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Characteristic Classes

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Introduction

Within Differential Geometry, a characteristic class is an isomorphism invariant of a vector bundle, most often constructed via curvature and suitable polynomials. It assigns a cohomology class to a vector bundle, used to distinguish non-isomorphic vector bundles and thus help classify vector bundles over a given manifold. The theory of characteristic classes yields some remarkable theorems, like the generalized Gauss-Bonnet Theorem, sometimes also called the Chern-Gauss-Bonnet Theorem (Theorem 3.10 in this paper).

The Chern-Gauss-Bonnet Theorem can be generalized to the Atiyah-Singer index theorem, which relates the analytical index to the topological index of an elliptic differential operator on a compact manifold, and finds many applications in theoretical physics (cf. [Nak18]). Although we will not cover index theorems and applications in Physics in this paper, these are main motivations to study characteristic classes.

In Chapter 1, we will study invariant polynomials and will find generators of the ring of invariant polynomials. Thus, when computing characteristic classes, we can restrict to these generators. Then, in Chapter 2, we will introduce connection and curvature matrices. These are another fundamental ingredient to construct a characteristic class. In Chapter 3, we will finally introduce characteristic classes. Pontrjagin, Euler and Chern classes are important special cases of characteristic classes, also discussed in this section. Lastly, in Chapter 4 we will motivate how the theory can be generalized to principal bundles.

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I would also like to acknowledge, that I mainly worked with the textbook of Loring W. Tu on Differential Geometry [Tu17]. I primarily summarized his exposition of characteristic classes in Chapter 5 and 6, added some details to some of the proofs and solved some exercises.

Contents

1 Invariant Polynomials

Characteristic classes are constructed via polynomials $P(X)$ in the entries of a square matrix $X = [x^i_j]$ of indeterminates that are invariant under conjugation by invertible matrices A, that is $P(X) = P(A^{-1}XA)$. We therefore have to understand these polynomials first, before studying characteristic classes.

1.1 Invariant Polynomials on $\mathfrak{gl}(r, \mathbb{R})$

We will start by showing some purely algebraic results about polynomials. For this, let R be a ring (here we always mean a commutative and unitary ring, even if we just write ring) and denote by $R[X_1, \ldots, X_n] = R[X]$ the polynomial ring in the variables $X_1, \ldots, X_n = X$. Evaluating a polynomial $P \in R[X]$ at a tupel of elements $r_1, \ldots, r_n \in R$ defines the polynomial function

$$\hat{P} : R^n \to R, r_1, \ldots, r_n \mapsto P(r_1, \ldots, r_n).$$

It is well known, that the map $P \mapsto \hat{P}$ is in general not injective. For example take $R = \mathbb{Z}/2\mathbb{Z}$ and the two distinct polynomials $P = 0$ and $Q = X^2 - X$, for which $\hat{P} = \hat{Q}$. However for infinite fields we have the following.

Theorem 1.1. *Let $\mathbb{F}$ be an infinite field. Consider the polynomial ring $\mathbb{F}[X]$ in the variables X over the field $\mathbb{F}$. Then the map $\mathbb{F}[X] \ni P \mapsto \hat{P} \in Fun(\mathbb{F}^n, \mathbb{F})$ is injective.*

Proof. Note that $P \mapsto \hat{P}$ is a ring homomorphism from the polynomial ring in n variables to the ring of functions $\mathbb{F}^n \to \mathbb{F}$. We show that the kernel of this map is trivial. We do so by induction over the number of variables n. For $n = 1$ let P be a polynomial in one variable such that $\hat{P} = 0$. Since $\mathbb{F}$ is infinite, this means that P has infinitely many roots and therefore has to be the trivial polynomial. Now suppose, that the statement is true for all polynomials up to $(n-1)$ variables. Let P be a polynomial of n variables such that $\hat{P}$ is the zero function. We write

$$P(X_1, \ldots, X_{n-1}, X_n) = \sum_{k=0}^{m} P_k(X_1, \ldots, X_{n-1}) X_n^k.$$

Then for fixed $a_1, \ldots, a_{n-1} \in \mathbb{F}$, $\hat{P}(a_1, \ldots, a_{n-1}, x_n)$ is the zero function in x_n and therefore $P(a_1, \ldots, a_{n-1}, X_n)$ has to be the trivial polynomial in one variable, that is all the coefficients $P_k(a_1, \ldots, a_{n-1})$ are zero. The point $(a_1, \ldots, a_{n-1}) \in \mathbb{F}^{n-1}$ was arbitrary, and hence $\hat{P}_k$ has to be the trivial function. By induction hypothesis, all the $P_k{}'s$ have to be trivial polynomials, and hence $\sum_{k=0}^{m} P_k(X_1, \ldots, X_{n-1}) X_n^k$ has to be trivial. □

We will use Theorem 1.1 to study polynomials that take a square matrix (as elements of a Lie Algebra) as input and are invariant under the adjoint representation of the Lie group. However, we will give a more general definition of these invariant polynomials. This generalization allows us to also generalize the theory of characteristic classes to principal bundles, cf. Chapter 4.

Let V be a vector space of dimension n over a field F and let $V^\star$ be the dual vector space. We denote by $Sym^k(V^\star)$ the k-th symmetric power of $V^\star$ and call its elements polynomials of degree k on V. Let $e_1,\dots,e_n$ be a basis of V with corresponding dual basis $\varepsilon^1,\dots,\varepsilon^n$ of $V^\star$. We call a function $f : V \to F$ a polynomial of degree k on V, if f can be expressed as

$$f = \sum_{\substack{I=(i_1,\dots,i_k)\\ 1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n}} a_I\varepsilon^I,$$

where ε^I denotes the symmetrization of $\varepsilon^{i_1}\cdots\varepsilon^{i_k}$.

Definition 1.2. *Let G be a Lie group with corresponding Lie algebra $\mathfrak{g}$. Let $f : \mathfrak{g} \to \mathbb{R}$ be a polynomial on the real vector space $\mathfrak{g}$. We say that f is $Ad(G)$-invariant, if for all $g \in G$ and all $X \in \mathfrak{g}$*

$$f((Ad(g)X) = f(X).$$

The Lie algebra of the matrix Lie group $GL(r,\mathbb{R})$ is the space of all real $r \times r$-matrices $\mathfrak{gl}(r,\mathbb{R}) = \mathbb{R}^{r\times r}$. A polynomial $P(X)$ on $\mathfrak{gl}(r,\mathbb{R})$ is a polynomial in the entries of $X = [x^i_j]$, an $r \times r$-matrix with indeterminate entries. $P(X)$ is $Ad(GL(r,\mathbb{R}))$-invariant if for all $A \in GL(r,\mathbb{R})$

$$P(X) = P(Ad(A^{-1})X) = P(A^{-1}XA).$$

Denote by $Inv(\mathfrak{gl}(r,\mathbb{R}))$ the algebra of all $Ad(GL(r,\mathbb{R}))$-invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$.

If $P(X) = P(A^{-1}XA)$ is true for all $A \in GL(r,\mathbb{R})$ and real $r \times r$-matrices X, that is it is true as an equality of polynomial functions in $Fun(\mathbb{R}^{r^2},\mathbb{R})$, then Theorem 1.1 implies that it is true as an equation of the polynomials, so P is $Ad(GL(r,\mathbb{R}))$-invariant. For the determinant this is clearly the case. For all real $r \times r$-matrices X and all $A \in GL(r,\mathbb{R})$ we have

$$det(A^{-1}XA) = det(A)^{-1}det(X)det(A) = det(X),$$

hence $det(X)$ is an $Ad(GL(r,\mathbb{R}))$-invariant polynomial on $\mathfrak{gl}(r,\mathbb{R})$. Therefore, also the characteristic polynomial $ch_{-X}(\lambda)$ of any $r \times r$-matrix $(-X)$ is $Ad(GL(r,\mathbb{R}))$-invariant

$$\begin{aligned} ch_{-X}(\lambda) &= det(\lambda\mathrm{I} - X) \\ &= det(A^{-1}(\lambda I - X)A) \\ &= det(A^{-1}\lambda\mathrm{I}A - A^{-1}XA) \\ &= det(\lambda\mathrm{I} - A^{-1}XA) \\ &= ch_{A^{-1}(-X)A}(\lambda). \end{aligned}$$

Note that

$$det(\lambda \mathrm{I} + X) = \sum_{\sigma \in S_r} sgn(\sigma) \prod_{i=1}^{r} (\lambda \mathrm{I} + X)_{i,\sigma i} = \sum_{k=0}^{r} f_k(X)\lambda^{r-k}$$

for some coefficient polynomials $f_k(X)$ on $\mathfrak{gl}(r,\mathbb{R})$. Because the determinant is invariant we have

$$\begin{aligned}\sum_{k=0}^{r} f_k(X)\lambda^{r-k} &= det(\lambda \mathrm{I} + X)\\ &= det(A^{-1}(\lambda \mathrm{I} + X)A)\\ &= \sum_{k=0}^{r} f_k(A^{-1}XA)\lambda^{r-k}\end{aligned}$$

and hence $f_k(X) = f_k(A^{-1}XA)$ for all k by comparison of coefficients. That is, the coefficient polynomials are also $Ad(GL(r,\mathbb{R})$-invariant. By the cyclic property of the trace $tr(ABC) = tr(BCA)$, we get that the trace polynomials $\Sigma_k(X) = tr(X^k)$ are also $Ad(GL(r,\mathbb{R}))$-invariant.

Example 1.1. *The constant coefficient of the characteristic polynomial $ch_{-X}(\lambda)$ of the $r \times r$-matrix $(-X)$ is just the determinant of X*

$$f_r(X) = det(X),$$

so in particular, the invariant polynomial $det(X)$ can be written as a polynomial with entries in $f_1(X), \ldots, f_r(X)$. Consider the 2×2-matix

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$$

of indeterminate entries with determinant $det(X) = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$. The first trace polynomial is $\Sigma_1(X) = x_{1,1} + x_{2,2}$. For the second trace polynomial we compute

$$X^2 = \begin{pmatrix} x_{1,1}^2 + x_{1,2}x_{2,1} & x_{1,1}x_{1,2} + x_{1,2}x_{2,2} \\ x_{2,1}x_{1,1} + x_{2,2}x_{2,1} & x_{2,1}x_{1,2} + x_{2,2}^2 \end{pmatrix},$$

so $\Sigma_2(X) = x_{1,1}^2 + 2x_{1,2}x_{2,1} + x_{2,2}^2$. Observe that

$$det(X) = \frac{1}{2}(\Sigma_1(X))^2 - \frac{1}{2}\Sigma_2(X),$$

so the invariant polynomial $det(X)$ (for a 2×2-matrix X) can also be written as a polynomial with entries in $\Sigma_1(X), \Sigma_2(X)$. The result holds true for $r \times r$-matrices X and trace polynomials up to order r for arbitrary r, but the computation gets very long.

In the next sections, we prove a generalization of this. $Inv(\mathfrak{gl}(r,\mathbb{R})$ is generated by the coefficient polynomials $f_1(X),\ldots,f_r(X)$ or by the trace polynomials $\Sigma_1(X),\ldots,\Sigma_r(X)$. Then, in Example 1.2, we will use the developed theory to compute a polynomial in the trace polynomials that equals the determinant of X for an $r\times r$-matrix X for arbitrary r.

1.2 Generators of $Inv(\mathfrak{gl}(r,\mathbb{C}))$

To find the generators of $Inv(\mathfrak{gl}(r,\mathbb{R}))$, we first prove that $Ad(GL(r,\mathbb{C}))$-invariant polynomials on $\mathfrak{gl}(r,\mathbb{C})$ are generated by the coefficient polynomials $f_1(X),\ldots,f_k(X)$ of the characteristic polynomial $det(\lambda\mathrm{I}+X)$. In the next section, we use this result to show the real case

$$Inv(gl(r,\mathbb{R}) = \mathbb{R}[f_1(X),\ldots,f_r(X)]$$

and use Newton's identities to show that the trace polynomials $\Sigma_1(X),\ldots,\Sigma_r(X)$ also generate $Inv(gl(r,\mathbb{R}))$.

Let $P(X)$ be an $Ad(GL(r,\mathbb{C}))$ invariant polynomial on $\mathfrak{gl}(r,\mathbb{C})$. Then the restriction of P to diagonal matrices

$$\widetilde{P}(t_1,\ldots,t_r) = P(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix})$$

defines a polynomial $\widetilde{P}$ in the variables $t_1,\ldots,t_r$. Consider the permutation matrix

$$S = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \mathrm{I}_{(r-2)\times(r-2)} \end{pmatrix} = S^{-1} \in GL(r,\mathbb{C}),$$

then because P is invariant, we get

$$\begin{aligned} \widetilde{P}(t_1,t_2,\ldots,t_r) &= P(diag(t_1,t_2,\ldots,t_r)) \\ &= P(S^{-1}diag(t_1,t_2,\ldots,t_r)S) \\ &= P(diag(t_2,t_1,\ldots,t_r)) \\ &= \widetilde{P}(t_2,t_1,\ldots,t_r) \end{aligned}$$

and similar for all other transpositions, and therefore for all permutations $\sigma\in S_r$, so $\widetilde{P}$ is a symmetric polynomial. We denote by $\mathbb{C}[t_1,\ldots,t_r]^{S_r}$ the algebra of symmetric polynomials in the variables $t_1,\ldots,t_r$ and by $Inv(\mathfrak{gl}(r,\mathbb{C}))$ the $Ad(GL(r,\mathbb{C}))$ invariant polynomials on $\mathfrak{gl}(r,\mathbb{C})$.

