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Moment Maps From Lie Groups

Semester Paper

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Abstract

An action of $\mathbb{R}$ on a symplectic manifold is called hamiltonian if the vector field corresponding to this action is hamiltonian. We discuss hamiltonian actions in a more general setting, namely we consider actions of arbitrary Lie groups G. In this setting, we say that an action is hamiltonian if there is a map from the manifold to the dual of the Lie algebra of G satisfying several properties related to this action. Such a map is then called a moment map. We will show some properties of moment maps and also give several examples of hamiltonian actions including computations of the corresponding moment maps. These include the action of a general Lie group on its coadjoint orbits as well as several examples of actions of the group of unitary matrices.

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Chapter 1

Introduction

The idea of moment maps first appeared in the work [Lie90] of Sophus Lie. It was then further developed in the 1960s by Bertram Kostant, Jean-Marie Souriau and Alexander Kirillow. Souriau was also first to give a formal definition and a name to the object we call today a moment map. In his paper [Sou70], which is written in French, he defined it as "moment". The name for this comes from the fact that an important first example of moment maps is the linear momentum, which is called "moment linéaire" in French. Then, Marsden and Weinstein translated this to English in their paper [MW74] as "moment". However, the English translation "moment map" of the French term "application moment" is physically incorrect, as it should rather be called momentum map. Nevertheless, the term "moment map" is widely used today. For more details on the history of moment maps, one can read [MR99, p. 369-370], which is also the source for this small introduction.

The main reference for this paper are the lecture notes [Sil06], which are a general introduction to symplectic geometry. We primarily used the lectures 1–2, 18 and 21–22 as well as the homeworks 17 and 19.

In chapter 2, we introduce important general concepts from differential geometry that we will need later. In chapter 3, we then move on to introduce symplectic manifolds. There, we will also give the definition of what a hamiltonian action and a moment map is. After that, we will discuss in chapter 4 two important properties of moment maps that will be used later. In chapter 5, we give an important example of a hamiltonian action, namely the coadjoint action of a Lie group on the coadjoint orbits. Finally, in chapter 6, we give further examples involving the group of unitary matrices $U(n)$.

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Chapter 2

Background From Differential Geometry

In this chapter, we will recall some important definitions and theorems from differential geometry that we will need later, including in particular the cotangent bundle, differential forms, the Lie derivative, Lie groups, Lie algebras, the adjoint and coadjoint representation of a Lie group and a few facts on orbits of smooth actions.

We will normally write M for a manifold and denote by m the dimension of M.

2.1 The Cotangent Bundle of a Manifold

Let M be a manifold. For $p \in M$ we write T_pM for the tangent space of M at p and T_p^*M for the cotangent space of M at p. Recall that then the cotangent bundle is defined as the disjoint union of the cotangent spaces

$$T^*M = \bigsqcup_{p \in M} T_p^*M.$$

We write $\pi\colon T^*M \to M$ for the map that sends T_p^*M to p. If (U, x) is a chart on M we have an associated chart $(\tilde{x}, \pi^{-1}(U))$ on T^*M that is defined by $\tilde{x}((p,\xi)) = (x(p), \sum_{i=1}^m dx_p^i(\xi)e_i)$. One can show that these chart indeed form an atlas on T^*M and that T^*M is thus a manifold of dimension $2m$.

2.2 Differential Forms

In this section, we will recall the definition of differential forms and state some important results that we will use later.

We will define the general notion of a differential k-form, but later we will be mostly working with 1- and 2-forms. The following definitions and theorems follow [Mer20], especially lectures 19, 21, 22 and 23.

Definition 2.1. A differential k-form or short k-form on a manifold M is a section of $\Lambda^k T^*M$. We denote by $\Omega^k(M)$ the set of all differential k-forms on M.

We will now state two characterisations of differential forms. The proofs can be found in [Mer20, Proposition 19.24] and [Mer20, Theorem 21.5] respectively. Recall that a multilinear map $\omega\colon V \times \cdots \times V \to \mathbb{R}$ is called (k-)alternating if it vanishes whenever two arguments are equal, where k denotes the number of arguments of ω.

Theorem 2.2. *If $\omega \in \Omega^k(M)$ and $p \in M$, then ω_p can be thought of as an alternating map $\omega_p\colon T_pM \times \dots T_pM \to \mathbb{R}$.*

Theorem 2.3. *There is a canonical identification between $\Omega^k(M)$ and alternating $C^\infty(M)$-multilinear functions*

$$\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M),$$

where $\mathfrak{X}(M)$ denotes the space of vector fields on M. This identification satisfies

$$\omega(X_1, \dots, X_k)(p) = \omega_p(X_1(p), \dots, X_k(p)).$$

Note that we write $C^\infty(M)$ for smooth functions from M to $\mathbb{R}$.

We will later need the following lemma relating the wedge product of two differential 1-forms with the original differential forms. It can be shown using the formula $\omega(X,Y)(p) = \omega_p(X(p), Y(p))$ from the theorem above and then [Mer20, Corollary 22.19].

Lemma 2.4. *Let $\omega, \theta \in \Omega^1(M)$. Then for $X, Y \in \mathfrak{X}(M)$ we have*

$$(\omega \wedge \theta)(X,Y) = \omega(X)\theta(Y) - \omega(Y)\theta(X).$$

We will now define three operations on differential forms, namely the Lie derivative, the exterior differential and the interior product. We will then state some important relations between them, which will be used later. We will also define the pullback of a differential form and give an equivalent definition of the Lie derivative. More details on these can be found in [Mer20], especially in lectures 21 and 22, which are also the main reference for the following. For all the definitions we use Theorem 2.3 from above.

Definition 2.5. For a vector field $X \in \mathfrak{X}(M)$ we define the Lie derivative $\mathcal{L}_X\colon \Omega^k(M) \to \Omega^k(M)$ as follows

$$\mathcal{L}_X\omega(Y_1, \dots, Y_k) = X(\omega(Y_1, \dots, Y_k)) - \sum_{i=1}^{k} \omega(Y_1, \dots, Y_{i-1}, [X, Y_i], Y_{i+1}, \dots, Y_k),$$

for $\omega \in \Omega^k(M)$ and $Y_1, \ldots, Y_k \in \mathfrak{X}(M)$.

Definition 2.6. We define the exterior differential $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ as follows

$$d\omega(Y_1, \ldots, Y_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} Y_i(\omega(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_k)) + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([Y_i, Y_j], Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_k),$$

for $Y_1, \ldots, Y_{k+1} \in \mathfrak{X}(M)$.

Definition 2.7. For a vector field $X \in \mathfrak{X}(M)$ we define the interior product $\iota_X\colon \Omega^k(M) \to \Omega^{k-1}(M)$ as follows

$$\iota_X \omega(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1}),$$

for $Y_1, \ldots, Y_{k-1} \in \mathfrak{X}(M)$. We also define $\iota_X\colon \Omega^0(M) \to \Omega^{-1}(M)$ to be the zero map.

We first give an alternative formula for the Lie derivative. A proof can be found in [Mer20, Theorem 23.12].

Lemma 2.8. *For $X \in \mathfrak{X}(M)$ we have*

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d.$$

We will later also need a formula for the interior product $\iota_{[X,Y]}$.

Lemma 2.9. *For $X, Y \in \mathfrak{X}(M)$ we have*

$$\iota_{[X,Y]} = \mathcal{L}_X \circ \iota_Y - \iota_Y \circ \mathcal{L}_X.$$

A proof of this lemma can be found in [Mer20, Corollary 23.11].

We will now define the pullback of a differential form. For this let $\varphi\colon M \to N$ be a smooth function and let $\omega \in \Omega^k(N)$.

Definition 2.10. For $k > 0$, the pullback of ω is the differential form $\varphi^*\omega \in \Omega^k(M)$ defined by

$$(\varphi^*\omega)_p(\xi_1, \ldots, \xi_k) = \omega_p(D\varphi(p)\xi_1, \ldots, D\varphi(p)\xi_k),$$

for $\xi_1, \ldots, \xi_k \in T_pM$. For 0-forms, we define the pullback $\varphi^*\omega$ as $\omega \circ \varphi$, which makes sense since $\Omega^0(M) = C^\infty(M)$.

We will later also need the following result about the Lie derivative, whose proof can be found in [Mer20, Proposition 22.14].

Lemma 2.11. *Let X be a vector field with flow Φ_t. Then for $\omega \in \Omega^k(M)$ the Lie derivative satisfies $\mathcal{L}_X\omega = \frac{d}{dt}\big|_{t=0} \Phi_t^*\omega$, where $\Phi_t^*\omega$ is the pullback of ω as defined above.*

If we apply this lemma to 0-forms, i.e. smooth functions, then we get the following equivalent definition of the Lie derivative applied to smooth functions.

Lemma 2.12. *For a vector field X the Lie derivative $\mathcal{L}_X \colon C^\infty(M) \to C^\infty(M)$ can be alternatively defined as*

$$(\mathcal{L}_X f)(p) = \lim_{t\to 0} \frac{f \circ \Phi_t(p) - f(p)}{t}.$$

2.3 Lie Derivative for Tensor Fields

One can also define the Lie derivative in a more general setting, namely for tensor fields. We do not need this in full generality, but we will need it for vector fields (i.e. tensors of type $(1,0)$). The reference for this section is [Mer20, Lecture 10].

Definition 2.13. Let X be a vector field with flow Φ_t. We define the Lie derivative of X on vector fields $Y \in \mathfrak{X}(M)$ as follows

$$(\mathcal{L}_X Y)(p) = \left.\frac{d}{dt}\right|_{t=0} D\Phi_{-t}(\Phi_t(p))Y(\Phi_t(p)).$$

One has the following equivalent definition of $\mathcal{L}_X Y$, whose proof can be found in [Mer20, Proposition 10.4].

Lemma 2.14. *For $X, Y \in \mathfrak{X}(M)$ we have*

$$\mathcal{L}_X(Y) = [X, Y].$$

2.4 Lie Groups and Lie Algebras

In this section, we will shortly repeat what a Lie group and a Lie algebra is and how we can get a Lie algebra from a Lie group. We start with two definitions. The reference for the following is again [Mer20], in particular lectures 8, 11 and 12.

Definition 2.15. A Lie group is a manifold G that is also a group in the algebraic sense, where multiplication and inversion are smooth maps.

