Eidgenössische Technische Hochschule Zürich
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A Topological Analysis of Contact Manifolds and Their Symplectic Fillings

Semester Paper

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Abstract

In this semester paper, we give definitions and examples of contact manifolds, open book decompositions, and symplectic fillings. We start by introducing contact manifolds and related concepts. We discuss the existence of contact structure, tight versus overtwisted dichotomy, and classification of overtwisted contact structures. Then, we investigate open book decompositions of contact manifolds and discuss the Giroux correspondence between open book decompositions and contact structures on 3-manifolds. Lastly, we discuss various types of symplectic fillings of contact manifolds and their connection to the topology of contact manifolds.

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Chapter 1

Introduction to Contact Manifolds

In this chapter, we introduce contact manifolds and related notions and explain some basic results and constructions about them.

In the first section, we will define contact and symplectic manifolds, their special submanifolds, and the vector fields on them. In Section 1.2, we prove the theorems of Gray [28] and Pfaff using Moser's Trick. These results describe the deformations of contact structures and their local forms. In Section 1.3, we define the characteristic foliation of a surface in a contact 3-manifold. That is, the singular foliation defined on a surface by the contact structure of the ambient manifold which gives information about the contact structure near the surface. In Section 1.4, we consider knots and their invariants in contact 3-manifolds. Lastly, in Section 1.5, we first consider the results of Martinet [35] and [34] about the existence of contact structures. We also introduce the fundamental dichotomy of tightness vs overtwistedness of contact structures and describe the classification of overtwisted structures on contact 3-manifolds by Eliashberg [8].

1.1 Basic Definitions and Examples

We start with the definitions of contact and symplectic manifolds.

Definition 1.1 *Let M be a smooth manifold of dimension (2n+1), for $n \geq 0$. A **contact structure** on M is a maximally nonintegrable hyperplane field $\xi \subset TM$. That is, locally $\xi = \ker \alpha$ for a 1-form $\alpha \in \Omega^1(M)$, with $\alpha \wedge (d\alpha)^n \neq 0$. Such a 1-form α on M is called a **contact 1-form**. The pair (M, ξ) is called a **contact manifold**.*

Observe that, if α is a contact 1-form on M, then $g\alpha$, where $g : M \to \mathbb{R} - \{0\}$ is a nonvanishing function, is also a contact form defining the same contact structure.

Unless otherwise stated, we will assume the contact structures are defined globally by a contact 1-form. This is the case precisely when TM/ξ is a trivial line bundle, that is, when ξ is **coorientable**. Not all contact structures are coorientable. For an example of a non-coorientable contact structure on $\mathbb{R}^{n+1} \times \mathbb{R}P^{n+1}$, see [22, Lemma 1.1.1 and Example 2.1.11].

Definition 1.2 *A 2-form ω on a smooth manifold X is a **symplectic form** if*

1. *ω is closed (that is, $d\omega = 0$) and,*
2. *ω is nondegenerate for all $p \in X$ (that is, at every point $p \in X$, for any nonzero tangent vector $v \in T_pX$, there is $w \in T_pX$ such that $\omega_p(v, w) \neq 0$).*

*The pair (X, ω) is called a **symplectic manifold**.*

By the nondegeneracy condition, a symplectic manifold is necessarily of even dimension. If (X, ω) is a symplectic 2n-manifold, we can reformulate the nondegeneracy condition for ω as $\omega^n \neq 0$. Thus, a symplectic form defines an orientation on X.

For a contact manifold $(M, \xi = \ker \alpha)$ of dimension $2n + 1$, we can also restate the nonintegrability condition $\alpha \wedge (d\alpha)^n \neq 0$ as $d\alpha|_\xi$ is nondegenerate.

By the contact condition, if α is a contact form, $\alpha \wedge (d\alpha)^n$ is a volume form, then a contact manifold M is necessarily orientable. In addition, if n is odd, the sign of $\alpha \wedge (d\alpha)^n$ only depends on the contact structure $\xi = \ker \alpha$. In this case, given an orientation on M, we can talk about a **positive contact structure** (respectively a **negative contact structure**) if $\alpha \wedge (d\alpha)^n > 0$ (respectively $\alpha \wedge (d\alpha)^n < 0$) on oriented frames.

Contact and symplectic manifolds are usually viewed as odd and even dimensional analogues of each other. We will investigate this relation to some extent in the following chapters.

For further properties and discussions about symplectic manifolds, one can consult [6] and [39].

Now we will give some non-examples and examples of contact manifolds:

Example 1.3 *By the previous discussions, non-orientable manifolds cannot be endowed with contact structures.*

Example 1.4 *Consider a 1-manifold M. Then, any nonvanishing 1-form on α satisfies $\alpha \wedge (d\alpha)^0 = \alpha \neq 0$. So, α is a contact form. Since α is nonvanishing, the contact structure $\xi = \ker \alpha$ is the zero section of the tangent bundle TM of M.*

Example 1.5 *We will define several contact structures on $\mathbb{R}^{2n+1}$ with coordinates $\{(x_1, y_1, \ldots, x_n, y_n, z)\}$ for $n \geq 1$:*

First, consider the 1-form

$$\alpha_1 = dz + \sum_{i=1}^{n} x_i dy_i$$

We have,

$$\begin{aligned}\alpha_1 \wedge (d\alpha_1)^n &= (dz + \sum_{i=1}^{n} x_i dy_i) \wedge (\sum_{i=1}^{n} dx_i \wedge dy_i)^n \\ &= n! dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n\end{aligned}$$

The form $dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ is a volume form on $\mathbb{R}^{2n+1}$, thus

$$\alpha \wedge d\alpha^n \neq 0$$

Therefore, α_1 is a global contact 1-form on $\mathbb{R}^{2n+1}$ that defines a contact structure

$$\xi_1 = \ker \alpha_1 = span\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, x_1\frac{\partial}{\partial z} - \frac{\partial}{\partial y_1}, \ldots, x_n\frac{\partial}{\partial z} - \frac{\partial}{\partial y_n}\}$$

and we get the contact manifold $(\mathbb{R}^{2n+1}, \xi_1 = \ker \alpha_1)$. For the case $2n+1 = 3$, Figure 1.1 describes this contact structure.

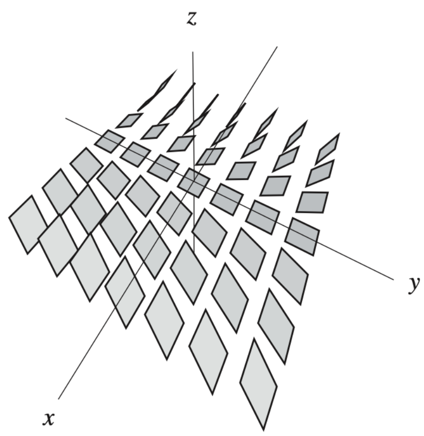

Figure 1.1: The contact structure $\xi_1 = \ker(dz + \sum_{i=1}^{n} x_i dy_i)$, [12].

Similarly, the 1-form

$$\alpha_2 = dz - \sum_{i=1}^{n} y_i dx_i$$

defines a contact manifold $(\mathbb{R}^{2n+1}, \xi_2 = \ker \alpha_2)$.

Lastly, consider the 1-form

$$\alpha_3 = dz + \sum_{i=1}^{n} x_i dy_i - y_i dx_i = dz + \sum_{i=1}^{n} r_i^2 d\varphi_i$$

where (r_i, φ_i) are the polar coordinates on respective (x_i, y_i) planes. Then,

$$\alpha_3 \wedge (d\alpha_3)^n = 2^n n! dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \neq 0$$

Thus, α_3 defines a contact structure

$$\xi_3 = span\{x_1 \frac{\partial}{\partial z} - \frac{\partial}{\partial y_1}, \ldots, x_n \frac{\partial}{\partial z} - \frac{\partial}{\partial y_n}, y_1 \frac{\partial}{\partial z} + \frac{\partial}{\partial x_1}, \ldots, y_n \frac{\partial}{\partial z} + \frac{\partial}{\partial x_n}\}$$

and we get a contact manifold $(\mathbb{R}^{2n+1}, \xi_3 = \ker \alpha_3)$.

Example 1.6 *By the same argument for α_3 described above, the solid torus $S^1 \times D^2$ with coordinates $(\theta, (r, \varphi))$ is a contact manifold with the contact structure defined by the form*

$$\alpha = d\theta + r^2 d\varphi$$

This can be viewed as $(\mathbb{R}^3, \alpha)$ wrapped around the z-axis, and restricted to the solid torus.

Example 1.7 *Consider $\mathbb{R}^3$ with cylindrical coordinates $\{(r, \theta, z)\}$. and the 1-form*

$$\alpha_{ot} = \cos r dz + r \sin r d\theta$$

We have

$$\begin{aligned} \alpha_{ot} \wedge d\alpha_{ot} &= (1 + \frac{\sin r \cos r}{r}) r dr \wedge d\theta \wedge dz \\ &= (1 + \frac{\sin r \cos r}{r}) dVol \end{aligned}$$

Thus, by extending $\frac{\sin r}{r}$ smoothly by 1 where $r = 0$ to adress differentiability of of $\frac{\cos r \sin r}{r}$, we see that $\alpha_{ot} \wedge d\alpha_{ot} \neq 0$ and α_{ot} defines a contact structure

$$\xi_{ot} = \ker \alpha_{ot} = span\{\frac{\partial}{\partial r}, r \tan r \frac{\partial}{\partial z} - \frac{\partial}{\partial \theta}\}$$

*called the **standard overtwisted contact structure on** $\mathbb{R}^3$. See Figure 1.2 for this contact structure. We will talk more about "overtwisted" contact structures in Sections 1.3 and 1.5.2.*

Example 1.8 *Consider the unit sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$, and the 1-form*

$$\alpha_{S^{2n+1}} = \sum_{i=1}^{n+1} x_i dy_i - y_i dx_i$$

where we use the coordinates $\{(x_1, y_1, \ldots, x_n, y_n, x_{n+1}, y_{n+1})\}$ for $\mathbb{R}^{2n+2}$.

Considering $r^2 = \sum_{i=1}^{n+1} x_i^2 + y_i^2$ where r is the radial coordinate on $\mathbb{R}^{2n+2}$, we get $r dr \wedge \alpha_{S^{2n+1}} \wedge (d\alpha_{S^{2n+1}})^n \neq 0$ for $r \neq 0$. Thus, since $S^{2n+1} \subset \mathbb{R}^{2n+2}$ is the level

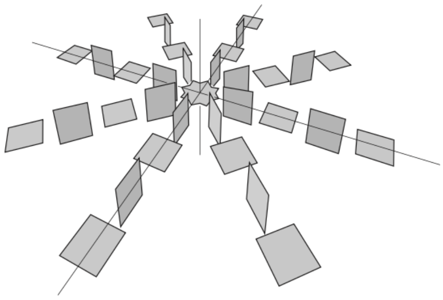

Figure 1.2: The contact structure ξ_{ot}, [22].

set $r = 1$*, the 1-form* $\alpha_{S^{2n+1}} \wedge (d\alpha_{S^{2n+1}})^n$ *is nonzero when restricted to the sphere. The contact structure* $\xi_{S^{2n+1}} = \ker \alpha_{S^{2n+1}}$ *on* S^{2n+1} *is called the* ***standard contact structure on*** S^{2n+1}

Here is an other description of this contact structure on S^{2n+1}*: Consider the smooth map* $f : \mathbb{R}^n \to \mathbb{R}$ *given by*

$$f(x_1, y_1, \ldots, x_n, y_n, x_{n+1}, y_{n+1}) = \sum_{i=1}^{n+1} x_i^2 + y_i^2$$

Then, $S^{2n+1} = f^{-1}(1)$ *and*

$$\begin{aligned} T_pS^{2n+1} &= kerdf_p \\ &= ker(2x_1dx_1 + 2y_1dy_1 + \cdots + 2x_{n+1}dx_{n+1} + 2y_{n+1}dy_{n+1}) \end{aligned}$$

for $p = (x_1, y_1, \ldots, x_n, y_n, x_{n+1}, y_{n+1}) \in S^{2n+1}$*. We identify* $\mathbb{R}^{2n+2}$ *with* $\mathbb{C}^{n+1}$ *to get a complex structure* J *on each tangent space, that is a linear map such that* $J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ *and* $J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$ *for all* $i = 1, \ldots, n+1$*.*

One can then check that the contact form is $\alpha_{S^{2n+1}} = -\frac{1}{2}df \circ J|_{S^{2n+1}}$ *and the contact structure is* $(\xi_{S^{2n+1}})_p = T_pS^{2n+1} \cap J(T_pS^{2n+1})$ *at each point* $p \in S^{2n+1}$*.*

Example 1.9 *Consider the 3-torus* $T^3 = \mathbb{R}^3/\mathbb{Z}^3$*. Then for each positive integer n, the 1-form* $\alpha_n = \sin(2\pi nz)dx + \cos(2\pi nz)dy$ *induces a contact structure on* T^3*. Contact structure is given by*

$$\xi_n = span\{\frac{\partial}{\partial z}, \cos(2\pi nz)\frac{\partial}{\partial x} - \sin(2\pi nz)\frac{\partial}{\partial y}\}$$

The circle $x = y = constant$ *is tangent to* ξ *and on this circle* ξ *makes n full twists. See Figure 1.3.*

Now we introduce the diffeomorphisms between contact manifolds that respect the contact structures.

