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This "mémoire" is dedicated to Giovanni D'Angelo, who if still alive would probably be proud of me!

Contents

Introduction

The aim of this note is to present a part of the research I have done during and after my Phd. The central argument of my research concerns the optimal transport problem and its applications, but I also worked on other subjects. Some of them, which I will describe here, are: the study of variational models for the incompressible Euler equations, Mather's theory, and some generalization of the Diperna-Lions theory for ODEs with non-smooth vector fields. The note is therefore structured in four independent parts.

In the first part, I will introduce the optimal transport problem, starting with some preliminaries. In Sections 1.2 and 1.3 I will describe some recent results, which I studied in [8, 10, 11, 17, 22, 24], concerning existence, uniqueness and properties of optimal transport maps in a Riemannian and sub-Riemannian setting.

I will then focus on an important problem in this area, which consists in studying the regularity of the optimal transport map. This is something I studied in [18, 25, 23]. In Section 1.4 I will state some of the obtained results. We will see in particular that there are some unexpected connections between regularity properties of the transport map on Riemannian manifolds, and the geometric structure of the manifold. As an example, as I showed with Rifford in [23], studying the regularity of the optimal transport one can prove as a corollary a convexity result on the cut-locus of the manifold.

We will then see some applications of the optimal transport, showing how one can apply it to prove some refined version of functional inequalities: in Section 1.5 we will see that the optimal transport allows to prove a sharpened isoperimetric inequality in $\mathbb{R}^n$, a result I did in [19] with Maggi and Pratelli. Moreover, always using the optimal transport, me and Ge were recently able to prove isoperimetric-type inequalities on manifolds with constant curvature [16].

Finally in Section 1.6 I will show a variant of the optimal transport that I studied in [15], and I called the "optimal partial transport problem".

The second part concerns some variational methods introduced by Brenier for the study of the incopressible Euler equations. These methods are based on a relaxation of Arnold's problem, which consists in looking at the Euler equations as geodesics in the space of volume preserving diffeomorphism. After introducing the models, in Section 2.2 I will describe some of the results obtained with Bernot and Santambrogio in [7], where we studied some particular generalized solutions in two dimensions. Then Section 2.3 is focused on giving sufficient and necessary conditions for being a generalized solution, a problem investigated with Ambrosio in [3, 4].

In the third part I will show a result obtained with Fathi and Rifford concerning the dimension of the quotient Aubry set [9]. Our results give a positive answer to Mather conjecture in many cases (in particular in dimension at most 3). Moreover, as I will explain in Section 3.2, this problem presents a deep connection with Sard Theorem, and this fact motivated a study I did in [12] on generalizations of the Sard Theorem in Sobolev spaces (see Section 3.3).

Finally, in the fourth part I will describe some recent generalizations of the DiPerna-Lions theory for non-smooth ODEs: we will see that one can develop a "weak" theory on existence and uniqueness of martingale solutions for non-smooth SDEs, and moreover one can adapt the finite dimensional techniques to the infinite dimensional case of an abstract Wiener space. This part concerns the results obtained in [13, 5].

Chapter 1

The optimal transport problem

The optimal transport problem (whose origin goes back to Monge [68]) is nowadays formulated in the following general form: given two probability measures μ and ν, defined on the measurable spaces X and Y, find a measurable map $T : X \to Y$ with $T_\sharp\mu = \nu$, i.e.

$$\nu(A) = \mu\left(T^{-1}(A)\right) \quad \forall A \subset Y \text{ measurable},$$

and in such a way that T minimizes the transportation cost. This last condition means

$$\int_X c(x, T(x))\, d\mu(x) = \min_{S_\sharp\mu=\nu} \left\{ \int_X c(x, S(x))\, d\mu(x) \right\},$$

where $c : X \times Y \to \mathbb{R}$ is some given cost function, and the minimum is taken over all measurable maps $S : X \to Y$ with $S_\sharp\mu = \nu$. When the transport condition $T_\sharp\mu = \nu$ is satisfied, we say that T is a *transport map*, and if T minimizes also the cost we call it an *optimal transport map*.

Even in Euclidean spaces, with the cost c equal to the Euclidean distance or its square, the problem of the existence of an optimal transport map is far from being trivial. Moreover, it is easy to build examples where the Monge problem is ill-posed simply because there is no transport map: this happens for instance when μ is a Dirac mass while ν is not. This means that one needs some restrictions on the measures μ and ν.

The major advance on this problem is due to Kantorovitch, who proposed in [55], [56] a notion of weak solution of the optimal transport problem. He suggested to look for *plans* instead of transport maps, that is probability measures γ in $X \times Y$ whose marginals are μ and ν, i.e.

$$(\pi_X)_\sharp\gamma = \mu \quad \text{and} \quad (\pi_Y)_\sharp\gamma = \nu,$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the canonical projections. Denoting by $\Pi(\mu, \nu)$ the set of plans, the new minimization problem becomes

$$C(\mu, \nu) = \min_{\gamma\in\Pi(\mu,\nu)} \left\{ \int_{M\times M} c(x, y)\, d\gamma(x, y) \right\}. \tag{1.0.1}$$

If γ is a minimizer for the Kantorovich formulation, we say that it is an *optimal plan*. Due to the linearity of the constraint $\gamma \in \Pi(\mu, \nu)$, it turns out that weak topologies can be used to provide existence of solutions to (1.0.1): this happens for instance whenever X and Y are Polish spaces and c is lower semicontinuous. The connection between the formulation of Kantorovich and that of Monge can be seen by noticing that any transport map T induces the plan defined by $(\mathrm{Id}_X \times T)_\sharp \mu$ which is concentrated on the graph of T. Thus, the problem of showing existence of optimal transport maps reduces to prove that an optimal transport plan is concentrated on a graph. It is however clear, from what we already said, that no such result can be expected without additional assumptions on the measures and on the cost.

1.1 Preliminary results

The first existence and uniqueness result is due to Brenier. In [37] he considers the case $X = Y = \mathbb{R}^n$, $c(x,y) = |x-y|^2$, and he shows the following:

Theorem 1.1.1 *Let μ and ν be two probability measures on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} |x|^2 \, d\mu(x) + \int_{\mathbb{R}^n} |y|^2 \, d\nu(y) < +\infty.$$

If μ is absolutely continuous with respect to $\mathscr{L}^n$, there exists a unique optimal transport map T. Moreover $T = \nabla\phi$, with $\phi : \mathbb{R}^n \to \mathbb{R}$ convex.

After this result, many researchers started to work on the problem, showing existence of optimal maps with more general costs, both in a Euclidean setting, in the case of compact manifolds, and in some particular classes on non-compact manifolds.

In particular, McCann was able to generalize Brenier's theorem to compact manifolds [66]:

Theorem 1.1.2 *Let (M,g) be a compact Riemannian manifold, take μ and ν two probability measures on M, and consider the optimal transport problem from μ to ν with cost $c(x,y) = d_g(x,y)^2$, where d_g denotes the Riemannian distance on M. If μ is absolutely continuous with respect to the volume measure, there exists a unique optimal transport map T. Moreover there exists a function $\varphi : M \to \mathbb{R}$ such that $T(x) = \exp_x(\nabla_x\varphi)$.*

Few years later, Ambrosio and Rigot proved the first existence and uniqueness result on optimal transport maps in a sub-Riemannian setting [29]. More precisely they consider the Heisenberg group $\mathbb{H}^n$, whose basis for the associated Lie Algebra of left-invariant vector fields is given by $\big(\mathbf{X}, \mathbf{Y}, \mathbf{T}\big) = \big(\mathbf{X}_1, \cdots, \mathbf{X}_n, \mathbf{Y}_1, \cdots, \mathbf{Y}_n, \mathbf{T}\big)$, where

$$\begin{aligned} \mathbf{X}_k &= \partial_{x_k} + 2y_k\partial_t \qquad \text{for } k = 1, \ldots, n \\ \mathbf{Y}_k &= \partial_{y_k} - 2x_k\partial_t \qquad \text{for } k = 1, \ldots, n \\ \mathbf{T} &= \partial_t. \end{aligned}$$

Then one has the following result:

Theorem 1.1.3 *Let μ_0 and μ_1 be two Borel probability measures on $\mathbb{H}^n$, where $\mathbb{H}^n$ denotes the Heisenberg group. Assume that μ_0 is absolutely continuous with respect to $\mathscr{L}^{2n+1}$ and that*

$$\int_{\mathbb{H}^n} d_C(0_{\mathbb{H}}, x)^2 d\mu_0(x) + \int_{\mathbb{H}^n} d_C(0_{\mathbb{H}}, y)^2 d\mu_1(y) < +\infty.$$

Then there exists a unique optimal transport plan from μ_0 to μ_1. Moreover there exists a function $\varphi : \mathbb{H}^n \to \mathbb{R}$ such that the optimal transport plan is concentrated on the graph of

$$T(x) := x \cdot \exp_{\mathbb{H}}(-\mathbf{X}\varphi(x) - \mathbf{i}\mathbf{Y}\varphi(x), -\mathbf{T}\varphi(x)).$$

1.2 Riemannian manifolds

With the aim of generalizing McCann's result to more general costs, and removing at the same time the compactenss assumption, in a joint work with Albert Fathi [8], we study the optimal transport problem on general non-compact manifolds with a "geometric" cost function:

$$c(x, y) := \inf_{\gamma(0)=x,\gamma(1)=y} \int_0^1 L(\gamma(t), \dot\gamma(t))\, dt,$$

where $L : TM \to \mathbb{R}$ is a Tonelli Lagrangian. In this general setting, without requiring any global assumption on the manifold (say, a bound on the sectional curvature), we were able to prove the following result:

Theorem 1.2.1 *Let L be a Tonelli Lagrangian on the connected manifold M. Let μ, ν be probability measures on M, with μ absolutely continuous with respect to volume measure, and assume that the infimum in the Kantorovitch problem (1.0.1) with cost c is finite. Then there exists a unique optimal transport map $T : M \to M$. Moreover there exists a function $\varphi : M \to \mathbb{R}$ such that*

$$T(x) = \pi^* \circ \phi_t^H(x, \tilde{d}_x\varphi),$$

where $\pi^* : T^*M \to M$ is the canonical projection, ϕ_t^H is the Hamiltonian flow of the Hamiltonian H associated to L, and $\tilde{d}_x\varphi$ denotes the approximate differential[1] of φ.

1.2.1 Weak regularity of the optimal transport map

To properly state the above result we needed to use the notion of approximate differential, which is a "measure theoretical" notion of differentiability. Thus in the above statement we are implicitly saying that the function φ is approximately differentiable a.e. As I showed in [11], this result can be sharpened: let us consider for simplicity the case $c(x, y) = \frac{1}{2}d(x, y)^2$ (which corresponds to the choice $L(x, v) = \frac{1}{2}|v|_x^2$). Then the optimal map is given by the formula $T(x) = \exp_x(\tilde\nabla_x\varphi)$, where $\tilde\nabla_x\varphi$ denotes the approximate gradient of φ. As proved in [11], the function φ is indeed twice approximate differentiable, so that we can define its approximate hessian $\tilde\nabla_x^2\varphi$. Thanks to this regularity property of φ, I could prove a change of variable formula, and the approximate differentiability of the transport map:

[1] We recall that $f : M \to \mathbb{R}$ has an *approximate differential* at $x \in M$ if there exists a function $h : M \to \mathbb{R}$ differentiable at x such that the set $\{f = h\}$ has density 1 at x with respect to the Lebesgue measure (this just means that the density is 1 in charts). In this case, the approximate value of f at x is defined as $\tilde f(x) = h(x)$, and the approximate differential of f at x is defined as $\tilde d_x f = d_x h$.

Theorem 1.2.2 *Assume that* $\mu = f\mathrm{vol}$, $\nu = g\mathrm{vol}$. *There exists a subset* $E \subset M$ *such that* $\mu(E) = 1$ *and, for each* $x \in E$, $Y(x) := d(\exp_x)_{\tilde{\nabla}_x \varphi}$ *and* $H(x) := \frac{1}{2}\mathrm{Hess}\, d(\cdot, T(x))^2|_{z=x}$ *both exist and we have*

$$f(x) = g(T(x)) \det[Y(x)(H(x) + \tilde{\nabla}^2_x \varphi)] \neq 0.$$

Moreover the transport map is approximatively differentiable for μ*-a.e.* x*, and its approximate differential is given by the formula*

$$\tilde{d}_x T = Y(x)\big(H(x) + \tilde{\nabla}^2_x \varphi\big),$$

In particular, if $A : [0 + \infty) \to \mathbb{R}$ *is a Borel function such that* $A(0) = 0$*, then*

$$\int_M A(g(y))\, d\mathrm{vol}(y) = \int_E A\left(\frac{f(x)}{J(x)}\right) J(x)\, d\mathrm{vol}(x),$$

where $J(x) := \det[Y(x)(H(x) + \tilde{\nabla}^2_x \varphi)] = \det\big(\tilde{d}_x T\big)$ *(either both integrals are undefined or both take the same value in* $\overline{\mathbb{R}}$*).*

1.2.2 Displacement convexity

The importance of the above theorem (which generalizes to non-compact manifolds the results in [44]) comes from the fact that it allows to study convexity properties of functionals along Wasserstein geodesic.

To explain this fact, let us consider the family of maps $T_t(x) := \exp_x(t\tilde{\nabla}_x \varphi)$. Observe that $T_0(x) = x$ and $T_1(x) = T(x)$, so that we can define a family of measures $\mu_t := (T_t)_\sharp \mu$ going from $\mu = \mu_0$ to $\nu = \mu_1$. By the results in [8, 11], we know that T_t coincides with the unique optimal map pushing μ forward to μ_t, and that μ_t is absolutely continuous with respect to vol for any $t \in [0, 1]$, so that we can write $\nu_t = \rho_t \mathrm{vol}$. Moreover μ_t is the *unique* geodesic between μ and ν with respect to the 2-Wasserstein distance (which is the square root of the optimal transport cost functional, when the cost function $c(x, y)$ coincides with the squared distance $d(x, y)^2$).