Definition 2.16. A Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear operation

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},\ (v, w) \mapsto [v, w],$$

called the Lie bracket, that is antisymmetric and satisfies the following identity

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0,\ \forall u, v, w \in \mathfrak{g},$$

which is also called the Jacobi identity.

Now, given a Lie group G we define its Lie algebra as follows. A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if $Dl_g(h)(X(h)) = X(gh)$ for all $g, h \in G$, where $l_g\colon G \to G$ is left multiplication by g. Then the Lie algebra of G is defined as the vector space of left-invariant vector fields. We denote it by $\mathfrak{g}$. The Lie bracket is the normal commutator of vector fields. We sometimes also denote this space of left invariant vector fields by $\mathfrak{X}^l(G)$.

We can alternatively define $\mathfrak{g}$ as the tangent of G at the identity element, i.e. $\mathfrak{g} = T_eG$. One can show that the map

$$\mathrm{eval}_e\colon \mathfrak{X}^l(G) \to T_eG,\ X \mapsto X(e)$$

is a vector space isomorphism and with this define a Lie bracket on T_eG, which then shows that one can indeed define $\mathfrak{g}$ as T_eG.

Later, we will use both definitions interchangeably as, depending on the context, one definition is easier to work with than the other.

We will also need the Lie algebra of the unitary matrices. Recall for this that

$$\mathrm{U}(n) := \left\{ A \in \mathbb{C}^{n^2} \mid A^*A = I_n \right\},$$

where A^* denotes the conjugate transpose of A and I_n is the $n \times n$ identity matrix. First of all, we need that this is a Lie group.

Lemma 2.17. *$U(n)$ is a Lie subgroup of $GL(n, \mathbb{C})$.*

Recall that a Lie subgroup is a subgroup that is in addition a Lie group itself and an immersed submanifold.

Proof. This can for example be shown using the Closed Subgroup Theorem that states that the closed subgroups of a Lie group are exactly the embedded Lie subgroups. A proof of this theorem can be found in [Lee13, Theorem 20.12].

Another proof would be to use a similar technique as in [Mer20, Proposition 10.18], that is to consider the map

$$\varphi\colon \mathrm{GL}(n, \mathbb{C}) \to \mathrm{Sym}(n, \mathbb{C}),\ A \mapsto AA^*,$$

where $\mathrm{Sym}(n, \mathbb{C})$ are all matrices $A \in \mathbb{C}^{n \times n}$ with $A = A^*$. Then one can show that I_n is a regular value and thus $\mathrm{U}(n) = \varphi^{-1}(I_n)$ is an embedded submanifold of $\mathrm{GL}(n, \mathbb{C})$ by the Implicit Function Theorem. From this it then follows that it is a Lie subgroup, e.g. by [Mer20, Proposition 10.15]. The strategy to prove that I_n is a regular value is to show that φ has constant rank and then it is enough to show that $D\varphi(I_n)$ is surjective. For this, we can identify $T_{I_n}\mathrm{GL}(n, \mathbb{C})$ with $T_{I_n}\mathbb{C}^{n \times n} \cong \mathbb{C}^{n \times n}$ as $\mathrm{GL}(n, \mathbb{C})$ is an open subset of the vector space $\mathbb{C}^{n \times n}$. Also, one can identify $T_I\mathrm{Sym}(n, \mathbb{C})$ with $\mathrm{Sym}(n, \mathbb{C})$ as it is a vector space. Then one can show that under these identifications φ maps $A \in \mathbb{C}^{n \times n}$ to $A + A^* \in \mathrm{Sym}(n, \mathbb{C})$, which is surjective as for $A \in \mathrm{Sym}(n, \mathbb{C})$ we have $A = \varphi(\frac{1}{2}A)$. ∎

We denote the Lie algebra of $\mathrm{U}(n)$ as $\mathfrak{u}(n)$. As $\mathrm{U}(n)$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$, $\mathfrak{u}(n)$ can be seen as a Lie subalgebra of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$, i.e. a vector subspace that is closed under the Lie bracket. As we have seen in the proof above, $\mathfrak{gl}(n, \mathbb{C})$ can be identified with all matrices $\mathbb{C}^{n\times n}$. Thus, it makes sense to describe the Lie algebra $\mathfrak{u}(n)$ as set of matrices in $\mathbb{C}^{n\times n}$.

Lemma 2.18. *The Lie algebra $\mathfrak{u}(n)$ is the set of all skew-hermitian matrices, i.e. all $A \in \mathbb{C}^{n\times n}$ with $A^* = -A$.*

Proof. As in the proof above, we consider the map

$$\varphi\colon \mathrm{GL}(n, \mathbb{C}) \to \mathrm{Sym}(n, \mathbb{C}),\ A \mapsto A + A^*.$$

We have argued there that I_n is a regular value. Using now [Mer20, Proposition 6.15] we get that the tangent space $T_{I_n}(\mathrm{U}(n))$, which is the Lie algebra $\mathfrak{u}(n)$, can be identified with the kernel of the map $D\varphi(I_n)$. Using the same identification as above, $D\varphi(I_n)$ is $A \mapsto A+A^*$ and thus this kernel is the set of skew-hermitian matrices as claimed. □

2.5 Adjoint and Coadjoint Representation

Let G be a Lie group. Then G acts on itself by conjugation, that is we consider the action $G \to \mathrm{Diff}(G)$, $c_g(h) = ghg^{-1}$. The derivative of this action at the identity gives the so-called adjoint representation, a linear action of G on its Lie algebra.

Definition 2.19. The action $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g = Dc_g(e)$ is called the adjoint action or adjoint representation.

By dualising this, we get an action of G on $\mathfrak{g}^*$, the coadjoint representation. Here and also later we will write $\langle \xi, X\rangle = \xi(X)$ for $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. With this we can now give the formal definition of the coadjoint representation.

Definition 2.20. The coadjoint action or coadjoint representation $\mathrm{Ad}^*\colon G \to \mathrm{GL}(\mathfrak{g}^*)$ is defined as

$$\langle \mathrm{Ad}^*_g \xi, X\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X\rangle$$

for $g \in G$, $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

In the case of matrix Lie groups we get the following lemma.

Lemma 2.21. *For G a matrix Lie group (i.e. a Lie subgroup of $GL(n)$), the adjoint action Ad_g is the conjugation $A \mapsto gAg^{-1}$.*

Proof. Let $A \in \mathfrak{g}$. Let γ be an curve with $A = \frac{d}{dt}\big|_{t=0} \gamma(t)$. Then

$$\begin{aligned} \mathrm{Ad}_g A &= Dc_g(e) \left.\frac{d}{dt}\right|_{t=0} \gamma(t) \\ &= \left.\frac{d}{dt}\right|_{t=0} (c_g \circ \gamma)(t) \\ &= \left.\frac{d}{dt}\right|_{t=0} g\gamma(t)g^{-1} \\ &= g \left(\left.\frac{d}{dt}\right|_{t=0} \gamma(t) \right) g^{-1} \\ &= gAg^{-1}, \end{aligned}$$

where in the second to last equality we used that G is a Lie subalgebra of $\mathrm{GL}(n)$ and thus the multiplication is normal matrix multiplication and we can take g respectively g^{-1} out of the derivative. □

2.6 Orbits of Smooth Actions

In this section we will look at the orbits of a smooth action $\Psi\colon G \times M \to M$. The reference for this section is [Wan16], especially Lemma 1.3 and Proposition 2.2 of it. In the following, our goal is to show that the orbit

$$G \cdot p = \{\Psi(g, p) \mid g \in G\}$$

is a manifold and describe the tangent vectors.

For $p \in M$ and $g \in G$, we consider the maps

$$\Psi^p\colon G \to M,\ \Psi^p(h) = \Psi(h, p)$$

and

$$\Psi_g\colon M \to M,\ \Psi_g(q) = \Psi(g, q).$$

Remark. We will use this notation, especially Ψ_g, also later in this paper, often without comment. Also, we will usually denote actions by Ψ.

We now want to argue that the orbits are manifolds. For this, note that we have the following equality of functions

$$\Psi^p \circ l_g = \Psi_g \circ \Psi^p.$$

By taking the derivative at e and noting that $Dl_g(e)$ and $D\Psi_g(\Psi^p(g))$ are diffeomorphisms, we get that $D\Psi^p(g)$ and $D\Psi^p(e)$ have the same rank. Thus, we can apply the Constant Rank Theorem to the function $\Psi^p\colon G \to M$ and conclude that the image of it is indeed a manifold. The Constant Rank Theorem can for example be found in [Mer20, Corollary 6.19]. As the image of Ψ^p is the orbit of p, we get the following lemma.

Lemma 2.22. *The orbit $G \cdot p$ is a manifold.*

Now that we know that the orbit is a manifold, we can say even more. It in fact holds that the orbit $G \cdot p$ is diffeomorphic to G/G_p, where $G_p = \{g \in G \mid \Psi_g(p) = p\}$ denotes the stabiliser of p. The diffeomorphism Φ is given by $\Phi(gG_p) = \Psi_g(p)$. All this follows from a more general theorem that can be found in [Mer20, Theorem 13.12]. We also state this as a lemma, so that we can reference to it later.

Lemma 2.23. *The orbit $G \cdot p$ is diffeomorphic to G/G_p with the diffeomorphism being $gG_p \mapsto \Psi_g(p)$.*

Using this, we get that the map $\Psi^p \colon G \to G \cdot p$ is the concatenation of the quotient map $G \to G/H$ and the diffeomorphism from above. Thus, we get that Ψ^p is a submersion and hence the tangent space $T_p(G \cdot p)$ is the image of $T_e(G) \cong \mathfrak{g}$ under the map $D\Psi^p$. So, let $X \in \mathfrak{X}^l(G)$. Using the isomorphism $\mathfrak{X}^l(G) \to T_e(G)$ we get that elements of $T_p(G \cdot p)$ are of the form

$$\begin{aligned} D\Psi^p(X(e))(f) &= X(e)(f \circ \Psi^p) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\Psi_{\exp(tX)}(p)) \\ &= \left(\left.\frac{d}{dt}\right|_{t=0} \Psi_{\exp(tX)}(p) \right)(f), \end{aligned}$$

for $f \in C^\infty(G \cdot p)$. Note that here and also later, we denote by $\exp(tX)$ the integral curve of X starting at the identity. To get the second equality note that we have $X(e) = \dot{\gamma}(0)$, where $\gamma(t) = \exp(tX)$, and thus we get

$$X(e)(f \circ \Psi^p) = \dot{\gamma}(0)(f \circ \Psi^p) = \left.\frac{d}{dt}\right|_{t=0} f \circ \Psi^p \circ \gamma(t) = \left.\frac{d}{dt}\right|_{t=0} f(\Psi_{\exp(tX)}(p)).$$

We also state this as a lemma.