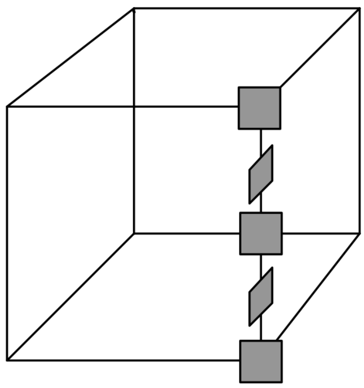

Figure 1.3: The contact structure on 3-torus

Definition 1.10 *Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are called **contactomorphic** if there is a diffeomorphism $f : M_1 \to M_2$ such that $f_* : TM_1 \to TM_2$ maps ξ_1 to ξ_2, that is $f_*(\xi_1) = \xi_2$. Such an f is called a **contactomorphism**. Equivalently, there exists a nowhere zero function $\lambda : M_1 \to \mathbb{R} - \{0\}$ such that $f^*(\alpha_2) = \lambda\alpha_1$ where α_i is a contact form defining ξ_i for $i = 1, 2$.*

*A contactomorphism is called **strict** when it preserves the contact structure. That is, when $f^*(\alpha_2) = \alpha_1$.*

*Two contact structures ξ_1 and ξ_2 on M are **isotopic** if there is a contactomorphism $f : (M, \xi_1) \to (M, \xi_2)$ isotopic to identity.*

Example 1.11 *Three contact structures on $\mathbb{R}^{2n+1}$ described in Example 1.5 are all contactomorphic. The contact structures ξ_1 and ξ_2 are related by a rotation about the z-axis, and the structures ξ_1 and ξ_3 are related by the map*

$$(\mathbb{R}^{2n+1}, \xi_1) \to (\mathbb{R}^{2n+1}, \xi_3)$$

defined as

$$(x_1, y_1, \ldots, x_n, y_n, z) \mapsto \\ (\frac{(x_1 + y_1)}{2}, \frac{(y_1 - x_1)}{2}, \ldots, \frac{(x_n + y_n)}{2}, \frac{(y_n - x_n)}{2}, z + \frac{\sum x_i y_i}{2})$$

*Any of these structures is called the **standard contact structure** ξ_{st} on $\mathbb{R}^{2n+1}$*

Example 1.12 *The contact manifold $(S^{2n+1} - \{p\}, \xi_{S^{2n+1}}|_{S^{2n+1}-\{p\}})$ described in Example 1.8 is contactomorphic to $\mathbb{R}^{2n+1}$ with its standard contact structure (see [22] and [21] for constructions of contactomorphisms based on stereographic projection or maps of complex domains). This contactomorphism justifies calling both of these* *contact structures "standard". Accordingly, we sometimes also use ξ_{st} to denote the standard contact structure on S^{2n+1}.*

We will also need to define some special submanifolds of contact manifolds:

Definition 1.13 *Let (M, ξ) be a contact manifold of dimension 2n+1 for $n \geq 0$.*

A submanifold $N_1 \subset M$ with a contact structure ξ_1 is called a ***contact submanifold*** *if $TN_1 \cap \xi|_{N_1} = \xi_1$.*

A submanifold N_2 is called an ***isotropic submanifold*** *if $T_pN_2 \subset \xi_p$ for all $p \in N_2$.*

A submanifold N_3 is called a ***Legendrian submanifold*** *if it is isotropic and of dimension n.*

By nondegeneracy of $d\alpha|_\xi$, an isotropic submanifold is necessarily of dimension at most n. Thus, Legendrian submanifolds are isotropic submanifolds of maximal dimension.

In dimension $2n + 1 = 3$, closed Legendrian submanifolds N are 1- dimensional. Hence, $N \cong S^1$. We call these **Legendrian knots**. We will discuss more about Legendrian knots later.

We can also define an important type of submanifolds of symplectic manifolds:

Definition 1.14 *Let (X, ω) be a symplectic manifold. A submanifold $Y \subset X$ is called a* ***Lagrangian submanifold*** *if $\omega_p|_{T_pY} = 0$ and $\dim T_pY = \frac{1}{2} \dim T_pM$ for all $p \in Y$.*

By nondegeneracy of ω, Lagrangian submanifolds are the submanifolds $Y \subset X$ of maximal dimension such that $\omega_p|_{T_pY} = 0$ for all $p \in Y$.

1.1.1 Contact, Reeb, and Liouville Vector Fields

Now, we will define some vector fields defined on contact and symplectic manifolds that will be of special importance in the following discussions.

Let $(M, \xi = \ker \alpha)$ be a contact manifold of dimension $2n + 1$. Since $d\alpha$ is nondegenerate on ξ by the nonintegrability condition, its kernel defines a unique line field, that is a unique vector field R on M up to scaling which satisfies $\alpha(R) \neq 0$. If we normalize R by the condition $\alpha(R) = 1$, we get a unique vector field associated to a contact form α:

Definition 1.15 *Let $(M, \xi = \ker \alpha)$ be a contact manifold. The* ***Reeb vector field*** *R_α is the unique vector field defined by the equations:*

1. $d\alpha(R_\alpha, -) \equiv 0$
2. $\alpha(R_\alpha) = 1$

We also have the notion of vector fields on $(M, \xi = \ker \alpha)$ that preserve the contact structure:

Definition 1.16 *Let $(M, \xi = \ker \alpha)$ be a contact manifold. A vector field v on M is a* ***contact vector field*** *if its flow φ_t is a contactomorphism, that is $(\varphi_t)_* \xi = \xi$, for all t. Equivalently, a vector field v is a contact vector field if $\mathcal{L}_v \alpha = g\alpha$ for some function $g : M \to \mathbb{R}$.*

We have $\mathcal{L}_R \alpha = i_{R_\alpha} d\alpha + d(\alpha(R_\alpha)) = 0$. So the Reeb vector field associated to a contact form is in particular a contact vector field.

Remark 1.17 *While contact vector fields are associated to* contact structures, *Reeb vector fields are associated to* contact forms. *In general, Reeb vector fields that are associated to contact forms that define the same contact structure might be distinct.*

Here are some examples of the Reeb vector fields on the contact manifolds we defined in previous examples:

Example 1.18 *For the standard contact structure α_1 described in Example 1.5, the Reeb vector field is $\frac{\partial}{\partial z}$. Indeed,*

$$\alpha_1(\frac{\partial}{\partial z}) = (dz + \sum_{i=1}^{n} x_i dy_i)(\frac{\partial}{\partial z}) = 1$$

and

$$d\alpha_1(\frac{\partial}{\partial z}, -) = (\sum_{i=1}^{n} dx_i \wedge dy_i)(\frac{\partial}{\partial z}, -) = 0$$

The flow of this vector field is translation along z-coordinate, which preserves the contact structure.

Example 1.19 *For the standard overtwisted contact structure α_{ot} described in Example 1.7, the Reeb vector field is*

$$\frac{1}{(r \sin r + (1 + r \cot r) \cos r)} \frac{\partial}{\partial \theta} + \frac{(1 + r \cot r)}{(r \sin r + (1 + r \cot r) \cos r)} \frac{\partial}{\partial r}$$

Example 1.20 *For the standard contact structure $\alpha_{S^{2n+1}}$ on the unit sphere described in Example 1.8, the Reeb vector field is*

$$\sum_{i=1}^{n+1} (x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i})$$

Lastly, we will define Liouville vector fields on symplectic manifolds.

Definition 1.21 *Let (X, ω) be a symplectic manifold. A vector field u on X is a* ***Liouville vector field*** *if the Lie derivative along u preserves the symplectic form, that is, $\mathcal{L}_u \omega = \omega$.*

Lemma 1.22 *Let (X, ω) be a symplectic manifold of dimension $2n+2$ and vector field u on X be a Liouville vector field. Then, $\alpha = i_u\omega = \omega(u, -)$ is a contact form when restricted to any hypersurface M transverse to u.*

Proof By Cartan's formula for the Lie derivative, we have

$$\omega = \mathcal{L}_u\omega = i_u d\omega + d(\omega(u, -)) = d(\omega(u, -))$$

because $d\omega = 0$. Therefore,

$$\begin{aligned} \alpha \wedge d\alpha^n &= i_u\omega \wedge d(\omega(u, -))^n \\ &= i_u\omega \wedge \omega^n \\ &= (n)^{-1} i_u(\omega^{n+1}) \end{aligned}$$

Thus, by nondegenerecy of ω, if M is transverse to u, then $\alpha \wedge d\alpha^n \neq 0$. □

Such a hypersurface M is called of **contact type**.

Example 1.23 *Let $(M, \xi = ker\alpha)$ be a contact manifold of dimension 2n-1 for $n \geq 1$. Define $X = \mathbb{R} \times M$ and $\omega = d(e^t\alpha)$ where t is the $\mathbb{R}$ coordinate.*

The pair (X, ω) is a symplectic manifold, called the ***symplectization*** *of M. The 'vertical' vector field $\frac{\partial}{\partial t}$ is a Liouville vector field.*

The submanifold L is a Legendrian submanifold of M if and only if $\mathbb{R} \times L$ is a Lagrangian submanifold of X.

The notion of a Liouville vector field gives another description of the standard contact structure described in Example 1.8

Example 1.24 *Consider $\mathbb{R}^{2n+2}$ with coordinates $(x_1, y_1, \ldots, x_{n+1}, y_{n+1})$ and the* ***standard symplectic structure****, that is*

$$\omega_{st} = \sum_{i=1}^{n+1} dx_i \wedge dy_i$$

Consider the radial vector field vector field

$$u = r\frac{\partial}{\partial r} = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$$

One can see by direct computation that u is a Liouville vector field on $\mathbb{R}^{2n+2}$ and transverse to the unit sphere S^{2n+1}. Thus, S^{2n+1} is a hypersurface of contact type in $\mathbb{R}^{2n+2}$ with the contact form

$$i_u\omega_{st} = \sum_{i=1}^{n+1} x_i dy_i - y_i dx_i = \alpha_{S^{2n+1}}$$

that is with the standard contact structure on S^{2n+1}.

1.2 Moser's Trick, Gray's Stability Theorem and Darboux Theorem

In this section, we prove Gray's stability theorem, which says that any smooth family of contact forms on a manifold can be connected through an isotopy of the manifold, and Darboux's theorem for contact manifolds, which shows that all contact structures locally look like the standard contact structure ξ_{st} on $\mathbb{R}^{2n+1}$.

First we start with a lemma concerning families of differential forms:

Lemma 1.25 *For $t \in [0,1]$, let ω_t be a smooth family of differential k–forms on a manifold M and $(\psi_t : M \to M)_{t\in[0,1]}$ be an isotopy. Define a family of vector fields X_t on M so that ψ_t is the flow of X_t, Then*

$$\frac{d}{dt}(\psi_t^*\omega_t) = \psi_t^*(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t)$$

Proof For a differential form ω on M, we have

$$\frac{d}{dt}(\psi_t^*\omega) = \psi_t^*(\mathcal{L}_{X_t}\omega)$$

From this, we compute

$$\begin{aligned}\frac{d}{dt}(\psi_t^*\omega_t) &= \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_{t+h} - \psi_t^*\omega_t}{h}\\ &= \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_{t+h} - \psi_{t+h}^*\omega_t + \psi_{t+h}^*\omega_t - \psi_t^*\omega_t}{h}\\ &= \lim_{h\to 0}\psi_{t+h}^*(\frac{\omega_{t+h}-\omega_t}{h}) + \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_t - \psi_t^*\omega_t}{h}\\ &= \psi_t^*(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t)\end{aligned}$$ □

Now we are ready to prove Gray's stability theorem, the contact analogue of Moser's stability theorem concerning symplectic structures (see [6]). Notably, in contrast to Moser's theorem, Gray's theorem does not have a cohomological condition.

We will use a technique called **Moser's trick** for the proof. Rougly, we will assume an isotopy $(\varphi_t)_{t\in[0,1]}$ is induced by a time dependent vector field $(X_t)_{t\in[0,1]}$ and we will find conditions on $(X_t)_{t\in[0,1]}$. Then by integrating X_t on the closed manifold M, we get the desired isotopy $(\varphi_t)_{t\in[0,1]}$. This technique will be used to construct and extend isotopies with various properties.

Theorem 1.26 (Gray [28]) *Let $(\xi_t)_{t\in[0,1]}$, be a smooth family of contact structures on a closed manifold M . Then, there is an isotopy $(\varphi_t)_{t\in[0,1]}$ of M such that*

$$(\varphi_t)_*(\xi_0) = \xi_t$$

for all $t \in [0,1]$.

