Chern-Weil Theory

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Abstract

In the following notes we will discuss the basics of Chern-Weil theory, and will define all common characteristic classes. Namely, we will discuss Chern, Pontryagin, and Euler classes, and will also mention Stiefel-Whitney classes. Moreover, we will discuss the relationship between connections on principal bundles and the associated vector bundles, as well as the reduction of structure groups.

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CHAPTER 1

Connection Forms And the Chern-Weil Homomorphism

In this chapter we begin by discussing connection forms on principal bundle, and introduce the most important concept in Chern-Weil Theory, the so called Chern-Weil homomorphism.

We assume the reader is familiar with vector bundles, but as most introductory courses on Differential Geometry do not cover more general (smooth) fiber bundles, we begin this Chapter by recalling a few key definitions.

Definition 1.1. Let M, E and L be smooth manifolds, and suppose $\pi : E \to M$ is a smooth surjective map. We say that $\pi : E \to M$ is a **fiber bundle over** M **with fiber** L if for every $p \in M$ there exists a neighborhood U_p of p, and a smooth map $\epsilon : \pi^{-1}(U) \to L$ such that

$$(\pi, \epsilon) : \pi^{-1}(U) \to U \times L$$

is a diffeomorphism. A collection of **bundle charts** $(U_a, \epsilon_a)_{a \in A}$ that is charts as above, covering M is called a **bundle atlas**.

As we are considering fiber bundles from the smooth, that is from the Differential Geometry stand point, fiber bundles as defined above are tough to work with. To see why, we need the following definition.

Definition 1.2. Let $\pi : E \to M$ be a fiber bundle with fiber L. Let U_a and U_b be two bundle charts such that $U_a \cap U_b \neq \varnothing$. For $p \in U_a \cap U_b$ the charts ϵ_a and ϵ_b restrict to diffeomorphisms from $E_p = \pi^{-1}(p)$ to L. From this we get a well-defined map

$$\epsilon_{ab} : U_a \cap U_b \to \mathrm{Diff}(L), \qquad p \mapsto (\epsilon_a)_{|p} \circ (\epsilon_b)^{-1}_{|p}.$$

We call ϵ_{ab} the **transition function from** ϵ_b **to** ϵ_a.

The reason why fiber bundles are tough to work with is because the transition functions take values in the infinite-dimensional manifold $\mathrm{Diff}(L)$. We will therefore only consider a certain kind of fiber bundles.

Definition 1.3. Let $\pi : E \to M$ be a fiber bundle with fiber L. Suppose a Lie group G acts on L via an effective (left or right) action σ. We now say that a bundle atlas $(U_a, \epsilon_a)_{a \in A}$ is a (G, σ)**-bundle atlas** if the transition functions take values in G, that is, whenever $U_a \cap U_b \neq \varnothing$, there exists a smooth map $g_{ab} : U_a \cap U_b \to G$ such that

$$\epsilon_{ab}(p) = \sigma_{g_{ab}(p)}.$$

If such a bundle atlas exists, the bundle E is called a (G, σ)**-fiber bundle**. The group G is then referred to as the **structure group** of E.

Now, a principal bundle is a specific kind of (G, σ)-fiber bundle:

Definition 1.4. Let M be a smooth manifold, and suppose G is a Lie group. A **G-principal bundle over** M is a (G, σ)-fiber bundle with fiber G and where σ is the left multiplication in G. We will usually denote a principal bundle by P, and will often use the term principal bundle instead of G-principal bundle.

Remark 1.5. Later on, we will focus our attention on the case $G = U(n)$ for some $n \in \mathbb{N}$, but at the start of this Chapter we will aim for maximal generality.

There is another useful characterization of principal bundles.

Proposition 1.6. Let $\pi : P \to M$ be a surjective submersion, and let G be a Lie group. Then it holds that P is a G-principal bundle if and only if there exists a free right action τ of G on P which is fiber preserving and transitive on the fibers.

Proof. This is discussed in Chapter 5 of [6]. □

This characterization of a principal bundle is useful for the definition of a connection on a principal bundle, as connections on principal bundles rely on the notion of *fundamental vector fields.*

Definition 1.7. Let $\pi : P \to M$ be a G-principal bundle. Denote the Lie algebra of G by $\mathfrak{g}$. We define the **fundamental vector fields on** P as follows. Given $\xi \in \mathfrak{g}$ we define, for $p \in P$,

$$Z_\xi(p) = \frac{d}{dt}_{|t=0} \tau_{\exp(t\xi)}(p) \in T_pP,$$

where $\exp : \mathfrak{g} \to G$ is the usual exponential map.

We can now give the definition of a connection.

Definition 1.8. Let $\pi : P \to M$ be a G-principal bundle. A **connection** on the principal bundle P is a $\mathfrak{g}$-valued 1-form, that is an element $\omega \in \Omega^1(P, \mathfrak{g})$, satisfying:

1. It holds that $\tau_g^*\omega = Ad_{g^{-1}}(\omega)$.
2. It holds that $\omega(Z_\xi) = \xi$.

Here, $Ad : G \to \mathfrak{g}$ denotes the adjoint representation of G on $\mathfrak{g}$.

Remark 1.9. Sometimes a 1-form $\omega \in \Omega(P, \mathfrak{g})$ is called a **connection form** instead of a connection. This is because there are multiple ways to describe a connection on a principal bundle, as we shall see further down below.

The final ingredient we will need to start our discussion of characteristic classes is the *curvature form* on principal bundles.

Definition 1.10. Let $\pi : P \to M$ be a G-principal bundle. Given a connection $\omega \in \Omega^1(P, \mathfrak{g})$ on P we define the **curvature form** $\Omega \in \Omega^2(P, \mathfrak{g})$ **of** ω as the element

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega],$$

where for two vector fields $X, Y \in \mathfrak{X}(P)$ we have

$$[\omega, \omega](X, Y) = [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)].$$

Remark 1.11. Of course one has to show that the curvature form is well-defined. This follows from the linearity of the Lie bracket.

We can now start discussing Chern-Weil Theory for principal bundles. We start by considering some Lie group G.

Definition 1.12. Denote by $I^k(G)$ the set of symmetric k-linear maps $f : \mathfrak{g}\times\cdots\times\mathfrak{g} \to \mathbb{R}$, that satisfy

$$f(Ad_g(\xi_1), \cdots, Ad_g(\xi_k)) = f(\xi_1, \cdots, \xi_k).$$

We say f is **(Ad)-invariant**. We denote by $I(G)$ the $\mathbb{R}$-algebra formed by the direct sum of all $I^k(G)$ equipped with pointwise multiplication.

These symmetric multilinear maps allow us to define differential forms on P induced by the connection form. More concretely, let f be an element of $I(G)$ and suppose we are given a G-principal bundle $\pi : P \to M$ with connection ω. We now want to use the wedge product for $\mathfrak{g}$-valued differential forms to construct a $2k$-form from Ω. It is important to note that for vector-valued differential forms the definition of the wedge product is slightly more tricky than the definition of the usual wedge product. Recall that vector-valued differential forms are defined by

$$\Omega^n(M, V) = \Omega^n(M) \otimes V,$$

where V is some vector space. Given two vector-valued differential forms $\xi_1 = \omega_1 \otimes v_1$, and $\xi_2 = \omega_2 \otimes v_2$ taking values in V_1 and V_2 respectively, we can write the wedge product as

$$\omega_1 \otimes v_1 \bar{\wedge} \omega_2 \otimes v_2.$$

Now we need to determine how to handle v_1 and v_2. To do this, we need a bilinear map $\beta : V_1 \times V_2 \to W$ for a third vector space W. In this case we can define

$$\omega_1 \otimes v_1 \bar{\wedge} \omega_2 \otimes v_2 := \omega_1 \wedge \omega_2 \otimes \beta(v_1, v_2),$$

where $\wedge$ is the usual wedge product. Luckily, for any two vector spaces we can always choose $W = V_1 \otimes V_2$ and let β be induced by the identity map $V_1 \times V_2 \to V_1 \times V_2$. We will now denote the wedge product $\bar{\wedge}$ simply by $\wedge$ - the context will make it clear which wedge product we are using. Now for the $2k$-form we mentioned earlier.

Definition 1.13. Let $\pi : P \to M$ be a G-principal bundle, with connection ω and curvature form Ω. Suppose $f \in I^k(G)$. Using the above wedge product, let

$$\Omega^k = \Omega \wedge \cdots \wedge \Omega \in \Omega^{2k}(P, \otimes_{i=1}^k \mathfrak{g}).$$

The map f is a symmetric k-linear map on $\mathfrak{g}$. As such, it corresponds to an element of $\bigotimes_{i=0}^k g^*$. We can thus define a $2k$-form $f(\Omega) \in \Omega^{2k}(P)$ by

$$f(\Omega)_p(\xi_1, \cdots, \xi_{2k}) = f(\Omega_p^k(\xi_1, \cdots, \xi_{2k})),$$

for $p \in P$ and $\xi_1, \cdots, \xi_{2k} \in T_pP$. More explicitly, this can be expressed as

$$f(\Omega)_p(\xi_1, \cdots, \xi_{2k}) = \frac{1}{(2k)!} \sum_{\sigma \in S_{2k}} \text{sign}(\sigma) f(\Omega_p(\xi_{\sigma(1)}, \xi_{\sigma(2)}), \cdots, \Omega_p(\xi_{\sigma(2k-1)}, \xi_{\sigma(2k)})).$$

We call such a $2k$-form a **characteristic form**.

We now state the first important result about characteristic forms.

Theorem 1.14. Let $\pi : P \to M$ be a G-principal bundle, with connection ω and curvature form Ω. Then the following are true:

1. For each $f \in I(G)$ the characteristic form $f(\Omega)$ projects to a unique closed $2k$-form $\overline{f(\Omega)} \in \Omega^{2k}(M)$, i.e. $\pi^*(\overline{f(\Omega)}) = f(\Omega)$.
2. The deRham cohomology class $[\overline{f(\Omega)}] \in H^{2k}_{dR}(M;\mathbb{R})$ is independent of the connection ω, and the map

$$CW_P : I(G) \to H^*_{dR}(M;\mathbb{R}), \quad f \mapsto [\overline{f(\Omega)}]$$

   is an algebra homomorphism.

We call the map CW_P the **Chern-Weil homomorphism**.

Proof. A proof can be found in Chapter 12 of *Foundations of Differential Geometry, Vol. 2* by Kobayashi and Nomizu [7]. □

With this Theorem in hand we can now define characteristic classes.

Definition 1.15. Let $\pi : P \to M$ be a G-principal bundle. For $f \in I(G)$ we define its **characteristic class** as the image of f under the Chern-Weil homomorphism.

Instead of considering characteristic classes defined by symmetric multilinear forms, we can also define characteristic classes using homogeneous polynomials. First, we define homogeneous polynomials.

Definition 1.16. Consider the $\mathbb{R}$-algebra $\mathbb{R}[x_1, \cdots, x_n]$ of polynomials in n indeterminates. We say $q \in \mathbb{R}[x_1, \cdots, x_n]$ is **homogeneous of degree k** if we can write

$$q(x_1, \cdots, x_n) = \sum c_{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k},$$

where we sum over all n^k tuples $(i_1, \cdots, i_k)$.

We now generalize this notion to arbitrary vector spaces:

Definition 1.17. For a vector space V of dimension n a **homogeneous polynomial of degree** k on V is a map $f : V \to \mathbb{R}$ such that for a basis $(e^i)_{i=1}^n$ of V^* we can write

$$f(v) = q(e^1, \ldots, e^n)(v) = \sum c_{i_1 \cdots i_k} e^{i_1}(v) \cdots e^{i_k}(v),$$

for some (unique) $q \in \mathbb{R}[x_1, \cdots, x_n]$ and all $v \in V$. We denote the space of such homogeneous polynomials by $\mathbb{P}^k(V)$.

As was the case with symmetric multilinear forms, the direct sum of the spaces $\mathbb{P}^k(V)$ gives rise to an algebra $\mathbb{P}(V)$.

Definition 1.18. Let $f \in \mathbb{P}(\mathfrak{g})$. Then we say f is Ad-invariant, if

$$f(Ad_g \xi) = f(\xi), \quad \forall g \in G, \ \forall \xi \in \mathfrak{g}.$$

Remark 1.19. Recall that for matrix Lie groups the adjoint representation is given by conjugation, that is for $A \in G$ we have that $Ad_A : \mathfrak{g} \to \mathfrak{g}$ is given by $Ad_A(B) = ABA^{-1}$.

The set of Ad-invariant homogeneous polynomials on $\mathfrak{g}$ is obviously a subalgebra of $P(\mathfrak{g})$ which we will denote by $\mathbb{P}_{inv}(\mathfrak{g})$ Now, the reason why these homogeneous polynomials on $\mathfrak{g}$ are of interest to us is the following Theorem:

Theorem 1.20. It holds that the algebras $I(\mathfrak{g})$ and $\mathbb{P}_{inv}(\mathfrak{g})$ are isomorphic.

Proof. This is Corllary 2.3 in Chapter 12 of [7]. □

Why this Theorem is useful will be come apparent in a bit. However, it is immediately clear from Theorem 1.20 that characteristic classes can be equivalently defined using invariant homogeneous polynomials. After having defined characteristic classes for G-principal bundles, we will now turn our attention towards vector bundles. Historically, this is the area where characteristic classes were first discussed and utilized.

CHAPTER 2

Chern classes and Pontryagin classes

We will start our discussion of characteristic classes by talking about *Chern classes.* For this, consider a complex vector bundle $\pi : E \to M$ of rank n over M. For each such vector bundle, there exists an associated principal bundle, called the *frame bundle of* E.

Definition 2.1. Let $\pi : E \to M$ be a complex vector bundle of rank n. Denote by $Fr(E_p)$ the set of isomorphisms $A : \mathbb{C}^n \to E_p$. The **frame bundle of** E is the $GL(n, \mathbb{C})$-principal bundle $Fr(E)$ with fiber $Fr(E_p)$.

Of course, just because we define $Fr(E)$ as a principal bundle does not mean it actually is a principal bundle. We will rectify this problem with the following Proposition.

Proposition 2.2. Let $\pi : E \to M$ be a complex vector bundle of rank n. Then there is a structure of a $GL(n, \mathbb{C})$-principal bundle on the frame bundle E.

Proof. We defined $Fr(E)$ to be the space

$$Fr(E) = \coprod_{p \in M} Fr(E_p).$$

Suppose we have two elements $l_1, l_2 \in Fr(E_p)$. Then these two only differ by an automorphism of $\mathbb{C}^n$, that is there exists $A \in GL(n, \mathbb{C})$ such that $l_1 = l_2 \circ A$. Thus, fixing an element $l \in Fr(E_p)$, we get a bijection $\phi : GL(n, \mathbb{C}) \to Fr(E_p)$. Now ϕ induces a topology on $Fr(E_p)$ by defining a set $U \subset Fr(E_p)$ to be open if and only if $\phi^{-1}(U)$ is open. Then ϕ is a homeomorphism. Now the smooth structure on $GL(n, \mathbb{C})$ induces a smooth structure on $Fr(E_p)$. Thus we have established that $Fr(E_p)$ is a manifold.

