Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Family name:	Department:
First name:	ETH ID No.:

For the grading:

	1K	2K	Points	Comments:
1				
2				
3				
4				
5				
6				
7-26				
Total				

MATHEMATICS I AND II EXAM

for students of Agricultural Science, Earth Sciences, Environmental Sciences, and Food Science

Important:

- Please fill the header on the cover page and lay your ETH-card visible on the table.
- Please write neatly with a non erasable blue or black pen, in particular not with a pencil. Beware that something that is too hard to read could be ignored.
- Please leave some empty space on the margins for the correction.
- This exam has 26 questions and lasts for 180 minutes.

For questions 1-6:

- Please write down all intermediate steps of your calculations and solutions.
- Write your name and ETH ID / Legi-Nr. on each additional sheet.
- The maximal score of each exercise part is given in the right margin.

For questions 7-26:

- Mark your answers clearly.
- There is always only one correct answer and 2 points per question.

Permitted aids:

- Written notes up to 40 A4-Pages, one English dictionary,
- **no** calculator, **no** mobile phone, **no** laptop.
- Please switch off your mobile phone and stow it away.

Good Luck!

1. Consider the function

$$f(x) = \sqrt{x^2+5} \quad \text{for } x \in \mathbb{R}.$$

a) Determine the linearization of $f(x)$ in $x_0 = 2$. 4 points

b) Determine the range of $f(x)$. 3 points

c) Let $F(x)$ be the solution of the initial value problem

$$\begin{cases} F'(x) = f(x) \\ F(0) = 33. \end{cases}$$

Is $F(1)$ bigger or smaller than 33? You do not have to compute $F(x)$. 3 points

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2. Determine the general solution of each of the following differential equations:

a) $y'' = 6y' - 10y$ 5 points

b) $3y' = (y-1)(y+2)$ 5 points

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3. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 4 & 4 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

a) Is the system

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

solvable? 3 points

b) Determine a basis of the solution set of the matrix equation $A\vec{x} = \vec{0}$. 4 points

c) Determine a basis of the space of all vectors $\vec{v}$ for which the matrix equation $A\vec{x} = \vec{v}$ is solvable. 3 points

4. Consider the function

$$f(x, y) = x^2 + y^2 - xy + 2x + 2y + 5$$

and its gradient $\vec{F} = \mathrm{grad}(f)$.

a) Determine the vector field $\vec{F}$. 2 points

b) Determine and classify the critical points of f (as local minimum, local maximum or saddle point). 3 points

c) The equation

$$f(x, y) = 4$$

defines a differentiable function $x = x(y)$ in a neighborhood of the point $(x, y) = (1, 0)$. Calculate the derivative $x'(0)$. 2 points

d) Calculate the line integral of $\vec{F}$ along the straight line C from the point $(1, 0)$ to the point $(1, 1)$. 3 points

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5. Consider the vector field

$$\vec{F}(x, y, z) = \begin{pmatrix} -y^3 \\ x^3 \\ z^3 \end{pmatrix}$$

and the surface A given by

$$z = x^2 + y^2, \ 0 \le z \le 1.$$

a) Parametrize A. 2 points

b) Parametrize the boundary curve of A (in an arbitrary direction). 3 points

c) Determine $\mathrm{rot}(\vec{F})$. 3 points

d) Determine the flux of $\mathrm{rot}(\vec{F})$ through A downwards (i.e. the z-component of the normal vector is negative),

$$\iint_A \left(\mathrm{rot}(\vec{F})\right) \cdot \vec{n} \, dA.$$

4 points

6. We consider problems of the form

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = u(0,y) = u(1,y) = 0 \\ u(x,1) = f(x) \end{cases}$$

for an unknown function $u(x,y)$ and $0 \leq x \leq 1$, $0 \leq y \leq 1$.

a) With the ansatz $u(x,y) = X(x)Y(y)$ the PDE can be separated into a system of ODEs for $X(x)$ and $Y(y)$ which depend on a parameter $k \in \mathbb{R}$. Determine that system of ODEs.
You do **not** have to solve that system of ODEs. 4 points

b) Determine the solution $u(x,y)$ of the problem with

$$f(x) = 5\sin(2\pi x).$$

You may use relevant eigenfunctions **without** deriving them. 4 points

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For exercises 7-26: Each question gives 2 points. Wrong or multiple answers give 0 points. Mark your answers on this exam.