Proof Suppose we have the isotopy φ_t and it is induced as the flow of the time-dependent vector field X_t

The condition $(\varphi_t)_*(\xi_0) = \xi_t$ can be restated as

$$(\varphi_t)^*(\alpha_t) = \lambda_t \alpha_0$$

where λ_t is a positive function on M and $\xi_t = \ker \alpha_t$ for all t. By differentiating we get

$$\varphi_t^*(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t) = \frac{d}{dt}(\lambda_t)\alpha_0 = \frac{\frac{d}{dt}(\lambda_t)}{\lambda_t}(\varphi_t)^*(\alpha_t)$$

by the previous lemma. Using Cartan's formula for the Lie derivative and setting $\mu_t = \frac{d}{dt}(\log \lambda_t) \circ \varphi_t^{-1}$ we get

$$\varphi_t^*(\frac{d}{dt}\alpha_t + d(\alpha_t(X_t)) + i_{X_t} d\alpha_t) = \varphi_t^*(\mu_t \alpha_t)$$

Therefore, a vector field $X_t \in \xi_t = \ker \alpha_t$ solves this equation provided that

$$\frac{d}{dt}\alpha_t + i_{X_t} d\alpha_t = \mu_t \alpha_t$$

Take the Reeb vector field R_{α_t} of α_t and evaluate to define the function μ_t as

$$\frac{d}{dt}\alpha_t(R_{\alpha_t}) = \mu_t$$

From these conditions and the nondegeneracy of $d\alpha_t$, we can uniquely find $X_t \in \xi_t = \ker \alpha_t$ and integrate it to get the desired isotopy $(\varphi_t)_{t\in[0,1]}$. □

Remark 1.27 *Original proof of this theorem in [28] by Gray used deformation theory.*

We will again use Moser's trick to prove the contact version of Darboux's theorem (also known as Pfaff's Theorem):

Theorem 1.28 (Pfaff's Theorem) *Every contact manifold $(M, \xi = ker\alpha)$ of dimension 2n+1 locally looks like $(\mathbb{R}^{2n+1}, \xi_{st} = \ker \alpha_{st})$. That is, for all $p \in M$ there exist open neighbourhoods U of p, V of $\mathbf{0}$, and a contactomorphism $\varphi : (U, \xi) \to (V, \xi_{st})$, with $\varphi(p) = \mathbf{0}$.*

Proof Let $(x_1, y_1, \ldots, x_n, y_n, z)$ be the local coordinates on M around p such that $\alpha(\frac{\partial}{\partial z}) = 1$, $d\alpha(\frac{\partial}{\partial z}, -) = 0$, $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \in \ker \alpha$, and $d\alpha = \sum dx_i \wedge dy_i$. This is possible by the contact condition.

Recall that $\alpha_{st} = dz + \sum x_i dy_i$ and consider the family

$$\alpha_t = (1-t)\alpha_{st} + t\alpha$$

for $t \in [0,1]$ on $\mathbb{R}^{2n+1}$. By our choice of coordinates, α_t and $d\alpha_t$ agree with α at 0. Hence, α_t are contact forms in a neighbourhood of 0.

Now, we will use Moser's trick and the previous notation. By differentiating $(\varphi_t)^*(\alpha_t) = \alpha_{st}$ we get

$$\varphi_t^*(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t) = 0$$

so X_t needs to satisfy

$$\frac{d}{dt}\alpha_t + d(\alpha_t(X_t)) + i_{X_t}d\alpha_t = 0$$

Writing $X_t = h_t R_t + Y_t$ where R_t is the Reeb field of α_t, $Y_t \in \ker \alpha_t$, and h_t is a family of smooth functions on M. Plugging R_t in, we get

$$\frac{d}{dt}\alpha_t(R_t) + dh_t(R_t) = 0$$

From this we can find a family of functions h_t by integration. Since $\frac{d}{dt}\alpha_t = 0$ at 0, we may also assume $h_t(0) = 0$ and $dh_t|0 = 0$.

Once h_t is chosen, Y_t and X_t can be found in a neighbourhood of 0. By the assumptions on h_t, we have $X_t(0) = 0$ for all t.

When we integrate X_t, we get the desired isotopy φ_t that fixes 0 of $\mathbb{R}^{2n+1}$, defined around a neighbourhood of origin. This gives the desired local diffeomorphism by composing φ_1 with local coordinate maps. □

Remark 1.29 *Observe that, for the proof of Darboux's theorem we use the equation $(\varphi_t)^*(\alpha_t) = \alpha_{st}$, that is we find an isotopy that connects contact* forms. *On the other hand, for Gray's theorem we use the equation $(\varphi_t)_*(\xi_1) = \xi_2$, that is we find an isotopy that connects contact* structures. *In general, one cannot find an isotopy such that $(\varphi_t)^*(\alpha_2) = \alpha_1$ in Gray's theorem. One can see this using Reeb vector fields, see [22, Example 2.2.5].*

By Darboux's Theorem, contact manifolds of the same dimension are all locally contactomorphic. That is, there are no local invariants of contact manifolds. This is in contrast to Riemannian geometry, where there exist local invariants such as curvature.

1.3 Characteristic Foliation of a Surface

Let (M, ξ) be a contact 3-manifold. Let Σ be an oriented surface in (M, ξ). At each point $x \in \Sigma$ consider the subspace $l_x = T_x\Sigma \cap \xi_x$. We call the points x such that $l_x = T_x M$ **singular points**. By integrating we get a singular foliation Σ_ξ, that is the disjoint union of singular points and 1- manifolds called **leaves** that are tangent to l_x at x. This singular foliation Σ_ξ is called the **characteristic foliation** of Σ.

Example 1.30 *Consider the unit sphere S^2 in $(\mathbb{R}^4, \xi_3)$ described in Example 1.5. Only singular points are $(0,0,\pm 1)$ and away from singular points the characteristic foliation is defined by the vector field $(y - xz)\frac{\partial}{\partial x} - (x + yz)\frac{\partial}{\partial y} + (x^2 + y^2)\frac{\partial}{\partial z}$*

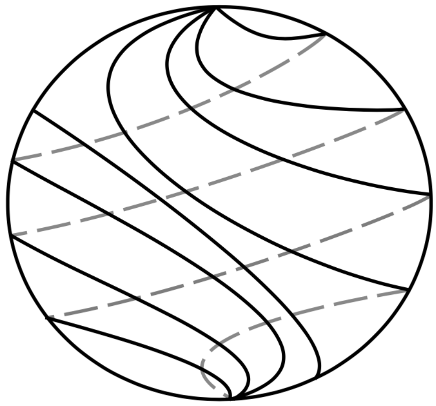

Figure 1.4: The characteristic foliation on the sphere, [12].

This foliation can be visualized as spirals connecting north and south poles, as in Figure 1.4

Example 1.31 *Consider the disk D of radius π on (r, θ) plane in $(\mathbb{R}^3, \xi_{ot})$ described in Example 1.7. Then the center of the disk and every point at the boundary is singular since contact planes are horizontal. The leaves are segments joining the center to the boundary. If we push the interior of D slightly in the z direction, the contact planes no longer coincide with the tangent planes on the boundary and the boundary becomes a leaf of the foliation. In this case, the characteristic foliation becomes a singular point at the center, boundary as a leaf, and spirals from the center approaching the boundary. See figure 1.5*

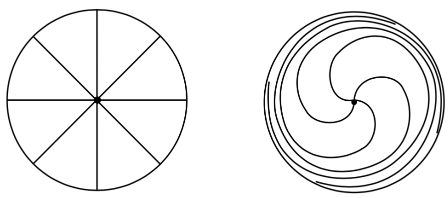

Figure 1.5: Overtwisted disks, [12].

Either of these disks with its characteristic foliation is called an ***overtwisted disk****.*

In general, any surface may be perturbed by a C^∞-small isotopy so that the singularities of the characteristic foliation are 'generic', that is one of the types described in Figure 1.6.

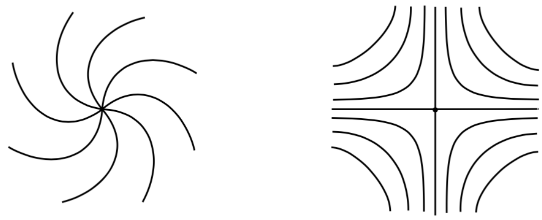

Figure 1.6: Generic singularities, [12]

The characteristic foliation on a surface determines the contact structure in a neighborhood of the surface. More precisely, we have

Theorem 1.32 *Let (M_i, ξ_i) be contact 3-manifolds and Σ_i be a closed surfaces in M_i for $i = 0, 1$. If there is a diffeomorphism $f : \Sigma_0 \to \Sigma_1$ that preserves the characteristic foliation, that is $f((\Sigma_0)_{\xi_0}) = (\Sigma_1)_{\xi_1}$, then f may be extended to a contactomorphism in some neighborhood of $(\Sigma_0)_{\xi_0}$. Moreover, if f was already defined on a neighborhood of Σ_0, f is isotopic to a contactomorphism in some (possibly) smaller neighborhood.*

This theorem is proved by Moser's trick. See Theorem 2.39 in [21] for further details.

1.4 Knots in Contact 3-Manifolds

We will consider 1-dimensional submanifolds, namely knots in contact 3-manifolds (M, ξ).

Definition 1.33 *Let (M, ξ) be a contact 3-manifold. Then, a knot $\gamma : S^1 \to M$ is called a **Legendrian knot** if $\gamma(S^1)$ is tangent to ξ at every point of $\gamma(S^1) \subset M$. That is, if $\gamma(S^1)$ is a Legendarian submanifold of M.*

*A knot $\gamma : S^1 \to M$ is called a **transverse knot** if $\gamma(S^1)$ is transverse to ξ at every point of $\gamma(S^1) \subset M$.*

If $\xi = \ker \alpha$, we can restate the conditions equivalently as $\alpha(\dot{\gamma}) = 0$ for a Legendrian knot, and $\alpha(\dot{\gamma}) \neq 0$ for a transverse knot, where $\dot{\gamma}$ is the tangent vector to γ. We call γ **positively transverse** (resp. **negatively transverse**) if $\alpha(\dot{\gamma}) > 0$ (resp. $\alpha(\dot{\gamma}) < 0$).

Usually, we will not distinguish between a knot and its image. Also, we will mostly discuss knots in $(\mathbb{R}^3, \ker(dz + xdy))$. The **front projection** of a knot is the projection of the knot to yz-plane.

Suppose γ is a Legendrian knot in $(\mathbb{R}^3, \xi = \ker \alpha_{st})$ where $\alpha = dz + xdy$. Then we have the condition $x = -\frac{dz}{dy}$ by the condition $\alpha(\dot{\gamma}) = 0$ and γ can be determined completely from its front projection by integrating. Moreover, the front projection of a Legendrian knot satisfies:

- There are no vertical tangencies, instead there are cusps when the tangency changes directions in the y-direction, and
- at all the crossings, the strand of γ with the smaller slope lies in front of the strand with the larger slope.

That is, the front projection looks like Figure 1.7.

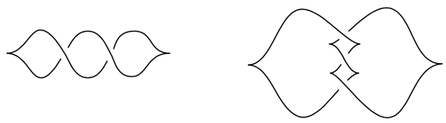

Figure 1.7: Front projections of some Legendrian knots, [12].

Conversely, a curve on the yz-plane that satisfy these conditions can be lifted to a Legendrian knot. Thus, approximating the front projection of any knot in $\mathbb{R}^n$ by "zig-zags" as in Figure 1.8, we get

Proposition 1.34 *Let γ be a knot in (R^3, ξ_{st}). Then γ can be C^0-approximated by a Legendrian knot isotopic to γ.*

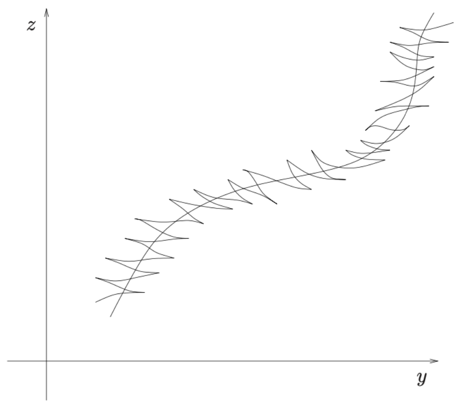

Figure 1.8: Approximation by a Legendrian knot, [22].

By similar arguments, and approximating by "loops" on the xy-plane, instead of "zig-zags" on yz-plane, we get

Proposition 1.35 *Let γ be a knot in (R^3, ξ_{st}). Then γ can be C^0-approximated by a transverse knot isotopic to γ.*

For more details, see [22].

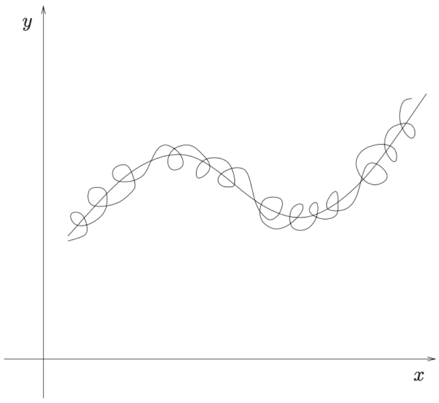

Figure 1.9: Approximation by a transverse knot, [22].

1.4.1 The Classical Invariants of Knots in Contact 3-Manifolds

Now we will define some invariants of Legendrian and transverse knots in contact 3-manifolds.

First, let γ be a Legendrian knot and assume it bounds a surface Σ [1]. Let v be a vector field along γ transverse to ξ and let γ' be the knot obtained by pushing γ along v. Now, the **Thurston-Bennequin invariant** $tb(\gamma)$ of γ is the linking number of γ and γ', that is the signed intersection of γ' with Σ.

For the second invariant of a Legendrian knot, let γ be an oriented Legendrian knot. Trivialize ξ_{st} along γ. the **rotation number** $r(\gamma)$ of γ is the winding number of γ with respect to this trivialization.

Lastly, we will define an invariant of a transverse knot. let γ be a transverse knot. Trivialize ξ_{st} along γ by a vector field u along γ, and γ' be the knot obtained by pushing γ along u. Then, the **self-linking number** $sl(\gamma)$ of γ is the linking number of γ and γ'.