We now want to show the $Fr(E_p)$ assemble to a principal bundle. Consider a vector bundle atlas $\{(U_a, \epsilon_a)\}_{a \in A}$. Then $\epsilon_a : E_{|U_a} \to U_a \times \mathbb{C}^n$ induces a bijection $\bar{\epsilon}_a : Fr(E)_{|U_a} \to U_a \times Fr(\mathbb{C}^n)$ by composition with $(\epsilon_a)_{|p} : E_p \to \mathbb{C}^n$. That is we have

$$(\bar{\epsilon}_a)_{|p}((\epsilon_a)_{|p}^{-1} \circ A) = A \in GL(n, \mathbb{C}),$$

where we used that $(\epsilon_a)_{|p}$ is an isomorphism from E_p to $\mathbb{C}^n$ by assumption. This bijection can, as above, be used to induce a a topology and the structure of a smooth manifold on $Fr(E)$. Note that $\pi : Fr(E) \to M$ given by $\pi(Fr(E)_p) = \pi(Fr(E_p)) = p$ is a smooth submersion. Now all that is left to show is that the transition functions are given by left multiplication in $GL(n, \mathbb{C})$. To see this note that for $A \in GL(n, \mathbb{C})$ it holds that

$$\begin{aligned}
\bar{\epsilon}(p)_{ab}(A) &= (\bar{\epsilon}_a)_{|p} \circ (\bar{\epsilon}_b^{-1})_{|p}(A) \\
&= (\bar{\epsilon}_a)_{|p}((\epsilon_b)_{|p}^{-1} \circ A) \\
&= \epsilon_{ab}(p) \circ A \\
&= l_{\epsilon_{ab}(p)}(A).
\end{aligned}$$

This shows that the transition functions are given by left multiplication in $GL(n, \mathbb{C})$, as desired, and it follows that $Fr(E)$ is a $GL(n, \mathbb{C})$-principal bundle. ☐

This way, we can associate to every complex vector bundle E a principal bundle. Using this, we can define characteristic classes for complex vector bundles. Consider an element $A \in \mathfrak{gl}(n, \mathbb{C})$, and note that we can write

$$\det(tId - \frac{1}{2\pi i}A) = \sum_{k=0}^{r} f_k(A)t^{r-k},$$

for polynomials $f_i \in \mathbb{P}_{inv}(\mathfrak{gl}(n, \mathbb{C}))$. The invariance under Ad_g follows from the fact that determinants are invariant under inner automorphisms.

Definition 2.3. Let $\pi : E \to M$ be a complex vector bundle of rank n. We define the **kth Chern class of E** to be

$$c_k(E) := [\overline{f_k(\Omega)}] \in H^{2k}_{dR}(M).$$

Here Ω denotes a curvature form on the frame bundle $Fr(E)$ associated to E.

Remark 2.4. Characteristic classes can also be defined using *classifying spaces* instead of Chern-Weil Theory. The two approaches yield different results in some instances, as we will discuss further down below. In the case of Chern classes this is not the case. Usually Chern classes are defined, either via induction (cf. [10]) or axiomatically (cf.[5]) as integer cohomology classes. However, one can show that the Chern classes as defined above are in fact integer classes (cf. Chapter 12, Theorem 3.1 in [7]).

We now want to show that the Chern classes generate the characteristic classes as defined above.

Theorem 2.5. Let $\pi : E \to M$ be a complex vector bundle of rank n. The algebra of characteristic classes of E is generated by the Chern classes $\{c_k(E)\}_{k=0}^n$.

To show this, we will utilize our observation that characteristic classes are defined equivalently by invariant polynomials as stated in Theorem 1.20. We will need the following result:

Lemma 2.6. Let $\pi : E \to M$ be a complex vector bundle of rank n, and suppose that P denotes the associated frame bundle from Definition 2.1. Then the structure group $GL(n, \mathbb{C})$ of P can be reduced to $U(n)$.

Let us explain what we mean by *reducing the structure group of* P. Consider a G-principal bundle F. In particular, F is a (G, σ)-fiber bundle with fiber G and σ given by left multiplication. Then by definition the transition functions ϵ_{ab} of a bundle atlas $\{(U_a, \epsilon_a)\}_{a \in A}$ take value in the group G. If there exists a subgroup $H \subset G$ and a bundle atlas $\{(U'_a, \bar{\epsilon}'_a)\}_{a' \in A}$ such that the transition functions $\bar{\epsilon}_{ab}$ take value in H, then this defines a $(H, \sigma_{|H})$-fiber bundle atlas on the G-principal bundle P. In this case we say that the **structure group of F can be reduced to H**. The existence of such a reduction has a useful consequence.

Proposition 2.7. Let $\pi : P \to M$ denote a G-principal bundle. Let $H \subset G$ be a subgroup. Then there exists a reduction of the structure group of P to H if and only if there is a H-principal bundle P', that is an embedded submanifold of P. In particular, P' is a subbundle of P.

Proof. This is the content of Proposition $I.5.3$ in [6]. $\square$

We refer to the bundle P' above as the **reduced bundle**. To show Lemma 2.6 we need to define what a Hermitian metric is.

Definition 2.8. Let E be a complex vector bundle over a manifold M. A **Hermitian metric** on E assigns to each fiber E_p a Hermitian inner product.

Lemma 2.9. Any complex vector bundle admits a Hermitian metric.

Proof. The proof is analogous to the existence of a Riemannian metric on a real vector bundle. Take a cover $\{U_a\}_{a\in A}$ of M such that E is trivial over each U_a. Then define the Hermitian metric over the trivial bundle $U_a \times \mathbb{C}^n$ using the standard inner product on $\mathbb{C}^n$. Note that here n is the rank of the complex vector bundle E. Then we can conclude using a partition of unity argument. □

We can now proof Lemma 2.6:

Proof(Lemma 2.6). It is a well-known fact from Riemannian geometry that the structure group of a real vector bundle can be reduced to $O(n)$ using the Riemannian metric of the vector bundle. A similar argument will work here. Fix a vector bundle atlas $\{U_a, \epsilon_a\}_{a\in A}$, and, using Lemma 2.9, fix a Hermitian metric h on E. We now want to modify the transition functions so that they are elements of $U(n)$, where n denotes the rank of the vector bundle. Take a local frame e_i^a over U_a, that is sections $e_i^a \in \Gamma(U_a, E)$ so that $(e_1^a(p), \dots, e_n^a(p))$ is basis of E_p for all $p \in U$ (this is equivalent to the existence of a vector bundle chart, just as it is in the real case). Using Gram-Schmidt and h we can produce a unitary basis from $(e_1^a(p), \dots, e_n^a(p))$. The family of change of basis matrices $A^a(p) : E_p \to E_p$ depends smoothly on p by smoothness of the sections and the smoothness of the Gram-Schmidt process. Thus the matrices assemble to a smooth map $A^a : \pi^{-1}(U_a) \to \pi^{-1}(U_a)$. Doing this for all vector bundle charts, and replacing ϵ_a by $\overline{\epsilon_a} = \epsilon_a \circ (A^a)^{-1}$, we find that the transition functions $\overline{\epsilon_{ab}}(p) : E_p \to E_p$ actually map unitary bases to unitary bases, so that $\overline{\epsilon_{ab}}$ takes values in $U(n)$. This yields the claim. □

Remark 2.10. Note that in the above, what we have actually shown is that the structure group of the $GL(n, \mathbb{C})$-fiber bundle E can be reduced to $U(n)$. This however is equivalent to the reduction of the structure group of $Fr(E)$ from $GL(n, \mathbb{C})$ to $U(n)$. To see this, simply recall the construction of the transition functions on the frame bundle in Proposition 2.2.

With this in hand we can now proof the claim we made about Chern classes generating the characteristic classes earlier. We start with the following result, which will make it obvious why we talked about reduction of structure groups above.

Proposition 2.11. For $A \in \mathfrak{u}(n)$ define the polynomial functions $g_1, ..., g_n$ on $\mathfrak{u}(n)$ by

$$\det(tId + iA) = t^n - f_1(A)t^{n-1} + \cdots + (-1)^n f_n(A).$$

Then the f_i are independent and generate the algebra $\mathbb{P}_{inv}(\mathfrak{u}(n))$.

Proof. This is Theorem 2.5 in [7]. □

Now we can proof Theorem 2.5.

Proof (Theorem 2.5). Let $P = Fr(E)$ be the associated $GL(n, \mathbb{C})$-principal bundle to E. By Lemma 2.6 we can reduce the structure group of P to $U(n)$. Denote this new bundle by Q. Choose a connection ω_Q on Q, and denote by Ω_Q the associated curavture. The connection form ω_Q can be extended to a connection form ω on P (for the details see Theorem 2.1 in Chapter 2 of [6]). Denote by Ω the curavture form of ω. For $f \in \mathbb{P}_{inv}(\mathfrak{gl}(n, \mathbb{C}))$ it holds that its restriction f_Q to $\mathfrak{u}(n)$ is an element of $\mathbb{P}_{inv}(\mathfrak{u}(n))$. By construction we have that the restriction of $f(\Omega)$ to Q is equal to $f_Q(\Omega_Q)$. Thus they determine the same characteristic class. The claim now follows from Proposition 2.11, and because the Chern-Weil homomorphism is an algebra homomorphism. □

After having established this, we will now discuss few more special characteristic classes. We start by defining *Pontryagin classes.* These can be defined for real vector bundles as well.

Definition 2.12. Let $\pi : E \to M$ be a real vector bundle of rank n. Then the k**th Pontryagin class of** E is given by

$$p_k(E) := (-1)^k c_{2k}(E \otimes \mathbb{C}),$$

where $E \otimes \mathbb{C}$ is the **complexification of** E.

A more explicit definition of Pontryagin classes for real vector bundles can be given using $\mathfrak{gl}(n, \mathbb{R})$-valued polynomials:

Proposition 2.13. For $A \in \mathfrak{gl}(n, \mathbb{R})$ define the polynomial functions $h_0, h_1, ..., h_n$ on $\mathfrak{gl}(n, \mathbb{R})$ by

$$\det(tId - \frac{1}{2\pi}A) = \sum_{k=0}^{n} h_k(A)t^{n-k}.$$

Then it holds that

$$p_k(E) = [\overline{h_k(\Omega)}].$$

Proof. This is Theorem 4.1 in Chapter 12 in [7]. □

In a similar manner we can now define Pontryagin classes for complex vector bundles:

Definition 2.14. Let $\pi : E \to M$ be a complex vector bundle of rank n. Then the k**th Pontryagin class of** E is given by

$$p_k(E) := (-1)^k c_{2k}(E_{\mathbb{R}} \otimes \mathbb{C}),$$

where $E_{\mathbb{R}}$ is E regarded as a real vector bundle of rank $2n$.

CHAPTER 3

Ehresmann connections

So far, we have defined characteristic classes for vector bundles using the associated frame bundles. Because characteristic classes are defined using connection forms, one might ask whether or not it's possible to define characteristic classes for vector bundles using the theory of connections on a vector bundle. The answer to this question is yes - but as it turns out the characteristic classes defined in this way coincide with the characteristic classes defined using the frame bundle. In the following, we want to illuminate this fact. For this, we will show that the connections on a vector bundle are in one-to-one correspondence with the connections on the associated vector bundle.

Before we can do this, we need to discuss another, equivalent, characterization of a connection form on a principal bundle. For this, we will use distributions. We begin by recalling what a distribution is.

Definition 3.1. Let M be a manifold. A **distribution** is a subbundle of the tangent bundle TM.

In the following we will be interested in two very special kinds of distributions called *horizontal* and *vertical distributions.* Together, these will allow us to split the tangent bundle of a principal bundle P into a direct sum. We start by defining the vertical subbundle.

Definition 3.2. Let $\pi : P \to M$ be a G-principal bundle. The map π is a submersion, that is, the derivative $D\pi$ is surjective. If we denote by V_p the kernel of the derivative at $p \in P$, then we get the short exact sequence

$$0 \longrightarrow V_p \longrightarrow T_pP \xrightarrow{D\pi(p)} T_{\pi(p)}M \longrightarrow 0,$$

and we get that $\dim V_p = \dim T_pP - \dim T_{\pi(p)}M$ (recall that a short exact sequence of vector spaces always splits). We say that V_p is the vertical tangent subspace at p. A tangent vector $\xi \in V_p$ is called a vertical tangent vector at p.

These vertical subspaces are in fact isomorphic to the Lie algebra $\mathfrak{g}$ of the Lie group G. For this recall Definition 1.7 of a fundamental vector field. Observe that we have

$$\begin{aligned} Z_\xi(u) &= \frac{d}{dt}_{|t=0} \tau_{\exp(t\xi)}(u) \\ &= \frac{d}{dt}_{|t=0} \tau^u(\exp(t\xi)) \\ &= D\tau^u(e)\xi. \end{aligned} \tag{3.1}$$

With this we get the following result:

Lemma 3.3. Let $\pi : P \to M$ be a G-principal bundle. Then for any $\xi \in \mathfrak{g}$ the fundamental vector field Z_ξ is vertical, and the derivative $D\tau^u(e)$ of the orbit map τ^u is an isomorphism $D\tau^u(e) : \mathfrak{g} \to V_uP$ for any $u \in P$.

Proof. For the first claim observe that by the above calculation we see that

$$D\pi(u)Z_\xi(u) = D\pi(u)D\tau^u(e)\xi = D(\pi \circ \tau^u)(e)\xi = 0,$$

because the composition $\pi \circ \tau^u$ is constant. This follows from the fact that τ is fiber-preserving.

Note that for that the above already implies that $D\tau^u(e)$ takes image in V_uP. To see this is an isomorphism, first note that the dimension of $\mathfrak{g}$ as well as the dimension V_uP are equal to $\dim G$. Thus it is enough to show that the map is injective. So suppose $\xi \in \ker D\tau^u(e)$. By the uniqueness of integral curves Equation (3.1) implies that u is a fixed point of $\tau_{\exp(t\xi)}$. This is because

$$\begin{aligned} 0 &= D\tau^u(e)\xi \\ &= Z_\xi(u) \\ &= \frac{d}{dt}_{|t=0}\tau_{\exp(t\xi)}(u) \\ &= \dot{\gamma}_u(0), \end{aligned}$$

where γ_u is the integral curve of Z_ξ with initial value u. But this is clearly also satisfied by the constant integral curve. Moreover, the action τ is also free so that this implies that $\xi = 0$. It follows that $\ker D\tau^u(e) = \{0\}$ so that $D\tau^u(e)$ is indeed an isomorphism. □

This Lemma is useful because it let's us conclude that the vertical tangent subspaces assemble to a subbundle VP of the tangent bundle TP. More explicitly, for a basis $\{\xi_i\}_{i=1}^{\dim G}$ of $\mathfrak{g}$ the associated fundamental vector fields $\{Z_{\xi_i}\}_{i=1}^{\dim G}$ define a global frame for the bundle

$$VP = \coprod_{u \in P} V_uP.$$

This implies that VP is a trivial vector subbundle of TP. We call this bundle the vertical bundle of P. The derivative $D\pi$ induces a vector bundle map $\overline{D\pi} : TP \to \pi^*TM$ over P, which is given by $\overline{D\pi}(u, \xi) = (u, D\pi(u)\xi)$ and so that $\pi_{TP} = \pi_{\pi^*TM} \circ \overline{D\pi}$. This is because the pullback is, categorically speaking, a limit. For more details, and a proof not using Category Theory, see Section 20.4 of [14]. The map $\overline{D\pi}$ is surjective because it sends the space T_uP to $(\pi^*TM)_u$, and by definition of the pullback bundle, $(\pi^*TM)_u$ is isomorphic to $T_{\pi(u)}M$, for which $D\pi$ is onto. Clearly, the kernel of this map is VP, and as a result, we get a short exact sequence of vector bundles:

$$0 \longrightarrow V \longrightarrow TP \xrightarrow{\overline{D\pi}} \pi^*TM \longrightarrow 0. \tag{3.2}$$

Definition 3.4. Let $\pi : P \to M$ be a G-principal bundle. A **horizontal distribution Δ on** P is a subbundle Δ of TP such that $TP = \Delta \oplus VP$ as vector bundles.

We now want to relate horizontal distributions on P to the short exact sequence in Equation (3.2). For this we proof the following Proposition.