Theorem 1.3. *The map*

$$\phi : Inv(\mathfrak{gl}(r,\mathbb{C})) \to \mathbb{C}[t_1,\ldots,t_r]^{S_r}$$

$$P(X) \mapsto \widetilde{P}(t_1, \ldots, t_r) := P\left(\operatorname{diag}(t_1, \ldots, t_r)\right),$$

where $\operatorname{diag}(t_1, \ldots, t_r)$ *denotes the diagonal matrix with entries* $t_1, \ldots, t_r$*, is a* $\mathbb{C}$*-algebra isomorphism.*

For the proof we will need the following two lemmas.

Lemma 1.4. *The set of diagonalizable complex* $r \times r$ *matrices is dense in* $\mathbb{C}^{r\times r} = \mathfrak{gl}(r,\mathbb{C})$*.*

The proof can be found in the appendix of Tu's book [Tu17, p. 312, proof of (1)].

Lemma 1.5 (Fundamental theorem of symmetric polynomials)**.** *For every symmetric polynomial* $F(t_1, \ldots, t_r)$ *over a ring* R *there exists a unique polynomial* $G(u_1, \ldots, u_r)$ *over* R*, such that* $F(t_1, \ldots, t_r) = G(\sigma_1, \ldots, \sigma_r)$*, where*

$$\sigma_0 = 1, \quad \sigma_1 = \sum_{i=1}^{r} t_i, \quad \sigma_2 = \sum_{i<j} t_i t_j, \quad \ldots \quad, \sigma_r = \prod_{i=1}^{r} t_i$$

are the elementary symmetric polynomials.

The proof is adapted from Prof. Dr. Richard Pinks lecture on algebra [Pin23, Thm. 7.3.8, lecture from May 15 2023].

Proof. Every polynomial F can uniquely be written as a finite sum

$$F = \sum_{d \geqslant 0} F_d,$$

where F_d are homogeneous polynomials of degree d. In the case where F is symmetric, every F_d is symmetric, so it suffices to prove the Lemma for symmetric F_d. If the statement is correct, then G is isobaric with weight d, where u_i has weight i. This suggests that we define $V := R[u_1, \ldots, u_r]_{\text{isobaric with weight d}}$ and $W := R[t_1, \ldots, t_r]_{\text{symmetric and homogeneous of degree d}}$,

$$\phi_d : V \to W, G \mapsto G(\sigma_1, \ldots, \sigma_r),$$

and prove that ϕ_d is a module isomorphism. To show that this linear map is bijective, we define an order on the monomials $t^i = t_1^{i_1} \cdots t_r^{i_r}$ in the following way

$$i > j :\Longleftrightarrow \exists 1 \leqslant \mu \leqslant r : \{\forall \nu > \mu : i_\nu = j_\nu \text{ and } i_\mu > j_\mu\}.$$

Then the smallest monomial in $\sigma_1^{i_1} \cdots \sigma_r^{i_r}$ is $t_1^{i_1+\cdots+i_r} \cdot t_2^{i_2+\cdots+i_r} \cdots t_r^{i_r}$. We will use this and induction to show bijectivity. For surjectivity, observe that $\phi_d(0) = 0$, so assume $0 \neq F_d \in W$, write $F_d = \sum a_i t^i$ and let j be the minimal multiindex with $a_j \neq 0$. By symmetry of F_d and minimality of j we must have $j_1 \geqslant j_2 \geqslant \cdots \geqslant j_r$ (otherwise we could switch two indices and get a smaller j). This implies that there exists $i = (i_1, \ldots, i_r)$ such that

$$j_1 = i_1 + \dots i_r, \quad j_2 = i_2 + \dots + i_r, \quad j_r = i_r.$$

Now define $G(u_1,\dots,u_r) = a_j u_1^{i_1} \cdots u_r^{i_r}$, isobaric with weight

$$i_1 + 2i_2 + \dots + ri_r = (i_1 + \dots + i_r) + (i_2 + \dots + i_r) + \dots + i_r = j_1 + \dots + j_r = d$$

and observe that

$$F_d(t_1,\dots,t_r) - \phi_d(G) = F_d - a_j \sigma_1^{i_1} \cdots \sigma_r^{i_r}$$

does not contain the monomial t^j, and because its the smallest monomial in F_d, the result does only contain bigger monomials. Since the number of multiindices $j = (j_1,\dots,j_r)$ with $\sum j_i = d$ is finite, by induction, we find finitely many $G_1,\dots,G_k$ such that

$$F_d - \phi_d(G_1) - \dots - \phi_d(G_k) = F_d - \phi_d(G_1 + \dots + G_k) = 0,$$

that is, ϕ_d is surjective. Injectivity is proven similar. Let $G \in ker(\phi_d)$ and write $G(u_1,\dots,u_r) = \sum b_i u^i$, where the multiindices are so that $i_1 + 2i_2 + \dots + ri_r = d$. Assume by contradiction that there is a $b_i \neq 0$, and let j be the minimal multiindex with $b_j \neq 0$. Then for all $i \neq j$ in the sum we have $i > j$ and therefore

$$(i_1 + \dots + i_r, i_2 + \dots + i_r, \dots, i_r) > (j_1 + \dots + j_r, j_2 + \dots + j_r, \dots, j_r) := J,$$

hence for G we have

$$G(\sigma_1,\dots,\sigma_r) = \sum_{i\neq j} b_i \sigma_1^{i_1} \cdots \sigma_r^{i_r} + b_j \sigma_1^{j_1} \cdots \sigma_r^{j_r},$$

where in $\sum_{i\neq j} b_i \sigma_1^{i_1} \cdots \sigma_r^{i_r}$ only monomials strictly bigger than t^J occur, and in $b_j \sigma_1^{j_1} \cdots \sigma_r^{j_r}$ the smallest monomial is t^J. By hypothesis that $G(\sigma_1,\dots,\sigma_r) = 0$ we conclude that $b_j = 0$. This implies $b_i = 0$ for all multiindices i and therefore $G = 0$. This proves injectivity and thus proves the lemma. □

Proof of Theorem 1.3. Suppose $\phi(P) = 0$, that is the $Ad(GL(r,\mathbb{C}))$- invariant polynomial vanishes on all diagonal matrices $X \in \mathfrak{gl}(r,\mathbb{C})$, so by its invariance under conjugation by $A \in GL(r,\mathbb{C})$ it vanishes on all diagonalizable matrices $X \in \mathfrak{gl}(r,\mathbb{C})$. By continuity, and since diagonalizable matrices $X \in \mathfrak{gl}(r,\mathbb{C})$ are dense in $\mathfrak{gl}(r,\mathbb{C})$, $P = 0$ as a polynomial function on $\mathfrak{gl}(r,\mathbb{C})$. By Theorem 1.1, $P = 0$ as an $Ad(GL(r,\mathbb{C}))$- invariant polynomial on $\mathfrak{gl}(r,\mathbb{C})$, that is $ker(\phi) = 0$.
Consider the characteristic polynomial of $(-X) \in \mathfrak{gl}(r,\mathbb{C}))$

$$P_\lambda(X) = ch_{-X}(\lambda) = det(\lambda \mathrm{I} + X) = \sum_{k=0}^{r} f_k(X)\lambda^{r-k},$$

then

$$\begin{aligned}\sum_{k=0}^{r} \phi(f_k(X))\lambda^{r-k} &= \phi(P_\lambda(X)) \\ &= P_\lambda(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix}) = det(\lambda \mathrm{I} + \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix}) \\ &= \prod_{i=1}^{r}(\lambda + t_i) \\ &= \sum_{k=0}^{r}\left(\sum_{i_1<\cdots<i_k} t_{i_1}\cdots t_{i_k}\right)\lambda^{r-k} \\ &= \sum_{k=0}^{r}\sigma_k(t)\lambda^{r-k}.\end{aligned}$$

So by comparison of coefficients we observe $\phi(f_k(X)) = \sigma_k(t)$. Let $F \in \mathbb{C}[t_1,\ldots,t_r]^{S_r}$, so we can write

$$F(t_1,\ldots,t_r) = G(\sigma_1,\ldots,\sigma_r).$$

Define $P(X) = G(f_1(X),\ldots,f_r(X))$ in $Inv(\mathfrak{gl}(r,\mathbb{C}))$, then

$$\begin{aligned}\phi(P(X)) &= \phi(G(f_1(X),\ldots,f_r(X))) \\ &= G(\phi(f_1(X)),\ldots,\phi(f_r(X))) \\ &= G(\sigma_1,\ldots,\sigma_r) \\ &= F(t_1,\ldots,t_r),\end{aligned}$$

so ϕ is also surjective. □

The isomorphism ϕ gives us generators of $Inv(\mathfrak{gl}(r,\mathbb{C}))$.

Theorem 1.6. *$Inv(\mathfrak{gl}(r,\mathbb{C}))$ as an algebra over $\mathbb{C}$ is generated by the coefficient polynomials $f_1(X),\ldots,f_r(X)$ of the characteristic polynomial $det(\lambda\mathrm{I} + X)$, that is*

$$Inv(\mathfrak{gl}(r,\mathbb{C}))\cong\mathbb{C}[f_1(X),\ldots,f_r(X)].$$

Proof. By the isomorphism ϕ in Theorem 1.3 and by Lemma 1.5 we have

$$Inv(\mathfrak{gl}(r,\mathbb{C})) \overset{\phi}{\cong} \mathbb{C}[t_1,\ldots,t_r]^{S_r} \overset{1.5}{\cong} \mathbb{C}[\sigma_1(t),\ldots,\sigma_r(t)] \overset{\phi}{\cong} \mathbb{C}[f_1(X),\ldots,f_r(X)].$$ □

We will use this result to study the real case.

1.3 Generators of $Inv(\mathfrak{gl}(r,\mathbb{R}))$

Proposition 1.7. *A real homogeneous polynomial $P(X)$ on $\mathfrak{gl}(r,\mathbb{R})$ that is invariant under conjugation by $GL(r,\mathbb{R})$ is invariant under conjugation by $GL(r,\mathbb{C})$. Since the converse is trivial we have*

$$Inv(\mathfrak{gl}(r,\mathbb{R})) = Inv(\mathfrak{gl}(r,\mathbb{C})) \cap \mathbb{R}[X],$$

and we have the following inclusion

$$Inv(\mathfrak{gl}(r,\mathbb{R})) \hookrightarrow Inv(\mathfrak{gl}(r,\mathbb{C})).$$

Proof. Suppose $P(X)$ has degree k and $A \in GL(r,\mathbb{R})$. We have $A^{-1} = \frac{1}{detA}A^\star$, where $A^\star$ denotes the adjugate of A. By homogeneity and invariance we have

$$P(AXA^\star) = (detA)^k P(AXA^{-1}) = (detA)^k P(X),$$

or

$$q(A,X) := P(AXA^\star) - (detA)^k P(X) = 0.$$

Now observe that q is defined via $A^\star$ instead of A^{-1}, so the formula makes sense for all $A, X \in \mathbb{C}^{r\times r}$. As a holomorphic function (polynomials are holomorphic) on $\mathbb{C}^{r\times r} \times \mathbb{C}^{r\times r}$, q has to be identically zero, since the roots of non-zero holomorphic functions are isolated, and q is identically zero on $GL(r,\mathbb{R}) \times \mathbb{R}^{r\times r}$. So for all $(A,X) \in GL(r,\mathbb{C}) \times \mathbb{C}^{r\times r}$

$$0 = \frac{1}{(detA)^k} q(A,X) = P(AXA^{-1}) - P(X). \qquad \square$$

Theorem 1.8. *The isomorphism $\phi : Inv(\mathfrak{gl}(r,\mathbb{C})) \to \mathbb{C}[t_1,\dots,t_r]^{S_r}$ in Theorem 1.3 restricts to an isomorphism of $\mathbb{R}$-algebras*

$$\begin{aligned} \phi_\mathbb{R} : Inv(\mathfrak{gl}(r,\mathbb{R})) &\to \mathbb{R}[t_1,\dots,t_r]^{S_r} \\ P(X) &\mapsto \widetilde{P}(t_1,\dots,t_r). \end{aligned}$$

Proof. The inclusions $\mathbb{R}[t_1,\dots,t_r]^{S_r} \hookrightarrow \mathbb{C}[t_1,\dots,t_r]^{S_r}$ and $Inv(\mathfrak{gl}(r,\mathbb{R})) \hookrightarrow Inv(\mathfrak{gl}(r,\mathbb{C}))$ give the commutative diagram

$$\begin{array}{ccc} Inv(\mathfrak{gl}(r,\mathbb{C})) & \xrightarrow{\phi} & \mathbb{C}[t_1,\dots,t_r]^{S_r} \\ {\scriptstyle \iota_1}\uparrow & & \uparrow{\scriptstyle \iota_2} \\ Inv(\mathfrak{gl}(r,\mathbb{R})) & \xrightarrow{\phi_\mathbb{R}} & \mathbb{R}[t_1,\dots,t_r]^{S_r}. \end{array}$$

$\phi \circ \iota_1$ is injective, so $\phi_\mathbb{R}$ is also injective.
Surjectivity of $\phi_\mathbb{R}$ is shown in the same way as surjectivity of ϕ. Let $F \in \mathbb{R}[t_1,\dots,t_r]^{S_r}$.

