④ Contact Betti numbers of Gorenstein toric contact manifolds

- Toric diagram $D \subset \mathbb{R}^n \rightsquigarrow (M_D^{2n+1}, \xi_D) \circlearrowleft \mathbb{T}^{n+1}$ and $(W_D^{2(n+1)}, \omega_D, X_D) \circlearrowleft \mathbb{T}^{n+1}$

- [Martelli - Sparks - Yau, 2006]

$\mathrm{int}(D) \subset \mathbb{R}^n$ parametrizes (normalized) toric Reeb vector fields:

$$\upsilon = (r_1, \ldots, r_n) \in \mathrm{int}(D) \rightsquigarrow \nu = (r_1, \ldots, r_n, 1) \in \mathrm{Lie}(\mathbb{T}^{n+1})$$
$$\rightsquigarrow R_\nu$$

[Follows from (S^{2d-1}, ξ_{std}) case of Lecture 4:

$$R_{S^{2d+1}} = \sum_{j=1}^{d} a_j e_j \qquad (a_j > 0,\ \{e_j\}_j \text{ basis of } \mathbb{R}^d)$$

$$\xRightarrow[\beta]{} R_{M_D^{2n+1}} = \sum_{j=1}^{d} a_j \beta(e_j) = \sum_{j=1}^{d} a_j \nu_j,\ a_j > 0$$

$$\xRightarrow[\text{normalization}]{c_1 = 0} \quad \nu_j = (\upsilon_j, \underline{1}) \text{ and } R_{M_D} = (\upsilon, \underline{1}) \text{ with}$$

$$\upsilon = \sum_{j=1}^{d} a_j \nu_j,\ a_j > 0 \text{ and } \sum_{j=1}^{d} a_j = 1,\ \text{i.e. } \upsilon \in \mathrm{int}(D).\ ]$$

• [A.-Macarini, A.-Macarini-Moreira]

R_ν non-deg. $\Leftrightarrow \{r_1, \dots, r_n, 1\}$ $\mathbb{Q}$-indep.

In that case:

(i) Simple closed R_ν-orbits $\xleftrightarrow{1:1}$ facets of D
$\gamma_1, \dots, \gamma_m$

(ii) $\deg(\gamma_\ell^N) := \mu_{CZ}(\gamma_\ell^N) + n - 2 \equiv \text{even} \in \mathbb{N}_0,$

$\forall\, \ell \in \{1, \dots, m\},\ N \in \mathbb{N}$

[Again, "follows" from (S^{2d+1}, ξ_{std}) case of Lecture 4:
γ = one of those orbits, $\{\nu_1 = (\underline{v_1}, 1), \dots, \nu_n = (\underline{v_n}, 1)\} =$
= <u>vertices</u> of corresponding facet of D,
$\eta = (h, 1)$ s.t. $\{\nu_1, \dots, \nu_n, \eta\}$ = $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$,
write $R_\nu = \sum_{j=1}^n b_j \nu_j + b\eta$, $b_j \in \mathbb{R}$, $b = 1 - \sum_{j=1}^n b_j \neq 0$.
then

$$\deg(\gamma^N) = \mu_{CZ}(\gamma^N) + (n-2) =$$

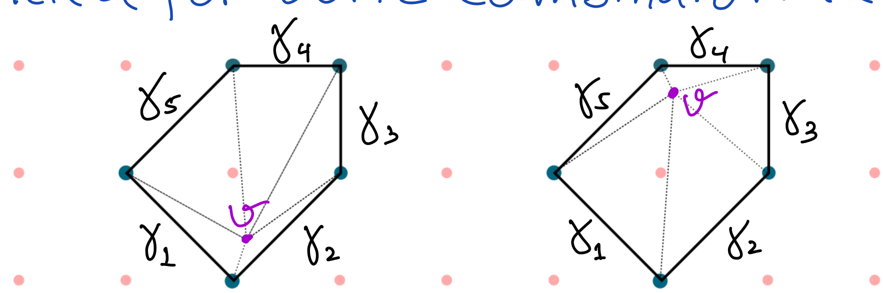
$$\overset{\text{"}}{=} \left[2\left(\sum_{j=1}^n \left\lfloor N \frac{b_j}{|b|} \right\rfloor + N \frac{b}{|b|} \right) + n \right]^{\text{"}} + (n-2). \quad //\ ]$$