Proposition 3.5. Let $\pi : P \to M$ be a G-principal bundle. Then the horizontal distributions on P are in one-to-one correspondence with the splittings of the short exact sequence

$$0 \longrightarrow V \longrightarrow TP \xrightarrow{\overline{D\pi}} \pi^*TM \longrightarrow 0.$$

Proof. Suppose Δ is a subbundle of TP such that $VP \oplus \Delta = TP$. Then there is a vector bundle isomorphism $\Delta \cong TP/VP \cong (\pi^*TM)$. It follows that there is an isomorphism from (π^*TM) onto Δ. This defines a section of $\overline{D\pi}$, and it follows that the short exact sequence is split.

Conversely, suppose there exists a section $s : (\pi^*TM) \to TP$ of $\overline{D\pi}$. Under these circumstances $s(\pi^*TM)$ is a subbundle of TP. Moreover, if we have some $(u, \xi) \in VP \cap s(\pi^*TM)$ then

$$(u, 0) = \overline{D\pi}(u, \xi) = \overline{D\pi} \circ s((u, \eta)) = (u, \eta),$$

for some $(u, \eta) \in \pi^*TM$. It follows that $VP \cap s(\pi^*TM) = \{0\}$. To see that $TP = VP + s(\pi^*TM)$ choose an arbitrary $(u, \xi) \in TP$. Then

$$\overline{D\pi}((u, \xi) - s(\overline{D\pi}((u, \xi))) = \overline{D\pi}(u, \xi) - \overline{D\pi}(u, \xi) = 0,$$

so that by construction $(u, \xi) - s(\overline{D\pi}((u, \xi))$ is an element of VP. Thus we can write (u, ξ) as a sum of elements of VP and $s(\pi^*TM)$ as desired. It follows that $s(\pi^*TM)$ is a horizontal distribution. □

We now show that there is a one-to-one correspondence between connection forms on a principal bundle P, and a certain kind of horizontal distributions on P .

Proposition 3.6. Let $\pi : P \to M$ be a G-principal bundle. Then any horizontal distribution Δ on P which satisfies

$$D\tau_g(u)(\Delta_u) = \Delta_{\tau_g(u)} \tag{$\star$}$$

induces a connection form on P.

Proof. We've seen above that given a horizontal distribution, we can decompose the tangent space T_uP as $T_uP = V_uP \oplus \Delta_u$. Denote by $\pi_{V_uP} : T_uP \to V_uP$ the projection. Then, using Lemma 3.3 we define a 1-form with values in $\mathfrak{g}$ by $\omega_u(\xi) = (D\tau^u)^{-1}(e)(\pi_{V_uP}(\xi))$. To see that this is a connection form, we have to verify that this is indeed a smooth 1-form, that $\omega_u(Z_\xi(u)) = \xi$ and that ω satisfies $\tau_g^*\omega = Ad_{g^{-1}}\omega$. We will smoothness last. The fact that $\omega_u(Z_\xi(u)) = \xi$ is immediate from Lemma 3.3. For the last claim, we must show that

$$\omega_{\tau_g(u)}(D\tau_g(u)\xi) = Ad_{g^{-1}}(\omega_u(\xi)).$$

Using that $T_uP = V_uP \oplus \Delta_u$ we can write ξ as the sum of its *vertical part* and its *horizontal part.* By of the expressions involved we thus have to show the above equation only in the case where ξ is either in V_uP or in Δ_u. If ξ is in Δ_u then $D\tau_g(u)\xi$ is in $\Delta_{\tau_g(u)}$ by assumption. Thus in this case the projection onto the vertical tangent subspaces is zero, so both sides vanish. If on the other hand we assume ξ is in V_uP, then by Lemma 3.3 there exists $\eta \in \mathfrak{g}$ such that $\xi = Z_\eta(u)$. Now observe that

$$D\tau_g(u)Z_\eta(u) = Z_{Ad_{g^{-1}}(\eta)}(\tau_g(u)).$$

This follows from differentiating the expression

$$\tau_g \circ \tau^u(h) = \tau^{\tau_g(u)}(g^{-1}hg)$$

at $h = e$. With this in mind we have

$$\begin{aligned}\omega_{\tau_g(u)}(D\tau_g(u)\xi) &= \omega_{\tau_g(u)}(Z_{Ad_{g^{-1}(\eta)}}(\tau_g(u))) \\ &= Ad_{g^{-1}}(\eta) \\ &= Ad_{g^{-1}}(\omega_u(Z_\eta(u)))\end{aligned}$$

which yields the desired claim.

Finally, to see that ω as defined above is smooth, note that because $\Delta \oplus VP$ we can choose a basis $\{\xi_i\}_{i=1}^{\dim G}$ of $\mathfrak{g}$ and a local frame $\{X_i\}_{i=1}^{n-\dim G}$ for Δ, so that $\{Z_{\xi_i}, X_j\}$ locally span TP. For an arbitrary vector field Y on P we can locally express Y as

$$Y = \sum_{i=1}^{\dim G} f_i Z_{\xi_i} + \sum_{j=1}^{n-\dim G} g_j X_j,$$

for smooth functions f_i, g_j on P. Applying ω yields $\omega(Z) = \sum_{i=1}^{\dim G} f_i \xi_i$, and we conclude that ω is indeed smooth, as desired. □

We now want to show the converse as well.

Proposition 3.7. Let ω be a connection on a G-principal bundle $\pi : P \to M$. Then $\ker \omega$ is a horizontal distribution on P and satisfies the equation $(\star)$.

Remark 3.8. The kernel $\ker \omega$ should be understood fiberwise.

Proof. We have to verify three things. First, that $TP = \ker \omega \oplus VP$, secondly that $\ker \omega$ satisfies $D\tau_g(\ker \omega) = \ker \omega$, and finally that $\ker \omega$ is actually a subbundle.

For the first claim, note that by assumption a connection form satisfies $\omega(Z_\xi) = Z_\xi$, that is ω is surjective. It follows that for all $u \in P$ we have $\dim \ker \omega_u = m$. At the same time, $\omega(Z_\xi) = Z_\xi$ also implies that $\ker \omega_u \cap V_u P = \{0\}$. As $V_u P$ has dimension equal to $\dim G$ this implies that $\ker \omega_u \oplus V_u P$, and as a result that $\ker \omega \oplus VP = TP$, showing the first claim. To verify the second claim, note that we have

$$Ad_{g^{-1}}(\omega_u(\xi)) = \tau_g^* \omega_u(\xi) = \omega_{\tau_g(u)}(D\tau_g(u)\xi),$$

so that $\xi \in \ker \omega_u$ implies $D\tau_g(u)\xi \in \ker \omega_{\tau_g(u)}$. This implies $D\tau_g(\ker \omega) \subset \ker \omega$. Using that τ_g is a diffeomorphism, we get the inclusion $\ker \omega \subset D\tau_g(\ker \omega)$ by applying $D\tau_{g^{-1}}$. This establishes the second claim. For the final claim, note that by the first step of the proof, the differential form ω has constant rank. Thus the fact that $\ker \omega$ is a subbundle follows the following more general result:

Let $\phi : E \to F$ be a vector bundle map between vector bundles E and F over a manifold M. If ϕ has constant rank, then $\ker \phi$ is a smooth subbundle of E.

This yields the claim for general differential k-forms ω on P with constant rank by letting $E = TP$ and setting F to be the exterior bundle of power $k-1$, i.e. $\wedge^{k-1} T^* M$. The bundle map I_ω in this case is given by sending $(u, \xi) \in TP$ to $\iota_\xi \omega_u$, where ι denotes the interior product. Then $\ker \omega = \ker I_\omega$, and the claim follows. As this also works for vector- and bundled-valued differential forms, this finishes the proof. For a proof of the above result, see for example Theorem 10.34 in [9]. □

We have now established that a connection on a G-principal bundle can equivalently be described by a $\mathfrak{g}$-valued differential form as in Definition 1.8, or by fixing a horizontal distribution on TP. Thus we give a special name to such a horizontal distribution:

Definition 3.9. Let P be a G-principal bundle. A horizontal distribution on TP satisfying the equation $(\star)$ is called an **Ehresmann connection**.

With this knowledge in hand, we are now ready to prove that there is a one-to-one correspondence between connections on a vector bundle, and connections on the frame bundle associated to the vector bundle.

Remark 3.10. As just remarked we have seen two equivalent ways for describing a connection on a principal bundle. Similarly, a connection on a vector bundle can be described in multiple equivalent ways. In the rest of these notes, unless otherwise specified we will always consider a connection on a vector bundle to be a covariant derivative.

CHAPTER 4

Connections On Vector Bundles And Frame Bundles

In this chapter we want to show that there is a one-to-one correspondence between connection forms (or equivalently Ehresmann connections) on the frame bundle and covariant derivatives on the corresponding vector bundle. Moreover, we will discuss discuss the concept of the *reduction of a structure group* for principal bundles as well as for vector bundles.

We begin by showing how a connection form on the frame bundle induces a covariant derivative on the corresponding vector bundle.

Theorem 4.1. Let $\pi : E \to M$ be a vector bundle over a manifold M. Let ω be a connection form on the frame bundle $Fr(E)$ of E. Then ω induces a covariant derivative on the vector bundle E.

Before we can begin with the proof we have do give another couple of definitions.

Definition 4.2. Let $\pi : P \to M$ be a G-principal bundle. Let $\omega \in \Omega^k(P, V)$ be a vector-valued differential form. We say that ω is **horizontal**, if whenever one of the arguments of ω is an element of the vertical tangent subspace, ω vanishes.

Recall that a G-principal bundle comes equipped with a right action τ of G on P (see Propositon 1.6).

Definition 4.3. Let $\pi : P \to M$ be a G-principal bundle. Let $\omega \in \Omega^k(P, V)$ be a vector-valued differential form and suppose that σ is a representation of G on V. We say that ω is (τ, σ)**-equivariant** if for all $g \in G$ we have

$$\tau_g^*\omega = \sigma_{g^{-1}}\omega.$$

We will denote by

$$\Omega_G^k(P, V) = \{\omega \in \Omega^k(P, V) \mid \omega \text{ is horizontal and } (\tau, \sigma)\text{-equivariant } \}$$

the subspace of horizontal (G-)equivariant differential forms. The proof of Theorem 4.1 will rely on the following auxiliary Lemma, which relates the space of horizontal equivariant differential forms on the frame bundle to bundle-valued forms on the manifold.

Lemma 4.4. Let $\pi : E \to M$ be a vector bundle. Then there exists an isomorphism

$$\Omega^k(Fr(E), V) \to \Omega^k(M, E),$$

where V denotes the fiber of the vector bundle E.

Remark 4.5. Recall that in the case of a vector bundle E the group $GL(V)$ acts on V by the canonical representation, that is, by left multiplication.

Proof. We begin by constructing a bundle valued form on M starting with a form on $Fr(E)$. Let $\omega \in \Omega^k_G(Fr(E), V)$. Fix $p \in M$, and let $\xi_1, \cdots, \xi_k$ be vectors on T_pM. Because $\pi_{Fr(E)} : Fr(E) \to M$ is a submersion, there exist $\eta_1, \cdots, \eta_n$ such that $D\pi_{Fr(E)}(u)\eta_i = \xi_i$ for any $u \in P_p$. Now by assumption $\omega_u(\eta_1, \cdots, \eta_k)$ is an element of V. Because E is a vector bundle with fiber V every element $u \in P_p$ is an isomorphism $u : V \to E_p$. Thus, applying u to $\omega_u(\eta_1, \cdots, \eta_k)$ yields an element of $E_{\pi_{Fr(E)}(u)} = E_p$. We denote this element by $\breve{\omega}_p(\xi_1, \cdots, \xi_k)$. We now claim that $\breve{\omega}$ is the desired E-valued differential form on M. To see this we need to check that it is independent of the choice of $u \in P_p$ and of the choice of vectors η_i in T_uP. For the last claim, simply observe that ω is horizontal by assumption and that if two vectors mapping to ξ_i under $D\pi(u)$ differ by a vertical vector. Then the claim follows from linearity. For the first claim notice that $\pi \circ \tau_g = \pi$ because τ is fiber preserving. Thus we have

$$D\pi(\tau_g(u)) \circ D\tau_g(u)\eta_i = D\pi(u)\eta_i = \xi_i,$$

so that $D\tau_g(u)\eta_i$ is an element of $T_{\tau_g(u)}$ mapping to ξ_i. Because τ is moreover transitive, it is enough to show that

$$u(\omega_u(\eta_1, \cdots, \eta_k)) = \tau_g(u)(\omega_{\tau_g(u)}(D\tau_g(u)\eta_1, \cdots, D\tau_g(u)\eta_k)).$$

By assumption ω is equivariant so that we get

$$\begin{aligned}\omega_{\tau_g(u)}(D\tau_g(u)\eta_1, \cdots, D\tau_g(u)\eta_k) &= (\tau_g^*\omega)_u(\eta_1, \cdots, \eta_k)\\ &= l_{g^{-1}}(\omega_u(\eta_1, \cdots, \eta_k)).\end{aligned}$$

Thus the claim follows if we can show that $\tau_g(u) = u \circ l_g$. But this follows immediately, as the action τ in the case of the frame bundle is given by right multiplication with $g \in GL(V)$, so that both sides are equal. Thus we have established that the map $\omega \mapsto \breve{\omega}$ is well-defined. To see that it is also an isomorphism we construct an explicit inverse. Given a differential form $\alpha \in \Omega^k(M, E)$ we define a differential form $\hat{\alpha} \in \Omega^k_G(Fr(E), V)$ by

$$\hat{\alpha}_u(\eta_1, \cdots, \eta_k) = u^{-1}(\alpha_p(D\pi_{Fr(E)}(u)\eta_1, \cdots, D\pi_{Fr(E)}(u)\eta_k),$$

for $u \in P_p$ and $\eta_1, \cdots, \eta_k \in T_uP$. This is clearly a horizontal form. To see that is is also equivariant observe that

$$\begin{aligned}(\tau_g^*\hat{\alpha})_u(\eta_1, \cdots, \eta_k) &= \hat{\alpha}_{\tau_g(u)}(D\tau_g(u)\eta_1, \cdots, D\tau_g(u)\eta_k)\\ &= \tau_g(u)^{-1}(\alpha_p(D\pi(\tau_g(u)) \circ D\tau_g(u)\eta_1, \cdots, D\pi(\tau_g(u)) \circ D\tau_g(u)\eta_k))\\ &= l_g^{-1} \circ u^{-1}(\alpha_p(D\pi(u)\eta_1, \cdots, D\pi(u)\eta_k))\\ &= l_g^{-1}(\hat{\alpha}_u(\eta_1, \cdots, \eta_k)),\end{aligned}$$

where we used again that $\pi \circ \tau_g = \pi$, and $\tau_g(u) = u \circ l_g$ from above. It is clear from construction that the two maps are inverses of one another. Thus the Proposition follows. □

With this Lemma in hand we can now proof Theorem 4.1.

