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Symplectic Toric Manifolds and Delzant's Theorem

Semester Paper

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Abstract

We provide a brief introduction to symplectic geometry with the goal to understand Delzant's correspondence theorem, which gives a bijection between symplectic toric manifolds and Delzant polytopes.

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Chapter 0

Introduction

Delzant's theorem [9] classifies (equivalence classes of) symplectic toric manifolds in terms of the combinatorial data encoded by a Delzant polytope. A symplectic toric manifold $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ is a connected and compact (smooth) manifold M^{2n} of dimension $2n$ carrying a symplectic structure given by the closed, nondegenerate 2-form ω, paired with an effective Hamiltonian action of the standard n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

Just as any n-dimensional *topological* manifold looks locally like $\mathbb{R}^n$, any $2n$-dimensional *symplectic* manifold looks locally like $\mathbb{R}^{2n}$ equipped with a symplectic form

$$\omega = \sum_{i=1}^{n} x_i \wedge y_i,$$

where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are local coordinates in $\mathbb{R}^{2n}$. This local classification of symplectic manifolds is made precise in the theorem of Darboux, Theorem 1.2.1. It shows an important similarity between symplectic structures and structures based on the complex numbers: The symplectic structure "weaves together" pairs of coordinates (x, y) in an analogous way as a complex number $z = x + iy$ combines two real numbers x and y. On a historical note, it is due to such similarities to complex structures that symplectic structures received their name. As clarified in [21], the word *symplectic* is coined by the German mathematician Hermann Weyl [23, p.165] in 1939, who replaced the Latin roots in the word "complex" (*com* "with, together" + *plectere* " to weave, braid, twine") with the Greek ones (*syn* + *plekein*).

An action of a standard torus $\mathbb{T}^n$ on a symplectic manifold (M, ω) is Hamiltonian if it preserves the symplectic form ω and if each restriction onto a $\mathbb{S}^1$-factor admits a Hamiltonian function $H\colon M \to \mathbb{R}$ preserved by the action, satisfying $\iota_X \omega = -dH$, where X is a vector field on M generated by the action. Putting these Hamiltonian functions together gives rise to a moment map $\mu\colon M \to \mathbb{R}^n$ which characterises the Hamiltonian action. The presence of such an action (which is, in addition, effective) on a compact and connected manifold allows us to replace the local classification with a global one. Indeed, it turns out that the image of the moment map is a convex polytope $\Delta_M = \mu(M)$, called the moment polytope, see Theorem 1.6.1. Delzant [9] showed that this polytope Δ_M completely characterises the diffeomorphism type of the manifold M and its symplectic structure. Thus, we are able to associate to the relatively complicated $2n$-dimensional object $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ a relatively simple geometric one, the n-dimensional convex polytope Δ_M.

In addition, Delzant's theorem also characterises the associated polytopes that arise, called Delzant polytopes. There are rather stringent requirements on the edges meeting at each vertex of a Delzant polytope. We will also see how to construct a symplectic toric manifold corresponding to a convex polytope satisfying those requirements. A crucial tool for that is the symplectic reduction by Marsden and Weinstein [15], and Meyer [19]. It shows that under appropriate conditions, the quotient of a symplectic manifold with respect to a group action (that is, the space of all orbits) not only results in a new manifold but also inherits an induced symplectic structure.

Our main goal is to define the bijective map $(M, \omega, \mathbb{T}^n, \mu) \mapsto \Delta_M$ from (equivalence classes of) symplectic toric manifolds to (translation classes of) Delzant polytopes and to show its surjectivity. Throughout the paper we provide some examples that should help illustrate the definitions and should lead to a more intuitive understanding of them.

Guided by this goal, this semester paper provides an introduction to symplectic geometry aimed at a reader familiar with basics of differential geometry. For the part on Delzant polytopes, some knowledge about polytopes might be helpful but is not required.

This project is largely based on the book *Lectures on Symplectic Geometry* by Prof. Ana Cannas da Silva [5] and her paper on symplectic toric manifolds [7]. The main proofs we present follow the ones of these books with added details.

Roadmap In Chapter 1, we provide some background in symplectic geometry starting from basic properties of symplectic manifolds and ending with Hamiltonian actions and their moment polytopes. Chapter 2 is devoted to symplectic reduction. In Section 3.1, we define the class of Delzant polytopes, formulate Delzant's theorem and prove the surjectivity part of the bijectivity statement. Lastly, in Appendix A, we discuss the Fubini-Study form.

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Chapter 1

Symplectic Toric Manifolds

In this chapter, we provide the background in symplectic geometry, covering the definition of symplectic manifolds, Darboux charts, symplectic and Hamiltonian actions, and moment polytopes.

1.1 Symplectic Manifolds

We start with some basic definitions, properties, and examples. The basis of symplectic geometry relies on symplectic manifolds, that is, (smooth) manifolds equipped with a certain differential structure.

Definition 1.1.1 (Symplectic manifold) *A **symplectic manifold** (M, ω) is a (smooth) manifold M paired with a closed 2-form $\omega \in \Omega^2(M)$ such that for each $p \in M$ the bilinear form ω_p is **nondegenerate**. That is, for each nonzero tangent vector $\xi \in T_pM$ there exists an $\eta \in T_pM$ such that $\omega_p(\xi, \eta) \neq 0$. Such a 2-form ω is called a **symplectic form**.*

Any symplectic manifold (M, ω) gives rise to the symplectic vector space[1] (T_pM, ω_p) at any point $p \in M$. The following linear algebra argument shows that symplectic vector spaces have to be even dimensional. A direct corollary of this is that a symplectic manifold has to be even dimensional.

Lemma 1.1.2 (Standard Form for Skew-Symmetric Bilinear Maps) *Let V be an m-dimensional vector space and $\omega \colon V \times V \to \mathbb{R}$ be a bilinear map. Assume that ω is skew-symmetric, that is, $\omega(v, w) = -\omega(w, v), \forall v, w \in V$. Then there exists a basis $\{u_1, \dots u_k, e_1, \dots, e_n, f_1, \dots, f_n\}$ of V such that*

$$\begin{aligned}
&\omega(u_i, v) = 0, && \forall i = 1, \dots, k \text{ and } v \in V \\
&\omega(e_i, e_j) = 0 = \omega(f_i, f_j), && \forall i, j = 1, \dots, n \\
&\omega(e_i, f_j) = \delta_{i,j}, && \forall i, j = 1, \dots, n
\end{aligned}$$

[1] That is, a vector space equipped with a nondegenerate skew-symmetric bilinear form.

Proof We prove the lemma by induction using a skew-symmetric version of the Gram-Schmidt process.

Let $U := \{u \in V \mid \omega(u,v) = 0 \ \forall v \in V\}$ and let $u_1, \dots, u_k$ be a basis of U. Choose a complementary space W of U in V, that is

$$V = U \oplus W.$$

Note that $\omega|_W$ is nondegenerate. We now construct a symplectic basis of W, that is, a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ of W satisfying $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$.

If W is not zero-dimensional, we pick some nonzero vector $e_1 \in W$ and since $\omega|_W$ is nondegenerate and bilinear, there exists some $f_1 \in W$ with $\omega(e_1, f_1) = 1$. Since ω is antisymmetric and bilinear, e_1 and f_1 have to be linearly independent. If $\dim(W) = 2$, then $\{e_1, f_1\}$ is the desired symplectic basis and we are done. Otherwise $\dim(W) > 2$. Then set $W_1 := \operatorname{span}(\{e_1, f_1\})$ and define $W_1^\omega := \{v \in W \mid \omega(v, w) = 0 \ \forall w \in W_1\}$[2]. Then W_1^ω is a linear vector space and $\omega|_{W_1^\omega}$ is still nondegenerate. Moreover, for arbitrary $v \in W$, we have $v = w_1 + w_2$ where $w_1 := \omega(v, f_1)e_1 - \omega(v, e_1)f_1 \in W_1$, and $w_2 = v - w_1 \in W_1^\omega$. Together with $W_1 \cap W_1^\omega = \{0\}$ this implies $W = W_1 \oplus W_1^\omega$. By proceeding as above on W_1^ω, we get a symplectic basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ on W. □

Note that if a bilinear form $\omega\colon V \times V \to \mathbb{R}$ is in particular symplectic (that is, skew-symmetric and nondegenerate) then Lemma 1.1.2 provides a so called **symplectic basis** that is, a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ of V with $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$.

We are now ready to capture the first important consequence of the definition of symplectic manifolds.

Proposition 1.1.3 *Let (M, ω) be a symplectic manifold. Then M is even dimensional and orientable.*

Proof As we argued in Lemma 1.1.2, it follows immediately that M has to be even dimensional since the associated symplectic basis is even dimensional. We now show that M is orientable by proving that $\omega^n = \omega \wedge \cdots \wedge \omega$ is a volume form. Since ω^n is already a top-dimensional form, it is enough to show that ω^n is nowhere vanishing. Let $p \in M$ be arbitrary and $e_{1,p}, \dots, e_{n,p}, f_{1,p}, \dots, f_{n,p}$ be a symplectic basis of T_pM with respect to ω and with dual basis $e^*_{1,p}, \dots, e^*_{n,p}$, $f^*_{1,p}, \dots, f^*_{n,p}$. Then $\omega_p = \sum_{i=1}^n e^*_{i,p} \wedge f^*_{i,p}$ and thus

$$\omega_p^n = n! e^*_{1,p} \wedge f^*_{1,p} \wedge \cdots \wedge e^*_{n,p} \wedge f^*_{n,p}$$

and hence $\omega_p^n(e_{1,p}, f_{1,p}, \dots, e_{n,p}, f_{n,p}) = n! \det(I_n) = n! \neq 0$. □

[2]This is called the symplectic complement of W_1.

We now give some standard examples of symplectic manifolds that we will encounter throughout the chapters.

Example 1.1.4

1. *Let* $M = \mathbb{R}^{2n}$ *with local coordinates* x_i *and* $y_i, i = 1, \ldots, n$. *The standard symplectic form on* M *is*

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

To see that ω_0 *is indeed symplectic, note that for each* $p \in M$, $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}_{i=1}^{n}$ *satisfies the condition of a symplectic basis of* $T_pM = \mathbb{R}^n$ *with respect to* ω_0. *Thus* ω_0 *is nondegenerate. And it is closed since* $\omega = d\alpha$ *for* $\alpha = \sum_{i=1}^{n} x_i dy_i$.

2. *Let* $M = \mathbb{C}^n$ *with local coordinates* $z_j, j = 1, \ldots, n$. *The standard symplectic form on* M *is given by*

$$\omega_0 = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

Under the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ *and* $z_j = x_j + iy_j$, *this expression boils down to the example above.*

Translated from the linear coordinates z_j *to polar coordinates* (r_j, θ_j) *with* $z_j = r_j e^{i\theta_j}$, ω_0 *has the form:*

$$\begin{aligned}
\omega_0 &= \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \\
&= \frac{i}{2} \sum_{j=1}^{n} \left(dr_j e^{i\theta_j} + ir_j e^{i\theta_j} d\theta_j\right) \wedge \left(dr_j e^{-i\theta_j} - ir_j e^{-i\theta_j} d\theta_j\right) \\
&= \frac{i}{2} \sum_{j=1}^{n} ir_j d\theta_j \wedge dr_j - ir_j dr_j \wedge d\theta_j \\
&= \sum_{j=1}^{n} r_j dr_j \wedge d\theta_j
\end{aligned}$$

3. *Let* $M = \mathbb{S}^2$ *be the 2-sphere. For some point* $p \in \mathbb{S}^2$, *one can identify* $T_p\mathbb{S}^2$ *with vectors orthogonal to* p. *Under this identification, the standard symplectic form on* $\mathbb{S}^2$ *is given by*

$$\omega_p(u, v) := \langle p, u \times v \rangle$$

for $u, v \in T_p\mathbb{S}^2$. This form is closed since it is of top degree and it is nondegenerate since for $u \neq 0$, for example, $v := u \times p$ gives a nonzero pairing.

One can check that away from the poles the standard symplectic form on $\mathbb{S}^2$ is given by

$$\omega = d\theta \wedge dh$$

where $h\colon \mathbb{S}^2 \to (-1, 1)$ is the height function and θ is the angle coordinate.

4. *Let $M = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the standard torus with angle coordinates θ_1, θ_2. The standard symplectic form on M is given by*

$$\omega = d\theta_1 \wedge d\theta_2.$$

To see that ω is actually symplectic, note that ω is top dimensional and thus closed and it is nondegenerate since $\{\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}\}$ is a symplectic basis of $T\mathbb{T}^2$.

5. *Let M be a manifold and let T^*M be its cotangent bundle with projection $\pi\colon T^*M \to M$. Then there is a canonical 1-form α on T^*M, called the Liouville form, given by*

$$\alpha_{(p,\eta)}(\xi) = \eta\big(D_p\pi(\xi)\big)$$

for $p \in M, \eta \in T^*M$ and $\xi \in T_{(p,\eta)}T^*M$.

Then the 2-form $\omega := -d\alpha$ is exact and in particular closed. For local coordinates $(x_1, \dots, x_n)$ on M with "dual" coordinates $(y_1, \dots, y_n)$ satisfying $y_1 = dx_1$, then $(x_1, \dots, x_n, y_1, \dots, y_n)$ is a system of local coordinates on T^*M in which α and ω are of the form:

$$\alpha = \sum_{i=1}^{n} y_i dy_i \qquad\qquad \omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

This shows that ω is nondegenerate.

We will encounter the special case $M = \mathbb{R}^n$ with $T^*M = \mathbb{R}^n \times (\mathbb{R}^n)^*$ in Section 3.4.

Since submanifolds of symplectic manifolds may not be even dimensional, we cannot hope that they always inherit a symplectic structure. The restriction of ω to a submanifold may become degenerate or even vanish. Submanifolds on which the restriction of ω vanishes are called **isotropic**. By Lemma 1.1.2, the maximal dimension of an isotropic submanifold is half the dimension of the original manifold. Since such maximal isotropic submanifolds turn out to be quite important in symplectic geometry, they deserve a special name.[3]

Definition 1.1.5 (Lagrangian submanifold) *Let (M^{2n}, ω) be a symplectic manifold. A **Lagrangian submanifold** of M is an n-dimensional isotropic submanifold L, that is, $\omega|_L \equiv 0$ or equivalently $i^*\omega = 0$ where $i\colon L \hookrightarrow M$ is the inclusion map.*

Example 1.1.6

1. *A standard Lagrangian submanifold of $(\mathbb{R}^{2n}, \omega_0)$ is given by an embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{2n}$ fixing the values for half of the coordinates. Indeed, closed Lagrangian submanifolds do all look like this locally. See [12, Proposition 2.6.4] for further details.*

2. *In particular, the Lagrangian submanifolds of a two-dimensional manifold are all its one-dimensional submanifolds (that is, all embedded curves).*

The following definition provides a natural notion of equivalence in the symplectic category.

Definition 1.1.7 (Symplectomorphism) *Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds and $\varphi\colon M_1 \to M_2$ be a diffeomorphism. Then φ is a **symplectomorphism**[4] if $\varphi^*\omega_2 = \omega_1$.*

We denote the group of symplectomorphisms of (M, ω) by $\mathrm{Symp}(M, \omega)$.

