② Toric Contact Manifolds [Lerman '03]

(2.1) Symplectic cones

Def. 1: (W, ω, X) is a symp. cone if:

(i) (W, ω) is a connected symp. mfld;

(ii) $X \in \mathfrak{X}(W)$ generates a free $\mathbb{R}$-action $\rho_t : W \to W$, $t \in \mathbb{R}$, s.t. $\rho_t^*(\omega) = e^t \omega$

$(\Leftrightarrow \mathcal{L}_X \omega = \omega \Leftrightarrow \omega = d(X \lrcorner \omega))$

In particular, ω is exact.

A closed symp. cone is a symp. c. s.t. $W/\mathbb{R}$ is closed.

Def. 2: $(M, \xi, [\alpha])$ is a (co-oriented) Contact manifold if

(i) $\xi \subset TM$ is an hyperplane distribution,

(ii) $\alpha \in \Omega^1(M)$ is s.t. $\xi = \mathrm{Ker}(\alpha)$ and $d\alpha|_\xi$ is non-degenerate.

$([\alpha] := \{ e^f \alpha : f \in C^\infty(M) \})$

$$\boxed{\text{symplectic cones} \xleftrightarrow{1:1} \text{(co-oriented) contact manifolds}}$$

$$(W,\omega,X) \longmapsto \left(M = W/\mathbb{R},\ \xi = \pi_*(\ker(X \lrcorner \omega)),\ [s^*(X \lrcorner \omega)]\right)$$

where $W \underset{s}{\overset{\pi}{\rightleftarrows}} W/\mathbb{R}$

s = any global section

$$\left(W = M\times\mathbb{R},\ \omega = d(e^t\alpha),\ X = \frac{\partial}{\partial t}\right) \longleftarrow (M,\xi,[\alpha])$$

symplectization

Exercise: Show that the following are equivalent:

(i) choice of a contact form for $(M,\xi,[\alpha])$;

(ii) choice of a global section of $\pi: W \to M$;

(iii) choice of an $\mathbb{R}$-equivariant splitting $W \cong M\times\mathbb{R}$

$\Downarrow$

Reeb vector field R_α

$(W = M\times\mathbb{R},\ \omega = d(e^t\alpha),\ X = \partial/\partial t)$ $\qquad$ (M,ξ,α)

$R_\alpha = X_{e^t}$, i.e. $R_\alpha \lrcorner \omega = -d(e^t)$ $\qquad$ $R_\alpha \lrcorner d\alpha \equiv 0$

$\alpha(R_\alpha) \equiv 1$

Exercise: $[R_\alpha, X] = 0.$

Examples

1) $(W, \omega, X) = \left(\mathbb{R}^{2(n+1)} \setminus \{0\}, \omega_{st} = du \wedge dv, X_{st} = \frac{1}{2}\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\right)$

coordinates $(u_1, \dots, u_{n+1}, v_1, \dots, v_{n+1})$

$\updownarrow$

(S^{2n+1}, ξ_{st}), $S^{2n+1} \subset \mathbb{C}^{n+1}$ unit sphere,

$\xi_{st} = TS^{2n+1} \cap iTS^{2n+1}$

$\alpha_{st} := X_{st} \lrcorner \omega_{st} \Rightarrow (R_{st})_s(z) = e^{is} z$

2) (B, ω) symp. mfld s.t. $\frac{[\omega]}{2\pi} \in H^2(M, \mathbb{Z})$

$\rightsquigarrow$ S^1-bundle $\pi : M \to B$ w/ $c_1(M) = \frac{[\omega]}{2\pi}$

$\rightsquigarrow$ $(M, \xi = \mathrm{Ker}(\alpha), [\alpha])$ where α is a

[Boothy-Wang] connection 1-form s.t.

$d\alpha = \pi^*(\omega).$

R_α generates natural S^1-action on M

$\updownarrow$

Symplectic cone is the total space of the corresponding $\mathbb{C}$-line bundle $L \to B$ without it's 0-section

<u>Exercise</u>: $(B,\omega) = (\mathbb{P}^n, 2\omega_{FS}) \rightsquigarrow$ Example 1

3) <u>Co-sphere bundles</u>

Q smooth manifold, (T^*Q, ω_{can}) symp. mfld where $\omega_{can} = -d\lambda_{can}$ and $\lambda_{can} \in \Omega^1(T^*Q)$ is the tautological 1-form

<u>Recall</u>: any vector bundle $E \to Q$ has an <u>Euler vector field</u> $\mathcal{E} \in \mathfrak{X}(E)$ whose flow $\rho_t : E \to E$ is given by

$$(q, v_q) \mapsto (q, e^t v_q).$$

<u>Prop.</u>: $(W = T^*Q \setminus \{0\text{-section}\}, \omega_{can} = -d\lambda_{can}, X = \mathcal{E})$ is a symplectic cone.

<u>Proof</u>: λ_{can} homog. of deg. 1 along fibers

$\Leftrightarrow (\rho_t)^* \lambda_{can} = e^t \lambda_{can} \Rightarrow \rho_t^*(\omega) = e^t \omega$. Q.E.D.