Again by Lemma 1.5, we can write

$$F(t_1, \dots, t_r) = G(\sigma_1, \dots, \sigma_r)$$

for a unique real polynomial G. Define $P(X) = G(f_1(X), \dots, f_r(X))$ in $Inv(\mathfrak{gl}(r, \mathbb{R}))$, then

$$\begin{aligned} \phi_{\mathbb{R}}(P(X)) &= \phi_{\mathbb{R}}(G(f_1(X), \dots, f_r(X)) \\ &= G(\phi_{\mathbb{R}}(f_1(X)), \dots, \phi_{\mathbb{R}}(f_r(X))) \\ &= G(\sigma_1, \dots, \sigma_r) \\ &= F(t_1, \dots, t_r). \end{aligned}$$ □

Analogously to Theorem 1.6 we get

Theorem 1.9. *$Inv(\mathfrak{gl}(r, \mathbb{R}))$ as an algebra over $\mathbb{R}$ is generated by the coefficient polynomials $f_1(X), \dots, f_r(X)$ of the characteristic polynomial $det(\lambda \mathrm{I} - X)$, that is*

$$Inv(\mathfrak{gl}(r, \mathbb{R})) = \mathbb{R}[f_1(X), \dots, f_r(X)].$$

Proof. By the isomorphism $\phi_{\mathbb{R}}$ in Theorem 1.8 and by Lemma 1.5 we have

$$Inv(\mathfrak{gl}(r, \mathbb{R})) \overset{\phi_{\mathbb{R}}}{\cong} \mathbb{R}[t_1, \dots, t_r]^{S_r} \overset{1.5}{\cong} \mathbb{R}[\sigma_1(t), \dots, \sigma_r(t)] \overset{\phi_{\mathbb{R}}}{\cong} \mathbb{R}[f_1(X), \dots, f_r(X)].$$ □

Finally we want to prove that $Inv(\mathfrak{gl}(r, \mathbb{R}))$ is not only generated by $f_1(X), \dots, f_r(X)$ over $\mathbb{R}$, but also by the trace polynomials $\Sigma_1(X), \dots, \Sigma_r(X)$ over $\mathbb{R}$. For this we consider the power sums

$$s_k(t_1, \dots, t_r) = \sum_{i=1}^{r} t_i^k,$$

which are symmetric polynomials in $t_1, \dots, t_r$. An algebraic result relates the elementary symmetric polynomials $\sigma_1, \dots, \sigma_r$ to the power sums $s_1, \dots, s_r$. For each $k \geqslant 1$,

$$s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} - \cdots + (-1)^{k-1} \sigma_{k-1} s_1 + (-1)^k k \sigma_k = 0.$$

These are Newton's identities. A proof can be found in [Tu17, Thm. B.14]. By induction we find polynomials F_k and G_k such that

$$\sigma_k = F_k(s_1, \dots, s_k), \quad s_k = G_k(\sigma_1, \dots, \sigma_k),$$

hence, the power sums also generate the symmetric polynomials over $\mathbb{R}$

$$\mathbb{R}[s_1, \dots, s_r] = \mathbb{R}[\sigma_1, \dots, \sigma_r] = \mathbb{R}[t_1, \dots, t_r]^{S_r}. \tag{1}$$

Furthermore, the power sums correspond to the trace polynomials under the isomorphism $\phi_\mathbb{R}$

$$\phi_R(\Sigma_k(X)) = \widetilde{\Sigma_k}(X) = \Sigma_k(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix}) = tr(\begin{pmatrix} t_1^k & & \\ & \ddots & \\ & & t_r^k \end{pmatrix}) = \sum_{i=1}^r t_i^k = s_k.$$

By this observation, Theorem 1.8 and equation (1) we get the following.

Theorem 1.10. *$Inv(\mathfrak{gl}(r,\mathbb{R}))$ as an algebra over $\mathbb{R}$ is generated by the trace polynomials $\Sigma_1(X), \ldots, \Sigma_r(X)$ of X, that is*

$$Inv(\mathfrak{gl}(r,\mathbb{R})) = \mathbb{R}[\Sigma_1(X), \ldots, \Sigma_r(X)].$$

To summarize Theorem 1.9 and Theorem 1.10 we just say, that every $Ad\ GL(r,\mathbb{R})$-invariant polynomial on $\mathfrak{gl}(r,\mathbb{R}))$ can be written as a polynomial in $f_1(X), \ldots, f_r(X)$ or as a polynomial in $\Sigma_1(X), \ldots, \Sigma_r(X)$.

At this point we understand enough about invariant polynomials to continue with Example 1.1.

Example 1.2. *Let again X be an $r \times r$-matrix. The $Ad(GL(r,\mathbb{R}))$-invariant polynomial $det(X)$ is given by the constant coefficient of the characteristic polynomial of $(-X)$*

$$f_r(X) = det(X).$$

The isomorphism $\phi_\mathbb{R}$ in Theorem 1.8 sends this to the r-th elementary symmetric polynomial

$$\phi_\mathbb{R}(det(X)) = \phi_\mathbb{R}(f_r(X)) = \sigma_r,$$

which in turn is related to the power sums by Newton's identities

$$s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} - \cdots + (-1)^{k-1}\sigma_{k-1}s_1 + (-1)^k k\sigma_k = 0.$$

Rewriting this yields

$$\sigma_k = \frac{(-1)^{k+1}}{k}\left(s_k - \sigma_1 s_{k-1} - \sigma_2 s_{k-2} + \cdots + (-1)^{k-1}\sigma_{k-1}s_1\right).$$

So inductively we get

$$\begin{aligned}
\sigma_1 &= s_1 = F_1(s_1)\\
\sigma_2 &= \frac{-1}{2}(s_2 - s_1^2) = F_2(s_1, s_2)\\
\sigma_3 &= \frac{1}{3}s_3 - \frac{1}{2}s_1 s_2 + \frac{1}{6}s_1^3 = F_3(s_1, s_2, s_3)
\end{aligned}$$

$$\begin{aligned}\sigma_4 &= F_4(s_1,\ldots,s_4)\\ &\vdots\\ \sigma_r &= F_r(s_1,\ldots,s_r).\end{aligned}$$

Now, using $\phi_{\mathbb{R}}(\Sigma_k(X)) = s_k$, we get a formula for the determinant of X in terms of the trace polynomials

$$\begin{aligned}det(X) &= f_r(X)\\ &= \phi_{\mathbb{R}}^{-1}\left(\phi_{\mathbb{R}}(f_r(X))\right)\\ &= \phi_{\mathbb{R}}^{-1}(\sigma_r)\\ &= \phi_{\mathbb{R}}^{-1}(F_r(s_1,\ldots,s_r))\\ &= F_r(\phi_{\mathbb{R}}^{-1}(s_1),\ldots,\phi_{\mathbb{R}}^{-1}(s_r))\\ &= F_r(\Sigma_1(X),\ldots,\Sigma_r(X)).\end{aligned}$$

So for example, in the case where

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3}\\ x_{2,1} & x_{2,2} & x_{2,3}\\ x_{3,1} & x_{3,2} & x_{3,3}\end{pmatrix}$$

is a 3×3*-matrix, we get*

$$det(X) = F_3(\Sigma_1(X),\Sigma_2(X),\Sigma_3(X)) = \frac{1}{3}\Sigma_3(X) - \frac{1}{2}\Sigma_1(X)\Sigma_2(X) + \frac{1}{6}\Sigma_1(X)^3.$$

This can be checked by directly computing the determinant and the trace polynomials of X.

2 Prerequisites from Differential Geometry

A characteristic class is the cohomology class of $P(\Omega)$, where $P(X)$ is an invariant polynomial and Ω is the curvature form of a connection ∇ on a vector bundle $E \to M$. We will briefly review vector bundles and connections, then we will introduce connection and curvature forms and discuss some important properties, that are needed to study characteristic classes. From now on, M will denote a smooth manifold of dimension m. That is a second countable Hausdorff topological space equipped with a maximal smooth atlas [1]. The Hausdorff property and second countability ensure the existence of a partition of unity subordinate to any open cover (see [Lee12, Thm. 2.23]). The existence of a partition of unity then allows us to define a Riemannian metric on every smooth manifold.

2.1 Connections and Curvature

A vector bundle $E \to M$ is a family of vector spaces of same dimension, smoothly parametrized over a base manifold M, so that each point in M has a corresponding vector space attached to it. Let us recall the basic definitions.

Definition 2.1. *A real vector bundle of rank r, denoted by $\pi : E \to M$, consists of a smooth manifold E of dimension $m + r$ and a smooth projection $\pi : E \to M$, such that*

i) for all $p \in M$, the fiber $E_p = \pi^{-1}(\{p\})$ is an r-dimensional real vector space, and

ii) for all $p \in M$, there exists an open neighborhood $U \ni p$ in M and a smooth diffeomorphism

$$\psi : \pi^{-1}(U) \to U \times \mathbb{R}^r,$$

such that $\psi_{E_q} : E_q \to \{q\} \times \mathbb{R}^r$ is a linear isomorphism for all $q \in U$.

A complex vector bundle is defined in the same way, but the fibers E_p carry the structure of an r-dimensional complex vector space. We will denote by $\Gamma(E)$ the space of smooth sections of E, that is the space of all smooth functions $s : M \to E$, such that $\pi \circ s = id_M$. Two simple examples for vector bundles are the trivial one $M \times \mathbb{R}^r$ and the tangent space of a manifold TM. The next step is to define a connection on a vector bundle.

Definition 2.2. *Let $\pi : E \to M$ be a vector bundle over M. A connection on E is an $\mathbb{R}$-bilinear map*

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E), (X, s) \mapsto \nabla(X, s) = \nabla_X s,$$

that satisfies

i) $\nabla_{fX} s = f\nabla_X s$, and

[1] We want to avoid spaces that behave unexpectedly: The real line with two origins is not Hausdorff and the space of uncountable copies of $\mathbb{R}^n$ is not second countable.

ii) $\nabla_X(fs) = (Xf)s + f\nabla_X s$,

for all $f \in C^\infty(M)$, $X \in \Gamma(TM)$ *and* $s \in \Gamma(E)$.

A connection on a complex vector bundle is defined the same way, except that it is $\mathbb{C}$-linear in the second argument. A connection on a complex vector bundle over a complex manifolds is $\mathbb{C}$-linear in both arguments.

Note that ∇ is a local operator, that is it can be restricted to open subsets of M. Analogously to the Riemann curvature tensor we define for any connection ∇ on E the multilinear map

$$\begin{aligned} R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) &\to \Gamma(E) \\ (X, Y, s) &\mapsto R(X, Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s. \end{aligned}$$

Let $f \in C^\infty(M)$ be any smooth function on M, then

$$\begin{aligned} R(fX, Y)s &= \nabla_{fX} \nabla_Y s - \nabla_Y \nabla_{fX} s - \nabla_{[fX,Y]} s \\ &= f\nabla_X \nabla_Y s - \nabla_Y f \nabla_X s - \nabla_{f[X,Y]-(Yf)X} s \\ &= f\nabla_X \nabla_Y s - f\nabla_Y \nabla_X s - (Yf)\nabla_X s - f\nabla_{[X,Y]} s + (Yf)\nabla_X s \\ &= fR(X, Y)s, \end{aligned}$$

and similar for the second argument. Further

$$\begin{aligned} R(X, Y)fs &= \nabla_X \nabla_Y fs - \nabla_Y \nabla_X fs - \nabla_{[X,Y]} fs \\ &= \nabla_X (Yf)s + \nabla_X f \nabla_Y s - \nabla_y (Xf)s - \nabla_Y f \nabla_X s - ([X, Y]f)s - f\nabla_{[X,Y]} s \\ &= (XYf)s + (Yf)\nabla_X s + (Xf)\nabla_Y s + f\nabla_X \nabla_Y s \\ &\quad - (YXf)s - (Xf)\nabla_Y s - (Yf)\nabla_X s - f\nabla_Y \nabla_X s \\ &\quad - (XYf)s + (YXf)s - f\nabla_{[X,Y]} s \\ &= fR(X, Y)s, \end{aligned}$$

so R is tensorial in all three arguments and therefore R is also a local operator. We call it the curvature tensor of the connection ∇ (when ∇ is the Levi-Civita connection on $TM \to M$, R is the Riemann curvature tensor). The curvature tensor is also defined for complex vector bundles and is given by the same formula.

Now suppose that $U \subset M$ is open and the rank r vector bundle E is trivial on U, that is there is a frame $e_1, \ldots, e_r$ for E on U. Then every section s on U can be written as a linear combination $a^i e_i$. Here we used Einstein's summation convention: We sum over repeated indices: $a^i e_i := \sum_{i=1}^r a^i e_i$. We will use this convention frequently. Let $X \in \Gamma(TM)|_U$ be a smooth vector field on U. Then for every $i \in \{1, \ldots, r\}$, $\nabla_X e_i$ defines a smooth section on U, and therefore can be written as $\nabla_X e_i = \omega_i^j e_j$, where the coefficients $\omega_i^j = \omega_i^j(X)$ are C^∞-linear in X. Hence the coefficients define a matrix of differential 1-forms $[\omega_i^j]$ on U.