We now want to consider the behavior of the functional

$$U(\rho) := \int_M A(\rho(x))\, d\mathrm{vol}(x)$$

along the path $t \mapsto \rho_t$. In Euclidean spaces, this path is called *displacement interpolation* and the functional U is said to be *displacement convex* if

$$[0, 1] \ni t \mapsto U(\rho_t) \quad \text{is convex for every } \rho_0, \rho_1.$$

As shown by McCann [64, 65], sufficient condition for the displacement convexity of U in $\mathbb{R}^n$ is that $A : [0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ satisfies

$$(0, +\infty) \ni s \mapsto s^n A(s^{-n}) \text{ is convex and nonincreasing, with } A(0) = 0. \tag{1.2.1}$$

Typical examples include the entropy $A(\rho) = \rho \log \rho$ and the L^q-norm $A(\rho) = \frac{1}{q-1}\rho^q$ for $q \geq (n-1)/n$.

Thanks to the (weak) regularity properties of the transport map stated above, I could prove that the displacement convexity of U is still true on Ricci nonnegative manifolds under the assumption (1.2.1) [11]:

Theorem 1.2.3 *If* Ric ≥ 0 *and* A *satisfies* (1.2.1)*, then* U *is displacement convex.*

In the above case, we have considered functional defined only on probability measures which are absolutely continuous with respect to vol. The big advantage of this fact is that, in this case, the curve μ_t we defined above is the *unique* Wasserstein geodesic between μ and ν. On the other hand, if we do not make any absolute continuity assuptions, the Wasserstein geodesic (which always exists) is in general not unique. One can therefore introduce two different notions of dispacement convexity, a strong and a weak one: the *strong* notion consists in asking that a functional defined in the space of probability measure on M is convex among *all* Wasserstein geodesics connecting two measures μ and ν; the *weak* one is that, for all μ and ν, there is *some* Wasserstein geodesic connecting them along which the functional is convex. The importance of introducing a weaker notion comes from the fact that it is more stable under passage to the limit, and so it is particularly suitable when one wants to recast lower bounds on the Ricci curvature tensor in terms of displacement convexity properties of certain nonlinear functionals [59, 73, 74]. However, as shown in collaboration with Cédric Villani, on Riemannian manifolds these notions are equivalent [24].

1.3 Sub-Riemannian manifolds

A sub-Riemannian manifold is given by a triple (M, Δ, g) where M denotes a smooth complete connected manifold of dimension n, Δ is a smooth nonholonomic distribution of rank $m < n$ on M, and g is a Riemannian metric on M. We recall that a smooth distribution of rank m on M is a rank m subbundle of TM. This means that, for every $x \in M$, there exist a neighborhood $\mathcal{V}_x$ of x in M, and a m-tuple $(f_1^x, \dots, f_m^x)$ of smooth vector fields on $\mathcal{V}_x$, linearly independent on $\mathcal{V}_x$, such that

$$\Delta(z) = \text{Span}\left\{f_1^x(z), \dots, f_m^x(z)\right\} \qquad \forall z \in \mathcal{V}_x.$$

One says that the m-tuple of vector fields $(f_1^x, \dots, f_m^x)$ represents locally the distribution Δ. We assume that the distribution Δ is *nonholonomic*, i.e. for every $x \in M$ there is a m-tuple $(f_1^x, \dots, f_m^x)$ of smooth vector fields on $\mathcal{V}_x$ which represents locally the distribution and such that

$$\text{Lie}\left\{f_1^x, \dots, f_m^x\right\}(z) = T_zM \qquad \forall z \in \mathcal{V}_x,$$

that is, such that the Lie algebra[2] spanned by $f_1^x, \dots, f_m^x$, is equal to the whole tangent space T_zM at every point $z \in \mathcal{V}_x$. This Lie algebra property is often called *Hörmander's condition.*

A curve $\gamma : [0, 1] \to M$ is called a *horizontal path* with respect to Δ if it belongs to $W^{1,2}([0, 1], M)$ and satisfies

$$\dot\gamma(t) \in \Delta(\gamma(t)) \qquad \text{for a.e. } t \in [0, 1].$$

[2]We recall that, for any family $\mathcal{F}$ of smooth vector fields on M, the Lie algebra of vector fields generated by $\mathcal{F}$, denoted by Lie($\mathcal{F}$), is the smallest vector space S satisfying

$$[X, Y] \subset S \qquad \forall X \in \mathcal{F}, \quad \forall Y \in S,$$

where $[X, Y]$ is the Lie bracket of X and Y.

According to the classical Chow-Rashevsky Theorem, since the distribution is nonholonomic on M, any two points of M can be joined by a horizontal path. That is, for every $x, y \in M$, there is a horizontal path $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The *length* of a path $\gamma \in \Omega_\Delta(x)$ is then defined by

$$\text{length}_g(\gamma) := \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt. \tag{1.3.1}$$

The *sub-Riemannian distance* $d_{SR}(x, y)$ (also called Carnot-Carathéodory distance) between two points x, y of M is the infimum over the lengths of the horizontal paths joining x and y. According to the Chow-Rashevsky Theorem, since the distribution is nonholonomic on M, the sub-Riemannian distance is finite and continuous on $M \times M$.

Assuming that (M, d_{SR}) is complete, denote by T^*M the cotangent bundle of M, by ω the canonical symplectic form on T^*M, and by $\pi : T^*M \to M$ the canonical projection. The *sub-Riemannian Hamiltonian* $H : T^*M \to \mathbb{R}$ which is canonically associated with the sub-Riemannian structure is defined as follows: for every $x \in M$, the restriction of H to the fiber T_x^*M is given by the nonnegative quadratic form

$$p \longmapsto \frac{1}{2} \max \left\{ \frac{p(v)^2}{g_x(v, v)} \mid v \in \Delta(x) \setminus \{0\} \right\}. \tag{1.3.2}$$

Let $\overrightarrow{H}$ denote the Hamiltonian vector field on T^*M associated to H, that is $\iota_{\overrightarrow{H}}\omega = -dH$. A *normal extremal* is an integral curve of $\overrightarrow{H}$ defined on $[0,1]$, i.e. a curve $\psi(\cdot) : [0,1] \to T^*M$ satisfying

$$\dot{\psi}(t) = \overrightarrow{H}(\psi(t)), \qquad \forall t \in [0,1].$$

Note that the projection of a normal extremal is a horizontal path with respect to Δ. For every $x \in M$, the *exponential mapping* with respect to x is defined by

$$\begin{aligned} \exp_x : T_x^*M &\longrightarrow M \\ p &\longmapsto \psi(1), \end{aligned}$$

where ψ is the normal extremal such that $\psi(0) = (x, p)$ in local coordinates. We observe that, unlike the Riemannian setting, the sub-Riemannian exponential mapping with respect to x is defined on the cotangent space at x.

1.3.1 Statement of the results

When one studies the optimal transport problem on sub-Riemannian manifolds with cost $c(x, y) = \frac{1}{2} d_{SR}(x, y)^2$, the main difficulty one encounters comes from the fact that the sub-Riemannian distance is singular along the diagonal $D := \{(x, y) \in M \times M \,|\, x = y\}$. To deal with this problem, in a joint work with Ludovic Rifford [22] we study the set of points which are in the support of an optimal γ for the Kantorovitch problem (1.0.1), and we analyze separately points on the diagonal and points outside the diagonal. Thanks to this refined analysis, we can prove an existence and uniqueness result assuming only a semi-concavity property of the function d_{SR}^2 in the complement of the diagonal (which is an assumption always satisfied in the Riemannian case, and it holds true on many sub-Riemannian manifolds).

Theorem 1.3.1 *Let μ and ν be (compactly supported) probability measures, with μ absolutely continuous with respect to* vol. *Assume that there exists an open set $\Omega \subset M \times M$ such that* $\mathrm{supp}(\mu \times \nu) \subset \Omega$, *and d_{SR}^2 is locally semiconcave (resp. locally Lipschitz) on $\Omega \setminus D$. Then there exists a function $\phi : M \to \mathbb{R}$, and an open set $A \subset M$, such that ϕ is locally semiconcave inside A, and the unique optimal transport map is given by*

$$T(x) := \begin{cases} \exp_x(-\,d\phi(x)) & \text{if } x \in A, \\ x & \text{if } x \in \ M \setminus A. \end{cases}$$

We see that we can recover more or less the standard Riemannian result, simply splitting the transport map in two sets suitably chosen. The key points with respect to previous results in the sub-Riemannian setting are:

1) We do not make any assumption of regularity on the sub-Riemannian distance on the diagonal (all previous results assumed at least the function d_{SR}^2 to be Lipschitz on the diagonal, like in the Heisenberg group).

2) We can prove a "second order regularity" of the function ϕ appearing in the formula for the transport map (recall indeed that semiconcave function are twice differentiable a.e.)

In particular, thanks to 2), we can prove a weak regularity property of the optimal transport map, as in the Riemannian case. This allows for instance to write (for the first time, to our knowledge) a weak formulation of Monge-Ampère equation in a sub-Riemannian setting:

Theorem 1.3.2 *With the same assumption of Theorem 1.3.1, the optimal transport map is differentiable μ-a.e. inside A, it is approximately differentiable at μ-a.e. x. Moreover*

$$Y(x) := d(\exp_x)_{-d\phi(x)} \quad \text{and} \quad H(x) := \frac{1}{2}\mathrm{Hess}\, d_{SR}(\cdot, T(x))^2|_{z=x}$$

exists for μ-a.e. $x \in A$, and the approximate differential of T is given by the formula

$$\tilde{d}_x T = \begin{cases} Y(x)\big(H(x) - \mathrm{Hess}\,\phi(x)\big) & \text{if } x \in A, \\ Id & \text{if } x \in M \setminus A, \end{cases}$$

where $Id : T_xM \to T_xM$ denotes the identity map.

Finally, assuming both μ and ν absolutely continuous with respect to the volume measure, and denoting by f and g their respective density, the following Jacobian identity holds:

$$\big|\det\big(\tilde{d}_x T\big)\big| = \frac{f(x)}{g(T(x))} \neq 0 \qquad \mu\text{-a.e.} \tag{1.3.3}$$

In particular ϕ satisfies in a weak sense the Monge-Ampère type equation

$$\det\big(H(x) - \mathrm{Hess}\,\phi(x)\big) = \frac{f(x)}{|\det(Y(x))| g(T(x))} \qquad \text{for } \mu\text{-a.e. } x \in A.$$

As another byproduct of our regularity result, we can prove the absolute continuity of Wasserstein geodesics. This fact was stated as an open problem in the case of the Heisenberg group [29], and solved in the Heisenberg group and in Alexandrov spaces in a joint work with Nicolas Juillet [17]:

Theorem 1.3.3 *With the same assumption of Theorem 1.3.1, there exists a unique Wasserstein geodesic* $(\mu_t)_{t\in[0,1]}$ *joining* $\mu = \mu_0$ *to* $\nu = \mu_1$*, which is given by* $\mu_t := (T_t)_\# \mu$ *for* $t \in [0, 1]$*, with*

$$T_t(x) := \begin{cases} \exp_x(-t\, d\phi(x)) & \text{if } x \in A, \\ x & \text{if } x \in M \setminus A. \end{cases}$$

Moreover, if Ω *is totally geodesically convex, then* μ_t *is absolutely continuous for all* $t \in [0, 1)$.

1.4 Regularity of the optimal transport on Riemannian manifolds

Let (M, g) be a compact connected Riemannian manifold, let $\mu(dx) = f(x)\mathrm{vol}(dx)$ and $\nu(dy) = g(y)\mathrm{vol}(dy)$ be probability measures on M, and consider the cost $c(x, y) = \frac{1}{2} d(x, y)^2$. Assume f and g to be C^∞ and strictly positive on M. A natural question is whether the optimal map T is smooth or not.

To understand a bit the problem, we start from the Jacobian equation

$$\big|\det(d_x T)\big| = \frac{f(x)\mathrm{vol}_x}{g(T(x))\mathrm{vol}_{T(x)}},$$

and the relation $T(x) = \exp_x\big(\nabla\varphi(x)\big)$. We now write a PDE for φ. Indeed, since

$$\nabla\varphi(x) + \nabla_x c(x, T(x)) = 0,$$

differentiating with respect to x and using the Jacobian equation we get

$$\det\big(\nabla^2\varphi(x) + \nabla_x^2 c\big(x, \exp_x\big(\nabla\varphi(x)\big)\big)\big) = \frac{f(x)\mathrm{vol}_x}{g(T(x))\mathrm{vol}_{T(x)}\big|\det(d_{\nabla\varphi(x)}\exp_x)\big|} =: h(x, \nabla\varphi(x)).$$

We see that φ solves a Monge-Ampère type equation with a perturbation $\nabla_x^2 c\big(x, \exp_x\big(\nabla\varphi(x)\big)\big)$ which is of first order in φ. Unfortunately, for Monge-Ampère type equations lower order terms do matter, and it turns out that it is exactly the term $\nabla_x^2 c\big(x, \exp_x\big(\nabla\varphi(x)\big)\big)$ which can create obstractions to the smoothness.

In [63], the authors found a mysterious forth-order conditions on the cost functions, which turned out to be sufficient to prove regularity results. The idea was to differentiate twice the above PDE for φ in order to get a linear PDE for the second derivatives of φ, and then try to prove an a priori estimate. In this computation, one ends up at a certain moment with a term which needs to have a sign in order to make the equation elliptic. This term is what now is called the Ma-Trudinger-Wang tensor (in short MTW tensor):

$$\mathfrak{S}_{(x,y)}(\xi, \eta) := \frac{3}{2} \sum_{ijklrs} \left(c_{ij,r} c^{r,s} c_{s,kl} - c_{ij,kl}\right) \xi^i \xi^j \eta^k \eta^l, \qquad \xi \in T_x M,\ \eta \in T_y M.$$

In the above formula the cost function is evaluated at (x, y), and we used the notation $c_j = \frac{\partial c}{\partial x^j}$, $c_{jk} = \frac{\partial^2 c}{\partial x^j \partial x^k}$, $c_{i,j} = \frac{\partial^2 c}{\partial x^i \partial y^j}$, $c^{i,j} = (c_{i,j})^{-1}$, and so on. The condition to impose on $\mathfrak{S}_{(x,y)}(\xi, \eta)$ is

$$\mathfrak{S}_{(x,y)}(\xi, \eta) \geq 0 \qquad \text{whenever } \sum_{ij} c_{i,j}\xi^i\eta^j = 0$$

(this is called the MTW condition).

