③ Periodic Orbits of Reeb flows on S^{2n+1}

3.1 Toric Reeb flows on S^{2n+1}

$$W = \mathbb{R}^{2n+2} \setminus \{0\} = \mathbb{C}^{n+1} \setminus \{0\} \cong S^{2n+1} \times \mathbb{R}$$

$$R_t : W \longrightarrow W, \quad t \in \mathbb{R}$$

$$\boxed{R_t(z_0, \dots, z_n) = (e^{ia_0 t} z_0, \dots, e^{ia_n t} z_n)}$$

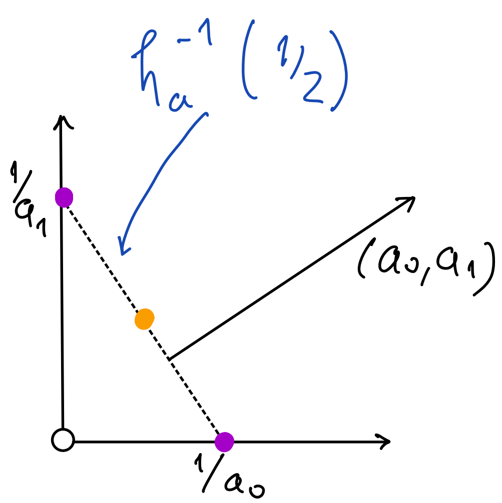
for some given constants $a_0, \dots, a_n \in \mathbb{R}^+$

$$R_t = (X_{h_a})^t, \quad h_a(z_0, \dots, z_n) = \frac{1}{2}\left(a_0 |z_0|^2 + \dots + a_n |z_n|^2\right)$$

• Example $(n = 1)$

$$\mu_{\mathbb{T}^2} : W \to \mathbb{R}^2$$

$$(z_0, z_1) \longmapsto \frac{1}{2}\left(|z_0|^2, |z_1|^2\right)$$



$$\mu^{-1}_{\mathbb{T}^2}(\bullet) \cong S^1 \times S^1$$

$$\mu^{-1}_{\mathbb{T}^2}(\bullet) \cong S^1$$

$$\boxed{\mu_{\mathbb{T}^2} \circ R_t = \mu_{\mathbb{T}^2}}$$

$R_t\big|_{\mu^{-1}_{\mathbb{T}^2}(\bullet)}$ has either ∞-many ($a_1/a_0 \in \mathbb{Q}$) or zero ($a_1/a_0 \in \mathbb{R}\setminus\mathbb{Q}$) periodic orbits

$\mu^{-1}_{\mathbb{T}^2}(\bullet)$ is a periodic orbit of R_t

- Proposition R_t restricted to a level set $h_a^{-1}(\lambda)$, $\lambda > 0$, has either $n+1$ or ∞-many distinct periodic orbits.

- Conley-Zehnder index of $\bullet$-orbits

Measure of "flowtwisting" around periodic orbit. For $\Gamma : [0,T] \to U(1)$, given by $\Gamma(t) = e^{2\pi i t}$, we have that

$$\boxed{\mu_{CZ}(\Gamma) = \begin{cases} 2T, & \text{if } T \in \mathbb{N} \\ 2\lfloor T \rfloor + 1, & \text{otherwise} \end{cases}}$$

$\left(\lfloor T \rfloor := \max\{ n \in \mathbb{Z} : n \leq T \} \right)$

For $\mathbb{Q}$-independent $a_0, \dots, a_n$ and $\bullet$-orbit γ_k that passes at the point $z_j = 0$, $j \neq k$, we have that

$$\mu_{CZ}(\gamma_k^N) = 2\sum_{j=0}^{n} \left\lfloor N \frac{a_j}{a_k} \right\rfloor + n$$

and

$$\boxed{\begin{aligned} \deg(\gamma_k^N) &:= \mu_{CZ}(\gamma_k^N) + n - 2 \\ &= 2\sum_{j=0}^{n} \left\lfloor N \frac{a_j}{a_k} \right\rfloor + 2(n-1) \end{aligned}}$$

<u>Example</u> $(n=1)$

(i) $a_0 = 1$, $a_1 = 1+\varepsilon$

$$\deg(\gamma_0^N) = 2\left(\lfloor N \rfloor + \lfloor N(1+\varepsilon) \rfloor\right) = 4N \quad \text{for } N < 1/\varepsilon$$

$$\deg(\gamma_1^N) = 2\left(\left\lfloor N \frac{1}{1+\varepsilon} \right\rfloor + \lfloor N \rfloor\right) = 4N-2 \quad \text{for } N-1 < 1/\varepsilon$$

Suppose $10 < 1/\varepsilon < 11$. Then

$\deg(\gamma_0^9) = 36$, $\deg(\gamma_0^{10}) = 40$, $\deg(\gamma_0^{11}) = 46$
$\deg(\gamma_1^{10}) = 38$, $\deg(\gamma_1^{11}) = 42$, $\deg(\gamma_1^{12}) = 44$