Example 1.1.8

1. *Any translation by some constant $c \in \mathbb{R}^{2n}$ is a symplectomorphism of the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$, see [17] .*

2. *Any linear symplectomorphism Ψ on $(\mathbb{R}^{2n}, \omega_0)$ has a matrix representation A, we write $\Psi = T_A$. Then the symplecticity condition $T_A^*\omega_0 = \omega_0$, is equivalent to the matrix condition $A^T J A = J$ where J is the $2n \times 2n$ block matrix $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. And thus T_A is a symplectomorphism if and only if $A \in \mathrm{SP}(2n)$.*

   A concrete example is

   $$\psi(x_1, y_1, \ldots, x_n, y_n) = \left(c_1 x_1, \frac{1}{c_1} y_1, \ldots c_n x_n, \frac{1}{c_n}\right)$$

   for given nonzero constants $c_1, \ldots, c_n \in \mathbb{R}$.

[3] For Weinstein [22] Lagrangian submanifolds are so important that he calls *everything is a Lagrangian submanifold* the "symplectic creed", meaning that one should try to express objects and constructions on symplectic geometry in terms of Lagrangian submanifolds.

[4] In literature (as for example [17]) this term is occasionally reserved for linear symplectic maps on symplectic vector spaces and is instead called symplectic diffeomorphism.

1.2 Darboux's Theorem

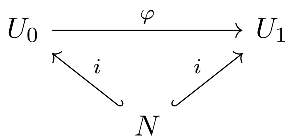
The following theorem shows that there exists a fundamental local classification of symplectic manifolds in terms of a unique invariant: the dimension. It states that every symplectic manifold is locally symplectomorphic to the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$. Namely, the theorem shows that the prototypical local piece of a symplectic manifold $M^{2n}, \omega)$ is $M = \mathbb{R}^{2n}$ with local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$.

Theorem 1.2.1 (Darboux) *Let (M, ω) be a $2n$-dimensional symplectic manifold and let $p \in M$ be arbitrary. Then we can find a coordinate system $(U, x_1, \dots, x_n, y_1, \dots, y_n)$ centred at p such that on U:*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Proof (Sketch) Here we will follow [6]. The modern proof of Darboux's theorem was first noted by Moser [20] and can be broken down into two key pieces, Lemma 1.1.2 coming from linear algebra and an argument from Moser called the *Moser trick* which turns the problem into solving a differential equation and gives rise to the following theorem.

Theorem 1.2.2 (Moser Theorem) *Let M be a manifold with compact submanifold N and inclusion map $i\colon N \hookrightarrow M$. Let ω_1, ω_0 be two symplectic forms in M. Suppose that $\omega_0|_p = \omega_1|_p$ for all $p \in N$. Then there exist neighbourhoods U_0 and U_1 of N in M and a diffeomorphism $\varphi\colon U_0 \to U_1$ such that $\varphi^*\omega_0 = \omega$ and the following diagram commutes:*



Using Lemma 1.1.2, we get a symplectic basis for T_pM with respect to ω_p. We can use them to construct coordinate charts $x'_1, \dots, x'_n, y'_1, \dots, y'_n$ centred at p such that $\omega|_p = \sum_{i=1}^{n} dx'_i \wedge dy'_i|_p$. Then we apply Theorem 1.2.2 with $N = \{p\}$ to get neighbourhoods U, V of p and a diffeomorphism $\varphi\colon U \to V$ such that $\varphi(p) = p$ and

$$\omega = \varphi^*\Big(\sum_{i=1}^{n} dx'_i \wedge dy'_i\Big) = \sum_{i=1}^{n} d(x_i \circ \varphi) \wedge d(y_i \circ \varphi).$$

So, the coordinates $x_i := x'_i \circ \varphi$ and $y_i := y'_i \circ \varphi$ work.

For a full proof, we refer the reader to [5, Theorem 8.1]. □

Definition 1.2.3 (Darboux chart) *Such a chart as in Theorem 1.2.1 is called a **Darboux chart** for M.*

1.3 Hamiltonian Vector Fields

In this section, we introduce the concept of Hamiltonian vector fields and show some properties of them. Such vector fields will be important when defining Hamiltonian actions.

Definition 1.3.1 (Hamiltonian vector fields) *Let (M, ω) be a symplectic manifold. A vector field $X \in \mathfrak{X}(M)$ is* ***symplectic*** *if its contraction is closed, that is, if $d\iota_X\omega = 0$. If the contraction is in addition exact, that is*

$$\iota_X\omega = -dH \tag{1.3.1}$$

for some smooth function $H\colon M \to \mathbb{R}$ (we say $H \in C^\infty(M)$), then the vector field X is called ***Hamiltonian****. Such a function H is called a* ***Hamiltonian function*** *for X.*[5]

Note that a Hamiltonian function H corresponding to a Hamiltonian vector field is unique up to a locally constant function.

The following proposition shows that symplectic and Hamiltonian vector fields preserve the symplectic structure.

Proposition 1.3.2 *Let (M, ω) be a symplectic manifold. Let $X \in \mathfrak{X}(M)$ be a symplectic vector field with flow Φ. Then Φ preserves the symplectic form ω, that is, $\Phi_t^*\omega = \omega, \forall t \in \mathbb{R}$.*

Let X be, in addition, Hamiltonian with a Hamiltonian function H. Then Φ preserves H, too.

Proof First, let X be a symplectic vector field. Using Cartan's magic formula[6], we get the following equation of 2-forms:

$$\mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\omega = 0.$$

Now consider the function

$$\begin{aligned} f\colon \mathbb{R} &\to \Omega^2(M) \\ t &\mapsto \Phi_t^*\omega \end{aligned}$$

Using an explicit formula of the Lie derivative, proven in [18, Proposition 22.14], we get for any $t_0 \in \mathbb{R}$ that

$$\lim_{t\to t_0} \frac{\Phi_t^*\omega - \omega}{t - t_0} = \Phi_{t_0}^*(\mathcal{L}_X\omega) = 0 = \frac{d}{dt}|_{t_0} f.$$

[5]Sign conventions vary depending on the text.

[6]Recall that Cartan's magic formula states: $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d$ for any vector field X. For a proof, see for example [18, Theorem 23.12]

Hence f is a constant function and thus $\varphi_t^*\omega = \omega, \forall t \in \mathbb{R}$.

Now let X be a Hamiltonian vector field. To show that Φ preserves the Hamiltonian function H observe:

$$\mathcal{L}_X H = X(H) = dH(X) = \iota_X dH = -\iota_X \circ \iota_X \omega = 0. \qquad \square$$

By using the definition only, it is hard to show that a particular vector field is *not* Hamiltonian since one has to prove that no such Hamiltonian function exists. If the vector field is on a compact manifold, then the following observation, proven in [12, Remark 2.4.5], might provide a simple argument.

Remark 1.3.3 *Let (M, ω) be a compact symplectic manifold and $X \in \mathfrak{X}(M)$ a Hamiltonian vector field with Hamiltonian function $H \in C^\infty(M)$. Then X has at least two zeros on M. (Observe that $X(p) = 0$ at $p \in M$ if and only if $dH_p = 0$.)*

Example 1.3.4

1. *On the symplectic 2-sphere $(\mathbb{S}^2, d\theta \wedge dh)$, where (θ, h) are the cylindrical coordinates away from the poles, the vector field $X = \frac{\partial}{\partial \theta}$ is Hamiltonian with Hamiltonian function $H = -h$. Observe that the flow generated by X is rotation about the z-axis. Consistent to Proposition 1.3.2, the flow of this vector field preserves both the are and the (negative) height function h.*

   The vector field $Y = \frac{\partial}{\partial h}$, on the other hand, is not even symplectic. The flow generated by Y goes from one pole to the other and thus does not preserve the area.

2. *On the torus $(\mathbb{T}^2, d\theta^1 \wedge d\theta^2)$ with local angle coordinates θ_1, θ_2, the vector fields $X_i = \frac{\partial}{\partial \theta_i}, i = 1, 2$ are symplectic but not Hamiltonian. Observe that the flows are again rotations which preserve the area.*

   Note that both vector fields are nowhere zero on the compact manifold $\mathbb{T}^2$. And thus by Remark 1.3.3 they cannot be Hamiltonian.

Proposition 1.3.5 *Let (M, ω) be a symplectic manifold. Then for any map $H \in C^\infty(M)$, there exists a unique Hamiltonian vector field X_H with H as Hamiltonian function.*

Proof We have to find a unique vector field $X_H \in \mathfrak{X}(N)$ solving Equation (1.3.1).

For this, let $p \in M$ be arbitrary and $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a coordinate system around p corresponding to a Darboux chart. Then we can define the smooth local solution:

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i} - \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i}.$$

We now argue that Equation (1.3.1) has a unique pointwise solution. Then local solution from above gives rise to a unique solution $X_H \in \mathfrak{X}(M)$.

Note that nondegeneracy of ω means that for each $p \in M$, the bilinear form ω_p is a nondegenerate. It follows from linear algebra that the map

$$\begin{aligned} \Psi\colon T_pM &\to T_p^*M \\ \xi &\mapsto \omega_p(\xi, \circ) \end{aligned}$$

is a linear isomorphism which implies that Equation (1.3.1) has a unique pointwise solution. □

Let us consider a simple example from [17].

Example 1.3.6 *Let*

$$\begin{aligned} H\colon \mathbb{R}^4 &\to \mathbb{R} \\ (x_1, y_1, x_2, y_2) &\mapsto \frac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2) \end{aligned}$$

be the Hamiltonian function. Then the corresponding Hamiltonian vector field is given by

$$X_H = x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2}.$$

Definition 1.3.7 (Hamiltonian system) *A symplectic manifold (M, ω) together with a Hamiltonian function H is called a* ***Hamiltonian system****.*

Lemma 1.3.8 *Let (M, ω) be a symplectic manifold, $f, g \in C^\infty(M)$ and let $X_f, X_g \in \mathfrak{X}(M)$ be the corresponding Hamiltonian vector fields. Then*

$$X_{\omega(X_f, X_g)} = [X_f, X_g].$$

Proof We have to show that $\iota_{[X_g, X_f]}\omega = -d\omega(X_f, X_g)$. For this, we use Cartan's magic formula, a formula for the contraction of a Lie bracket proven in [18, Corollary 13.11] and the closedness of ω and $\iota_X\omega$.

$$\begin{aligned} \iota_{[X_f, X_g]}\omega &= \mathcal{L}_{X_f} \circ \iota_{X_g}\omega - \iota_{X_g} \circ \mathcal{L}_{X_f}\omega \\ &= d\iota_{X_f} \circ \iota_{X_g}\omega + \iota_{X_f} \circ d\iota_{X_g}\omega - \iota_{X_g} \circ d\iota_{X_f}\omega - \iota_{X_g} \circ \iota_{X_f} d\omega \\ &= d\iota_{X_f} \circ \iota_{X_g}\omega \\ &= d\omega(X_g, X_f) \\ &= -d\omega(X_f, X_g) \end{aligned}$$

□

Definition 1.3.9 (Poisson bracket) *Let (M, ω) be a symplectic manifold. For $f, g \in C^\infty(M)$ we define the Poisson bracket of f and g by the following function:*

$$\{f, g\} := \omega(X_f, X_g).$$

If the Poisson bracket is equal to zero, then f and g are said to be in ***involution****.*

A short calculation shows that the Poisson bracket satisfies the Leibniz rule:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Moreover, using Lemma 1.3.8, we get

$$X_{\{f,g\}} = [X_f, X_g].$$

Remark 1.3.10 *A straightforward calculation, as done in [17, Example 7.4.13], shows that on* $(\mathbb{R}^{2n}, \omega_0)$ *for* $f, g \in C^\infty$ *we have*

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}.$$

1.4 Hamiltonian Actions

In this project, the manifolds of interest are not only equipped with a symplectic structure but also admit a special group action. The goal of this section is to discuss actions of a Lie group on manifolds and define such special actions, called Hamiltonian actions.

Recall the definition of a Lie group acting (smoothly) on a manifold:

Definition 1.4.1 (Action) *Let* G *be a Lie group and* M *be a manifold and let* σ *be a smooth map* $\sigma\colon G \times M \to M$ *satisfying:*

$$\sigma_{gh} = \sigma_g \circ \sigma_h, \forall g, h \in G \tag{1.4.1}$$

$$\sigma_e = \mathrm{id} \tag{1.4.2}$$

where σ_g *denotes the smooth map* $\sigma_g\colon M \to M, p \mapsto \sigma(g, p)$ *fixing some element* $g \in G$*. Then we call* σ *a* ***smooth left action of*** $\boldsymbol{G}$ ***on*** $\boldsymbol{M}$*.*

For some fixed point $p \in M$*, we call the smooth map*

$$\begin{aligned} \sigma^p\colon g &\to M \\ g &\mapsto \sigma(g, p) =: g \cdot p \end{aligned}$$

the ***orbit map*** *and its image* $\mathrm{orb}_\sigma(p) := \{\sigma_g(p) \mid g \in G\}$ *the* ***orbit*** *of* G *through* p*.*

The ***stabiliser*** *of a point* $p \in M$ *is the set of group elements fixing this point, that is,* $\mathrm{stab}_\sigma(p) := \{g \in G \mid \sigma_g(p) = p\}$*.*

In what follows, we always assume that a group action is a smooth left action.

Note that combining Equation (1.4.1) and Equation (1.4.2) gives

$$\sigma_{g^{-1}} = \sigma_g^{-1}.$$

This shows in particular that, for every $g \in G$, the map σ_g is a diffeomorphism of M. Using Equation (1.4.1) again, we conclude that

$$\begin{aligned} \psi_\sigma \colon G &\to \mathrm{Diff}(M) \\ g &\mapsto \psi_g := \sigma_g \end{aligned}$$

is a group homomorphism.

The converse also works: any group homomorphism $\psi \colon G \to \mathrm{Diff}(M)$ has a smooth associated **evaluation map**

$$\begin{aligned} \mathrm{ev}_\psi \colon G \times M &\to M \\ (g, p) &\mapsto \psi_g(p) \end{aligned}$$

then ev_ψ is a group action and moreover, $\psi = \psi_{\mathrm{ev}_\psi}$.

This gives us a bijective correspondence between σ and ψ_σ or between ψ and ev_ψ, respectively. Under this correspondence, we obtain a second definition of group actions which takes on a different point of view on actions and is for example, used in [7], [5], and [3]. Throughout the text, we will mainly use this second definition.

Definition 1.4.2 (Action) *A* ***smooth left action*** *of a Lie group G on a manifold M is a group homomorphism*

$$\begin{aligned} \psi \colon G &\to \mathrm{Diff}(M) \\ g &\mapsto \psi_g \end{aligned}$$

such that the associated evaluation map ev_ψ is smooth. For $p \in M$ we denote the corresponding orbit map by $\psi^p \colon G \to M, g \mapsto \psi(g)(p)$

We now give a list of properties such an action might have and that we will use later on.

