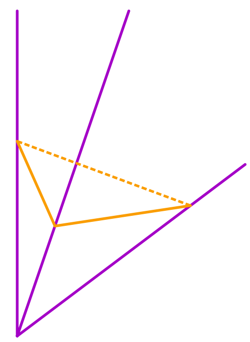
⓪ Introduction

Basic example:

$$\begin{tikzcd} S^{2n+1} \arrow[r, "\text{symplectization}"'] \arrow[d, "\text{quotient}"'] & \mathbb{R}^{2(n+1)} \setminus \{0\} \arrow[dl, "\text{symplectic reduction}"] \\ \mathbb{CP}^n = S^{2n+1} \big/ \text{diagonal } S^1\text{-action} & \end{tikzcd}$$



General example:

$$\begin{tikzcd} \text{toric contact mfld } M^{2n+1} \arrow[r] \arrow[d] & W^{2(n+1)} \cong M \times \mathbb{R} \text{ toric symp. cone} \arrow[dl] \\ \text{toric symp. orbifold } B^{2n} = M/\mathbb{R} = \text{Reeb} & \end{tikzcd}$$

Two main goals:

- Hands-on geometrical properties ($c_1, \pi_1, \ldots$)
- Hands-on Reeb dynamics properties (# periodic orbits, HC_*, ...)

- Plan

Lecture 1: toric symplectic manifolds

Lectures 2&3: toric contact manifolds

Lectures 3&4: toric Reeb dynamics and contact invariants

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① Toric Symplectic Manifolds

Recall: (B,ω) symp. mfld, $h: B \to \mathbb{R}$

$\rightsquigarrow \exists!\ X_h \in \mathfrak{X}(B)$ s.t. $X_h \lrcorner\, \omega = -dh$

$X_h \equiv$ hamiltonian v.f. of h

$\rightsquigarrow \mathcal{L}_{X_h}\omega = (X_h \lrcorner\, \overbrace{d\omega}^{=0}) + d(X_h \lrcorner\, \omega) = -d(dh) = 0$

$\rightsquigarrow \varphi_t := (X_h)_t \in \mathrm{Symp}(B,\omega),\ \forall t$

Note: $h \rightsquigarrow h_t$ and same argument applies

$\rightsquigarrow \mathrm{Ham}(B,\omega) \subset \mathrm{Symp}(B,\omega)$.

1.1 Hamiltonian torus actions, convexity and symplectic reduction

- $\mathbb{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}^m = (\mathbb{R}/2\pi\mathbb{Z}) \times \cdots \times (\mathbb{R}/2\pi\mathbb{Z})$

$(\theta_1, \ldots, \theta_m)$

$\cong S^1 \times \cdots \times S^1$ compact abelian group

$(e^{i\theta_1}, \ldots, e^{i\theta_m})$

Def.: A hamiltonian $\mathbb{T}^m$-action on (B,ω) is a group homomorphism $\mathcal{I} : \mathbb{T}^m \longrightarrow \mathrm{Ham}(B,\omega)$

$\rightsquigarrow$ $d\mathcal{I}(e_j) = X_{h_j}$, $j = 1, \ldots, m$, $\{e_1, \ldots, e_m\}$ std basis

$\rightsquigarrow$ $\mu := (h_1, \ldots, h_m) : B \longrightarrow \mathbb{R}^m$, moment map for the action (unique up to constants:

$\mu : B \to (\mathbb{Z}^m)^*$, $d(\mu(v)) = -d\mathcal{I}(v) \lrcorner\, \omega$.)

Exercises:

1) $\omega(X_{h_k}, X_{h_\ell}) = \{h_k, h_\ell\} \equiv 0$, $\forall k, \ell \in \{1, \ldots, m\}$, i.e. $\mathbb{T}^m$-orbits are isotropic ($\omega|_{\mathbb{T}^m\text{-orbit}} \equiv 0$).

Hence, when $\mathbb{T}^m$-action is effective, i.e. ι is injective or $\exists$ m-dim'l orbits, then $\underline{m \leq n}$.

2) $\forall \lambda \in \mathbb{R}^m$, $\mu^{-1}(\lambda)$ is $\mathbb{T}^m$-invariant.

- Atiyah-Guillemin-Sternberg Convexity

Compact, connected $(B,\omega) \circlearrowleft \mathbb{T}^m$, $\mu : B \to \mathbb{R}^m$. Then

(i) level sets $\mu^{-1}(\lambda)$ are connected, $\forall \lambda \in \mathbb{R}^m$.