We will call this matrix $\omega = [\omega_i^j]$ the connection matrix of the connection ∇ relative to the frame $e_1, \ldots, e_r$ on U. We can view ω as a matrix-valued 1-form on M, i.e. a smooth and alternating function $\omega : \Gamma(TM) \to \Gamma(GL(r, \mathbb{R}))$. Proceeding in the same manner, but with a complex vector bundle, we get a matrix of complex valued 1-forms, which also can be viewed as a matrix-valued 1-form $\omega : \Gamma(TM) \to \Gamma(GL(r, \mathbb{C}))$.

Similarly we define the curvature matrix of Ω of the connection ∇ relative to the frame $e_1, \ldots, e_r$ on U. Since $R(X,Y)e_i$ is alternating and C^∞-linear in X and Y, $R(X,Y)e_i = \Omega_i^j(X,Y)e_j$ defines a matrix of differential 2-forms $\Omega = [\Omega_i^j]$ on U. And for a complex vector bundle we get a matrix of complex valued 2-forms. Of course, both of these can be viewed as matrix-valued 2-forms $\Omega : \Gamma(TM) \times \Gamma(TM) \to \Gamma(GL(r, \mathbb{R}))$ or $\Gamma(GL(r, \mathbb{C}))$.

2.2 Properties of Connection and Curvature Matrices

By definition, the curvature tensor R is related to the connection ∇, so one might expect that there is a relation between Ω and ω. This is the content of the next theorem. To state this, we define the wedge product $\wedge$ and exterior derivative d for matrices of differential forms. For a (l,m)-matrix $a = [a_i^j]$ of differential r-forms and a (m,n)-matrix $b = [b_i^j]$ of differential s forms, we define the (l,n)-matrix $a \wedge b$ of differential $(r+s)$-forms via

$$(a \wedge b)_i^j = a_k^j \wedge b_i^k,$$

and the (l,m)-matrix da of differential $(r+1)$ forms via

$$(da)_i^j = d(a_i^j).$$

Proposition 2.3. *For the wedge product and exterior derivative of matrices of differential forms we have the basic algebraic results,*

i) $(a \wedge b)^T = (-1)^{rs} b^T \wedge a^T$

ii) If $l = n$, *then* $tr(a \wedge b) = (-1)^{rs} tr(b \wedge a)$

iii) If $l = m$, *then* $d\, tr(a) = tr(da)$.

Proof. A short and direct computation. □

Theorem 2.4. *Let* $\pi : E \to M$ *be a vector bundle of rank* r *and* $U \subset M$ *a trivializing open set with frame* $e_1, \ldots, e_r$. *Let* ω *and* Ω *be the connection and curvature matrix of* ∇ *relative to the frame* $e_1, \ldots, e_r$ *on* U *respectively. Then we have the relation*

$$\Omega_i^j = d\omega_i^j + \omega_k^j \wedge \omega_i^k,$$

or, in matrix notation,

$$\Omega = d\omega + \omega \wedge \omega.$$

This is called the second structural equation. It is proved by direct computation ([Tu17, Thm. 11.1]). There is also a first structural equation, but this is not of interest in this work.

The connection and curvature matrices of a connection ∇ on TM relative to a frame $e_1, \dots, e_m$ on a trivializing open subset $U \subset M$ additionally satisfy the following relations for any integer $k \geqslant 1$

i) (Second Bianchi identity) $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$,

ii) (Generalized second Bianchi identity) $d(\Omega^k) = \Omega^k \wedge \omega - \omega \wedge \Omega^k$.

There is a first Bianchi identity as well, but we will not need it. The second Bianchi identity follows directly from the second structural equation

$$\begin{aligned} d\Omega &= d\left(d\omega + \omega \wedge \omega\right) \\ &= dd\omega + dw \wedge \omega - \omega \wedge d\omega \\ &= (\Omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (\Omega - \omega \wedge \omega) \\ &= \Omega \wedge \omega - \omega \wedge \Omega. \end{aligned}$$

The generalized second Bianchi identity follows from the second Bianchi identity and by induction: Assume the formula is correct for $k-1$, then

$$\begin{aligned} d(\Omega^k) &= d(\Omega^{k-1} \wedge \Omega) \\ &= d\Omega^{k-1} \wedge \Omega + (-1)^{2(k-1)}\Omega^{k-1} \wedge d\Omega \\ &= \left(\Omega^{k-1} \wedge \omega - \omega \wedge \Omega^{k-1}\right) \wedge \Omega + \Omega^{k-1} \wedge (\Omega \wedge \omega - \omega \wedge \Omega) \\ &= \Omega^k \wedge \omega - \omega \wedge \Omega^k. \end{aligned}$$

Of course there is not just one frame $e_1, \dots, e_r$ on a trivializing open subset $U \subset M$ that we could choose to define connection and curvature matrices. Suppose $\widetilde{e}_1, \dots, \widetilde{e}_r$ is another frame of E on U. Then at each point $p \in U$ the two frames are related to each other by multiplication with an invertible matrix a

$$\widetilde{e}_i(p) = e_j(p) a_i^j(p).$$

Note that $\widetilde{e}_i$ is smooth on U for all $i \in \{1, \dots, r\}$, so the coefficients a_i^j with respect to the smooth frame $e_1, \dots, e_r$ have to be smooth on U, thus we get a matrix $a = [a_i^j]$ of smooth functions on U. We may write each frame as row vectors $\widetilde{e} = [\widetilde{e}_1, \dots, \widetilde{e}_r]$, $e = [e_1, \dots, e_r]$ and therefore in matrix notation

$$\widetilde{e} = ea$$

as smooth functions of p on U. (We will interchangeably write e for the frame $e_1, \dots, e_r$ and for the row vector $[e_1, \dots, e_r]$. The exact meaning will be clear from context.) This leads to the following change of basis formulae of connection and curvature matrices.

Theorem 2.5. *Let $\pi : E \to M$ be a vector bundle of rank r and $U \subset M$ a trivializing open set with frames e and $\tilde{e}$ such that $\tilde{e} = ea$ for a matrix a of smooth functions on U. Let ω and $\tilde{\omega}$ be the connection matrices of ∇ relative to e and $\tilde{e}$ and let Ω and $\tilde{\Omega}$ be the curvature matrices of ∇ relative to e and $\tilde{e}$, then*

i) $\tilde{\omega} = a^{-1}\omega a + a^{-1}da$,

ii) $\tilde{\Omega} = a^{-1}\Omega a$.

A direct computation proves Theorem 2.5, see [Tu17, Thm 22.1] for a detailed proof.

We have seen that, on a trivializing set $U \subset M$ with frame e, a connection ∇ defines an $r \times r$ matrix of 1-forms ω, called the connection matrix. Conversely, assume that a trivializing set U with frame e and an $r \times r$ matrix of 1-forms ω are given. For a vector field $X \in \Gamma(TU)$ and a section $s \in \Gamma(E|_U)$, $s = s^j e_j$, define

$$\nabla^\star_X s := \left(X(s^j) + s^i \omega_i^j(X)\right) e_j. \tag{2}$$

$\nabla^\star$ is clearly $\mathbb{R}$-linear in X and s, since ω and X are $\mathbb{R}$-linear in X and s respectively. Further we can compute for any C^∞ function f und U

$$\begin{aligned}
\nabla^\star_{fx} s &= \left(fX(s^j) + s^i\omega_i^j(fX))\right) e_j \\
&= f\left(X(s^j) + s^i\omega_i^j(X)\right) e_j \\
&= f\nabla^\star_X s \\
\nabla^\star_X fs &= \left(X(fs^j) + fs^i\omega_i^j(X)\right) e_j \\
&= X(f)s + f\left(X(s^j) + s^i\omega_i^j(X)\right) e_j \\
&= X(f)s + f\nabla^\star_X s.
\end{aligned}$$

We have shown that $\nabla^\star$ defines a connection on U. It is an easy check, that the connection matrix of $\nabla^\star$ is ω. Thus, from now on, we will denote this connection just by ∇. Further suppose that $\tilde{e}$ is another frame of U such that $\tilde{e} = ea$ for an invertible matrix a and $\tilde{\omega} = a^{-1}\omega a + a^{-1}da$. Then Theorem 2.5 implies that the two (a priori) different connections defined via (2) by ω and $\tilde{\omega}$ coincide. Therefore, given a cover of M by trivializing subsets $\{U_i\}$ with frames e_i and $r \times r$ matrices of 1-forms ω_i such that pairwise on intersections all these matrices satisfy i), we can define a connection on M with this data.

Given a smooth map $f : N \to M$ between two smooth manifolds, we can pull back the connection matrices of a connection and use these to define a unique connection on the pullback bundle.

Definition 2.6. *Let $\pi : E \to M$ be a vector bundle over M and $f : N \to M$ be a smooth* *map between smooth manifolds. Define the total space*

$$f^\star(E) := \{(n, e) \in N \times E | f(n) = \pi(e)\}$$

and endow it with the subspace topology. Then $f^\star(E)$ together with the projection map $\rho : f^\star(E) \to N, (n, e) \mapsto n$ defines a vector bundle over N, called the pullback bundle of E by f.

[Tu17, Thm. 20.6] proves that $\rho : f^\star(E) \to N$ indeed defines a vector bundle.

Theorem 2.7. *Let $f : N \to M$ be a smooth map between smooth manifolds. Let $E \to M$ be a vector bundle over M with connection ∇ on E, and denote by $f^\star E \to N$ the pullback bundle over N. Let $\{U_\alpha\}_\alpha$ be an open cover of M with frames e_α and connection matrices ω_α relative to e_α. Then there exists a unique connection on $f^\star E$ with connection matrices $f^\star(\omega_\alpha)$ relative to $f^\star(e_\alpha)$.*

Proof. Write $e_\beta = e_\alpha a$ for and invertible matrix a on $U_\beta \cap U_\alpha$. Then $f^\star(e_\beta) = f^\star(e_\alpha) f^\star(a)$ on $f^{-1}(U_\beta \cap U_\alpha)$, and

$$\begin{aligned} f^\star(\omega_\beta) &= f^\star(a^{-1}\omega_\alpha a + a^{-1}da) \\ &= f^\star(a^{-1})f^\star(\omega_\alpha)f^\star(a) + f^\star(a^{-1})df^\star(a), \end{aligned}$$

on $f^{-1}(U_\beta \cap U_\alpha)$, that is $f^\star(\omega_\alpha)$ satisfies i). Therefore, by the previous paragraph, it induces a unique connection on the pullback bundle $f^\star E$ with connection matrices $f^\star(\omega_\alpha)$. □

In the proof of the main theorem about characteristic classes (Theorem 3.1), we will also need to work with differential forms that vary smoothly in time. Let $\{\omega_t\}_t$ be a family of smooth k-forms on M that varies smoothly with t. That is, locally

$$\omega_t = \sum_I a_I(x, t) dx^I$$

for smooth functions $a_I(x, t)$ and increasing multi-indices I. For every $p \in M$ we define the map $t \mapsto \omega_{t,p}$ and

$$\dot{\omega}_{t,p} = \left(\frac{d\omega_t}{dt}\right)_p = \frac{d}{dt}\omega_{t,p}.$$

We also define for $a < b$

$$\left(\int_a^b \omega_t dt\right)_p = \int_a^b \omega_{t,p} dt.$$

Locally, this is

$$\frac{d}{dt}\omega_t = \sum_I \frac{\partial a_I}{\partial t}(x, t) dx^I,$$

$$\int_a^b \omega_t dt = \sum_I \left(\int_a^b a_I(x,t)dt \right) dx^I.$$

These definitions extend entry by entry to matrices of differential forms that depend smoothly on a parameter t.

Proposition 2.8. *Let ω_t and τ_t be matrices of smooth forms on M that depend smoothly on t. Then*

i) If ω is a square matrix, $\frac{d}{dt}(\mathrm{tr}\ \omega_t) = \mathrm{tr}\left(\frac{d\omega_t}{dt}\right)$.

ii) If $\omega \wedge \tau$ is defined, $\frac{d}{dt}(\omega \wedge \tau) = \dot{\omega} \wedge \tau + \omega \wedge \dot{\tau}$.

iii) $\frac{d}{dt}(d\omega) = d\left(\frac{d}{dt}\omega\right)$.

iv) $\int_a^b d\omega_t dt = d\left(\int_a^b \omega_t dt\right)$.