Definition 1.4.3 (Some properties of actions) *Let ψ be an action of a Lie group G acting on a manifold M. This action is called:*

- ***effective*** *if $\psi \colon G \to \mathrm{Diff}(M)$ is injective. That is, each non-trivial element of G moves at least one element from M,*
- ***transitive*** *if there is just one orbit. That is, for each $p \in M$ we have $\mathrm{orb}_\sigma(p) = M$,*
- ***free*** *if all stabilisers are trivial. That is, for each $p \in M$ we have $\mathrm{stab}_\sigma(p) = \{e\}$, and*
- ***proper*** *if the map $(\sigma, \mathrm{id}) \colon G \times M \to M \times M$ defined by $(p, g) \mapsto (\sigma_g(p), p)$ is proper[7].*

[7] A continous map is proper if the preimage of a compact set is compact.

When the manifold M admits a symplectic structure, the second view on group actions leads us to a natural definition of a symplectic action.

Definition 1.4.4 (Symplectic action) *Let G be a Lie group and (M, ω) be a symplectic manifold. An action $\psi\colon G \to \mathrm{Diff}(M)$ of G on M is called **symplectic** if it satisfies:*

$$\psi\colon G \to \mathrm{Symp}(M, \omega) \subseteq \mathrm{Diff}(M).$$

Example 1.4.5

1. *Consider the sphere $\mathbb{S}^2$ with the symplectic form $\omega = d\theta \wedge dh$ in cylindrical coordinates. The action of the circle $\mathbb{S}^1$ on the sphere $\mathbb{S}^2$ given by*

$$\psi_t(\theta, h) = (\theta + t, h), t \in \mathbb{S}^1$$

   is symplectic since it preserves the area.

2. *Similarly, on the torus $\mathbb{T}^2$ with the symplectic form $d\theta_1 \wedge d\theta_2$ in angle coordinates there are two symplectic actions of $\mathbb{S}^1$ given by*

$$\psi_{1,t}(\theta_1, \theta_2) = (\theta_1 + t, \theta_2), t \in \mathbb{S}^1$$

   and

$$\psi_{2,t}(\theta_1, \theta_2) = (\theta_1, \theta_2 + t), t \in \mathbb{S}^1$$

   respectively.

Complete vector fields[8] on a manifold M are in bijective correspondence with actions of $\mathbb{R}$ on M. A proof of this can be found, for example, in [18, Proposition 9.15]. We note that the action corresponding to a complete vector field is given by its flow Φ_t. If M admits a symplectic structure, the same correspondence holds for complete symplectic vector fields and symplectic actions of $\mathbb{R}$ on M.

Proposition 1.4.6 *Let (M, ω) be a symplectic manifold. Then the symplectic actions of $\mathbb{R}$ on M are in a bijective correspondence with complete symplectic vector fields on M.*

Proof Let ψ be a symplectic action of $\mathbb{R}$ on M. Let $X \in \mathfrak{X}(M)$ be the corresponding complete vector field. We only need to show that this vector field is symplectic. Since ψ is symplectic, we get $\psi_t^*\omega = \omega$ for all $t \in \mathbb{R}$. Using the explicit formula of the Lie derivative as in Proposition 1.3.2, this shows $\mathcal{L}_X\omega = 0$ and by applying Cartan's magic formula, we conclude: $d\iota_X\omega = \mathcal{L}_X\omega - \iota_X d\omega = 0$. That is, $\iota_X\omega$ is closed and, thus, X is a symplectic vector field.

[8] A vector field is **complete** if its flow is defined on all $\mathbb{R} \times M$.

Now let $X \in \mathfrak{X}(M)$ be a symplectic vector field with flow Φ_t. Then

$$\psi : \mathbb{R} \to \mathrm{Diff}(M)$$
$$t \mapsto \Phi_t$$

is an action of $\mathbb{R}$ on M. It follows directly from Proposition 1.3.2 that ψ is symplectic, that is, $\psi_t = \Phi_t$ preserves ω for every $t \in \mathbb{R}$. □

Motivated by the correspondence in Proposition 1.4.6, we introduce the definition of Hamiltonian torus actions, starting with Hamiltonian actions of the real line $\mathbb{R}$.

Definition 1.4.7 (Hamiltonian action of $\mathbb{R}$) *A symplectic action ψ of $\mathbb{R}$ on a symplectic manifold (M, ω) is **Hamiltonian** if the vector field X generated by ψ is Hamiltonian. That is, there exists a Hamiltonian function $H\colon M \to \mathbb{R}$ with $\iota_X \omega = -dH$.*

Thus, by definition, Hamiltonian actions of $\mathbb{R}$ are in bijective correspondence to complete Hamiltonian vector fields.

If $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ acts on a symplectic manifold (M, ω), we can adapt the above definition. Denote the quotient map of $\mathbb{R}/\mathbb{Z}$ by $\rho\colon \mathbb{R} \to \mathbb{S}^1$. Then the action ψ can be seen as 2π-periodic $\mathbb{R}$-action with underlying $\mathbb{R}$-action $\bar{\psi}_t := \psi_{\rho(t)}$. This is indeed an action since $\rho(t) + \rho(s) = \rho(t+s)$ for all $t, s \in \mathbb{R}$ and since ρ is smooth and $\mathrm{ev}_{\bar{\psi}}(t, p) = \mathrm{ev}_{\psi}(\rho(t), p)$.

Example 1.4.8

1. *Consider the symplectic action on $(\mathbb{S}^2, d\theta \wedge dh)$ from Example 1.4.5 given by*
$$\psi_t(\theta, h) = (\theta + t, h), t \in \mathbb{S}^1.$$
Note that the vector field generated by this action is given by $X = \frac{\partial}{\partial \theta}$. In Example 1.3.4 we have seen that X is a Hamiltonian vector field with Hamiltonian function $-h$. Thus, the action is Hamiltonian.

2. *The two symplectic actions on $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$ given by rotation on one of the factors, however, are not Hamiltonian. This is because the corresponding vector fields are given by $X_i = \frac{\partial}{\partial \theta_i}$. In Example 1.3.4 we have seen that they are symplectic but not Hamiltonian.*

Now we can extend Definition 1.4.7 to **torus actions**, actions by a torus Lie group $G = \mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ for some $n \in \mathbb{Z}^+$.

Definition 1.4.9 (Hamiltonian torus action) *Let ψ be a symplectic action of the standard n-torus $\mathbb{T}^n$ on a symplectic manifold (M^{2n}, ω). For $i = 1, \ldots, n$, let*
$$\psi_i\colon \mathbb{S}^1 \to \mathrm{Symp}(M, \omega)$$

*be the restriction of ψ to the ith factor and let X_i be the vector field generated by $\bar{\psi}_i$ as defined above. Then ψ is **Hamiltonian** if there exists $H_i \in C^\infty(M)$ such that $\iota_{X_i}\omega = -dH_i$ and $H_i \circ \psi_g = H_i$ for each $g \in \mathbb{T}^n$.*

In short, a symplectic toric action is Hamiltonian if each restriction onto a factor is Hamiltonian with a Hamiltonian function that is preserved by the action of the torus.

*We can combine the Hamiltonian functions $H_i, i = 1, \ldots, n$, into one function $H\colon M \to \mathbb{R}^n$. We call this function a **moment map** of the action ψ and denote it by μ.*

Note that the moment map is unique only up to adding a locally constant function.

Example 1.4.10

1. *Consider $\mathbb{C}^n$ with the standard symplectic form ω_0. There is a Hamiltonian action of $\mathbb{S}^1$, regarded as the unit circle in $\mathbb{C}$, given by*

$$\psi_t(z_1, \ldots, z_n) = (z_1 t, \ldots, z_n t)$$

   This is a Hamiltonian action with moment map $\mu(z) = \frac{1}{2}\sum_{i=1}^n |z_i|^2$.

2. *Let $k_1 \ldots, k_n \in \mathbb{Z}$ be fixed and let the standard torus $\mathbb{T}^n$ act diagonally on $\mathbb{C}^n$ via:*

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (z_1 t_1^{k_1}, \ldots, z_n t_n^{k_n}).$$

   This is a Hamiltonian torus action and

$$\mu(z_1, \ldots, z_n) = \frac{1}{2}\Big(k_1|z_1|^2, \ldots, k_n|z_n|^2\Big)$$

   is a moment map.

   To see this, we use the translation from linear coordinates to polar coordinates in each factor as seen in Example 1.1.4. Then

$$\begin{aligned} \omega_0 &= \sum_{j=1}^n r_j dr_j \wedge d\theta_j \\ (e^{i\tau_1}, \ldots, e^{i\tau_n}) \cdot (r_1, \theta_1, \ldots, r_n, \theta_n) &= (r_1, \theta_1 + k_1\tau_1, \ldots, r_n, \theta_n + k_n\tau_n) \\ \mu(r_1, \theta_1, \ldots, r_n, \theta_n) &= \frac{1}{2}(k_1 r_1^2, \ldots, k_n r_n^2) \end{aligned}$$

   Note that μ is preserved by the action since it does not depend on the angle coordinates.

   Then the action of the jth $\mathbb{S}^1$-factor is given by

$$e^{i\tau_j} \cdot (r_1, \theta_1, \ldots, r_j, \theta_j, \ldots, r_n, \theta_n) = (r_1, \theta_1, \ldots, r_j, \theta_j + k_j\tau_j, \ldots, r_n, \theta_n).$$

The generated vector field is given by $X_j = k_j \frac{\partial}{\partial \theta_j}$. This is Hamiltonian with Hamiltonian function $H_j = \frac{1}{2} k_j r_j^2$ since $\iota_{X_j} \omega_0 = -dH_j$ as ω_0 is given by $\omega_0 = \sum_{j=1}^n r_j dr_j \wedge d\theta_j$.

In order to define Hamiltonian actions of an arbitrary Lie group G on a symplectic manifold (M, ω), we need to generalise the idea of a vector field generated by an action.

First, we give two equivalent definitions for the **time-t-map**. For the first one, we identify the Lie algebra $\mathfrak{g}$ of G with the tangent space at the group unit e, that is, $\mathfrak{g} = T_e G$. Then we define the **exponential map** as

$$\begin{aligned} \exp\colon \mathfrak{g} &\to G \\ \xi &\mapsto \gamma^\xi(1) \end{aligned}$$

where γ^ξ is the integral curve starting at e of the unique left-invariant[9] vector field X_ξ such that $X_\xi(e) = \xi$.[10] Then for some fixed $\xi \in \mathfrak{g}$, the time-t-map defined by $t \mapsto \exp(t\xi)$ is given by the one-parameter subgroup $\gamma^\xi(t)$. For a proof of this and other properties of the exponential map, see for example [18, Proposition 12.2]. For the second one, we identify the Lie algebra with the vector space of left-invariant vector fields of G, that is, $\mathfrak{g} = \mathfrak{X}_l(G)$. Now we denote an element of $\mathfrak{g}$ by X. And we directly define the time-t-map $t \mapsto \exp(tX)$ as the flow of the vector field X.

In the following, we use the notation in the second of the two equivalent definitions of the time-t-map. Let ψ be an action of G on M and let $X \in \mathfrak{g}$ be an element of the Lie algebra. We now define the **fundamental vector field** associated with X, denoted by $X^\#$, to be the vector field corresponding the $\mathbb{R}$-action $\phi\colon \mathbb{R} \to \mathrm{Diff}(M)$ with $\phi_t(p) := \psi_{ev}(p, \exp tX)$ for $t \in \mathbb{R}$.

A different approach to the definition of $X^\#$ is given in [3]. For this, we consider the differential of the orbit map ψ^p at e given by the map

$$D_e\psi^p\colon \mathfrak{g} \to T_pM.$$

Fixing $X \in \mathfrak{g} = T_eG$ and varying $p \in M$ gives rise to the vector field

$$X^\#(p) := D_e\psi^p X.$$

Definition 1.4.11 (Hamiltonian action) *Let ψ be an action of a Lie group G on a symplectic manifold (M, ω). Then ψ is a **Hamiltonian action** if there exists a so-called **moment map** $\mu\colon M \to g^*$ which satisfies the following two conditions:*

[9] A vector field $X \in \mathfrak{X}(G)$ is left-invariant if $(l_g)_* X = X$ for all $g \in G$ and where l_g denotes left multiplication.

[10] In fact, $X_\xi(g) = D_{l_g}\xi(e)$.

1. *For each* $X \in \mathfrak{g}$ *denote by*

$$\begin{aligned} \mu^X : \mathbb{R} &\to M \\ p &\mapsto \langle \mu(p), X \rangle = \mu_p(X) \end{aligned}$$

the component of μ *along* X*. Then* μ^X *is a Hamiltonian function of the fundamental vector field* $X^\#$*. That is,*

$$d\mu^X = -\iota_{X^\#}\omega.$$

2. *The map* μ *is equivariant with respect to the given action* ψ *on* G *and the coadjoint action*[11] Ad^* *of* G *on* $\mathfrak{g}^*$*. That is,*

$$\mu \circ \psi_g = \mathrm{Ad}^*_g \circ \mu, \text{ for all } g \in G.$$

Then (M, ω, G, μ) is called a **Hamiltonian *G*-space** and in the case where G is a torus, it is called a **Hamiltonian torus space**.

Beware that, unless G is an abelian group, we can no longer safely add a constant to the moment map to obtain another one. Note that the coadjoint action of an abelian group and thus in particular of a torus is trivial. After identifying the Lie algebra of a torus $\mathbb{T}^n$ and its dual with $\mathbb{R}^n$, we recover Definition 1.4.9.

Remark 1.4.12 *If the group* G *is connected, the second condition in Definition 1.4.11 can be replaced with the following condition: The map*

$$\begin{aligned} \mathfrak{g} &\to C^\infty(M) \\ X &\mapsto \mu^X \end{aligned}$$

is a Lie algebra homomorphism with respect to the Poisson bracket. That is,

$$\{\mu^X, \mu^Y\} = \mu^{[X,Y]}$$

for all $X, Y \in \mathfrak{g}$*. For a proof, see [16, Lemma 5.2.1].*

The following lemma is both a good exercise to get familiar with Hamiltonian actions and a useful result in the proof of Delzant's Theorem.

Lemma 1.4.13 *Let* G *be any compact Lie group and* H *a closed subgroup of* G *with* $\mathfrak{g}$ *and* $\mathfrak{h}$ *the respective Lie algebras. The inclusion* $i\colon \mathfrak{h} \hookrightarrow \mathfrak{g}$ *is dual to the projection* $i^*\colon \mathfrak{g}^* \to \mathfrak{h}^*$*. Suppose that* (M, ω, G, μ) *is a Hamiltonian* G*-space. Then the restriction of the* G*-action to* H *is Hamiltonian with moment map* $i^* \circ \mu\colon M \to \mathfrak{h}^*$*.*

[11] For a definition of the coadjoint action see [5, Section 21.5].

Proof Denote the inclusion by $j\colon H \hookrightarrow G$. Notice that $i = D_e j$. Let $\psi_G\colon G \to \mathrm{Diff}(M)$ be the G-action with restriction to H given by $\psi_H = \psi_G \circ j$.