(ii) $\mu(B) = \operatorname{conv}(\mu(B^{\mathbb{T}^m}))$ fixed pts

$\equiv$ moment polytope

- Marsden-Weinstein symplectic reduction

If $\lambda \in \mathbb{R}^m$ is a μ-reg. value, then $B_{red} := \mu^{-1}(\lambda)/\mathbb{T}^m$ is a symp. orbifold w/ symp. form ω_{red} characterized by

$$\begin{array}{ccc} \mu^{-1}(\lambda) & \xrightarrow{\iota} & (B,\omega) \\ {\scriptstyle pr}\downarrow & & \\ (B_{red}, \omega_{red}) & & \end{array}$$

$$\boxed{pr^*(\omega_{red}) = \iota^*(\omega)}$$

λ = reduction level

• <u>Example</u>: $\mathbb{R}^{2(n+1)} \simeq \mathbb{C}^{n+1}$, $z = u + iv = (z_0, \dots, z_n)$

$\omega_0 = du \wedge dv = \frac{i}{2} dz \wedge d\bar{z}$

$\rightsquigarrow$ $\mathbb{T}^{n+1}$- action: $\theta \cdot z = e^{i\theta} z \; (e^{i\theta_0} z_0, \dots, e^{i\theta_n} z_n)$

$$\mu_{\mathbb{T}^{n+1}}(z_0, \dots, z_n) = \frac{1}{2}\left(|z_0|^2, \dots, |z_n|^2\right)$$

$\rightsquigarrow$ diagonal S^1- action:

$$\theta \cdot (z_0, \dots, z_n) = (e^{i\theta} z_0, \dots, e^{i\theta} z_n)$$

$$\mu_{S^1}(z_0, \dots, z_n) = \frac{1}{2}\left(|z_0|^2 + \dots + |z_n|^2\right) = \frac{1}{2}\|z\|^2$$

<u>Exercise</u>: $(B^{2n}, \omega) \circlearrowleft \mathbb{T}^m$, $\mu_{\mathbb{T}^m} : B \to \mathbb{R}^m$

$A : (\mathbb{R}^k, \mathbb{Z}^k) \hookrightarrow (\mathbb{R}^m, \mathbb{Z}^m)$ linear inclusion inducing $\mathbb{T}^k \overset{A}{\hookrightarrow} \mathbb{T}^m$. Then

$$\mu_{\mathbb{T}^k} = A^t \circ \mu_{\mathbb{T}^m} : B \to \mathbb{R}^k.$$

$\rightsquigarrow$ Any $\lambda > 0$ is μ_{S^1}-reg. value and

$$B_{red} := \mu_{S^1}^{-1}(\lambda)/S^1 = S^{2n+1}_{\sqrt{2\lambda}}/S^1 \cong \mathbb{P}^n.$$

In fact: $(B_{red}, \omega_{red}) \cong (\mathbb{P}^n, 2\lambda\, \omega_{FS})$

Fubini-Study

$\rightsquigarrow$ $\mathbb{T}^{n+1}/S^1 \cong \mathbb{T}^n \cong (0, \theta_1, \dots, \theta_n) \in \mathbb{T}^{n+1}$

acts on $\mathbb{P}^n = \{ z \in \mathbb{C}^{n+1} : \|z\|^2 = 2\lambda \}/S^1$

with moment map

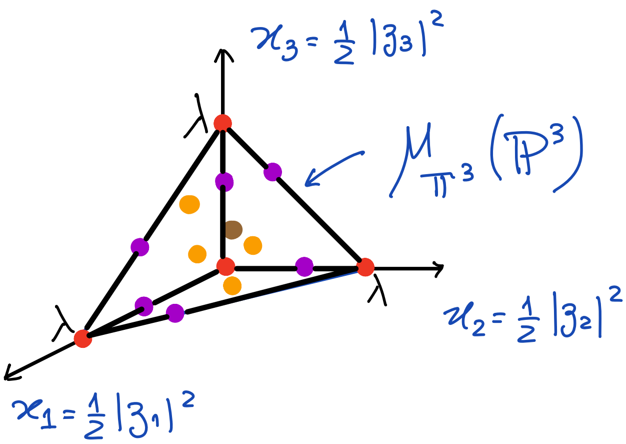
$$\mu_{\mathbb{T}^n}([z]) = \frac{1}{2}\left(|z_1|^2, \dots, |z_n|^2\right)$$

Fixed points : $p_k = \{[z] : z_j = 0, \forall j \neq k\}$,
$k = 0, \dots, n$

Moment polytope:

$$\mu_{\mathbb{T}^n}(\mathbb{P}^n) = \left\{ x \in \mathbb{R}^n : x_j \geq 0 \ ,\ j = 1, \dots, n, \ -\sum_{j=1}^{n} x_j + \lambda \geq 0 \right\}$$

$\boxed{n = 3}$



$\mu^{-1}_{\mathbb{T}^3}(\bullet) = S^1 \times S^1 \times S^1 = \mathbb{T}^3$

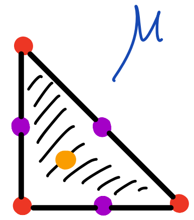
$\mu^{-1}_{\mathbb{T}^3}(\bullet) = S^1 \times S^1 = \mathbb{T}^2$

$\mu^{-1}_{\mathbb{T}^3}(\bullet) = S^1 = \mathbb{T}^1$

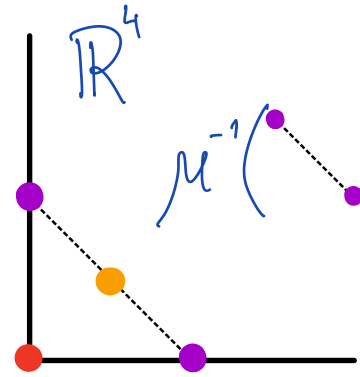
$\mu^{-1}_{\mathbb{T}^3}(\bullet) =$ fixed point

$\boxed{n = 2}$

$\mathbb{P}^2$



$\mu^{-1}(\bullet\!-\!\bullet) = \mathbb{P}^1$
$= S^2$



$\mu^{-1}(\bullet \cdots \bullet) = S^3$

1.2 Toric Symplectic Manifolds

- **Def.**: connected $(B^{2n}, \omega) \circlearrowleft \mathbb{T}^n$ effective, $\mu : B^{2n} \to \mathbb{R}^n$, $P := \mu(B) \subset \mathbb{R}^n$.

