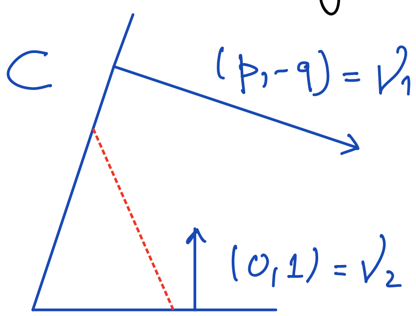
(2.2) Toric symplectic cones (cont.)

- **Examples**

1) **3-dim'l good toric contact manifolds**



$$\beta : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$$

$$\begin{bmatrix} p & 0 \\ -q & 1 \end{bmatrix}$$

$$K = \ker(\beta) = \langle (1/p, q/p) \rangle \cong \mathbb{Z}/p\mathbb{Z} \equiv \mathbb{Z}_p$$

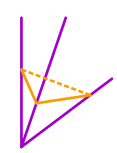
$\Rightarrow W_C$ = symplectization of lens space

$$L^3_p(1,q) := S^3 \Big/ \langle e^{\frac{2\pi i}{p}}, e^{\frac{2\pi i q}{p}} \rangle$$

2) **Pre-quantizations of integral toric symplectic manifolds**

$P \subset \mathbb{R}^n$ integral Delzant polytope

$$\Downarrow$$



$$C := \{ t(x,1) \in \mathbb{R}^n \times \mathbb{R} : x \in P, t \geq 0 \} \subset \mathbb{R}^{n+1}$$

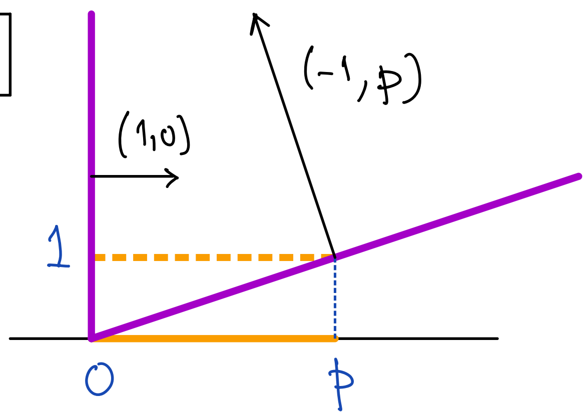
(i) C is a good cone $\rightsquigarrow (W_C, \omega_C, X_C)$.

(ii) $(B_P^{2n}, \omega_P, \mu_P)$ is the $S^1 \cong \{\mathbb{1}\} \times S^1 \subset \mathbb{T}^{n+1}$ symp. reduction of (W_C, ω_C, X_C) at level one.

(iii) $\left(M_C^{2n+1} := \mu_C^{-1}(\mathbb{R}^n \times \{1\}),\ \alpha_C := ((X_C \lrcorner \omega_C)|_{M_C})\right) \circlearrowleft \mathbb{T}^{n+1}$ is the Boothby–Wang mfld of (B_P, ω_P).

(iv) (W_C, ω_C, X_C) is the symplectization of (M_C, α_C)

$\boxed{n=1}$



$$\beta = \begin{bmatrix} 1 & -1 \\ 0 & p \end{bmatrix}$$

$$K = \langle (1/p, 1/p) \rangle$$

$$M_C = L_p^3(1,1)$$

- <u>Remark</u>: any toric symplectic cone can be seen as the pre-quantization of an integral toric symplectic <u>orbifold</u> (in ∞-many different ways).

- <u>First Chern class and toric diagrams</u>

(W_c, ω_c, X_c), $C \subset \mathbb{R}^{n+1}$ with normals $\nu_1, \dots, \nu_d \in \mathbb{Z}^{n+1}$ and facets $F_1, \dots, F_d$.