Proof. Write $\omega_t = [\omega^i_j(t)]$, then

$$\frac{d}{dt}(\mathrm{tr}\ \omega_t) = \frac{d}{dt}\omega^i_i(t) = \sum \frac{d\omega^i_i}{dt}(t) = \mathrm{tr}\ \frac{d\omega_t}{dt}.$$

To prove the second equation, write the entries of ω and τ in coordinates

$$\begin{aligned} \omega^i_j(t) &= \sum_I a^i_{j,I}(x,t)dx^I \\ \tau^i_j(t) &= \sum_I b^i_{j,I}(x,t)dx^I, \end{aligned}$$

then

$$\begin{aligned} \frac{d}{dt}(\omega \wedge \tau)^i_j &= \frac{d}{dt}\omega^i_k(t) \wedge \tau^k_j(t) \\ &= \frac{d}{dt}\sum_{I,J} a^i_{k,I}(x,t)b^k_{j,J}(x,t)dx^I \wedge dx^J \\ &= \sum_{I,J}\left(\dot{a}^i_{k,I}(x,t)b^k_{j,J}(x,t) + a^i_{k,I}(x,t)\dot{b}^k_{j,J}(x,t)\right) dx^I \wedge dx^J \\ &= \dot{\omega}^i_k(t) \wedge \tau^k_j(t) + \omega^i_k(t) \wedge \dot{\tau}^k_j(t) \\ &= (\dot{\omega} \wedge \tau)^i_j + (\omega \wedge \dot{\tau})^i_j. \end{aligned}$$

For the third equation, compute again in coordinates

$$\begin{aligned} \frac{d}{dt}\left(d\omega^i_j(t)\right) &= \frac{d}{dt} d \sum_I a^i_{j,I}(x,t)dx^I \\ &= \frac{d}{dt}\sum_k \sum_I \frac{\partial}{\partial x^k} a^i_{j,I}(x,t)dx^k \wedge dx^I \end{aligned}$$

$$= \sum_k \sum_I \frac{\partial}{\partial x^k} \frac{\partial}{\partial t} a^i_{j,I}(x,t) dx^k \wedge dx^I$$
$$= d\left(\frac{d}{dt}\omega^i_j(t)\right).$$

Lastly,

$$\begin{aligned}\left(\int_a^b d\omega_t dt\right)^i_j &= \sum_k \sum_I \left(\int_a^b \frac{\partial}{\partial x^k} a^i_{j,I}(x,t)dt\right) dx^k \wedge dx^I \\ &= \sum_k \sum_I \frac{\partial}{\partial x^k}\left(\int_a^b a^i_{j,I}(x,t)dt\right) dx^k \wedge dx^I \\ &= d\left(\sum_I \left(\int_a^b a^i_{j,I}(x,t)dt\right) dx^I\right) \\ &= d\left(\int_a^b \omega_t dt\right)^i_j \\ &= \left(d\int_a^b \omega_t dt\right)^i_j,\end{aligned}$$

where we are allowed to interchange differentiation and integration, because the interval $[a,b]$ is compact and $a^i_{j,I}$ is smooth. □

Everything we said above about the connection and curvature matrices and matrices of differential forms, as well as about connections and the pullback of connections has a direct translation to the complex case.

2.3 Metrics on a Vector Bundle

Later we will need more structure on the vector bundle $E \to M$. Let g be a map that assigns to all $p \in M$ an inner product $g(p)(\cdot,\cdot) : E_p \times E_p \to \mathbb{R}$ on the fiber E_p over p in a smooth way. By smooth, we mean that for all smooth sections $s,t : M \to E$, the map $M \ni p \mapsto g(p)(s(p),t(p)) \in \mathbb{R}$ is smooth. Then we call g a Riemannian metric on E and the tupel (E,g) a Riemannian bundle. The existence of such a Riemannian metric is proved in the same way as in the case $E = TM$, using a partition of unity. Analogously to the case $E = TM$, we say that a connection ∇ on E is compatible with the metric g, if for all vector fields X and for all sections $s,t : M \to E$

$$Xg(s,t) = g(\nabla_X s, t) + g(s, \nabla_X t)$$

as functions $M \to \mathbb{R}$. As in the case $E = TM$, a metric connection exists. A proof can be found in [Tu17, Ch. 10.5]. Similarly, in the complex case, a Hermitian metric is a map g that assigns to all points $p \in M$ a complex inner product $g(p)$ in a smooth way. That is, $g(p)$ satisfies

i) $g(p)(s_p, s_p) \geqslant 0$, with equality if and only if $s_p = 0$,

ii) $g(p)(s_p, t_p) = \overline{g(p)(t_p, s_p)}$ for all $s, t \in \Gamma(E)$,

iii) $g(p)(\lambda s_p + \mu t_p, u_p) = \lambda g(p)(s_p, u_p) + \mu g(p)(t_p, u_p)$ for all $\lambda, \mu \in \mathbb{C}$ and all $s, t, u \in \Gamma(E)$.

Existence of such a Hermitian metric is proved as in the real case via a partition of unity. A complex vector bundle together with a Hermitian metric is called a Hermitian bundle. A complex connection on a Hermitian bundle is compatible with the metric, if the same property as in the real case is satisfied, that is

$$Xg(s,t) = g(\nabla_X s, t) + g(s, \nabla_X t),$$

for any real tangent vector field X and all sections $s, t \in \Gamma(E)$.

Now let $e_1, \ldots, e_r$ be an orthonormal frame on some open set $U \subset M$ of the vector bundle $\pi : E \to M$. Then, for a metric connection ∇ and a smooth vector field X over U, we have for all $1 \leqslant i, j \leqslant r$

$$\begin{aligned} 0 &= Xg(e_i, e_j) \\ &= g(\nabla_X e_i, e_j) + g(e_i, \nabla_X e_j) \\ &= g(\omega_i^k(X) e_k, e_j) + g(e_i, \omega_j^k(X) e_k) \\ &= \omega_i^j(X) + \omega_j^i(X), \end{aligned}$$

so ω is skew symmetric. Thus, Ω is skew symmetric as well

$$\begin{aligned} \Omega_j^i &= d\omega_j^i + \omega_k^i \wedge \omega_j^k \\ &= d(-\omega_i^j) + (-\omega_i^k) \wedge (-\omega_k^j) \\ &= -d\omega_i^j - (-\omega_k^j) \wedge (-\omega_i^k) \\ &= -\Omega_i^j. \end{aligned}$$

So in this case, we can view the connection form and the curvature form as matrix-valued differential forms, where the matrix is skew symmetric. In the complex case, we have

$$\omega = -\overline{\omega}^T, \quad \Omega = -\overline{\Omega}^T,$$

thus, the curvature and connection matrix are skew hermitian.

3 Characteristic Classes

Finally, we can explain how the curvature of a connection ∇ on a vector bundle $E \to M$, together with an invariant polynomial $P(X)$ defines a closed differential form $P(\Omega)$. We will also prove that the cohomology class $[P(\Omega)]$ is independent of the connection ∇. This is exactly the content of Theorem 3.1. This theorem immediately leads to the Chern-Weil homomorphism (3). We will also look at characteristic classes from a more abstract point of view and introduce Pontrjagin, Euler and Chern classes, which are important special cases of characteristic classes. Along this, we will state a generalization of the famous Gauss-Bonnet theorem, which uses Euler classes. Lastly, we will compute the Chern class of a simple example.

3.1 The Chern-Weil Homomorphism

Let $E \to M$ be a smooth vector bundle over M of rank r, and let ∇ be a connection on E. Let $\{U_\alpha\}_\alpha$ be a trivializing open cover and let $e^\alpha = [e_1^\alpha, \ldots, e_r^\alpha]$ be a frame of E over U_α. Denote by Ω_α the curvature matrix of ∇ over U_α relative to e^α. Now consider a homogeneous invariant polynomial $P(X)$ of degree k on $\mathfrak{gl}(r, \mathbb{R})$. Then for all α, β, $P(\Omega_\alpha)$ is a $2k$-form on U_α and on overlaps $U_\alpha \cap U_\beta$ the two forms $P(\Omega_\alpha)$ and $P(\Omega_\beta)$ coincide, because of invariance of P and the change of basis formula for Ω. Therefore, P and ∇ induce a global $2k$-form $P(\Omega)$.

Theorem 3.1. *In the above setting, the global $2k$-form $P(\Omega)$ is closed, and its cohomology class in $H^{2k}(M)$ is independent of the connection ∇.*

Proof. Recall that we proved that the $Ad\ GL(r, \mathbb{R})$-invariant polynomials on $\mathfrak{gl}(r, \mathbb{R})$ are generated by the trace polynomials $\Sigma_1(X), \ldots, \Sigma_r(X)$. Therefore it suffices to prove the theorem for the trace polynomials.
By Proposition 2.3 and the generalized second Bianchi identity we can compute

$$\begin{aligned} d\ \mathrm{tr}(\Omega^k) &= \mathrm{tr}(d\Omega^k) \\ &= \mathrm{tr}(\Omega^k \wedge \omega - \omega \wedge \Omega^k) \\ &= \mathrm{tr}(\Omega^k \wedge \omega) - \mathrm{tr}(\omega \wedge \Omega^k) \\ &= \mathrm{tr}(\Omega^k \wedge \omega) - (-1)^{2k}\mathrm{tr}(\Omega^k \wedge \omega) \\ &= 0, \end{aligned}$$

since Ω^k is a matrix of differential forms of even degree $2k$. This proves the first part of the theorem.
Next, we consider two connections ∇^0, ∇^1 on E. Given a frame $e = e_1, \ldots, e_r$ of E on an open set U, we denote by ω_0, ω_1 and Ω_0, Ω_1 the connection and curvature matrices of ∇^0

and ∇^1 relative to e. Observe that

$$\nabla^t := (1-t)\nabla^0 + t\nabla^1$$

defines a connection on E for every $t \in [0,1]$ and denote by ω and Ω the connection and curvature matrix of ∇^t relative to e. For any smooth vector field X on U,

$$\begin{aligned}\nabla^t_X e_j &= \left((1-t)\nabla^0_X + t\nabla^1_X\right) e_j \\ &= \left((1-t)(\omega_0)^i_j + t(\omega_1)^i_j\right)(X) e_i\end{aligned}$$

shows that $\omega = (1-t)\omega_0 + t\omega_1$ varies smoothly with t. So by the structural equation,

$$\Omega = d\omega + \omega \wedge \omega$$

also varies smoothly with t and we can differentiate the trace polynomials Σ_k of Ω

$$\begin{aligned}\frac{d}{dt}\Sigma_k(\Omega) &= \frac{d}{dt}\mathrm{tr}(\Omega^k) \\ &= \mathrm{tr}\left(\frac{d}{dt}\Omega^k\right) \\ &= \mathrm{tr}\left(\dot{\Omega} \wedge \Omega^{k-1} + \Omega \wedge \dot{\Omega} \wedge \Omega^{k-2+} \cdots + \Omega^{k-1} \wedge \dot{\Omega}\right) \\ &= k \ \mathrm{tr}\left(\Omega^{k-1} \wedge \dot{\Omega}\right)\end{aligned}$$

by Proposition 2.8 and Proposition 2.3 because Ω is a matrix of 2-forms. Also by Proposition 2.8, the exterior derivative and taking the trace commute with the derivative in time, so again by the structural equation

$$\begin{aligned}\mathrm{tr}\left(\Omega^{k-1} \wedge \dot{\Omega}\right) &= \mathrm{tr}\left(\Omega^{k-1} \wedge \frac{d}{dt}(d\omega + \omega \wedge \omega)\right) \\ &= \mathrm{tr}\left(\Omega^{k-1} \wedge d\dot{\omega} + \Omega^{k-1} \wedge \dot{\omega} \wedge \omega + \Omega^{k-1} \wedge \omega \wedge \dot{\omega}\right) \\ &= \mathrm{tr}\left(\Omega^{k-1} \wedge d\dot{\omega} - \omega \wedge \Omega^{k-1} \wedge \dot{\omega} + \Omega^{k-1} \wedge \omega \wedge \dot{\omega}\right) \\ &= \mathrm{tr}\left(\Omega^{k-1} \wedge d\dot{\omega} + \left(d\Omega^{k-1}\right) \wedge \dot{\omega}\right) \\ &= \mathrm{tr}\left(d\left(\Omega^{k-1} \wedge \dot{\omega}\right)\right) \\ &= d\left(\mathrm{tr}\left(\Omega^{k-1} \wedge \dot{\omega}\right)\right).\end{aligned}$$

Here, we also used the generalized second Bianchi identity and that ω is a matrix of 1-forms. Integration over time yields

$$\Sigma_k(\Omega_1) - \Sigma_k(\Omega_0) = \int_0^1 \frac{d}{dt}\Sigma_k(\Omega)dt$$

$$= \int_0^1 d\left(k \text{ tr}\left(\Omega^{k-1} \wedge \dot{\omega}\right)\right)$$
$$= d\int_0^1 k \text{ tr}\left(\Omega^{k-1} \wedge \dot{\omega}\right).$$

Note that under a change of frame $\tilde{e} = ea$, we have $\dot{\tilde{\omega}} = a^{-1}\dot{\omega}a$, because a is independent of time. Hence $\tilde{\Omega}^{k-1} \wedge \dot{\tilde{\omega}} = a^{-1}\left(\Omega^{k-1} \wedge \dot{\omega}\right) a$, and since the trace polynomials are invariant under conjugation by invertible matrices, $\text{tr}\left(\Omega^{k-1} \wedge \dot{\omega}\right)$ induces a global form. That is, $\Sigma_k(\Omega_1) - \Sigma_k(\Omega_0) = d\int_0^1 k \text{ tr}\left(\Omega^{k-1} \wedge \dot{\omega}\right)$ is a global equation, so the cohomology classes of $\Sigma_k(\Omega_0)$ and $\Sigma_k(\Omega_1)$ coincide. □

The theorem shows that for any vector bundle $E \to M$ of rank r, the map

$$\begin{aligned} c_E : Inv(\mathfrak{gl}(r,\mathbb{R}) &\to H^\star(M), \\ P(X) &\mapsto [P(\Omega)], \end{aligned} \tag{3}$$

is a well-defined algebra homomorphism, called the Chern-Weil homomorphism.