Let $X \in \mathfrak{h} \subseteq \mathfrak{g}$ be arbitrary. Then the fundamental vector field of X with respect to the G-action is defined as

$$X^{\#}(p) = D_e \psi_G^p X$$

and the one with respect to the H-action is defined as

$$D_e \psi_H^p X = D_e \psi_G^p D_e j X = D_e \psi_G^p \circ i(X) = (i(X))^{\#}.$$

By definition of the dual i^*, we can calculate:

$$(i^* \circ \mu)^X(p) = (i^* \circ \mu)_p(X) = \mu_p\big(i(X)\big) = \mu^{i(X)}(p).$$

And thus $d((i^* \circ \mu)^X) = d(\mu^{i(X)})$. Finally, using that μ is a moment map of the G-action, we get:

$$\iota_{(i(X))^{\#}}\omega = -d\mu^{i(X)} = -d\big((i^* \circ \mu)^X\big).$$

Hence, $i^* \circ \mu$ satisfies the first property of being a moment map of the H-action.

For $g \in G$, denote by Ad_g the adjoint of $g \in G$. Then the coadjoint of $h \in H$ is given by $i^* \circ \mathrm{Ad}^*_{j(h)} \circ (i^{-1})^*$ where i^{-1} is the inverse of i on its image.

Using the second property of the moment map μ, we get for $h \in H$:

$$i^* \circ \mu \circ (\psi_H)_h = i^* \circ \mu \circ (\psi_G)_{j(h)} = i^* \circ \mathrm{Ad}^*_{i(h)} \circ (i^{-1})^* \circ i^* \mu.$$

We conclude that $i^* \circ \mu$ is a moment map of the H-action. □

1.5 Darboux's Theorem revisited

If a symplectic manifold is equipped with a symplectic action of a compact Lie group, there is a description of standard neighbourhoods of fixed point similar to Darboux's theorem, Theorem 1.2.1. The proof of this relies on an equivariant version of the Moser trick and can be found in [11].

Theorem 1.5.1 (Darboux — equivariant version) *Let $(M, \omega, \mathbb{T}^k, \mu)$ be a $2n$-dimensional Hamiltonian torus space and $p \in M$ be a fixed point of the action.*

Then there is a $\mathbb{T}^k$-invariant neighbourhood U of p in M, coordinate functions $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ centred at p, and weights $\lambda^1, \ldots, \lambda^n \in \mathbb{Z}^k$ with respect to which we have:

1. $\omega|_U = \sum_{j=1}^n dx_j \wedge dy_j$

2. *the action becomes the linear action of* $\mathbb{T}^k$

$$(e^{i\theta_1}, \dots, e^{i\theta_k}) \cdot (z_1, \dots, z_n) = \left(e^{i\langle\lambda^1,\theta\rangle} z_1, \dots, e^{i\langle\lambda^n,\theta\rangle} z_n\right)$$

where $\theta = (\theta_1, \dots, \theta_k)$ *and* $z_j = x_j + iy_j$, *and*

3. *the moment map becomes*

$$\mu|_U = \mu(p) + \frac{1}{2}\sum_{j=1}^{n} \lambda^j |z_j|^2.$$

One can show that the weights are given by the so-called **isotropy representation** at p, that is, the induced action of the group at the tangent space at the fixed point p.[12]

The following proposition shows a property of an effective linear torus action. It will be used to show that the map in Delzant's theorem is well-defined.

Proposition 1.5.2 *If the action of* $\mathbb{T}^k$ *on* $\mathbb{C}^n$ *given by*

$$(e^{i\theta_1}, \dots, e^{i\theta_k}) \cdot (z_1, \dots, z_n) = \left(e^{i\langle\lambda^1,\theta\rangle} z_1, \dots, e^{i\langle\lambda^n,\theta\rangle} z_n\right)$$

is effective, then $k \leq n$ *and the weights* $\lambda^1, \dots, \lambda^n$ $\mathbb{Z}$*-span* $\mathbb{Z}^k$.

Proof If the action is effective, then every non-trivial element of $\mathbb{T}^k$ moves at least one element from $\mathbb{C}^n$. In particular, for each $\theta \in \mathbb{R}^k \setminus \{0\}$, there is some λ^i with $\langle\lambda^i, \theta\rangle \neq 0$.

Namely, the orthogonal complement of the $\mathbb{R}$-span of the λ^i is trivial and thus they $\mathbb{R}$-span the vector space $\mathbb{R}^k$. We conclude that $k \leq n$ and after permuting the indices if necessary, we can assume that $\{\lambda^i \mid i = 1, \dots, k\}$ is an $\mathbb{R}$-basis of $\mathbb{R}^k$. Thus the matrix Λ with $\lambda^1, \dots, \lambda^k$ as columns has full rank and, therefore, its rows are a basis of $\mathbb{R}^k$. Since Λ has integer entries, we get that the standard basis vectors $e_i, i = 1, \dots, k$, are in the $\mathbb{Q}$-span of the rows. Equivalently, there is some integer matrix Q whose rows are primitive (meaning, that the gcd of its entries is 1) such that $Q\Lambda = M$ where M is some diagonal matrix with nonzero integer diagonal entries. And in particular $Q\lambda^i = m_i e_i$ for some $m_i \in \mathbb{Z}$.

Now observe that

$$\langle\lambda^j, Q^T e_i\rangle = \langle Q\lambda^j, e_i\rangle = m_i\delta_{i,j}.$$

If there exists some i with $m_i \notin \{\pm 1\}$, then let $v = \frac{2\pi}{m_i} Q^T e_i$. Then $\langle\lambda^j, v\rangle$ is an integer multiple of m for each j and thus $e^{i\langle\lambda^j,\theta\rangle} = 1$ which shows that v does not move any point. On the other hand, since the rows of Q are primitive, v is not the trivial element of the torus and this contradicts with the assumption that the action is effective. □

[12] For a proof of the existence of this isotropy representation, see for example [18, Proposition 13.11].

1.6 The Moment Polytope

A compact connected Hamiltonian torus space has the property that the image of its moment map is a convex polytope. This result is a first big step towards proving Delzant's correspondence theorem.

Theorem 1.6.1 (Atiyah [2], Guillemin-Sternberg [10]) *Let (M, ω) be a compact connected symplectic manifold, and let $\psi\colon \mathbb{T}^m \to \mathrm{Symp}(M, \omega)$ be a Hamiltonian torus action with moment map $\mu\colon M \to \mathbb{R}^m$. Then:*

1. *the levels of μ are connected;*
2. *the image of μ is convex;*
3. *the image of μ is the convex hull of the images of the fixed points of ψ;*
4. *the fixed point set of ψ is a finite union of pairwise disjoint connected symplectic submanifolds and the moment map is constant on each component.*

It follows that the image $\mu(M)$ of μ is a convex polytope; it is called the **moment polytope**.

The proof of Guillemin and Sternberg first uses the equivariant Darboux's theorem, Theorem 1.5.1, to prove a local convexity theorem. They then use that each component of a moment map has a unique local maximum to extend the local convexity theorem to a global one. A proof of Theorem 1.6.1 following Atiyah can be found in [16] and goes by induction over the dimension m of the torus and uses the following observation.

Remark 1.6.2 *Denote the components of the moment map $\mu\colon M \to \mathbb{R}^m$ by*

$$\mu(p) = \big(\mu_1(p), \dots, \mu_m(p)\big).$$

*We call μ **irreducible** if the 1-forms $d\mu_1, \dots, d\mu_m$ are linearly independent and **reducible**, otherwise.*

Assume that μ is reducible, then μ induces an action of $\mathbb{T}^{m-1}$ as follows. Since μ is reducible, the function

$$H_\theta = \sum_{i=1}^{m} \theta_i \mu_i$$

is constant for some $\theta \in \mathbb{R}^m \setminus \{0\}$. Then there is an action

$$\begin{aligned} \psi'\colon \mathbb{T}^{m-1} &\to \mathrm{Symp}(M, \omega) \\ t &\mapsto \psi'_t \end{aligned}$$

with moment map $\mu'\colon M \to \mathbb{R}^{m-1}$ and there is a matrix $A \in \mathbb{Z}^{(m-1)\times m}$ with

$$\psi_t = \psi'_{At} \qquad\qquad \mu(p) = A^T \mu'(p)$$

for $t \in \mathbb{T}^m$ and $p \in M$.

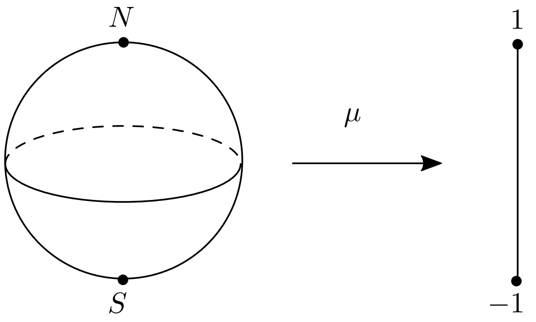


Figure 1.1: The action on the 2- sphere and its moment map.

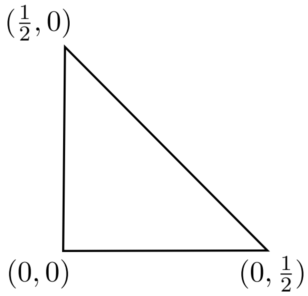


Figure 1.2: Moment polytope of the action on the projective plane.

Example 1.6.3

1. *Recall from Example 1.4.8 the Hamiltonian action on* $(\mathbb{S}^2, d\theta \wedge dh)$ *given by rotation*
$$\psi_t(\theta, h) = (\theta + t, h), t \in \mathbb{S}^1$$
with moment map $\mu = -h$.

   By Theorem 1.6.1, the image of μ *is a convex polytope in* $\mathbb{R}$*. And indeed* $\mu(\mathbb{S}^2) = [-1, 1]$*. Moreover, observe that the fixed points of* ψ *are exactly given by the two poles which are mapped to the vertices* $\{\pm 1\}$*. This is illustrated in Figure 1.1.*

2. *The first example can be generalised by taking products. Let the* n*-dimensional torus* $\mathbb{T}^n$ *act on* $\mathbb{S}^2 \times \cdots \times \mathbb{S}^2 =: M$ *by letting each* $\mathbb{S}^1$ *factor of* $\mathbb{T}^n$ *act on a* $\mathbb{S}^2$ *factor as above. This results in a Hamiltonian action with moment map given by* $\mu^n \colon M \to \mathbb{R}^n$*. Then the corresponding moment polytope is given by the* n*-dimensional cube.*

3. *Another classical example involves a symplectic structure on* $\mathbb{CP}^n$ *called the Fubini-Study form* ω_{FS}*, which we will treat in more details in Appendix A. Let* $\mathbb{T}^2$ *act on* $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ *by*
$$(e^{i\theta_1}, e^{i\theta_2}) \cdot ([z_0 : z_1 : z_2]) = [z_0 : z_1 e^{i\theta_1} : z_2 e^{i\theta_2}]$$
We will see that a moment map is given by:
$$\mu([z_0 : z_1 : z_2]) = \frac{1}{2}\Big(\frac{|z_1|^2}{|z|^2}, \frac{|z_2|^2}{|z|^2}\Big).$$
Then, the fixed points of ψ *are* $[1 : 0 : 0], [0 : 1 : 0]$*, and* $[0 : 0 : 1]$ *which are mapped to* $(0, 0), (\frac{1}{2}, 0)$*, and* $(0, \frac{1}{2})$*, respectively. This moment polytope is show in Figure 1.2.*

1.7 Symplectic Toric Manifolds

The following two results use the fact that any effective action $\mathbb{T}^m \to \mathrm{Diff}(M)$ has at least one orbit of dimension m: A proof may be found in [4, Corollary 5.4]

Lemma 1.7.1 *Let ψ be an effective Hamiltonian $\mathbb{T}^m$-action on a compact manifold. Then ψ has to have at least $m+1$ fixed points.*

Proof (sketch) A complete proof can be found in [7, Corollary 1.5.2]. The main idea is to use that the orbit map is a submersion on a point in an orbit. Choosing some m-dimensional orbit $\mathcal{O}$, this implies that a point $p \in \mathcal{O}$ is an interior point of the moment polytope and thus $\mu(M) \subseteq \mathbb{R}^m$ is nondegenerate and has at least $m+1$ vertices which are the images of fixed points of ψ. □

Theorem 1.7.2 *Let $(M, \omega, \mathbb{T}^m, \mu)$ be a Hamiltonian $\mathbb{T}^m$-space. If the underlying Hamiltonian action is effective, then $\dim(M) \geq 2m$.*

Proof (sketch) By utilising the idea that the moment map μ is constant on orbits of the Hamiltonian torus action, one can show that such an orbit $\mathcal{O}$ is an isotropic submanifold of M, which implies $\dim(\mathcal{O}) \leq \frac{1}{2}\dim(M)$. On can then apply this on an orbit $\mathcal{O}$ of dimension m to complete the proof. A proof can be found in [7, Theorem 1.5.3]. □

Definition 1.7.3 (Symplectic toric manifold) *A **symplectic toric manifold** is a compact connected symplectic manifold (M^{2n}, ω) of dimension $2n$ equipped with an effective Hamiltonian action of an n-dimensional torus $\mathbb{T}^n$ and with a choice of moment map μ.*

Example 1.7.4

1. *Recall, from Example 1.4.8, the Hamiltonian action on $(\mathbb{S}^2, d\theta \wedge dh)$ given by rotation*
$$\psi_t(\theta, h) = (\theta + t, h), t \in \mathbb{S}^1$$
with moment map $-h$, the negative height function. This action is effective since for $t \in \mathbb{S}^1, t \neq 0$ (or, equivalentely, $t \in \mathbb{R} \setminus 2\pi\mathbb{Z}$) we have $\psi_t \neq \mathrm{id}$. We have also seen that the negative height function h is a moment map of this action. Thus $(\mathbb{S}^2, d\theta \wedge dh, \mathbb{S}^1, -h)$ is a symplectic toric manifold.

2. *Recall from Example 1.6.3 the Hamiltonian action of $\mathbb{T}^2$ on $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ given by*
$$(e^{i\theta_1}, e^{i\theta_2}) \cdot ([z_0 : z_1 : z_2]) = [z_0 : z_1 e^{i\theta_1} : z_2 e^{i\theta_2}]$$
with moment map
$$\mu([z_0 : z_1 : z_2]) = \frac{1}{2}\Big(\frac{|z_1|^2}{|z|^2}, \frac{|z_2|^2}{|z|^2}\Big).$$

The action is effective. Since for a non trivial torus element $(e^{i\theta_1}, e^{i\theta_2})$ *one of* θ_1, θ_2 *is not in* $2\pi\mathbb{Z}$*, assume it is* θ_1*, and therefore for* $z_1 \neq 0$ *we get* $z_1 e^{i\theta_1} \neq z_1$ *and therefore* $[z_0 : z_1 : z_2] \neq [z_0 : z_1 e^{i\theta_1} : z_2 e^{i\theta_2}]$ *for* $z_0, z_1 \neq 0$*. Then* $(\mathbb{CP}^2, \omega_{\mathrm{FS}}, \mathbb{T}^2, \mu)$ *is a symplectic toric manifold.*

For symplectic toric manifolds there is a stronger equivalence condition than being symplectomorphic. Such equivalent toric manifolds are often undistinguished.