$\mu_c : W_c \to C$, $D_j := \mu^{-1}(F_j) =$ toric divisor

$\rightsquigarrow [D_j] \in H^2(W_c; \mathbb{Z})$ and

$$c_1(TW_c) = \sum_{j=1}^{d} [D_j] \in H^2(W_c; \mathbb{Z})$$

Moreover,

$$0 \longrightarrow \mathbb{Z}^{n+1} \xrightarrow{\beta^t} \mathbb{Z}^d \longrightarrow H^2(W_c; \mathbb{Z}) \longrightarrow 0$$

$$e_j \longmapsto [D_j]$$

Hence,

$\boxed{c_1(TW_c) = 0} \Leftrightarrow (1, \dots, 1)^t = \beta^t \nu$ for some $\nu \in \mathbb{Z}^{n+1} \Leftrightarrow \nu^*(\nu_j) = 1$ for some $\nu^* \in (\mathbb{Z}^{n+1})^*$ and all $j = 1, \dots, d$

$$\Updownarrow$$

$\exists$ integral basis of $\mathbb{Z}^{n+1}$ for which $\nu_1, \dots, \nu_d \in \mathbb{Z}^{n+1}$ are of the form

$$\boxed{\nu_j = (v_j, 1),\ v_j \in \mathbb{Z}^n,\ j = 1, \dots, d} \, .$$

In this case, define

$$\boxed{D := \operatorname{conv}(v_1, \dots, v_D) \subset \mathbb{R}^n}$$

We have that

C good $\Leftrightarrow$ D is an integral, simplicial polytope with unimodular facets (i.e. $\mathrm{Aff}(n,\mathbb{Z})$-equiv. to $\operatorname{conv}(e_1, \dots, e_n)$)

$\underset{\text{def.}}{\Longleftrightarrow}$ $\boxed{D \text{ is a toric diagram}}$

Prop.: Good toric symplectic cones $(W_C^{2(n+1)}, \omega_C, X_C)$ with $c_1(TW_C) = 0$ (i.e. <u>Gorenstein</u> toric symp. cones)

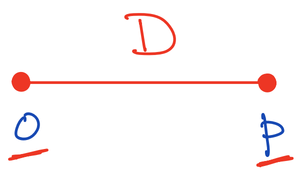
$$\Big\updownarrow 1{:}1$$

toric diagrams $D \subset \mathbb{R}^n$

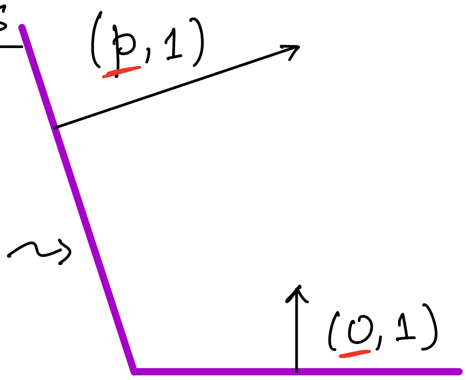
<u>Notation</u>: $D \subset \mathbb{R}^n \rightsquigarrow (M_D^{2n+1}, \xi_D), (W_D^{2(n+1)}, \omega_D, X_D)$

• Examples

1) $\boxed{n=1}$



$\rightsquigarrow$



$$\beta = \begin{bmatrix} p & 0 \\ 1 & 1 \end{bmatrix}$$

$$K = \langle (1/p , \tfrac{p-1}{p}) \rangle$$

$$M_D = L^3_p(1, p-1)$$

Exercise: $c_1\left(L^{2n+1}_p(\ell_0, \dots, \ell_n)\right) = 0$

$\iff \ell_0 + \cdots + \ell_n \equiv 0 \mod p$

In particular, $c_1(\mathbb{RP}^{2n+1}) = 0 \iff$

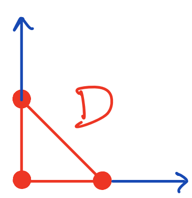
$\iff 1 + \cdots + 1 = n+1 \equiv 0 \mod 2$

$\iff n$ is odd.

2) $\boxed{n=2}$ $D \subset \mathbb{R}^2$ is a toric diagram

$\Updownarrow$

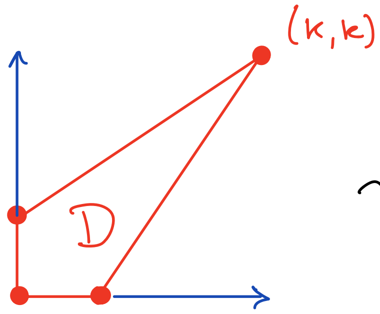
Vertices are its only integral boundary points



$\rightsquigarrow$

$v_1 = (1, 0, 1)$
$v_2 = (0, 1, 1)$
$v_3 = (0, 0, 1)$

$\rightsquigarrow (M_D, \xi_D) = (S^5, \xi_{st})$



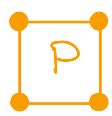
$$\rightsquigarrow \quad (M_D, \xi_D) = (S^3 \times S^2, \xi_k)$$