These invariants can be computed from the front projections of a knot and they are useful to classify such knots to some extent (see [22]).

Moreover, these invariants describe some properties of the underlying contact structure, as one can see while discussing convex surfaces in contact manifolds.

[1] Let γ be a nullhomologous knot in an oriented 3–manifold. An embedded connected, compact, orientable surface Σ with boundary $\partial\Sigma = \gamma$ is called a **Seifert surface** for γ.

1.5 Contact Structures on 3-Manifolds

In this section, we will state some results about the existence and classification of contact structures on 3-manifolds, and we will introduce the tight vs. overtwisted dichotomy in contact 3-manifolds. Unless otherwise stated, we will consider closed, orientable 3-manifolds and positive contact structures, that is $\alpha \wedge d\alpha > 0$.

1.5.1 Existence of Contact Structures on 3-Manifolds

The main existence result is due to Martinet [35] and Lutz [34]:

Theorem 1.36 *Every closed, orientable 3-manifold admits a contact structure in each homotopy class of 2-plane fields.*

There are several proofs of the existence part of this theorem. One due to Thurston and Winkelnkemper [45], uses open book decompositions of 3-manifolds. We will discuss that proof more in-depth in Chapter 2.

The original proof of existence by Martinet uses a surgery description of 3-manifolds due to Lickorish [33] and Wallace [46]: every closed, orientable 3-manifold can be obtained from S^3 by Dehn surgery, that is, by removing a solid torus and gluing it back by a diffeomorphism of its boundary. One proves that given a contact structure near the boundary of a solid torus, one can extend it to the whole solid torus by finding a suitable contact form $\alpha = h_1(r)d\theta + h_1(r)d\varphi$ modeled after the one described in Example 1.6. Therefore, starting with S^3 and a contact structure on it, we can perform Dehn surgery along transverse knots to get the desired manifolds and endow the resulting 3-manifolds with contact structures.

By using the so-called Lutz twists, a topologically trivial Dehn surgery, we may produce a new contact structure on a given manifold, possibly not homotopic (as plane fields) to the one we started with. We show that by Lutz twists, we obtain contact structures in each homotopy class of plane fields.

The full details of this proof can be found in [22, Chapter 4].

1.5.2 Tightness and Overtwistedness

In this section, we will define tight and overtwisted contact 3-manifolds. Tightness versus overtwistedness is a fundamental dichotomy of contact manifolds. While overtwisted structures are "relatively easy to find" in the sense we will discuss in the next subsection, not all 3-manifolds can have a tight contact structure (see [17]).

Definition 1.37 *A contact 3-manifold (M, ξ) is* ***overtwisted*** *if it contains an overwisted disk. That is, if there is an embedded disk in M whose characteristic* *foliation is homeomorphic to the characteristic foliation of an overtwisted disk (the characteristic foliations described in Example 1.31). Otherwise, it is* ***tight****.*

Figure 1.10 shows the disk of radius π that is the overtwisted disk in the standard overtwisted structure ξ_{ot} on $\mathbb{R}^3$.

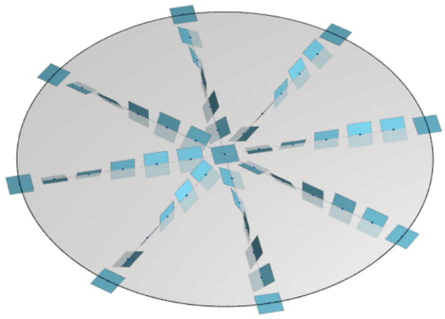

Figure 1.10: Overtwisted disk in $(\mathbb{R}^3, \xi_o t)$, [37].

The existence of a tight contact structure gives information about the properties of the contact manifold. We will give some conditions and consequences of tightness related to convexity and symplectic fillings in the following chapters.

In the meantime, we will give some examples of tight and overtwisted contact structures.

Example 1.38 *By definition, the standard overtwisted structure defined in Example 1.7 on $\mathbb{R}^3$ is overtwisted.*

Example 1.39 *The standard contact structure on S^3 defined in Example 1.8 is tight. In fact, Eliashberg proved that the standard structure on S^3 is the unique tight structure on S^3 up to isotopy.*

Example 1.40 *For each n, the contact structure*

$$(T^3, \xi_n = \ker(\sin(2\pi nz)dx + \cos(2\pi nz)dy))$$

defined in Example 1.9 is a distinct tight structure.

1.5.3 Classification of Overtwisted Contact Structures

Let $(M, \xi = \ker \alpha)$ be a contact manifold. In fact, by a Lutz twist in M, one can always obtain an overtwisted contact structure ξ' homotopic to ξ. See [22] for details. Therefore, every homotopy class of plane fields M contains an overtwisted contact structure.

Eliashberg gave a much stronger classification of overtwisted contact structures in [8] that reads as follows:

Let $Cont^{ot}(M, D)$ be the set of cooriented, positive, overtwisted contact structures ξ on M such that the characteristic foliations on an overtwisted disk D agree. Also let $Distr(M, D)$ be the set of cooriented plane fields that are tangent (with matching orientations) to D at the center. Consider both spaces with C^∞ topology.

Theorem 1.41 ([8]) *The inclusion of*

$$i : Cont^{ot}(M, D) \to Distr(M, D)$$

is a weak homotopy equivalence.

Remark 1.42 *The spaces $Cont^{ot}(M, D)$, $Distr(M, D)$ have homotopy types of CW-complexes. Thus, the inclusion is a homotopy equivalence by Whitehead theorem.*

By considering path components of $Cont^{ot}(M, D)$ and $Distr(M, D)$, as a corollary we have:

Corollary 1.43 ([8]) *Two overtwisted contact structures on a contact manifold $(M, \xi = \ker \alpha)$ are isotopic if and only if they are homotopic as oriented 2-plane fields. Moreover, every homotopy class of oriented 2-plane fields on M contains an overtwisted contact structure.*

The existence and classification of overtwisted contact structures in higher dimensions is due to Eliashberg, Borman, and Murphy [3].

On the other hand, there is no complete classification of tight contact structures on contact 3-manifolds. However, there are classifications of tight structures on certain families of 3-manifolds (for example, see [30], [31]).

Chapter 2

Open Book Decompositions

In this chapter, we introduce open book decompositions of contact manifolds. The topological notion of open book decompositions of manifolds gives us tools to investigate contact manifolds through the concept of an open book supporting a contact structure.

In Section 2.1, we introduce open book decompositions of manifolds and ways to obtain other open book decompositions from given ones. Namely, Murasugi sums and stabilizations. In Section 2.2, we consider the relationship between contact 3-manifolds and open book decomposition. We define open book decompositions supporting a contact structure and prove the result of Thurston and Winkelnkemper [45] about the existence of contact structures adapted to open book decompositions of 3-manifolds. Then, we sketch the proofs of the results of Giroux [27] about the existence of open book decompositions supporting a given contact structure. When combined, these results give the celebrated Giroux correspondence between open book decompositions and contact structures on 3-manifolds. Lastly, we mention the analogous higher dimensional result due to Giroux and Mohsen [27].

2.1 Open Book Decompositions of 3-Manifolds

In this section, we will consider the topological notion of "open books" and related constructions that we will use to investigate manifolds.

Definition 2.1 *An* ***open book decomposition*** *of a closed, oriented smooth n-manifold M is a pair (B, π) where:*

- *B is a codimension 2 submanifold of M,*
- *$\pi : M - B \to S^1$ is a smooth fibration, such that on a tubular neighborhood $B \times D^2 \subseteq M$ of $B = B \times \{0\}$, the map π restricts to angular coordinates on D^2*

The closures of fibers of π are called ***pages****. The submanifold B is called the* ***binding****.*

We will consider closed, oriented 3-manifolds unless otherwise stated. In this case, B is an oriented link, and the page $\overline{\pi^{-1}(\theta)}$ is a compact oriented surface $\Sigma_\theta = \Sigma$ in M with boundary $\partial\Sigma_\theta = B$ for any $\theta \in S^1$. Going forward, we will interchangeably use the closures of fibers, and the fibers themselves for the *pages*.

As a related notion, we can define abstract open books using the information about the page.

Definition 2.2 *An* ***abstract open book*** *is a pair (Σ, φ) where:*

- *Σ is a manifold with nonempty boundary,*
- *$\varphi : \Sigma \to \Sigma$ is a diffeomorphism that is identity in a neighborhood of the boundary $\partial\Sigma$*

The map φ is called the ***monodromy****.*

Given an abstract open book (Σ, φ) where Σ is a 2-manifold, we define a 3-manifold M_φ as follows:

$$\begin{aligned} M_\varphi &= \Sigma_\varphi \cup_{id_{\partial\Sigma}} (\partial\Sigma \times D^2) \\ &\cong \Sigma_\varphi \cup_\psi (\coprod_{|\partial\Sigma|} S^1 \times D^2) \end{aligned}$$

where $|\partial\Sigma|$ is the number of components of $\partial\Sigma$, and

$$\Sigma_\varphi = \Sigma \times [0,1]/(x,1) \sim (\varphi(x), 0)$$

That is, Σ_φ is the mapping torus of φ. The gluing map identifies the boundary of the mapping torus $\partial\Sigma_\varphi = \partial\Sigma \times S^1$ with the boundaries of the solid tori $\partial(\partial\Sigma \times D^2) = \partial\Sigma \times \partial D^2 \cong \coprod_{|\partial\Sigma|} S^1 \times D^2$. We denote the union $\coprod_{|\partial\Sigma|} S^1 \times \{0\}$ of the cores of solid tori $\coprod_{|\partial\Sigma|} S^1 \times D^2$ by B_φ.

Also, define a map $\pi_\varphi : M_\varphi - B_\varphi \to S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ by

$$\pi_\varphi([x, \theta]) = [\theta]$$

for $[x, \theta] \in \Sigma_\varphi$ and

$$\pi_\varphi(\theta, e^{i\rho}) = [\rho]$$

for $(\theta, e^{i\rho}) \in S^1 \times D^2 \subset \partial\Sigma \times D^2$.

Two abstract open book decompositions (Σ_1, φ_1) and (Σ_2, φ_2) are called equivalent if there is a diffeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h \circ \varphi_1 = \varphi_2 \circ h$.

We will state the following lemma that can be deduced from the constructions above, which gives the relationship between open book decompositions and abstract open books:

Lemma 2.3 *Using the notation above, we have the following facts about open books and abstract open books:*

1. *Equivalent abstract open books give diffeomorphic 3-manifolds.*
2. *An abstract open book determines M_φ and an open book (B_φ, π_φ) up to diffeomorphism.*
3. *Conversely, an open book decomposition (B, π) of M gives an abstract open book $(\Sigma_\pi, \varphi_\pi)$ such that $(M_{\varphi_\pi}, B_{\varphi_\pi})$ is diffeomorphic to (M, B).*

Because of this lemma, we usually will not differentiate between a (non-abstract) open book decompositions of a 3-manifold M and the corresponding abstract open books. However, observe that when discussing open books we can discuss the binding and pages up to *isotopy* in M, whereas when discussing abstract open books we can only discuss them up to *diffeomorphism*.

Due to the following result of Alexander [2], open book decompositions give us a general way to investigate 3-manifolds.

Theorem 2.4 *Every closed, oriented 3-manifold has an open book decomposition.*

There are several proofs of this result, some of which can be found in [13], [22], [44].

Remark 2.5 *The proof given in [44] also shows that the decomposition can be produced so that pages are genus 0 surfaces, i.e. planar surfaces. A brief discussion on these open books can be found in Section 2.2.2.*

Here are some examples of open book decomposition of $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$ with coordinates $(z_1, z_2) = (r_1 e^{\theta_1}, r_2 e^{\theta_2}) = (x_1 + iy_1, x_2 + iy_2)$ used interchangeably:

Example 2.6 *Let $U = \{z_1 = 0\} \subset S^3$ be the unknot in S^3. We have the fibration $\pi_U : S^3 - U \to S_1$ given by $\pi_U(z_1, z_2) = z_1/|z_1|$ or in polar coordinates by $\pi_U(r_1 e^{\theta_1}, r_2 e^{\theta_2}) = \theta_1$. We have the pages as*

$$\overline{\pi_U^{-1}(\theta)} = \{|z_2| < 1, z_1 = \sqrt{1 - |z_2|^2} e^{i\theta}\} \cong D^2$$

This fibration can be visualized as in Figure 2.1.

Example 2.7 *Define*

$$H^+ = \{(z1, z2) \in S^3 : z_1 z_2 = 0\} \subset S^3$$

and

$$H^- = \{(z1, z2) \in S^3 : z_1 \overline{z_2} = 0\} \subset S^3$$

These subsets are called ***positive and negative Hopf links*** *respectively.*

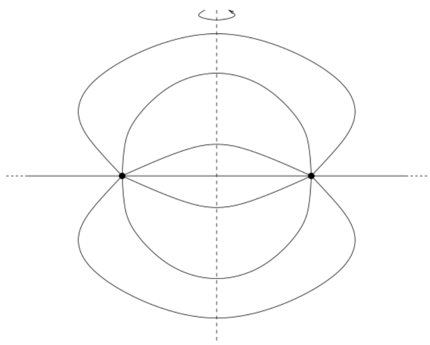

Figure 2.1: The 3–sphere with the open book decomposition (U, π_U), [22].