7. The determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 1 \\ 4 & 5 & 1 & 3 \\ 2 & 6 & 0 & 0 \end{pmatrix} \text{ is}$$

is

(a) -2.

(b) -1.

(c) 1.

(d) 2.

8. The vector $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix

$$\begin{pmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix}.$$

What is the eigenvalue belonging to $\vec{v}$?

(a) -2 (b) -1 (c) 1 (d) 2

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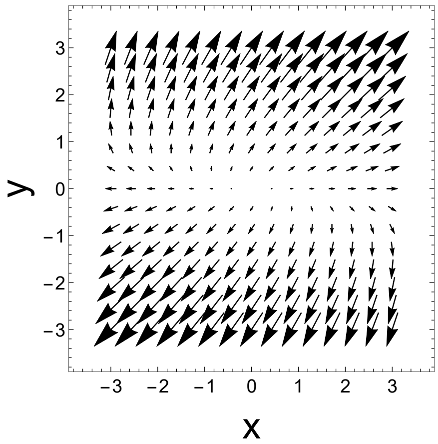
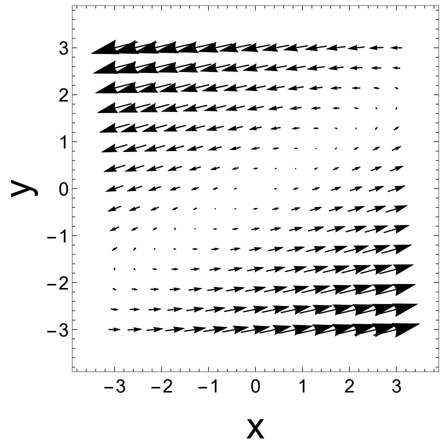
9. Which picture shows the phase portrait of the system

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} \vec{x} \quad ?$$

(a)


(c)


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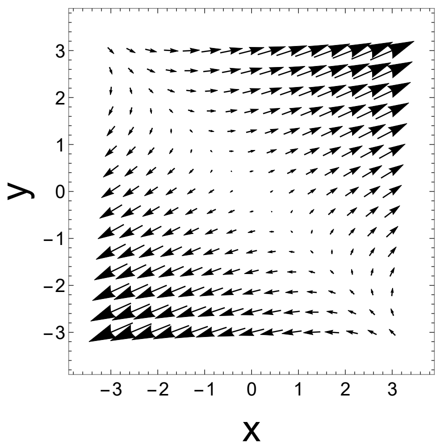

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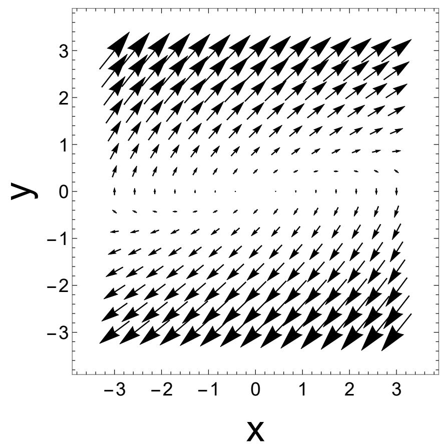

10. The limit

$$\lim_{x\to+\infty} x^{\frac{2}{x}}$$

is given by

(a) 0 (b) 1 (c) 2 (d) $+\infty$

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11. Consider the function

$$f(x) = \int_0^x \ln(t^2 + e^3)\, dt.$$

Then $f'(0)$ is given by

(a) 0 (b) 1 (c) 2 (d) 3

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12. Let $g(y)$ be the inverse function of the function

$$y = f(x) = e^{(3-x)^3-1}.$$

Consider $g(y)$ at the point $y = f(2) = 1$. Then the derivative $g'(1)$ is given by

(a) -3 (b) $-\frac{1}{3}$ (c) $\frac{1}{3}$ (d) 3

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13. The zeros of the polynomial $p(\lambda) = \lambda^4 + 1$ are given by:

(a) $-1, 1, -i, i$

(b) $-1, 1, e^{i\frac{\pi}{4}}, e^{-i\frac{\pi}{4}}$

(c) $e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$

(d) $e^{i\frac{\pi}{4}}, e^{i\frac{5\pi}{4}}, -i, i$

14. Let $f(x)$ be a function that is differentiable for all $x \in \mathbb{R}$. Which of the following statements are always true?