Proof of Thoerem 4.1. Let V be the fiber of the vector bundle E, and let Δ denote an Ehresmann connection on $Fr(E)$. Suppose we start with an arbitrary vector-valued differential form $\omega \in \Omega^k(Fr(E), V)$. We then define the **horizontal component** of ω to be the differential form defined by

$$\omega_u^h(\eta_1, \cdots, \eta_k) = \omega_u(\eta_1^h, \cdots, \eta_k^h),$$

where η_i^h denotes the horizontal component of η_i, that is, if we write η_i using the direct sum decomposition $T_uP = \Delta_u \oplus V_uFr(E)$, then η_i^h is the projection onto Δ_u. Clearly the differential form ω^h is a horizontal form. We now claim that if $\omega \in \Omega_G^k(Fr(E), V)$ then $(d\omega)^h \in \Omega_G^{k+1}(Fr(E), V)$. To see this we only need to show equivariance. We do this in two steps. First, assume that a differential form ω is equivariant. Then it holds that $d\omega$ is equivariant as well, because

$$\begin{aligned}\tau^*(d\omega) &= d\tau^*(\omega)\\ &= d(l_{g^{-1}}(\omega))\\ &= l_{g^{-1}}(\omega).\end{aligned}$$

For the first inequality we use that the pullback commutes with the exterior differential. In the second line we used equivariance of ω. The last line follows from the fact that $l_{g^{-1}}$ is linear map. If one goes through the definitions of vector-valued differential forms, and of the exterior differential for vector-valued differential forms, one sees that this yields that $l_{g^{-1}}$ commutes with the exterior differential as well. Now we want to show that if ω is equivariant, then so is ω^h. This is another simple, although slightly longer, calculation. Fix $u \in P, g \in G$ and let $\eta_1, \cdots, \eta_k \in T_uP$, and let $\omega \in \Omega^k(P, V)$. Then we have

$$\begin{aligned}(\tau_g^*\omega^h)_u(\eta_1, \cdots, \eta_k) &= \omega^h_{\tau_g(u)}(D\tau_g(u)\eta_1, \cdots, D\tau_g(u)\eta_k)\\ &= \omega_{\tau_g(u)}((D\tau_g(u)\eta_1)^h, \cdots, (D\tau_g(u)\eta_k)^h)\\ &= \omega_{\tau_g(u)}(D\tau_g(u)\eta_1^h, \cdots, D\tau_g(u)\eta_k^h)\\ &= (\tau_g^*\omega)_u(\eta_1^h, \cdots, \eta_k^h)\\ &= l_{g^{-1}}\omega_u(\eta_1^h, \cdots, \eta_k^h)\\ &= l_{g^{-1}}\omega_u^h(\eta_1, \cdots, \eta_k).\end{aligned}$$

The fact that $(D\tau_g(u)\eta_i)^h = D\tau_g(u)\eta_i^h$ is a consequence of the fact that an Ehresmann connection satisfies $D\tau_g(u)\Delta_u = \Delta_{\tau_g(u)}$. With these observations it now follows that for any $\omega \in \Omega_G^k(Fr(E), V)$ we have that $(d\omega)^h \in \Omega_G^{k+1}(Fr(E), V)$. With this knowledge we can define what is called an *exterior covariant differential*. An exterior covariant differential is an analogue of the exterior differential for bundle-valued forms, that is it is an operator $d^\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$. We define it by d^∇

$$d^\nabla(\alpha) := \widetilde{(d\hat{\alpha})^h}.$$

In particular, in degree $k = 0$, where $\Omega^k(M, E) = \Gamma(M, E)$, we write

$$d^\nabla(s) = \nabla s.$$

This is a covariant derivative - the only non-trivial property to check is the product formula $\nabla_X(fs) = X(f)s + f\nabla_X s$, for $f \in C^\infty(M)$ and $s \in \Gamma(M, E)$. To see that the product formula holds, observe that for $\xi \in T_pM$ we have

$$
\begin{aligned}
\nabla_\xi(fs) &= (\widehat{dfs})^h(\xi) \\
&= (d\widehat{((f \circ \pi_{Fr(E)})\hat{s})})^h(\xi) \\
&= (\widehat{d(f \circ \pi_{Fr(E)})\hat{s} + (f \circ \pi_{Fr(E)})d\hat{s}})^h(\xi) \\
&= u((d(f \circ \pi_{Fr(E)})\hat{s})^h)(\xi) + u(((f \circ \pi_{Fr(E)})d\hat{s})^h)(\xi) \\
&= \xi(f)s + f\nabla_\xi s,
\end{aligned}
$$

which is just a careful application of the construction of the maps involved. This finishes the proof of the Theorem. □

Remark 4.6. The above construction is just a special case of a more general fact. Given a G-principal bundle, and any representation σ of G on some vector space V. Then there is a vector bundle called an **associated bundle of P** and denoted by $P \times_G V$, for which the analogue statements of Lemma 4.4 and Theorem 4.1 hold. For more details on the construction of the associated bundle we refer the interested reader to the book *The Topology of Fibre Bundles* by Norman Steenrod [13]. Another account is giving in Chapter 31 of Tu's *Differential Geometry: Connections, Curvature, and Characteristic Classes* [14].

We have now shown that any connection on the frame bundle corresponds to a covariant derivative on the corresponding vector bundle. The next step is showing that the converse is also true. Note that this converse direction is not true for the more general setup described in the above Remark. We begin this direction with a few more definitions as well.

As remarked on earlier, there are different ways of describing connections on vector bundles. Above we used the notion of a covariant derivative - the usual definition as a map $\mathfrak{X}(M) \times \Gamma(M, E) \to \Gamma(M, E)$ is just a special case of the more general notion of **covariant derivative along a map** $\phi : N \to M$, where ϕ is a smooth map between smooth manifolds. A covariant derivative along a map ϕ is a map

$$\nabla^\phi : \mathfrak{X}(N) \times \Gamma_\phi(E) \to \Gamma_\phi(E),$$

where $\Gamma_\phi(E)$ denotes sections of E along ϕ. It is an easy exercise that $\Gamma_\phi(E) \cong \Gamma(\phi^*E)$, so that we can write this also as

$$\nabla^\phi : \mathfrak{X}(N) \times \Gamma(\phi^*E) \to \Gamma(\phi^*E).$$

The usual covariant derivative is just a covariant derivative along the identity map. We will use this only in the context where ϕ is a smooth path $\phi : [0, 1] \to M$ - because of this we will denote ϕ by γ from now on. For details of this special case, called **covariant derivative along a curve** (or **path**), we refer the reader to Chapter 4 in John M. Lee's book *Introduction to Riemannian Manifolds* [8] or Chapter 13 in Tu's book on Differential Geometry [14]. In both instances this is shown for the special case of $E = TM$, but the proofs are analogous for arbitrary E.

Definition 4.7. Let $\pi : E \to M$ be a vector bundle with connection ∇. Let $\gamma : [0, 1] \to M$ be a smooth curve. Then the we say a section $s \in \Gamma(\gamma^*E)$ is parallel along γ if $\nabla^\gamma s \equiv 0$. If E has rank n, then we say a collection $\{s_i\}_{i=1}^n$ of sections along γ is a **parallel frame** if all the s_i are parallel along γ and for every $t \in [0, 1]$ we have that $\{s_i(t)\}_{i=1}^n$ is a basis of the vector space $(\gamma^*E)_t \cong E_{\gamma(t)}$.

To show that we get an induced connection on the frame bundle $Fr(E)$, we will construct a horizontal distribution on $Fr(E)$ using the connection ∇ on E. The first step in this direction is the following definition.

Definition 4.8. Let E be a vector bundle over a manifold M with connection ∇, and denote by $\pi : Fr(E) \to M$ the associated frame bundle of E. A curve $\bar{\gamma} : [0,1] \to Fr(E)$ is called a **lift** of a curve $\gamma : [0,1] \to M$ if $\gamma = \pi \circ \bar{\gamma}$. If moreover $\bar{\gamma}$ is a parallel frame along γ, then we say that $\bar{\gamma}$ is a **horizontal lift** of γ.

An important result is the following Lemma.

Lemma 4.9. Let E be a vector bundle over a manifold M with covariant derivative ∇. For every curve $\gamma : [0,1] \to M$ and every $u \in (Fr(E))_{\gamma(0)}$ there exists a unique horizontal lift $\bar{\gamma} : [0,1] \to M$ which satisfies $\bar{\gamma}(0) = u$.

Proof. This is explained in Section 29.1 of [14]. □

The horizontal distribution we will construct on $Fr(E)$ will be made up of tangent vectors that *come from* horizontal lifts. More precisely, we give the following definition:

Definition 4.10. Let E be a vector bundle over a manifold M. Suppose ∇ is a connection on E. For $p \in M$ and $u \in (Fr(E))_p$ we say that $\xi \in T_uP$ is a **horizontal tangent vector** if there is a curve $\gamma : [0,1] \to M$ such that $\xi = \dot{\bar{\gamma}}(0)$ for the horizontal lift $\bar{\gamma}$ of γ.

What is useful about these horizontal tangent vectors is that the set of horizontal tangent vectors at $u \in T_uP$ is a subspace, which we will denote by Δ_u, and that this subspace is isomorphic to the tangent space of M at $\pi_{Fr(E)}(u)$. We will not prove these facts here, but refer the reader to the account given in Section 29.2 of [**8**]. However, these properties of Δ_u is exactly what we need to show that it is complementary to the vertical tangent subbundle $VFr(E)$. To see this, first consider the case at a fixed $u \in Fr(E)$. Recall that $V_uFr(E)$ is the kernel of the surjective map $D\pi_{Fr(E)}(u)$. Thus $V_uFr(E)$ must have dimension $\dim Fr(E) - \dim M$, so that the dimensions of Δ_u and $V_uFr(E)$ are complementary. Moreover, because Δ_u is isomorphic to T_uP it follows that $V_uFr(E) \cap \Delta_u = \{0\}$. which establishes that, at least locally over every fiber, we have the desired decomposition

$$T_uFr(E) = \Delta_u \oplus V_uFr(E).$$

What is left to show is that these vector spaces Δ_u assemble to a vector subbundle of TM. To show this, we introduce the following definition.

Definition 4.11. Let $\pi : E \to M$ be a vector bundle with covariant derivative ∇. Let $X \in \mathfrak{X}(M)$ be a vector field. As by construction $D\pi_{Fr(E)}(u) : \Delta_u \to T_{\pi_{Fr(E)}(u)}M$ is an isomorphism, it holds that for any $p \in M$ there exists a unique vector $\eta \in \Delta_u$ such that $D\pi_{Fr(E)}(u)\eta = \mathfrak{X}(p)$. The vector field of these vectors is called the **horizontal lift of** X. We will denote the horizontal lift of X by $\bar{X}$.

Of course one has to show that $\bar{X}$ is smooth.

Lemma 4.12. Let $\pi : E \to M$ be a vector bundle with covariant derivative ∇. The horizontal lift of a vector field $X \in \mathfrak{X}(M)$ to the frame bundle $Fr(E)$ is a smooth vector field.

Proof. We will not give all details of the proof here, for that we refer the reader once more to Tu's Differential Geometry book [14]. More explicitly, this is Proposition 29.8 in the book. However, we will quickly discuss the main ideas.

Because the property of being smooth is a local property, we can without loss of generality assume that the frame bundle $Fr(E) = M \times G$ is a trivial bundle. The next property we need is that the horizontal lift is right-invariant:

$$\bar{X}(x,g) = Dr_g(x,e)\bar{X}(x,e).$$

To see this, observe that

$$\begin{aligned} D\pi(x,g) \circ Dr_g(x,1)\bar{X}(x,e) &= D\pi(x,1)\bar{X}(x,e) \\ &= X(\pi(x,e)) \\ &= X(\pi(x,g)) \\ &= D\pi(x,g)\bar{X}(x,g) \end{aligned}$$

where we used that $\pi \circ r_g = \pi$ once more - the claim now follows. It holds that, if $s \in \Gamma(U, Fr(E))$ is a local frame, the horizontal lift of X can be written as a difference between the image of X under $Ds(p) : T_pM \to T_uFr(E)$ and a fundamental vector field on $Fr(E)$. More precisely, for a smooth curve $\gamma : [0,1] \to M$, with $\gamma(0) = p$, let $\bar{\gamma}$ be the horizontal lift of γ with initial value u. If we now take s to be a local frame around p such that $s(p) = u$, then there exists a smooth path of matrices $a(t) \in GL(n,\mathbb{R})$ such that $s(\gamma(t)) = \bar{\gamma}(t)a(t)$, where $a(0) = \text{id}$. The derivative of $a(t)$ at 0 is an element of $T_e(GL(n,\mathbb{R})) = \mathfrak{gl}(n,\mathbb{R})$. Then it holds that

$$Ds(\gamma(0))\dot{\gamma}(0) = \dot{\bar{\gamma}}(0) + Z_{a'(0)}(u).$$

Because every tangent vector in T_pM can be represented as the derivative of a curve through p and because every horizontal tangent vector in Δ_u is the derivative of a horizontal lift of some curve, we can use the above expression to express the horizontal lift $\bar{X}$ of X as the difference between the pushforward of X and a fundamental vector field.

Now viewing X as a derivation on $C^\infty(M)$ one shows, using the right-invariance of the horizontal lift $\bar{X}$ as well as the right-invariance of the fundamental vector fields (see the proof Proposition 3.6 for the second claim), that $X(f)$ is a smooth function, which, as is well-known, is equivalent to X being a smooth vector field (for readers unfamiliar with this viewpoint, we recommend Chapter 8 of [9]). □

This result will allow us to prove that the set

$$\Delta = \coprod_{u \in Fr(E)} \Delta_u$$

has indeed a natural structure of a subbundle, from which it will follow, that Δ is a horizontal distribution, and hence an Ehresmann connection. Then we have finally shown, that any covariant derivative on a vector bundle induces a connection on the frame bundle.

Theorem 4.13. Let $\pi : E \to M$ be a vector bundle with covariant derivative ∇. Then ∇ induces an Ehresmann connection (or equivalently a connection form) on the frame bundle $Fr(E)$. More explicitly, the Ehresmann connection induced on $Fr(E)$ at a point $u \in Fr(E)$ is given by the set of horizontal tangent vectors.

Proof. To prove the claim we need to show three things. First, we need Δ, as defined above, to be a subbundle. Secondly, we need that $TFr(E) = \Delta \oplus VFr(E)$, which is equivalent to showing $T_uFr(E) = \Delta_u \oplus V_uFr(E)$ for all $u \in P$. This is already been shown above. The last thing to show is that

$$\Delta_{r_g(u)} = Dr_g(u)\Delta_u$$

for all $u \in P$. We start by showing that Δ is a subbundle. For this, we will once more utilize Proposition 20.4 from [14], which tells us that it is enough to show that for any open set $\bar{U}$ in P there exist $\dim M$ local sections which span $\Delta_{|\bar{U}}$. For this, let $\{X_1, \ldots, X_n\}$ be a local frame for TM. By the previous Proposition the lifts $\{\bar{X}_1, \cdots, \bar{X}_n\}$ are all smooth vector fields on P defined on $\bar{U} := \pi_{Fr(E)}(U)$. Because $D\pi_{Fr(E)}(u)$ is an isomorphism for every $u \in \bar{U}$, it follows that for each $u \in \bar{U}$ the set $\{\bar{X}_1(u), \cdots, \bar{X}_n(u)\}$ is a basis for Δ_u. Thus the sections $\{\bar{X}_1, \cdots, \bar{X}_n\}$ span Δ over $\bar{U}$. By Proposition 20.4 it follows that Δ is a subbundle of TP, as desired. For the final claim, let $u \in Fr(E)$ and fix some $\dot{\bar{\gamma}}(0) \in \Delta_u$, where $\bar{\gamma}$ is again the lift of some smooth curve γ in M, with initial value $\bar{\gamma}(0) = u$. Write $\bar{\gamma}(t) = [v_1(t), \cdots, v_k(t)]$, where k is the rank of the vector bundle E, and each $v_i(t)$ is a column vector representing the ith column of the matrix $\bar{\gamma}(t) \in Fr(E_{\gamma(t)})$. As we assume that $\bar{\gamma}$ is a horizontal lift, it holds that $\nabla^\gamma v_i \equiv 0$. Recall that the action of $GL(k, \mathbb{R})$ on $Fr(E)$ is given by right multiplication, so we can get for $A \in GL(k, \mathbb{R})$ that

$$\bar{\gamma}(t)A = [\sum_{i=1}^{r} g_1^i v_i(t), \cdots, \sum_{i=1}^{r} g_k^i v_i(t)],$$

and by linearity of ∇^γ it follows that each component of $\bar{\gamma}(t)A$ is a parallel section. Thus $\bar{\gamma}(t)A$ is a horizontal lift of $\gamma(t)$ with initial value $\bar{\gamma}(0)A$. Observe that

$$\frac{d}{dt}_{|t=0}\bar{\gamma}(0)A = Dr_A(\bar{\gamma}(0))\dot{\bar{\gamma}}(0) \in \Delta_{r_A(\bar{\gamma}(0))} = \Delta_{uA}.$$

This shows $Dr_A(u)\Delta_u \subset \Delta_{uA}$ - but r_A is a diffeomorphism, and thus $Dr_A(u)$ is an isomorphism for every $u \in Fr(E)$. Thus we get equality, which finishes the proof. □

This brings our discussion of the correspondence between covariant derivatives of a vector bundle E and Ehresmann connections on the principal bundle $Fr(E)$ to an end. Observe that the functions constructed in Theorem 4.1 and in Theorem 4.13 above are injective. Thus the correspondence is one-to-one. We summarize the above results in the following Corollary.