As shown by Loeper [57], the MTW tensor satisfies the following remarkable identity:

$$\mathfrak{S}_{(x,x)}(\xi, \eta) = -\frac{3}{2}\frac{\partial^2}{\partial s^2}|_{s=0}\frac{\partial^2}{\partial t^2}|_{t=0}F(t, s) = \mathrm{Sect}_x([\xi, \eta]),$$

where $\xi, \eta \in T_xM$ are two orthogonal unit vectors, $F(t, s) := \frac{1}{2}d\big(\exp_x(t\xi), \exp_x(s\eta)\big)^2$, and $\mathrm{Sect}_x([\xi, \eta])$ denotes the sectional curvature of the plane generated by ξ and η. This fact shows that the MTW tensor is a non-local version of the sectional curvature, and the MTW condition implies non-negative sectional curvature. Loeper also showed that the MTW condition is indeed a necessary condition for the regularity of the optimal map. In particular, regularity cannot hold on manifolds which have a point x where the sectional curvature of a plane in T_xM is negative.

In collaboration with Gregoire Loeper [18], I proved a regularity result in two dimension for optimal maps under weak assumptions on the measures:

Theorem 1.4.1 *Let (M, g) be a two-dimensional manifold. Assume that the* MTW *condition holds, that $f \leq A$ and $g \geq a$ for some $A, a > 0$ on their respective support, and that the cost function $c(x, y)$ is smooth on the set* $\mathrm{supp}(\mu) \times \mathrm{supp}(\nu)$. *Finally suppose that* $(\exp_x)^{-1}(\mathrm{supp}(\nu)) \subset T_xM$ *is convex for any $x \in \mathrm{supp}(\mu)$. Then T is continuous.*

We remark that this result is "local", in the sense that the assumption that c is smooth on $\mathrm{supp}(\mu) \times \mathrm{supp}(\nu)$ means that we stay away from the cut locus.

In the general case one has to deal with singularity of the cost function, which makes things much more complicated. It turns out that the convexity of cut-loci is useful to prove regularity and stability results. For this reason we give the following definition:

Definition 1.4.2 *Given $x \in M$ and $v \in T_xM$ we define the cut time as*

$$t_c(x, v) := \inf\{t > 0 \,|\, s \mapsto \exp_x(sv) \textit{ is not minimizing between } x \textit{ and } \exp_x(tv)\}.$$

We say that (M, g) satisfies CTIL *(Convexity of the Tangent Injectivity Loci) if, for all $x \in M$, the set*

$$\mathrm{TIL}(x) := \{tv \in T_xM \,|\, 0 \leq t < t_c(x, v)\} \subset T_xM$$

is convex.

Combining (a strengthened version of) the MTW condition (called MTW(K, C)) with CTIL, Loeper and Villani proved the continuity of the optimal map [58]:

Theorem 1.4.3 *Let (M, g) be a (compact) Riemannian manifold satisfying* $\mathrm{MTW}(K, C)$ *with* $K > 0$. *Assume morever that all* $\mathrm{TIL}(x)$ *are uniformly convex, and let f and g be two probability densities on M such that $f \leq A$ and $g \geq a$ for some $A, a > 0$. Then the optimal map is continuous.*

As noted in [58], it seems reasonable to conjecture that the MTW condition implies CTIL, so that in general one should expect that regularity results hold assuming only the MTW condition. This conjecture has been proved by Loeper and Villani [58] assuming that there is no focalization at the cut locus (i.e., $d_{t_c(x,v)v} \exp_x$ is invertible for all x, v). However big complications arise when one tries to prove this result in general, due to the complicated structure of the cut locus.

In [23], in collaboration with Ludovic Rifford, we studied the case of the perturbation of the 2-sphere. First of all, we prove that perturbations of the standard sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ satisfy a variant of the MTW condition. Then we prove that this new condition actually implies CTIL. Therefore we get the following:

Theorem 1.4.4 *If (M, g) is a C^4-perturbation of $\mathbb{S}^2$, then* CTIL *holds. Moreover, for any f and g probability densities on M such that $f \leq A$ and $g \geq a$ for some $A, a > 0$, the optimal map is continuous.*

Thus we see that an interesting (and unexpected) feature appears in the study of the regularity issue: the MTW condition, although it was introduced as a necessary condition for the regularity, turns out to be a geometric condition which gives new kind of geometric informations on the manifolds. It would be therefore interesting to understand which manifolds satisfies the MTW condition, and which geometric informations it implies. This is one of the subjects of my present research.

1.5 The (anisotropic) isoperimetric inequality

The anisotropic isoperimetric inequality arises in connection with a natural generalization of the Euclidean notion of perimeter. In dimension $n \geq 2$, consider an open, bounded, convex set K of $\mathbb{R}^n$, containing the origin. Starting from K, define a weight function on directions through the Euclidean scalar product

$$\|\nu\|_* := \sup\{x \cdot \nu : x \in K\}, \quad \nu \in \mathbb{S}^{n-1},$$

where $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$. Let E be an open subset of $\mathbb{R}^n$, with smooth or polyhedral boundary ∂E oriented by its outer unit normal vector ν_E, and let $\mathcal{H}^{n-1}$ stand for the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$. The anisotropic perimeter of E is defined as

$$P_K(E) := \int_{\partial E} \|\nu_E(x)\|_* d\mathcal{H}^{n-1}(x). \tag{1.5.1}$$

This notion of perimeter obeys the scaling law $P_K(\lambda E) = \lambda^{n-1} P_K(E)$, $\lambda > 0$, and it is invariant under translations. However, at variance with the Euclidean perimeter, P_K is not invariant by the action of $O(n)$, or even of $SO(n)$, and in fact it may even happen that $P_K(E) \neq P_K(\mathbb{R}^n \setminus E)$, provided K is not symmetric with respect to the origin. When K is the Euclidean unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ of $\mathbb{R}^n$, then $\|\nu\|_* = 1$ for every $\nu \in \mathbb{S}^{n-1}$, and therefore $P_K(E)$ coincides with the Euclidean perimeter of E.

Apart from its intrinsic geometric interest, the anisotropic perimeter P_K arises as a model for surface tension in the study of equilibrium configurations of solid crystals with sufficiently small grains, and constitutes the basic model for surface energies in phase transitions. In the former setting, one is naturally led to minimize $P_K(E)$ under a volume constraint. This is of course equivalent to study the isoperimetric problem

$$\inf \left\{ \frac{P_K(E)}{|E|^{(n-1)/n}} : 0 < |E| < \infty \right\}, \tag{1.5.2}$$

where $|E|$ is the Lebesgue measure of E. As conjectured by Wulff [75] back to 1901, the unique minimizer (modulo the invariance group of the functional, that consists of translations and scalings) is the set K itself. In particular the anisotropic isoperimetric inequality holds:

$$P_K(E) \geq n|K|^{1/n}|E|^{(n-1)/n}, \qquad \text{if } |E| < \infty. \tag{1.5.3}$$

It was Dinghas [50] to show how to derive (1.5.3) from the Brunn-Minkowski inequality

$$|E + F|^{1/n} \geq |E|^{1/n} + |F|^{1/n}, \quad \forall\, E, F \subseteq \mathbb{R}^n. \tag{1.5.4}$$

The formal argument is well known. Indeed, (1.5.4) implies that

$$\frac{|E + \varepsilon K| - |E|}{\varepsilon} \geq \frac{(|E|^{1/n} + \varepsilon |K|^{1/n})^n - |E|}{\varepsilon}, \qquad \forall\, \varepsilon > 0.$$

As $\varepsilon \to 0^+$, the right hand side converges to $n|K|^{1/n}|E|^{(n-1)/n}$, while, if E is regular enough, the left hand side has $P_K(E)$ as its limit.

Gromov's proof of the anisotropic isoperimetric inequality

Although Gromov's proof [67] was originally based on the use of the Knothe map M between E and K, his argument works with any other transport map having suitable structure properties, like the Brenier map. This is a well-known, common feature of all the proofs of geometric-functional inequalities based on mass transportation [45]. However it seems that, in the study of stability, Brenier map is more efficient.

We now want to give the proof of the anisotropic isoperimetric inequality, without caring about regularity issues.

Let us apply Theorem 1.1.1 to the measures $\mu = \frac{1}{|E|}\chi_E\, dx$, $\nu = \frac{1}{|K|}\chi_K\, dy$. Then we know that there exists a transport map T which takes E into K and such that

$$\det \nabla T = \frac{|K|}{|E|} \quad \text{on } E.$$

Moreover T is the gradient of a convex function and has positive Jacobian, so $\nabla T(x)$ is a symmetric and positive definite $n \times n$ matrix, with n-positive eigenvalues $0 < \lambda_k(x) \le \lambda_{k+1}(x)$, $1 \le k \le n-1$, such that

$$\nabla T(x) = \sum_{k=1}^{n} \lambda_k(x) e_k(x) \otimes e_k(x)$$

for a suitable orthonormal basis $\{e_k(x)\}_{k=1}^n$ of $\mathbb{R}^n$. In particular

$$\operatorname{div} T(x) = \sum_{i=1}^{n} \lambda_i(x), \qquad \big(\det \nabla T(x)\big)^{1/n} = \left(\prod_{i=1}^{n} \lambda_i(x)\right)^{1/n},$$

and the arithmetic-geometric mean inequality, applied to the λ_k's, gives

$$\operatorname{div} T(x) \ge n\big(\det \nabla T(x)\big)^{1/n} = n\left(\frac{|K|}{|E|}\right)^{1/n}. \tag{1.5.5}$$

Let us now define, for every $x \in \mathbb{R}^n$,

$$\|x\| = \inf\{\lambda > 0 : \lambda x \notin K\}.$$

Note that this quantity fails to define a norm only because, in general, $\|x\| \ne \|-x\|$ (indeed, K needs not to be symmetric with respect to the origin). Then, the set K can be characterized as

$$K = \{x \in \mathbb{R}^n : \|x\| < 1\}, \tag{1.5.6}$$

and $\|T\| \le 1$ on ∂E as $T(x) \in K$ for $x \in E$. Moreover, by the definition of $\|\cdot\|_*$, we have

$$\|\nu\|_* = \sup\{x \cdot \nu : \|x\| = 1\},$$

and therefore the following Cauchy-Schwarz type inequality holds:

$$x \cdot y \le \|x\| \|y\|_*, \quad \forall x, y \in \mathbb{R}^n. \tag{1.5.7}$$

Combining all together, and applying the Divergence Theorem, we get

$$\begin{aligned} P_K(E) &\ge \int_{\partial E} \|T\| \|\nu_E\|_* \, d\mathcal{H}^{n-1} \ge \int_{\partial E} T \cdot \nu_E \, d\mathcal{H}^{n-1} \\ &= \int_E \operatorname{div} T(x)\, dx \ge n\left(\frac{|K|}{|E|}\right)^{1/n} \int_E dx = n|K|^{1/n}|E|^{(n-1)/n}, \end{aligned}$$

and the isoperimetric inequality is proved.

1.5.1 Stability of isoperimetric problems

A quantitative version of the anisotropic

Whenever $0 < |E| < \infty$, we introduce the *isoperimetric deficit of* E,

$$\delta(E) := \frac{P_K(E)}{n|K|^{1/n}|E|^{(n-1)/n}} - 1\,.$$

This functional is invariant under translations, dilations and modifications on a set of measure zero of E. Moreover, $\delta(E) = 0$ if and only if, modulo these operations, E is equal to K (as a consequence of the characterization of equality cases of isoperimetric inequality). Thus $\delta(E)$ measures, in terms of the relative size of the perimeter and of the measure of E, the deviation of E itself from being optimal in (1.5.3). The stability problem consists in quantitatively relating this deviation to a more direct notion of distance from the family of optimal sets. To this end we introduce the *asymmetry index* of E,

$$A(E) := \inf_{x\in\mathbb{R}^n}\left\{\frac{|E\Delta(x+rK)|}{|E|}\;:\; r^n|K| = |E|\right\}\,,$$

where $E\Delta F$ denotes the symmetric difference between the sets E and F. The asymmetry is invariant under the same operations that leave the deficit unchanged. We look for constants C and α, depending on n and K only, such that the following quantitative form of (1.5.3) holds true:

$$P_K(E) \geq n|K|^{1/n}|E|^{(n-1)/n}\left\{1+\left(\frac{A(E)}{C}\right)^{\alpha}\right\}\,, \tag{1.5.8}$$

i.e. $A(E) \leq C\,\delta(E)^{1/\alpha}$. This problem has been thoroughly studied in the Euclidean case $K = B$, starting from the two dimensional case, already considered by Bernstein [33] and Bonnesen [35]. They prove (1.5.8) with the exponent $\alpha = 2$, that is optimal concerning the decay rate at zero of the asymmetry in terms of the deficit. Concerning the higher dimensional case, it was recently shown in [54] that (1.5.8) holds with the sharp exponent $\alpha = 2$.

The main technique behind these proofs is to use *quantitative* symmetrization inequalities, that of course reveal useful due to the complete symmetry of B. However, if K is a generic convex set, then the study of uniqueness and stability for the corresponding isoperimetric inequality requires the employment of different ideas. The first stability result for (1.5.3) is due to Esposito, Fusco and Trombetti in [53] with some constant $C = C(n,K)$ and for the exponent

$$\alpha(2) = \frac{9}{2}, \qquad \alpha(n) = \frac{n(n+1)}{2}, \quad n \geq 3.$$

This remarkable result leaves however the space for a substantial improvement concerning the decay rate at zero of the asymmetry index in terms of the isoperimetric deficit. In collaboration with Francesco Maggi and Aldo Pratelli, we could indeed prove the result with the sharp decay rate [19]:

Theorem 1.5.1 *Let E be a set of finite perimeter with $|E| < \infty$, then*

$$P_K(E) \geq n|K|^{1/n}|E|^{(n-1)/n}\left\{1+\left(\frac{A(E)}{C_0(n)}\right)^2\right\},$$

or, equivalently,

$$A(E) \leq C_0(n)\sqrt{\delta(E)},$$

with a constant $C_0(n)$ depending on the dimension only. Moreover $C_0(n)$ can be computed explicitly, and we have $C_0(n) = \frac{61\,n^7}{(2-2^{(n-1)/n})^{3/2}}$.