(iii) $cb_j(D,\nu) := \#\{\text{closed } R_\nu\text{-orbits with degree} = j\}$
<u>should</u> be a <u>contact invariant</u> of (M_D, ξ_D)

$= \operatorname{rank} HC_j(M_D, \xi_D) =:$ <u>contact Betti #</u>

In particular, each $cb_j(D,\nu)$ <u>should</u> be <u>independent of ν</u> and give rise to the <u>contact Betti numbers</u> $\boxed{cb_*(D)}$

(iv) General idea for toric combinatorial proof:



γ_1 and γ_2 with much bigger degrees than γ_3, γ_4 and γ_5

γ_4 and γ_5 with much bigger degrees than γ_1, γ_2 and γ_3

(4.1) <u>Ehrhart polynomials</u>

- $L_D(t) := \#(D \cap \frac{1}{t}\mathbb{Z}^n), \quad t \in \mathbb{N}$

- <u>Properties</u>:

(i) L_D is a polynomial of degree n,

i.e. $$L_D(t) = \sum_{k=0}^{n} c_k(D)\, t^k, \quad \text{with}$$

(ii) $c_0(D) = 1$ and $c_n(D) = \mathrm{vol}(D)$.

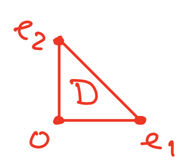
(iii) [Stanley] The coefficients of L_D in the polynomial basis $\left\{ \binom{t-k+n}{n},\ k=0,\dots,n \right\}$ are <u>non-negative integers</u>:

$$L_D(t) = \sum_{k=0}^{n} \delta_k(D) \binom{t-k+n}{n}, \quad \text{w/ } \underline{\delta_k(D) \in \mathbb{N}_0}$$

Note: define $\delta_k = 0$ for $k<0$ and $k>n$.

- <u>Examples</u>

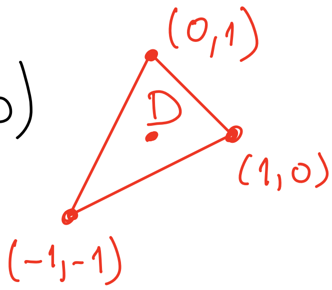
a) $D = \mathrm{conv}\{0, e_1, \dots, e_n\} \subset \mathbb{R}^n$



$$\left[(M_D, \xi_D) = (S^{2n+1}, \xi_{std}) \right]$$

$$L_D(t) = \frac{(t+1)\cdots(t+n)}{n!} = \binom{t+n}{n}$$

$$\Longrightarrow \quad \delta_k(D) = \begin{cases} 1, & k=0 \\ 0, & \text{otherwise} \end{cases}$$

b) 

$$\left[(M_D, \xi_D) = (S^5/\mathbb{Z}_3, \xi_{std}) = L^5_3(1,1,1) \right]$$

$$L_D(t) = \frac{1}{2}(3t^2 + 3t + 2) =$$

$$= \delta_0(D)\binom{t+2}{2} + \delta_1(D)\binom{t+1}{2} + \delta_2(D)\binom{t}{2}$$

$$= \delta_0(D)\frac{(t+2)(t+1)}{2} + \delta_1(D)\frac{(t+1)t}{2} + \delta_2(D)\frac{t(t-1)}{2}$$

with $\delta_0(D) = \delta_1(D) = \delta_2(D) = 1$.

4.2 Main result [A.-Macarini-Moreira]

Thm.: $\boxed{cb_{2j}(D,\nu) - cb_{2(j-1)}(D,\nu) = \delta_{n-j}(D),}$

i.e. $cb_*(D)$ are combinatorically given by

$$\boxed{cb_{2j}(D) = \sum_{k=0}^{j} \delta_{n-k}(D), \quad j \in \mathbb{N}_0.}$$

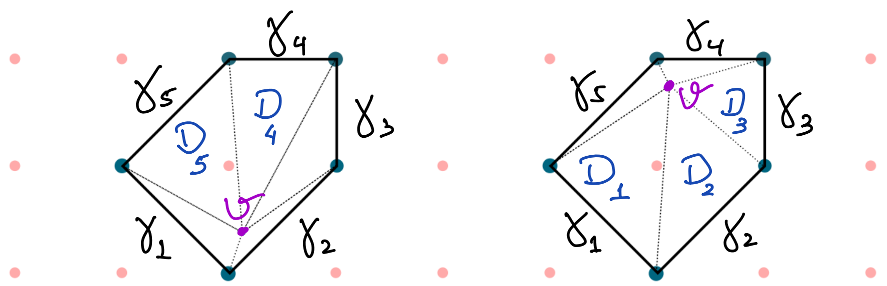
Cor.:

(i) The sequence $\{cb_{2j}(D)\}_{j\in\mathbb{Z}}$ is non-decreasing and $\boxed{cb_{2j}(D) = n!\,vol(D),\ \forall j \geq n}$

(ii) $\boxed{cb_0(D) = S_n(D) = \#(int(D) \cap \mathbb{Z}^n).}$

"Proof":

Lemma: Given $n, S \in \mathbb{N}$, the number of solutions of $\sum_{i=1}^{n} m_i = S$, with $m_i \in \mathbb{N}_0$, is given by $\binom{S+n-1}{n-1}$.



$$\Rightarrow L_{int D_\ell}(t) = \sum_{N \geq 1} \binom{t - \frac{1}{2}\deg \gamma_\ell^N + n - 2}{n-1}$$

$$= \sum_{j \in \mathbb{Z}} \left(\#\{N \geq 1 : \deg \gamma_\ell^N = 2j\}\right) \binom{t - j + n - 2}{n-1}$$

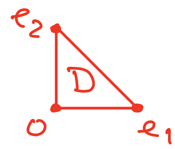
$$\Rightarrow \; L_{\mathrm{int}\,D}(t) = \sum_{j\in\mathbb{Z}} cb_{2j}(D,\nu)\binom{t-j+n-2}{n-1}$$

$$\overset{\text{Ehrhart reciprocity}}{=} (-1)^n L_D(-t)$$

"Q.E.D."

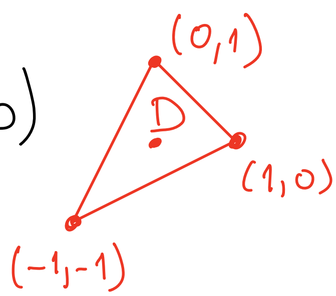
• <u>Examples</u>

a) $D = \mathrm{conv}\{0, e_1, \dots, e_n\} \subset \mathbb{R}^n$



$$(M_D, \xi_D) = (S^{2n+1}, \xi_{std})$$

$$\delta_k(D) = \begin{cases} 1, & k=0 \\ 0, & \text{otherwise} \end{cases} \Rightarrow cb_*(S^{2n+1}) = \begin{cases} 1, & *=2k\geq 2n \\ 0, & \text{otherwise} \end{cases}$$

b) 

$$\left[(M_D, \xi_D) = (S^5/\mathbb{Z}_3, \xi_{std}) = L_3^5(1,1,1) \right]$$

$$\begin{aligned} &\delta_0(D) = \delta_1(D) = \delta_2(D) = 1 \\ &\delta_k(D) = 0 \quad \text{otherwise} \end{aligned} \Rightarrow cb_*(L_3^5(1,1,1)) = \begin{cases} 1, & *=0 \\ 2, & *=2 \\ 3, & *=2k\geq 4 \\ 0, & \text{otherwise} \end{cases}$$

c) $M = L_p^{2n+1}(l_0, \dots, l_n)$, $a \in \pi_1(M)$ generator

[$H^2(M;\mathbb{Z}) \cong \mathbb{Z}_p$ and $c_1(M, \xi_{std}) = l_0 + \dots + l_n \pmod p$]

Prop.: If $m \cdot c_1(M, \xi_{std}) = 0$ then

$$cb_a^*(M) = \begin{cases} 1, & * = k_a + 2k, \quad k \in \mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases}$$

where $k_a := \min\{k \in \overset{\mathbb{Q}}{\mathbb{Z}} : HC_a^k(M) \neq 0\}$ is $\overset{>-1}{\geq 0}$ and can be computed directly from $p, l_0, \dots, l_n$.

This plays a relevant role in the proof of the following

Thm. [A. – Macarini – H. Liu, arXiv:2211.16470]

Any dyn. convex Reeb flow on a Lens space (M, ξ_{std}) has at least 2 simple periodic orbits with homotopy class a.

[Generalizes Ekeland - Hofer (1987, $p=1$, convex), D. Zhang (2013, convex), A. – Macarini (2017, $p=1$), H. Liu - L. Zhang (2022, $p=2$).]