Corollary 4.14. Let M be a manifold and suppose $\pi : E \to M$ is a vector bundle over M. Then there is a one-to-one correspondence between covariant derivatives on the vector bundle E and Ehresmann connections on the associated frame bundle $Fr(E)$.

With this acquired knowledge in hand, the answer to our initial question, whether one can define characteristic classes using covariant derivatives, and the associated curvature, on the vector bundles, and if so whether they coincide with the characteristic classes defined via the frame bundle, can be answered with yes.

Remark 4.15. Using this, we see that characteristic classes defined on vector bundles are independent of the choice of connection as well.

4.1 Reduction of Structure Groups

Before we return to the discussion of characteristic classes however, we want to discuss the reduction of structure groups once more - now with the new acquired knowledge about the correspondence of connections from above. Above we described how the choice of a Riemannian metric on vector bundle lets us reduce the structure group of the vector bundle to $O(n)$. We did this by showing that the transition functions can be chosen such that they take values in $O(n)$. By Theorem 2.7 this is equivalent to the existence of an $O(n)$-subbundle of the frame bundle $Fr(E)$. We now want to explicitly construct this $O(n)$-subbundle and show that there is indeed a one-to-one correspondence between such subbundles and the Riemannian metrics on the vector bundle E. After this, we will show that the connections on an $O(n)$-subbundle associated to a Riemannian metric g on E correspond to those covariant derivatives on E, which are metric with respect to g.

Let E be a vector bundle, and suppose that we start with an $O(n)$-subbundle of the frame bundle $Fr(E)$. Let us quickly recap what this actually means. Suppose we have a G-principal bundle P over a manifold M. Let $H \subset G$ be a (Lie) subgroup, and suppose that Q is a H-principal bundle over M. A **homomorphism of principal bundles** between Q and P now consists of a Lie group morphism $\phi : H \to G$ and a smooth map $f : Q \to P$, such that

$$f(\tau_h^Q q) = \tau_{\phi(h)}^P f(q) \quad \forall q \in Q \text{ and } h \in H,$$

where τ^Q and τ^P are the respective right actions on the principal bundles. We denote such a morphism by (f, ϕ).

Remark 4.16. We call the above pair of maps a homomorphism instead of a morphism, because both P and Q are bundles over the same manifold M. One can also define morphisms between bundles over different manifolds.

We say that a principal bundle homomorphism (f, g) is an **embedding** if the map $f : Q \to P$ is an embedding, and if the Lie group morphism ϕ is a monomorphism. By identifying Q with $f(Q)$ and H with $\phi(H)$, we say that Q is an H-subbundle of P.

Remark 4.17. Put slightly differently, we can say that Q is an H-subbundle, if Q is an embedded submanifold of P, together with an H-action, so that the inclusions $\iota_Q : Q \to P$ and $\iota_H : H \to G$ assemble to an embedding (ι_Q, ι_H) of principal bundles.

Now, let us get back to the special case of P being the frame bundle, and Q being some $O(n)$-subbundle. Choose $u \in Q$, and note that for any $p \in M$ the map u_p is an orthogonal map $u_p : R^n \to E_p$. Define a map

$$g : E \times E \to \mathbb{R}, \qquad g_p(\xi, \eta) = \langle u_p^{-1}\xi, u_p^{-1}\eta \rangle,$$

where $\langle , \rangle$ denotes the Euclidean inner product. From Linear Algebra we know that the Euclidean inner product is invariant under multiplication by elements of $O(n)$ - thus the map g is independent of the choice of u. By construction g defines an inner product on each fiber E_p of the vector bundle E. In this way Q determines a unique canonical Riemannian metric on E. Conversely, let g be a Riemannian metric on E. We will now construct an $O(n)$-subbundle of $Fr(E)$. Let Q denote the subset of $Fr(E)$ consisting of those $u \in Fr(E)$ which are orthonormal with respect to g - that is Q is made up of those u satisfying

$$g_p(u_p(\xi), u_p(\eta)) = \langle \xi, \eta \rangle \quad \forall p \in M \text{ and } \xi, \eta \in \mathbb{R}^n.$$

The verification that this bundle Q is a smooth $O(n)$-principal bundle can be done analogous to Proposition 2.2. That this is moreover a subbundle can quickly be verified. Also, the constructions are clearly inverses of one another. This establishes a bijective correspondence between Riemannian metrics g on the vector bundle E and $O(n)$-subbundles of the frame bundle $Fr(E)$ of E.

As indicated above, we now want to discuss how the connections on such an $O(n)$-subbundle correspond, under the identification established in Corollary 4.14 above, to covariant derivatives on E which are metric with respect to g. For this discussion, we need to introduce a few more definitions. We start by discussing *reduced connections.* Before we can define what a reduced connection is, we first have to discuss how principal bundle (homo)morphisms interact with connections.

Theorem 4.18. Let Q and P denote two principal bundles over a manifold M. Denote by (f, ϕ) a homomorphism from Q to P. Suppose that ω^Q is a connection form on Q, with curvature form Ω^Q, and let Δ^Q denote the corresponding Ehresmann connection from Propositon 3.7. Then the following holds:

1. There is a unique Ehresmann connection Δ^P on P so that
$$Df(u)\Delta^Q_u \subset \Delta^P_{f(u)}.$$
2. If ω^P and Ω^P are the connection form and the curvature form corresponding to Δ^P, then
$$f^*\omega^P = \phi(\omega^Q) \text{ and } f^*\Omega^P = \phi(\Omega^Q).$$

Proof. The result can be found in [6] as Proposition 6.1 in Chapter II. We will quickly explain the main ideas here. For the first claim, choose $p \in M$ and let $u \in P_p$ be arbitrary. As the right action on P is transitive on the fibers, choose some $u' \in Q_p$ such that $\tau^P_g f(u') = u$. The horizontal distribution Δ^P at u will now be defined as

$$\Delta^P_u = D\tau^P_g(f(u'))Df(u')\Delta^Q_{u'}.$$

A simple calculation yields independence of the choice of $u' \in Q$ and $g \in G$. The same holds true for establishing that Δ^P is really an Ehresmann connection. A quick calculation yields invariance under $D\tau^P_g$, and to see that $\Delta^P_u \oplus V_u P$ observe the following: Working locally, we can without loss of generality assume that P is trivial. In this case we only have to show that Δ^P_u is isomorphic to the tangent space $T_{\pi_Q(u)}M$, where $\pi_Q : Q \to M$ is the submersion associated to the principal bundle Q. To further simplify the notation, assume with out loss generality, this time using the invariance of the distribution under τ^P, that $u = f(u')$. Consider the commutative diagram

$$\begin{array}{ccc} \Delta^Q_{u'} & \xrightarrow{\quad Df(u) \quad} & \Delta^P_u \\ {\scriptstyle D\pi_Q}\big\downarrow & & \big\downarrow{\scriptstyle D\pi_P} \\ T_{\pi_Q(u')}M & \xrightarrow[D\,\mathrm{id}(\pi_Q(u'))]{} & T_{\pi_P(u)}M \end{array}$$

and note that $D\pi_Q(u')$ and $D\,\mathrm{id}(\pi_Q(u'))$ are isomorphism by assumption. The commutativity of the above diagram then yields that the other two maps must be isomorphisms as well. This finishes the proof of the first claim. The second claim can be verified by direct calculation. For this one uses linearity to consider the cases of horizontal and vertical vectors independently. For the proof of the vertical part Cartan's structure equation

$$d\omega = \Omega + \frac{1}{2}[\omega, \omega]$$

is used. □

Definition 4.19. Let $\pi : P \to M$ be a G-principal bundle. Suppose Q is an H-principal subbundle of P. We say a connection form ω on P is reducible to H if the connection form arises from a connection form ω' on Q under the above correspondence.

Now fix some metric g on E and let us denote by Q the reduced subbundle of $Fr(E)$. Choose a connection form ω' on denote by ω the unique connection form induced by ω' on $Fr(E)$. As we know this connection form ω induces a covariant derivative on E. Let us denote this covariant derivative by ∇ in the following. The most common way to express that a covariant derivative is metric, is in the form of the equation

$$X(g(r,s)) = g(\nabla_X r, s) + g(r, \nabla_X s)$$

for all sections $r, s \in \Gamma(E)$ and vector fields $X \in \mathfrak{X}(M)$. We now want to rephrase this condition in a slightly different language. Our first observation is that we can view a Riemannian metric on E as a section of the bundle $E^* \otimes E^*$, which has the property that it defines an inner product on E_p for each p. It is well-known that a covariant derivative on E induces covariant derivatives on all tensor bundles $T^{(h,k)}(E) = E \otimes \cdots E \otimes E^* \cdots E^*$, where h equals the number of copies of E, while k equals the number of copies of E^*. Using this notation a Riemannian metric g on E is a section of the bundle $T^{(0,2)}(E)$. By the definition of the induced covariant derivative on $T^{(0,2)}(E)$ it holds that ∇ is metric with respect to g if and only if

$$\nabla_X^{E^* \otimes E^*} g = 0, \quad \forall X \in \mathfrak{X}(M). \tag{4.1}$$

We will use this formulation of being metric to show that the induced covariant derivative is metric with respect to g. The next notion we will need is *parallel transport*. Let $\gamma : [0,1] \to M$ be a smooth path. Recall from above the definition of a parallel section, that is, $s \in \Gamma(\gamma^* E)$ is parallel if and only if $\nabla_X^\gamma s = 0 \quad \forall X \in \mathfrak{X}(M)$. If s is such a parallel section, and we let $a := s(0) \in E_{\gamma(0)}$ and $b := s(1) \in E_{\gamma(1)}$ then we define **parallel transport of a along γ** to be b. This notion of parallel transport is interesting because of the following result:

Proposition 4.20. Let $\pi : E \to M$ be a vector bundle. Then for any smooth curve $\gamma : [0,1] \to M$ there exists an isomorphism $\mathbb{P}_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ given by parallel transport along γ.

Proof. For a full proof see Section 29.1 of [14]. The main idea is as follows, one first establishes that a parallel section is uniquely determined by its initial value, and that, at least locally, there exists a section for any initial value in $E_{\gamma(0)}$. This follows from the existence and uniqueness of solutions to ordinary differential equations. Now, consider a basis $e_1, \cdots, e_n \in E_{\gamma(t_0)}$ for some $t_0 \in [0,1]$. By the existence result above, we can find an $\epsilon > 0$ such that we can parallel transport each basis vector e_i along the interval $(t_0 - \epsilon, t_0 + \epsilon)$. By linearity, this means that we can parallel transport any vector in $T_{\gamma(t_0)}$ along $(t_0 - \epsilon, t_0 + \epsilon)$. Using the inverse parameterization of γ we find an inverse for $\mathbb{P}_{\gamma_{|(t_0-\epsilon, t_0+\epsilon)}}$ so that it is indeed an isomorphism. Thus we find for any $t \in [0,1]$ an open interval such that parallel transport is locally defined and an isomorphism on this interval. Because $[0,1]$ is compact, it is covered by a finite number of such open intervals - the claim follows. □

This notion of parallel transport is interesting for us, because the parallel transport determines the covariant derivative. We make this precise in the following Proposition.

Proposition 4.21. Let $\pi : E \to M$ be a vector bundle. Suppose ∇ is a covariant derivative on E. If $X \in \mathfrak{X}(M)$ is a vector field, and $s \in \Gamma(E)$ is a section, then for every $p \in M$ it holds that

$$\nabla_X s(p) = \lim_{t\to 0} \frac{\mathbb{P}_{\gamma_{|[0,t]}} s(\gamma(t)) - s(\gamma(0))}{t},$$

where $\gamma : [0,1] \to M$ is any curve satisfying $\dot{\gamma}(0) = X(\gamma(0))$.

Proof. This is proved in [8] as Proposition 4.36. Although it is shown in the case $E = TM$ there, the general case for arbitrary vector bundles E can be carried out, mutatis mutandis, analogously. □

To show that the Riemannian metric g is a parallel section, we want to use Equation (4.1) and parallel transport. What we will now establish is that a section of $E^* \otimes E^*$ is parallel in the sense of Equation (4.1) if and only if it is constant along parallel sections of E. Because we are interested in studying the frame bundle, the first step is determining the relation between parallel transport and the frame bundle.

Recall that a parallel frame along a curve γ is a collection of parallel sections $\{s_i\}_{i=1}^n$ for E such that $\{s_i(\gamma(t))\}$ is a basis for each $E_{\gamma(t)}$. Also, denote by $\bar{\gamma}$ the horizontal lift of γ. We now claim that a section $s \in \Gamma(\gamma^* E)$ is parallel if and only if s is of the form

$$s(t) = \bar{\gamma}(t)\xi,$$

for some $\xi \in \mathbb{R}^n$. One direction is clear from the definition of a horizontal lift.

To see the only if part, one proceeds as follows. First, define parallel transport by the right hand side. Then, using the relation from Proposition 4.21, it is a straightforward calculation that what we defined as parallel transport does indeed coincide with the definition of a parallel transport along a curve with respect to the induced covariant derivative ∇ on E. For the details we refer the reader to the book *A Comprehensive Introduction to Differential Geometry, Volume 2* by Michael Spivak [12]. More specifically, see Chapter 8 Lemma 8. To apply this, notice that for the induced covariant derivative we have

$$\begin{aligned}
\nabla_\xi s &= \widetilde{(d\hat{s})^h}(\xi) \\
&= u(\hat{d(s)}^h(\zeta)) \\
&= u(\hat{d(s)}(\zeta^h)) \\
&= u(\zeta^h(\hat{(s)}(u))) \\
&= u(\zeta^h(u^{-1}(s(p))))
\end{aligned}$$

With this we now know the relation between the parallel transport on a vector bundle and the corresponding frame bundle, or rather, how the parallel transport of a covariant derivative induced on a vector bundle, by an Ehresmann connection on the frame bundle, is defined. Now, back to our main point - the discussion of metric covariant derivatives. To this end, let λ denote a section of the dual bundle E^*. As stated above, we claim that λ is parallel if and only if it is constant along a parallel section of E. The if part is obvious. For the only if part, consider a parallel dual frame, i.e. a parallel frame of the dual bundle, whose existence can be shown in the same fashion as the existence of a parallel frame. Using linearity it follows that if λ is constant along this parallel dual frame, then the covariant derivative must vanish (along the curve).

Remark 4.22. Here we use the canonical covariant derivative on E^* induced by the covariant derivative on E. That is, the connection on E^* satisfying the Leibniz rule with respect to the covariant derivative on E.

Now, with this and the rest of the above discussion in mind, it follows from Equation (4.1), that for a Riemannian metric g on a vector bundle E, a covariant derivative ∇ is metric with respect to g if and only if

$$g(\bar{\gamma}(0)\xi, \bar{\gamma}(0)\zeta) = g(\bar{\gamma}(1)\xi, \bar{\gamma}(1)\zeta).$$

With this established, we can now prove our initial claim:

Theorem 4.23. Let $\pi : E \to M$ be a vector bundle. Denote by g a Riemannian metric on E, and denote by Q the $O(n)$-subbundle of the frame bundle $Fr(E)$ determined by the choice of g. Then a covariant derivative on E induced by a connection on Q is metric with respect to g.