Assume that $F \to M$ is a vector bundle, isomorphic to E. That is there exist smooth maps $\varphi : E \to F$, $\psi : F \to E$ such that

i) the diagrams

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow[id_M]{} & M \end{array}$$

and

$$\begin{array}{ccc} F & \xrightarrow{\psi} & E \\ {\scriptstyle \pi_F}\downarrow & & \downarrow{\scriptstyle \pi_E} \\ M & \xrightarrow[id_M]{} & M \end{array}$$

commute,

ii) φ and ψ restrict to linear maps $\varphi_p : E_p \to F_p$ and $\psi_p : F_p \to E_p$ for all $p \in M$,

iii) $\varphi \circ \psi = id_F$ and $\psi \circ \varphi = id_E$.

Now let $\{U_\alpha\}_\alpha$ be a trivializing cover of M with frames e_α for E and denote by ω_α the connection matrices of a connection ∇^E on E relative to e_α. Then, $id_M^\star(e_\alpha)$ defines a frame on $id_M^{-1}(U_\alpha) = U_\alpha$ for F and the pullbacks $id_M^\star(\omega_\alpha) = \omega_\alpha$ of the connection matrices, define a connection ∇^F on F with the same connection matrices as the connection ∇^E. Therefore, the curvature matrices Ω_α^E of ∇^E and the curvature matrices Ω_α^F of ∇^F coincide. Hence for a homogeneous invariant polynomial $P(X)$, we have $P(\Omega^E) = P(\Omega^F)$. That is, the cohomology class $[P(\Omega)]$ does only depend on the isomorphism class of the vector bundle $E \to M$. This leads us to the following more general definition of a characteristic class.

Definition 3.2. *A characteristic class on real vector bundles associates to each manifold a map*

$$c_M : \left\{ \begin{matrix} \text{isomorphism classes of real} \\ \text{vector bundles over } M \end{matrix} \right\} \to H^\star(M),$$

such that for any smooth map $f : N \to M$ between smooth manifolds and any vector bundle $E \to M$

$$c_N (f^\star E) = f^\star c_M(E).$$

Denote by $Vect_r(M)$ the set of all isomorphism classes of rank r vector bundles over M. Then the definition is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} Vect_r(M) & \xrightarrow{c_M} & H^r(M) \\ \big\downarrow f^\star & & \big\downarrow f^\star \\ Vect_r(N) & \xrightarrow{c_N} & H^r(N), \end{array}$$

also called naturality property.

We will prove that the cohomology class $[P(\Omega)]$ is a characteristic class. That is, the map that assigns the cohomology class $[P(\Omega)]$ to a vector bundle E induces such a commutative diagram, that is, it is natural. For this let $E \to M$ be a vector bundle of rank r over M with connection ∇^E over E. Denote by ω_α^E the connection matrices and by Ω_α^E the curvature matrices over frames e_α for E. Let $f : N \to M$ be a smooth map between smooth manifolds and denote by $f^\star E$ the pullback bundle over N. By Theorem 2.7 there exists a unique connection $\nabla^{f^\star E}$ on $f^\star E$ with connection matrices $f^\star(\omega_\alpha)$ relative to the frames $f^\star(e_\alpha)$. Hence, the curvature matrices $\Omega_\alpha^{f^\star E}$ relative to $f^\star(e_\alpha)$ are

$$\begin{aligned} \Omega_\alpha^{f^\star E} &= df^\star(\omega_\alpha) + f^\star(\omega_\alpha) \wedge f^\star(\omega_\alpha) \\ &= f^\star(d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha) \\ &= f^\star(\Omega_\alpha^E). \end{aligned}$$

Therefore, since $f^\star$ is a homomorphism, $P\left(\Omega_\alpha^{f^\star E}\right) = P(f^\star \left(\Omega_\alpha^E\right) = f^\star P\left(\Omega_\alpha^E\right)$, or as a diagram

$$\begin{array}{ccccccc} E & \longmapsto & & & & & [P(\Omega^E)] \\ \big\downarrow & & & & & & \big\downarrow \\ f^\star E & \longmapsto & [P(\Omega^{f^\star E})] & = & [f^\star P(\Omega^E)] & = & f^\star [P(\Omega^E)] \end{array}$$

3.2 Pontrjagin Classes

We proved that the ring $Inv(\mathfrak{gl}(r,\mathbb{R}))$ of invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$ is generated by the coefficients $f_k(X)$ of the characteristic polynomial of $(-X)$. This motivates to study the characteristic classes of these polynomials. The characteristic classes associated to these coefficient polynomials are called Pontrjagin classes.

For every vector bundle $E \to M$, there exist a Riemannian metric g and a connection ∇ that is compatible with g. As proved in Theorem 3.1 the cohomology class $[P(\Omega)]$ is independent of the connection. Hence, without loss of generality, we can always use the Riemannian structure and compute characteristic classes with a metric connection. In this case, the curvature matrix Ω relative to an orthonormal frame $e_1, \ldots, e_r$ is skew-symmetric $\Omega^i_j = -\Omega^j_i$, thus, the trace vanishes $0 = \operatorname{tr}(\Omega) = f_1(\Omega)$. For odd k, Ω^k is also skew-symmetric

$$\Omega^k = (-\Omega^T)^k = (-1)^k (\Omega^k)^T = -(\Omega^k)^T.$$

Therefore, for odd k

$$\operatorname{tr}(\Omega^k) = 0.$$

Let $P(X)$ be a homogeneous invariant polynomial on $\mathfrak{gl}(r, \mathbb{R})$ with odd degree k. Since $Inv(\mathfrak{gl}(r, \mathbb{R}))$ is generated by the trace polynomials, we can write

$$P(\Omega) = Q(\Sigma_1(\Omega), \ldots, \Sigma_r(\Omega)).$$

Because P is homogeneous with odd degree, every monomial in Q must contain a trace polynomial $\Sigma_j(\Omega)$ of odd degree j. But for odd j, we have $\Sigma_j(\Omega) = \operatorname{tr}(\Omega^j) = 0$, hence $P(\Omega) = 0$. Therefore $f_j(\Omega) = 0$ for odd j, because these polynomials are homogeneous and invariant. Thus the ring of characteristic classes on E is generated by

$$[\Sigma_2(\Omega)], [\Sigma_4(\Omega)], \ldots$$

or by

$$[f_2(\Omega)], [f_4(\Omega)], \ldots$$

Definition 3.3. *The k-th Pontrjagin class $p_k(E)$ of a vector bundle $E \to M$ is*

$$p_k(E) = \left[f_{2k}\left(\frac{i}{2\pi}\Omega \right) \right] \in H^{4k}(M),$$

where the factor $\frac{i}{2\pi}$ is introduced to make other formulas sign free and ensure that $p_k(E)$ is represented by a form that gives an integer when integrated over any submanifold of M of dimension $4k$. This property is also called integrality.

By definition of the coefficient polynomials f_k we have

$$\det\left(\mathrm{I} + \frac{i}{2\pi}\Omega \right) = 1 + f_1\left(\frac{i}{2\pi}\Omega \right) + \cdots + f_r\left(\frac{i}{2\pi}\Omega \right),$$

so when we pass to cohomology

$$p(E) := \left[\det\left(\mathrm{I} + \frac{i}{2\pi}\Omega\right)\right] = [1] + p_1(E) + \cdots + p_{\lfloor \frac{r}{2} \rfloor}(E),$$

and we call the expression $p(E)$ the total Pontrjagin class of E.

In the case when M is a $4m$-dimensional compact oriented manifold, a monomial $f_2(\frac{i}{2\pi}\Omega)^{a_2} f_4(\frac{i}{2\pi}\Omega)^{a_4} \cdots f_{2\lfloor \frac{r}{2} \rfloor}(\frac{i}{2\pi}\Omega)^{a_{2\lfloor \frac{r}{2} \rfloor}}$ with degree

$$2\left(a_2 + a_4 + \cdots + a_{2\lfloor \frac{r}{2} \rfloor}\right) = 4m,$$

can be integrated and the resulting number is called a Pontrjagin number of E. In the case $E = TM$, these numbers are simply called Pontrjagin numbers of M.

For direct sums of vector bundles we have the following formula.

Theorem 3.4. *If E' and E'' are vector bundles over M, then*

$$p(E' \oplus E'') = p(E')p(E''). \tag{4}$$

The proof is a direct computation and can be found in [Tu17, Thm 24.6]. So, under the assumption that a vector bundle $E \to M$ is a direct product of two vector bundles $E = F \oplus P$, the product formula yields an easy way of computing $p(E)$.

3.3 Euler Classes

We will add more structure to our theory. Namely, we will consider an oriented Riemannian vector bundle $E \to M$ of rank r and polynomials that are invariant under actions that respect this orientation. This will lead to a generalization of the well known Gauss-Bonnet Theorem.

As for the tangent bundle, if there exists a nowhere vanishing section $s \in \Gamma(\Lambda^r E)$, then we call E orientable. We then call two nowhere vanishing sections $s, s' \in \Gamma(\Lambda^r E)$ equivalent, if $s' = fs$ for a positive smooth function f. Then, the two resulting equivalence classes are called orientations of E. We have the following simple criterion for orientability of E.

Proposition 3.5. *A vector bundle E of rank r is orientable if and only if the line bundle $\Lambda^r E$ is trivial.*

Proof. If the line bundle $\Lambda^r E$ is trivial, i.e. it is diffeomorphic to $M \times \mathbb{R}$, then we can define the nowhere vanishing section

$$s : M \to M \times \mathbb{R}, p \mapsto (p, 1).$$

If there exists a nowhere vanishing section $s \in \Gamma(\Lambda^r E)$, then we can define

$$\psi : \Lambda^r E \to M \times \mathbb{R}, x \mapsto (\pi(x), \lambda(x)),$$

where $\lambda(x)$ is the unique real number such that $x = s(\pi(x))\lambda(x)$. Since s is nowhere vanishing and x and $s(\pi(x))$ are both elements of $\Lambda^r E_{\pi(x)} \cong \mathbb{R}^r$, there exists such a $\lambda(x)$ and it is unique. The inverse of ψ is given by

$$\psi^{-1} : M \times \mathbb{R} \to \Lambda^r E, (p, r) \mapsto s(p)r.$$

This can be checked by direct computation. Also, ψ and ψ^{-1} are smooth, because π and s are. Hence, ψ is a diffeomorphism and $\Lambda^r E$ is trivial. □

From now on, we assume that E is orientable and fix an orientation of E. Let $\eta \in \Gamma(\Lambda^r E)$ be a nowhere vanishing form that represents this orientation. A frame $[e_1, \ldots, e_r]$ on an open set $U \subset M$ is said to be positively oriented, if $e_1 \wedge \cdots \wedge e_r = f\eta|_U$ for some positive smooth function f on U. Now let us also fix a Riemannian metric g on E and a connection ∇ that is compatible with the metric. Restrict the computation of the connection and curvature matrices to positively oriented orthonormal frames only. Then for two such frames $\tilde{e}$ and e on an open set U, that are related by a special orthogonal matrix $a : U \to SO(r)$, $\tilde{e} = ea$, we have the basis change formula

$$\tilde{\Omega} = a^{-1}\Omega a.$$

Recall that the curvature matrix of a metric connection is skew symmetric $\Omega^i_j = -\Omega^j_i$. The space of all real skew symmetric $r \times r$ matrices is is the Lie algebra $\mathfrak{so}(r)$. A polynomial on $\mathfrak{so}(r)$ is a polynomial in the entries of $X = [x^i_j]$, where X is a matrix with indeterminate entries that satisfy $x^i_j = -x^j_i$ for all $1 \leqslant i, j \leqslant r$. Denote by $Inv(\mathfrak{so}(r))$ the ring of $SO(r)$-invariant polynomials on $\mathfrak{so}(r)$. As we did before, we want to find generators of $Inv(\mathfrak{so}(r))$. Since $Ad\ GL(r, \mathbb{R})$-invariant polynomials on $\mathfrak{gl}(r, \mathbb{R})$ are also $Ad\ SO(r)$-invariant polynomials on $\mathfrak{so}(r)$, the trace polynomials are a subset of generators of $Inv(\mathfrak{so}(r))$. We will show that for r even, we get an additional generator of $Inv(\mathfrak{so}(r))$, called the Pfaffian. For r odd, the trace polynomials already generate $Inv(\mathfrak{so}(r))$. A proof of this can be found in [KN69, Thm. XII. 2.7]. We will continue by studying the Pfaffian.