Lemma 2.24. *The elements of the tangent space $T_p(G \cdot p)$ are of the form*

$$\left.\frac{d}{dt}\right|_{t=0} \Psi_{exp(tX)}(p)$$

for $X \in \mathfrak{g}$.

Chapter 3

Symplectic Manifolds and Actions

In this chapter, we will first define symplectic vector spaces and manifolds and then move on to define symplectic actions and moment maps.

3.1 Symplectic Vector Spaces

Let V be a vector space. Recall that a bilinear map $\Omega\colon V \times V \to \mathbb{R}$ is called skew-symmetric if for all $u, v \in V$ we have $\Omega(u, v) = -\Omega(v, u)$. Note that skew-symmetric is the same as 2-alternating. Skew-symmetric maps can be written in a standard form as the following theorem shows. A proof can be found in [Sil06, Theorem 1.1].

Theorem 3.1. *Let $\Omega\colon V \times V \to \mathbb{R}$ be skew-symmetric. Then there is a basis $e_1, \ldots, e_n, f_1, \ldots, f_n, u_1, \ldots, u_k$ of V such that*

$$\Omega(e_i, f_j) = \delta_{ij}$$
$$\Omega(e_i, u_l) = 0$$
$$\Omega(f_j, u_l) = 0$$

for $1 \leq i, j \leq n$ and $1 \leq l \leq k$.

Definition 3.2. A skew-symmetric bilinear map $\Omega\colon V \times V \to \mathbb{R}$ is called symplectic or non-degenerate if there is no non-zero $u \in V$ such that for all $v \in V$ we have $\Omega(u, v) = 0$, i.e. the set $U := \{u \in V \mid \omega(u, v) = 0 \,\forall v \in V\}$ is equal to $\{0\}$. The vector space V is then called a symplectic vector space.

Using Theorem 3.1 we can show the following important property of symplectic vector spaces.

Corollary 3.3. *A symplectic vector space V is even-dimensional.*

Proof. Take the standard form from Theorem 3.1. Then

$$\{0\} = U = \text{span}(\{u_1, \ldots, u_k\}).$$

Thus, $k = 0$ and

$$\dim(V) = 2n + k = 2n$$

is even. □

We will now give an example of a symplectic vector space.

Example 3.4. Consider the vector space $\mathbb{R}^{2n}$. Then we can define the following symplectic map on $\mathbb{R}^{2n}$

$$\Omega \colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R},\ \Omega(v_i, v_j) = \begin{cases} 1 & \text{if } j = i + n \\ -1 & \text{if } j = i - n \\ 0 & \text{otherwise} \end{cases},$$

where $\{v_i\}_{i=1}^{2n}$ is the standard basis of $\mathbb{R}^{2n}$, i.e. $(v_i)_j = \delta_{ij}$. The basis from Theorem 3.1 would then be $e_i = v_i$ and $f_i = v_{i+n}$.

3.2 Symplectic Manifolds

We can now define what a symplectic manifold is. For this, we will first define symplectic 2-forms.

Definition 3.5. A 2-form $\omega \in \Omega^2(M)$ is called symplectic if ω is closed (i.e. $d\omega = 0$) and $\omega_p \colon T_pM \times T_pM \to \mathbb{R}$ is a symplectic map for every $p \in M$. (M, ω) is then called a symplectic manifold.

We will also often omit the map ω and just call M a symplectic manifold. We can now show the following.

Corollary 3.6. *Symplectic manifolds are even-dimensional.*

Proof. Let M be a symplectic manifold. Using Corollary 3.3, we get that the dimension of T_pM is even. As the dimensions of T_pM and M are equal, we get that also M is even-dimensional. □

Example 3.7. Similar to Example 3.4 we can give the manifold $\mathbb{R}^{2n}$ the structure of a symplectic manifold. Namely, we can define the following 2-form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dx^{i+n}.$$

Then ω is closed and ω_p is a symplectic map. A basis for T_pM as in Theorem 3.1 is for example

$$e_i = \left.\frac{\partial}{\partial x^i}\right|_p \text{ and } f_i = \left.\frac{\partial}{\partial x^{i+n}}\right|_p.$$

The following example gives another important example of symplectic manifolds, namely we will show that any cotangent space is a symplectic manifold. The example follows [Sil06, Lecture 2].

Example 3.8. Recall from Section 2.1 that the cotangent bundle T^*M is a manifold and that we can construct a chart a follows. If (U, x) is a chart on M, then for $p \in M$ and $\xi \in T_p^*M$ we can write ξ as sum $\xi = \sum_{i=1}^n \xi_i (dx^i)_p$. This defines functions $\xi_i \colon T^*U \to \mathbb{R}$. Then $(x^1, \ldots, x^m, \xi_1, \ldots, \xi_m)$ is a coordinate chart on T^*M. We now claim that we can define a 2-form on T^*M using the definition

$$\omega = \sum_{i=1}^m dx^i \wedge d\xi_i$$

on U. For this we first need to show that the definition of ω does not depend on the chart that we choose. For this, note first that

$$\omega = -d\left(\sum_{i=1}^m \xi_i dx^i\right).$$

We define

$$\alpha \coloneqq \sum_{i=1}^m \xi_i dx^i$$

and claim that α is independent of the choice of chart. This will then imply that ω is and directly show that ω is closed.

So let (U, x) and (V, y) be two coordinate charts on M with $U \cap V \neq \emptyset$. Also let $(x^1, \ldots, x^m, \xi_1, \ldots, \xi_m)$ be the chart on T^*M we get using (U, x) and let $(y^1, \ldots, y^m, \zeta_1, \ldots, \zeta_m)$ be the one we get using (V, y). Then we have

$$\zeta_j = \sum_{i=1}^m \xi_i \left(\frac{\partial x^i}{\partial y^j}\right)$$

and

$$dx^i = \sum_{j=1}^m \left(\frac{\partial x^i}{\partial y^j}\right) dy^j.$$

From this it follows that

$$\sum_{i=1}^m \xi_i dx^i = \sum_{j=1}^m \zeta_j dy^j$$

and thus α is independent of the chart we use. It remains to show that ω_p is non-degenerate but this follows as $\omega_p(\zeta, \zeta') = \sum_{i=1}^m (d\xi_i \wedge dx^i)_p(\zeta, \zeta')$ can be zero for all ζ' only if $\zeta = 0$, which shows exactly non-degeneracy of ω.

The following definition describes what it means for a diffeomorphism between symplectic manifolds to preserve the symplectic structure.

Definition 3.9. A diffeomorphism φ between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is called a symplectomorphism if in addition $\varphi^* \omega_2 = \omega_1$. The manifolds are then called symplectomorphic. We denote by $\mathrm{Sympl}(M, \omega)$ the group of symplectomorphism from M to itself.

3.3 Symplectic and Hamiltonian Vector Fields

In this section, we are going to define symplectic and hamiltonian vector fields.

So let (M, ω) be a symplectic manifold and let $H\colon M \to \mathbb{R}$ be a smooth function. Then dH is an element of $\Omega^1(H)$. By non-degeneracy of the 2-form ω, there is a unique vector field $X_H \in \mathfrak{X}(M)$ with $\iota_{X_H}\omega = dH$.

Definition 3.10. The vector field X_H is called a hamiltonian vector field with hamiltonian function H.

A vector field that is preserving the symplectic form is called symplectic.

Definition 3.11. The vector field X is called symplectic if $\mathcal{L}_X \omega = 0$.

There are several equivalent definition of symplectic and hamiltonian vector fields that we will discuss in the following.

For hamiltonian vector fields we can give the following lemma, whose proof is immediate from the definition.

Lemma 3.12. *A vector field X is hamiltonian if and only if $\iota_X \omega$ is exact.*

For the case of symplectic vector fields we can show the following.

Lemma 3.13. *Let (M, ω) be a symplectic manifold and let $X \in \mathfrak{X}(M)$. Also, let $\rho_t\colon M \to M$ be the flow of X. Then the following are equivalent:*

1. *X is symplectic, i.e. $\mathcal{L}_X \omega = 0$,*
2. *$\rho_t^* \omega = \omega$ for all t, and*
3. *$\iota_X \omega$ is closed.*

Proof.

“1. ⇒ 2.” By Lemma 2.11, we get that

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=t_0} \rho_t^* \omega &= \left.\frac{d}{dt}\right|_{t=0} \rho_{t_0+t}^* \omega \\ &= \rho_{t_0}^* \left.\frac{d}{dt}\right|_{t=0} \rho_t^* \omega \\ &= \rho_{t_0}^* \mathcal{L}_X \omega = 0. \end{aligned}$$

As $\rho_0^* \omega = \omega$, we can conclude that $\rho_t^* \omega = \omega$ for all t.

"2. $\Rightarrow$ 1." Similar as above, we have

$$0 = \left.\frac{d}{dt}\right|_{t=0} \rho_t^* \omega = \mathcal{L}_X \omega,$$

as wanted.

"1. $\Rightarrow$ 3." Using Lemma 2.8 and the fact that $d\omega = 0$, we get

$$0 = \mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega = d\iota_X \omega,$$

so $d\iota_X \omega$ is closed.

"3. $\Rightarrow$ 1." As above, by closedness of $\iota_X \omega$, we have

$$0 = d\iota_X \omega = \mathcal{L}_X \omega,$$

as wanted. □

So, in particular, we get that hamiltonian vector fields are also symplectic. We denote the symplectic respectively hamiltonian vector field by $\mathfrak{X}^{\mathrm{sympl}}(M)$ and $\mathfrak{X}^{\mathrm{ham}}(M)$ respectively.

We can prove the following lemma.

