Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Family name:	**Department:**
First name:	**ETH ID No.:**

For the grading:

	1K	**2K**	**Points**	**Comments:**
1				
2				
3				
4-13				
Total				

MATHEMATICS II EXAM

for students of Agricultural Science, Earth Sciences, Environmental Sciences, and Food Science

Important:

- Please fill the header on the cover page and lay your ETH-card visible on the table.
- Please write neatly with a non erasable blue or black pen, in particular not with a pencil. Beware that something that is too hard to read could be ignored.
- Please leave some empty space on the margins for the correction.
- This exam has 13 questions and lasts for 90 minutes.

For questions 1-3:

- Please write down all intermediate steps of your calculations and solutions.
- Write your name and ETH ID / Legi-Nr. on each additional sheet.
- The maximal score of each exercise part is given in the right margin.

For questions 4-13:

- Mark your answers clearly.
- There is always only one correct answer and 2 points per question.

Permitted aids:

- Written notes up to 20 A4-Pages, one English dictionary,
- **no** calculator, **no** mobile phone, **no** laptop.
- Please switch off your mobile phone and stow it away.

Good Luck!

1. Consider the function

$$f(x, y) = x^2 + y^2 - xy + 2x + 2y + 5$$

and its gradient $\vec{F} = \mathrm{grad}(f)$.

a) Determine the vector field $\vec{F}$. 2 points

b) Determine and classify the critical points of f (as local minimum, local maximum or saddle point). 3 points

c) The equation

$$f(x, y) = 4$$

defines a differentiable function $x = x(y)$ in a neighborhood of the point $(x, y) = (1, 0)$. Calculate the derivative $x'(0)$. 2 points

d) Calculate the line integral of $\vec{F}$ along the straight line C from the point $(1, 0)$ to the point $(1, 1)$. 3 points

---

2. Consider the vector field

$$\vec{F}(x, y, z) = \begin{pmatrix} -y^3 \\ x^3 \\ z^3 \end{pmatrix}$$

and the surface A given by

$$z = x^2 + y^2,\ 0 \leq z \leq 1.$$

a) Parametrize A. 2 points

b) Parametrize the boundary curve of A (in an arbitrary direction). 3 points

c) Determine $\mathrm{rot}(\vec{F})$. 3 points

d) Determine the flux of $\mathrm{rot}(\vec{F})$ through A downwards (i.e. the z-component of the normal vector is negative),

$$\iint\limits_A \left(\mathrm{rot}(\vec{F})\right) \cdot \vec{n}\ dA.$$

4 points

3. We consider problems of the form

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = u(0,y) = u(1,y) = 0 \\ u(x,1) = f(x) \end{cases}$$

for an unknown function $u(x,y)$ and $0 \le x \le 1$, $0 \le y \le 1$.

a) With the ansatz $u(x,y) = X(x)Y(y)$ the PDE can be separated into a system of ODEs for $X(x)$ and $Y(y)$ which depend on a parameter $k \in \mathbb{R}$. Determine that system of ODEs.
You do **not** have to solve that system of ODEs. 4 points

b) Determine the solution $u(x,y)$ of the problem with

$$f(x) = 5\sin(2\pi x).$$

You may use relevant eigenfunctions **without** deriving them. 4 points

---

For exercises 4-13: Each question gives 2 points. Wrong or multiple answers give 0 points. Mark your answers on this exam.

4. The trajectory of a moving point mass satisfies the following initial value problem:

$$\begin{cases} \dfrac{d\vec{r}}{dt} = \begin{pmatrix} 4t \\ 1 - \sin(t) \end{pmatrix} \\ \vec{r}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases}$$

The position vector of the point mass at time $t = 1$ is given by:

(a) $\vec{r}(1) = \begin{pmatrix} 2 \\ 1 - \cos(1) \end{pmatrix}$

(b) $\vec{r}(1) = \begin{pmatrix} 2 \\ 1 + \cos(1) \end{pmatrix}$

(c) $\vec{r}(1) = \begin{pmatrix} 3 \\ 1 - \cos(1) \end{pmatrix}$

(d) $\vec{r}(1) = \begin{pmatrix} 3 \\ 1 + \cos(1) \end{pmatrix}$

5. What is the arc length of the curve that is parametrized by

$$x = -e^t \sin(t), \qquad y = e^t \cos(t),\ 0 \le t \le 1 \quad ?$$

(a) $e - 2$. (b) $2\sqrt{2}e$. (c) $e - \sqrt{2}$. (d) $\sqrt{2}e - \sqrt{2}$.

---

6. What is the domain D and the range W of the function

$$f(x, y) = \tan\left(\frac{1}{1 + x^2 + y^2}\right) \quad ?$$

(a) $D = \mathbb{R}^2,\ W =]0, \tan(1)]$.

(b) $D = \mathbb{R}^2,\ W = [\tan(1), +\infty[$.

(c) $D = \{(x, y) \mid \tan(1 + x^2 + y^2) \neq 1\},\ W =]0, \tan(1)]$.