Proof. Recall that the subbundle Q consists of those frames, which are orthonormal with respect to g. Any connection form on Q is a reduced connection form of $Fr(E)$ by Theorem 4.18, and any such connection form on Q induces a connection form on $Fr(E)$. This connection form on $Fr(E)$ in turn induces a covariant derivative on E. By the above, a section s of E along a curve $\gamma : [0,1] \to M$ is of the form $s(t) = \bar{\gamma}(t)\xi$ for some $\xi \in \mathbb{R}^n$. As noted right before this Theorem, to show that the induced covariant derivative is metric, we need to argue why

$$g(\bar{\gamma}(0)\xi, \bar{\gamma}(0)\zeta) = g(\bar{\gamma}(1)\xi, \bar{\gamma}(1)\zeta).$$

But by the definition of the induced connection form on $Fr(E)$ it follows that $\bar{\gamma}(t) \in Q$ for all $t \in [0,1]$. By construction, all elements of Q are orthonormal with respect to g - it follows that

$$g(\bar{\gamma}(0)\xi, \bar{\gamma}(0)\zeta) = \langle \xi, \zeta \rangle = g(\bar{\gamma}(1)\xi, \bar{\gamma}(1)\zeta),$$

where $\langle , \rangle$ again denotes the Euclidean inner product. The claim now follows. □

With this, we will end our discussion of connections and their relations, and will return to the discussion of characteristic classes.

CHAPTER 5

Euler classes

Another important type of characteristic classes associated to vector bundles are *Euler classes*. Euler classes may arguably be the most well-known type of characteristic classes. However, they are only defined for *orientable* vector bundles. Let us quickly recall the definition of an orientable vector bundle.

Definition 5.1. Let $\pi : E \to M$ be a real vector bundle of rank n. We say E is an **orientable vector bundle** if the structure group of E can be reduced to $GL_+(n, \mathbb{R})$.

There are a few different definitions of orientability, all of which agree with the one just given. We will discuss these in the following. For this we first need to discuss the orientation of a vector space.

Definition 5.2. Let V be a vector space. If V is a one-dimensional vector space then $V \backslash \{0\}$ has two components. Then an orientation on V is a choice of one of these components. We call this choice **positive component**. A **positive basis** is a choice of non-zero vector belonging to our choice of positive component. The second component is then called the **negative component** and any non-zero vector belonging to the negative component is called a **negative basis**.

We now want to extend this definition to the case where V is not one-dimensional. For this we define a new vector space $\det V$ arising from V.

Definition 5.3. Let V be an n-dimensional vector space. The **determinant of** V is the vector space

$$\det V = \bigwedge^n V.$$

This determinant will be useful in defining orientations on arbitrary finite-dimensional vector spaces because, by basic Multilinear Algebra, the space $\det V$ is always one-dimensional. Thus we can give the following definition:

Definition 5.4. Let V a finite-dimensional vector space. An **orientation of** V is an orientation on the determinant of V. We say that a basis $\{e_i\}_{i=1}^n$ is a positive (or negative) basis if the vector $e_1 \wedge \cdots \wedge e_n$ is a positive (or negative) basis of $\det V$. Note that in particular a vector space has exactly two orientations.

There is something called the *standard orientation* on $\mathbb{R}^n$.

Definition 5.5. The **standard orientation** on $\mathbb{R}^n$ is given by declaring that the standard basis is a positive basis.

Remark 5.6. The name *determinant of a vector space* seems strange at first glance. It is not immediately clear what, if there is any, the connection to the usual determinant from Linear Algebra is. Start by considering a linear map $A : V \to W$ where V and W are vector spaces of the same dimension n. Then the map A induces a map between the two determinants $\det V$ and $\det W$. Explicitly this map $A_{\det}$ is given by sending an element $v_1 \wedge \cdots \wedge v_n$ of V to the element $A(v_1) \wedge \cdots \wedge A(v_n)$ in W. As the vector spaces are one-dimensional any map $B : \det V \to \det W$, after choosing bases, is simply multiplication by a scalar. One can quickly check that this scalar is non-zero if and only if A is an isomorphism. In that case one says that A is orientation-preserving if A maps the positive component to the positive component. We call A orientation-reversing if it is not orientation-preserving. In the special case of $V = W$, it turns out that if we equip both the source and the target with the same orientation, then the scalar is given precisely by the determinant of A.

We now want to extend this definition to vector bundles over a manifold. To mimic the above construction in the case of vector bundles we define the *determinant line bundle*, which be a substitute for the determinant of a vector space.

Definition 5.7. Let M be a manifold, and suppose E is a vector bundle of rank n over M. The **determinant line bundle of** E, denoted by $\det E$, is the vector bundle over M which has fiber $(\det E)_p = \det E_p$.

The determinant line bundle gives two additional ways how an orientability can be characterized. This is the content the the next Proposition. However, for this we will utilize the *dual bundle* of a vector bundle.

Definition 5.8. Let $\pi : E \to M$ be a vector bundle of rank n over a manifold M. Then the **dual bundle** E^* **of** E is the vector bundle over M with fiber

$$(E^*)_p = Hom(E_p, \mathbb{R}).$$

Of course, one has to verify that the dual bundle is indeed a vector bundle as well.

Lemma 5.9. For a vector bundle E of rank n over a manifold M the dual bundle E^* is a vector bundle.

Proof. Let (U, ϵ) be a vector bundle chart on E. This gives rise to a vector bundle chart on E^* in the following way. Over U we define a vector bundle chart $\epsilon^* : (\pi^*)^{-1}(U) \to U_a \times (\mathbb{R}^n)^*$, which over each fiber $E*_p$ for $p \in U_a$ is given by

$$\epsilon_p^*(\lambda) = \lambda \circ (\epsilon_p)^{-1}.$$

Usually a bundle chart is defined as a map towards a product of an open subset U of the manifold M with $\mathbb{R}^n$. Above we define it as a map to a product with $(\mathbb{R}^n)*$. However, this is not a problem because there is an isomorphism between $\mathbb{R}^n$ and $(\mathbb{R}^n)*$ using the standard Euclidean inner product. Similar to the proof of Proposition 2.2 one can now show that we get an induced structure of a vector bundle (first a topology, then a smooth structure, both using E, and finally a vector bundle structure using the charts defined above) on E^*. We will however quickly determine the relation between the transition functions on E and on E^*. Recall from Linear Algebra that the dual map induced by a linear map is given by the pullback. However, in general this construction is contravariant, so that the pullback of $\epsilon_p : E_p \to \mathbb{R}^n$ would yield a map $(\mathbb{R}^n)^* \to (E^*)_p$ which is not what we want. Because ϵ_p is an isomorphism we can counteract this contravariance by using $(\epsilon_p)^{-1}$ for the construction of the charts as above. With this definition we have that for two bundle charts ϵ_a and ϵ_b and the corresponding dual charts we get for $(\epsilon_{ab})^* : U_a \cap U_b \to GL((\mathbb{R}^n)^*)$

$$\begin{aligned}(\epsilon_{ab})^*_p(l) &= (\epsilon^*_a)_p \circ (\epsilon^*_b)^{-1}_p(l)\\ &= l \circ (\epsilon_b)_p \circ (\epsilon_a)^{-1}_p.\end{aligned}$$

With this observation it follows that

$$(\epsilon_{ab})^*_p(l)((\epsilon_{ab})_p(v)) = l(v),$$

for all $l \in (\mathbb{R}^n)^*$ and $v \in \mathbb{R}^n$. Writing this using the canonical pairing between a vector space and its dual the above yields for $u = (\epsilon_{ab})_p(v)$

$$\begin{aligned}&\langle(\epsilon_{ab})^*_p(l), (\epsilon_{ab})_p(v)\rangle = \langle l, v\rangle\\ \iff &\langle(\epsilon_{ab})^*_p(l), u\rangle = \langle l, v\rangle = \langle l, ((\epsilon_{ab})_p)^{-1}(u)\rangle\\ \iff &\langle(\epsilon_{ab})^{-T}_p(l), u\rangle = \langle(\epsilon_{ab})^*_p(l), u\rangle\end{aligned}$$

where in the last step we used the definition of the transpose of a linear map. Thus the image of the transition functions on E^* is given by by the inverse transpose of the image of transition functions on E. If we denote the transition functions on E by ϵ^E_{ab}, and the transition functions on E^* by ϵ^*_{ab}, then this can be expressed as

$$\epsilon^*_{ab} = (\epsilon^E_{ba})^*,$$

where the $*$ on the right denotes the usual dual map. □

Proposition 5.10. Let M be a manifold, and suppose E is a vector bundle of rank n over M. Then the following are equivalent:

1. There is a nowhere vanishing section $\mu \in \Gamma(\det E^*)$ on the dual bundle of E.
2. The structure group of E can be reduced to $GL_+(n, \mathbb{R})$.
3. The determinant line bundle of E^* is trivial.

Proof. We start by showing that 1. implies 2.. First, let $\{U_a, \epsilon_a\}_{a\in A}$ be a vector bundle atlas for E. Then over any U_a there exists a local frame, that is a collection of n sections $e^a_i \in \Gamma(U_a, E)$ such that $\{e^a_i(p)\}^n_{i=1}$ is a basis for E_p for every $p \in U_a$. This is immediate from the existence of the bundle chart by setting $e^a_i(p) = (\epsilon_a)^{-1}_p(e_i)$, where the e_i on the right denotes the ith standard basis vector of $\mathbb{R}^n$. By assumption the function $\mu(e^a_1, \cdots, e^a_n) : U_a \to \mathbb{R}$ is non-vanishing, so it does not change sign. If this function is positive then we do nothing. If the function is negative, then we replace the section e^a_1 with $-e^a_1$. Thus without loss of generality we can assume that the function $\mu(e^a_1, \cdots, e^a_n)$ is positive. Now, we can also construct a local trivialization of the bundle E over U_a using these sections e^a_i by inverting the above construction. More explicitly, we can send $v \in \pi^{-1}(U_a)$, which can be uniquely written as $v = \sum^n_{i=1} a_i e_i(p)$ for some $p \in U_a$, to the vector $(a_1, \cdots, a_n) \in \mathbb{R}^n$. Doing this for every $a \in A$ we get a new bundle atlas. To get the claim we have to show that the transition functions of this new bundle atlas are contained in $GL_+(n, \mathbb{R})$. To see this, observe that if $U_a \cap U_b \neq \varnothing$, then there are smooth functions $f^i_j : U_a \cap U_b \to \mathbb{R}$ so that

$$e^b_j(p) = \sum^n_{i=1} f^i_j(p) e^a_i(p).$$

More explicitly, the function f^i_j is equal to the (i,j)th entry of the transition matrix at p given by $(\epsilon_{ba})(p)$. Unraveling the definitions we find that this implies

$$\begin{aligned}\mu(e^b_1, \cdots, e^b_n) &= \mu(\sum_{i=1}^n f^i_1 e^a_i, \cdots, \sum_{i=1}^n f^i_j e^a_i) \\ &= \det \epsilon_{ba}(p) \mu(e^a_1, \cdots, e^a_n).\end{aligned}$$

As μ is positive it follows that ϵ_{ba} takes values in $GL_+(n,\mathbb{R})$. As a and b were arbitrary, this yields the claim and shows that 1. implies 2..

To see that 2. implies 1. we start with a $GL_+(n,\mathbb{R})$ vector bundle atlas $\{U_a, \epsilon_a\}_{a\in A}$ on E. As above we can then find a local frame $\{e^a_i\}^n_{i=1}$ over U_a. There now exists a local frame $\{e^i_a\}^n_{i=1}$ associated to $\{e^a_i\}^n_{i=1}$ using the construction of the dual bundle above. Let $\{f_a\}_{a\in A}$ be a partition of unity subordinate to our cover $\{U_a\}_{a\in A}$. We now define a section

$$\mu : M \to \det E^*, \qquad \mu := \sum_{a\in A} f_a e^1_a \wedge \cdots \wedge e^n_a.$$

To see that μ is nowhere vanishing let $p \in M$ be arbitrary. Choose $b \in A$ such that $p \in U_b$. Then for $e^b_1(p) \wedge \cdots \wedge e^b_n(p) \in (\det E)_p$ we find

$$\mu(e^b_1(p) \wedge \cdots \wedge e^b_n(p)) = \sum_{a\in A} (\det \epsilon_{ba}(p)) f_a(p),$$

which is positive by the assumption that the transition functions take value in $GL_+(n,\mathbb{R})$.

Finally we show that 1. and 3. are equivalent. However, this is immediate from our proof that 1. implies 2.. More explicitly, because the determinant line bundle is one-dimensional, the existence of a global local frame, and thus the existence of a global vector bundle chart, is equivalent to the existence of a nowhere vanishing section. This finishes the proof. □

Remark 5.11. Some authors, see for example Loring W. Tu's *Differential Geometry* [14], define the orientability of a vector bundle as the existence of a nowhere vanishing section on the bundle $\det E$. We can quickly see that this is equivalent to our definition. For this, recall from the proof of Lemma 5.9 how the transition functions on the dual bundle E^* are related to the transition functions on the bundle E itself. Then it is immediate that the bundle E is orientable if and only if E^* is orientable. If we now note that $(E^*)^*$ is again the bundle E it follows from the above Proposition that a nowhere vanishing section of $\det E^*$ exists if and only if a nowhere vanishing section of $\det E$ exists. We prefer the above definition because it works more nicely with the definition of orientability of a manifold via a nowhere vanishing volume form. A nowhere vanishing volume form is a nowhere toplevel differential form, that is a nowhere vanishing section of the bundle $\det TM^*$.

With this discussion of orientable vector bundles behind us we can now finally define Euler classes. We will define Euler classes for even dimensional real vector bundles only. We will discuss the reason for this further down below.

Remark 5.12. In the following we will use local representations of the curvature on a vector bundle. Locally the curvature of a vector bundle can be represented by a $2n \times 2n$ of 2-forms. For readers unfamiliar with this, we recommend Section 11.1 in [14]. If the connection is metric, then the resulting matrices are skew-symmetric. This is discussed in Section 11.4 of [14].

The Euler class of a vector bundle is defined using the *Pfaffian*. The Pfaffian is defined for any $2n \times 2n$ antisymmetric matrix. More explicitly:

Definition 5.13. Let $A \in Mat(2n, \mathbb{R})$ be an antisymmetric matrix. The **Pfaffian of** A is defined to be

$$\mathrm{Pf}(A) = \frac{1}{2^n 2!} \sum_{\sigma in S_{2n}} \mathrm{sign}(\sigma) \prod_{i=1}^{n} A_{\sigma(2i-1)\sigma(2i)},$$

where S_{2n} denotes the symmetric group on $2n$ elements.

The Pfaffian satisfies a number of useful properties. Observe, for example, that $\det A = (\mathrm{Pf}\, A)^2$. The most important property for us will be the following:

Proposition 5.14. Let $A \in Mat(2n, \mathbb{R})$ be an antisymmetric matrix. If $B \in Mat(2n, \mathbb{R})$ is arbitrary, then the Pfaffian of A satisfies

$$\mathrm{Pf}(BAB^T) = \det(B)\,\mathrm{Pf}(A).$$

Proof. This is a straightforward application of the definition of the Pfaffian and the determinant of a matrix. ☐

We are mainly interested in this property because of the following Corollary.

Corollary 5.15. The Pfaffian is invariant under conjugation by special orthogonal matrices.