Lemma 3.14. *If $X, Y \in \mathfrak{X}^{sympl}(M)$, then $[X, Y] \in \mathfrak{X}^{ham}(M)$.*

Proof. Using Lemma 3.12, it is enough to show that $\iota_{[X,Y]}\omega$ is exact. For this, we compute, using Lemmas 2.8 and 2.9,

$$\iota_{[X,Y]}\omega = \mathcal{L}_X \iota_Y \omega - \iota_Y \mathcal{L}_X \omega = d(\iota_X \iota_Y \omega) + \iota_X d(\iota_Y \omega) - \iota_Y d(\iota_X \omega) - \iota_Y \iota_X d\omega.$$

By Lemma 3.13, $\iota_X \omega$ and $\iota_Y \omega$ are closed. Also, ω is closed, so we get that

$$\iota_{[X,Y]}\omega = d\iota_X \iota_Y \omega$$

is exact, which completes the proof. □

3.4 Symplectic and Hamiltonian Actions and Moment Maps

In this section, we will define symplectic and hamiltonian actions of Lie groups on symplectic manifolds.

Definition 3.15. Let G be a Lie group and (M, ω) a symplectic manifold. An action $\Psi \colon G \to \mathrm{Diff}(M)$ is called symplectic if the image of Ψ is contained in the subgroup $\mathrm{Sympl}(M, \omega) \subseteq \mathrm{Diff}(M)$. We also say that G acts by symplectomorphisms.

We have a bijective correspondence between complete symplectic vector fields on M and symplectic actions of $\mathbb{R}$ on M. The correspondence is given by

$$X \longmapsto (t \mapsto (\Phi_X)_t) \text{ and } X_p = \frac{d\Psi_t(p)}{dt}\bigg|_{t=0} \longleftarrow \Psi.$$

Here, we denote the flow of the vector field X at time t by $(\Phi_X)_t$. Recall that we denote the integral curve of X starting at the identity by $\exp(tX)$, i.e. we have

$$\exp(tX) = (\Phi_X)_t(e).$$

Using this, we can define a hamiltonian action for the special case $G = \mathbb{R}$.

Definition 3.16. A symplectic action Ψ of $\mathbb{R}$ on a symplectic manifold (M, ω) is called hamiltonian if the corresponding vector field X as above is hamiltonian.

By Lemma 3.12, this is equivalent to the existence of a function $H : M \to \mathbb{R}$ with $dH = \iota_X \omega$. We now want to generalise this definition of hamiltonian actions to general Lie groups G. We get the following definition.

Definition 3.17. Let G be a Lie group and (M, ω) a symplectic manifold. A symplectic action $\Psi : G \to \mathrm{Sympl}(M, \omega)$ is called hamiltonian if there exists a map $\mu : M \to \mathfrak{g}^*$ such that the following two conditions are satisfied.

1. For any $X \in \mathfrak{g}$, let $\mu^X : M \to \mathbb{R}$, $\mu^X(p) = \langle \mu(p), X \rangle$ be the component of μ along X and let $X^\#$ be the vector field with $X^\#_p = \frac{d}{dt}\big|_{t=0} \Psi_{\exp(tX)}(p)$. Then $d\mu^X = \iota_{X^\#}\omega$, i.e. μ^X is a hamiltonian function for $X^\#$.
2. μ is (Ψ, Ad^*)-equivariant, i.e. $\mu \circ \Psi_g = \mathrm{Ad}^*_g \circ \mu$ for any $g \in G$.

Then the tuple (M, ω, G, μ) is called hamiltonian G-space and the map μ is called a moment map.

This indeed generalises the definition from above for $G = \mathbb{R}$ as we argue in the following. In this case, $\mathfrak{g}$ and $\mathfrak{g}^*$ are both isomorphic to $\mathbb{R}$. By linearity, it is enough to check condition 1 for $X = 1$. For this X we have $\mu^X = \mu$ using the identification $\mathfrak{g}^* \cong \mathbb{R}$. Also, we have that $X^\#$ is the vector field as above in Definition 3.16 as $\exp(tX)$ is the integral curve that goes through 0 for $t = 0$ and has derivative 1, which gives $\exp(tX) = t$ in this case. So summarised, condition 1 is in this case that there is a function μ such that $d\mu = \iota_{X^\#}\omega$, where $X^\#$ is the vector field as above in Definition 3.16. So, by Lemma 3.12, we get that the existence of such a function μ that satisfies condition 1 is equivalent to the action being hamiltonian in the sense of Definition 3.16. Also, here the coadjoint action is trivial as $\mathbb{R}$ is commutative, thus the only thing that is left to check is that an action that is hamiltonian in the sense of Definition 3.16 satisfies $\mu \circ \Psi_t = \mu$, which is the same as $\mathcal{L}_{X^\#}\mu = 0$ by Lemma 2.12. But this is true as by Lemma 2.8

$$\mathcal{L}_{X^\#}\mu = d\iota_{X^\#}\mu + \iota_{X^\#}d\mu = 0 + \iota_{X^\#}\iota_{X^\#}\omega = 0.$$

So, Definition 3.17 indeed generalises Definition 3.16.

Chapter 4

Properties of Moment Maps

In this chapter, we discuss two properties of moment maps. The first propertry we will consider is the action of a Lie group G on a product manifold $M_1 \times M_2$, where we have a hamiltonian action of G on the manifolds M_i. Then, we will also consider the action of a product Lie group $G \times H$ on some manifold M, where we have a hamiltonian action of G and H on the manifold M. Note that this action is a priori not well-defined and we will need the condition that the actions commute in order to define this action.

4.1 Moment Map for Product Action

So, let G be a Lie group that acts on two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) and assume that these action are hamiltonian with moment maps μ_1 and μ_2 respectively. We want to prove the following lemma.

Lemma 4.1. *The action of G on $M_1 \times M_2$ given by $g \cdot (p_1, p_2) = (g \cdot p_1, g \cdot p_2)$ is hamiltonian with moment map given by*

$$\mu \colon M_1 \times M_2 \to \mathfrak{g}^*, \ (p_1, p_2) \mapsto \mu_1(p_1) + \mu_2(p_2),$$

where $M_1 \times M_2$ is a symplectic manifold with symplectic form $\omega = \rho_1^* \omega_1 + \rho_2^* \omega_2$ for $\rho_i \colon M_1 \times M_2 \to M_i$ the projections.

Proof. For $p = (p_1, p_2) \in M_1 \times M_2$ and $X \in \mathfrak{g}$ we compute the function μ^X

$$\mu^X(p) = \langle \mu(p), X \rangle = \langle \mu_1(p_1), X \rangle + \langle \mu_2(p_2), X \rangle = \mu_1^X(p_1) + \mu_2^X(p_2).$$

The vector field $X^\#$ as in Definition 3.17 satisfies

$$X^\# = X^{1,\#} \times X^{2,\#},$$

where the $X^{i,\#}$ are the vector fields of Definition 3.17 for the actions on M_i. This is true as the flow of a product vector field is given by the product of the flows, so $X^{1,\#} \times X^{2,\#}$ has flow $(\Psi_1 \times \Psi_2)_{\exp(tX)}$, as wanted.

Now, let $Y_1 \in \mathfrak{X}(M_1)$ and $Y_2 \in \mathfrak{X}(M_2)$. Then we can compute, using the formula for μ^X from above,

$$d\mu^X(Y_1 \times Y_2) = d\mu_1^X(Y_1) + d\mu_2^X(Y_2).$$

Also, using the definition of the symplectic form on $M_1 \times M_2$, we can compute

$$\iota_{X^\#}\omega = \iota_{X^{1,\#} \times X^{2,\#}}(\rho_1^*\omega_1 + \rho_2^*\omega_2),$$

which evaluated at $Y_1 \times Y_2$ gives

$$\omega_1(X^{1,\#}, Y_1) + \omega_2(X^{2,\#}, Y_2).$$

Now, as the original actions were hamiltonian, we get that

$$d\mu_i^X = \iota_{X^{i,\#}}\omega_i.$$

Using that the Y_i were arbitrary, we can conclude from this that

$$d\mu^X = \iota_{X^\#}\omega,$$

which shows that the action of G on $M_1 \times M_2$ is also hamiltonian with the claimed moment map. □

4.2 Moment Map for Action of Product Group

Let G and H be Lie groups that act on the same symplectic manifold (M, ω). Call these action Ψ^1 and Ψ^2 and assume that both are hamiltonian with moment maps μ_1 and μ_2 respectively. Assume furthermore that the actions commute. Then the action of the product group $G \times H$ on M defined by $(g, h) \cdot p = \Psi_g^1(\Psi_h^2(p))$ is well-defined. We now want to show the following lemma.

Lemma 4.2. *If the moment maps are invariant with respect to the other action (i.e. $\mu_1 \circ \Psi_h^2 = \mu_1$ and $\mu_2 \circ \Psi_g^1 = \mu_2$), then the action of the product group as defined above is hamiltonian with moment map*

$$\mu\colon M \to (\mathfrak{g} \times \mathfrak{h})^* \cong \mathfrak{g}^* \times \mathfrak{h}^*,\ p \mapsto \mu_1(p) \oplus \mu_2(p).$$

Proof. We want to verify the conditions of Definition 3.17. So let $X_1 \times X_2 \in \mathfrak{g} \times \mathfrak{h}$. Then the vector field $X^\#$ has flow $\Psi_{\exp(t(X_1 \times X_2))} = \Psi_{(\exp(tX_1),\, \exp(tX_2))}$, so we get

$$X_p^\# = \left.\frac{d}{dt}\right|_{t=0} \Psi^1_{\exp(tX_1)}(\Psi^2_{\exp(tX_2)}(p)).$$

We now use the chain rule in the form that for a function $F(x, y)$ the derivative $\frac{d}{dt}\big|_{t=0} F(t,t) = \frac{\partial F}{\partial x}(0,0) + \frac{\partial F}{\partial y}(0,0)$ together with the fact that the actions commute to get

$$X_p^{\#} = \left(\frac{d}{dt}\bigg|_{t=0} \Psi^1_{\exp(tX_1)} \right) (\Psi^2_{\exp(0X_2)}(p)) + \left(\frac{d}{dt}\bigg|_{t=0} \Psi^2_{\exp(tX_2)} \right) (\Psi^1_{\exp(0X_1)}(p)).$$

