1. Consider the function

$$f(x,y) = x^2 + 2y^3 - 3y^2 + 1 \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

a) Determine and classify the critical points of f (as local maximum, local minimum or saddle point). 4 points

b) We consider the composition $f(x(t), y(t))$ of $f(x,y)$ with the following parametrization of the unit circle:

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \qquad \text{for } t \in [0, 2\pi].$$

Determine the derivative of this composition for $t = \pi$. 3 points

c) Let $\vec{F} = \text{grad}(f)$. Determine the line integral of the vector field $\vec{F}$ along the *quarter circle* parametrized by

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \qquad \text{for } t \in [0, \tfrac{\pi}{2}].$$

3 points

2. The plane with the equation $y = 2z$ intersects the solid straight circular cylinder $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4\}$ in a surface S.

a) Parametrize S using cylindrical coordinates. 2 points

b) Determine the area of S. 4 points

c) Using Stokes Theorem, determine the work $\oint_C \vec{F} \cdot d\vec{r}$ done by the vector field

$$\vec{F}(x,y,z) = \begin{pmatrix} yz \\ 2xz \\ x \end{pmatrix} \qquad \text{for } (x,y,z) \in \mathbb{R}^3,$$

when going once around S along the boundary curve C of S in positive direction, when observed from above. 4 points

3. Consider the following vector field

$$\vec{F}(x,y) = \begin{pmatrix} y - \frac{y}{x^2+y^2} \\ x + \frac{x}{x^2+y^2} \end{pmatrix},$$

which is defined for $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

a) Is $\vec{F}$ a gradient field on the first quadrant 3 points

$$Q := \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x, y > 0\right\}?$$

Yes: ☐ No: ☐

Justification:

b) Is $\vec{F}$ a gradient field on $\mathbb{R}^2 \setminus \{(0,0)\}$? 3 points

Yes: ☐ No: ☐

Justification:

For exercises 4-14: Each question gives two points. Wrong or multiple answers give zero points. **Only** answers **on the answer sheet** count.

4. Which integral computes the arc length of the helix with polar equation

$$r = \theta^2 , \quad 0 \le \theta \le 3 \ ?$$

(a) $\displaystyle\int_0^3 \theta\sqrt{\theta^2+4}\, d\theta$

(b) $\displaystyle\int_0^3 \theta^2\sqrt{\theta^2+4}\, d\theta$

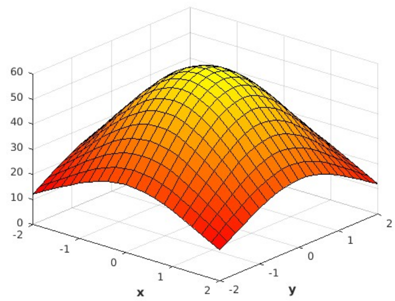
(c) $\displaystyle\int_0^9 \theta\sqrt{9-\theta}\, d\theta$

(d) $\displaystyle\int_0^9 \theta^2\sqrt{9-\theta}\, d\theta$

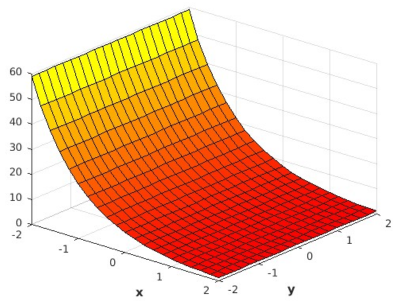
5. Which picture shows the graph of the function

$$f(x,y) = e^{-x-y} \ ?$$

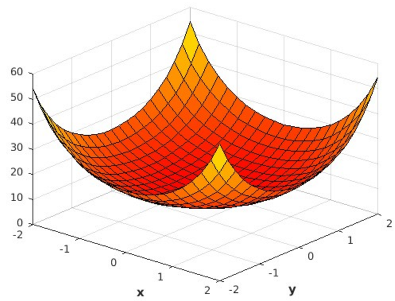
(a)



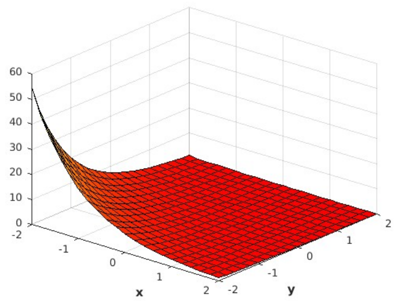
(c)



(b)



(d)



6. For which value of the parameter b does the equation

$$2x + by + 3z = 11$$

describe the tangent plane of the graph of the function

$$f(x, y) = 4 - \sqrt{1 + x^2 + y^2}$$

at the point $(x_0, y_0) = (2, -2)$?

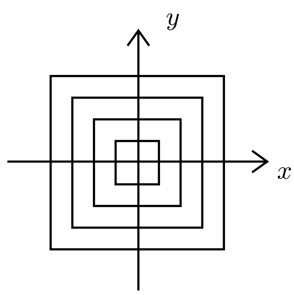
(a) $b = -4$

(b) $b = -2$

(c) $b = 3$

(d) $b = 5$

7. The following picture shows the level sets of a function $f : \mathbb{R}^2 \to \mathbb{R}$.



Which of the following functions has level sets as in the picture?

(a) $f(x, y) = |x| + |y|$

(b) $f(x, y) = |x| - |y|$

(c) $f(x, y) = |x + y| + |x - y|$

(d) $f(x, y) = |x + y| - |x - y|$

8. The lemiscate $y^2(1 - y^2) = x^2$ can be described...

(a) near the point $(1, 0)$ as a graph of a function in x.