Now by construction, when A is a matrix of indeterminates, the Pfaffian of A is a polynomial. Moreover, recalling that $\mathfrak{so}(2n)$ is the vector space of antisymmetric matrices, and that by the above Corollary Pf is invariant under conjugation by $SO(2n)$, we get that the Pfaffian is an invariant polynomial in $P_{inv}(so(2n))$. Thus we can define a characteristic class using the Pfaffian for any $SO(2n)$-principal bundle. Thus, by our discussion above, it follows that for a vector bundle to have an Euler class, it is not enough to be merely orientable. To reduce the structure group of a vector bundle to $SO(2n)$ we also need a choice of Riemannian metric. So, let us now assume that we have an even dimensional oriented Riemannian vector bundle (E, g). In this case, as remarked on in Remark 5.12 the curvature of the vector bundle itself can locally be represented by an antisymmetric matrix. We use this to define the Euler class.

Definition 5.16. Let (E, g) be an oriented Riemannian vector bundle over a manifold M. The Euler class $e(E)$ is defined to be

$$e(E) := \left[\frac{1}{(2\pi)^n} \mathrm{Pf}(\Omega) \right],$$

where Ω is the curvature of a metric connection with respect to g.

An obvious difference between the Euler class and other characteristic classes we have so far discussed is that there is only one Euler class, and not multiple Euler classes. The Euler class is a single cohomology class, with degree equal to the rank of the Riemannian vector bundle under consideration. Compared to the other characteristic classes, the definition of the Euler class involves a choice of Riemannian metric, so not only do we have to establish that the Euler class is independent of the chosen (metric) connection, but that it is independent of the chosen metric as well - only then will the characteristic class be an invariant of the vector bundle E. However, this turns out to be simple.

Suppose we have a fixed metric g. Then the key observation is that for two metric connections ∇^0 and ∇^1 it holds that for any $t \in [0, 1]$ the connection

$$(1 - t)\nabla^0 + t\nabla^1$$

is metric as well. With this, one can prove independence of the metric connection analogous to showing that the deRham cohomology class we get from the Chern-Weil homomorphism is independent of the chosen connection.
Now suppose we have two different Riemannian metrics g_0 and g_1. The same principle applies: one can show, that for any $t \in [0, 1]$ the sum

$$(1 - t)g_0 + tg_1$$

is a metric as well. Then the argument from above can be modified to prove the independence of the chosen metric.

Remark 5.17. Note that we in fact did not define the Euler class as the image of an invariant polynomial under the Chern-Weil homomorphism. Thus it is not immediately clear that the differential form used to define the Euler class is indeed closed. However, this can be done without too much trouble using the Binachi identity. For details we refer the reader to the Appendix of [10].

It is not apparent why we have to choose a metric connection in the definition in the of the Euler class. Of course one could define an Euler class for arbitrary metrics. However, as we will discuss below there are other ways in which one can construct characteristic classes. If one wants the different constructions to coincide, then it is necessary to restrict to metric connections. Readers already familiar with the classical definition of characteristic classes can find a counterexample for non-metric connections in the Appendix of [10].

One of the most fundamental and well-known results from the Differential Geometry of surfaces is the Gauss-Bonnet Theorem. In two dimensions it relates the (Gaussian) curvature of a surface to the Euler characteristic of the surface, thus linking the Geometry and Topology of surfaces. It can be stated as follows:

Theorem 5.18. *Let S denote a compact Riemannian manifold of dimension 2. The Gaussian curvature K and the Euler characteristic $\chi(S)$ satisfy:*

$$\int_S K dA = 2\pi\chi(S).$$

Using Euler classes we can generalize this statement for higher dimensional compact orientable Riemannian manifolds. Historically, the generalization of the Gauss-Bonnet Theorem was one of the main driving forces for Chern and Weil to develop their Chern-Weil Theory. The generalized result is the following.

Theorem 5.19. *Let M be an oriented $2n$-dimensional compact Riemannian manifold. Then the Euler class $e(M) := e(TM)$ and the Euler characteristic of $\chi(M)$ of M are related by*

$$\int_M e(M) = \chi(M).$$

Note that we need the manifold to be of even dimension, and be orientable, because the Euler class is only defined for even-dimensional and orientable vector bundles. The first proof of this general fact was given by Chern in 1944 [2]. Hence the Theorem is today called the *Chern-Gauss-Bonnet Theorem.*

As we have hinted at in our discussion so far, there are other ways to construct characteristic classes. In particular, it is possible to construct characteristic classes for general (topological) principal and vector bundles. These more generally defined characteristic classes have some advantages over the characteristic classes defined via Chern-Weil Theory, as we will now discuss. For example, there is another very useful type of characteristic class, called the *Stiefel-Whitney class*, which can not be defined using Chern-Weil Theory. Why Stiefel-Whitney classes are useful will be discussed further down below. The fact that Stiefel-Whitney classes can not be defined using Chern-Weil Theory is a result of a more general observation. By construction, all of our characteristic classes are elements of the deRham cohomology with coefficients in $\mathbb{R}$. The more general construction of characteristic classes, as is done in, for example, Milnor's book [10], of course uses singular cohomology, and defines characteristic classes as singular cohomology classes with coefficients in $\mathbb{Z}$. By the well-known deRham Theorem, there is an isomorphism between the de Rham cohomology and the singular cohomology with real coefficients. As we have mentioned before, one can show that the Chern-Weil homomorphism yields classes with integral coefficients. More precisely, what we mean by that is the following. There is an obvious inclusion from the integers into the real numbers. This inclusion induces an inclusion

$$H^p(M;\mathbb{Z}) \to H^p(M;\mathbb{R})$$

on the (co)homology level as well. Then the above statement says that the Chern-Weil homomorphism takes image in the image of $H^p(M;\mathbb{Z})$ under this inclusion. It is important to note that in general this map will not be injective. In particular, if the group $H^p(M;\mathbb{Z})$ has torsion elements, these will be in the kernel of the map. In this sense the integral characteristic classes are stronger than the real characteristic classes. As the integer classes are defined using only topological data, one could refer to them as **topological characteristic classes**.

Remark 5.20. Topological Euler classes are not only defined for vector bundles of even rank, but also for bundles with odd rank. However, one can show that these are always torsion elements, so that these are in the kernel of the inclusion from above. This explains why we only defined Euler classes for vector bundles of even rank.

We will not give a detailed description of Stiefel-Whitney classes, but rather refer the interested reader once more to Milnor's book on characteristic classes [10]. To see why we can not define these in terms of Chern-Weil Theory it suffices to know that Stiefel-Whitney classes are cohomology classes with coefficients in $\mathbb{Z}_2$. As such, Stiefel-Whitney classes only exist as topological characteristic classes.

CHAPTER 6

Properties of Characteristic Classes And Some Applications

In the last chapter of these notes we will give an overview of some further useful results about characteristic classes, and also about the application of characteristic classes.

In general, characteristic classes might not always be easy to calculate explicitly. In the case of complex vector bundles, it is possible to simplify the calculation of Chern classes (and hence, by Theorem 2.5, of all characteristic classes) to the calculation of Chern classes for direct sums of (complex) line bundles. The main result is the following:

Theorem 6.1. Let $\pi : E \to M$ be a complex vector bundle of rank n over a manifold. There exists a manifold $F(E)$, and a map $p : F(E) \to M$ such that

1. The pullback of E to $F(E)$ splits into a direct sum of line bundles:
$$p^*E = L_1 \oplus \cdots \oplus L_k.$$
2. The map $p^* : H^*(M) \to H^*(F(E))$ induced by p in cohomology is injective.

Proof. The proof is done by induction on the rank of E. For the details we refer to Raoul Bott and Loring Tu's book *Differential Forms in Algebraic Topology* [1]. □

One can then iterate this construction, and given a finite tuple $(E_1, \cdots, E_n)$ of vector bundles over the manifold M, prove existence of a manifold N, and a map $p : N \to M$ such that each pullback bundle p^*E_i is a direct sum of line bundles and so that $p^* : H^*(M) \to H^*(N)$ is an injection. This result is usually referred to as the **Splitting Lemma**. It has the following useful Corollary.

Corollary 6.2. To prove a polynomial identity in the Chern classes of complex vector bundles, it suffices to prove the identity under the assumption that the involved vector bundles are direct sums of line bundles.

Proof. We illustrate the Corollary in the case of two vector bundles E and F over and their tensor product $E \otimes F$ over a manifold M. If we denote a polynomial identity in their Chern classes by $P(c(E), c(F), c(E \otimes F) = 0$, then because by construction the Chern classes are natural, it holds that

$$p^*(P(c(E), c(F), c(E \otimes F)) = P(c(p^*E), c(p^*F), c(p^*(E \otimes F))).$$

Now $p^{-1}E$ and $p^{-1}F$ are direct sums of line bundles, and thus so is

$$p^{-1}(E \otimes F) = p^*E \otimes p^*F.$$

Thus if the polynomial identity holds for direct sums of line bundles we get

$$p^*(P(c(E), c(F), c(E \otimes F)) = 0.$$

By injectivity this implies $P(c(E), c(F), c(E \otimes F) = 0$ as desired. □

Remark 6.3. The splitting principle holds true for coefficients in any commutative ring R.

Another question one might ask is whether characteristic classes classify isomorphism classes of bundles in some way. In general, this classification of isomorphism classes of (principal) bundles is a complex topic. A nice description of such isomorphism classes can be given using sheaves and Cech cohomology. A detailed discussion is contained in Torsten Wedhorn's book *Manifolds, Sheaves and Cohomology.* However, as sheaves and Cech cohomology warrant a discussion of their own, they do not have a place in these notes, and we will therefore abstain from a discussion of this general case. Instead, we will focus on the more approachable case of complex and real vector bundles, where these results can be shown using more basic techniques and results. For the proofs and a detailed discussion the reader is referred to Allen Hatcher's books on Algebraic Topology (see [3]) and on K-Theory (see). Although we will not give all details for this case either, the above sources are very thorough and should be approachable to any graduate student in mathematics.

One important property of characteristic classes is *stability.* To understand this concept, we have to first define a weaker notion of isomorphism for vector bundles.

Definition 6.4. Let E and F be vector bundles over a manifold M. We say that E and F are **stably isomorphic** if they become isomorphic after taking the direct sum of E and F with trivial bundles. More precisely, E and F are stably isomorphic if there exists a trivial vector bundle ϵ^n of rank n and a trivial vector bundle ϵ^m of rank m such that

$$E \oplus \epsilon^n \cong F \oplus \epsilon^m.$$

We say that a vector bundle is stably trivial, if it is stably isomorphic to a trivial bundle.

To illustrate this, note the following example.

Example 6.5. The tangent bundle $T\mathbb{S}^2$ of the sphere is non-trivial, as can be seen from the Hairy Ball Theorem. However, if one takes the direct sum between the tangent bundle $T\mathbb{S}^2$ and the normal bundle of the sphere, then the elements of this direct sum are triples $(x, v, tx) \in \mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{R}^3$ which satisfy that x and v are orthogonal. The map

$$(x, v, tx) \mapsto (x, v + tx)$$

is a vector bundle isomorphism between this direct sum bundle and the trivial bundle $\mathbb{S}^2 \times \mathbb{R}^3$, and observe that the normal bundle is trivial. Thus the tangent bundle $T\mathbb{S}^2$ is stably trivial.

We say that a characteristic class is stable, if stably isomorphic vector bundles have the same characteristic class. It turns out that all characteristic classes we have discussed and mentioned, with the exception of Euler classes, are stable. Thus it is immediately clear, that generally none but the Euler classes can be considered to classify isomorphism classes of vector bundles. However, as it turns out, in the special case of real line bundles, the Stiefel-Whitney classes are non-stable and are in fact in one-to-one correspondence with the isomorphism classes of real line bundles. A similar result holds true for complex line bundles and the Euler classes. A more detailed explanation of these facts can be found in Sections 1.1, 1.2, and 3.1 of [4]. We summarize this in the following Theorem.

Theorem 6.6. Denote by $\mathrm{Vect}^1_{\mathbb{R}}(M)$ and by $\mathrm{Vect}^1_{\mathbb{C}}(M)$ the isomorphism classes of real and complex line bundles over a manifold M respectively. Then the first Stiefel-Whitney class induces an isomorphism

$$\mathrm{Vect}^1_{\mathbb{R}}(M) \to H^1(M; \mathbb{Z}_2)$$

and the Euler class induces an isomorphism

$$\mathrm{Vect}^1_{\mathbb{C}}(M) \to H^2(M; \mathbb{Z}).$$

Here the isomorphism classes form groups with respect to the tensor product, and the induced maps are in fact group isomorphisms.

This is a powerful application of characteristic classes. Not only does this show that the question whether line bundles over a manifold are isomorphic can be solved using cohomology, but moreover, that for any homology class there exists a vector bundle representing it. The case of vector bundles of higher rank is more complex, but again expanded on in Hatcher's book.

The main ingredient to show this is to realize that there is a connection between the isomorphism classes of vector bundles and a certain set of homotopy classes of maps. We start by observing the following property of vector bundles.

Theorem 6.7. Let $\pi : E \to M$ be a vector bundle over a manifold. Suppose $f_0, f_1 : N \to M$ are homotopic maps from a manifold N to M. Then the pullback bundles $f_0^* E$ and $f_1^* E$ are isomorphic.

Proof. This is almost immediate from the fact that for a (topological) vector bundle $\pi : E \to X \times [0, 1]$ over some topological space X the bundles $X \times \{0\}$ and $X \times \{1\}$ are isomorphic as bundles over X. This result is slightly technical and can be found as Propositon 1.7 in [4]. □

Remark 6.8. The above Theorem is a purely topological result and does not only hold for manifolds but rather holds for any paracompact Hausdorff spaces.

A useful Corollary of the Theorem is that any vector bundle over a contractible manifold must be trivial. We moreover get the following result:

Corollary 6.9. A homotopy equivalence $f : N \to M$ between manifolds induces a bijection

$$f^*_{\mathrm{Vect}} : \mathrm{Vect}^k_{\mathbb{R}}(M) \to \mathrm{Vect}^k_{\mathbb{R}}(N),$$

where $\mathrm{Vect}^k_{\mathbb{R}}(-)$ denotes the isomorphism classes of rank k vector bundles. The same holds true for complex vector bundles.

Proof. This is immediate from the definition of a homotopy equivalence. □

With this in mind, the most important observation is that there exists a so called *universal bundle*.

Definition 6.10. A vector bundle $\pi_n : EGL(n) \to BGL(n)$ is a **universal real vector bundle** if for all manifolds M there exists a natural bijection between the set of isomorphism classes of real vector bundles of rank n over M and the set of homotopy classes of maps $f : M \to BGL(n)$, which is usually denoted by $[M, BGL(n)]$. There is an analogous universal complex vector bundle denoted by $\pi_n^{\mathbb{C}} : EU(n) \to BU(n)$.

Remark 6.11. As the notation for a universal vector bundle already hints at, these universal bundles exist more generally for G-prinicpal bundles. In this case one denotes a **universal G-principal bundle** by $\pi_G : EG \to BG$. The space BG is referred to as the **classifying space**.

In general, the sets $[M, BGL(n)]$ and $[M, BU(n)]$ are difficult to calculate, but as we already stated above in Theorem 6.6 they have a nice form for line bundles, that is for $n = 1$.

We will end these notes with two examples. In the first one, we will illustrate why, knowing that the first Stiefel-Whitney classes classify real line bundles, it makes sense geometrically, that we can not determine Stiefel-Whitney classes using the curvature of a line bundle. For this, consider the circle $\mathbb{S}^1$. Using the well-known fact that $H_1(\mathbb{S}^1; \mathbb{Z})$ and $H_0(\mathbb{S}^1; \mathbb{Z})$ are isomorphic to $\mathbb{Z}$ it follows from the Universal Coefficients Theorem that

$$H^1(\mathbb{S}^1; \mathbb{Z}_2) \cong \mathbb{Z}_2,$$

and thus by Theorem 6.6 it follows that there are exactly two isomorphism classes of real line bundles over the circle. Representatives of these isomorphism classes are the trivial bundle, and the *Möbius bundle*. The Möbius bundle can be thought of as a Möbius strip with infinite width. It can be constructed explicitly as follows.