As $\exp(0)$ is the identity and $\frac{d}{dt}\big|_{t=0} \Psi^i_{\exp(tX_i)}(p) = X_p^{i,\#}$, where $X^{i,\#}$ denotes the vector field of Definition 3.17 for the action Ψ^i, we get

$$X_p^{\#} = X_p^{1,\#} + X_p^{2,\#}.$$

On the other hand,

$$\mu^{X_1 \times X_2}(p) = \langle \mu(p), X_1 \times X_2 \rangle = \mu_1^{X_1}(p) + \mu_2^{X_2}(p)$$

by definition of the map μ. So, together we get that

$$d\mu^X = d\mu_1^{X_1} + d\mu_2^{X_2} = \iota_{X^{1,\#}}\omega + \iota_{X^{2,\#}}\omega = \iota_{X^{1,\#}+X^{2,\#}}\omega = \iota_{X^{\#}}\omega$$

as the actions Ψ^i are hamiltonian. So it remains to show that the map μ is equivariant with respect to Ψ and Ad^*. For $(g,h) \in G \times H$ and $p \in M$, we get, using commutativity of the actions and invariance of the moment maps with respect to the other action,

$$\begin{aligned} \mu \circ \Psi_{(g,h)}(p) &= \mu(\Psi_g^1(\Psi_h^2(p))) \\ &= \mu_1(\Psi_h^2(\Psi_g^1(p))) \oplus \mu_2(\Psi_g^1(\Psi_h^2(p))) \\ &= \mu_1(\Psi_g^1(p)) \oplus \mu_2(\Psi_h^2(p)). \end{aligned}$$

As the original actions are hamiltonian, we get furthermore that

$$\begin{aligned} \mu_1(\Psi_g^1(p)) \oplus \mu_2(\Psi_h^2(p)) &= \mathrm{Ad}_g^*(\mu_1(p)) \oplus \mathrm{Ad}_h^*(\mu_2(p)) \\ &= (\mathrm{Ad}_g^* \times \mathrm{Ad}_h^*)(\mu_1(p) \oplus \mu_2(p)) \\ &= \mathrm{Ad}_{(g,h)}^* \circ \mu(p). \end{aligned}$$

So, summarised we get, as p was arbitrary,

$$\mu \circ \Psi_{(g,h)} = \mathrm{Ad}_{(g,h)}^* \circ \mu,$$

which is exactly what we wanted to show. □

Chapter 5

Coadjoint Orbits

In this chapter, we want to show that the coadjoint action of a Lie group G on the coadjoint orbits is hamiltonian with the moment map being the inclusion map. For this, we will first consider the vector fields $X^{\#}$ as in Definition 3.17. Then we will show that the coadjoint orbits are symplectic manifolds and write down explicitly a symplectic form. In the end, we can then show that the action is indeed hamiltonian. For the whole chapter, let G be any Lie group and let $X \in \mathfrak{g}$ be a left-invariant vector field. The reference for this chapter is [Sil06, Homework 17, ex. 1-5, Homework 19, ex. 3].

5.1 The Vector Fields $X^{\#}$

We define $X^{\#} \in \mathfrak{X}(\mathfrak{g}^*)$ as in the definition of moment maps

$$X^{\#}_{\xi} = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(tX)}\xi,\ \xi \in \mathfrak{g}^*.$$

We also need a similar vector field $X_{\#} \in \mathfrak{X}(\mathfrak{g})$

$$(X_{\#})_Y = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tX)}Y,\ Y \in \mathfrak{g}.$$

We will now show two lemmas that give a more explicit description of these vector fields.

Lemma 5.1.

$$(X_{\#})_Y = [X, Y].$$

Proof. Let $Y \in \mathfrak{g}$. Then, by definition we have

$$X_{\#}(Y) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tX)}Y(e) \in T_e\mathfrak{g} \simeq \mathfrak{g},$$

where we identify $Y \in \mathfrak{X}^l(G)$ with $Y(e) \in T_eG$ to apply the adjoint action. Now, by the definition of this action, we have

$$\mathrm{Ad}_{\exp(tX)} = Dc_{\exp(tX)}(e),$$

where $c_g = r_{g^{-1}} \circ l_g \colon G \to G$ denotes the conjugation by g. Thus, applying the chain rule, we get

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tX)} Y(e) = \left.\frac{d}{dt}\right|_{t=0} Dr_{\exp(tX)^{-1}}(\exp(tX)) \circ Dl_{\exp(tX)}(e)Y(e).$$

As $Y \in \mathfrak{g}$ is left-invariant, we get that

$$Dl_{\exp(tX)}(e)Y(e) = Y(\exp(tX)).$$

Thus, we get

$$\begin{aligned} X_{\#}(Y) &= \left.\frac{d}{dt}\right|_{t=0} Dr_{\exp(tX)^{-1}}(\exp(tX)) \circ Dl_{\exp(tX)}(e)Y(e) \\ &= \left.\frac{d}{dt}\right|_{t=0} Dr_{\exp(tX)^{-1}}(\exp(tX))Y(\exp(tX)), \end{aligned}$$

which is equal to $\mathcal{L}_X(Y)(e) = [X,Y](e)$ by Lemma 2.14 and the fact that the flow of the left invariant vector field X is $r_{\exp(tX)}$. A proof of this fact can be found in [Mer20, Proposition 12.2(iv)]. Hence, identifying again the tangent vector $[X,Y](e) \in T_eG$ with its associated left-invariant vector field $[X,Y]$, we conclude that $X_{\#}(Y) = [X,Y]$. □

For the next lemma note that $T_\xi(\mathfrak{g}^*) \cong \mathfrak{g}^*$ as $\mathfrak{g}^*$ is a vector space. For $Z \in T_\xi(\mathfrak{g}^*)$ and $Y \in \mathfrak{g}$ we then write $\langle Z, Y\rangle := Z(Y)$, where we think of Z as an element in $\mathfrak{g}^*$.

Lemma 5.2.

$$\left\langle X_\xi^{\#}, Y \right\rangle = \langle \xi, [Y,X]\rangle, \ \forall Y \in \mathfrak{g}.$$

Proof. We have $X_\xi^{\#} = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(tX)}\xi$ by definition. Now, for any $Y \in \mathfrak{g}$ we can compute

$$\begin{aligned} \left\langle X_\xi^{\#}, Y \right\rangle = & \left\langle \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(tX)}\xi, Y \right\rangle = \left.\frac{d}{dt}\right|_{t=0} \left\langle \mathrm{Ad}^*_{\exp(tX)}\xi, Y \right\rangle \\ = & \left.\frac{d}{dt}\right|_{t=0} \langle \xi, \mathrm{Ad}_{\exp(-tX)} Y \rangle = \left\langle \xi, \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t(-X))} Y \right\rangle, \end{aligned}$$

where we used in the second to last equality the fact that $\exp(tX)^{-1} = \exp(-tX)$. Now, using Lemma 5.1, we get that

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t(-X))} Y = [-X, Y] = [Y, X].$$

So we can conclude that

$$\left\langle X_\xi^{\#}, Y \right\rangle = \left\langle \xi, \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t(-X))} Y \right\rangle = \langle \xi, [Y, X] \rangle \,,$$

as wanted. □

We will later study the restriction of the coadjoint action on the coadjoint orbits $G \cdot \xi$, for $\xi \in \mathfrak{g}^*$. For this, we also need the following consequence of Lemma 2.24.

Lemma 5.3. *The space of vector fields $\mathfrak{X}(G \cdot \xi)$ is $C^\infty(G \cdot \xi)$-generated by the vector fields $\left\{X^{\#} \mid X \in \mathfrak{g}\right\}$.*

Proof. In order to prove this, note that the expression for the tangent space $T_\xi(G \cdot \xi)$ from Lemma 2.24 specialises to the following in the case of the coadjoint action

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(tX)} \xi,$$

which is exactly the definition of $X_\xi^{\#}$. So we get that the tangent space $T_\xi(G \cdot \xi)$ is generated by $\left\{X_\xi^{\#} \mid X \in \mathfrak{g}\right\}$. If we now take a basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{g}$, then we get that $T_\xi(G \cdot \xi)$ is generated by $\left\{(X_1)_\xi^{\#}, \ldots, (X_n)_\xi^{\#}\right\}$. Hence, any vector field $X \in \mathfrak{X}(G \cdot \xi)$ can be written as

$$X = \sum_{i=1}^{n} f_i X_i^{\#},$$

where the $f_i \colon G \cdot \xi \to \mathbb{R}$ are smooth functions, which is exactly what we wanted. □

5.2 A Symplectic Form on the Coadjoint Orbits

In this section, we want to show that the coadjoint orbit $G \cdot \xi$ for $\xi \in \mathfrak{g}^*$ is a symplectic manifold. The orbit of a smooth action is in general a manifold by Lemma 2.22, so it remains to show that we can define a symplectic form on it. For this, we first define the following skew-symmetric bilinear form on $\mathfrak{g}$

$$\omega_\xi \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},\ \omega_\xi(X, Y) \coloneqq \langle \xi, [X, Y] \rangle \,.$$

Denote by G_ξ the stabiliser of ξ. This is in fact a Lie subgroup of G. For a proof of this, see for example [Mer20, Proposition 13.3]. We now show that the kernel of the map $X \mapsto \omega_\xi(X, \cdot)$ is the Lie algebra $\mathfrak{g}_\xi$ of the Lie group G_ξ.

For fixed $X \in \mathfrak{g}$, using Lemma 5.2, we have that

$$\begin{aligned}
\omega_\xi(X,Y) = 0 \quad \forall Y \in \mathfrak{g} \quad &\iff \quad \left\langle X_\xi^\#, Y \right\rangle = 0 \quad \forall Y \in \mathfrak{g} \\
&\iff \quad \left\langle \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*_{\exp(tX)}\xi, Y \right\rangle = 0 \quad \forall Y \in \mathfrak{g} \\
&\iff \quad \left\langle \left.\frac{d}{dt}\right|_{t=t_0} \mathrm{Ad}^*_{\exp(tX)}\xi, Y \right\rangle = 0 \quad \forall Y \in \mathfrak{g},\ \forall t_0 \\
&\iff \quad \left.\frac{d}{dt}\right|_{t=t_0} \mathrm{Ad}^*_{\exp(tX)}\xi = 0 \quad \forall t_0,
\end{aligned}$$

where in the second to last equivalence we used that

$$\left\langle \mathrm{Ad}^*_{\exp((t_0+t)X)}\xi, Y \right\rangle = \left\langle \mathrm{Ad}^*_{\exp(tX)}\xi, \mathrm{Ad}_{\exp(-t_0X)}Y \right\rangle.$$

Now, we furthermore have that

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0} \mathrm{Ad}^*_{\exp(tX)}\xi = 0 \quad \forall t_0 \quad &\iff \quad \mathrm{Ad}^*_{\exp(tX)}\xi = \xi \quad \forall t \\
&\iff \quad \exp(tX) \in G_\xi \quad \forall t.
\end{aligned}$$

Using the fact that $\exp(tX)$ is the flow of X, we get that

$$\exp(tX) \in G_\xi \quad \forall t \quad \iff \quad X \in \mathfrak{g}_\xi.$$

Summarised, we get that

$$\omega_\xi(X,Y) = 0 \quad \forall Y \in \mathfrak{g} \quad \iff \quad X \in \mathfrak{g}_\xi,$$

as wanted.