We have the fibrations $\pi_\pm : S^3 - H^\pm \to S_1$ *given by* $\pi_+(z_1, z_2) = z_1 z_2/|z_1 z_2|$ *and* $\pi_-(z_1, z_2) = z_1\overline{z_2}/|z_1\overline{z_2}|$ *or in polar coordinates by* $\pi_\pm(r_1 e^{\theta_1}, r_2 e^{\theta_2}) = \theta_1 \pm \theta_2$. *We have the pages for* H^+ *as*

$$\overline{\pi^{-1}(\theta)} = \{(\sqrt{1-r_1^2}e^{i(\theta-\rho)}, r_1 e^{i\rho})\} \cong [0,1] \times S^1$$

and similarly for H^- *as*

$$\overline{\pi^{-1}(\theta)} = \{(\sqrt{1-r_1^2}e^{i(\theta+\rho)}, r_1 e^{i\rho})\} \cong [0,1] \times S^1$$

This fibration gives another open book decomposition of S^3 *with binding* $H^\pm$ *and pages as in Figure 2.2.*

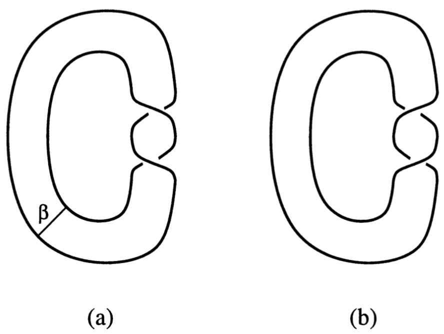

Figure 2.2: The pages in the open book decomposition of S^3 with binding $H^\pm$, [42].

Remark 2.8 *The Examples 2.6 and 2.7 can be generalized using the notion of a "Milnor fibration" to give other open book decompositions of* S^3. *See [13] and [40] for details.*

Now, we will define ways to obtain new open books from given open books:

Definition 2.9 *Given two abstract open books* (Σ_i, φ_i)*,* $i = 0, 1$*, let* c_i *be an arc properly embedded in* Σ_i *and* R_i *a rectangular neighborhood of* c_i*,* $R_i = c_i \times [-1, 1]$*. The* ***Murasugi sum*** *of* (Σ_0, φ_0) *and* (Σ_1, φ_1) *is the open book* $(\Sigma_0, \varphi_0) \star (\Sigma_1, \varphi_1)$ *with page* $\Sigma_0 \star \Sigma_1 = \Sigma_0 \cup_{R1=R2} \Sigma_1$*, where* R_0 *and* R_1 *are identified so that* $c_0 \times \{-1, 1\} = (\partial c_1) \times [-1, 1]$*, and the monodromy* $\varphi_0 \circ \varphi_1$*.*

One can view the Murasugi sum as in Figure 2.3

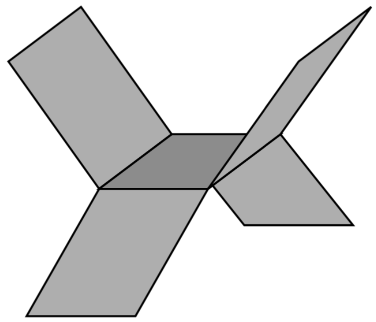

Figure 2.3: Murasugi sum of the pages of an open book.

The following result due to Gabai [19] relates connected sums and Murasugi sums. See [13] for a proof sketch.

Proposition 2.10 $M_{(\Sigma_0,\varphi_0)\star(\Sigma_1,\varphi_1)}$ *is diffeomorphic to* $M_{(\Sigma_0,\varphi_0)} \# M_{(\Sigma_1,\varphi_1)}$

To define stabilizations, first we will define Dehn twists which we previously mentioned in Section 1.5.1. Given an embedded closed curve c in an oriented surface Σ let $N = c \times [0, 1]$ be a neighborhood (oriented as a product consistently with Σ) of the curve. We then define the **right-handed Dehn twist along** c, denoted D_c , to be the diffeomorphism of Σ that is the identity on $\Sigma - N$ and is given by $(\theta, t) \to (\theta + 2\pi t, t)$ on N , where θ is the coordinate on $c = S^1$ and t is the coordinate on $[0, 1]$. The **left-handed Dehn twist along** c is defined to be D_c^{-1}.

Definition 2.11 *A* ***positive (resp. negative) stabilization*** *of an abstract open book* (Σ, φ) *is the open book*

1. *with page* $\Sigma' = \Sigma \cup$ *1-handle, and*
2. *monodromy* $\varphi' = \varphi \circ D$ *where* D *is a right- (resp. left-) handed Dehn twist along a curve* c *in* Σ' *that intersects the co-core of the 1-handle exactly one time.*

We denote this stabilization by $S_{(a,\pm)}(\Sigma, \varphi)$ *where* $a = c \cap \Sigma$ *and* $\pm$ *refers to the positivity or negativity of the stabilization. (We omit the curve* a *if it is unimportant in a given context.)*

Stabilizations of an open book can be visualized as in Figure 2.4.

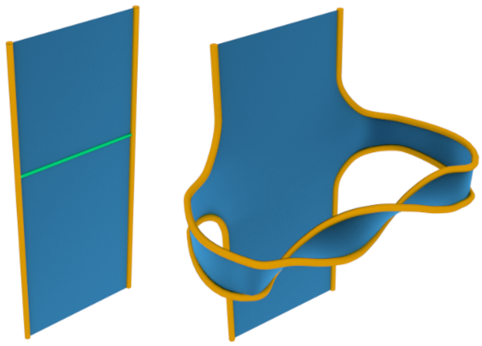

Figure 2.4: The stabilization of an open book, [36].

The stabilization operations are related to Murasugi sums by the following result:

Proposition 2.12 *The positive (resp. negative) stabilization of an open book is the Murasugi sum with* (H^+, π_+) *(resp.* (H^-, π_-)*) defined in Example 2.7. That is,*

$$S_\pm(\Sigma, \varphi) = (\Sigma, \varphi) \star (H^\pm, \pi_\pm)$$

Using Propositions 2.10 and, 2.12, we get:

Proposition 2.13 $M_{S_\pm(\Sigma,\varphi)} \cong M_{(\Sigma,\varphi)}$

Thus, stabilizations give us a way to obtain new open book decompositions of a 3-manifold M from a given open book decomposition.

2.2 Contact Structures and Open Book Decompositions of 3-Manifolds

Given an open book decomposition (B, π) of a 3-manifold M, we can define contact structures that are "compatible with" the open book decomposition (B, π) as follows:

Definition 2.14 *A contact structure* ξ *on M is* ***supported by (or adapted to) an open book decomposition*** (B, π) *of M if* ξ *can be isotoped through contact structures so that there is a contact 1-form* α *for* ξ *such that*

1. $d\alpha$ *is a positive area form on each page* Σ_θ *of the open book, and*
2. $\alpha > 0$ *on B.*

Here are some examples of contact structures supported by open books:

Example 2.15 *Consider the standard contact structure on the sphere:*

$$(S^3, \xi = \ker \alpha)$$

where $\alpha = (r_1^2 d\theta_1 + r_2^2 d\theta_2)$

The open book decomposition from Example 2.6 supports ξ*:*

On the binding $U = \{r_1 = 0\}$*, the tangent is* $\frac{\partial}{\partial\theta_2}$*, so* $\alpha(\frac{\partial}{\partial\theta_2}) > 0$*.*

Moreover, we can parametrize the page $\pi_U^{-1}(\theta_1)$ *by* D^2 *with coordinates* (r, θ) *as*

$$f(r,\theta) = (\sqrt{1-r^2}, \theta_1, r, \theta)$$

Thus, we have $df^*(\alpha) = 2rdrd\theta$ *which is a positive volume form on* D^2*, so* $d\alpha$ *is a positive volume form on pages*

Example 2.16 *Again, consider the standard contact structure on the sphere. The open book decomposition* H^+ *from Example 2.7 supports* ξ*:*

On the binding H^+*, the tangent is* $\frac{\partial}{\partial\theta_1}$ *on one component and* $\frac{\partial}{\partial\theta_2}$ *on the other, so* $\alpha > 0$ *on* H^+*.*

Moreover, we can parametrize the page $\pi_+^{-1}(\rho)$ *by* $(0,1) \times S^1$ *with coordinates* (s, θ) *as*

$$f(s,\theta) = (\sqrt{1-s^2}, \theta - \rho, s, \rho)$$

Thus, we have $df^*(\alpha) = 4sdsd\theta$ *which is a positive volume form on* $(0,1) \times S^1$*, so* $d\alpha$ *is a positive volume form on pages.*

On the other hand, for the open book decomposition (H^-, π_-)*,* $d\alpha$ *is a negative volume form. Thus* $(S^3, \xi = \ker \alpha)$ *is not supported by* (H^-, π_-)*.*

Now, we prove the theorem of Thurston and Winkelnkemper [45] that we mentioned in Section 1.5, concerning the existence of contact structures adapted to open books.

Theorem 2.17 *Every open book decomposition* (Σ, φ) *supports a contact structure* $\xi_\varphi = \ker \alpha_\varphi$ *on* M_φ*.*

Proof We will first define a contact form α_K on Σ_φ. For ease of notation, assume $\partial\Sigma$ is connected, and hence $\partial\Sigma \cong S^1$, with coordinates $\theta_1 \in S^1$. Let t be the collar parameter for a collar neighbourhood $[0,1] \times S^1$ of $\partial\Sigma$ in Σ.

Consider the set S of 1-forms λ on Σ such that

1. $\lambda = (1-t)d\theta_1$ near $\partial\Sigma$,
2. $d\lambda$ is a volume form on Σ.

It can be checked by computation that S is a convex set and if $\lambda \in S$, then $\varphi^*\lambda$ is also in S. Therefore, if S is nonempty, then we can define a 1-form

$$\lambda_{[0,1]\times\Sigma} = (1-s)\lambda + s\varphi^*\lambda$$

on $[0,1]\times\Sigma$ where we denote the first component by s, such that its restriction to each $\Sigma \cong \{s\}\times\Sigma$ is again in S. Then, for sufficiently large K,

$$\alpha_K = \lambda_{[0,1]\times\Sigma} + Kdt$$

becomes a contact form, and it descends to a contact form on Σ_φ.

Now to show that S is nonempty, let λ_1 be any 1-form on Σ satisfying the conditions 1 and 2 near $\partial\Sigma$. Also, let ω be a volume form on Σ with total volume 2π. Then,

$$\int_\Sigma \omega - d\lambda_1 = \int_\Sigma \omega - \int_\Sigma d\lambda_1 = 2\pi - \int_{S^1} \lambda_1 = 2\pi - 2\pi = 0$$

Hence, there exists a 1-form λ which vanishes near the boundary, such that $d\lambda = \omega - d\lambda_1$. Then, $\lambda + \lambda_1$ is a form in S. Therefore, S is nonempty and we can indeed define a contact form α_K on Σ_φ.

Now, we will extend α_K to $\partial\Sigma \times D^2$. Use coordinates (r,θ_2) on $D^2 \subset \mathbb{R}^2$.

The manifold M_φ can be represented as $M_\varphi = \Sigma_\varphi \cup_\psi (\partial\Sigma \times D_2^2)$, where D_R^2 is the disk of radius R. With this setup, we can identify the gluing map ψ with

$$\psi(t,\theta_1,\theta_2) = (\theta_1, r = 1+t, \theta_2)$$

where $(t,\theta_1,\theta_2) \in [0,1]\times\partial\Sigma\times S^1$ is in the mapping torus of a collar neighborhood $[0,1]\times\partial\Sigma$, and $(\theta_1, r = 1+t, \theta_2) \in \partial\Sigma \times (D_2^2 - D^2)$ where we identify $\partial\Sigma \times (D_2^2 - D^2)$ with $\partial\Sigma \times [1,2] \times S^1$.

With this map defined near the boundary, we can extend α_K near the boundary of $\partial\Sigma \times D_2^2$ as $rd\theta_1 + Kd\theta_2$. On the other hand, near the core $\partial\Sigma \times \{0\}$ of $\partial\Sigma \times D_2^2$, the form $d\theta_1 + r^2 d\theta_2$ defines a contact form.

Define $\alpha_\varphi = h_1(r)d\theta_1 + h_2(r)d\theta_2$ on $S^1 \times D^2$ with coordinates (θ_1, r, θ_2). If the functions h_1, h_2 satisfy

1. $h_1 = 1$ and $h_2 = r^2$ near the core $\partial\Sigma \times \{0\}$ of $\partial\Sigma \times D_2^2$,
2. $h_1(r) = r$ and $h_2(r) = K$ for $r \in [1,2]$,
3. $(h_1(r), h_2(r))$ is never parallel to $(h_1'(r), h_2'(r))$ for $r \neq 0$,

then α_φ defines a contact form on $S^1 \times D^2$ that agrees with α_K on the boundary. Such a pair of functions can be found. The figure 2.5 describes such $(h_1(r), h_2(r))$.

Therefore, we get a contact form α_φ defined on M_φ that defines a contact structure supported by the open book decomposition (Σ, φ). □

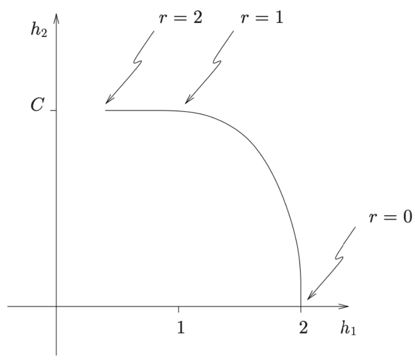

Figure 2.5: Extension of the contact structure to the solid torus

Combined with Theorem 2.4, Theorem 2.17 gives a proof of the fact that every closed orientable 3-manifold admits a contact structure.