The strategy of the proof is to carefully look at Gromov's proof, and understand which informations can be recovered from each inequality which appears along the proof.

A refined Brunn-Minkowski inequality

As a corollary of this result, we could also prove a refined version of the Brunn-Minkowski on convex sets: the Brunn-Minkowski inequality states that, given two sets E and F, one has

$$|E+F|^{1/n} \geq |E|^{1/n} + |F|^{1/n}.$$

It is well-known that, whenever E and F are open bounded convex sets, equality holds in the Brunn-Minkowski inequality if and only if there exist $r > 0$ and $x_0 \in \mathbb{R}^n$ such that $E = x_0 + rF$. One can use Theorem 1.5.1 to infer an optimal result concerning the stability problem with respect to the *relative asymmetry index of E and F*, defined as

$$A(E,F) = \inf_{x\in\mathbb{R}^n}\left\{\frac{|E\Delta(x+rF)|}{|E|} \,:\, r^n|F| = |E|\right\}.$$

To this end, it is convenient to introduce the *Brunn-Minkowski deficit of E and F*,

$$\beta(E,F) := \frac{|E+F|^{1/n}}{|E|^{1/n}+|F|^{1/n}} - 1\,,$$

and the *relative size factor of E and F*, defined as

$$\sigma(E,F) := \max\left\{\frac{|F|}{|E|}, \frac{|E|}{|F|}\right\}.$$

Theorem 1.5.2 *If E and F are open bounded convex sets, then*

$$|E+F|^{1/n} \geq (|E|^{1/n}+|F|^{1/n})\left\{1+\frac{1}{\sigma(E,F)^{1/n}}\left(\frac{A(E,F)}{C(n)}\right)^2\right\}$$

or, equivalently,

$$C(n)\sqrt{\beta(E,F)\sigma(E,F)^{1/n}} \geq A(E,F).$$

An admissible value for $C(n)$ is $C(n) = 2C_0(n)$, where $C_0(n)$ is the constant defined in Theorem 1.5.1.

We remark that, as we showed in [19] by suitable examples, the decay rate of A in terms of β and σ provided by the above theorem is sharp.

An application to Cheeger sets

A Cheeger set E for an open subset $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is any minimizer of the variational problem

$$c_m(\Omega) := \inf \left\{ \frac{P(E)}{|E|^m} \mid E \subset \Omega, 0 < |E| < \infty \right\}.$$

In order to avoid trivial situations, it is assumed that Ω has finite measure and that the parameter m satisfies the constraint

$$m > \frac{n-1}{n}. \tag{1.5.9}$$

An interesting question is how to provide lower bounds on $c_m(\Omega)$ in terms of geometric properties of Ω. The basic estimate in this direction is the *Cheeger inequality*,

$$|\Omega|^{m-(n-1)/n} c_m(\Omega) \geq |B|^{m-(n-1)/n} c_m(B)\,, \tag{1.5.10}$$

where B is the Euclidean unit ball. The bound is sharp, in the sense that equality holds in (1.5.10) if and only if $\Omega = x_0 + rB$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$. In [20] we strengthen this lower bound in terms of the *Fraenkel asymmetry* of Ω

$$A(\Omega) := \inf_{x \in \mathbb{R}^n} \left\{ \frac{|\Omega \Delta B_r(x)|}{|E|} \,:\, |B_r| = |E| \right\},$$

Theorem 1.5.3 *Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, with $|\Omega| < \infty$, and let m satisfy* (1.5.9). *Then*

$$|\Omega|^{m-(n-1)/n} c_m(\Omega) \geq |B|^{m-(n-1)/n} c_m(B) \left\{ 1 + \left(\frac{A(\Omega)}{C(n,m)} \right)^2 \right\},$$

where $C(n,m)$ is a constant depending only on n and m. A possible value for $C(m,n)$ is given by

$$C(n,m) = \frac{2}{m-(n-1)/n} + C_0(n),$$

where $C_0(n)$ is the constant defined in Theorem 1.5.1.

1.5.2 An isoperimetric-type inequality on constant curvature manifolds

In the case of a Riemannian manifold (M,g), one can try to mimic Gromov's proof to obtain an isoperimetric type inequality. However in this case things become extremely more complicated, since many computations which are trivial on $\mathbb{R}^n$ involves second derivatives of the distance, and so in particular Jacobi fields. In [16], in collaboration with Yuxin Ge, we succeeded in adapting Gromov's argument to the case of the sphere and the hyperbolic space.

More precisely, let $M^n(K)$ denote the n-dimensional simply connected Riemannian manifold with constant sectional curvature $K \in \mathbb{R}$. Set $c(x,y) := \frac{1}{2} d_g(x,y)^2$, where $d_g(x,y)$ is the geodesic distance between x and y on M, and for $K \in \mathbb{R}$ define

$$G_K(r) := \begin{cases} \left(\frac{(\sqrt{K} r) \cos(\sqrt{K} r)}{\sin(\sqrt{K} r)} \right) & \text{if } K > 0, \\ 1 & \text{if } K = 0, \\ \left(\frac{(\sqrt{|K|} r) \cosh(\sqrt{|K|} r)}{\sinh(\sqrt{|K|} r)} \right) & \text{if } K < 0, \end{cases}$$

$$\ell_K(r) := \begin{cases} \frac{\sqrt{K}r}{\sin(\sqrt{K}r)}, & \text{if } K > 0, \\ 1, & \text{if } K = 0, \\ \frac{\sqrt{|K|}r}{\sinh(\sqrt{|K|}r)}, & \text{if } K < 0. \end{cases}$$

We denote by ω_n the volume of the unit ball in the Euclidean space $\mathbb{R}^n$, and we fix $N \in M^n(K)$ (for example, the north pole of the sphere when $K > 0$, and define $r_x := d_g(x, N)$).

Our isoperimetric-type inequality can be read as follows:

Theorem 1.5.4 *Let $E \subset M^n(K)$ be set with finite perimeter such that $d(\cdot, N) : M \to \mathbb{R}$ is smooth in a neighborhood of E . Then*

$$\int_{\partial^* E} e^{(n-1)[G_K(0)-G_K(r_x)]} \left|\nabla_x \nabla_y c(x, N) \cdot n_x\right| d\sigma(x) \geq n\, \omega_n^{1/n} \Big(\int_E e^{n[G_K(0)-G_K(r_x)]} \ell_K(r_x)\, \mathrm{dvol}(x) \Big)^{(n-1)/n}.$$

Furthermore equality holds if and only if E is a geodesic ball (centered at N if $K \neq 0$).

The assumption that $d(\cdot, N) : M \to \mathbb{R}$ is smooth in a neighborhood of E is always satisfied if $K \leq 0$, while for $K > 0$ it amounts to say that E is at positive distance from the point antipodal to N.

Moreover, since for $K \leq 0$ one has $\left|\nabla_x \nabla_y c(x, N) \cdot n_x\right| \leq 1$ with equality when E is a geodesic ball centered at N, we get the following

Corollary 1.5.5 *If $K \leq 0$, then*

$$\int_{\partial^* E} e^{(n-1)[G_K(0)-G_K(r_x)]}\, d\sigma(x) \geq n\, \omega_n^{1/n} \Big(\int_E e^{n[G_K(0)-G_K(r_x)]} \ell_K(r_x)\, \mathrm{dvol}(x) \Big)^{(n-1)/n}.$$

for all $E \subset M^n(K)$ with finite perimeter. Furthermore equality holds if and only if E is a geodesic ball (centered at N if $K < 0$).

The above inequalities, read on the tangent space $T_N M$ on sets $\tilde{E} = (\exp_N)^{-1}(E)$ such that $\partial \tilde{E} = \{f(\theta)\theta \mid \theta \in \mathbb{S}^{n-1}\}$ with $f : \mathbb{S}^{n-1} \to (0, +\infty)$ smooth, give:

- if $K > 0$,

$$\int_{\mathbb{S}^{n-1}} e^{-(n-1)G_K(f)} \Big(\frac{\sin(\sqrt{K} f)}{\sqrt{K}} \Big)^{n-1} \sqrt{1 + \ell_K(f)^4 \frac{|\nabla f|^2}{f^2}}\, d\mathscr{H}^{n-1} \geq (n\omega_n)^{1/n} \Big(\int_{\mathbb{S}^{n-1}} e^{-nG_K(f)} \Big(\frac{\sin(\sqrt{K} f)}{\sqrt{K}} \Big)^n d\mathscr{H}^{n-1} \Big)^{(n-1)/n}; \quad (1.5.11)$$

- if $K = 0$,

$$\int_{\mathbb{S}^{n-1}} f^{n-1} \sqrt{1 + \frac{|\nabla f|^2}{f^2}}\, d\mathscr{H}^{n-1} \geq (n\omega_n)^{1/n} \Big(\int_{\mathbb{S}^{n-1}} f^n\, d\mathscr{H}^{n-1} \Big)^{(n-1)/n}; \quad (1.5.12)$$

- if $K < 0$,

$$\int_{\mathbb{S}^{n-1}} e^{-(n-1)G_K(f)}\Big(\frac{\sinh(\sqrt{|K|}f)}{\sqrt{|K|}}\Big)^{n-1}\sqrt{1+\ell_K(f)^4\frac{|\nabla f|^2}{f^2}}\,d\mathscr{H}^{n-1}$$
$$\geq (n\omega_n)^{1/n}\bigg(\int_{\mathbb{S}^{n-1}} e^{-nG_K(f)}\Big(\frac{\sinh(\sqrt{|K|}f)}{\sqrt{|K|}}\Big)^{n}\,d\mathscr{H}^{n-1}\bigg)^{(n-1)/n}. \quad (1.5.13)$$

All these results show how optimal transport reveals to be an extremely powerful instrument for (im)proving functional inequalities. One of my projects is to try to see how to apply these strategy to other cases, for instance to improve log-Sobolev inequalities.

1.6 The optimal partial transport problem

The optimal partial transport problem is a variant of the classical optimal transport problem: given two densities f and g, we want to transport a fraction $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ of the mass of f onto g minimizing the transportation cost $c(x,y) = |x-y|^2$. More precisely, let $f, g \in L^1(\mathbb{R}^n)$ be two nonnegative functions, and denote by $\Gamma_{\leq}(f,g)$ the set of nonnegative Borel measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose first and second marginals are dominated by f and g respectively. Fix a certain amount $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ which represents the mass one wants to transport, and consider the following partial transport problem:

$$\text{minimize } C(\gamma) := \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\,d\gamma(x,y)$$

among all $\gamma \in \Gamma_{\leq}(f,g)$ with $\int d\gamma = m$.

Using weak topologies, it is simple to prove existence of minimizers for any fixed amount of mass $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$. We remark however that in general one cannot expect uniqueness of minimizers: if $m \leq \int_{\mathbb{R}^n} f \wedge g$, any γ supported on the diagonal $\{x = y\}$ with marginals dominated by $f \wedge g$ is a minimizer with zero cost. To ensure uniqueness, in [43] Caffarelli and McCann assume f and g to have disjoint supports. Under this assumption they are able to prove (as in the classical Monge-Kantorovich problem) that there exists a (unique) convex function φ such that the unique minimizer is concentrated on the graph of $\nabla\varphi$. This φ is also shown to solve in a weak sense a Monge-Ampère double obstacle problem. Then, strengthening the disjointness assumption into the hypothesis on the existence of a hyperplane separating the supports of the two measures, they can prove a semiconvexity result on the free boundaries. Furthermore, under some classical regularity assumptions on the measures and on their supports, local $C^{1,\alpha}$ regularity of φ and on the free boundaries of the active regions is shown.

In [15], I studied what happens if one removes the disjointness assumption. Although minimizers are non-unique for $m < \int_{\mathbb{R}^n} f \wedge g$ (but in this case the set of minimizers can be trivially described), uniqueness holds for any $m \geq \int_{\mathbb{R}^n} f \wedge g$. Moreover, exactly as in [43], the unique minimizer is concentrated on the graph of the gradient of a convex function.

Moreover, I showed that the marginals of the minimizers always dominate the common mass $f \wedge g$ (that is all the common mass is both source and target). This property, which has an interest on its own, plays also a crucial role in the regularity of the free boundaries. Indeed, I proved that the free boundary has zero Lebesgue measure under some mild assumptions on the supports of the two densities, and as a consequence of this fact I could apply Caffarelli's regularity theory for the Monge-Ampère equation whenever the support of g is assumed to be convex, and f and g are bounded away from zero and infinity on their respective support. This allows to deduce local $C^{0,\alpha}$ regularity of the transport map, and to prove that it extends to an homeomorphism up to the boundary if both supports are assumed to be strictly convex.

On the other hand, in this situation where the supports of f and g can intersect, something new happens: usually, assuming C^∞ regularity on the density of f and g (together with some convexity assumption on their supports), one can show that the transport map is C^∞ too. In our case, the $C^{0,\alpha}_{\mathrm{loc}}$ regularity is in some sense optimal: I constructed two C^∞ densities on $\mathbb{R}$, supported on two bounded intervals and bounded away from zero on their supports, such that the transport map is not C^1.