Consider the space $X := [0, 1] \times \mathbb{R}$ and define an equivalence relation on X by $(0, t) \equiv (1, -t)$. Then the space $E := X$ is the Möbius bundle. There are of course also other constructions of the Möbius bundle, for example via a quotient of $\mathbb{R}^2$ or, for those readers familiar with it, via the Fiber Bundle Construction Theorem, but by the above we already know that, as all of these constructions are non-trivial bundles, they must all be isomorphic. That these are the only two isomorphism classes over the circle can also be shown in a more direct fashion, but the above approach highlights the power of characteristic classes. We could deduce this fact about line bundles using only algebraic arguments.

Now, to illustrate why curvature alone fails to capture the Stiefel-Whitney classes of a line bundle, recall that the curvature of a vector bundle is represented by a 2-form. Under the Chern-Weil homomorphism any non-trivial polynomial maps to the cohomology class of an even-dimensional form - as $\mathbb{S}^1$ is a one-dimensional manifold, there are no differential forms of larger degree than 1. Thus the Chern-Weil homomorphism is trivial in this case. This shows that we can not recover the Stiefel-Whitney classes from Chern-Weil Theory.

Although we already mentioned above, that in general it is not possible to calculate $[M, BGL(n)]$ for $n > 1$ for arbitrary manifolds M, there is a straightforward argument to do this in the case where M is $\mathbb{S}^1$. This is mainly due to the simple nature of $\mathbb{S}^1$ as the quotient of the unit interval.

Consider the map $f : [0, 1] \to \mathbb{S}^1$ given by

$$f(t) := exp(2\pi i t).$$

If we denote by E some vector bundle over $\mathbb{S}^1$, then f^*E is a vector bundle over $[0, 1]$. As we have seen above any vector bundle over a contractible space is trivial, so that f^*E must be trivial as well. Consider the following commutative diagram we get from the pullback:

$$\begin{array}{ccc} f^*E & \xrightarrow{F} & E \\ \big\downarrow & & \big\downarrow \pi_E \\ I & \xrightarrow{f} & \mathbb{S}^1 \end{array}$$

where I denotes the unit interval. Because f^*E is given by $[0,1] \times \mathbb{R}^k$, where k equals the rank of E, it follows that F is a fiber-wise isomorphism. Thus we can construct a linear isomorphism $A : \mathbb{R}^k \to \mathbb{R}^k$ such that

$$F(0,v) = F(1, A(v))$$

for all $v \in \mathbb{R}^k$. From this we can deduce that all vector bundles on $\mathbb{S}^1$ can be obtained from the trivial bundle over $[0,1]$ by identifying points $(0,v) \cong (1, A(v))$ for some linear isomorphism A of $\mathbb{R}^k$. To conclude, we now observe that $GL(n)$ has exactly two path components, and thus any two linear isomorphisms A as above are homotopic if they are in the same path component. Because homotopic maps result in isomorphic vector bundles by Theorem 6.7 it follows that for any $k \in \mathbb{N}$ there are exactly two isomorphism classes of rank k real vector bundles over $\mathbb{S}^1$.

For the final example of how characteristic classes can be useful, we will prove that $\mathbb{C}P^2$ can not be embedded into $\mathbb{R}^5$. More explicitly, we will use Pontryagin classes to do this. For this, we will use the *Whitney Product Formula*, sometimes also called the *Whitney Sum Formula*. It is given by the equation

$$c(E \oplus F) = c(E) \cup c(F),$$

where $c(E) := 1 + c_1(E) + c_2(E) + \cdots$ is the **total Chern class of** E. Here the cup product is the multiplication in the singular cohomology ring. Translated into deRham cohomology we can also express this relation as

$$c_k(E \oplus F) = \sum_{i=0}^{k} p_i(E) \wedge p_{k-i}(F).$$

Here we use the convention that $p_0 = 1$. To prove this, one utilizes the Splitting Lemma. The proof for a direct sum of line bundles is straightforward - details can be found in [1].

We now want to discuss the tangent bundle of $\mathbb{C}P^2$, or more generally $\mathbb{C}P^n$. For this we first need to define the so called *Hopf line bundle over* $\mathbb{C}P^n$. To construct it we consider the trivial bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$. Now, an arbitrary element ℓ of $\mathbb{C}P^n$ is a line through the origin in $\mathbb{C}^{n+1}$ and as such a 1-dimensional subspace. Now consider the subset

$$L = \{(\ell, z) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \mid z \in \ell\}.$$

It is straightforward to verify that this is a 1-dimensional subbundle of the trivial bundle, and is hence a (complex) line bundle.

The Hopf line bundle is useful because a direct sum of the dual bundles of L is stably isomorphic to the tangent bundle of $\mathbb{C}P^n$. More explicitly we have the following Lemma:

Lemma 6.12. Let L be the Hopf line bundle over $\mathbb{C}P^n$, and denote by ϵ_1 the trivial line bundle over $\mathbb{C}P^n$. Then there is a vector bundle isomorphism

$$T\mathbb{C}P^n \oplus \epsilon_1 \cong L^* \oplus \cdots \oplus L^*,$$

where we have $(n+1)$ summands on the right hand side.

Proof. The proof is not particularly hard, but it involves some technicalities which we do not want to discuss here, so refer the reader to Section 5.1 in [11]. □

We will need a couple more auxiliary results. We start with an isomorphism between the *conjugate bundle* of a complex vector bundle and the dual bundle.

Definition 6.13. Let E be a complex vector bundle over a manifold M. For every $p \in M$ we define a new complex vector space $\bar{E}_p$ by defining multiplication by $(a+ib) \in \mathbb{C}$ for $v \in E_p$ by $(a+ib)v = av - ibv$. The vector bundle $\bar{E}$ with fiber $\bar{E}_p$ over $p \in M$ is called the **conjugate bundle of** E.

Now for the isomorphism:

Lemma 6.14. Let E be a complex vector bundle. Then the conjugate bundle $\bar{E}$ is isomorphic to the dual bundle E^* of E.

Proof. Take a Hermitian metric on E, which exists by Lemma 2.9. For each $p \in M$ we define the linear map

$$v \in \bar{E}_p \mapsto l(v) \in E_p^*, \quad l(v)u = \langle u, v\rangle \in \mathbb{C}.$$

It is easy to verify that this is indeed an isomorphism, and hence that these maps assemble to a vector bundle isomorphism. □

The next step is to relate the characteristic classes of a complex vector bundle to the characteristic classes of its conjugate bundle, and hence its dual bundle.

Lemma 6.15. The Chern classes of the conjugate bundle $\bar{E}$ of some complex vector bundle are given by the formula

$$c_k(\bar{E}) = (-1)^k c_k(E).$$

Proof. This is follows from the fact that any connection ∇ on E is also a connection on $\bar{E}$. However, the curvature form of $\bar{E}$ is given by $\bar{\Omega}$, where Ω denotes the curvature form of ∇ on E. Because the Chern classes are real cohomology classes, it follows that the matrix representing the curvature at a point is skew-hermitian. Thus it follows that $\bar{\Omega} = -\Omega^T$. Now the claim is immediate from the definition of Chern classes. □

Above, we have defined the Pontryagin classes using the Chern classes. Thus it not suprising that there is a useful formula relating the two.

Proposition 6.16. Let E be an n-dimensional complex vector bundle. Then if we denote by p_i the ith Pontryagin class of E, and by c_i the ith Chern class of E we find that

$$\sum_{i=0}^{n}(-1)^i p_i = (\sum_{i=0}^{n} c_i) \cup (\sum_{i=0}^{n}(-1)^i c_i),$$

where we use the conventions $p_0 = 1$ and $c_0 = 1$.

Proof. This can be seen as follows. If we denote by $E_{\mathbb{R}}$ the vector bundle E regarded as a real bundle, then there is an isomorphism

$$E_{\mathbb{R}} \otimes \mathbb{C} \cong E \oplus \bar{E},$$

where $E_{\mathbb{R}} \otimes \mathbb{C}$ is an $2n$-dimensional complex vector bundle. For the details of this isomorphism see Propositon 5.41 in [11].

Now, by definition we have that

$$p_k(E) = (-1)^k c_{2k}(E_{\mathbb{R}} \otimes \mathbb{C}),$$

while the Whitney product formula combined with the above isomorphism yields

$$c(E_{\mathbb{R}} \otimes \mathbb{C}) = c(E) \cup c(\bar{E}).$$

Combining these two expressions, and observing the result in Lemma 6.15 above, we get the desired result. □

We have already used that a complex vector bundle can be viewed as a real vector bundle as well multiple times already in these notes. As a result, a complex vector bundle has Chern classes as well as Euler classes. The relationship between the two is very simple.

Theorem 6.17. Let E be a complex vector bundle of rank n. Then it holds that

$$e(E_{\mathbb{R}}) = c_n(E).$$

Proof. This is a straightforward, but lengthly calculation. It uses the fact that a Hermitian metric on E can be viewed as a Riemannian metric on $E_{\mathbb{R}}$. For the details of this calculation we refer the reader to Propositon 5.43 in [11]. □

Before calculating the Chern classes of $\mathbb{C}P^n$, for which we will utilize Lemma 6.12, we will calculate the first Chern class of L. For this, let us quickly recall some basic facts about the (co)homology of $\mathbb{C}P^n$.

The space $\mathbb{C}P^n$ has a very simple CW-complex structure. It has exactly one cell in each degree $i = 0, 2, \cdots, 2n$. Using cellular homology it is then straightforward to calculate the homology of $\mathbb{C}P^n$. From this, it is also easy to determine that the cohomology groups are given by

$$H^i(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \leqslant i = \text{even} \leqslant 2n \\ 0 & \text{otherwise} \end{cases}$$

If we denote by $x \in H^2(\mathbb{C}P^n)$ a generator, then using induction one can show that the generators x^{2j} of $H^{2j}(\mathbb{C}P^n)$ are of the form

$$x^{2j} = x \cup \cdots \cup x,$$

that is, x^{2j} is the cup product of x with itself j times. Furthermore, one can then go on and show that the cohomology ring of $\mathbb{C}P^n$ is given by

$$H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1}).$$

With these observations in mind, let us now calculate the first Chern class of L.

Lemma 6.18. Let L denote the Hopf line bundle over $\mathbb{C}P^n$. Then it holds that

$$c_1(L) = -x.$$

Proof. Denote by $\iota : \mathbb{C}P^1 \to \mathbb{C}P^n$ the inclusion. Then it holds that $\iota^* L = L$. Thus it suffices to only consider the case $n = 1$. By Lemma 6.15 it suffices to show that $c_1(L^*) = x$. Because $H^2(\mathbb{C}P^1) \cong \mathbb{Z}$ we know that $c_1(L^*) = kx$ for some $k \in \mathbb{Z}$. We will not show that $k = 1$ holds. By Lemma 6.12 we have that

$$T\mathbb{C}P^1 \oplus \epsilon_1 \cong L^* \oplus L^*.$$

From the Whitney Product Formula it follows that

$$c_1(\mathbb{C}P^1) = c_1(T\mathbb{C}P^1) = c_1(T\mathbb{C}P^1 \oplus \epsilon_1) = c_1(L^* \oplus L^*) = 2c_1(L^*) = 2kx,$$

where we used that Chern classes are stable. But now using the Chern-Gauss-Bonnet Theorem, and the Proposition above, telling us that the Euler class equals the top Chern class, we find

$$c_1(\mathbb{C}P^1) = e((\mathbb{C}P^1)_{\mathbb{R}}) = e(S^2) = \chi(S^2)x = 2x,$$

where we used that $\mathbb{C}P^1$ is the Riemann sphere, so that viewed as a real manifold it is the 2-sphere. From this it follows that $k = 1$, as desired. □

We can now calculate the Chern classes of $\mathbb{C}P^n$.

Theorem 6.19. The Chern classes of the complex projective space $\mathbb{C}P^n$ are given by

$$c(\mathbb{C}P^n) = (1+x)^{n+1}, \qquad c_i(\mathbb{C}P^n) = \binom{n+1}{k} x^k.$$

Proof. It is enough to prove the expression for the total Chern class, the rest will follow from our above discussion of generators. It holds that

$$c(\mathbb{C}P^n) = c(T\mathbb{C}P^n) = c(T\mathbb{C}P^n \oplus \epsilon_1) = c_1((n+1)L^*) = (1+x)^{n+1},$$

where we used stability of Chern classes, Lemma 6.12, the Whitney Product Formula, and finally applied Lemma 6.18 from above. □

From this result it is easy to calculate the Pontryagin classes of $\mathbb{C}P^n$ as well.

Theorem 6.20. The Pontryagin classes of the complex projective space $\mathbb{C}P^n$ are given by

$$p(\mathbb{C}P^n) = (1+x^2)^{n+1}, \qquad p_i(\mathbb{C}P^n) = \binom{n+1}{k} x^{2k}.$$

Proof. This is a quick calculation using Theorem 6.19 as well as Propositon 6.16. □

Now that we have determined the Pontryagin classes of $\mathbb{C}P^n$, we can finally show that $\mathbb{C}P^2$ does not embed in $\mathbb{R}^5$. First, we need the following result:

Lemma 6.21. Suppose M is a compact manifold of dimension m which can be embedded in $\mathbb{R}^{m+1}$. Then $p_k(M) = 0$ for all $k > 0$.

Proof. If M can be embedded in $\mathbb{R}^{m+1}$ then the normal bundle $\mathrm{Norm}(M)$ of of M is one-dimensional. However, in this case it holds that $p_k(\mathrm{Norm}(M)) = 0$ for all $k > 0$. To see this, note that the characteristic classes are independent of the chosen connection. If we choose a metric connection, then the curvature is skew-symmetric. But because the bundle is one-dimensional, this implies that the curvature must be zero, and hence the characteristic classes, and in particular the Pontryagin classes, vanish.

Recall that the Chern-Weil homomorphism is natural with respect to map, so that the characteristic classes are natural with respect to maps as well. Consider the vector bundle isomorphism

$$T\mathbb{R}^{m+1}_{|M} \cong TM \oplus \mathrm{Norm}(M),$$

and use the naturality of the Pontryagin classes applied to the embedding $M \hookrightarrow \mathbb{R}^{m+1}$. Then it follows that $p_k(T\mathbb{R}^{m+1}_{|M}) = 0$ for all $k > 0$ as well. From the Whitney Product Formula we can deduce that the same must hold true for TM as well, so that $p_k(M) = 0$ for all $k > 0$, as desired. □

Now the fact that $\mathbb{C}P^n$ can not be embedded in R^5 is immediate:

Corollary 6.22. The complex projective space $\mathbb{C}P^2$, viewed as a four-dimensional real manifold, does not embed in $\mathbb{R}^5$.

Proof. From Theorem 6.20 we get that the first Pontryagin class has the form $3x^2$ where x is a generator of $H^2(\mathbb{C}P^2)$, so that in particular it is not zero. By the previous Lemma we get the claim. □

This example again illustrates how we can use characteristic classes, and algebraic arguments, to deduce topological properties of manifolds (or more generally topological spaces). With this second application example of how to use characteristic classes we will end our notes. There are of course a lot more useful applications of characteristic classes, and there is more to learn about them in general. In these notes, we illustrated how in the case of smooth manifolds, we can use the curvature of a connection to find (topological) invariants using Chern-Weil theory, and illustrated how they can be useful. As we discussed, there is a more general theory of characteristic classes for topological spaces, which coincides with our geometric approach for smooth manifolds. For the interested reader there is a plethora of books which discuss the more classical approach to characteristic classes of topological space. One seminal work is the book *Characteristic Classes* by Milnor and Stasheff which we have cited multiple times in these notes as well. It is a good starting point to learn more about characteristic classes. On our journey through Chern-Weil theory we also discussed the connection between vector bundles and principal bundles, different types of connections, and how these types of connections are related to one another. In particular, we discussed how connections on principal bundles and connections on vector bundles are related.

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