The goal in the following is to show that ω_ξ in fact defines a 2-form on the tangent space of $G\cdot\xi$. So we would like ω_ξ to define a map $T_\xi(G\cdot\xi)\times T_\xi(G\cdot\xi) \to \mathbb{R}$. For this, note that the orbit through ξ is diffeomorphic to G/G_ξ; see Lemma 2.23. Using this, we get that

$$T_\xi(G\cdot\xi) \cong T_e(G/G_\xi) \cong \mathfrak{g}/\mathfrak{g}_\xi.$$

By the argumentation above, we have that $\omega_\xi \colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ is 0 on $\mathfrak{g}_\xi \times \mathfrak{g}$ and $\mathfrak{g}\times\mathfrak{g}_\xi$, we get that ω_ξ induces a map $\mathfrak{g}/\mathfrak{g}_\xi \times \mathfrak{g}/\mathfrak{g}_\xi \to \mathbb{R}$, as wanted. Now, the kernel of the original map $X \mapsto \omega_\xi(X,\cdot)$ was exactly $\mathfrak{g}_\xi$, so we get that the map

$$\omega_\xi \colon \mathfrak{g}/\mathfrak{g}_\xi \times \mathfrak{g}/\mathfrak{g}_\xi \to \mathbb{R}$$

is in fact non-degenerate. As this works for any element in the orbit of ξ, we have defined a 2-form $\omega \in \Omega^2(G\cdot\xi)$ with ω_ξ being non-degenerate.

To show that ω is a symplectic form, it remains to show the following lemma, namely that ω is closed.

Lemma 5.4. *The 2-form ω as defined above is closed.*

Proof. Using Theorem 2.3 and Lemma 5.3, it is enough to show that

$$d\omega(X^{\#}, Y^{\#}, Z^{\#}) = 0$$

for any $X, Y, Z \in \mathfrak{g}$. Using Definition 2.6, we get that for $\xi \in \mathfrak{g}^*$

$$\begin{aligned} d\omega_\xi(X^{\#}, Y^{\#}, Z^{\#}) = {} & X^{\#}_\xi(\omega(Y^{\#}, Z^{\#})) - Y^{\#}_\xi(\omega(X^{\#}, Z^{\#})) + Z^{\#}_\xi(\omega(X^{\#}, Y^{\#})) \\ & - \omega_\xi([X^{\#}, Y^{\#}], Z^{\#}) + \omega_\xi([X^{\#}, Z^{\#}], Y^{\#}) - \omega_\xi([Y^{\#}, Z^{\#}], X^{\#}). \end{aligned}$$

For the second line we can compute, using the definition of ω_ξ,

$$\omega_\xi([X^{\#}, Y^{\#}], Z^{\#}) = \left\langle \xi, [[X^{\#}, Y^{\#}], Z^{\#}] \right\rangle.$$

We can use this also for the other two terms on the second line and thus get that the second line in the above expression is equal to zero as by the Jacobi identity we have

$$\begin{aligned} & - [[X^{\#}, Y^{\#}], Z^{\#}] + [[X^{\#}, Z^{\#}], Y^{\#}] - [[Y^{\#}, Z^{\#}], X^{\#}] \\ & \qquad = - \big([[X^{\#}, Y^{\#}], Z^{\#}] + [[Y^{\#}, Z^{\#}], X^{\#}] + [[Z^{\#}, X^{\#}], Y^{\#}]\big) = 0. \end{aligned}$$

So, it remains to show that

$$X^{\#}_\xi(\omega(Y^{\#}, Z^{\#})) - Y^{\#}_\xi(\omega(X^{\#}, Z^{\#})) + Z^{\#}_\xi(\omega(X^{\#}, Y^{\#})) = 0.$$

For this we compute

$$\begin{aligned} X^{\#}_\xi(\omega(Y^{\#}, Z^{\#})) &= \frac{d}{dt}\bigg|_{t=0} \omega(Y^{\#}, Z^{\#}) \circ \mathrm{Ad}^*_{\exp(tX)}(\xi) && \text{(definition of } X^{\#}) \\ &= \frac{d}{dt}\bigg|_{t=0} \left\langle \mathrm{Ad}^*_{\exp(tX)}(\xi), [Y, Z] \right\rangle && \text{(definition of } \omega) \\ &= \frac{d}{dt}\bigg|_{t=0} \left\langle \xi, \mathrm{Ad}_{\exp(-tX)}[Y, Z] \right\rangle && \text{(definition of } \mathrm{Ad}^*) \\ &= \left\langle \xi, (-X_{\#})_{[Y,Z]} \right\rangle && \text{(definition of } X_{\#}) \\ &= \langle \xi, [-X, [Y, Z]] \rangle && \text{(Lemma 5.1)} \\ &= \langle \xi, [[Y, Z], X] \rangle. \end{aligned}$$

Now, by a similar argument as for the second line, also the first line of the expression for $d\omega_\xi(X^{\#}, Y^{\#}, Z^{\#})$ vanishes by the Jacobi identity. So we have indeed that

$$d\omega(X^{\#}, Y^{\#}, Z^{\#}) = 0,$$

which shows that the 2-form ω is closed. □

Summarised, we get the following theorem.

Theorem 5.5. *The orbit $G \cdot \xi$ of the coadjoint action is a symplectic manifold with the symplectic form $\omega \in \Omega^2(G \cdot \xi)$ defined by*

$$\omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle ,$$

where $X, Y \in \mathfrak{g}$ and we use the identification $T_\xi(G \cdot \xi) \cong \mathfrak{g}/\mathfrak{g}_\xi$.

5.3 The Inclusion Map as Moment Map

In this section, we show that the inclusion map $i\colon G \cdot \xi \to g^*$ is a moment map for the coadjoint action. For this we check the two properties in Definition 3.17. The second one is clear as the action we consider is (the restriction of) the coadjoint action. So it remains to check the first condition.

For $X \in \mathfrak{g}$, we define the map $i^X : G \cdot \xi \to \mathbb{R}$ as in Definition 3.17, i.e.

$$i^X(\eta) = \langle i(\eta), X \rangle, \quad \eta \in G \cdot \xi.$$

We now need to compute di^X. For $Y^\# \in \mathfrak{X}(G \cdot \xi)$ as in Section 5.1 and $\eta \in G \cdot \xi$ we get

$$\begin{aligned} di^X(Y^\#)(\eta) &= Y^\#_\eta(i^X) \\ &= \frac{d}{dt}\bigg|_{t=0} \left(\mathrm{Ad}^*_{\exp(tY)}\eta\right)(i^X) \\ &= \frac{d}{dt}\bigg|_{t=0} i^X \circ \mathrm{Ad}^*_{\exp(tY)}\eta \\ &= \frac{d}{dt}\bigg|_{t=0} \left\langle i(\mathrm{Ad}^*_{\exp(tY)}\eta), X \right\rangle \\ &= \left\langle Y^\#(\eta), X \right\rangle \\ &= \langle \eta, [X, Y] \rangle , \end{aligned}$$

where in the second to last equality we used that i is the inclusion and in the last one we used Lemma 5.2. On the other hand, we have

$$\iota_{X^\#}\omega(Y^\#)(\eta) = \omega_\eta(X^\#, Y^\#) = \langle \eta, [X, Y] \rangle,$$

where the last equality used the definition of ω_η. As $\eta \in G \cdot \xi$ was arbitrary and the vector fields $Y^\#$ $C^\infty(G \cdot \xi)$-generate $\mathfrak{X}(G \cdot \xi)$ by Lemma 5.3, we get that

$$di^X = \iota_{X^\#}\omega,$$

as required in Definition 3.17.

Summarised, we get the following theorem, which is what we wanted to show in this chapter.

Theorem 5.6. *The coadjoint action on the coadjoint orbits is hamiltonian with moment map the inclusion $i\colon G \cdot \xi \to \mathfrak{g}^*$. Thus, $(G \cdot \xi, \omega, G, i)$ is a hamiltonian G-space.*

Chapter 6

Actions of $U(n)$

In this chapter we are going to look at some natural actions of the unitary matrices $U(n)$ and show that they are hamiltonian. Recall from Lemma 2.18 that the Lie algebra of $U(n)$ is the set of skew-hermitian matrices, i.e. $A \in \mathbb{C}^{n\times n}$ with $A^* = -A$, and is denoted by $\mathfrak{u}(n)$.

6.1 Natural Action on $\mathbb{C}^n$

In this section we consider the action of $U(n)$ on $\mathbb{C}^n$, where $A \cdot p = Ap$ is given by normal matrix-vector multiplication. $\mathbb{C}^n$ is a symplectic manifold via the identification

$$\mathbb{C}^n \cong \mathbb{R}^{2n},\ p + iq \mapsto \begin{pmatrix} p \\ q \end{pmatrix}.$$

If we denote the real coordinates by x^j and the imaginary coordinates by y^j, i.e. $x^j(p + iq) = p_j$ and $y^j(p + iq) = q_j$, a symplectic form on $\mathbb{C}^n$ is given by

$$\omega \coloneqq \sum_{j=1}^{n} dx^j \wedge dy^j.$$

We want to prove the theorem.

Theorem 6.1. *The action of $U(n)$ on $\mathbb{C}^n$ as described above is hamiltonian with moment map*

$$\mu\colon \mathbb{C}^n \to \mathfrak{u}(n)^*,\ z \mapsto \frac{i}{2} tr(zz^* \cdot),$$

where tr denotes the trace of a matrix.