Chapter 2

Variational methods for the Euler equations

The velocity field of an incompressible fluid moving inside a smooth domain $D \subset \mathbb{R}^d$ is classically represented by a time-dependent and divergence-free vector field $\boldsymbol{u}(t,x)$ which is parallel to the boundary ∂D. The Euler equations for incompressible fluids describing the evolution of such a velocity field $\boldsymbol{u}$ in terms of the pressure field p are

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p & \text{in } [0,T]\times D, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } [0,T]\times D, \\ \boldsymbol{u}\cdot n = 0 & \text{on } [0,T]\times \partial D. \end{cases} \tag{2.0.1}$$

If we assume that $\boldsymbol{u}$ is smooth, the trajectory of a particle initially at position x is obtained by solving

$$\begin{cases} \dot g(t,x) = \boldsymbol{u}(t, g(t,x)), \\ g(0,x) = x. \end{cases}$$

Since $\boldsymbol{u}$ is divergence free, for each time t the map $g(t,\cdot) : D \to D$ is a measure-preserving diffeomorphism of D (say $g(t,\cdot) \in \mathrm{SDiff}(D)$), which means

$$g(t,\cdot)_\# \mathscr{L}^d_{\lfloor D} = \mathscr{L}^d_{\lfloor D}$$

where $\mathscr{L}^d_{\lfloor D}$ denotes the Lebesgue measure inside D. Writing Euler equations in terms of g, we get

$$\begin{cases} \ddot g(t,x) = -\nabla p\,(t, g(t,x)) & \text{in } [0,T]\times D, \\ g(0,x) = x & \text{in } D, \\ g(t,\cdot) \in \mathrm{SDiff}(D) & \text{for } t\in[0,T]. \end{cases} \tag{2.0.2}$$

2.1 Arnorld's interpretation and Brenier's relaxation

In [30], Arnold interpreted the equation above, and therefore (2.0.1), as a *geodesic* equation on the space $\mathrm{SDiff}(D)$, viewed as an infinite-dimensional manifold with the metric inherited from the embedding in $L^2(D)$ and with tangent space corresponding to the divergence-free vector fields. According to this interpretation, one can look for solutions of (2.0.2) by minimizing

$$\int_0^T \int_D \frac{1}{2}|\dot{g}(t,x)|^2 \, d\mathscr{L}^d_{\lfloor D}(x)\, dt \tag{2.1.1}$$

among all paths $g(t,\cdot) : [0,T] \to \mathrm{SDiff}(D)$ with $g(0,\cdot) = f$ and $g(T,\cdot) = h$ prescribed (typically, by right invariance, f is taken as the identity map $\boldsymbol{i}$). In this way the pressure field arises as a Lagrange multiplier from the incompressibility constraint.

Although in the traditional approach to (2.0.1) the initial velocity is prescribed, while in the minimization of (2.1.1) is not, this variational problem has an independent interest and leads to deep mathematical questions, namely existence of relaxed solutions, gap phenomena, and necessary and sufficient optimality conditions. Such problems have been investigated in a joint work with Luigi Ambrosio [3]. We also remark that *no* existence result of distributional solutions of (2.0.1) is known when $d > 2$ (the case $d = 2$ is different, thanks to the vorticity formulation of (2.0.1)).

On the positive side, Ebin and Marsden proved in [52] that, when D is a smooth compact manifold with no boundary, the minimization of (2.1.1) leads to a unique solution, corresponding also to a solution to Euler equations, if f and h are sufficiently close in a suitable Sobolev norm.

On the negative side, Shnirelman proved in [71, 72] that when $d \geq 3$ the infimum is not attained in general, and that when $d = 2$ there exists $h \in \mathrm{SDiff}(D)$ which cannot be connected to $\boldsymbol{i}$ by a path with finite action. These "negative" results motivate the study of relaxed versions of Arnold's problem.

The first relaxed version of Arnold's minimization problem was introduced by Brenier in [36]: he considered probability measures $\boldsymbol{\eta}$ in $\Omega(D)$, the space of continuous paths $\omega : [0,T] \to D$, and solved the variational problem

$$\text{minimize} \qquad \mathscr{A}_T(\boldsymbol{\eta}) := \int_{\Omega(D)} \int_0^T \frac{1}{2}|\dot{\omega}(\tau)|^2 \, d\tau \, d\boldsymbol{\eta}(\omega), \tag{2.1.2}$$

with the constraints

$$(e_0, e_T)_\# \boldsymbol{\eta} = (\boldsymbol{i}, h)_\# \mathscr{L}^d_{\lfloor D}, \qquad (e_t)_\# \boldsymbol{\eta} = \mathscr{L}^d_{\lfloor D} \qquad \forall\, t \in [0,T] \tag{2.1.3}$$

(where $e_t(\omega) := \omega(t)$ denote the evaluation maps at time t). Brenier called these $\boldsymbol{\eta}$ *generalized incompressible flows* in $[0,T]$ between $\boldsymbol{i}$ and h. The existence of a minimizing $\boldsymbol{\eta}$ is a consequence of the coercivity and lower semicontinuity of the action, provided that there exists at least a generalized flow $\boldsymbol{\eta}$ with finite action (see [36]). This is the case for instance if $D = [0,1]^d$, or if D is the unit ball $B_1(0)$ (as follows from the results in [36, 40] and by [3, Theorem 3.3]).

We observe that any sufficiently regular path $g(t,\cdot) : [0,1] \to \mathrm{SDiff}(D)$ induces a generalized incompressible flow $\boldsymbol{\eta} = (\Phi_g)_\# \mathscr{L}^d_{\lfloor D}$, where $\Phi_g : D \to \Omega(D)$ is given by $\Phi_g(x) = g(\cdot, x)$, but the converse is far from being true: in the case of generalized flows, particles starting from different points are allowed to cross at a later time, and particles starting from the same point are allowed to split, which is of course forbidden by classical flows. Although this crossing/splitting phenomenon could seem strange, it arises naturally if one looks for example at the hydrodynamic limit of the Euler equation. Indeed, the above model allows to describe the limits obtained by solving the Euler equations in $D \times [0, \varepsilon] \subset \mathbb{R}^{d+1}$ and, after a suitable change of variable, letting $\varepsilon \to 0$ (see for instance [41]).

In [36], a consistency result was proved: smooth solutions to (2.0.1) are optimal even in the larger class of the generalized incompressible flows, provided the pressure field p satisfies

$$T^2 \sup_{t\in[0,T]} \sup_{x\in D} \nabla_x^2 p(t, x) \le \pi^2 I_d \tag{2.1.4}$$

(here I_d denotes the identity matrix in $\mathbb{R}^d$), and are the unique ones if the above inequality is strict.

2.2 A study of generalized solutions in 2 dimensions

In [7], in collaboration with Marc Bernot and Filippo Santambrogio, we considered Problem (2.1.2)-(2.1.3) in the particular cases where $D = B_1(0)$ or D is an annulus, in dimension 2.

If $D = B_1(0) \subset \mathbb{R}^2$ is the unit ball, the following situation arises: an explicit solution of Euler equations is given by the transformation $g(t, x) = \boldsymbol{R}_t x$, where $\boldsymbol{R}_t : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counterclockwise rotation of an angle t. Indeed the maps $g(t, \cdot) : D \to D$ are clearly measure preserving, and moreover we have

$$\ddot{g}(t, x) = -g(t, x),$$

so that $\boldsymbol{v}(t, x) = \dot{g}(t, y)|_{y=g^{-1}(t,x)}$ is a solution to the Euler equations with the pressure field given by $p(x) = |x|^2/2$ (so that $\nabla p(x) = x$). Thus, thanks to (2.1.4) and by what we said above, the generalized incompressible flow induced by g is optimal if $T \le \pi$, and is the unique one if $T < \pi$. This implies in particular that there exists a unique minimizing geodesic from $\boldsymbol{i}$ to the rotation $\boldsymbol{R}_T$ if $0 < T < \pi$. On the contrary, for $T = \pi$ more than one optimal solution exists, as both the clockwise and the counterclockwise rotation of an angle π are optimal (this shows for instance that the upper bound (2.1.4) is sharp). Moreover, Brenier found in [36, Section 6] an example of action-minimizing path $\boldsymbol{\eta}$ connecting $\boldsymbol{i}$ to $-\boldsymbol{i}$ in time π which is not induced by a classical solution of the Euler equations (and it cannot be simply constructed using the two opposite rotations):

$$\int_{\Omega(D)} \varphi(\omega)\, d\boldsymbol{\eta}(\omega) := \int_{D\times\mathbb{R}^d} \varphi\big(t \mapsto x\cos(t) + v\sin(t)\big)\, d\mu(x, v) \qquad \forall \varphi \in C(\Omega),$$

with μ given by

$$\mu(dx, dv) = \frac{1}{2\pi\sqrt{1 - |x|^2}} \left[\mathscr{H}^1_{\lfloor\{|v|=\sqrt{1-|x|^2}\}}(dv) \right] \otimes \mathscr{L}^2_{\lfloor D}(dx).$$

What is interestingly shown by the solution constructed by Brenier is the following: when $\boldsymbol{\eta}$ is of the form $\boldsymbol{\eta} = (\Phi_g)_\# \mathscr{L}^d_{\lfloor D}$ for a certain map g, one can always recover $g(t, \cdot)$ from $\boldsymbol{\eta}$ using the identity

$$(e_0, e_t)_\# \boldsymbol{\eta} = (\boldsymbol{i}, g(t, \cdot))_\# \mathscr{L}^d_{\lfloor D}, \qquad \forall\, t \in [0, T].$$

In the example found by Brenier no such representation is possible (i.e. $(e_0, e_t)_\# \boldsymbol{\eta}$ is not a graph), which implies that the splitting of fluid paths starting at the same point is actually possible for optimal flows (in this case, we will say that these flows are *non-deterministic*). We moreover observe that this solution is in some sense the most isotropic: each particle starting at a point x splits uniformly in all directions and reaches the point $-x$ in time π. Due to this isotropy, it was conjectured that this solution was an extremal point in the set of minimizing geodesic [42]. However in [7] we showed that this is not the case: the decomposition of μ as the sum of its clockwise and an anticlockwise components gives rise to two new geodesics which, in addition to being non-deterministic, they induce two non-trivial stationary solutions to Euler equations with a new "macroscopic" pressure field (see the discussion below). More in general, in [7] we were able to construct and classify a large class of generalized solutions. Moreover all the constructed solutions have the interesting feature of inducing stationary and non-stationary solutions to Euler equations.

To explain this fact, we recall that, as shown by Brenier [38], there exists a "unique" gradient of the pressure field p which satisfies the distributional relation

$$\nabla p(t,x) = -\partial_t \overline{\boldsymbol{v}}_t(x) - \operatorname{div}\left(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t(x)\right). \tag{2.2.1}$$

Here $\overline{\boldsymbol{v}}_t(x)$ is the "effective velocity", defined by $(e_t)_\#(\dot\omega(t)\boldsymbol{\eta}) = \overline{\boldsymbol{v}}_t \mathscr{L}^d_{\lfloor D}$, and $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t$ is the quadratic effective velocity, defined by $(e_t)_\#(\dot\omega(t)\otimes\dot\omega(t)\boldsymbol{\eta}) = \overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t \mathscr{L}^d_{\lfloor D}$ (to define $\overline{\boldsymbol{v}}$ and $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}$, one can use *any* minimizer $\boldsymbol{\eta}$). The proof of this fact is based on the so-called dual least action principle: if $\boldsymbol{\eta}$ is optimal, we have

$$\mathscr{A}_T(\boldsymbol{\nu}) \geq \mathscr{A}_T(\boldsymbol{\eta}) + \langle p, \rho^{\boldsymbol{\nu}} - 1\rangle \tag{2.2.2}$$

for any measure $\boldsymbol{\nu}$ in $\Omega(D)$ such that $(e_0, e_T)_\#\boldsymbol{\nu} = (\boldsymbol{i}, h)_\# \mathscr{L}^d_{\lfloor D}$ and $\|\rho^{\boldsymbol{\nu}} - 1\|_{C^1} \leq 1/2$. Here $\rho^{\boldsymbol{\nu}}$ is the (absolutely continuous) density produced by the flow $\boldsymbol{\nu}$, defined by $\rho^{\boldsymbol{\nu}}(t,\cdot)\mathscr{L}^d_{\lfloor D} = (e_t)_\#\boldsymbol{\nu}$. In this way, the incompressibility constraint can be slightly relaxed and one can work with the augmented functional (still minimized by $\boldsymbol{\eta}$)

$$\boldsymbol{\nu} \mapsto \mathscr{A}_T(\boldsymbol{\nu}) - \langle p, \rho^{\boldsymbol{\nu}} - 1\rangle,$$

whose first variation leads to (2.2.1).

The fact that in general $\overline{\boldsymbol{v}\otimes\boldsymbol{v}} \neq \overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}$ shows that generalized solutions *do not* necessarily induce classical solutions to the Euler equations. On the other hand, if the difference $\overline{\boldsymbol{v}\otimes\boldsymbol{v}} - \overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}$ is a gradient, one indeed gets a solution to the Euler equations with a different pressure field (what we called above "macroscopic" pressure field).

2.3 A second relaxed model and the optimality conditions

A few years later, Brenier introduced in [40] a new relaxed version of Arnold's problem of a mixed Eulerian-Lagrangian nature: the idea is to add to the Eulerian variable x a Lagrangian one a representing, at least when $f = \boldsymbol{i}$, the initial position of the particle; then, one minimizes a functional of the Eulerian variables (density and velocity), depending also on a. Brenier's motivation for looking at the new model was that this formalism allows to show much stronger regularity results for the pressure field, namely $\partial_{x_i} p$ are locally finite measures in $(0,T) \times D$. Let us assume $D = \mathbb{T}^d$, the d-dimensional torus. A first result achieved in [3] in collaboration with Luigi Ambrosio was to show that this model is basically equivalent to the one described before. This allows to show that the pressure fields of the two models (both arising via the dual least action pronciple) are the same. Moreover, as I showed with Ambrosio in [4], the pressure field of the second model is not only a distibution, but is indeed a function belonging to the space $L^2_{loc}\big((0,T), BV(\mathbb{T}^d)\big)$. We can therefore transfer the regularity informations on the pressure field up to the Lagrangian model, thus obtaining the validity of (2.2.2) for a much larger class of generalized flows $\boldsymbol{\nu}$. This is crucial for the study of the necessary and sufficient optimality conditions for the geodesic problem (which strongly require that the pressure field p is a function and not only a distribution).