Definition 1.7.5 (Isomorphic symplectic toric manifolds) *Two symplectic toric manifolds* $(M_i, \omega_i, \mathbb{T}_i, \mu_i)$ *for* $i = 1, 2$*, are called* ***isomorphic*** *if there exists an isomorphism* $\lambda\colon \mathbb{T}_1 \to \mathbb{T}_2$ *and a symplectomorphism* $\varphi\colon M_1 \to M_2$ *satisfying* $\mu_1 = \mu_2 \circ \varphi$*.*[13]

Note that in particular, the images of the moment maps μ_1, μ_2 are the same. Since the moment map of a Hamiltonian torus action is unique up to a constant, we conclude that the moment polytope of an equivalence class of symplectic toric manifolds is unique up to translations.

[13]Such a symplectomorphism is called an **equivariant symplectomorphism**.

Chapter 2

Symplectic Reduction

We have seen in Theorem 1.6.1 that the image of the moment map of a symplectic toric manifold is a convex polytope. In Delzant's correspondence theorem, we would like to study the other direction. In this chapter, we present a tool to construct a symplectic toric manifold out of the data encoded in a given convex polytope: The symplectic reduction of an (old) symplectic manifold equipped with a Hamiltonian group action into a (new) symplectic manifold by taking quotients in the symplectic sense. The construction we will discuss was independently formulated by Meyer [19] on one side and by Marsden and Weinstein [15] on the other side.

Definition 2.0.1 (Orbit space) *Let ψ be a G-action on a manifold M. Then the **orbit space** is the set of orbits defined as $M/G := M/\sim$ equipped with the quotient topology*[1]*. Here $\sim$ is the orbit equivalence relation*

$$p \sim q \iff p \in \mathrm{orb}_\Psi(q).$$

*We denote by Π the **point-orbit projection**, given by*

$$\begin{aligned} \Pi\colon M &\to M/G \\ p &\mapsto [p]_\sim \end{aligned}$$

Note that the quotient topology is the weakest topology such that the map Π is continuous. In general, the orbit space is no longer a manifold; it does not even need to be Hausdorff.[2] However, if the action is free and proper, the orbit space is well-behaved.

Theorem 2.0.2 (Quotient manifold theorem [18]) *Let ψ be a proper and free action of G on M. Then the quotient space M/G admits the structure of*

[1] In the quotient topology, a set $U \subseteq M/G$ is open if and only if $\Pi^{-1}(U) \subseteq M$ is open.
[2] An example of a non-Hausdorff orbit space can be found in [13, Example 21.2]

a smooth manifold of dimension $\dim(M) - \dim(G)$ *such that the point-orbit projection* Π *is a smooth submersion. Moreover,* $\Pi\colon M \to M/G$ *is a principal* G*-bundle.*

Theorem 2.0.3 (Marsden-Weinstein [15], Meyer [19]) *Let* (M, ω, G, μ) *be a Hamiltonian* G*-space for a compact Lie group* G*. Let* $i\colon \mu^{-1}\{0\} \hookrightarrow M$ *be the inclusion map. If* G *acts freely on* $N = \mu^{-1}\{0\}$*, then:*

- *the orbit space* $M_{\text{red}} := N/G$ *is a manifold of dimension* $\dim(M) - 2\dim(G)$*,*
- $\Pi\colon N \to M_{\text{red}}$ *is a principal* G*-bundle, and*
- *there is a symplectic form* ω_{red} *on* M_{red} *satisfying* $i^*\omega = \Pi^*\omega_{\text{red}}$*.*

Definition 2.0.4 (Symplectic reduction) *The pair* $(M_{\text{red}}, \omega_{\text{red}})$ *is called the* ***symplectic reduction*** *of* (M, ω) *with respect to* G *and* μ*.*

The proof of Theorem 2.0.3 uses two lemmas. The first one is once again a statement from linear algebra.

Lemma 2.0.5 *Let* (V, ω) *be a symplectic vector space with isotropic subspace* I*. Then* ω *induces a canonical symplectic form* Ω *on* I^ω/I*.*

Proof Let $\iota\colon I^\omega \hookrightarrow V$ denote the inclusion. For $v, w \in I^\omega$, define

$$\Omega([v], [w]) := \omega(\iota(v), \iota(w)).$$

Then Ω is well-defined since

$$\omega(v + i, w + j) = \omega(v, w) + \omega(v, j) + \omega(i, w) + \omega(i, j) = \omega(v, w)$$

for each $i, j \in I$ by definition of the symplectic complement I^ω and since $\omega|_I \equiv 0$.

Moreover, Ω is nondegenerate. Suppose that there is some $[v] \in V/I$ with $\Omega([v], [w]) = 0, \forall [w] \in I^\omega/I$. Then $\omega(\iota(v), \iota(w)) = 0, \forall w \in I^\omega$ and thus $v \in I^\omega \cap (I^\omega)^\omega = \{0\}$. □

The second lemma concerns the kernel and image of $d\mu_p$. Because its proof is rather technical, we refer the reader to [5]. A crucial observation made in the proof is that the tangent space at an orbit is generated by the fundamental vector fields.

Lemma 2.0.6 *Let* (M, ω, G, μ) *be a Hamiltonian* G*-space for a compact Lie group* G*. Denote by* $\mathfrak{g}_p$ *the Lie algebra of the stabiliser of* $p \in M$*. Then:*

$$\ker(d\mu_p) = (T_p \operatorname{orb}_\psi(p))^{\omega_p} \tag{2.0.1}$$

$$\operatorname{im}(d\mu_p) = \mathfrak{g}_p^0 = \{\eta \in \mathfrak{g}^* \mid \langle \eta, X \rangle = 0, \forall X \in \mathfrak{g}_p\} \tag{2.0.2}$$

Moreover, if G acts freely on $N = \mu^{-1}\{0\}$, then N is a submanifold of codimension $\dim(G)$. And the G-orbit at a point $p \in N$ is an isotropic submanifold. That is, $T_p \operatorname{orb}_\psi(p)$ is an isotropic subspace of T_pM.

Proof (of Theorem 2.0.3) From Lemma 2.0.6, we know that $N = \mu^{-1}\{0\}$ is a manifold of dimension $\dim(M) - \dim(G)$. Since G is compact, its action on N is proper. If it is in addition free, then we can apply Theorem 2.0.2 to conclude the first two parts of the theorem.

Moreover, we know from Lemma 2.0.6 that $T_p \operatorname{orb}_\psi(p) \subseteq T_pM$ is isotropic for each $p \in N$ with $(T_p \operatorname{orb}_\psi(p))^{\omega_p} = T_pN$. Then Lemma 2.0.5 gives a canonical symplectic structure on $T_pN/T_p \operatorname{orb}_\psi(p)$.

Since $\Pi\colon N \to N/G$ is a submersion, again by Theorem 2.0.2, we have that $D_p\Pi\colon T_pN \to T_{[p]}(N/G)$ is a surjective linear map whose kernel has dimension $\dim(G)$. Now observe that for each $p \in N$, the orbit $\operatorname{orb}_\psi(p)$ at p is a $\dim(G)$-dimensional subspace on which Π is constant. We conclude that $T_p \operatorname{orb}_\psi(p) = \ker(D\Pi)$ and that the map

$$\begin{aligned} T_pN/T_p \operatorname{orb}_\psi(p) &\to T_{[p]}(N/G) \\ [\xi] &\mapsto D\Pi(\xi) \end{aligned}$$

is an isomorphism.

By Lemma 2.0.5, ω_p together with this ismorphism then induces a symplectic form $(\omega_{\text{red}})_p$ on the vector space $T_{[p]}(N/G)$. Since ω is G-invariant, the values depend only on the orbit of p and thus ω_{red} is well-defined for points in N/G. It is left to show that this pointwise defined ω_{red} is a closed 2-form.

To show that ω_{red} is indeed a 2-form, we use the differential Form criterion ([18, Theorem 21.5]). For this, let $X, Y \in \mathfrak{X}(N/G)$ be two vector fields. Then $X = D\Pi[X']$ and $Y = D\Pi[Y']$ for $X', Y' \in \mathfrak{X}(N)$. Then, by definition, $\omega_{\text{red}}(X, Y) = \omega(X', Y') \in C^\infty(N/G)$.

Closedness follows from the fact that the exterior differential commutes with the pullback of a map (see [18, Lemma 23.4]).

Finally,

$$\begin{aligned} i^*\omega(\xi, \zeta) &= \omega_p(D_pi\xi, D_pi\zeta) \\ &= (\omega_{\text{red}})_p([\xi], [\zeta]) \\ &= (\omega_{\text{red}})_p(D_p\Pi\xi, D_p\Pi\zeta) \\ &= \Pi^*(\omega_{\text{red}})_p(\xi, \zeta) \end{aligned}$$

for all $\xi, \zeta \in T_pN$ and thus $i^*\omega = \Pi^*\omega_{\text{red}}$. □

Example 2.0.7 *Consider the standard symplectic space $(\mathbb{C}^{n+1}, \omega_0)$ with an $\mathbb{S}^1$-action given by multiplying with $e^{i\theta}$. This action is Hamiltonian with moment map $\mu(\theta) = \frac{1}{2}|z|^2 - \frac{1}{2}$.*

Its zero level set is $N = \mu^{-1}(\{0\}) = \{z \in \mathbb{C}^{n+1} \mid |z| = 1\}$. *Since the action of* $\mathbb{S}^1$ *on* N *is given by multiplication with with* $e^{i\theta}$, *we conclude that the quotient space* $N/\mathbb{S}^1$ *is isomorpic to* $\mathbb{CP}^n$. *Moreover, as any* $z \in N$ *has some non-zero coordinate, then* $e^{i\theta}z \neq z$ *for any* $\theta \notin 2\pi\mathbb{Z}$. *Ths implies that* $\mathbb{S}^1$ *acts freely on* N.

By Theorem 2.0.3, the quotient manifold $M_{\text{red}} = N/\mathbb{S}^1 \cong \mathbb{CP}^n$ *admits a symplectic form* ω_{FS}. *In the appendix, it is shown that this symplectic form is equal to the Fubini-Study form* ω_{FS}, *which we already mentioned in Example 1.6.3.*

Proposition 2.0.8 *Let* H *be another Lie group acting on* (M, ω) *in a Hamiltonian way with moment map* $\eta \colon M \to \mathfrak{h}^*$. *If the* H*-action commutes with the* G*-action and if* η *is* G*-invariant, then* $M_{\text{red}} = N/G$ *inherits a Hamiltonian action of* H *and a moment map* $\eta_{\text{red}} \colon M_{\text{red}} \to \mathfrak{h}^*$ *satisfying* $\eta_{\text{red}} \circ \Pi = \eta \circ i$, *where* i *denotes the inclusion map and* Π *the projection map.*

Proof The H-action ψ induces an H-action ψ_{red} on M_{red} given by:

$$(\psi_{\text{red}})_h([p]) := \Pi\Big(\psi_h\big(i(p)\big)\Big).$$

for $h \in H$. It is well-defined since the two actions commute and the images under ψ_h of two elements in the same G-orbit end up in the same G-orbit.

Define $\eta_{red} \colon M_{\text{red}} \to \mathfrak{h}^*$ by $\eta_{\text{red}}([p]) := \eta(i(p))$. This is well-defined since η is by assumption G-invariant and is thus constant on G-orbits. And by definition $\eta_{\text{red}} \circ \Pi = \eta \circ i$.

It is left to show that η_{red} is indeed a moment map for the induced H-action on M_{red}.

Let $X \in \mathfrak{h}$. We denote by $X^\#$ be the fundamental vector field associated to X with respect to ψ and by $X^\#_{\text{red}}$ the one with respect to ψ_{red}. Then

$$X^\#_{\text{red}}\big([p]\big) = D_e\psi^{[p]}_{\text{red}}X = D_e\big(\Pi \circ \psi^{i(p)}\big)X = D_{[p]}\Pi \circ D_e\psi^{i(p)}X$$

for each $[p] \in M_{\text{red}}$.

By using $i^*\omega = \Pi^*\omega_{\text{red}}$ and $d\eta^X = \iota_{X^\#}\omega$, we get for $p \in N$ and $\xi \in T_pN$:

$$\begin{aligned}
\iota_{X^\#_{\text{red}}}(\omega_{\text{red}})_{[p]}\big(D_{[p]}\Pi\xi\big) &= (\omega_{\text{red}})_{[p]}\big(X^\#_{\text{red}}([p]), D_{[p]}\Pi\xi\big) \\
&= (\omega_{\text{red}})_{[p]}\big(D_e\psi^{[p]}_{\text{red}}X, D_{[p]}\Pi\xi\big) \\
&= (\omega_{\text{red}})_{[p]}\big(D_{[p]}(\Pi) \circ D_e\psi^{i(p)}X, D\Pi\xi\big) \\
&= (\Pi^*\omega_{\text{red}})_p\big(D_e\psi^{i(p)}, \xi\big) \\
&= (i^*\omega)_p\big(X^\#, \xi\big) \\
&= d\eta^X(\xi) \\
&= d\eta^X_{\text{red}}(D\Pi\xi)
\end{aligned}$$

In the proof of Theorem 2.0.3, we have seen that every element of $T_{[p]}M_{\text{red}}$ is of the form $D_{[p]}\Pi\xi$ for some $\xi \in T_pN$. And thus $\iota_{X^{\#}_{\text{red}}}\omega = d\eta^X_{\text{red}}$.

Since the G-action commutes with the H-action, η_{red} directly inherits the second condition of a moment map from η. $\square$

Example 2.0.9 *Here we revisit Example 1.6.3 and calculate the moment map of the action on the projective plane. For this, we consider the Hamiltonian $\mathbb{S}^1$-space $(\mathbb{C}^4, \omega_0, \mathbb{S}^1, \mu)$ visited in Example 2.0.7 with moment map*

$$\mu(z) = \frac{1}{2}|z|^2 - \frac{1}{2}$$

such that the reduced space is given by $(\mathbb{CP}^2, \omega_{\text{FS}})$. Then we look at the $\mathbb{T}^2$-action on $\mathbb{C}^3$ given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_0, z_1, z_2) = (z_0, z_1 e^{i\theta_1}, z_2 e^{i\theta_2}).$$

This is a special case of the action encountered in Example 1.4.10 and has moment the map

$$\eta(z_0, z_1, z_2) = \frac{1}{2}(|z_1|^2, |z_2|^2).$$

Both actions commute and η is invariant under the $\mathbb{S}^1$-action since it only depends on the absolute values of the entries.

Then $\mu^{-1}(\{0\}) = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 \mid |z|^2 = 1\}$ and under the identification $\mu^{-1}(\{0\}) \cong \mathbb{CP}^2 \setminus \{0\}$, we get $\eta_{\text{red}}([z_0, z_1, z_2]) = \frac{1}{2}\left(\frac{|z_1|}{|z|}, \frac{|z_2|}{|z|}\right)$.

Remark 2.0.10 *A special case of Proposition 2.0.8 is the reduction for product groups. Let $G = G_1 \times G_2$ be the product of two compact and connected Lie groups G_1 and G_2. Suppose that (M, ω, G, μ) is a Hamiltonian G-space with moment map*

$$\mu\colon M \to \mathfrak{g}^* \cong \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*.$$

Then μ is of the form $\mu = (\mu_1, \mu_2)$ where $\mu_i\colon M \to \mathfrak{g}_i^*$ for $i = 1, 2$. Since conjugation by $g_1 \in G_1$ is equal to the identity on G_2, $\text{Ad}^*_{g_1}$ is the identity on $\mathfrak{g}_2^*$ and therefore equivariance of μ implies that μ_2 is invariant under the action of G_1. Similarly, μ_1 is invariant under the action of G_2.