The Pfaffian is a square root of the determinant of an even dimensional skew symmetric matrix. The existence follows from a basic theorem about skew symmetric matrices in symplectic geometry (cf. [Can08, Thm. 1.1]). For this, let X be an $r \times r$ skew symmetric matrix over a field F and let V be the vector space F^r. Define the bilinear form

$$b : V \times V \to F, (x, y) \mapsto x^T X y.$$

This bilinear form is antisymmetric

$$b(x, y) = x^T X y = \left(x^T X y\right)^T = y^T X^T x = -y^T X x = -b(y, x).$$

Define the subspace $O := \{v \in V | b(v, \cdot) = 0\} \subset V$ and choose a basis $(o_1, \ldots, o_m)$ of O. Let $U_0 \subset V$ be a complementary subspace of O. Then for any $u_1 \in U_0$, we can find $w_1 \in U_0$ such that $b(u_1, w_1) = 1 = -b(w_1, u_1)$. Set $U_1 := span\{u_1, w_1\}$ and $\overline{U_1} := \{w \in U_0 | b(w, u) = 0 \text{ for all } u \in U_1\}$. Then we have $U_0 = U_1 \oplus \overline{U_1}$. We can proceed by induction and get a basis $(u_1, w_1, \ldots, u_m, w_m, o_1, \ldots, o_n)$ of V with $2m + n = r$. Now, define the matrix

$$A := \begin{pmatrix} | & | & & | & | & | & & | \\ u_1 & w_1 & \ldots & u_m & w_m & o_1 & \ldots & o_n \\ | & | & & | & | & | & & | \end{pmatrix}.$$

So by changing the basis via A, the matrix that represents the bilinear form b is given by

$$A^T X A = \begin{pmatrix} S & & & & & \\ & \ddots & & & & \\ & & S & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

with

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If b is nondegenerate, then $O = \{0\}$ and

$$A^T X A = \begin{pmatrix} S & & \\ & \ddots & \\ & & S \end{pmatrix} =: J_{2m},$$

moreover,

$$det(A)^2 det(X) = det(A^T X A) = det(S)^m = 1$$

which implies

$$det(X) = \frac{1}{det(A)^2}.$$

Note that $\frac{1}{det(A)} \in F$, that is, $det(X)$ is a perfect square in F. The following theorem generalizes this result.

Theorem 3.6. *Let $X = [x^i_j]$ be an even dimensional skew-symmetric matrix of indeter-*

minates. Then there exists a polynomial $Q(X) \in \mathbb{Z}[x^i_j]$ *such that*

$$det(X) = Q(X)^2.$$

This polynomial $Q(X)$ is unique up to the sign

$$det(X) = Q(X)^2 = (-Q(X))^2$$

We define the Pfaffian of X to be the unique polynomial $Pf(X)$, that satisfies $det(X) = Pf(X)^2$ and the normalization condition $Pf(J_{2m}) = 1$. A proof of Theorem 3.6 can be found in [Tu17, Thm. 25.3].

Example 3.1. *The determinant of the* 4×4 *skew-smmetric matrix*

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

is given by the polynomial

$$a^2f^2 + b^2e^2 + c^2d^2 + 2acdf - 2abef - 2bcde = (af - be + cd)^2 = Q(X)^2,$$

with $Q(X) := af - be + cd$*. If we evaluate at* $a = f = 1, b = c = d = e = 0$*, we get* $X = J_4$ *and* $Q(X) = 1$*. Hence by the normalization condition, the Pfaffian of* X *is*

$$Pf(X) = Q(X).$$

Proposition 3.7. *For* $2m \times 2m$ *matrices of indeterminates* $A = [a^i_j]$ *and* $X = [x^i_j]$ *with* X *skew-symmetric, we have*

$$Pf(A^TXA) = det(A)Pf(X)$$

as polynomials in $\mathbb{Z}[a^i_j, x^i_j]$.

Hence for $A \in SO(r)$ and $X \in \mathfrak{so}(r)$ we have

$$Pf(A^{-1}XA) = Pf(A^TXA) = det(A)Pf(X) = Pf(X),$$

that is, $Pf(X) \in Inv(\mathfrak{so}(r))$.

Proof. For the Pfaffian of the skew-symmetric matrix A^TXA we have

$$Pf(A^TXA)^2 = det(A^TXA) = det(A)^2det(X) = det(A)^2Pf(X)^2,$$

thus

$$Pf(A^T XA) = \pm det(A)Pf(X).$$

$Pf(A^T XA)$ and $det(A)Pf(X)$ are uniquely defined by A and X. Hence, either

$$Pf(A^T XA) = +det(A)Pf(X),$$

or

$$Pf(A^T XA) = -det(A)Pf(X),$$

is true for all such A and X and it suffices to evaluate at one particular A and one particular X to determine the sign. For $A = \mathrm{I}$ and $X = J_{2m}$ we get

$$Pf(\mathrm{I}^T J_{2m}\mathrm{I}) = Pf(J_{2m}) = 1 = det(\mathrm{I})Pf(J_{2m}).$$

So the sign is positive and the proposition follows. □

We have shown that for an oriented Riemannian vector bundle E, of even rank r, with connection ∇ that is compatible with the metric, the Pfaffian of the skew-symmetric curvature matrix Ω induces a global r-form on M. We denote this global form by $Pf(\Omega)$. This form is closed and its cohomology class is independent of the connection. For this, see Chapter 4.4. We define the Euler class of the oriented Riemannian bundle E as

$$e(E) := [Pf(\frac{1}{2\pi}\Omega)].$$

Again, the factor $\frac{1}{2\pi}$ ensures integrality of the Euler class. The Euler class allows us to formulate a generalized Gauss-Bonnet Theorem. Let us recall the classical result.

Theorem 3.8. *Let M be a compact oriented 2-dimensional manifold embedded into $\mathbb{R}^3$. Then*

$$\int_M KdS = 2\pi\chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M and $\int_M KdS$ is the surface integral of the Gaussian curvature.

Definition 3.9. *The Euler characteristic of a compact topological manifold M is the alternating sum of the Betti numbers*

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^i b_i(M) = \sum_{i=0}^{\dim M} (-1)^i \mathrm{rk}(H_i(M)).$$

Here, $H_i(M)$ denotes the i'th singular homology group of M.

The generalized result is the following.

Theorem 3.10. *Let M be a compact, oriented Riemannian manifold of even dimension with a connection on the tangent bundle TM that is compatible with the metric. Let $Pf(\Omega)$ be the Pfaffian of the curvature matrices relative to prositively oriented orthonormal frames. Then*

$$\int_M Pf(\frac{1}{2\pi}\Omega) = \chi(M).$$

3.4 Chern Classes

The Chern Classes are the complex analogon of Pontrjagin classes. Let $\pi : E \to M$ be a complex vector bundle of rank r with a connection ∇. Also let $Q(X)$ be a homogeneous $Ad(GL(r,\mathbb{C}))$-invariant polynomial of degree k on $\mathfrak{gl}(r,\mathbb{C})$. As in the real case, this invariant polynomial induces a global complex valued $2k$-form $Q(\Omega)$, which is closed and its cohomology class is independent of the connection. For this, see Chapter 4.4. This leads to the definition of Chern Classes $c_i(E)$

$$\left[det(\mathrm{I} + \frac{i}{2\pi}\Omega)\right] = 1 + c_1(E) + \cdots + c_r(E).$$

The Chern Classes $c_i(E)$ are natural and satisfy (4).

The top Chern class $c_r(E) = \left[det(\frac{i}{2\pi}\Omega)\right]$ may be identified with the Euler class $c(E)$ of E, when viewing E as a real orientable vector bundle of rank $2r$. Since E is orientable, we may restrict to orthonormal frames when calculating Ω, which yields that Ω is skew hermitian

$$\Omega = -\overline{\Omega}^T.$$

Skew hermitian matrices can be diagonalized by a unitary matrix A

$$A^{-1}\Omega A = \begin{pmatrix} i\lambda_1 & & \\ & \ddots & \\ & & i\lambda_r \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_r$ are real by the skew hermitian property. Hence $det(i\Omega) = (-1)^r\lambda_1 \cdots \lambda_r$. Then the associated real skew symmetric $2r \times 2r$-matrix is

$$\Omega^{\mathbb{R}} := \begin{pmatrix} 0 & -\lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\lambda_r \\ & & & \lambda_r & 0 \end{pmatrix}.$$

With Proposition 3.7, we can compute the Pfaffian

$$Pf(\Omega^{\mathbb{R}}) = Pf(B^T(-\Omega^{\mathbb{R}})B) = det(B)Pf(-\Omega^{\mathbb{R}}) = (-1)^r Pf(-\Omega^{\mathbb{R}}) = (-1)^r\lambda_1 \cdots \lambda_r,$$

where

$$B = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$$

is a $2r \times 2r$-matrix with determinant $(-1)^r$. Hence,

$$c_r(E) = det\left(\frac{i}{2\pi}\Omega\right) = \frac{1}{(2\pi)^r}(-1)^r\lambda_1 \cdots \lambda_r = Pf\left(\frac{1}{2\pi}\Omega^{\mathbb{R}}\right) = e(E).$$

3.5 The Tautological Line Bundle over $\mathbb{CP}^1$

In this section we will apply the theory to the tautological line bundle over the complex projective line and compute its first Chern Class.

On $\mathbb{C}^2\backslash\{0\}$ define the equivalence relation $\sim$ by

$$(z_1, z_2) \sim \lambda(z_1, z_2)$$

for all $\lambda \in \mathbb{C}\backslash\{0\}$. Denote the equivalence class of (z_1, z_2) by $[z_1 : z_2]$. Then the complex projective line is the quotient space

$$\mathbb{CP}^1 := \mathbb{C}^2\backslash\{0\}/_\sim = \{[z_1 : z_2] | (z_1, z_2) \in \mathbb{C}^2\backslash\{0\}\}$$

with the quotient topology. $\mathbb{CP}^1$ is a 1-dimensional complex manifold. For this, define

$$U_1 := \{[z_1 : z_2] | z_1 \neq 0\} \qquad \phi_1 : U_1 \to \mathbb{C}, [z_1 : z_2] \mapsto \frac{z_2}{z_1}$$
$$U_2 := \{[z_1 : z_2] | z_2 \neq 0\} \qquad \phi_2 : U_2 \to \mathbb{C}, [z_1 : z_2] \mapsto \frac{z_1}{z_2},$$

with inverse maps

$$\phi_1^{-1} : \mathbb{C} \to U_1, z \mapsto [1 : z]$$
$$\phi_2^{-1} : \mathbb{C} \to U_2, w \mapsto [w : 1].$$

Then $\{U_i, \phi_i\}_{i=1,2}$ defines a holomorphic atlas on $\mathbb{CP}^1$ with transition functions

$$\phi_2|_{U_1 \cap U_2} \circ \phi_1|^{-1}_{U_1 \cap U_2} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}, z \mapsto \frac{1}{z},$$
$$\phi_1|_{U_1 \cap U_2} \circ \phi_2|^{-1}_{U_1 \cap U_2} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}, w \mapsto \frac{1}{w}.$$

For every point $p = [z_1 : z_2] \in \mathbb{CP}^1$ define the line $L_p = \{\lambda(z_1, z_2) | \lambda \in \mathbb{C}\} \subset \mathbb{C}^2$ and define the tautological line bundle as the total space

$$L := \{(p, v) | p \in \mathbb{CP}^1, v \in L_p\}$$

with projection map $(p, v) \mapsto p$. This defines a complex 1-dim vector bundle over $\mathbb{CP}^1$, called the tautological line bundle.

For a 1-dimensional vector bundle like L, the curvature form Ω is a 1×1-matrix of 2-forms, i.e. just a 2-form. Hence the Chern Class $c_1(L)$ is just a multiple of $[\Omega]$, which follows directly from the formula

$$1 + c_1(L) = \left[det \left(\mathrm{I} + \frac{i}{2\pi}\Omega \right) \right] = 1 + \frac{i}{2\pi} [\Omega] .$$

Thus, we just have to compute Ω to compute the first chern class

$$c_1(L) = \frac{i}{2\pi}[\Omega].$$

L is a subbundle of the product vector bundle $\mathbb{CP}^1 \times \mathbb{C}^2$ and therefore, the standard hermitian metric on $\mathbb{CP}^1 \times \mathbb{C}^2$ restricts to a hermitian metric h on L. To compute h in local coordinates, consider the frames

$$\left([1 : z], \begin{pmatrix} 1 \\ z \end{pmatrix} \right), \text{ and } \left([w : 1], \begin{pmatrix} w \\ 1 \end{pmatrix} \right)$$

of L over U_1 and U_2, defined using local coordinates z, w given by ϕ_1 and ϕ_2 respectively. So, on U_1, h is given by the function

$$h(z) = 1 + z\overline{z} = 1 + |z|^2,$$

and on U_2, h is given by

$$h(w) = 1 + w\overline{w} = 1 + |w|^2.$$

On U_1 with frame $s = \left([1 : z], \begin{pmatrix} 1 \\ z \end{pmatrix} \right)$, we can compute the connection matrix ω_1 of the metric connection. For this, let $X = a\frac{\partial}{\partial z} + b\frac{\partial}{\partial \overline{z}}$ be a tangent vector field over U_1, where $a(z), b(z)$ are smooth complex functions on $\mathbb{C}$. Write $z = x + iy$, for x, y real. Then

$$T_{\phi_1^{-1}(z)}\mathbb{CP}^1 = \text{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}.$$

This implies $a(z) = \overline{b}(z)$. So by imposing metric compatibility of the connection ∇ and by the definition of the connection matrix $\nabla_x s = \omega(X)s$, we get

$$dh(s, s)(X) = \partial(1 + z\overline{z})(X) + \overline{\partial}(1 + z\overline{z})(X)$$

$$
\begin{aligned}
&= \overline{z}dz\left(a\frac{\partial}{\partial z} + b\frac{\partial}{\partial \overline{z}}\right) + zd\overline{z}\left(a\frac{\partial}{\partial z} + b\frac{\partial}{\partial \overline{z}}\right) \\
&= a\overline{z} + bz \\
&= a\overline{z} + \overline{a}z \\
&\overset{\star}{=} h(\nabla_X s, s) + h(s, \nabla_X s) \\
&= h(\omega_1(X)s, s) + h(s, \omega_1(X)s) \\
&= \left(\omega_1(X) + \overline{\omega_1(X)}\right) h(s,s).
\end{aligned}
$$

Here we imposed metric compatibility of ∇ in $\star$. Rearranging this, yields

$$\omega_1(X) + \overline{\omega_1(X)} = \frac{a\overline{z} + \overline{a}z}{1 + z\overline{z}}.$$

By [Kob87, Prop. 1.3.9 and Prop. 1.4.9], ω can be chosen to be a $(1,0)$ form. Thus,

$$\omega_1 = \frac{\overline{z}dz}{1 + z\overline{z}}.$$

Since U_1 is just $\mathbb{CP}^1$ minus a point, this extends uniquely to a 1-form ω on the whole manifold. In the same way, we can compute ω_2 on U_2 and check that ω_1 and ω_2 satisfy Theorem 2.5.