(ii) $a_0 = 1$, $a_1 = \varepsilon$

$$\deg(\gamma_0^N) = 2\left(\lfloor N \rfloor + \lfloor N\varepsilon \rfloor\right) = 2N \text{ for } N < 1/\varepsilon$$
$$\deg(\gamma_1^N) = 2\left(\left\lfloor N\tfrac{1}{\varepsilon}\right\rfloor + \lfloor N \rfloor\right) = 2N + 2\lfloor N/\varepsilon \rfloor$$

Suppose $10 < 1/\varepsilon < 11$. Then

$\deg(\gamma_0^{10}) = 20$, $\deg(\gamma_0^{11}) = 24$, $\deg(\gamma_1^{1}) = 22$,

- Proposition: For any $\mathbb{Q}$-independent $a_0, \dots, a_n \in \mathbb{R}^+$ and even $2m + 2(n-1)$, $m \in \mathbb{N}$, there exist unique $k \in \{0, \dots, n\}$ and $N \in \mathbb{N}$ such that

$$\boxed{\deg(\gamma_k^N) = 2m + 2(n-1)}$$

In Lecture 5 we will state and "prove" a result of this type for any Gorenstein TCM.

(3.2) Reeb flows on S^{2n+1}

- $h : \mathbb{R}^{2(n+1)} \longrightarrow \mathbb{R}, \quad (u,v) \longmapsto h(u,v)$

$\rightsquigarrow$ $\boxed{\dot{u} = -\frac{\partial h}{\partial v}, \quad \dot{v} = \frac{\partial h}{\partial u}}$ Hamilton's equations

$\rightsquigarrow$ solutions $(u(t), v(t))$ are such that $\frac{d}{dt}\left(h(u(t), v(t))\right) = 0$

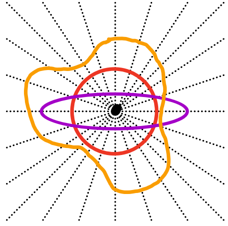
$\rightsquigarrow$ flow of X_h (Hamiltonian flow)

- Examples

(1) $h(u,v) = \frac{1}{2} \sum_{j=0}^{n} a_j \left(u_j^2 + v_j^2\right), \quad a_j > 0$

$\rightsquigarrow$ toric Reeb flows $z(t) = (e^{ia_0 t} z_0, \dots, e^{ia_n t} z_n)$

(2) $f : S^{2n+1} \hookrightarrow \mathbb{R}^{2(n+1)} \setminus \{0\}$ s.t. $\Sigma_f := f(S^{2n+1})$ bounds a star-shapped domain w.r.t. origin.



Let $h_f : \mathbb{R}^{2(n+1)} \setminus \{0\} \to \mathbb{R}$ be the homogeneous of degree 2 function s.t. $h_f^{-1}(1) = \Sigma_f$.

$\rightsquigarrow$ corresponding hamiltonian flow on $\Sigma_f \cong S^{2n+1}$ $\equiv$ Reeb flow

[toric Reeb flows correspond to ellipsoids
$\frac{1}{2} \sum_j a_j |z_j|^2 = 1$]

- Conjecture: any Reeb flow on S^{2n+1} has at least $n+1$ geometrically distinct periodic orbits.

Some results:

P. Rabinowitz (1978): at least 1.

Ekeland-Hofer (1987): at least 2 for convex.

Y. Long - C. Zhu (2002): at least $\lfloor \frac{n+1}{2} \rfloor + 1$ for convex & at least $n+1$ for convex + non-deg.

J. Gutt - J. Kang (2016): at least $n+1$ for dynamical convex + non-degenerate.

A. - L. Macarini (2017): at least 2 for dynamical convex. [...]

- Main tools

1) Variational Principle

Action Functional: $\mathcal{A}: C^\infty(S^1, \mathbb{R}^{2(n+1)}) \to \mathbb{R}$

$[\gamma(e^{it}) = (u(t), v(t))] \mapsto \mathcal{A}(\gamma) = \int_0^{2\pi} \left(\sum_j u_j(t)\, \dot{v}_j(t)\right) dt$

Any critical point γ_0 of $\mathcal{A}$, subject to the constraint $\int_0^{2\pi} h_f(\gamma_0(t))\, dt = 2\pi$ with nonzero Lag. multiplier λ, is such that $\gamma_0(e^{it/\lambda})$ is a $2\pi\lambda$ periodic orbit of Reeb flow on Σ_f.

Bad variational problem: $\mathcal{A}$ is not bounded from above or below, critical points have ∞ index and coindex, etc.

Rabinowitz (1978): first to overcome some of these problems.

2) Floer homology $\equiv$ "Morse homology" of $\mathcal{A}$ graded by $\deg = \mu_{CZ} + n - 2$.

Theorem: For any non-degenerate Reeb flow on S^{2n+1} and any even $2m+2(n-1)$, $m\in\mathbb{N}$, there exists at least one periodic orbit γ_m such that $\deg(\gamma_m) = 2m+2(n-1)$.
["Morse inequalities"]

Note: this only guarantees 1 geom. distinct periodic orbit, since apriori we could have a single γ_1 such that
$$\deg(\gamma_1^m) = 2m+2(n-1).$$

3) Index theory

R. Bott (1956): index theory in the context of geodesic flows.

Y. Long (1990's, ...): cf. book "Index theory for symplectic paths with applications", 2002.