Throughout the proof, we will write $g = h + ik$ for a matrix in $U(n)$ and $X = V + iW$ for a matrix in $\mathfrak{u}(n)$. Note that, as X is skew-hermitian, we have

$V = -V^T$ and $W = W^T$. Also, we will sometimes identify $z = p + iq \in \mathbb{C}^n$ with the vector $\begin{pmatrix} p \\ q \end{pmatrix}$, depending on what is better for the calculation we want to do.

We first prove the following lemma.

Lemma 6.2. *The map μ^X as in Definition 3.17 satisfies*

$$\mu^X(z) = z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} z.$$

Proof. We compute

$$\mu^X(z) = \langle \mu(z), X \rangle = \frac{i}{2}\mathrm{tr}(zz^*X).$$

Note that

$$\overline{\mathrm{tr}(zz^*X)} = \mathrm{tr}((zz^*X)^*) = \mathrm{tr}(X^*zz^*) = \mathrm{tr}(zz^*X^*) = -\mathrm{tr}(zz^*X)$$

as the trace is invariant under cyclic permutations and as X is skew-hermitian. Thus, $\mathrm{tr}(zz^*X)$ is imaginary. This also shows that the map μ as defined in Theorem 6.1 is well-defined, i.e. evaluated at an element in $\mathfrak{u}(n)$ it gives a real number.

We have, using again the invariance of the trace under cyclic permutation and using that the trace of a complex number is the number itself,

$$\mathrm{tr}(zz^*X) = \mathrm{tr}(z^*Xz) = z^*Xz = (p - iq)^T(V + iW)(p + iq).$$

Now, as this is imaginary, note that only the terms with an i remain, so we get

$$z^*Xz = p^T(iVq + iWp) + (-i)q^T(Vp - Wq).$$

Thus, for $\mu^X(z)$ we get

$$\begin{aligned} \mu^X(z) &= \frac{i}{2}\mathrm{tr}(zz^*X) \\ &= \frac{i}{2}\left(p^T(iVq + iWp) + (-i)q^T(Vp - Wq)\right) \\ &= -\frac{1}{2}p^TVq - \frac{1}{2}p^TWp + \frac{1}{2}q^TVp - \frac{1}{2}q^TWq. \end{aligned}$$

Now, note that $p = \begin{pmatrix} I_n & 0 \end{pmatrix} z$ and $q = \begin{pmatrix} 0 & I_n \end{pmatrix} z$. So we can rewrite the above formula further to get

$$p^TVq = z^T \begin{pmatrix} I_n \\ 0 \end{pmatrix} V \begin{pmatrix} 0 & I_n \end{pmatrix} z = z^T \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} z.$$

Similarly, we get for the other three terms

$$p^TWp = z^T \begin{pmatrix} I_n \\ 0 \end{pmatrix} W \begin{pmatrix} I_n & 0 \end{pmatrix} z = z^T \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} z,$$

$$q^T V p = z^T \begin{pmatrix} 0 \\ I_n \end{pmatrix} V \begin{pmatrix} I_n & 0 \end{pmatrix} z = z^T \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} z,$$

$$q^T W q = z^T \begin{pmatrix} 0 \\ I_n \end{pmatrix} W \begin{pmatrix} 0 & I_n \end{pmatrix} z = z^T \begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix} z.$$

Together, we have

$$\begin{aligned} \mu^X(z) &= -\frac{1}{2} p^T V q - \frac{1}{2} p^T W p + q^T V p - q^t W q \\ &= \frac{1}{2} z^T \left(-\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix} \right) z \\ &= \frac{1}{2} z^T \begin{pmatrix} -W & -V \\ V & -W \end{pmatrix} z, \end{aligned}$$

as claimed. □

We can now prove the theorem.

Proof of Theorem 6.1. Recall that in order to show that the action is hamiltonian, we need to show that $\iota_{X^\#}\omega = d\mu^X$ and that μ is equivariant. In the following we will first compute these two terms separately and show that they are equal. Afterwards, we will then also show that μ is equivariant.

For the action we consider here the vector field $X^\#$ as in Definition 3.17 has flow $\Psi_{\exp(tX)} = \Psi_{\exp(t(V+iW))}$. Thus the flow at a point z is $\exp(t(V+iW))\, z$. In the case of matrix Lie groups, exp is normal matrix exponentiation. A proof of this fact can for example be found in [Mer20, Proposition 12.7]. Thus, $X^\#(z)$ is

$$\begin{aligned} X^\#(z) &= \left.\frac{d}{dt}\right|_{t=0} \exp(t(V+iW))\, z \\ &= (V+iW) \exp(0(V+iW))\, z \\ &= \begin{pmatrix} V & -W \\ W & V \end{pmatrix} z, \end{aligned}$$

where in the last step we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Thus, we can now compute $\iota_{X^\#}\omega$. For this let $Y \in \mathfrak{X}(\mathbb{C}^n)$. Denote Y_z by $\tilde{z} \in \mathbb{R}^{2n}$. Then we get, using

Lemma 2.4,

$$\begin{aligned}
(\iota_{X^\#}\omega(Y))(z) &= \omega(X^\#, Y)(z) \\
&= \sum_{j=1}^{n} dx^j \wedge dy^j (X^\#, Y)(z) \\
&= \sum_{j=1}^{n} dx^j(X^\#)(z) dy^j(Y)(z) - dx^j(Y)(z) dy^j(X^\#)(z) \\
&= \sum_{j=1}^{n} \left(\begin{pmatrix} V & -W \\ W & V \end{pmatrix} z \right)_j \tilde{z}_{j+n} - \left(\begin{pmatrix} V & -W \\ W & V \end{pmatrix} z \right)_{j+n} \tilde{z}_j \\
&= \sum_{j=1}^{n} \left(\begin{pmatrix} V & -W \end{pmatrix} z \right)_j \tilde{z}_{j+n} - \left(\begin{pmatrix} W & V \end{pmatrix} z \right)_j \tilde{z}_j \\
&= \sum_{j=1}^{n} \left(\begin{pmatrix} V & -W \end{pmatrix} z \right)_j \tilde{z}_{j+n} + \left(\begin{pmatrix} -W & -V \end{pmatrix} z \right)_j \tilde{z}_j \\
&= \sum_{j=1}^{2n} \left(\begin{pmatrix} -W & -V \\ V & -W \end{pmatrix} z \right)_j \tilde{z}_j \\
&= z^T \begin{pmatrix} -W & -V \\ V & -W \end{pmatrix}^T \tilde{z} \\
&= z^T \begin{pmatrix} -W & -V \\ V & -W \end{pmatrix} \tilde{z}.
\end{aligned}$$

Now, we want to compute $d\mu^X$. Note that we can write Y_z as $\frac{d}{dt}\big|_{t=0} z + t\tilde{z}$. Then we get, using Lemma 6.2,

$$\begin{aligned}
(d\mu^X(Y))(z) &= Y_z(\mu^X) \\
&= \left(\frac{d}{dt}\bigg|_{t=0} z + t\tilde{z} \right) (\mu^X) \\
&= \frac{d}{dt}\bigg|_{t=0} \mu^X(z + t\tilde{z}) \\
&= \frac{d}{dt}\bigg|_{t=0} (z + t\tilde{z})^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} (z + t\tilde{z}) \\
&= z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} \tilde{z} + \tilde{z}^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} z
\end{aligned}$$

Now note that these are real numbers, so they are equal to their transpose, so we get that

$$
\begin{aligned}
(d\mu^X(Y))(z) &= z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} \tilde{z} + \tilde{z}^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} z \\
&= z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} \tilde{z} + z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix}^T \tilde{z} \\
&= z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} \tilde{z} + z^T \begin{pmatrix} -\frac{1}{2}W & -\frac{1}{2}V \\ \frac{1}{2}V & -\frac{1}{2}W \end{pmatrix} \tilde{z} \\
&= z^T \begin{pmatrix} -W & -V \\ V & -W \end{pmatrix} \tilde{z}
\end{aligned}
$$

Comparing this with our expression for $(\iota_{X^\#}\omega(Y))(z)$, we get

$$(\iota_{X^\#}\omega(Y))(z) = (d\mu^X(Y))(z).$$

As $z \in \mathbb{C}^n$ and $Y \in \mathfrak{X}(\mathbb{C}^n)$ were arbirary, we can conclude that

$$\iota_{X^\#}\omega = d\mu^X.$$

It remains now to show that μ is equivariant. Let $g \in U(n)$, $z \in \mathbb{C}^n$ and $X \in \mathfrak{u}(n)$. Then we can compute

$$\mu(\Psi_g(z))(X) = \mu(gz)(X) = \frac{i}{2}\mathrm{tr}(gz(gz)^*X) = \frac{i}{2}\mathrm{tr}(gzz^*g^*X).$$

On the other hand, using Lemma 2.21 and the definition of the coadjoint action, we get that

$$
\begin{aligned}
\mathrm{Ad}_g^*(\mu(z))(X) &= \langle \mathrm{Ad}_g^*(\mu(z)), X\rangle \\
&= \langle \mu(z), \mathrm{Ad}_{g^{-1}}X\rangle \\
&= \langle \mu(z), g^{-1}Xg\rangle \\
&= \frac{i}{2}\mathrm{tr}(zz^*g^{-1}Xg).
\end{aligned}
$$

Now, using that the trace is invariant under cyclic permutations and that $g^* = g^{-1}$ as $g \in U(n)$, we can conclude that

$$\mu(\Psi_g(z))(X) = \mathrm{Ad}_g^*(\mu(z))(X).$$

As X and z were arbitrary, this shows that

$$\mu \circ \Psi_g = \mathrm{Ad}_g^* \circ \mu.$$

Hence, μ is equivariant and thus the action is hamiltonian with the claimed moment map. □

6.2 Natural Action on $\mathbb{C}^{n\times k}$

In this section, we will look at a generalisation of what we did before. We will specifically look at the natural action of $U(n)$ on the set of $(n\times k)$-matrices over $\mathbb{C}$, i.e. $g\cdot A = gA$ is just normal matrix multiplication. We want to prove the following theorem, whose proof will follow from Lemma 4.1 and Theorem 6.1.