To describe the conditions we found in [3], we first observe that by the Sobolev embeddings $p \in L^2_{\mathrm{loc}}\big((0,T); L^{d/(d-1)}(\mathbb{T}^d)\big)$. Hence, taking into account that the pressure field in (2.2.2) is uniquely determined up to additive time-dependent constants, we may assume that $\int_{\mathbb{T}^d} p(t,\cdot)\, d\mathscr{L}^d = 0$ for almost all $t \in (0,T)$.
The first elementary remark is that any integrable function q in $(0,T)\times\mathbb{T}^d$ with $\int_{\mathbb{T}^d} q(t,\cdot)\, d\mathscr{L}^d = 0$ for almost all $t \in (0,T)$ provides us with a null-lagrangian for the geodesic problem, as the incompressibility constraint gives

$$\int_{\Omega(\mathbb{T}^d)} \int_0^T q(t,\omega(t))\, dt\, d\boldsymbol{\nu}(\omega) = \int_0^T \int_{\mathbb{T}^d} q(t,x)\, d\mathscr{L}^d(x)\, dt = 0$$

for any generalized incompressible flow $\boldsymbol{\nu}$. Taking also the constraint $(e_0, e_T)_\# \boldsymbol{\nu} = (\boldsymbol{i}, h)_\# \mu$ into account, we get

$$\mathscr{A}_T(\boldsymbol{\nu}) = T \int_{\Omega(\mathbb{T}^d)} \left(\int_0^T \frac{1}{2} |\dot\omega(t)|^2 - q(t,\omega)\, dt \right) d\boldsymbol{\nu}(\omega) \geq \int_{\mathbb{T}^d} c_q^T(x, h(x))\, d\mathscr{L}^d(x),$$

where $c_q^T(x,y)$ is the minimal cost associated with the Lagrangian $T\int_0^T \frac{1}{2}|\dot\omega(t)|^2 - q(t,\omega)\, dt$. Since this lower bound depends only on h, we obtain that any $\boldsymbol{\eta}$ satisfying (2.1.3) and concentrated on c_q-minimal paths, for some $q \in L^1$, is optimal, and $\overline{\delta}^2(\boldsymbol{i}, h) = \int c_q^T(\boldsymbol{i}, h)\, d\mathscr{L}^d$. This is basically the argument used by Brenier in [36] to show the minimality of smooth solutions to (2.0.1), under assumption (2.1.4): indeed, this condition guarantees that solutions of $\ddot\omega(t) = -\nabla p(t,\omega)$ (i.e. stationary paths for the Lagrangian, with $q = p$) are also minimal.

We are able to show that basically this condition is *necessary and sufficient* for optimality if the pressure field is globally integrable. However, since no global in time regularity result for the pressure field is presently known, we have also been looking for necessary and sufficient optimality conditions that don't require the global integrability of the pressure field. Using the regularity $p \in L^1_{\mathrm{loc}}((0,T); L^r(D))$ for some $r > 1$, we show that any optimal $\boldsymbol{\eta}$ is concentrated on *locally minimizing* paths for the Lagrangian

$$\mathcal{L}_p(\omega) := \int \frac{1}{2}|\dot\omega(t)|^2 - p(t,\omega)\, dt. \tag{2.3.1}$$

Since we need to integrate p along curves, this statement is not invariant under modifications of p in negligible sets, and the choice of a *specific* representative $\bar{p}(t,x) := \liminf_{\varepsilon\downarrow 0} p(t,\cdot) * \phi_\varepsilon(x)$ in the Lebesgue equivalence class is needed. Moreover, the necessity of pointwise uniform estimates on p_ε requires the integrability of $Mp(t,x)$, the maximal function of $p(t,\cdot)$ at x.

In addition, we identify a second necessary (and more hidden) optimality condition. In order to state it, let us consider an interval $[s,t] \subset (0,T)$ and the cost function

$$c_p^{s,t}(x,y) := \inf\left\{\int_s^t \frac{1}{2}|\dot{\omega}(\tau)|^2 - p(\tau,\omega)\,d\tau :\ \omega(s)=x,\ \omega(t)=y,\ Mp(\tau,\omega)\in L^1(s,t)\right\} \quad (2.3.2)$$

(the assumption $Mp(\tau,\omega) \in L^1(s,t)$ is forced by technical reasons). Recall that, according to the theory of optimal transportation, a probability measure λ in $\mathbb{T}^d \times \mathbb{T}^d$ is said to be c-optimal if

$$\int_{\mathbb{T}^d\times\mathbb{T}^d} c(x,y)\,d\lambda' \geq \int_{\mathbb{T}^d\times\mathbb{T}^d} c(x,y)\,d\lambda$$

for any probability measure λ' having the same marginals μ_1, μ_2 of λ. We shall also denote $W_c(\mu_1,\mu_2)$ the minimal value, i.e. $\int_{\mathbb{T}^d\times\mathbb{T}^d} c\,d\lambda$, with λ c-optimal. Now, let $\boldsymbol{\eta}$ be an optimal generalized incompressible flow between $\boldsymbol{i}$ and h; according to the disintegration theorem, we can represent $\boldsymbol{\eta} = \int \boldsymbol{\eta}_a\,d\mathscr{L}^d_{\lfloor D}(a)$, with $\boldsymbol{\eta}_a$ concentrated on curves starting at a (and ending, since our final conditions is deterministic, at $h(a)$), and consider the plans $\lambda_a^{s,t} = (e_s,e_t)_\#\boldsymbol{\eta}_a$. We show that

$$\text{for all } [s,t]\subset(0,T), \quad \lambda_a^{s,t} \text{ is } c_p^{s,t}\text{-optimal for } \mathscr{L}^d\text{-a.e. } a\in\mathbb{T}^d. \quad (2.3.3)$$

Roughly speaking, this condition tells us that one has not only to move mass from x to y achieving $c_p^{s,t}$, but also to optimize the distribution of mass between time s and time t. In the "deterministic" case when either $(e_0,e_s)_\#\boldsymbol{\eta}$ or $(e_0,e_t)_\#\boldsymbol{\eta}$ are induced by a transport map g, the plan $\lambda_a^{s,t}$ has $\delta_{g(a)}$ either as first or as second marginal, and therefore it is uniquely determined by its marginals (it is indeed the product of them). This is the reason why condition (2.3.3) does not show up in the deterministic case.

Finally, we show that the two conditions are also sufficient, even on general manifolds D: if, for some $r>1$ and $q \in L^1_{\rm loc}\left((0,T);L^r(D)\right)$, a generalized incompressible flow $\boldsymbol{\eta}$ concentrated on locally minimizing curves for the Lagrangian $\mathcal{L}_q$ satisfies

$$\text{for all } [s,t]\subset(0,T), \quad \lambda_a^{s,t} \text{ is } c_q^{s,t}\text{-optimal for } \mathscr{L}^d_{\lfloor D}\text{-a.e. } a\in D,$$

then $\boldsymbol{\eta}$ is optimal in $[0,T]$, and q is the pressure field.

These results show a somehow unexpected connection between the variational theory of incompressible flows and the theory developed by Bernard-Buffoni [32] of measures in the space of action-minimizing curves; in this framework one can fit Mather's theory as well as optimal transportation problems on manifolds, with a geometric cost. In our case the only difference is that the Lagrangian is possibly nonsmoooth (but hopefully not so bad), and not given *a priori*, but generated by the problem itself. Our approach also yields a new variational characterization of the pressure field, as a maximizer of the family of functionals (for $[s,t]\subset(0,T)$)

$$q \mapsto \int_{\mathbb{T}^d} W_{c_q^{s,t}}(\eta_a^s,\gamma_a^t)\,d\mathscr{L}^d(a), \qquad Mq\in L^1\left([s,t]\times\mathbb{T}^d\right),$$

where η_a^s, γ_a^t are the marginals of $\lambda_a^{s,t}$.

Chapter 3

Mather quotient and Sard Theorem

Let (M, g) be a smooth complete Riemannian manifold without boundary, and denote by $d(x, y)$ the Riemannian distance from x to y. For $v \in T_xM$ the norm $\|v\|_x$ is given by $g_x(v, v)^{1/2}$, and we also denote by $\|\cdot\|_x$ the dual norm on T^*M.

We assume that $H : T^*M \to \mathbb{R}$ is a Hamiltonian of class $C^{k,\alpha}$, with $k \geq 2, \alpha \in [0, 1]$, which satisfies the three following conditions:

(H1) **C^2-strict convexity:** $\forall (x, p) \in T^*M$, the second derivative along the fibers $\frac{\partial^2 H}{\partial p^2}(x, p)$ is strictly positive definite;

(H2) **uniform superlinearity:** for every $K \geq 0$ there exists a finite constant $C(K)$ such that

$$H(x, p) \geq K\|p\|_x + C(K), \qquad \forall\, (x, p) \in T^*M;$$

(H3) **uniform boundedness in the fibers:** for every $R \geq 0$, we have

$$\sup_{x \in M} \{H(x, p) \mid \|p\|_x \leq R\} < +\infty.$$

By the Weak KAM Theorem it is known that, under the above conditions, there is $c(H) \in \mathbb{R}$ such that the Hamilton-Jacobi equation

$$H(x, d_x u) = c$$

admits a global viscosity solution $u : M \to \mathbb{R}$ for $c = c(H)$ and does not admit such solution for $c < c(H)$. In fact, for $c < c(H)$, the Hamilton-Jacobi equation does not admit any viscosity subsolution. Moreover, if M is assumed to be compact, then $c(H)$ is the only value of c for which the Hamilton-Jacobi equation above admits a viscosity solution. The constant $c(H)$ is called the *critical value*, or the *Mañé critical value* of H. In the sequel, a viscosity solution $u : M \to \mathbb{R}$ of $H(x, d_x u) = c(H)$ will be called a *critical viscosity solution* or a *weak KAM solution*, while a viscosity subsolution u of $H(x, d_x u) = c(H)$ will be called a *critical viscosity subsolution* (or *critical subsolution* if u is at least C^1).

The Lagrangian $L : TM \to \mathbb{R}$ associated to the Hamiltonian H is defined by

$$\forall (x,v) \in TM, \quad L(x,v) = \max_{p \in T_x^* M} \{p(v) - H(x,p)\}.$$

Since H is of class C^k, with $k \geq 2$, and satisfies the three conditions (H1)-(H3), it is well-known that L is finite everywhere of class C^k, and is a Tonelli Lagrangian, i.e. satisfies the analogous of conditions (H1)-(H3). Moreover, the Hamiltonian H can be recovered from L by

$$\forall (x,p) \in T_x^* M, \quad H(x,p) = \max_{v \in T_x M} \{p(v) - L(x,v)\}.$$

Therefore the following inequality is always satisfied

$$p(v) \leq L(x,v) + H(x,p).$$

This inequality is called the Fenchel inequality. Moreover, due to the strict convexity of L, we have equality in the Fenchel inequality if and only if

$$(x,p) = \mathcal{L}(x,v),$$

where $\mathcal{L} : TM \to T^*M$ denotes the Legendre transform defined as

$$\mathcal{L}(x,v) = \left(x, \frac{\partial L}{\partial v}(x,v) \right).$$

Under our assumption $\mathcal{L}$ is a diffeomorphism of class at least C^1. We will denote by ϕ_t^L the Euler-Lagrange flow of L, and by X_L the vector field on TM that generates the flow ϕ_t^L. If we denote by ϕ_t^H the Hamiltonian flow of H on T^*M, then as is well-known this flow ϕ_t^H is conjugate to ϕ_t^L by the Legendre transform $\mathcal{L}$. Moreover, thanks to assumptions (H1)-(H3), the flow ϕ_t^H (and so also ϕ_t^L) is complete.

As done by Mather in [60], it is convenient to introduce for $t > 0$ fixed, the function $h_t : M \times M \to \mathbb{R}$ defined by

$$h_t(x,y) = \inf \int_0^t L(\gamma(s), \dot\gamma(s))\, ds, \qquad \forall\, x,y \in M$$

where the infimum is taken over all the absolutely continuous paths $\gamma : [0,t] \to M$ with $\gamma(0) = x$ and $\gamma(t) = y$. The *Peierls barrier* is the function $h : M \times M \to \mathbb{R}$ defined by

$$h(x,y) = \liminf_{t \to \infty} \{h_t(x,y) + c(H)t\}.$$

It is clear that this function satisfies for all $t > 0$

$$h(x,z) \leq h(x,y) + h_t(y,z) + c(H)t$$

$$h(x,z) \leq h_t(x,y) + c(H)t + h(y,z)$$

for any $x, y, z \in M$, and therefore it also satisfies the triangle inequality

$$h(x, z) \leq h(x, y) + h(y, z).$$

Moreover, given a weak KAM solution u, we have

$$u(y) - u(x) \leq h(x, y), \qquad \forall\, x, y \in M.$$

In particular, we have $h > -\infty$ everywhere. It follows, from the triangle inequality, that the function h is either identically $+\infty$ or it is finite everywhere. If M is compact, h is finite everywhere. In addition, if h is finite, then for each $x \in M$ the function $h_x(\cdot) = h(x, \cdot)$ is a critical viscosity solution. The *projected Aubry set* $\mathcal{A}$ is defined by

$$\mathcal{A} = \{x \in M \mid \; h(x, x) = 0\}.$$

As done by Mather (see [60, page 1370]), one can symmetrize h to define the function δ_M : $M \times M \to \mathbb{R}$ by

$$\forall x, y \in M, \quad \delta_M(x, y) = h(x, y) + h(y, x).$$

Since h satisfies the triangle inequality and $h(x, x) \geq 0$ everywhere, the function δ_M is symmetric, everywhere nonnegative and satisfies the triangle inequality. The restriction δ_M : $\mathcal{A} \times \mathcal{A} \to \mathbb{R}$ is a genuine semi-distance on the projected Aubry set. We call this function δ_M the *Mather semi-distance* (even when we consider it on M rather than on $\mathcal{A}$). We define the *Mather quotient* $(\mathcal{A}_M, \delta_M)$ to be the metric space obtained by identifying two points $x, y \in \mathcal{A}$ if their semi-distance $\delta_M(x, y)$ vanishes (we mention that this is set is also called *quotient Aubry set*). When we consider δ_M on the quotient space $\mathcal{A}_M$ we will call it the *Mather distance*.

3.1 The dimension of the Mather quotient

In [62], Mather formulated the following problem:

Mather's Problem. If L is C^∞, is the set $\mathcal{A}_M$ totally disconnected for the topology of δ_M, i.e. is each connected component of $\mathcal{A}_M$ reduced to a single point?