In fact, due to the following result of Giroux [27], the open book decomposition defines a unique contact structure up to isotopy.

Theorem 2.18 *Two contact structures ξ_i for $i = 1, 2$ on M, supported by the same open book (B, π) are isotopic.*

Proof Let $\partial\Sigma \times D^2_2$ be a neighborhood of the binding with coordinates (θ_1, r, θ_2) on each component, as in the previous proof. Since contact structures $\xi_i = \ker \alpha_i$ are supported by the contact structure (B, π), we have $\alpha_i(\frac{\partial}{\partial\theta_1}) > 0$. Let h be a function such that $h(0) = 0$, $h' \geq 0$, $h(r) = r^2$ near $r = 0$, and $h = 1$ outside $\partial\Sigma \times D^2_1$.

Observe that with such a function h, we have:

- $\alpha_i \wedge d\alpha_i > 0$ by the contact condition,
- $h(r)d\theta_2 \wedge d\alpha_i \geq 0$ since $d\alpha_i$ is an area form on pages,
- $h'(r)\alpha_i \wedge dr \wedge d\theta_2 \geq 0$ since $dr \wedge d\theta_2$ vanishes on $\frac{\partial}{\partial\theta_1}$ while $\alpha_i(\frac{\partial}{\partial\theta_1}) > 0$.

Therefore if we define

$$\alpha_{i,R} = \alpha_i + Rh(r)d\theta_2$$

for $R \in \mathbb{R}_{>0}$, we get

$$\alpha_{i,R} \wedge d\alpha_{i,R} = \alpha_i \wedge d\alpha_i + Rh(r)d\theta_2 \wedge d\alpha_i + Rh'(r)\alpha_i \wedge dr \wedge d\theta_2 > 0$$

That is, $\alpha_{i,R}$ is a contact form for any R and $i = 0, 1$.

Then, for sufficiently large R, the forms $(1-t)\alpha_{0,R} + t\alpha_{1,R}$ for $t \in [0, 1]$ are contact forms, and define the isotopy between ξ_0 and ξ_1. □

By this theorem, we may denote any contact structure that is supported by the open book (Σ, φ) or (B, π) by $\xi_{(\Sigma,\varphi)}$ or $\xi_{(B,\pi)}$.

From the proofs of Proposition 2.10 and Theorem 2.17, it also follows that:

Proposition 2.19 *We have* $\xi_{(\Sigma_0,\varphi_0)\star(\Sigma_1,\varphi_1)} = \xi_{(\Sigma_0,\varphi_0)}\#\xi_{(\Sigma_1,\varphi_1)}$

Note that, this equivalence is up to isotopy by the previous theorem.

Then, because of the Example 2.16, we have:

Theorem 2.20 *Let* (Σ, φ) *be an open book and* $a \subset \Sigma$ *any properly embedded arc. Then*

$$M_{S(\pm,a)(\Sigma,\varphi)} \textit{ is diffeomorphic to } M_{(\Sigma,\varphi)}$$

and,

$$\xi_{S(+,a)(\Sigma,\varphi)} \textit{ is isotopic to } \xi_{(\Sigma,\varphi)}$$

On the other hand, again by Example 2.16, $\xi_{S(-,a)(\Sigma,\varphi)}$ is *not* isotopic to $\xi_{(\Sigma,\varphi)}$.

2.2.1 Giroux Correspondence

In Theorem 2.17, we proved that every open book decomposition (B, π) of a compact, oriented 3-manifold M supports a contact structure $\xi = \ker\alpha$. In this section, we sketch the proof of the converse.

Theorem 2.21 (Giroux [27]) *Every oriented contact structure* ξ *on a closed oriented 3–manifold* M *is supported by an open book decomposition.*

The proof relies on the theory of convex surfaces. For an overview of convex surfaces and references, see the appendix.

For the discussion of the proof, we start with a definition and a lemma about cell decompositions of contact 3-manifolds:

Definition 2.22 *A* ***contact cell decomposition*** *of a contact 3–manifold*

$$(M, \xi = \ker\alpha)$$

is a finite CW-decomposition of M *such that*

1. *the 1-skeleton is a Legendrian graph, that is every 1-cell is a Legendrian curve in* M*,*
2. *each 2-cell* D *satisfies* $tb(\partial D) = -1$*, i.e., the contact planes twist negatively once along* ∂D *with respect to the surface* D*, and*
3. ξ *is tight when restricted to each* $3-$ *cell*

Lemma 2.23 *Every closed contact 3-manifold* (M, ξ) *has a contact cell decomposition.*

Proof We can cover M by finitely many Darboux balls such that 1-skeleton can be isotoped to a Legendrian graph and every 3-cell is contained in a Darboux ball. Since the standard contact structure on $\mathbb{R}^3$ is tight, restrictions to 3-cells are tight as desired. Lastly, we can perturb any 2-cell to assume it is convex and use the Legendrian Realization Principle (see the appendix) to get the 2-skeleton with desired properties by subdividing with Legendrian curves if necessary. □

Now we define the ribbon of the 1-skeleton, which will be used to construct the open book.

Definition 2.24 *Let G be the 1-skeleton of a contact cell decomposition of* (M, ξ)*. The* ***ribbon*** *R (or* R_G*) of G is a compactly embedded surface in M such that*

- *R retracts onto G,*
- $T_pR = \xi_P$ *for all* $p \in G$*, and,*
- $T_pR \neq \xi_P$ *for* $p \in R - G$*.*

The 1-skeleton G of a contact cell decomposition of (M, ξ) has a ribbon R. Let $B = \partial R$. By the definition of a ribbon, B is a transverse link in M.

To prove Theorem 2.21, it is enough to show that B is the binding of an open book decomposition that supports the contact structure ξ.

Since B is a transverse link, each component of B has a neighborhood N(B) contactomorphic to $(\mathbb{R}^3, ker(dz + r^2 d\theta))/(r, \theta, z) \sim (r, \theta, z + 1)$.

We define:

- $X(B) = \overline{M - N(B)}$
- $R_X = R \cap X(B)$
- $N(R) \cong R_X \times [-\delta, \delta]$ a neighbourhood of R_X such that $\partial R_X \times \{pt\}$ corresponds to a line in $N(B)$ with constant θ value.
- $X(R) = X(B) - N(R)$

Idea is to define the open book with pages R. For this, one shows that $X(R)$ is diffeomorphic to $R_X \times [-1, 1]$. Then, $X(B)$ can be obtained by identifying $R_X \times \{\pm 1\}$ with $R_X \times \{\pm\delta\}$. This gives a fibration over S^1 with fiber R_X, and this fibration can be extended to a fibration of $N(B) - B$, so that the boundary of fibers is B and the closure of the fibers is R.

To prove this, we write $\partial X(R) = A \cup F$ with $A = \partial X(R) \cap N(B)$ (that is, the "outside part" of $N(B)$, which constitutes an annulus for each component of B) and $F = F^+ \cup F^-$ where $F^\pm$ is identified with $R_X \times \{\pm\delta\}$. The surface $\partial X(R)$ is a convex surface after appropriate edge roundings with dividing set $\Gamma_{\partial X(R)}$ equal to cores of annuli in A, and such that $F^\pm \subset (\partial X(R))_\pm$

Let $D_1, \ldots, D_k$ be the 2-cells in the contact cell decomposition of M. Since the ribbon R twists with the contact structure and $tb(\partial D_i) = -1$, the curve B intersects D_i exactly twice for all i. Let $D_i' = D_i \cap X(R)$, then D_i' intersects the boundary region A in exactly two arcs that connect different boundary components of A.

Moreover, $X(R)$ cut along D_i' is a disjoint union of balls contained in 3-cells which are diffeomorphic to $D^2 \times [-1, 1]$, where $\partial D^2 \times [-1, 1]$ correspond to $A \subset X(R)$ and $D^2 \times \{-1, 1\}$ correspond to $F \subset X(R)$. Then, we can glue back these balls to obtain $X(R)$ showing it is diffeomorphic to $R_X \times [-1, 1]$ as desired.

Thus B is indeed the binding of an open book decomposition of M with pages R. For more in-depth description of how cutting and gluing these spaces work, see [32].

Now it remains to show that this open book supports $\xi = \ker \alpha$. For this, we use the following lemma:

Lemma 2.25 *Let (B, π) be an open book decomposition of $(M, \xi = \ker \alpha)$. If there exists a Reeb vector field X on M, such that X is positively tangent to B and X is positively transverse to pages, then (B, π) supports ξ.*

Proof The existence of such an X shows that on B, the contact form is $\alpha > 0$ since $\alpha(X) = 1$ and positive tangency shows tangent vectors to B are positive multiples of X. Moreover, if X is transverse to pages, we have $d\alpha = i_X \alpha \wedge d\alpha > 0$ on pages by the condition $d\alpha(X, -)$. □

The desired Reeb vector field is constructed as follows:

First, use $N(B) = (\mathbb{R}^3, ker(dz + r^2 d\theta))/(r, \theta, z) \sim (r, \theta, z + 1)$ of the neighborhood of B to define a Reeb vector field that is positively tangent to B and transverse to the pages in this neighborhood.

Now this defines a Reeb field in the neighborhood ∂R_X which can be extended to a transverse vector field R_X and this vector field can be used to construct the neighborhood $N(R) \cong R_X \times [-\delta, \delta]$. Therefore we can define the desired Reeb field R on $\partial(N(B) \cup N(R))$.

The Reeb vector field R is defined in a neighborhood of $\partial X(R)$, which is a convex surface as observed above. This allows us to first define the desired Reeb vector field on a tubular neighborhood of $\partial X(R)$, and then on the whole $X(R)$. Thus, we get the desired Reeb vector field R defined on whole M, proving that ξ is supported by the open book decomposition we constructed by the lemma.

For more details on the construction of the desired Reeb vector field, see [13].

Actually, one can prove the following theorem that is also due to Giroux [27]:

Theorem 2.26 *Two open books supporting the same contact manifold (M, ξ) are related by positive stabilizations.*

The proof of this result is similar to the proof of Theorem 2.21, see [13] for details.

Now, we can combine Theorems 2.17, 2.18, 2.21, and 2.26 to get the Giroux correspondence between contact structures on 3-manifolds and open book decompositions:

Theorem 2.27 (Giroux [27]) *Let M be a closed, oriented 3-manifold. Then there is a one-to-one correspondence between*

$$\{\text{oriented contact structures } \xi \text{ on } M \text{ up to isotopy}\}$$

and

$$\{\text{open book decompositions of } M \text{ up to positive stabilization}\}$$

The open book decomposition supporting a contact manifold (M, ξ) also encodes information about the tightness:

Theorem 2.28 *A contact 3-manifold (M, ξ) is overtwisted if and only if it is supported by an open book decomposition which is a negative stabilization of another open book decomposition.*

The proof uses the classification of overtwisted contact structures, Theorem 1.41. See [42] for the proof.

2.2.2 Planar Open Books

Definition 2.29 *An open book decomposition (Σ, φ) of a 3-manifold M is called **planar** if its pages have zero genus.*

Recall that in Remark 2.5, we mentioned every closed, oriented 3-manifold admits a planar open book decomposition. However, if (M, ξ) is a contact 3-manifold with the contact structure ξ supported by the open book (Σ, φ), then the page Σ of the open book is not necessarily planar. Therefore, we can ask which contact structures are supported by planar open book decompositions.

As a partial answer to this question, we have the following result due to Etnyre:

Theorem 2.30 (Etnyre [14]) *Any overtwisted contact structure on a closed 3–manifold is supported by a planar open book decomposition.*

For some tight contact structures on contact manifolds, there are also some conditions for having a planar open book decomposition. We will discuss some of these conditions related to symplectic fillings in the next chapter.

2.3 Higher Dimensions

For contact manifolds $(M, \xi = \ker \alpha)$ of arbitrary odd dimension $2n+1$ for $n > 1$, we can generalize the notion of a contact structure supported by an open book (Σ, φ):

Definition 2.31 *A contact structure $\xi = \ker \alpha$ on M is said to be supported by an open book (B, π) if it has the following properties:*

- *α induces a contact form on B;*
- *$d\alpha$ induces a symplectic form on each fiber X of π;*
- *The orientation on B defined by the contact form α coincides with its orientation as the boundary of the symplectic manifold $(X, d\alpha)$.*

For higher dimensional contact manifolds, Giroux and Mohsen (see [27]) proved the generalization of Theorem 2.21:

Theorem 2.32 *Any contact structure on a closed manifold M is supported by an open book, each fiber of which is a Weinstein manifold.*

A **Weinstein manifold** is a quadruple $(X, \omega = d\lambda, Z, \varphi)$ where $(X, d\lambda)$ is an open exact symplectic manifold, Z is a Liouville vector field on X pointing outwards, φ is a Morse function that is proper and bounded for which Z is gradient-like, and ∂X is a regular level set of φ.

We will not define "gradient-like" precisely. However, it essentially means that $d\varphi(Z) > 0$ away from critical points, and its decay is controlled near the critical points. See [39] for the details about Weinstein manifolds.

We will also encounter Weinstein manifolds in the next chapter in the context of symplectic fillings.