Having computed ω in coordinates on U_1, directly gives Ω on U_1, using the second structural equation

$$
\begin{aligned}
\Omega &= d\omega + \omega \wedge \omega \\
&= d\frac{\overline{z}dz}{1 + |z|^2} \\
&= \frac{\partial}{\partial z}\frac{\overline{z}}{1 + |z|^2}dz \wedge dz + \frac{\partial}{\partial \overline{z}}\frac{\overline{z}}{1 + |z|^2}d\overline{z} \wedge dz \\
&= \frac{1}{1 + |z|^2}d\overline{z} \wedge dz - \frac{z\overline{z}}{(1 + |z|^2)^2}d\overline{z} \wedge dz \\
&= \frac{1}{(1 + |z|^2)^2}d\overline{z} \wedge dz.
\end{aligned}
$$

Because U_1 is dense in $\mathbb{CP}^1$, this fully determines the Chern Class.

$$c_1(L) = \frac{i}{2\pi}\frac{1}{(1 + |z|^2)^2}[d\overline{z} \wedge dz].$$

Changing to polar coordinates $z = re^{i\theta}$ yields

$$
\begin{aligned}
d\overline{z} \wedge dz &= e^{-i\theta}(dr - ird\theta) \wedge e^{i\theta}(dr + ird\theta) \\
&= 2irdr \wedge d\theta,
\end{aligned}
$$

and thus

$$c_1(L) = -\frac{r}{\pi(1+r^2)^2}[dr \wedge d\theta].$$

Since U_1 is $\mathbb{CP}^1$ minus one point, integrating over $\mathbb{CP}^1$ is the same as integrating over $U_1 \cong \mathbb{C} \cong \mathbb{R}^2$

$$\int_{\mathbb{CP}^1} c_1(L) = -\frac{1}{\pi}\int_0^\infty dr \frac{r}{(1+r^2)^2}\int_0^{2\pi} d\theta = -1.$$

We see that the first Chern Class is indeed integral.

4 Generalization to Principal Bundles

In this chapter, we want to briefly motivate that the theory of connections on vector bundles can be seen as a special case of the theory of connections on principal bundles. We will just illustrate the main ideas and refer to [Tu17, Ch. 6] for the details. In this chapter, G will denote a Lie group with associated Lie algebra $\mathfrak{g}$.

4.1 Principal Bundles

We say that a smooth map

$$\mu : M \times G \to M$$

is a smooth right action of G on M, if for all $x \in M$ and all $g, h \in G$

i) $xe = x$ and

ii) $(xg)h = x(gh)$,

where we used the short notation $xg := \mu(x, g)$. A left action $\eta : G \times M \to M$ is analogously defined. The action is called free, if for every point $x \in M$, the stabilizer $\mathrm{Stab}(x)$ is the trivial subgroup $\{e\} < G$. A manifold M together with a left/right action of G on M is called a left/right G-manifold. A map $f : M \to N$ between two right G-manifolds is called right G-equivariant, if for all $x \in M$ and all $g \in G$

$$f(xg) = f(x)g,$$

and similarly we define left G-invariant maps. If $f : M \to N$ for a right G-manifold M and a left G-manifold N and for all $x \in M$ satisfies for all $g \in G$

$$f(xg) = g^{-1}f(x) =: f(x)g,$$

then we call f G-invariant and $g^{-1}p = pg$ defines a right action on $N \ni p$.

Let E, M, F be manifolds and $\pi : E \to M$ be a smooth surjection. A local trivialization with fiber F for π is an open cover $\{U_\alpha\}$ of M with a collection of fiber preserving diffeomorphisms $\phi_U : \pi^{-1}(U) \to U \times F$ for $U \in \{U_\alpha\}$, such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\rho} & \\ & U & \end{array} .$$

Here, $\rho : U \times F \to U$ denotes the projection onto the first coordinate. A fiber bundle with fiber F is a smooth surjection $\pi : E \to M$ with local trivializations with fiber F. Let $E \to M$ be such a fiber bundle, then at each $x \in M$, the fiber $E_x := \pi^{-1}(\{x\})$ is diffeomorphic to F.

A fiber bundle $\pi : P \to M$ with fiber G is called a principal G-bundle, if G acts smoothly and freely on P on the right and the fiber preserving diffeomorphisms

$$\phi_U : \pi^{-1}(U) \to U \times G$$

are G-equivariant. Note that G acts from the right on $U \times G$ by

$$(x, g)h = (x, gh),$$

for $x \in M$ and $g, h \in G$.

Example 4.1. *The simplest example for a principal G-bundle is the product bundle $M \times G$ with the right action $(x, g)h = (x, gh)$.*

4.2 The Frame Bundle of a Vector Bundle

Let V be a real vector space of dimension r and denote by $Fr(V)$ the set of all ordered basis. Write an element $v \in Fr(V)$ as a row vector of vectors $v_1, \ldots, v_r \in V$

$$v := [v_1, \ldots, v_r].$$

Then, multiplication from the right by elements from $GL(r, \mathbb{R})$ defines a free right action on $Fr(V)$. Fixing a $v \in Fr(V)$, this right action induces a bijection

$$\phi_v : GL(r, \mathbb{R}) \to Fr(V), A \mapsto vA,$$

by the orbit-stabilizer theorem. Let $\{U, \psi_U\}$ be a smooth atlas on $GL(r, \mathbb{R})$, then $\{\phi_v(U), \psi_U \circ \phi_v^{-1}|_{\phi_v(U)}\}$ defines a smooth atlas on $Fr(V)$, that makes ϕ_v a diffeomorphism. We say that the bijection ϕ_v transfers the manifold structure from $GL(r, \mathbb{R})$ to $Fr(V)$. One can show that this atlas is independent of the choice of v. We call $Fr(V)$ together with this smooth structure the frame manifold of V.

Doing this fiberwise for a rank r real vector bundle $E \to M$, defines the following principal $GL(r, \mathbb{R})$-bundle. Define

$$Fr(E) = \bigsqcup_{x \in M} Fr(E_x)$$

and the map

$$\pi : Fr(E) \to M, Fr(E_x) \mapsto x.$$

Then the local trivializations $\phi_U : E|_U \to U \times \mathbb{R}^r$ induce bijections

$$\widetilde{\phi|_U} : Fr(E)|_U \to U \times Fr(\mathbb{R}^r)$$

that transfer the manifold structure to $Fr(E)$. This give $\pi : Fr(E) \to M$ the structure of a fiber bundle with fibers $Fr(\mathbb{R}^r)$ which is diffeomorphic to $GL(r, \mathbb{R})$. One can check that $Fr(E)$ is indeed a principal $GL(r, \mathbb{R})$-bundle, which we call the frame bundle of E.

This procedure also works if we only consider frames with positive orientation (cf. Euler Class), or complex vector bundles (cf. Chern Class). In these cases we get a principal $SL(r)$-bundle or a principal $GL(r, \mathbb{C})$-bundle respectively.

4.3 Connections on Principal Bundles

A connection on a vector bundle can be described by a matrix-valued 1-form, the connection form. For a principle G-bundle $P \to M$, there exists a $\mathfrak{g}$-valued 1-form ω on P that generalizes the idea of a connection, called an Ehresmann connection. The generalization of the curvature form is a $\mathfrak{g}$-valued 2-form on P given by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega],$$

where $[\cdot, \cdot]$ denotes the Lie-bracket on $\mathfrak{g}$.

Consider a vector bundle $E \to M$ with connection ∇ on E and the associated frame bundle $Fr(E) \to M$. In this case, there is a way to construct an Ehresmann connection on $Fr(E)$, using the connection ∇. This Ehresmann connection is called the Ehresmann connection on $Fr(E)$ determined by ∇. The two resulting connection forms and also the two curvature forms are related to each other by the following theorem (cf. [Tu17, Thm. 29.10 and Thm. 30.2]).

Theorem 4.1. *Let ∇ be a connection on a vector bundle $E \to M$, ω the associated Ehresmann connection determined by ∇ and Ω the curvature of the Frame bundle $Fr(E)$. Then the connection matrix ω_e and the curvature matrix Ω_e relative to a frame e for E over an open set $U \subset M$ are given by the pullbacks*

$$\omega_e = e^\star \omega$$
$$\Omega_e = e^\star \Omega,$$

where e is viewed as a section $e : U \to Fr(E)$.

4.4 Characteristic Classes on Principal Bundles

Let $P \to M$ be a principal G-bundle for a Lie group G with Ehresmann connection ω and curvature Ω. If we fix a basis $e_1, \dots, e_r$ on the Lie algebra $\mathfrak{g}$, the connection can be written as the linear combination

$$\Omega = \Omega^i e_i.$$

Here the coefficients Ω^i are 2-forms. A real valued polynomial f of degree k on $\mathfrak{g}$ can be written as the linear combination

$$f = f_I \varepsilon^{i_1} \cdots \varepsilon^{i_k},$$

where the f_I are real coefficients and $\varepsilon^1, \ldots, \varepsilon^r$ is the dual basis to $e_1, \ldots, e_r$. Evaluating f at Ω then yields the $2k$-form

$$f(\Omega) = f_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k},$$

which is independent of the basis $e_1, \ldots, e_r$. To check this, let $\widetilde{e}_1, \ldots, \widetilde{e}_r$ be another basis of $\mathfrak{g}$. That is there exists an invertible matrix $a = [a_i^j]$ such that

$$\widetilde{e}_i = e_j a_i^j,$$

and with respect to this basis, we have the linear combination

$$\Omega = \widetilde{\Omega}^i \widetilde{e}_i = \widetilde{\Omega}^i e_j a_i^j,$$

which yields $\Omega^j = \widetilde{\Omega}^i a_i^j$. To compute the coefficients $\widetilde{f}_I$ in the $\sim$ basis, we use $\varepsilon^k = \widetilde{\varepsilon}^j a_j^k$ and get

$$\begin{aligned} f &= f_I \varepsilon^{i_1} \cdots \varepsilon^{i_k} \\ &= f_I \widetilde{\varepsilon}^{j_1} a_{j_1}^{i_1} \cdots \widetilde{\varepsilon}^{j_k} a_{j_k}^{i_k} \\ &= f_I a_{j_1}^{i_1} \cdots a_{j_k}^{i_k} \widetilde{\varepsilon}^{j_1} \cdots \widetilde{\varepsilon}^{j_k} \\ &= \widetilde{f}_J \widetilde{\varepsilon}^{j_1} \cdots \widetilde{\varepsilon}^{j_k}, \end{aligned}$$

hence $\widetilde{f}_J = f_I a_{j_1}^{i_1} \cdots a_{j_k}^{i_k}$. Combining this, shows that $f(\Omega)$ is independent of the chosen basis

$$\begin{aligned} f_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k} &= f_I \left(\widetilde{\Omega}^{j_1} a_{j_1}^{i_1} \right) \wedge \cdots \wedge \left(\widetilde{\Omega}^{j_k} a_{j_k}^{i_k} \right) \\ &= f_I a_{j_1}^{i_1} \cdots a_{j_k}^{i_k} \widetilde{\Omega}^{j_1} \wedge \cdots \wedge \widetilde{\Omega}^{j_k} \\ &= f_J \widetilde{\Omega}^{j_1} \wedge \cdots \wedge \widetilde{\Omega}^{j_k}. \end{aligned}$$

The generalization of Theorem 3.1 to principal bundles is the following (cf. [Tu17, Thm. 32.2]).

Theorem 4.2. *Let $\pi : P \to M$ be a principal G-bundle with connection ω and curvature Ω and let f be an $Ad(G)$-invariant polynomial on $\mathfrak{g}$, then*

i) There exists a $2k$-form Λ on M such that $f(\Omega)$ is the pullback $\pi^\star \Lambda$.

ii) Λ is closed.

iii) $[\Lambda]$ *is independent of the connection* ω.

This allows us to define the Chern-Weil homomorphism for principal bundles

$$\begin{aligned} w : Inv(\mathfrak{g}) &\to H^\star(M) \\ f &\mapsto [\Lambda], \text{ with } f(\Omega) = \pi^\star\Lambda. \end{aligned}$$

Remark. *Now we have two (a priori different) procedures of calculating characteristic classes of a vector bundle. We can use the theory from Chapter 3, or we can associate a frame bundle that respects the structure and use the theory from Chapter 4. It turns out, that both procedures of calculating characteristic classes are equivalent. Therefore, we don't have to show by hand, that Euler and Chern classes are well defined. Instead, we construct the associated frame bundle, that respects the given structure and then apply Theorem 4.2 to conclude, that the characteristic class is well defined. For more details, we refer to [Tu17, Ch. 32].*

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