Def.: The corresponding contact mfld is called the <u>co-sphere bundle of Q</u> and denoted by $\underline{S^*Q}$.

Remark: a Riemannian metric g on TQ, and hence on T^*Q, gives an $\mathbb{R}$-equiv. splitting $T^*Q \setminus \{0\text{-section}\} = S^*_{g=1}Q \times \mathbb{R}$, hence a contact form α_g and a Reeb vector field R_g. The flow of R_g is the (co) <u>geodesic flow</u> of (Q, g).

(2.2) Toric symplectic cones

Def.: $(W^{2(n+1)}, \omega, X) \circlearrowleft \mathbb{T}^{n+1}$ action, effective, X preserving, with moment map $\mu : W \to \mathbb{R}^{n+1}$ s.t. $\mu(\rho_t(w)) = e^t \mu(w)$, $\forall w \in W, t \in \mathbb{R}$.

Moment cone: $C := \mu(W) \cup \{0\} \subset \mathbb{R}^{n+1}$.

Example: $(\mathbb{R}^{2(n+1)} \setminus \{0\}, \omega_{st}, X_{st}) \circlearrowleft \mathbb{T}^{n+1}$

$$\theta \cdot z = e^{i\theta} z = (e^{i\theta_1} z_1, \ldots, e^{i\theta_{n+1}} z_{n+1})$$

$$\mu_{st}(z) = \frac{1}{2} \|z\|^2, \quad C = (\mathbb{R}_0^+)^{n+1} \subset \mathbb{R}^{n+1}.$$

Def. [Lerman] A cone $C \subset \mathbb{R}^{n+1}$ is good if $\exists$ minimal set of primitive $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$, $d \geq n+1$, s.t.

(i) $C = \bigcap_{j=1}^{d} \{ x \in \mathbb{R}^{n+1} : \ell_j(x) := \langle x, \nu_j \rangle \geq 0 \}$

and

(ii) any codim-k face of C, $1 \leq k \leq n$, is the intersection of exactly k facets whose set of normals can be completed to $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$.

Thm. [Banyaga-Molino, Boyer-Galicki, Lerman] For each good cone $C \subset \mathbb{R}^{n+1}$ there exists a unique $(W_C, \omega_C, X_C, \mu_C)$ with moment cone C.

Def.: These are the good toric symplectic cones (resp. good toric contact mflds).

Remark [Lerman] The toric contact manifolds that are not good are:

(i) Certain 3-dim'l contact structures on lens spaces, $S^1 \times S^2$ and $\mathbb{T}^3$;

(ii) a unique contact structure on each principal $\mathbb{T}^{n+1}$-bundle over S^n, $n \geq 2$ (free $\mathbb{T}^{n+1}$-action).

[A closed toric contact manifold of dim > 3 is good iff the torus action is not free.]

- Explicit models for good toric symp. cones

(symplectic reductions of $(\mathbb{R}^{2d} \setminus \{0\}, \omega_{st}, X_{st}) \circlearrowleft \mathbb{T}^d$)

$$C = \bigcap_{j=1}^{d} \left\{ x \in (\mathbb{R}^{n+1})^* : \ell_j(x) := \langle x, \nu_j \rangle \geq 0 \right\} \quad \text{good cone}$$

$$\Downarrow$$

$$\beta : \mathbb{R}^d \longrightarrow \mathbb{R}^{n+1}, \quad \beta(e_j) = \nu_j, \ j = 1, \dots, d,$$

i.e.

$$\beta = \left[\nu_1 \,\middle|\, \cdots \,\middle|\, \nu_d \right]$$

$$\Downarrow$$

$$0 \to \mathfrak{k} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^{n+1} \to 0, \qquad 0 \to (\mathbb{R}^{n+1})^* \xrightarrow{\beta^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} \mathfrak{k}^* \to 0$$

$$0 \to K \xrightarrow{\iota} \mathbb{T}^d \xrightarrow{\beta} \mathbb{T}^{n+1} \to 0$$

$$K = \left\{ [\theta] \in \mathbb{T}^d : \sum_{j=1}^{d} \theta_j \cdot \nu_j \in 2\pi \mathbb{Z}^n \right\}$$

[K need not be connected]

$$\Downarrow$$

$$(\mathbb{R}^{2d} \setminus \{0\}, \omega_{st}, X_{st}) \circlearrowleft \mathbb{T}^d, \quad \mu_{\mathbb{T}^d}(z) = \sum_{j=1}^{d} \frac{|z_j|^2}{2} e_j^* \in (\mathbb{R}^d)^*$$

$$K \subset \mathbb{T}^d, \quad \mu_K(z) = (\iota^* \circ \mu_{\mathbb{T}^d})(z) = \sum_{j=1}^{d} \frac{|z_j|^2}{2} \iota^*(e_j^*) \in \mathfrak{k}^*$$

$$\Downarrow$$

$$W_c = \left(Z := \left(\mu_K^{-1}(0) \setminus \{0\}\right)\right) / K \circlearrowleft \mathbb{T}^d / K \cong \mathbb{T}^{n+1}$$

$$\omega_c = [\omega_{st}]_{red}, \qquad X_c = [X_{st}]$$