If G_1 acts freely on $Z_1 := \mu_1^{-1}(\{0\})$, we can reduce (M, ω) with respect to the G_1-action and obtain the reduced space M_1 with the corresponding reduced symplectic form ω_1.

Since the induced Hamiltonian G_2-action on (M, ω) commutes with the G_1-action and μ_1 is G_2-invariant, Proposition 2.0.8 tells us that (M_1, ω_1) inherits a Hamiltonian action of G_2 with moment map $\eta_2\colon M_1 \to \mathfrak{g}_2^*$ satisfying $\eta_2 \circ \Pi_1 = \mu_2 \circ i_1$, where $\Pi_1\colon Z_1 \to M_1$ is the projection map and $i_1\colon Z_1 \hookrightarrow M$ is the inclusion map.

Chapter 3

Delzant's Theorem

In this chapter, we bring the ingredients of the previous chapters together to formulate Delzant's correspondence theorem and prove its surjectivity part.

Theorem 3.0.1 (Delzant [9]) *Equivalence classes of toric manifolds are classified by Delzant polytopes up to translation. More precisely, there is a bijective correspondence between toric manifolds and Delzant polytopes induced by the moment map:*

$$\begin{aligned} \{\textit{toric manifolds}\} &\to \{\textit{Delzant polytopes}\} \\ (\textit{mod equivalence}) &\qquad (\textit{mod translations}) \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) &\mapsto \mu(M) \end{aligned}$$

In order to understand this correspondence, we first define and discuss Delzant polytopes, a class of convex polytopes with special properties. In Theorem 1.6.1, we already saw that the image of a moment map is a convex polytope. However, it is left to show that it also fullfills the requirements to be a Delzant polytope for the map to be well-defined. Then we establish that we can map a symplectic toric manifold to a Delzant polytope. We will omit the proof of injectivity because we do not have the required tools for the proof. However, we will show that this map is surjective by constructing a symplectic toric manifold corresponding to a given Delzant polytope using symplectic reduction. Finally, we will discuss how this construction can be further interpreted.

3.1 Delzant Polytopes

First we introduce some definitions in the context of convex polytopes. A **convex polytope** $\mathcal{P}$ in $\mathbb{R}^n$ is the convex hull of a set P of finitely many

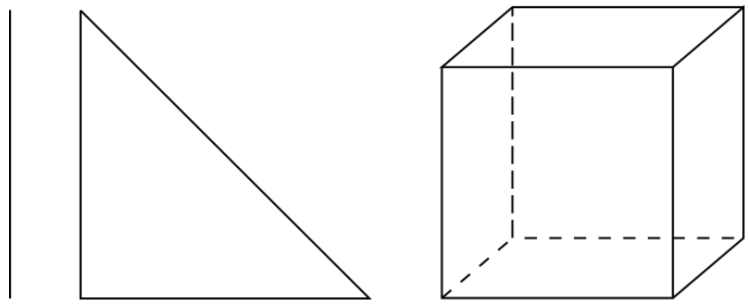


Figure 3.1: Examples of Delzant polytopes

points.[1] Equivalently[2], it is the bounded intersection of finitely many half-spaces. We call a polytope $\mathcal{P}$ d-dimensional if its affine hull is d-dimensional. The **faces** of a polytope $\mathcal{P}$ are the intersections of $\mathcal{P}$ with hyperplanes tangent to $\mathcal{P}$ and the dimension of a face is the dimension of its affine hull. Zero-dimensional faces are also called **vertices**, one-dimensional ones are called **edges** and $(\dim(\mathcal{P}) - 1)$-dimensional ones are called **facets**. The vertices of $\mathcal{P} := \text{conv}(P)$ can also be characterised as the set of points $p \in P$ such that $\text{conv}(P \setminus \{p\}) \neq \mathcal{P}$.

Definition 3.1.1 (Delzant polytope) *A **Delzant polytope** Δ in $\mathbb{R}^n$ is a polytope satisfying:*

1. ***simplicity**, that is, there are n edges meeting at each vertex.*
2. ***rationality**, that is, the edges meeting at a vertex τ are of the form $\{\tau + tu_k \mid 0 \leq t \leq T, T > 0, u_k \in \mathbb{Z}^n\}$.*
3. ***smoothness**[3], that is, for each vertex, the corresponding $u_1, \dots, u_n$ can be chosen to form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Notice that translating a Delzant polytope $\Delta \subseteq \mathbb{R}^n$ by some vector $v \in \mathbb{R}^n$ gives another Delzant polytope since the conditions at the vertices do not change. Also applying some linear transformation $A \in \text{GL}(n, \mathbb{Z})$ gives another Delzant polytope since A maps a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ onto another one. Moreover, observe that the smoothness condition is equivalent to the premise that the matrix having the u_i as columns has determinant equal to ± 1. The following examples are stated as exercises in [7] with additional hints in [6].

[1] There are many characterisations of the convex hull of a finite point set P. For instance, it is the set of all convex combinations of points in P.

[2] This is called the Minkowsky-Weyl theorem or the main theorem of polytopes, one of the most fundamental result in polytope theory. See [24] for its proof.

[3] A brief note on the name of this property: Delzant polytopes correspond to smooth projective toric varieties and this property corresponds to the smoothness of the variety for more, see [8].

Example 3.1.2

1. *In Figure 3.1 there are three examples of polytopes that are Delzant: a closed interval, an isosceles triangle, and a cube.*

2. *Up to translations and linear transformations in* $\mathrm{GL}(2,\mathbb{Z})$*, Delzant polytopes in* $\mathbb{R}^2$ *on three vertices are represented by the one-parameter family of (isosceles, rectangular) triangles* Δ_α *with vertices* $(0,0),(\alpha,0)$ *and* $(0,\alpha)$ *for* $\alpha > 0$*.*

   First notice that for each $\alpha > 0$*, the triangle* Δ_α *is indeed Delzant since it satisfies simplicity and the vectors* u_1, u_2 *and* u_3 *can be chosen to be* $\{(0,1),(1,0)\}$*,* $\{(-1,0),(-1,1)\}$*, and* $\{(0,-1),(1,-1)\}$*, respectively. Hence the rationality and smoothness conditions are both satisfied.*

   Let $\Delta \subseteq \mathbb{R}^2$ *be a Delzant triangle with vertices* p, q *and* r*. After translating* Δ*, we can assume* $p = (0,0)$ *and therefore we only need to find some* $A \in \mathrm{GL}(2,\mathbb{Z})$ *that transforms* Δ *into an isosceles triangle with a right angle at* p*. Let* u_1, u_2 *be* $\mathbb{Z}$*-vectors defining the edges at* p*, forming a* $\mathbb{Z}$*-basis. Then there exists an* $A \in \mathrm{GL}(2,\mathbb{Z})$ *that transforms the basis vectors into the standard basis vectors. That is* $Au_i = e_i$ *for* $i = 1,2$*. In particular after applying* A *to* Δ*, we get a right angle at* $p' = p$ *with edges in* $\{p + te_i \mid t \geq 0, i = 1,2\}$*.*

   Denote the lengths of the two edges meeting at p *by* a *and* b*. The new triangle* $\Delta' = A\Delta$ *is still Delzant with an edge between* q' *and* r' *in* $\{q' + t(-a,b) \mid t > 0\}$*. Thus the vector* $(-a,b)$ *has to be a multiple of the vector* u *corresponding to this edge. That is,*

   $$\alpha u := \alpha(-a', b') = (-a, b).$$

   In order to satisfy the smoothness condition at q' *and* r'*, each one of* $\{(0,1),(-a',b')\}$ *and* $\{(1,0),(a',-b')\}$ *has to form a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^2$*. This only works if* $a', b' \in \{\pm 1\}$ *and thus both edges meeting at* p *have the same length* α*. After multiplying* A *with some reflection, if necessary, the vertices of* Δ' *are given by* $(0,0),(\alpha,0)$*, and* $(0,\alpha)$ *as desired.*

3. *Consider the trapezoid shown in Figure 3.2. We claim that it is Delzant if and only if* $c = nb$ *for some* $n \in \mathbb{N}_0$*. Such a trapezoid is called a* ***Hirzebruch trapezoid****.*

   The first two conditions are clearly satisfied and so is the smoothness condition at the vertices $(0,0)$ *and* $(0,b)$*. In order to satisfy the smoothness condition at* (a,b) *and* $(a+c,0)$ *a* $\mathbb{Q}$*-multiple of* $(c,-b)$ *together with* $(-1,0)$ *has to be a* $\mathbb{Z}$*-basis. For this,* b *has to divide* c *and thus* $c = nb$ *for some* $n \in \mathbb{N}_0$*.*

4. *Up to translations and linear transformations in* $\mathrm{GL}(2,\mathbb{Z})$*, the Delzant polytopes in* $\mathbb{R}^2$ *on four vertices are given by the Hirzebruch trapezoids* $H_{a,b,n}$*, where* $a, b > 0$ *and* $n \in \mathbb{N}_0$*.*

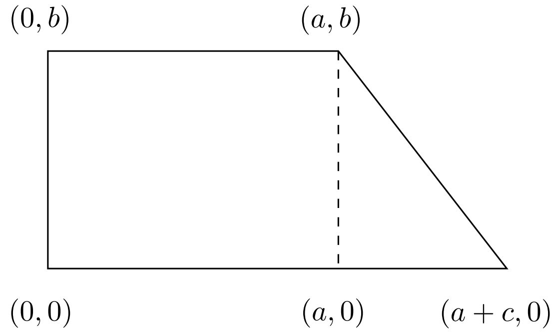


Figure 3.2: The trapezoid is a Hirzebruch trapezoid if $c = nb$

Let $\Delta \subseteq \mathbb{R}^2$ be a Delzant polytope with four vertices. After translation we can assume that one of them is $(0, 0)$. As in the triangular case, the smoothness condition at $(0, 0)$ implies that there is some $A \in \mathrm{GL}(2, \mathbb{Z})$ such that $A\Delta = \Delta'$ has a right angle at $(0, 0)$ and that the second vertices of the edges meeting at $(0, 0)$ are of the form $(0, b)$ and $(a, 0)$, respectively for some $a, b \in \mathbb{Z}^+$.

Using the smoothness condition at $(0, b)$, we conclude that the second vector has to complete a $\mathbb{Z}$-basis with $(0, 1)$ and thus is of the form $(1, B)$ for some $B \in \mathbb{Z}$. Observe that it cannot be -1 since the polygon is convex. Similarly, the second vector at $(a, 0)$ has to be of the form $(A, 1)$ for some $A \in \mathbb{Z}$.

Since the polygon is convex and satisfies the smoothness condition at the fourth vertex, the two edges meeting at the fourth vertex, pointing to this vertex, give rise to a positively oriented $\mathbb{Z}$-basis $\{(A, 1), (1, B)\}$. And thus

$$-1 = \det \begin{pmatrix} A & 1 \\ 1 & B \end{pmatrix} = BA - 1.$$

Therefore either $B = 0$ or $A = 0$ which implies that Δ' has to be a trapezoid, more precisely a Hirzebruch trapezoid.

5. *Figure 3.3 demonstrates three examples of polytopes that are not Delzant. The ocahedron fails the simplicity condition at each vertex, the rotated triangle fails the rationality condition at each vertex if $\theta \in \mathbb{R}$ is chosen carefully and the non-isosceles rectangular triangle fails the smoothness condition at $(0, 1)$ and $(2, 0)$.*

In order to construct a symplectic toric manifold out of a Delzant polytope, we convert the Delzant conditions into similar conditions in terms of normal vectors to the facets.

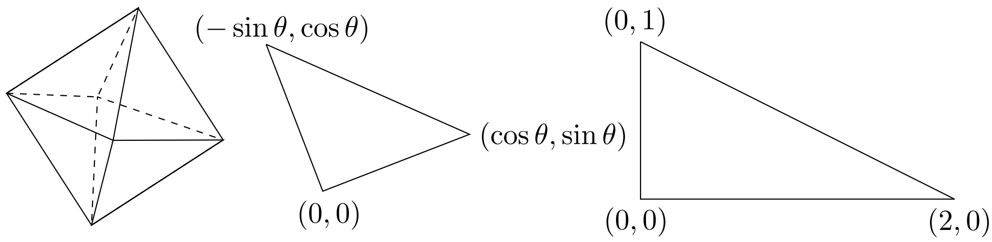


Figure 3.3: Examples for non Delzant polytopes

Definition 3.1.3 (Primitive vector) *A vector $v \in \mathbb{Z}^n$ is called **primitive** if it cannot be written in the form lv' for $l \in \mathbb{Z}, |l| > 1$ and $v' \in \mathbb{Z}^n$ or equivalently, if the gcd of entries is equal to one.*

Proposition 3.1.4 *Let $\Delta \subseteq \mathbb{R}^n$ be a Delzant polytope and v some vertex of Δ. Then there are n facets meeting at v and the primitive inward-pointing normal vectors to these facets form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Proof After a translation, we can assume that $v = 0$. Let $u_1, \dots, u_n$ be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ arising from the Delzant conditions. Then there exists a matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ such that $Au_i = e_i$ for $i = 1, \dots, n$, where $e_1, \dots, e_n$ is the standard basis of $\mathbb{Z}^n$. Denote by $\Delta' := A\Delta$ the transformed polytope.

One can show that Δ' lies inside the cone based at $v = 0$ spanned by the vectors e_i, see [24, Lemma 3.6]. Now let

$$H_i := \{x \in \mathbb{R}^n \mid \langle e_i, x\rangle = 0\}$$

be the hyperplane at $v = 0$ with primitive inner normal e_i. Then by the above statement, Δ' lies in the positive half-space bounded by H_i. Consider $F_i := \Delta' \cap H_i$. By convexity of Δ' and by definition of H_i, $v + \sum_{j\neq i} t_j e_j \in F_i$ for small enough $t_j > 0$. Moreover, the affine hull of F_i is given by $\{v + \sum_{j\neq i} t_j e_j \mid t_j \in \mathbb{R}\}$. Thus F_i is $n-1$-dimensional and therefore a facet of Δ'. The primitive inner unit normal at the facet F_i is e_i.

Since A is invertible, there are n facets of Δ meeting at v with outer or inner unit normals $A^{-1}e_i =: v_i$. Since $A \in \mathrm{GL}(n, \mathbb{Z})$ these v_i form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Let $v_i = lu_i$ for some $u_i \in \mathbb{Z}^n$. Then $e_i = Av_i = lAu_i$ and thus $l = \pm 1$ and we conclude that v_i is primitive. □

Example 3.1.5 *Consider a triangle with vertices $(0,0), (a,0)$, and $(0,b)$ where $a, b \in \mathbb{Z}^+$. Then the primitive inward-pointing normal vectors at the facets are given by: $(1,0), (0,1)$, and (b', a'), where $da' = a$ and $db' = b$ with $d = gcd(a,b)$. From Example 3.1.2 we know that the triangle is Delzant if and only if $a = b$ and indeed this is the only possibility for (a', b') to form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$ with either one of $(1,0)$ and $(0,1)$.*

3.2 Proof of Well-Definedness

We begin by showing that the map from Delzant's theorem is well-defined, following [6].

