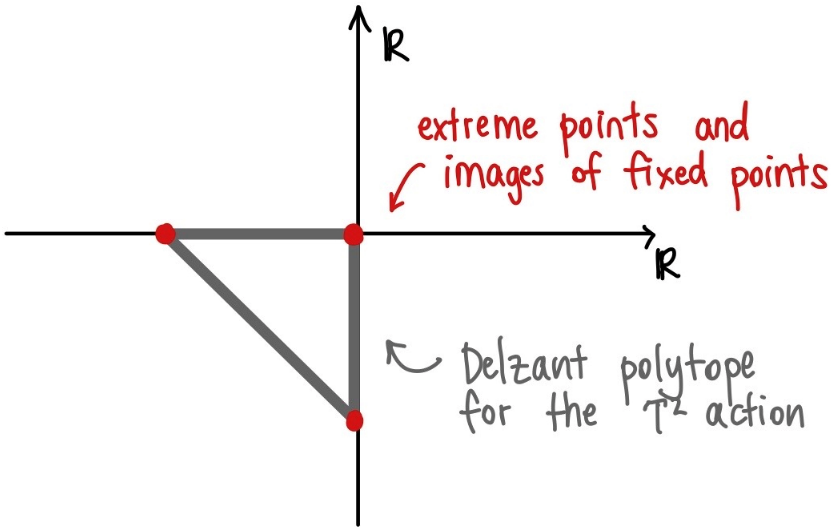
4.2.1 Example

Let us give a concrete example of how to apply Theorem 4.4.
Consider $(\mathbb{C}P^2, \omega_{\mathrm{FS}})$ with the $\mathbb{T}^2$ action

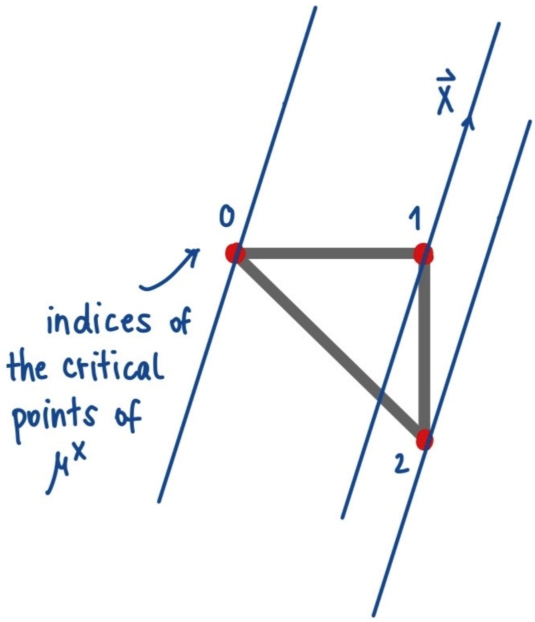
$$(e^{i\theta_1, e^{i\theta_2}}) \cdot [z_0 : z_1 : z_2] \mapsto [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$$

and associated moment map

$$\mu([z_0 : z_1 : z_2]) = -\frac{1}{2}\Big(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\Big).$$



The image above shows the moment polytope associated to this Hamiltonian toric action.
The vertices correspond to the fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$.
By Theorem 4.4 it suffices to project the moment map onto some irrational direction and consider the direction of the edges of the polytope to compute the homology of $\mathbb{C}P^2$.



Indeed, the three critical points of μ^X have index 0,1 and 2 respectively and hence

$$H_0(\mathbb{C}P^2) = \mathbb{R},$$

$$H_2(\mathbb{C}P^2) = \mathbb{R},$$

$$H_4(\mathbb{C}P^2) = \mathbb{R},$$

$$H_{\text{odd}}(\mathbb{C}P^2) = 0$$

as we expected.

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