(I) If f is not injective, then there exists a c with $f'(c) = 0$.

(II) If there exists a c with $f'(c) = 0$, then f is not injective.

(a) Both statements (I) and (II) are true.

(b) Statement (I) is true, but statement (II) is false.

(c) Statement (II) i true, but statement (I) is false.

(d) Both statements (I) and (II) are false.

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15. The trajectory of a moving point mass satisfies the following initial value problem:

$$\begin{cases} \dfrac{d\vec{r}}{dt} = \begin{pmatrix} 4t \\ 1 - \sin(t) \end{pmatrix} \\ \vec{r}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases}$$

The position vector of the point mass at time $t = 1$ is given by:

(a) $\vec{r}(1) = \begin{pmatrix} 2 \\ 1 - \cos(1) \end{pmatrix}$

(b) $\vec{r}(1) = \begin{pmatrix} 2 \\ 1 + \cos(1) \end{pmatrix}$

(c) $\vec{r}(1) = \begin{pmatrix} 3 \\ 1 - \cos(1) \end{pmatrix}$

(d) $\vec{r}(1) = \begin{pmatrix} 3 \\ 1 + \cos(1) \end{pmatrix}$

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16. What is the arc length of the curve that is parametrized by

$$x = -e^t \sin(t), \qquad y = e^t \cos(t), \; 0 \le t \le 1 \quad ?$$

(a) $e - 2$.

(b) $2\sqrt{2}e$.

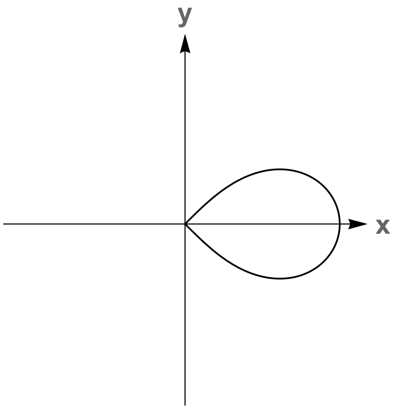
(c) $e - \sqrt{2}$.

(d) $\sqrt{2}e - \sqrt{2}$.

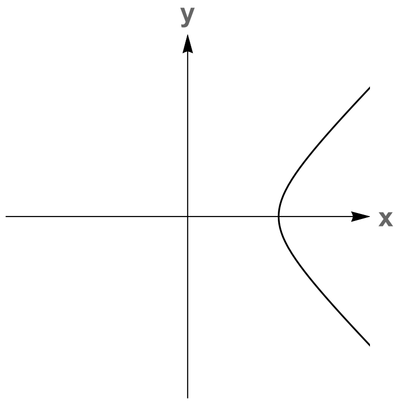
17. Which of the following pictures shows the curve described by the following equations in polar coordinates

$$r^2 = 4\cos(2\theta), \qquad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \quad ?$$

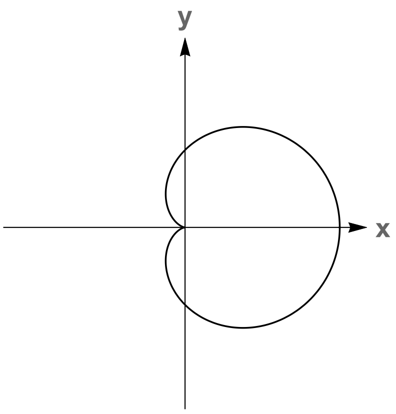
(a)