We mentioned earlier that the moment polytope of a class of symplectic toric manifolds is unique up to translations. Thus, it is left to show that the resulting moment polytope is a Delzant polytope.

Proposition 3.2.1 *Let $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold. Then the image Δ of μ is a Delzant polytope.*

Proof By the Atiyah-Guillemin-Sternberg convexity theorem, Theorem 1.6.1, we know that Δ is a convex polytope. More precisely, it is the convex hull of the images of the fixed points of the action. Let τ be a vertex of Δ and $p \in M$ be a fixed point satisfying $\mu(p) = \tau$. Then by the equivariant Darboux Theorem, Theorem 1.5.1, there is a $\mathbb{T}^n$-invariant neighbourhood U centred at p on which the action is linear. On this neighbourhood, the moment map is of the form

$$\mu_U(z) = \mu(p) + \frac{1}{2}\sum_{j=0}^{n} \lambda^j |z_j|^2$$

with weights λ^j.

Since the action is effective, we know from Proposition 1.5.2 that the weights $\lambda^j, j = 1, \ldots, n$, $\mathbb{Z}$-span $\mathbb{Z}^n$ and therefore form a basis for $\mathbb{Z}^n$.

Thus, the neighbourhood of $\tau = \mu(p)$ is of the form $\tau + \sum_{j=1}^{n} t_j \lambda^j$ with $t_j \geq 0$. And it, therefore, satisfies the simplicity, rationality, and smoothness conditions. As the vertex τ was arbitrary, Δ is a Delzant polytope. □

3.3 Surjectivity

The goal of this section is to use symplectic reduction for product groups to associate a symplectic toric manifold $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ to a Delzant polytope Δ with $\mu_\Delta(M_\Delta) = \Delta$. In what follows, we will describe Delzant's construction following [7].

The plan is to start with a standard action of $\mathbb{T}^d$ on $(\mathbb{C}^d, \omega_0)$ and a moment map depending on the Delzant polytope $\Delta \subset \mathbb{R}^n$. Here d is the number of facets of Δ and so in particular $d > n$. Then we construct a splitting short exact sequence[4].

$$\mathbb{1} \longrightarrow N \xrightarrow{\;i\;} \mathbb{T}^d \underset{\sigma}{\overset{\Pi}{\rightleftarrows}} \mathbb{T}^n \longrightarrow \mathbb{1}$$

[4]For several equivalent criteria for a short exact sequence to split, see [1, Theorem 3.9].

Using the resulting isomorphism $\mathbb{T}^d \cong N \times \mathbb{T}^n$, we will apply the reduction for product groups described in Remark 2.0.10. That is, we will show that we can reduce M with respect to the N-factor such that the obtained manifold M_Δ has a Hamiltonian action of the $\mathbb{T}^n$-factor and a moment map μ_Δ with the desired properties.

Let $\Delta \subseteq (\mathbb{R}^n)^*$ be a Delzant polytope with d facets and primitive inward-pointing normals $v_1, \ldots, v_d \in \mathbb{Z}^n$. Then

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \geq \lambda_i, i = 1, \ldots, d\}$$

for some $\lambda_i \in \mathbb{R}$ where

$$\langle \cdot, \cdot \rangle \colon (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$$

is the natural pairing.[5]

We consider the following standard effective torus action of $\mathbb{T}^d$ on $(\mathbb{C}^d, \omega_0)$ given by:

$$(e^{i\theta_1}, \ldots, e^{i\theta_d}) \cdot (z_1, \ldots, z_d) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_d} z_d)$$

with moment map

$$\begin{aligned} \mu \colon \mathbb{C}^d &\to (\mathbb{R}^d)^* \\ (z_1, \ldots, z_d) &\mapsto \frac{1}{2}\left(|z_1|^2, \ldots, |z_d|^2\right) + \lambda \end{aligned}$$

where $\lambda = (\lambda_1, \ldots, \lambda_d)$ is our choice of constant and is the part that depends on the polytope Δ.

We now move on to defining the short exact sequence. For this, consider the map Π defined on the standard basis of $\mathbb{R}^d$:

$$\begin{aligned} \Pi \colon \mathbb{R}^d &\to \mathbb{R}^n \\ e_i &\mapsto v_i \end{aligned}$$

It follows from Proposition 3.1.4 reformulating the Delzant conditions that Π is surjective and maps $\mathbb{Z}^d$ onto Z^n. Since Π is also a group homomorphism, it induces a surjecive group homomorphism, which we will still denote by Π, between tori:

$$\Pi \colon \mathbb{T}^d \cong \mathbb{R}^d/(2\pi\mathbb{Z}^d) \to \mathbb{T}^n \cong \mathbb{R}^n/(2\pi\mathbb{Z}^n).$$

Then its kernel $N := \ker(\Pi)$ is a connected, $(d-n)$-dimensional Lie subgroup of $\mathbb{T}^d$ with inclusion $i \colon N \hookrightarrow \mathbb{T}^d$ and Lie Algebra $\mathfrak{n}$. By surjectivity of Π and the definition of N, Π gives rise to a short exact sequence of Lie groups:

$$\mathbb{1} \longrightarrow N \xrightarrow{\ i\ } \mathbb{T}^d \xrightarrow{\ \Pi\ } \mathbb{T}^n \longrightarrow \mathbb{1}$$

[5]This is a corollary from the main theorem and is only true since Δ is fully dimensional (that is, $\Delta \subseteq \mathbb{R}^n$ is of dimension n).

Since Π is a surjective group homomorphism, it has constant rank and is therefore a smooth submersion.

In [13], it is stated that such a Lie group homomorphism induces a Lie algebra homomorphism by taking derivatives. Moreover, the kernel of the induced homomorphism i_* is equal to Lie algebra of the kernel of i. That is, $\ker(i_*) = \mathfrak{n}$ and hence the above short exact sequence of Lie groups gives rise to the following short exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{n} \xrightarrow{i_*} \mathbb{R}^d \xrightarrow{\Pi_*} \mathbb{R}^n \longrightarrow 0$$

Since the first map is an inclusion of Lie algebras and its dual is the restriction, it follows together with a theorem on the dual of short exact sequences stated and proven in [1, Theorem 3.10] that the dual of this short exact sequence is again a short exact sequence:

$$0 \longrightarrow (\mathbb{R}^\mathfrak{n})^* \xrightarrow{\Pi_*^*} (\mathbb{R}^d)^* \xrightarrow{i_*^*} \mathfrak{n}^* \longrightarrow 0$$

By Lemma 1.4.13, restricting the action of $\mathbb{T}^d$ gives a Hamiltonian action of N with moment map $i^* \circ \mu$. We now want to show that we can apply symplectic reduction with respect to this action. To show this, we make the following observation.

Claim 3.3.1 $\mu(Z) = \Pi^*(\Delta)$

Proof Let $y \in \mu(Z)$. Then using the dual short exact sequence, we get $y \in \ker(i^*) = \operatorname{im}(\Pi^*)$ and thus there is some $x \in (\mathbb{R}^n)^*$ with $y = \Pi^*(x)$. Since $y \in \operatorname{im}(\mu)$, it satisfies

$$\langle y, e_i \rangle = \frac{1}{2}|z_i|^2 + \lambda_i \geq \lambda_i.$$

Therefore,

$$\langle x, v_i \rangle = \langle x, \Pi(e_i) \rangle = \langle \Pi^*(x), e_i \rangle \geq \lambda_i$$

and hence $x \in \Delta$. Thus by definition $y = \Pi^*(x) \in \Delta'$. The converse follows analogously. □

Claim 3.3.2 *The induced action of N acts freely on the zero level set*

$$Z := (i^* \circ \mu)^{-1}\big(\{0\}\big).$$

Proof Let p be a vertex of Δ and let $z \in Z$ such that $\Pi^*(p) = \mu(z)$. By the simplicity of Δ, there are n facets with index set $I = \{i_1, \dots, i_n\}$ meeting at p. For $i \in I$,

$$\lambda_i = \langle p, v_i \rangle = \langle p, \Pi(e_i) \rangle = \langle \Pi^*(p), e_i \rangle = \langle \mu(z), e_i \rangle = \lambda_i + \frac{1}{2}|z_i|^2$$

Therefore, we obtain $z_i = 0$ for each $i \in I$. We get that the the vertices of Δ correspond to the points in Z having exactly n coordinates equal to zero. The stabiliser of z is

$$\text{stab}(z) = \{(t_1, \dots, t_d) \in \mathbb{T}^d \mid t_i = 0 \text{ for } i \notin I\}.$$

By Proposition 3.1.4, the restriction of Π to the indices in I is still surjective and thus, for dimensional reasons, bijective.[6] Therefore on the torus level, the restriction to $\text{stab}(z)$ is still bijective. Since $N = \ker(\Pi)$, we have

$$N \cap \text{ stab}(z) = \{e\},$$

thus N acts freely at each vertex. But since we have identified the vertices with the points in Z having n coordinates equal to zero, the stabiliser of every point has to be included in the stabiliser of some vertex. □

The Marsden-Weinstein-Meyer theorem, Theorem 2.0.3, then guarantees that the orbit space $M_\Delta := Z/N$ is a manifold of dimension $2d - 2(d-n) = 2n$ with a symplectic form ω_Δ satisfying $j^*\omega_0 = \rho^*\omega_\Delta$, where $j\colon Z \hookrightarrow \mathbb{C}^d$ is the inclusion map and $\rho\colon Z \to M_\Delta$ the point-orbit projection map.

Now given the Delzant polytope Δ, we have constructed a symplectic manifold $(M_\Delta, \omega_\Delta)$. Since we want a symplectic toric manifold, it is left to show that M_Δ is compact, connected and can be equipped with an effective Hamiltonian toric action whose moment polytope is Δ.

Claim 3.3.3 *The manifold M_Δ is compact and connected.*

Proof Since M_Δ is the image of Z under the continuous point-orbit projection, it suffices to argue that Z is connected and compact.

Observe that $(i^*)^{-1}(\{0\})$ is a linear subspace and each fibre of μ is the Cartesian product of (possibly degenerate) circles. Keeping all except one coordinate fixed, we get the product of an annulus with circles which is path connected. Thus, Z is connected.

To see that Z is compact, we use the Heine-Borel theorem. That is, since Z is closed, it suffices to show that it is bounded. To see this, note that by Claim 3.3.1 $\mu(Z) = \Pi^*(\Delta)$. Since μ is a proper map[7] and Δ' is compact, $Z \subseteq \mu^{-1}(\mu(Z))$ is bounded. □

Now that we established that we can reduce $\mathbb{C}^d$ with respect to the action of N, we finish the proof by showing that the short exact sequence splits and that this gives rise to an effective Hamiltonian $\mathbb{T}^n$-action on the reduced space.

[6]This is the crucial step where we used that the polytope is Delzant.
[7]To see that μ is proper, use Heine Borel, again.

Claim 3.3.4 *The manifold $(M_\Delta, \omega_\Delta)$ is a Hamiltonian $\mathbb{T}^n$-space with the moment map μ_Δ satisfying $\mu_\Delta(M_\Delta) = \Delta$.*

Proof Let p be a vertex of Δ and $z \in Z$ with $\mu(z) = \Pi^*(p)$. In the proof of Claim 3.3.2, we saw that the restriction of Π to $\operatorname{stab}(z) \cong \mathbb{T}^n$ is bijective. Then its inverse σ is a section of the short exact sequence.

$$\mathbb{1} \longrightarrow N \xrightarrow{\ i\ } \mathbb{T}^d \underset{\sigma}{\overset{\Pi}{\rightleftarrows}} \mathbb{T}^n \longrightarrow \mathbb{1}$$

Therefore, the sequence splits and we obtain an isomorphism

$$(i, \sigma)\colon N \times \mathbb{T}^n \to \mathbb{T}^d.$$

By the Property (1.4.1), the $\mathbb{T}^d$-action on M splits into two commuting Hamiltonian actions of N and $\mathbb{T}^n$, respectively, on M. By Lemma 1.4.13, the moment maps are given by $i^* \circ \mu$ and $\sigma^* \circ \mu$ respectively. Since the elements of $M_\Delta = Z/N$ are given by the orbits of the N-factor of the action, the commuting $\mathbb{T}^n$ factor descends to an action on M_Δ which remains effective.

By the reduction for product groups, Remark 2.0.10, there is a moment map μ_Δ satisfying

$$\mu_\Delta \circ p = \sigma^* \circ \mu \circ j.$$

Using this equation and

$$\sigma^* \circ \Pi^* = (\Pi \circ \sigma)^* = \mathrm{id}$$

we can finally calculate the moment polytope:

$$\mu_\Delta(M_\Delta) = \sigma^* \circ \mu \circ j(Z) = \sigma^* \circ \Pi^*(\Delta) = \Delta. \qquad \square$$

Example 3.3.5 *Let $\Delta = [-1, 1]$. Our goal is to show that $M_\Delta = \mathbb{CP}^1 = \mathbb{S}^2$. The primitive inward-pointing unit normals u_1 and u_2 are given by 1 and -1 respectively. Thus,*

$$\begin{aligned} \Pi\colon \mathbb{R}^2 &\to \mathbb{R} \\ e_i &\mapsto (-1)^i \end{aligned}$$

Then Π has kernel spanned by $e_1 + e_2$. On the torus level, Π has kernel N consisting of the diagonal elements and the moment map has the form

$$\begin{aligned} \mu\colon \mathbb{C}^2 &\to (\mathbb{R}^2)^* \\ (z_1, z_2) &\mapsto \frac{1}{2}(|z_1|^2, |z_2|^2) - (1, 1) \end{aligned}$$

The preimage of zero is given by $Z = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2| = \sqrt{2}\}$. The action on Z is given by multiplication with $\lambda \in \mathbb{C} \setminus \{0\}$.

As shown in [3, Example 3.2.18], this can be generalised. Let Δ be the simplex with vertices at the origin and at the points $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. The primitive inward-pointing normal vectors of Δ are given by the standard vectors $e_i, i = 1, \ldots, n$ and $(-1, \ldots, -1)$. Therefore Π has kernel $N = \{(\theta, \ldots, \theta) \mid \theta \in \mathbb{S}^1\}$ and N acts on $\mathbb{C}^n$ by multiplying with $e^{i\theta}$. By Example 1.4.10 this has moment map $i^* \circ \mu(z) = \frac{1}{2}\sum_{i=1}^{n} |z_i|^2 +$ constant and its preimage Z is a sphere. Then $M_\Delta = Z/N = \mathbb{CP}^n$.

3.4 About the Construction

The proof of the injectivity of the map is quite involved and requires the introduction of additional tools. A proof can be found in [3] (going back to [9]) and [14].

We now explain the main idea behind Delzant's construction following [7]. Then we discuss the obtained quotient space.