Chapter 3

Symplectic Fillings

In the first chapter, we described contact and symplectic manifolds as odd and even dimensional counterparts of each other. Symplectic fillings (and more generally symplectic cobordisms) give us a tool to investigate this connection and understand the properties of contact manifolds using symplectic tools.

In this chapter, we start by defining various types of symplectic fillings of contact manifolds. In Section 3.2, we discuss the differences between these various types of fillings. Then, in Section 3.3, we describe the relationship between the tightness of a contact manifold and its fillability. Finally, in the last section, we define the Lefschetz fibrations and open book decompositions induced by them, which gives information about the planarity and fillability of an open book supporting a contact structure.

3.1 Symplectic Fillings of Contact Manifolds

Let us start with the definition of symplectic fillings:

Definition 3.1 *Let $(M, \xi = \ker \alpha)$ be a contact manifold of dimension $2n-1$ for $n \geq 1$, where M and ξ are oriented by the forms $\alpha \wedge d\alpha^{n-1}$ and $d\alpha^{n-1}|_\xi$ respectively.*

1. *In the case that 2n-1 = 3, a compact symplectic 4-manifold (X, ω) is called a **weak symplectic filling** of (M, ξ) if $\partial X = M$ as oriented manifolds and $\omega|_\xi > 0$.*
2. *A compact symplectic 2n-manifold (X, ω) is called a **strong symplectic filling** of (M, ξ) if $\partial X = M$ as oriented manifolds and there is a Liouville vector field Y defined near ∂X, pointing outwards along ∂X, and satisfying $\xi = \ker(i_Y\omega|_{TM})$.*

   *If the vector field Y extends to a global Liouville vector field (equivalently, if the 1-form $i_Y\omega$ extends to a global primitive of ω on X), then we call (X, ω) an **exact symplectic filling** of (M, ξ)*

For the purposes of this paper, we omitted the definition of weak symplectic fillings in higher dimensions. See [38] for details.

By definition, every exact symplectic filling is a strong symplectic filling. Also, observe that, when Y is a Liouville vector field, the 1- form $i_Y\omega|_\xi$ is a contact form defining ξ and

$$\omega|_\xi = \mathcal{L}_Y\omega|_\xi = di_Y\omega|_\xi + i_Y d\omega|_\xi = di_Y\omega|_\xi$$

Therefore, by the contact condition, we have $\omega|_\xi > 0$. Hence, a strong symplectic filling is automatically a weak symplectic filling in dimension 3.

Admitting a strong symplectic filling determines the contact structure up to isotopy by Gray's theorem:

Lemma 3.2 *If (X, ω) is a strong filling of (M, ξ_0) and (M, ξ_1), then the contact structures ξ_0 and ξ_1 are isotopic.*

Proof Let Y_i be the Liouville vector field corresponding to ξ near $\partial X = M$ for $i = 0, 1$. Then, $Y_t = (1-t)Y_0 + Y_1$ is a Liouville vector field for all $t \in [0, 1]$ and $i_{Y_t}\omega$ is a homotopy of contact forms. Therefore, ξ_0 and ξ_1 are isotopic by Gray's Stability Theorem. □

Here are some examples of symplectic fillings of contact manifolds:

Example 3.3 *By Example 1.24, the symplectic manifold $(D^4, \omega_{st}|_{D^4})$ where ω_{st} is the standard symplectic structure on $\mathbb{R}^4$, is an exact symplectic filling of (S^3, ξ_{st}).*

In fact, this is the unique symplectic filling of (S^3, ξ_{st}) (up to symplectic deformation equivalence and blow-up). One proof of this fact that can be found in [1] uses Lefschetz fibrations that will be defined in the last section.

Example 3.4 *As observed by Giroux in [26], the manifold $T^2 \times D^2$ with a product symplectic structure can be viewed as the weak symplectic filling of each contact manifold (T^3, ξ_n) described in Example 1.9.*

In fact, Giroux also showed that each ξ_n is distinct. Thus, (T^3, ξ_n) show that the immediate analogue of Lemma 3.2 for weak symplectic fillings do not hold.

A related notion to symplectic fillings is the notion of *symplectic cobordisms*. Recall that a **cobordism** is a triple (X, M_0, M_1) of compact oriented manifolds (possibly empty), where $\partial X \cong M_0 \coprod M_1$ as oriented manifolds. In this case, we call M_0 and M_1 **cobordant** and X a **cobordism from** M_0 **to** M_1. Being cobordant defines an equivalence relation: reflexivity is obtained by considering the manifold $X = [0, 1] \times M$, symmetry is obtained by reversing the orientation of X, and transitivity is given by gluing cobordisms.

Now we can define symplectic cobordisms:

Definition 3.5 *Let $(M_\pm, \xi_\pm)$ be closed contact manifolds of dimension 2n-1 for $n \geq 1$, with cooriented contact structures, which induce the orientation of the respective manifold. A* ***strong symplectic cobordism*** *from (M_-, ξ_-) to (M_+, ξ_+) is a compact 2n-dimensional symplectic manifold (X, ω), oriented by the volume form ω^n, such that the following conditions hold:*

- *The oriented boundary of X equals $\partial X = M_+ \coprod -M_-$, where $-M_-$ stands for M_- with reversed orientation.*
- *In a neighbourhood of ∂X, there is a Liouville vector field Y for ω, transverse to the boundary and pointing outwards along M_+, inwards along M_-.*
- *The 1–form $i_Y\omega$ restricts to $M_\pm$ as a contact form for $\xi_\pm$.*

We call M_+ the ***convex boundary*** *of the cobordism W and M_- the* ***concave boundary****.*

If the vector field Y extends to a global Liouville vector field (equivalently, if the 1-form $i_Y\omega$ extends to a global primitive of ω on X), then we call (X, ω) an ***exact symplectic cobordism*** *from (M_-, ξ_-) to (M_+, ξ_+).*

Example 3.6 *A strong (resp. exact) symplectic filling (X, ω) of a contact manifold (M, ξ) is a strong (resp. exact) symplectic cobordism from the empty set to (M, ξ)*

We can similarly define a ***strong concave filling*** *(X, ω) of a contact manifold (M, ξ) as a strong symplectic cobordism from (M, ξ) to the empty set. (By Stokes' theorem, there are no exact concave fillings in this sense.)*

Example 3.7 *Similar to symplectization defined in Example 1.23, the symplectic manifold $([0,1] \times M, d(e^t\alpha))$ is an exact symplectic cobordism from $(M, \xi = \ker\alpha)$ to itself.*

Using the Lioville vector field we can define a symplectic collar neighborhood of the boundary. Similarly to the topological cobordisms, we can glue symplectic cobordisms together using this collar neighborhood:

Proposition 3.8 *Let (X_-, ω_-) be a symplectic cobordism from the contact manifold (M_-, ξ_-) to (M, ξ), and (X_+, ω_+) a symplectic cobordisms from (M, ξ) to (M_+, ξ_+). Then there is a symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+), which is topologically given by gluing X_- and X_+ along M.*

See [22] for a proof.

Remark 3.9 *The relation of being symplectically cobordant is reflexive and transitive by the previous example and proposition. However, in contrast with topological cobordisms, because of the orientations being induced by contact and symplectic structures, it is not always symmetric.*

As an example, due to the results of Etnyre, Honda [18] and Gay [20] every contact manifold has a strong concave filling. See [13] for a proof. On the other hand in the *following sections, we will see that not every contact manifold can admit a strong symplectic filling.*

Before closing this section, we also mention two more types of fillings:

Definition 3.10 *Let $(M, \xi = \ker \alpha)$ be a contact manifold of dimension $2n-1$, for $n \geq 1$, where M and ξ are oriented by the forms $\alpha \wedge d\alpha^{n-1}$ and $d\alpha^{n-1}|_\xi$ respectively.*

- *A **Weinstein filling** of (M, ξ) is an exact filling $(X, \omega = d\lambda)$ of (M, ξ) for which the global Liouville vector field Y is gradient-like (see Section 2.2.2) with respect to a Morse function $\varphi : X \to \mathbb{R}$ that is locally constant at the boundary.*
- *A Weinstein filling $(X, \omega = d\lambda, \varphi)$ is called a **Stein filling** if it carries an complex structure J that is compatible with ω and satisfies $\lambda = -d\varphi \circ J$.*

For the definition of compatible complex structures, see [6], [39] for details. For precise definitions and equivalent conditions for Stein and Weinstein fillings see [22], [42], and [7].

Due to the following result of Giroux [27], Stein fillability of a contact 3-manifold (M, ξ) is completely determined by the open book decomposition of (M, ξ):

Theorem 3.11 *A contact 3-manifold (M, ξ) is Stein fillable if and only if there is an open book decomposition for (M, ξ) whose monodromy can be written as a composition of right-handed Dehn twists.*

There are various proofs of this result. For a proof using handlebody decompositions of Stein manifolds, see [13]. For a proof using Lefschetz fibrations, see [42].

3.2 Types of Fillings of Contact Manifolds

By the Definitions 3.1, 3.10, and the definition of weak symplectic fillings in higher dimensions, we have the chain of inclusions:

$$\{\text{weakly}\} \supset \{\text{strongly}\} \supset \{\text{exactly}\} \supset \{\text{Weinstein}\} \supset \{\text{Stein}\}$$

One can ask whether these inclusions are proper. In this section, we will discuss the results in this direction.

First, we will consider the inclusion $\{\text{weakly fillable}\} \supset \{\text{strongly fillable}\}$.

In dimension 3, we already mentioned the observation of Giroux that (T^3, ξ_n) admit *weak* symplectic fillings for all $n \in \mathbb{N}$. On the other hand, in [10], Eliashberg showed that only (T^3, ξ_1) admits a *strong* symplectic filling. Therefore, (T^3, ξ_n) for $n \geq 2$ provide examples of contact manifolds that are weakly but not strongly fillable.

In higher dimensions, by using the argument of "Giroux Torsion" and a generalization of the open book construction, Massot, Niederkrüger, and Wendl showed in [38] that for certain manifolds M, manifolds of the form $M \times T^2$ can be endowed with distinct contact structures ξ_n for all $n \in \mathbb{N}$, such that only ξ_1 is strongly fillable. On the other hand, in dimension 5, all ξ_n are weakly fillable.

Therefore, in dimensions 3 and 5, strongly fillable contact manifolds are a proper subset of weakly fillable contact manifolds.

Now, we will discuss the inclusion $\{\text{strongly fillable}\} \supset \{\text{exactly fillable}\}$.

In dimension 3, in [23], Ghiggini showed that for each even integer $n \geq 2$, the Seifert fibered 3-manifold $-\Sigma(2,3,6n+5)$ admits a contact structure that is strongly but not exactly fillable. During the course of the proof, Ghiggini uses Ozsváth–Szabó invariants, which are Floer theoretic invariants of contact 3-manifolds constructed in [43] using the open book decomposition of contact manifolds.

In higher dimensions, a recent result of Zhou [24] shows that for every $n \geq 3$, there exists a $2n-1$ dimensional contact manifold which is strongly fillable but not exactly fillable. This shows that in all dimensions exactly fillable contact manifolds are a proper subset of strongly fillable contact manifolds.

Due to the work of Eliashberg and Cieliebak from [7], we actually have

$$\{\text{Weinstein fillable}\} = \{\text{Stein fillable}\}$$

So lastly, we will consider the inclusion $\{\text{strongly fillable}\} \supset \{\text{Stein fillable}\}$. In dimension 3, Bowden [4] showed that for all even $n \geq 2$, there exists a contact structure η on $\Sigma(2,3,6n+5)$ such that the contact connected sum

$$(\Sigma(2,3,6n+5),\eta)\#(-\Sigma(2,3,6n+5),\eta_0)$$

where η_0 is the contact structure constructed in [23], is exactly but not Stein fillable. In higher dimensions, Bowden, Crowley, and Stipsicz [5] showed that at every odd dimension greater than 3, there are exactly fillable contact manifolds that are not Stein fillable. Therefore, the inclusion $\{\text{strongly fillable}\} \supset \{\text{Stein fillable}\}$ is known to be proper in all dimensions.

As a summary, we have the chain of inclusions:

$$\{\text{weakly}\} \supset \{\text{strongly}\} \supset \{\text{exactly}\} \supset \{\text{Weinstein}\} = \{\text{Stein}\}$$

where all inclusions except $\{\text{weakly}\} \supset \{\text{strongly}\}$ in dimensions higher than 5, are known to be proper.

3.3 Tightness and Symplectic Fillings

As mentioned previously in Section 1.5.2, tightness of a contact 3-manifold (M, ξ) is related to the existence of a symplectic filling (X, ω) of (M, ξ).

One of the main results in this direction is:

Theorem 3.12 (Eliashberg [9], Gromov [29]) *A contact 3-manifold (M, ξ) that admits a weak symplectic filling is tight.*

The original proof of this theorem uses techniques related to complex structures on manifolds. A sketch can be found in [32].

An alternative argument that can be found in [42] uses the techniques of contact surgery and Seiberg-Witten theory.

An analogous theorem for higher dimensions is proved by Niederkrüger in [41] for higher dimensions.