Theorem 6.3. *The action of $U(n)$ on $\mathbb{C}^{n\times k}$ as described above is hamiltonian with moment map*

$$\mu\colon \mathbb{C}^{n\times k} \to \mathfrak{u}(n)^*,\ A \mapsto \frac{i}{2} tr(AA^*\cdot).$$

Proof. We have that $\mathbb{C}^{n\times k} = (\mathbb{C}^n)^k$ and the action of $U(n)$ on $(\mathbb{C}^n)^k$ is as the action in Lemma 4.1 where the action of $U(n)$ on $\mathbb{C}^n$ is as in Theorem 6.1. So we can apply this lemma inductively k times and get that this action is indeed hamiltonian and that the moment map is given by

$$A \in \mathbb{C}^{n\times k} \mapsto \mu(A) = \sum_{j=1}^{k} \mu'(A_j),$$

where we denote by μ' the moment map of Theorem 6.1 and A_j denotes the j-th column of A. Thus, we have

$$\mu(A) = \sum_{j=1}^{k} \frac{i}{2}\mathrm{tr}(A_j A_j^* \cdot) = \frac{i}{2}\mathrm{tr}(\sum_{j=1}^{k} A_j A_j^* \cdot).$$

Thus, it is enough to show that

$$\sum_{j=1}^{k} A_j A_j^* = AA^*.$$

For this, we compute the (a, b)-th entry of these matrices. We have

$$\left(\sum_{j=1}^{k} A_j A_j^*\right)_{ab} = \sum_{j=1}^{k} A_{aj}\overline{A}_{bj} = \sum_{j=1}^{k} A_{aj}A_{jb}^* = (AA^*)_{ab},$$

which finishes the proof. □

6.3 Conjugation Action on $\mathbb{C}^{n\times n}$

We now want to show that the action of $U(n)$ on $\mathbb{C}^{n\times n}$ by conjugation is hamiltonian. For this, we first want to look at the following action that is similar to the first two sections in this chapter

$$\Psi\colon U(n) \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n},\ g\cdot A = Ag^*.$$

We want to argue why this is hamiltonian with moment map $\mu(A) = -\frac{i}{2}\mathrm{tr}(A^*A\cdot)$.

For this, consider first the action of $U(n)$ on $\mathbb{C}^n$ defined by $g \cdot z = (z^T g^*)^T = \overline{g}z$. Similar to the Section 6.1, this action is hamiltonian with moment map $\mu(z) = \overline{\frac{i}{2}\mathrm{tr}(zz^*\cdot)} = -\frac{i}{2}\mathrm{tr}(\overline{z}z^T\cdot)$. Now, the action Ψ as above can be written as

$$g \cdot A = \begin{pmatrix} \left(\overline{g} \cdot A_1^T\right)^T \\ \vdots \\ \left(\overline{g} \cdot A_n^T\right)^T \end{pmatrix},$$

where the A_j are here the rows of A. Thus, we can as in Section 6.2 use Lemma 4.1 to get that the action is hamiltonian with moment map

$$\mu(A) = -\frac{i}{2}\sum_{j=1}^{n} \mathrm{tr}(\overline{A_j^T}A_j\cdot).$$

To prove our claim about the action Ψ, it remains to show that $\sum_{j=1}^n A_j^* A_j = A^*A$, but this works exactly the same as in the proof of Theorem 6.3. So, we have indeed that the action of $U(n)$ on $\mathbb{C}^{n\times n}$ given by $g \cdot A = Ag^*$ is hamiltonian with moment map

$$\mu(A) = -\frac{i}{2}\mathrm{tr}(A^*A\cdot).$$

We will now move on to prove the following theorem.

Theorem 6.4. *The action of $U(n)$ on $\mathbb{C}^{n\times n}$ by conjugation (i.e. $g \cdot A = gAg^*$) is hamiltonian with moment map*

$$\mu\colon \mathbb{C}^{n\times n} \to \mathfrak{u}(n)^*,\ A \mapsto \frac{i}{2}tr([A, A^*]\cdot).$$

Proof. We can write our action as the composition of the two commuting actions $g \cdot A = gA$ and $g \cdot A = Ag^*$. We have seen before that these are hamiltonian. We want to apply Lemma 4.2 to get that the action of $U(n) \times U(n)$ on $\mathbb{C}^{n\times n}$ given by $(g, h) \cdot A = gAh^*$ is also hamiltonian with moment map

$$\mu\colon \mathbb{C}^{n\times n} \to \mathfrak{u}(n)^* \times \mathfrak{u}(n)^*,\ A \mapsto \mu_1(A) \oplus \mu_2(A),$$

where μ_1 and μ_2 are the moment maps of the actions $g \cdot A = gA$ and $g \cdot A = Ag^*$. For this to work, we need to check that we can really apply Lemma 4.2, i.e. we need to show that the moment maps are invariant with respect to the other action. But this is true, as for $g \in U(n)$ we have

$$\begin{aligned}\mu_1(Ag^*) &= \frac{i}{2}\mathrm{tr}((Ag^*)(Ag^*)^*\cdot)\\ &= \frac{i}{2}\mathrm{tr}(Ag^*(g^*)^*A^*\cdot)\\ &= \frac{i}{2}\mathrm{tr}(AA^*\cdot)\\ &= \mu_1(A),\end{aligned}$$

since $g \in U(n)$ implies that $g^*(g^*)^* = g^*g$ is the identity matrix. Similarly,

$$\mu_2(gA) = -\frac{i}{2}\mathrm{tr}((gA)^*(gA)\cdot) = -\frac{i}{2}\mathrm{tr}(A^*g^*gA\cdot) = \mu_2(g),$$

as g^*g is again the identity matrix since $g \in U(n)$.

So we have indeed that the action of $U(n) \times U(n)$ on $\mathbb{C}^{n\times n}$ given by $(g,h)\cdot A = gAh^*$ is hamiltonian with moment map μ as defined above.

When we now restrict our action to the diagonal $D = \{(g,g) \mid g \in U(n)\}$, the action is still hamiltonian with moment map $\mu\colon \mathbb{C}^{n\times n} \to \mathfrak{g}^* \times \mathfrak{g}^* \hookrightarrow \mathfrak{d}^*$, where the second map is the inclusion of $\mathfrak{g}^* \times \mathfrak{g}^*$ into $\mathfrak{d}^*$, which makes sense as $\mathfrak{d}$ (the Lie algebra of D) is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{g}$.

Now we have that $D \cong G$ and $\mathfrak{d}^* \cong \mathfrak{g}^*$ via

$$f \in \mathfrak{d}^* \mapsto f \circ i \in \mathfrak{g}^*,$$

where $i\colon \mathfrak{g} \to \mathfrak{d}$ is the map $x \mapsto (x,x)$. Thus, the action as in the statement of the theorem of G on $\mathbb{C}^{n\times n}$ is also hamiltonian with moment map

$$\mu\colon \mathbb{C}^{n\times n} \to \mathfrak{g}^*,\ A \mapsto \mu_1(A) + \mu_2(A).$$

Now, we have

$$\mu_1(A) + \mu_2(A) = \frac{i}{2}\mathrm{tr}(AA^*\cdot) - \frac{i}{2}\mathrm{tr}(A^*A\cdot)$$

by Section 6.2 and the arguments at the beginning of this section. Furthermore, we have

$$\frac{i}{2}\mathrm{tr}(AA^*\cdot) - \frac{i}{2}\mathrm{tr}(A^*A\cdot) = \frac{i}{2}\mathrm{tr}([A,A^*]\cdot),$$

which finishes the proof of the theorem. □

6.4 Generalisation of These Examples

We want to shortly mention that we can generalise the examples before to more general actions. Namely, we have that any symplectic representation of $\mathbb{C}^n$ (i.e. a linear symplectic action on $\mathbb{C}^n$) is a hamiltonian action. Even more concretely, one gets the following theorem. A reference for this is [Mei00, Section 6.4.5], where there is an even more general statement.

Theorem 6.5. *Let $\rho\colon G \to \mathit{Sympl}(\mathbb{C}^n,\omega)$ be a symplectic representation of a Lie group G on $\mathbb{C}^n$ with the standard symplectic structure. Then ρ defines a hamiltonian action with moment map given by*

$$\mu\colon \mathbb{C}^n \to \mathfrak{g}^*,\ \mu(z)(X) = \frac{i}{4}\left(z^*\tilde{X}z - z^*\tilde{X}^*z\right),$$

where $\tilde{\rho}\colon \mathfrak{g} \to$ Lie algebra of $\mathit{Sympl}(\mathbb{C}^n,\omega)$, $X \mapsto \tilde{X}$ is the derivative of ρ at the identity.

We will now shortly sketch how to use this for the action of $U(n)$ on $\mathbb{C}^{n\times k}$ as in Section 6.2.

There, we have not just a symplectic representation but a unitary representation, thus the moment map gets $\frac{i}{2}z^*\tilde{X}z$, as $\tilde{X}$ is then skew-hermitian. We also need to identify $\mathbb{C}^{k\times n}$ with $\mathbb{C}^{nk}$. We do this by taking the columns of a matrix $A \in \mathbb{C}^{n\times k}$ and putting them below each other in a vector $z_A \in \mathbb{C}^{nk}$. Then our action ρ is mapping a matrix $g \in U(n)$ to a matrix $\rho(g) \in U(nk)$ that is a block diagonal matrix with k times the matrix g on the diagonal as g acts on the columns of a matrix A by left multiplication. Then, also $\tilde{\rho}$ maps a matrix in $\mathfrak{u}(n)$ to a block diagonal matrix with this matrix k times on the diagonal. By the above theorem we get as a moment map for the action

$$\mu(z_A)(X) = \frac{i}{2}z_A^*\tilde{X}z_A = \frac{i}{2}\sum_{j=1}^{k} A_j^* X A_j,$$

where the A_j are the columns of the matrix A. Then, we can use that for a complex number α, $\mathrm{tr}(\alpha) = \alpha$, and get, using also the invariance of the trace under cyclic permutations,

$$\mu(z_A)(X) = \frac{i}{2}\mathrm{tr}(\sum_{j=1}^{k} A_j^* X A_j) = \frac{i}{2}\mathrm{tr}(\sum_{j=1}^{k} A_j A_j^* X).$$

Now, as in Section 6.2, we have that

$$\sum_{j=1}^{k} A_j A_j^* = AA^*.$$

Thus, the above theorem gives us the moment map

$$\mu(z_A) = \frac{i}{2}\mathrm{tr}(AA^*\cdot),$$

which is the same as in Section 6.2.

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