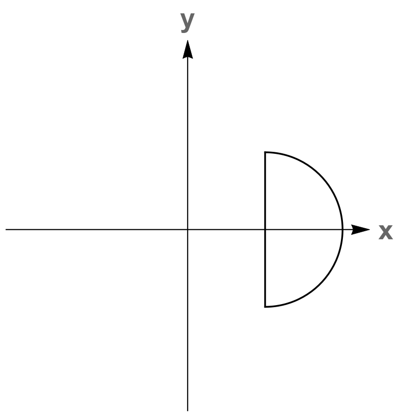
(c)



(b)



(d)



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18. What is the domain D and the range W of the function

$$f(x,y) = \tan\left(\frac{1}{1+x^2+y^2}\right) \quad ?$$

(a) $D = \mathbb{R}^2,\ W =]0, \tan(1)]$.

(b) $D = \mathbb{R}^2,\ W = [\tan(1), +\infty[$.

(c) $D = \{(x,y) \mid \tan(1+x^2+y^2) \neq 1\},\ W =]0, \tan(1)]$.

(d) $D = \{(x,y) \mid \tan(1+x^2+y^2) \neq 1\},\ W = [\tan(1), +\infty[$.

19. The directional derivative of the function $f(x,y) = xy$ at the point $(1,3)$ in the direction of the vector $(2,1)$ is equal to

(a) $\frac{1}{\sqrt{5}}$.

(b) $\frac{3}{\sqrt{5}}$.

(c) $\sqrt{5}$.

(d) $\frac{7}{\sqrt{5}}$.

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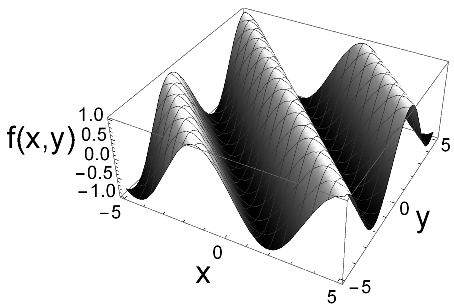
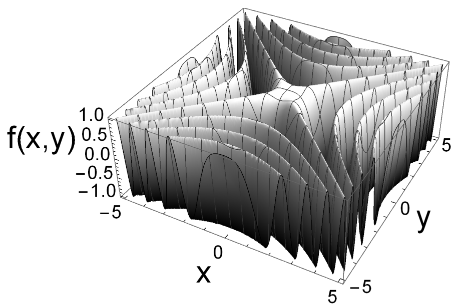
20. Which picture shows the graph of the function

$$f(x,y) = \cos(x^2 - y^2) \quad ?$$

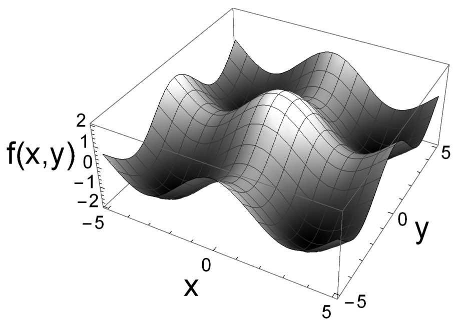
(a)



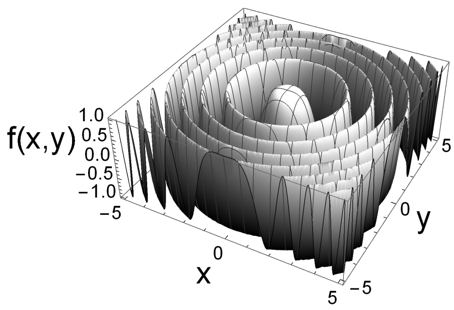
(c)



(b)



(d)