With this result we can prove the tightness of the standard contact structures on S^3 and on $\mathbb{R}^3$ that we mentioned in Section 1.5.2:

Example 3.13 *By Example 3.3, (S^3, ξ_{st}) is tight. Any open subset of a tight contact 3–manifold is again tight. Therefore, $(\mathbb{R}^3, \xi_{st}) \cong (S^3 - \{p\}, \xi_{st})$ is also tight.*

Example 3.14 *By Example 3.4, (T^3, ξ_n) is tight for all $n \in \mathbb{N}$ as mentioned in Example 1.40.*

However, the converse of the theorem of Eliashberg and Gromov does not hold. That is, there are contact 3-manifolds (M, ξ) where ξ is tight but there are no symplectic fillings of M. The first examples of such contact manifolds are due to Etnyre and Honda, and found in [16]. More examples and tools for systematic constructions of such examples can be found in [42]. Therefore, we have the strict inclusion

$$\{\text{tight}\} \supsetneq \{\text{weakly fillable}\}$$

There is also a relationship between overtwistedness and symplectic cobordisms. Overtwisted contact structures are "minimal" for exact cobordisms in the following sense:

Theorem 3.15 *If (M_-, ξ_-) and (M_+, ξ_+) are closed nonempty contact manifolds of the same dimension and ξ_- is overtwisted, then there exists an exact cobordism from (M_-, ξ_-) to (M_+, ξ_+).*

The 3-dimensional case of this theorem is due to Etnyre and Honda [18], and is actually proved for "Stein cobordisms" which can be defined analogously. The higher dimensional version of this theorem is due to Murphy and Eliashberg [11].

3.4 Lefschetz Fibrations and Open Books

In this section, we define Lefschetz fibrations which give us some tools to investigate symplectic fillings and open book decompositions of contact manifolds. We will follow the exposition of the survey [1]. For other constructions and definitions related to Lefschetz fibrations, see [42].

Definition 3.16 *A* ***Lefschetz fibration*** *on a 2n-dimensional manifold X with boundary and corners is a surjective map $f : X \to D^2$, where D^2 is a 2-disk, with finitely many nondegenerate critical points all of which lie in the interior of X.*

Near each critical point, one can choose complex coordinates $(z_1, \ldots, z_n)$ such that in these coordinates

$$f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$$

Away from critical values, f is a trivial fibration.

If the total space X is a symplectic manifold, for $f : X \to D^2$ to be a **symplectic Lefschetz fibration** we require:

- the generic fibers to be $(2n-2)$-dimensional symplectic submanifolds (that is, the restriction of the symplectic form to the fibers is nondegenerate) with boundaries away from critical points, and
- at the critical points the coordinates in which f looks locally like a complex Morse function are chosen to be holomorphic for some compatible almost complex structure.

We can write the boundary ∂X as the union

$$\partial X = \partial_v X \cup \partial_h X$$

where $\partial_v X = f^{-1}(\partial D^2)$ and $\partial_h X = \bigcup_{z \in D^2} \partial f^{-1}(z)$.

Observe that, since there are no critical values of f on $S^1 = \partial D^2$, the **vertical boundary** $\partial_v X$ is smoothly fibered over S^1. That is, $\partial_v X$ is the mapping torus of the fiber $f^{-1}(z)$ over $z \in \partial D^2$ for some monodromy map.

On the other hand, the **horizontal boundary** $\partial_h X$ is diffeomorphic to a disjoint union of the boundaries of fibers, that are fibered by D^2.

With these observations, after appropriate smoothings, the boundary ∂X induces an open book decomposition of ∂X where the pages are obtained from the fibers of the vertical boundary $\partial_v X$, and the binding is the central fiber $\partial f^{-1}(0)$ of the horizontal boundary $\partial_h X$.

The monodromy of the open book induced by a Lefschetz fibration is completely determined by the critical points of the fibration. If the Lefschetz fibration has no critical points, then the monodromy of the induced open book is trivial. On the other hand, if there are critical points, then the monodromy is nontrivial. See [42] for the construction of the monodromy.

The following result due to Wendl gives a connection between symplectic fillings of *planar* contact 3-manifolds and Lefschetz fibrations:

Theorem 3.17 (Wendl [47]) *Let (M, ξ) be a contact manifold that is supported by a planar open book decomposition (B, π). Then, every strong symplectic filling X of M admits a symplectic Lefschetz fibration $f : X \to D^2$ that induces the open book decomposition (B, π).*

As a corollary, Wendl shows that for planar contact manifolds, Stein and strong fillability is equivalent:

Corollary 3.18 *Strongly fillable contact manifolds that are supported by planar open books are Stein fillable.*

For higher dimensional contact manifolds, a generalization of planarity, called *iterated planarity* is also related to Lefschetz fibrations. For more references and results related to this relationship, see [1].

Appendix A

Convex Surfaces

Convex surfaces are introduced by Giroux and are central in the study of contact manifolds. In this appendix, we will introduce the basic definitions and properties related to convex surfaces in contact manifolds, mostly without proof. For detailed information about convex surfaces, see [15], [32], and [25].

Definition A.1 *Let Σ be a surface in a contact 3-manifold (M, ξ) that is either closed or compact with Legendrian boundary.*

*The surface Σ is called **convex** if there is a contact vector field v on M that is transverse to Σ.*

For a convex surface Σ with a transverse contact vector field v, the set

$$\Gamma_\Sigma = \{x \in \Sigma : v(x) \in \xi_x\}$$

*is called the **dividing set** of Σ.*

Throughout our discussion of convex surfaces, we will only consider contact 3-manifolds (M, ξ).

Note that, if v is a contact vector field, then $-v$ is also a contact vector field. So there is no distinguished side of Σ, and there is no concavity present in the usual sense.

Here is an example of a convex surface in $\mathbb{R}^3$:

Example A.2 *Consider standard contact structure on $\mathbb{R}^3$, that is*

$$(\mathbb{R}^3, \xi = \ker(dz + xdy - ydx))$$

The vector field $v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$ is a contact vector field since

$$\begin{aligned}\mathcal{L}_v(dz + xdy - ydx) &= i_v(d(dz + xdy - ydx)) + d(i_v(dz + xdy - ydx)) \\ &= i_v(2dx \wedge dy) + d(2z + xy - xy) \\ &= 2xdy - 2ydx + 2dz \\ &= 2(dz + xdy - ydx)\end{aligned}$$

The vector field v is transverse to S^2, thus $S^2 \subset \mathbb{R}^3$ is a convex surface.

The dividing set of S^2 is

$$\begin{aligned}\Gamma_{S^2} &= \{(x,y,z) \in S^2 : v(x,y,z) \in \xi_{(x,y,z)}\} \\ &= \{(x,y,z) \in S^2 : (dz + xdy - ydx)(v(x,y,z)) = 0\} \\ &= \{(x,y,z) \in S^2 : 2z = 0\} \\ &= \{(x,y,z) \in S^2 : z = 0\}\end{aligned}$$

That is the equator circle of S^2.

We can equivalently define convex surfaces as surfaces with a tubular neighborhood in which the contact structure is vertically invariant:

Lemma A.3 *A surface $\Sigma \subset (M, \xi = \ker \alpha)$ is convex with the transverse vector field v if and only if it has a neighborhood $N \cong \Sigma \times \mathbb{R}$ such that ξ is invariant in the $\mathbb{R}$ direction, that is, under the flow of $\frac{\partial}{\partial t}$ where t is the coordinate on $\mathbb{R}$. With this identification, the vector field v can be identified with $\frac{\partial}{\partial t}$. In the neighborhood $N \cong \Sigma \times \mathbb{R}$ the contact 1-form a can be written as $fdt + \beta$, where $f : \Sigma \to \mathbb{R}$ is a function and β is a 1-form on Σ.*

Using the notation of this result, we can write the dividing set as

$$\Gamma_\Sigma = \{x \in \Sigma : f(x) = 0\}$$

With this notation, the characteristic foliation on Σ is given by integrating $\ker \beta$.

Proposition A.4 *The dividing set Γ_Σ of a convex surface Σ has the following properties*

1. *Γ_Σ is nonempty,*
2. *Γ_Σ is a multi-curve on Σ, that is a properly embedded smooth 1-manifold,*
3. *Γ_Σ is transverse to the characteristic foliation Σ_ξ of Σ,*
4. *$\Sigma - \Gamma_\Sigma = \Sigma_+ \coprod \Sigma_-$ where $\Sigma_+ \subset \Sigma$ (resp. $\Sigma_- \subset \Sigma$) is the set of points x where the normal orientation to Σ given by $v(x)$ agrees with (resp. is opposite to) the normal orientation to ξ_x. Then, as we cross Γ_Σ (once, transversely), we move from $\Sigma_\pm$ to $\Sigma_\mp$,*

5. *there is a vector field w and volume form Ω on Σ such that*

    a) *w directs Σ_ξ (i.e. w is tangent to Σ_ξ where it is nonsingular and is zero where it is singular),*

    b) *the flow of w expands Ω on Σ_+ and contracts Ω on Σ_-,*

    c) *w points out of Σ_+.*

Proof We write $\alpha = fdt + d\beta$ as in Lemma A.3. In this case, the contact condition becomes

$$(fdt + \beta) \wedge (df \wedge dt + d\beta) = \beta \wedge df \wedge dt + fdt \wedge d\beta > 0$$

. We will prove the first 4 statements.

1. Assume Γ_Σ is empty. Then $f(x) \neq 0$ on Σ and we can divide by f to use the contact form $\alpha' = dt + \beta/f$. Then by the contact condition $d(\beta/f) > 0$.

   We have
   $$\int_\Sigma d\alpha' = \int_{\partial\Sigma} \alpha' = 0$$
   since $\partial\Sigma$ is empty or Legendrian. On the other hand,
   $$\int_\Sigma d\alpha' = \int_\Sigma d(\beta/f) > 0$$
   which gives a contradiction. Hence, Γ_Σ is nonempty.

2. By the contact condition, if $f(x) = 0$, then $df_x \neq 0$. Thus, 0 is a regular value and $f^{-1}(0) = \Gamma_\Sigma$ is a properly embedded smooth 1-manifold.

3. The characteristic foliation is given by $\ker\beta$ and the tangent space of Γ_Σ is given by $\ker df$. However, by the contact condition, $\beta \wedge df > 0$ on Σ. So if $u \in \ker df \cap \ker\beta$, then $u = 0$. Therefore, Γ_Σ and Σ_ξ intersect transversely.

4. The normal orientation to Σ is given by $\frac{\partial}{\partial t}$, Then
   $$\Sigma_+ = \alpha(\frac{\partial}{\partial t}) = f > 0$$
   and
   $$\Sigma_- = \alpha(\frac{\partial}{\partial t}) = f < 0$$
   and since 0 is a regular value, sign changes when passing through Γ_Σ.

For the proof of 5, see [22] or [12]. □

The above properties can be taken as a definition for a multi-curve that "divides" a characteristic foliation on a surface:

Definition A.5 *Let Γ be a multi-curve on Σ and let $\mathcal{F}$ be a singular foliation on Σ. The multi-curve Γ is said to* ***divide*** *$\mathcal{F}$ if*

1. *Γ is transverse to the $\mathcal{F}$,*
2. *$\Sigma - \Gamma = \Sigma_+ \coprod \Sigma_-$,*
3. *there is a vector field w and volume form Ω on Σ such that*
   a) *w is directs $\mathcal{F}$,*
   b) *the flow of w expands Ω on Σ_+ and contracts Ω on Σ_-,*
   c) *w points out of Σ_+.*

By the following propositions, we see that the convex surfaces are "generic" in contact manifolds:

Proposition A.6 *Any closed surface Σ in a contact manifold (M, ξ) is C^∞-close to a convex surface. Any surface Σ with Legendrian boundary satisfying $tb(l) \leq 0$ for all boundary components $l \subset \partial\Sigma$ may be C^0 small perturbed near the boundary and then C^∞-small perturbed on the interior so as to become convex.*

Proposition A.7 *If Σ is convex in (M, ξ), and $\mathcal{F}$ is any other foliation divided by Γ_Σ then there is an isotopy, through convex surfaces, of Σ to Σ' so that $\Sigma'_\xi = \mathcal{F}$.*

See [15] for the full proofs of these propositions.

The following results due to Honda (see [30], [31]), are important in the proof of the Giroux correspondence (Theorem 2.27), and in general for cutting and gluing contact manifolds:

Proposition A.8 *If c is a properly embedded arc or a closed curve on Σ, a convex surface, and all components of $\Sigma - c$ contain some component of $\Gamma_\Sigma - c$ then Γ may be isotoped through convex surfaces so that c is Legendrian. This is called* ***Legendrian realization principle****.*

Proposition A.9 *If Σ_1 and Σ_2 are convex surfaces, $\partial\Sigma_1 = \partial\Sigma_2$, and the surfaces meet transversely, then the dividing curves interlace as shown in Figure A.1, and we can round the corner to get a single smooth convex surface with dividing curves shown in Figure A.1.*

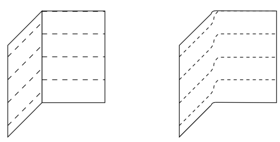

Figure A.1: Two convex surfaces meeting transversely and rounding of the edges.

Lastly, by the following proposition, convex surfaces and dividing curves in a contact manifold tell us about the tightness of the contact structure in a neighborhood of a convex surface.

Theorem A.10 (Giroux Tightness Criterion) *Let Σ be a convex surface in a contact manifold (M, ξ). A vertically invariant neighborhood of Σ is tight if and only if either $\Sigma \neq S^2$ and Γ_Σ contains no contractible curves, or $\Sigma = S^2$ and Γ_Σ is connected.*

The proof uses Legendrian Realization Principle.

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