In [61], Mather brought a positive answer to that problem in low dimension. More precisely, he proved that if M has dimension two, or if the Lagrangian is the kinetic energy associated to a Riemannian metric on M in dimension ≤ 3, then the Mather quotient is totally disconnected. Mather mentioned in [62, page 1668] that it would be even more interesting to be able to prove that the Mather quotient has vanishing one-dimensional Hausdorff measure, because this implies the upper semi-continuity of the mapping $H \mapsto \mathcal{A}$.

In [9], in a joint work with Albert Fathi and Ludovic, we were able to show that the vanishing of the one-dimensional Hausdorff measure of the Mather quotient is satisfied under various assumptions. Let us state our results.

Theorem 3.1.1 *If* $\dim M = 1, 2$ *and* H *of class* C^2 *or* $\dim M = 3$ *and* H *of class* $C^{k,1}$ *with* $k \geq 3$*, then the Mather quotient* $(\mathcal{A}_M, \delta_M)$ *has vanishing one-dimensional Hausdorff measure.*

Above the projected Aubry $\mathcal{A}$, there is a compact subset $\tilde{\mathcal{A}} \subset TM$ called the Aubry set. The projection $\pi : TM \to M$ induces a homeomorphism $\pi_{|\tilde{\mathcal{A}}}$ from $\tilde{\mathcal{A}}$ onto $\mathcal{A}$ (whose inverse is Lipschitz by a theorem due to Mather). The Aubry set can be defined as the set of $(x, v) \in TM$ such that $x \in \mathcal{A}$ and v is the unique element in T_xM such that $d_x u = \frac{\partial L}{\partial v}(x, v)$ for any critical viscosity subsolution u. The Aubry set is invariant under the Euler-Lagrange flow $\phi_t^L : TM \to TM$. Therefore, for each $x \in \mathcal{A}$, there is only one orbit of ϕ_t^L in $\tilde{\mathcal{A}}$ whose projection passes through x. We define the *stationary Aubry set* $\tilde{\mathcal{A}}^0 \subset \tilde{\mathcal{A}}$ as the set of points in $\tilde{\mathcal{A}}$ which are fixed points of the Euler-Lagrange flow $\phi_t(x, v)$, i.e.

$$\tilde{\mathcal{A}}^0 = \{(x, v) \in \tilde{\mathcal{A}} \mid \forall\, t \in \mathbb{R},\, \phi_t^L(x, v) = (x, v)\}.$$

In fact it can be shown, that $\tilde{\mathcal{A}}^0$ is the intersection of $\tilde{\mathcal{A}}$ with the zero section of TM, i.e. $\tilde{\mathcal{A}}^0 = \{(x, 0) \mid (x, 0) \in \tilde{\mathcal{A}}\}$.

We define the *projected stationary Aubry set* $\mathcal{A}^0$ as the projection on M of $\tilde{\mathcal{A}}^0$, that is $\mathcal{A}^0 = \{x \mid (x, 0) \in \tilde{\mathcal{A}}\}$. At the very end of his paper [61], Mather noticed that the argument he used in the case where L is a kinetic energy in dimension 3 proves the total disconnectedness of the Mather quotient in dimension 3 as long as $\mathcal{A}_M^0$ is empty. In fact, if we consider the restriction of δ_M to $\mathcal{A}^0$, we have the following result on the quotient metric space $(\mathcal{A}_M^0, \delta_M)$.

Theorem 3.1.2 *Suppose that L is at least C^2, and that the restriction $x \mapsto L(x, 0)$ of L to the zero section of TM is of class $C^{k,1}$. Then $(\mathcal{A}_M^0, \delta_M)$ has vanishing Hausdorff measure in dimension $2 \dim M/(k + 3)$. In particular, if $k \geq 2 \dim M - 3$ then $\mathscr{H}^1(\mathcal{A}_M^0, \delta_M) = 0$, and if $x \mapsto L(x, 0)$ is C^∞ then $(\mathcal{A}_M^0, \delta_M)$ has zero Hausdorff dimension.*

As a corollary, we have the following result which was more or less already mentioned by Mather in [62, §19 page 1722], and proved by Sorrentino [70].

Corollary 3.1.3 *Assume that H is of class C^2 and that its associated Lagrangian L satisfies the following conditions:*

1. *$\forall x \in M, \quad \min_{v \in T_xM} L(x, v) = L(x, 0)$;*
2. *the mapping $x \in M \mapsto L(x, 0)$ is of class $\mathrm{C}^{l,1}(M)$ with $l \geq 1$.*

If $\dim M = 1, 2$, or $\dim M \geq 3$ and $l \geq 2 \dim M - 3$, then $(\mathcal{A}_M, \delta_M)$ is totally disconnected. In particular, if $L(x, v) = \frac{1}{2}\|v\|_x^2 - V(x)$, with $V \in \mathrm{C}^{l,1}(M)$ and $l \geq 2 \dim M - 3$ ($V \in \mathrm{C}^2(M)$ if $\dim M = 1, 2$), then $(\mathcal{A}_M, \delta_M)$ is totally disconnected.

Since $\mathcal{A}^0$ is the projection of the subset $\tilde{\mathcal{A}}^0 \subset \tilde{\mathcal{A}}$ consisting of points in $\tilde{\mathcal{A}}$ which are fixed under the the Euler-Lagrange flow ϕ_t^L, it is natural to consider $\mathcal{A}^p$ the set of $x \in \mathcal{A}$ which are projection of a point $(x, v) \in \tilde{\mathcal{A}}$ whose orbit under the the Euler-Lagrange flow ϕ_t^L is periodic with strictly positive period. We call this set the *projected periodic Aubry set.* We have the following result:

Theorem 3.1.4 *If $\dim M \geq 2$ and H of class $C^{k,1}$ with $k \geq 2$, then $(\mathcal{A}_M^p, \delta_M)$ has vanishing Hausdorff measure in dimension $8 \dim M/(k + 8)$. In particular, if $k \geq 8 \dim M - 8$ then $\mathscr{H}^1(\mathcal{A}_M^p, \delta_M) = 0$, and if H is C^∞ then $(\mathcal{A}_M^p, \delta_M)$ has zero Hausdorff dimension.*

In the case of compact surfaces, using the finiteness of exceptional minimal sets of flows, we have:

Theorem 3.1.5 *If M is a compact surface of class C^∞ and H is of class C^∞, then $(\mathcal{A}_M, \delta_M)$ has zero Hausdorff dimension.*

Finally, always in [9], we give some applications of our result in dynamic, whose Theorem 3.1.6 below is a corollary. If X is a C^k vector field on M, with $k \geq 2$, the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated to X is defined by

$$L_X(x, v) = \frac{1}{2}\|v - X(x)\|_x^2, \quad \forall (x, v) \in TM.$$

We will denote by $\mathcal{A}_X$ the projected Aubry set of the Lagrangian L_X. The following question was raised by Albert Fathi (see `http://www.aimath.org/WWN/dynpde/articles/html/20a/`):

Problem. Let $L_X : TM \to \mathbb{R}$ be the Mañé Lagrangian associated to the C^k vector field X $(k \geq 2)$ on the compact connected manifold M.

(1) Is the set of chain-recurrent points of the flow of X on M equal to the projected Aubry set $\mathcal{A}_X$?

(2) Give a condition on the dynamics of X that insures that the only weak KAM solutions are the constants.

The above theorems, together with the applications in dynamics we developed in [9, Section 6], give an answer to this question when $\dim M \leq 3$.

Theorem 3.1.6 *Let X be a C^k vector field, with $k \geq 2$, on the compact connected C^∞ manifold M. Assume that one of the conditions hold:*

(1) The dimension of M is 1 or 2.

(2) The dimension of M is 3, and the vector field X never vanishes.

(3) The dimension of M is 3, and X is of class $\mathrm{C}^{3,1}$.

Then the projected Aubry set $\mathcal{A}_X$ of the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated to X is the set of chain-recurrent points of the flow of X on M. Moreover, the constants are the only weak KAM solutions for L_X if and only if every point of M is chain-recurrent under the flow of X.

3.2 The connection with Sard Theorem

To explain in a simpler way the connection between the above problem and Sard Theorem, we consider here the problem of proving that the Mather quotient is totally disconnected (we remark that having vanishing 1-dimensional Hausdorff dimension implies the total disconnectedness).

Let us call by $\mathcal{SS}^1$ the set of C^1 critical viscosity subsolutions. The following representation formula holds: for every $x, y \in \mathcal{A}$,

$$\delta_M(x,y) = \max_{u_1,u_2 \in \mathcal{SS}^1} \{(u_1 - u_2)(y) - (u_1 - u_2)(x)\}.$$

We remark that, since on the projected Aubry set the gradients of all critical viscosity subsolutions coincide, we have $d_x(u_1 - u_2) = 0$ on $\mathcal{A}$, that is $\mathcal{A}$ is contained in the set of critical points of $u_1 - u_2$.

Assume now that we can prove the the difference of two critical viscosity subsolution satisfies Sard Theorem, i.e. the set of critical values has zero Lebesgue measure. Consider two points $x, y \in \mathcal{A}$ such that $\delta_M(x,y) > 0$. By the above formula there exists two critical subsolutions u_1 and u_2 such that $0 < \delta_M(x,y) = v(x) - v(y)$, with $v := u_1 - u_2$. Since v satisfies Sard Theorem and $\mathcal{A}$ is contained in the set of critical points of v, we get $\mathscr{L}^1(v(\mathcal{A})) = 0$. Therefore there exists a value $t_0 \in \mathbb{R}$ such that $v(y) < t_0 < v(x)$, which implies that x and y are in two different connected components.

Thus we see that Mather's problem can be reduced to prove a Sard Theorem on viscosity subsolutions. Since critical subsolutions are in general not more regular than $C^{1,1}$, one cannot hope to apply just the classical Sard Theorem, but one has to use that u_1 and u_2 satisfy the Hamilton Jacobi equation, and take advantage of the regularity of the Hamiltonian. This is exactly what we did in [9].

3.3 A Sard Theorem in Sobolev spaces

During the study of Mather's problem, since there was a deep connection with Sard Theorem, I started to get interested in the proof of Sard Theorem and its generalization. Let me recall the classical result:

Theorem 3.3.1 (Sard) *Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^m$ be a C^{n-m+1} function, with $n \geq m$ (C^1 if $m > n$). Then the set of critical values of f has $\mathscr{L}^m$-measure zero.*

After that theorem, many generalizations have been proved and, at the same time, many counterexamples have been found in the case of not sufficient regularity. In particular, in [31] the same conclusion of the Morse-Sard Theorem has been proved under the only assumption of a $C^{n-m,1}$ regularity, while in [49] only a $W^{n-m+1,p}$ regularity, with $p > n$, is assumed. In [12] I gave a simple proof of the result in [49]. Moreover, as the proof is independent of Theorem 3.3.1, my result implies the classical Morse-Sard Theorem:

Theorem 3.3.2 *Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^m$ be a $W^{n-m+1,p}_{loc}$ function, with $p > n \geq m$. Then the set of critical values of f has $\mathscr{L}^m$-measure zero.*

We remark that $W^{n-m+1,p} \hookrightarrow C^{n-m,\alpha}$, with $\alpha = 1 - \frac{n}{p}$. However with the only assumption of $C^{n-m,\alpha}$ regularity with $\alpha < 1$ the result is false, and the key point is in fact the existence of another weak derivative summable enough.

Chapter 4

DiPerna-Lions theory for non-smooth ODEs

Recent research activity has been devoted to study transport equations with rough coefficients, showing that a well-posedness result for the transport equation in a certain subclass of functions allows to prove existence and uniqueness of a flow for the associated ODE. The first result in this direction is due to DiPerna and P.-L.Lions [51], where the authors study the connection between the transport equation and the associated ODE $\dot{\gamma} = b(t, \gamma)$.

Their result can be informally stated as follows: existence and uniqueness for the transport equation is equivalent to a sort of well-posedness of the ODE which says, roughly speaking, that the ODE has a unique solution for $\mathscr{L}^d$-almost every initial condition. In that paper they also show that the transport equation $\partial_t u + \sum_i b_i \partial_i u = c$ is well-posed in L^∞ if $b = (b_1, \ldots, b_n)$ is Sobolev and satisfies suitable global conditions (including L^∞-bounds on the spatial divergence), which yields the well-posedness of the ODE.

In [26], using a slightly different philosophy, Ambrosio studied the connection between the continuity equations $\partial_t u + \operatorname{div}(bu) = c$ and the ODE $\dot{\gamma} = b(t, \gamma)$. This different approach allows him to develop the general theory of the so-called Regular Lagrangian Flows (see [27, Remark 31] for a detailed comparison with the DiPerna-Lions axiomatization), which relates existence and uniqueness for the continuity equation with well-posedness of the ODE, without assuming any regularity on the vector field b. Indeed, since the transport equation is in a conservative form, it has a meaning in the sense of distributions even when b is only L^∞_{loc} and u is L^1_{loc}. Thus, as in the case of DiPerna-Lions, one shows that the continuity equation is equivalent to a sort of well-posedness of the ODE. After having proved this, in [26] the well-posedness of the continuity equations in L^∞ is proved in the case of vector fields with BV regularity whose distributional divergence belongs to L^∞.

4.1 A review of DiPerna-Lions and Ambrosio's theory

We now give a review of the theory. Since for the extentions to the stochastic case Ambrosio's framework seems to be more suitable, we will focus on the link between continuity equations and ODEs.

We recall that the continuity equation is an equation of the form

$$\partial_t \mu_t + \mathrm{div}(b\mu_t) = 0,$$

and the associated ODE is

$$\begin{cases} \dot{X}(t,x) = b(t, X(t,x)), \\ X(0,x) = 0. \end{cases}$$

Indeed, the classical theory for continuity equation with Lipschitz vector fields states that, if $b(t)$ is Lipschitz, then there exists a unique (measure-valued) solution of the PDE given by

$$\mu_t := X(t)_\# \mu_0,$$

where $X(t)$ denotes the (unique) flow of the ODE.