21. Let $f(x,y)$ be a continuous real function that is defined on $\mathbb{R}^2$. Which integral is in general equal to

$$\int_{-1}^{0}\int_{4+4x^3}^{4} f(x,y)\,dy\,dx \quad ?$$

(a) $\displaystyle\int_0^4\int_{-1}^{\sqrt[3]{\frac{y}{4}-1}} f(x,y)\,dx\,dy$

(b) $\displaystyle\int_0^4\int_{\sqrt[3]{\frac{y}{4}-1}}^{-1} f(x,y)\,dx\,dy$

(c) $\displaystyle\int_0^4\int_{\sqrt[3]{\frac{y}{4}-1}}^{0} f(x,y)\,dx\,dy$

(d) $\displaystyle\int_0^4\int_{0}^{\sqrt[3]{\frac{y}{4}-1}} f(x,y)\,dx\,dy$

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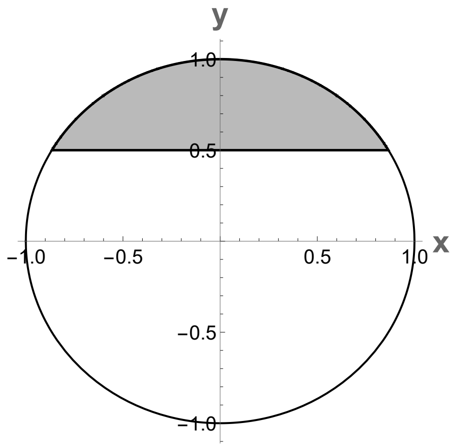
22. Consider the part of the disc



that is defined by $x^2+y^2 \le 1$, $y \ge \frac{1}{2}$. Which of the following inequalities describes the domain in polar coordinates?

(a) $\dfrac{1}{2\sin(\theta)} \le r \le 1,\ \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}.$

(b) $\dfrac{1}{2\sin(\theta)} \le r \le 1,\ \dfrac{\pi}{6} \le \theta \le \dfrac{5\pi}{6}.$

(c) $\dfrac{\sin(\theta)}{2} \le r \le 1,\ \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}.$

(d) $\dfrac{\sin(\theta)}{2} \le r \le 1,\ \dfrac{\pi}{6} \le \theta \le \dfrac{5\pi}{6}.$

23. Consider the integral

$$I = \iiint_V x\, dV$$

over the octant of the sphere with radius R

$$V = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq R^2, x, y, z \leq 0\right\}.$$

Which of the following formulas is correct?

(a) $I = \int_{\pi}^{\frac{3\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^R \rho^2 \sin(\varphi) \cos(\theta)\, d\rho\, d\varphi\, d\theta$

(b) $I = \int_{\frac{3\pi}{2}}^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^R \rho^2 \sin(\varphi) \cos(\theta)\, d\rho\, d\varphi\, d\theta$

(c) $I = \int_{\pi}^{\frac{3\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \int_0^R \rho^3 \sin^2(\varphi) \cos(\theta)\, d\rho\, d\varphi\, d\theta$

(d) $I = \int_{\frac{3\pi}{2}}^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \int_0^R \rho^3 \sin^2(\varphi) \cos(\theta)\, d\rho\, d\varphi\, d\theta$

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24. The volume of the bounded solid defined by

$$1 \leq z \leq \sqrt{2 - x^2 - y^2}$$

is equal to

(a) $\dfrac{4\sqrt{2} - 5}{3}\pi.$

(b) $\dfrac{5\sqrt{2} - 4}{3}\pi.$

(c) $(4\sqrt{2} - 5)\pi.$

(d) $(5\sqrt{2} - 4)\pi.$

25. Let $\vec{F}$ be a vector field that is at least twice differentiable in $\mathbb{R}^3$. Which of the following formulas does **not** make sense?

(a) $\operatorname{grad}(\operatorname{div}(\vec{F}))$.

(b) $\operatorname{div}(\operatorname{rot}(\vec{F}))$.

(c) $\operatorname{grad}(\operatorname{grad}(\vec{F}))$.

(d) $\operatorname{rot}(\operatorname{rot}(\vec{F}))$.

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26. What is the circulation to the vector field

$$\vec{F} = \begin{pmatrix} xy^2 - y \\ x^4 + x^2y + x \end{pmatrix}$$

along the boundary curve of the rectangle

$$-1 \leq x \leq 1,\ 0 \leq y \leq 1$$

in counterclockwise direction?

(a) -8.

(b) -4.

(c) 4.

(d) 8.