(b) near the point $(1, 0)$ as a graph of a function in y.

(c) near the point $(0, 1)$ as a graph of a function in x.

(d) near the point $(0, 1)$ as a graph of a function in y.

9. For which of the following differential equations is the function

$$f(t,x) = \cos(x - 3t) + e^{x+3t}$$

a solution?

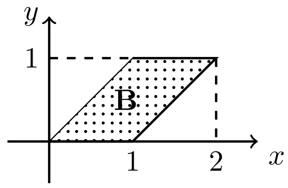
(a) $f_{tt} - 3f_x = 0$

(b) $f_{tt} + 3f_x = 0$

(c) $f_{tt} - 9f_{xx} = 0$

(d) $f_{tt} + 9f_{xx} = 0$

10. Which expression computes the integral of an arbitrary integrable function $f(x,y)$ over the domain B showed in the picture?



(a) $\int_0^1 \int_{1-x}^2 f(x,y)\,dy\,dx$

(b) $\int_0^1 \int_y^{1+y} f(x,y)\,dx\,dy$

(c) $\int_0^2 \int_x^{2-x} f(x,y)\,dy\,dx$

(d) $\int_0^2 \int_1^{y-1} f(x,y)\,dx\,dy$

11. Which integral is in general equal to

$$\int_0^{\sqrt{2}} \int_0^x f(x,y)\,dy\,dx\ ?$$

(a) $\int_0^{\frac{\pi}{4}} \int_0^{\frac{\sqrt{2}}{\cos\theta}} r f(r\cos\theta, r\sin\theta)\,dr\,d\theta.$

(b) $\int_0^{\frac{\pi}{4}} \int_{\frac{\sqrt{2}}{\sin\theta}}^2 r f(r\cos\theta, r\sin\theta)\,dr\,d\theta.$

(c) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\sqrt{2}}{\sin\theta}} r f(r\cos\theta, r\sin\theta)\,dr\,d\theta.$

(d) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\sqrt{2}}{\cos\theta}}^2 r f(r\cos\theta, r\sin\theta)\,dr\,d\theta.$

12. The three-dimensional domain V is described in cartesian coordinates by the following inequalities:

$$2 \leq x^2 + y^2 + z^2 \leq 4, \quad z^2 \leq x^2 + y^2.$$

Which of the following inequalities describes V in spherical coordinates?

(a) $2 \leq \rho \leq 4, \quad \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi.$

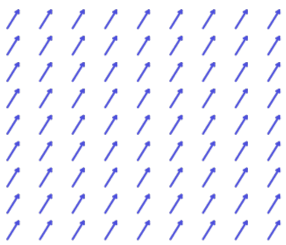
(b) $2 \leq \rho \leq 4, \quad \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, \quad 0 \leq \theta \leq 2\pi.$

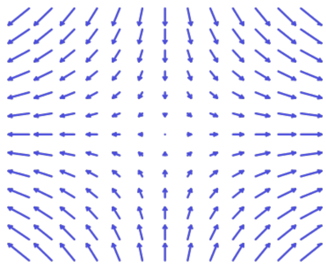
(c) $\sqrt{2} \leq \rho \leq 2, \quad \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi.$

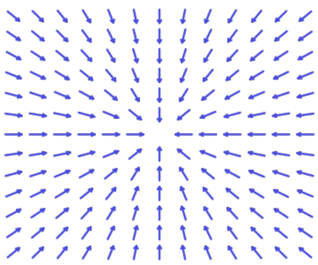
(d) $\sqrt{2} \leq \rho \leq 2, \quad \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, \quad 0 \leq \theta \leq 2\pi.$

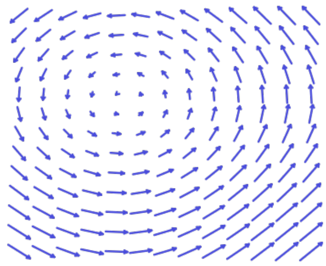
13. Which picture shows the vector field

$$\vec{F}(x,y) = \begin{pmatrix} x \\ -y \end{pmatrix} \ ?$$

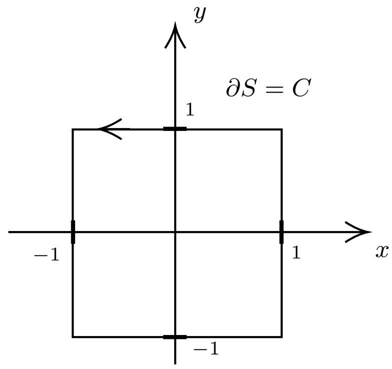
(a)


(c)


(b)


(d)


14. Let S be the square with boundary curve C as shown in the picture.



Let $\vec{F}_a$ be the a-dependent vector field

$$\vec{F}_a(x,y) = \begin{pmatrix} x - 4xy - 2y \\ 2a(3x - y) \end{pmatrix}.$$

For which a is the work of $\vec{F}_a$ along the curve C in counter-clockwise direction equal to 4? That is, for which a is $\oint_C \vec{F}_a \cdot d\vec{r} = 4$?

(a) $a = -\frac{1}{2}$

(b) $a = -\frac{1}{6}$

(c) $a = \frac{1}{6}$

(d) $a = \frac{1}{2}$