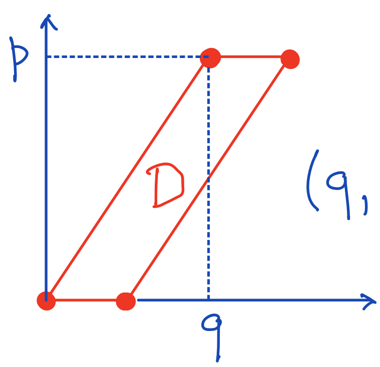
and

$$(S^3 \times S^2, \xi_1) = (S^* S^3, \xi_{st})$$

co-sphere bundle of S^3

<u>Exercise</u>: $(S^3 \times S^2, \xi_1)$ is also the pre-quantization of $(\mathbb{P}^1 \times \mathbb{P}^1, \omega_{FS} \times \omega_{FS})$, i.e. of





$$(q,p) = 1 \quad \rightsquigarrow \quad (M_D, \xi_D)$$

$$= (S^* L^3_p(1,q), \xi_{st})$$

3) Prequantizations of (B_P, ω_P) with $[\omega_P] = 2\pi\, c_1(TB_P)$, i.e

$$P = \bigcap_{j=1}^{d} \left\{ x \in (\mathbb{R}^n)^* : \langle x, v_j \rangle + \underline{\underline{1}} \geq 0 \right\}$$

(i.e. P is <u>reflexive</u>)

$$\rightsquigarrow \quad C = \bigcap_{j=1}^{d} \left\{ (x,t) \in (\mathbb{R}^n)^* \times (\mathbb{R})^* : \langle x, v_j \rangle + t \cdot \underline{\underline{1}} \geq 0 \right\}$$

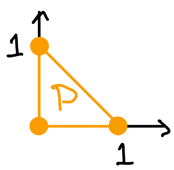
$$D = \operatorname{conv}(v_1, \ldots, v_d) \quad (= P^*)$$

E.g. prequantization of $(\mathbb{P}^n, \omega = 2\pi c_1(\mathbb{P}^n))$ is

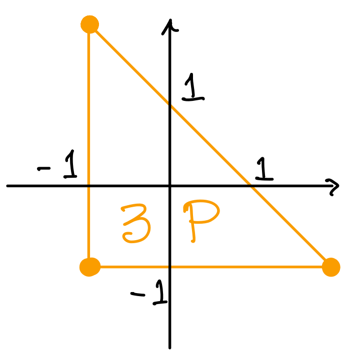
$$\left(S^{2n+1} \big/ \text{diag. } \mathbb{Z}_{n+1}\text{-action}, \ \xi_{st} \right)$$

<u>Note</u>: prequant. of integral (B_P, ω_P) is Gorenstein iff $\exists\, r \in \mathbb{N}$ such that rP is reflexive (up to translation).

E.g. prequantization of



is Gorenstein since



is reflexive.

- <u>Fundamental Group</u>

$C \subset (\mathbb{R}^{n+1})^*$ w/ normals $\nu_1, \ldots, \nu_d \in \mathbb{R}^{n+1}$

Lerman: $\pi_1(W_C) \cong \mathbb{Z}^{n+1} / \langle \nu_1, \ldots, \nu_d \rangle$.

In fact,

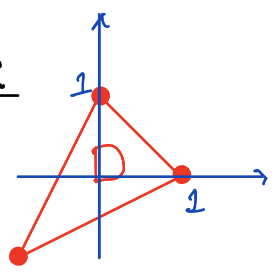
<u>Prop.</u>: $\pi_1(W_C)$ is cyclic of order N, where

$$N = \gcd \left\{ \left| \nu_{j_1} | \cdots | \nu_{j_{n+1}} \right| : 1 \le j_1 < \cdots < j_{n+1} \le d \right\}$$

In particular, when C is determined by a toric diagram $D = \mathrm{conv}(v_1, \ldots, v_d)$, then

$$N = \gcd \left\{ n!\, \mathrm{Vol}(\mathrm{conv}(v_{j_1}, \ldots, v_{j_{n+1}})) : 1 \le j_1 < \cdots < j_{n+1} \le d \right\}.$$

<u>Example:</u>



$$2!\, \mathrm{Vol}(D) = 2 \times \frac{3}{2} = 3$$

$$\Rightarrow \pi_1(M_D) \cong \mathbb{Z}_3$$