For $d > n$, the Euclidean space $\mathbb{R}^d$ is universal in the sense that for any n-dimensional (nondegenerate) polytope $\mathcal{P}$ with d facets, there is some affine plane A such that $\mathcal{P}$ is given by the intersection of A with the positive orthant $\mathbb{R}^d_+ = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i \geq 0\ \forall i\}$.[8]

Observe that $\mathbb{R}^d_+$ is the image of the moment map of the standard Hamiltonian action of $\mathbb{T}^d$ on $\mathbb{C}^d$. For a Delzant polytope, $\mu^{-1}(A)$ is then a compact manifold with inclusion $i\colon \mu^{-1}(A) \hookrightarrow \mathbb{C}^d$. This manifold has an induced closed 2-form $i^*\omega_0$ which is, however, degenerate. But with respect to this 2-form μ acts as "moment map" for the non-effective action of $\mathbb{T}^n = \mathbb{T}^d \times N$ with image given by $\mu(\mu^{-1}(A)) = \Delta$. For Delzant polytopes, the issues with $i^*\omega_0$ being degenerate can be fixed by quotienting out its kernel to obtain $M_\Delta = \mu^{-1}(A)/N$. And the remaining action of $\mathbb{T}^n$ on M_Δ satisfies the required properties.

The following observation generalises the statement that vertices of Δ are the fixed points of the action and is proven in [6]; a weaker version of it can be found in [7, Theorem 2.6.2].

Theorem 3.4.1 *For any $x \in \Delta$, $\mu_\Delta^{-1}(\{x\})$ is a single $\mathbb{T}^n$-orbit. Moreover, the dimension of the orbit is equal to the dimension of the smallest face to which x belongs.*

This also shows that for a given symplectic toric manifold, Δ is the orbit space.

[8]Similar calculations as in Claim 3.3.1 show that A is given by the image of $(\Pi^* - \lambda)$.

Lastly, we discuss the illustrative visualisation of $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ given in [14]. This description of the space is an interpretation of an alternative construction using the symplectic cutting technique from Lerman [14].

Let Δ be a Delzant polytope. The interior of Δ is the set of points given by strict inequalities:

$$\Delta^\circ = \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle > \lambda_i \ \forall i = 1, \ldots, d\}$$

This is a manifold as it is just an open subset of $\mathbb{R}^n$ and Δ itself is a manifold with corners.

Now consider the product $\mathbb{T}^n \times \Delta$. On points in the interior $\mathbb{T}^n \times \Delta^\circ$, the tangent space is isomorphic to $\mathbb{R}^n \times (\mathbb{R}^n)^*$ and one can define a closed non-degenerate 2-form induced by the natural pairing, see Example 1.1.4. At the corners, the directions in the tangent space of Δ, $T\Delta \cong (\mathbb{R}^n)^*$, normal to the incident facets are missing and a simple extension of the 2-form would be degenerate. In $T(\mathbb{T}^n \times \Delta)$, those missing directions pair up with the vectors v_k defining the facets meeting at the corner. Hence one can fix the degeneracy by eliminating the in the $\mathbb{R}^n$-component of $T(\mathbb{T}^n \times \Delta) \cong \mathbb{R}^n \times (\mathbb{R}^n)^*$ the directions of the vectors v_k. And this can be done by collapsing the orbit of the subgroup of $\mathbb{T}^n$ generated by those v_ks. As such one simultaneously gets rid of corners and singularities of ω.

$\mathbb{T}^n$ acts on $\mathbb{T}^n \times \Delta$ by multiplication of the T^n factor and has as moment map the projection onto the Δ factor.

Appendix A

On the Fubini-Study form

In this appendix, we discuss the Fubini-Study form on $\mathbb{CP}^n$. We show several equivalent definitions for the case $n = 1$ and we show that $(\mathbb{CP}^n, \omega_{\text{FS}})$ is the reduced space of $(\mathbb{C}^n, \omega_0)$ under the right action.

The Fubini-Study form on $\mathbb{C}^n$ is defined as

$$\omega_{\text{FS}} := \frac{i}{2}\partial\bar{\partial}\log\left(1 + |z|^2\right) \in \Omega^2(\mathbb{C}^n)$$

where $\partial\bar{\partial}f := \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$.

Recall that $\mathbb{CP}^n$ is obtained from $\mathbb{C}^{n+1} \setminus \{0\}$ by taking the quotient with respect to the equivalence relation $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n), \lambda \in \mathbb{C} \setminus \{0\}$. Such an equivalence class is denoted by $[z_0 : \cdots : z_n]$.

For $j = 0, \ldots, n$ let $V_j := \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid z_j \neq 0\}$ and $U_j := \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_j \neq 0\}$. Then the $V_j, j = 0, \ldots, n$ form an open cover of $\mathbb{C}^{n+1} \setminus \{0\}$ and the $U_j, j = 0, \ldots, n$ form an open cover of $\mathbb{CP}^n$ which are the domains of charts with coordinate maps

$$\begin{aligned} \varphi_j : U_i &\to \mathbb{C}^n \\ [z_0 : \cdots : z_n] &\mapsto \left(\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}\right) \end{aligned}$$

One can show that $\varphi_j^*\omega_{\text{FS}}$ and $\varphi_k^*\omega_{\text{FS}}$ are identical on the overlap $U_j \cap U_k$ for any j, k. Thus, the pull backs $\varphi_j^*\omega_{\text{FS}}$ "glue together" to define a symplectic form on $\mathbb{CP}^n$. This is called the Fubini-Study form on the complex projective space and is denoted by ω_{FS}.

We show that for $n = 1$ on the chart $U_0 = \{[z_0, z_1] \in \mathbb{CP}^1 \mid z_0 \neq 0\}$ and with coordinate $\frac{z_1}{z_0} = w = x + iy$ the Fubini-Study form is given by:

$$\omega_{\text{FS}} = \varphi_0^* \frac{i}{2}\partial\bar{\partial}\log(1 + |w|^2) = \frac{i}{2}\frac{dw \wedge d\bar{w}}{(1 + |w|^2)^2} = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}$$

$$\begin{aligned}
\frac{i}{2}\partial\bar\partial\log(1+|w|^2) &= \frac{i}{2}\frac{\partial^2\log(1+w\bar w)}{\partial w\partial\bar w}dw\wedge d\bar w &&= \frac{i}{2}\frac{\partial}{\partial w}\Big(\frac{w}{1+w\bar w}\Big)dw\wedge d\bar w\\
&= \frac{i}{2}\frac{1+w\bar w-w\bar w}{(1+|w|^2)^2}dw\wedge d\bar w &&= \frac{i}{2}\frac{dw\wedge d\bar w}{(1+|w|^2)^2}\\
&= \frac{i}{2}\frac{d(x+iy)\wedge d(x-iy)}{(1+(x+iy)(x-iy))^2} &&= \frac{i}{2}\frac{dx\wedge(-i)dy+idy\wedge dx}{(1+x^2+y^2)^2}\\
&= \frac{dx\wedge dy}{(1+x^2+y^2)^2}
\end{aligned}$$

Now we want to show that after converting the coordinates using a stereographic projection, ω_{FS} is related to $\omega_{\text{euc}} = d\theta\wedge dh$ as follows: $\omega_{\text{FS}} = \frac{1}{4}d\theta\wedge dh$.

For this we use the following (orientation reversing) stereographic projection with projection point N being the north pole:

$$\begin{aligned}
\phi\colon \mathbb{S}^2\setminus\{N\} &\to U_0\\
(\theta,h) &\mapsto [1 : \big(\cos(\theta)+i\sin(\theta)\big)f(h)]
\end{aligned}$$

where $f(h) := \sqrt{\frac{1}{0.5-0.5h}-1}$ for $h\in[-1,1)$. Then the map f satisfies: $f(-1)=0, f(0)=1$ and $\lim_{h\nearrow 1} f(h)=\infty$. Moreover

$$f'(h) = \Big((1-h)^2\sqrt{\frac{1}{0.5-0.5h}-1}\Big)^{-1} = \Big((1-h)^2 f(h)\Big)^{-1}.$$

Then we can write $D\phi$ with respect to basis vectors $\frac{\partial}{\partial\theta}, \frac{\partial}{\partial h}$ and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ respectively as:

$$D\phi = \begin{pmatrix} -\sin(\theta)f(h) & \cos(\theta)f'(h)\\ \cos(\theta)f(h) & \sin(\theta)f'(h)\end{pmatrix}.$$

Now we calculate:

$$\begin{aligned}
(\phi^*\omega_{\text{FS}})_{(\theta,h)}\Big(\frac{\partial}{\partial\theta},\frac{\partial}{\partial h}\Big) &= (\omega_{\text{FS}})_{\phi(\theta,h)}\Big(D\phi\frac{\partial}{\partial\theta}, D\phi\frac{\partial}{\partial h}\Big)\\
&= (\omega_{\text{FS}})_{\phi(\theta,h)}\Big(f(h)\big(-\sin(\theta)\frac{\partial}{\partial x}+\cos(\theta)\frac{\partial}{\partial y}\big),\\
&\qquad f'(h)\big(\cos(\theta)\frac{\partial}{\partial x}+\sin(\theta)\frac{\partial}{\partial y}\big)\Big)\\
&= \frac{1}{(1+f(h)^2)^2}\Big(\frac{-\sin(\theta)^2 f(h)}{(1-h)^2 f(h)} - \frac{\cos(\theta)^2 f(h)}{(1-h)^2 f(h)}\Big)\\
&= \frac{(1-h)^2}{4}\frac{-1}{(1-h)^2}\\
&= -\frac{1}{4}\\
&= -\frac{1}{4}d\theta\wedge dh\Big(\frac{\partial}{\partial\theta},\frac{\partial}{\partial h}\Big)
\end{aligned}$$

At first glance this does not show the claimed equality $\omega_{\text{FS}} = \frac{1}{4} d\theta \wedge dh$ because the sign is off. But the Jacobian of our projection ϕ has negative determinant and so it is orientation reversing. To fix this, one can introduce another diffeomorphism on U_0 reversing the orientation once again, which we won't do here.

Now consider the action of $\mathbb{S}^1$ on $(\mathbb{C}^{n+1}, \omega)$ by multiplying with $e^{i\theta}$. This action is Hamiltonian with moment map $\mu(\theta) = \frac{1}{2}|z|^2 - \frac{1}{2}$. In Example 2.0.7, we argued that the action is free on the zero level set with reduced space $M_{\text{red}} = \mathbb{CP}^n$. We now want to show that the reduced symplectic form ω_{red} equals ω_{FS}.

In the following, we use the variables $z_0, \ldots z_n$ for the coordinates in $\mathbb{C}^{n+1}$ and $w_1, \ldots, w_n$ for the coordinates in $\mathbb{C}^n$. Then the Fubini-Study form on the chart U_i is given by $\varphi_i^* \omega_{\text{FS}}$, where

$$\omega_{\text{FS}} = \frac{i}{2} \partial \bar{\partial} \log(1 + |w|^2).$$

For this, we first show that on V_i $\Pi^* \varphi_i^* \omega_{\text{FS}} = \frac{i}{2} \partial \bar{\partial} \log(|z|^2)$. For simplicity, we show it for $i = 0$.

Note that $w_i = \frac{z_i}{z_0}$ and thus

$$\begin{aligned}
dw_i &= \frac{\partial w_i}{\partial z_i} dz_i + \frac{\partial w_i}{\partial z_0} dz_0 = \frac{1}{z_0} dz_i + \frac{-z_i}{|z_0|^2} dz_0 \\
\frac{\partial}{\partial w_i} &= z_0 \frac{\partial}{\partial z_i} \\
|w|^2 &= \sum_{i=1}^{n} |w_i|^2 = \frac{|z|^2 - z_0^2}{z_0^2} = |z|^2 - 1 \\
\frac{\partial}{\partial z_0} &= -\sum_{i=1}^{n} \frac{z_i z_0}{|z_0|^2} \frac{\partial}{\partial z_i}
\end{aligned}$$

$$
\begin{aligned}
\Pi^*\varphi_0^*\omega_{\mathrm{FS}} &= \frac{i}{2}\partial\bar{\partial}\log(1+|w|^2) \\
&= \frac{i}{2}\sum_{j,k=1}^{n}\frac{\partial^2\log(1+|w|^2)}{\partial w_j\partial\bar{w}_k}dw_j\wedge d\bar{w}_k \\
&= \frac{i}{2}\sum_{j,k=1}^{n}\frac{\partial^2\log(|z|^2)}{\partial z_j\partial\bar{z}_k}|z_0|^2\Big(\frac{1}{z_0}dz_j+\frac{-z_j}{|z_0|^2}dz_0\Big)\wedge\Big(\frac{1}{\bar{z}_0}d\bar{z}_i+\frac{-\bar{z}_i}{|z_0|^2}d\bar{z}_0\Big) \\
&= \frac{i}{2}\sum_{j,k=1}^{n}\frac{\partial^2\log(|z|^2)}{\partial z_j\partial\bar{z}_k}dz_j\wedge d\bar{z}_k+\frac{i}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\log(|z|^2)}{\partial z_j\partial\bar{z}_k}\frac{-z_jz_0}{|z_0|^2}dz_0\wedge d\bar{z}_k \\
&\quad+\frac{i}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2\log(|z|^2)}{\partial z_j\partial\bar{z}_k}\frac{-\bar{z}_k\bar{z}_0}{|z_0|^2}dz_j\wedge d\bar{z}_0 \\
&\quad+\frac{i}{2}\sum_{j,k=1}^{n}\frac{\partial^2\log(|z|^2)}{\partial z_j\partial\bar{z}_k}\frac{z_j\bar{z}_k}{|z_0|^2}dz_0\wedge d\bar{z}_0 \\
&= \frac{i}{2}\sum_{j,k=0}^{n}\frac{\partial^2\log(|z|^2)}{\partial z_j\partial\bar{z}_k}dz_j\wedge d\bar{z}_k \\
&= \frac{i}{2}\partial\bar{\partial}\log(|z|^2)
\end{aligned}
$$

Now we want to show that $\Pi^*\varphi_0^*\omega_{\mathrm{FS}}$ has the same restriction to $\{(z_0,\dots,z_n)\mid |z|^2=1\}=\mu^{-1}(\{0\})$ as $\omega_0=\frac{i}{2}\sum_{j=0}^{n}dz_j\wedge d\bar{z}_j$.

$$
\partial\bar{\partial}\log(|z|^2)=\sum_{j=0}^{n}\frac{|z|^1-|z_j|^2}{|z|^4}dz_j\wedge d\bar{z}_j+\sum_{j,k=0,j\neq k}^{n}\frac{-\bar{z}_jz_k}{|z|^4}dz_j\wedge d\bar{z}_k
$$

Thus this boils down to show that on $\mu^{-1}(\{0\})$, we have

$$
\begin{aligned}
\sum_{j=0}^{n}|z_j|^2dz_j\wedge\bar{z}_j+\sum_{k,j=0,k\neq j}^{n}\bar{z}_jz_kdz_j\wedge d\bar{z}_k &= \sum_{j=0}^{n}\bar{z}_jdz_j\wedge\big(z_jd\bar{z}_j+\sum_{j\neq k=0}^{n}z_kd\bar{z}_k\big) \\
&= \sum_{j=0}^{n}\bar{z}_jdz_k\wedge\big(d(|z|^2)\big)=0
\end{aligned}
$$

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