(d) $D = \{(x, y) \mid \tan(1 + x^2 + y^2) \neq 1\},\ W = [\tan(1), +\infty[$.

---

7. The directional derivative of the function $f(x, y) = xy$ at the point $(1, 3)$ in the direction of the vector $(2, 1)$ is equal to

(a) $\frac{1}{\sqrt{5}}$. (b) $\frac{3}{\sqrt{5}}$. (c) $\sqrt{5}$. (d) $\frac{7}{\sqrt{5}}$.

---

8. Let $f(x, y)$ be a continuous real function that is defined on $\mathbb{R}^2$. Which integral is in general equal to

$$\int_{-1}^{0} \int_{4+4x^3}^{4} f(x, y)\, dy\, dx \quad ?$$

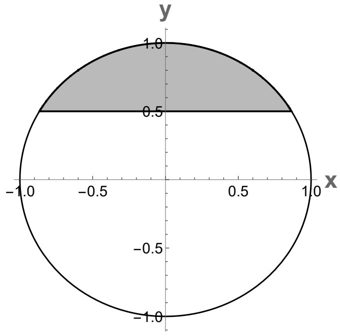
(a) $\displaystyle\int_0^4 \int_{-1}^{\sqrt[3]{\frac{y}{4}-1}} f(x, y)\, dx\, dy$

(b) $\displaystyle\int_0^4 \int_{\sqrt[3]{\frac{y}{4}-1}}^{-1} f(x, y)\, dx\, dy$

(c) $\displaystyle\int_0^4 \int_{\sqrt[3]{\frac{y}{4}-1}}^{0} f(x, y)\, dx\, dy$

(d) $\displaystyle\int_0^4 \int_0^{\sqrt[3]{\frac{y}{4}-1}} f(x, y)\, dx\, dy$

9. Consider the part of the disc



that is defined by $x^2+y^2 \leq 1$, $y \geq \frac{1}{2}$. Which of the following inequalities describes the domain in polar coordinates?

(a) $\dfrac{1}{2\sin(\theta)} \leq r \leq 1,\ \dfrac{\pi}{4} \leq \theta \leq \dfrac{3\pi}{4}$.

(b) $\dfrac{1}{2\sin(\theta)} \leq r \leq 1,\ \dfrac{\pi}{6} \leq \theta \leq \dfrac{5\pi}{6}$.

(c) $\dfrac{\sin(\theta)}{2} \leq r \leq 1,\ \dfrac{\pi}{4} \leq \theta \leq \dfrac{3\pi}{4}$.

(d) $\dfrac{\sin(\theta)}{2} \leq r \leq 1,\ \dfrac{\pi}{6} \leq \theta \leq \dfrac{5\pi}{6}$.

---

10. Consider the integral

$$I = \iiint\limits_V x\, dV$$

over the octant of the sphere with radius R

$$V = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq R^2, x,y,z \leq 0\right\}.$$

Which of the following formulas is correct?

(a) $I = \displaystyle\int_{\pi}^{\frac{3\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^R \rho^2 \sin(\varphi)\cos(\theta)\, d\rho\, d\varphi\, d\theta$

(b) $I = \displaystyle\int_{\frac{3\pi}{2}}^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^R \rho^2 \sin(\varphi)\cos(\theta)\, d\rho\, d\varphi\, d\theta$

(c) $I = \displaystyle\int_{\pi}^{\frac{3\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \int_0^R \rho^3 \sin^2(\varphi)\cos(\theta)\, d\rho\, d\varphi\, d\theta$

(d) $I = \displaystyle\int_{\frac{3\pi}{2}}^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \int_0^R \rho^3 \sin^2(\varphi)\cos(\theta)\, d\rho\, d\varphi\, d\theta$

11. The volume of the bounded solid defined by

$$1 \leq z \leq \sqrt{2 - x^2 - y^2}$$

is equal to

(a) $\dfrac{4\sqrt{2}-5}{3}\pi.$

(b) $\dfrac{5\sqrt{2}-4}{3}\pi.$

(c) $(4\sqrt{2}-5)\pi.$

(d) $(5\sqrt{2}-4)\pi.$

---

12. Let $\vec{F}$ be a vector field that is at least twice differentiable in $\mathbb{R}^3$. Which of the following formulas does **not** make sense?

(a) $\operatorname{grad}(\operatorname{div}(\vec{F})).$

(b) $\operatorname{div}(\operatorname{rot}(\vec{F})).$

(c) $\operatorname{grad}(\operatorname{grad}(\vec{F})).$

(d) $\operatorname{rot}(\operatorname{rot}(\vec{F})).$

---

13. What is the circulation to the vector field

$$\vec{F} = \begin{pmatrix} xy^2 - y \\ x^4 + x^2y + x \end{pmatrix}$$

along the boundary curve of the rectangle

$$-1 \leq x \leq 1, \ 0 \leq y \leq 1$$

in counterclockwise direction?

(a) $-8.$ (b) $-4.$ (c) $4.$ (d) $8.$