- **Examples**

  1) $(\mathbb{R}^{2n}, \omega_0) \circlearrowleft \mathbb{T}^n$ as before,
  $\mu : \mathbb{R}^{2n} = \mathbb{C}^n \to \mathbb{R}^n$, $\mu(z) = \frac{1}{2}(|z_1|^2, \dots, |z_n|^2)$
  $P = (\mathbb{R}_0^+)^n \subset \mathbb{R}^n$

  2) $(\mathbb{P}^n, 2\omega_{FS}) \circlearrowleft \mathbb{T}^n$ as before,
  $P = n$-simplex $\subset \mathbb{R}^n$

- **Delzant ('88)**: compact $(B^{2n}, \omega) \circlearrowleft \mathbb{T}^n$:
  Characterized convex polytopes $P = \mu(B)$ that arise [simple, integral, convex] and showed that any such $P \subset \mathbb{R}^n$ determines unique compact $(B^{2n}, \omega) \circlearrowleft \mathbb{T}^n$.

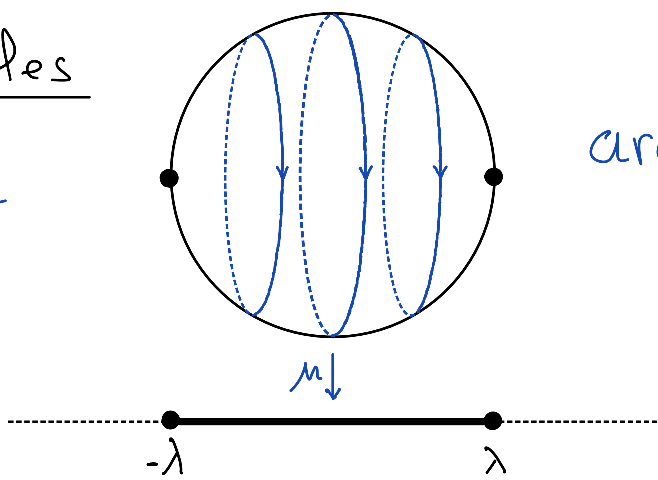
$$P \subset \mathbb{R}^n \longrightarrow (B_P, \omega_P) \circlearrowleft \mathbb{T}^n$$

$$\mu_P : B_P \longrightarrow P \subset \mathbb{R}^n$$

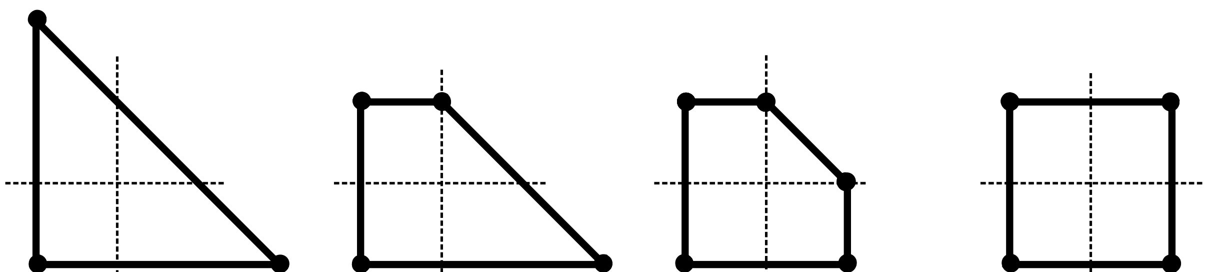
symplectic reduction of $(\mathbb{C}^N, \omega_0)$, where $N = \#$ of facets of P, with respect to subtorus $K_P \subset \mathbb{T}^N$, at level $\lambda_P \in \mathbb{R}^N$, and $\mathbb{T}^N / K_P \cong \mathbb{T}^n$ acts on quotient (B_P, ω_P) with $\mu_P : B_P \longrightarrow P$.

- Examples

$\mathbb{P}^1 = S^2$



$\text{area} = 2\lambda \times 2\pi$
$= 4\pi\lambda$



$\mathbb{P}^2$ | $\mathbb{P}^2 \# \overline{\mathbb{P}^2}$ | $\mathbb{P}^2 \# 2\overline{\mathbb{P}^2} = (\mathbb{P}^1 \times \mathbb{P}^1) \# \overline{\mathbb{P}^2}$ | $\mathbb{P}^1 \times \mathbb{P}^1$