Thus, in the classical theory, solutions of the continuity equations move along characteristics of the flow generated by b, and so the ODE gives information on the PDE. On the other hand, if $\gamma(t)$ satisfies $\dot{\gamma}(t) = b(t, \gamma(t))$, then

$$\mu_t := \delta_{\gamma(t)}$$

solves the PDE with $\mu_0 = \delta_{\gamma(0)}$. From this remark one can easily deduce that uniqueness of non-negative measure-valued solutions of the PDE implies uniqueness for the ODE.

On the other hand, the converse of this fact is also true. To show this, we need a representation formula for solution of the PDE. Let us denote by Γ_T the space $C([0,T], \mathbb{R}^d)$ of continuous paths in $\mathbb{R}^d$, and by $\mathcal{M}_+(\mathbb{R}^d)$ the set of non-negative finite measures on $\mathbb{R}^d$. Moreover assume for simplicity that b is bounded. Then the following holds [26]:

Theorem 4.1.1 *Let μ_t be a solution of the PDE such that $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ for any $t \in [0,T]$, with $\mu_t(\mathbb{R}^d) \leq C$ for any $t \in [0,T]$. Then there exists a measurable family of probability measures $\{\nu_x\}_{x\in\mathbb{R}^d}$ on Γ_T such that:*

- *ν_x is concentrated on integral curves of the ODE starting from x (at time 0) for μ_0-a.e. x;*

- *the following representation formula holds:*

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t = \int_{\mathbb{R}^d \times \Gamma_T} \varphi(\gamma(t)) \, d\nu_x(\gamma) \, d\mu_0(x).$$

From this result, it is not difficult to prove that uniqueness for the ODE implies uniqueness of non-negative measure-valued solutions of the PDE.

The idea is now the following: by what we just said, one has that existence and uniqueness for the PDE in $\mathcal{M}_+(\mathbb{R}^d)$ implies existence and uniqueness for the ODE (and viceversa). But in order to have existence and uniqueness for the PDE in $\mathcal{M}_+(\mathbb{R}^d)$ one needs strong requirements on b, for instance $b(t)$ Lipschitz.

Thus the hope is that, under weaker assumptions on b, one can still prove an existence and uniqueness result for the PDE in some smaller class, like $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and from this one would like to deduce existence and uniqueness for the ODE in the almost everywhere sense.

This is exaclty what DiPerna-Lions and Ambrosio were able to do in [51, 26]. To state a precise result, we need to introduce the concept of *Regular Lagrangian Flows (RLF)*. The idea is that, if there exists a flow which produces solutions in $L^1 \cap L^\infty$, it cannot concentrate. Therefore we expect that, if such a flow exists, it must be a RLF in the sense of the following definition:

Definition 4.1.2 *We say that $X(t,x)$ is a RLF (starting at time 0), if:*

(i) for $\mathscr{L}^d$-a.e. x, $X(\cdot,x)$ is an integral curve of the ODE starting from x (at time 0);

(ii) there exists a nonnegative constant C such that, for any $t \in [0,T]$,

$$X(t)_{\#}\mathscr{L}^d \leq C\mathscr{L}^d.$$

It is not hard to show that, because of condition (ii), this concept is indeed invariant under modifications of b, and so it is appropriate to deal with vector fields belonging to L^p spaces.

As proved in [26], the following existence and uniqueness result for RLF holds:

Theorem 4.1.3 *Assume that, for any $\mu_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique solution of the PDE in $L^\infty([0,T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$. Then there exists a unique RLF. Moreover the RLF is stable by smooth approximations.*

The well-posedness of the PDE in $L^\infty([0,T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ has been shown by DiPerna-Lions [51] under the assumption

$$b \in W^{1,p}(\mathbb{R}^d), \quad [\mathrm{div} b]^- \in L^\infty(\mathbb{R}^d),$$

and then generalized by Ambrosio [26] assuming only

$$b \in BV(\mathbb{R}^d), \quad \mathrm{div} b \in L^1(\mathbb{R}^d), \quad [\mathrm{div} b]^- \in L^\infty(\mathbb{R}^d).$$

This theory presents still many open interesting questions, like to understand better whether uniqueness holds under the above hypotheses in bigger classes like $L^\infty([0,T], L^1(\mathbb{R}^d))$ (so that the solution can be unbounded). Or at the level of the ODE to see whether, under one of the above assumptions on the vector field, one can prove a statement like: there exists a set $A \subset \mathbb{R}^n$, with $|A| = 0$, such that for all $x \notin A$ the solution of the ODE is unique. These are problems that I would like to attack in the future.

4.2 The stochastic extension

In the stocastic case, the continuity equation becomes the Fokker-Planck equation

$$\partial_t \mu_t + \sum_i \partial_i (b^i \mu_t) - \frac{1}{2} \sum_{ij} \partial_{ij}(a^{ij} \mu_t) = 0,$$

and its associated SDE is

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t))\, dB(t), \\ X(0) = 0. \end{cases}$$

Here $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^r, \mathbb{R}^d)$ are bounded, $a^{ij} = (\sigma\sigma^*)^{ij}$, and B is an r-dimensional Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Classical theory states that, if $b(t), \sigma(t)$ are Lipschitz, then there exists a unique flow $X(t) = X(t, x, \omega)$. Moreover there exists a unique solution of the PDE, which is given by the formula

$$\int f(x)\, d\mu_t(x) := \int \mathbb{E}[f(X(t, x, \omega))]\, d\mu_0(x) \quad \forall f \in C_c(\mathbb{R}^d).$$

Since the PDE can see only the law of the process solving the SDE and not the process itself, if we hope to deduce some information on the ODE from the PDE, one needs to introduce a weaker concept of solution, the one of "martingale solution". In this way we are able extend the deterministic theory of RLF in the stochastic setting.

First of all, I could prove a representation formula for non-negative solutions of the PDE as in the deterministic case [13]:

Theorem 4.2.1 *Let μ_t be a solution of the PDE such that $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ for any $t \in [0, T]$, with $\mu_t(\mathbb{R}^d) \leq C$ for any $t \in [0, T]$. Then there exists a measurable family of probability measures $\{\nu_x\}_{x \in \mathbb{R}^d}$ on Γ_T such that:*

- *ν_x is martingale solution of the ODE starting from x (at time 0) for μ_0-a.e. x;*
- *the following representation formula holds:*

$$\int_{\mathbb{R}^d} \varphi\, d\mu_t = \int_{\mathbb{R}^d \times \Gamma_T} \varphi(\gamma(t))\, d\nu_x(\gamma)\, d\mu_0(x).$$

Then, I replaced the concept of Regular Lagrangian Flow by the one of *Stochastic Lagrangian Flow (SLF)*:

Definition 4.2.2 *Given a measure $\mu_0 = \rho_0 \mathscr{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0 \in L^\infty(\mathbb{R}^d)$, we say that a measurable family of probability measures $\{\nu_x\}_{x \in \mathbb{R}^d}$ on Γ_T is a μ_0-SLF (starting at time 0), if:*

(i) for μ_0-a.e. x, ν_x is a martingale solution of the SDE starting from x (at time 0);

(ii) there exists a nonnegative constant C such that, for any $t \in [0, T]$,

$$\mu_t := (e_t)_\# \left(\int \nu_x\, d\mu_0(x) \right) \leq C \mathscr{L}^d.$$

Finally, assuming well-posedness for the PDE in $L^1 \cap L^\infty$, I could prove existence and uniqueness of the SLF [13] (in particular, the SLF is independent of the initial measure μ_0):

Theorem 4.2.3 *Assume that, for any $\mu_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, there exists a unique solution of the PDE in $L^\infty([0,T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$. Then there exists a unique SLF. Moreover the SLF is stable by smooth approximations.*

As I showed in [13], two non-trivial situations where the theory is applicable (i.e. when the PDE is well-posed but no uniqueness result at the level of the SDE is known) are when the diffusion coefficients are uniformly elliptic and Lipschitz in time, or when the noise is just additive and the vector field is BV:

1. (a) $a_{ij}, b_i \in L^\infty([0,T] \times \mathbb{R}^d)$ for $i,j = 1,\dots,d$;
   (b) $\sum_j \partial_j a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ for $i = 1,\dots,d$,
   (c) $\partial_t a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ for $i,j = 1,\dots,d$;
   (d) $(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij} \partial_{ij} a_{ij})^- \in L^\infty([0,T] \times \mathbb{R}^d)$;
   (e) $\langle \xi, a(t,x)\xi \rangle \geq \alpha|\xi|^2 \ \forall (t,x) \in [0,T] \times \mathbb{R}^d$, for some $\alpha > 0$;
   (f) $\frac{a}{1+|x|^2} \in L^2([0,T] \times \mathbb{R}^d)$, $\frac{b}{1+|x|} \in L^2([0,T] \times \mathbb{R}^d)$.
2. (a) $a_{ij}, b_i \in L^\infty([0,T] \times \mathbb{R}^d)$ for $i,j = 1,\dots,d$;
   (b) $b \in L^1([0,T], BV_{loc}(\mathbb{R}^d, \mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{loc}([0,T] \times \mathbb{R}^d)$;
   (c) $(\sum_i \partial_i b_i)^- \in L^1([0,T], L^\infty(\mathbb{R}^d))$.

4.3 The infinite dimensional case

Let $(E, \|\cdot\|)$ be a separable Banach space endowed with a centered Gaussian measure γ, and denote by $\mathcal{H} \subset E$ the Cameron Martin space associated to (E,γ)[1]; in the finite-dimensional theory ($E = \mathcal{H} = \mathbb{R}^N$) other reference measures γ could be considered as well (for instance the Lebesgue measure). As in the finite dimensional case, we introduce the concept of regular flows:

Definition 4.3.1 *Let $b : (0,T) \times E \to E$ be a Borel vector field. If $X : [0,T] \times E \to E$ is Borel and $1 \leq r \leq \infty$, we say that X is a L^r-*regular flow* associated to b if the following two conditions hold:*

(i) for γ-a.e. $x \in X$ the map $t \mapsto \|b(t, X(t,x))\|$ belongs to $L^1(0,T)$ and

$$X(t,x) = x + \int_0^t b(\tau, X(\tau,x))\, d\tau \qquad \forall t \in [0,T]. \tag{4.3.1}$$

(ii) for all $t \in [0,T]$, $X(t)_\# \gamma$ is absolutely continuous with respect to γ, with a density ρ_t in $L^r(\gamma)$, and $\sup_{t\in[0,T]} \|\rho_t\|_{L^r(\gamma)} < \infty$.

[1]We recall that $\mathcal{H}$ can be defined as

$$\mathcal{H} := \left\{ \int_E \phi(x) x \, d\gamma(x) \ : \ \phi \in L^2(\gamma) \right\}.$$

In (4.3.1), the integral is understood in Bochner's sense, namely

$$\langle e^*, X(t,x) - x\rangle = \int_0^t \langle e^*, b(\tau, X(\tau,x))\rangle\, d\tau \qquad \forall e^* \in E^*.$$

As before, using the theory of characteristics we want to link the ODE to the continuity equation. Moreover, we want to transfer well-posedness informations from the continuity equation to the ODE, getting existence and uniqueness results of the L^r-regular b-flows under suitable assumptions on b.

However in this case we have to take into account an intrinsic limitation of the theory of L^r-regular b-flows that is typical of infinite-dimensional spaces: even if $b(t,x) \equiv v$ were constant, the flow map $X(t,x) = x + tv$ would not leave γ quasi-invariant, unless v belongs to $\mathcal{H}$. So, from now on we shall assume that b takes its values in $\mathcal{H}$ (however, thanks to a suitable change of variable, we were also able to treat some non $\mathcal{H}$-valued vector fields, see [5] for more details).

We recall that $\mathcal{H}$ can be endowed with a canonical Hilbertian structure $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ that makes the inclusion of $\mathcal{H}$ in E compact; we fix an orthonormal basis (e_i) of $\mathcal{H}$ and we shall denote by b^i the components of b relative to this basis (however, our result is independent of the choice of (e_i)).

With this choice of the range of b, whenever $\mu_t = u_t\gamma$ the equation can be written in the weak sense as

$$\frac{d}{dt}\int_E u_t\, d\gamma = \int_E \langle b_t, \nabla\phi\rangle_{\mathcal{H}} u_t\, d\gamma \qquad \forall \phi \in \mathrm{Cyl}(E,\gamma), \tag{4.3.2}$$

where $\mathrm{Cyl}(E,\gamma)$ is a suitable space of cylindrical functions induced by (e_i)[2]. Furthermore, a Gaussian divergence operator $\mathrm{div}_\gamma \boldsymbol{c}$ can be defined as the adjoint in $L^2(\gamma)$ of the gradient along $\mathcal{H}$:

$$\int_E \langle c, \nabla\phi\rangle_{\mathcal{H}}\, d\gamma = -\int_E \phi\, \mathrm{div}_\gamma c\, d\gamma \qquad \forall \phi \in \mathrm{Cyl}(E,\gamma).$$

Another typical feature of our Gaussian framework is that L^∞-bounds on div_γ do not seem natural, unlike those on the Euclidean divergence in $\mathbb{R}^N$ when the reference measure is the Lebesgue measure: indeed, even if $b(t,x) = c(x)$, with $c : \mathbb{R}^N \to \mathbb{R}^N$ smooth and with bounded derivatives, we have $\mathrm{div}_\gamma c = \mathrm{div} c - \langle c, x\rangle$ which is unbounded, but exponentially integrable with respect to γ.

The main result in this framework, proved in collaboration with Luigi Ambrosio in [5], is the following:

Theorem 4.3.2 *Let p, $q > 1$ and let $b : (0,T) \times E \to \mathcal{H}$ be satisfying:*

(i) $\|b_t\|_{\mathcal{H}} \in L^1\big((0,T); L^p(\gamma)\big)$*;*

[2] We recall that $\phi : E \to \mathbb{R}$ is cylindrical if

$$\phi(x) = \psi\big(\langle e_1^*, x\rangle, \ldots, \langle e_N^*, x\rangle\big) \tag{4.3.3}$$

for some integer N and some $\psi \in C_b^\infty(\mathbb{R}^N)$, where $C_b^\infty(\mathbb{R}^N)$ is the space of smooth functions in $\mathbb{R}^N$, bounded together with all their derivatives.

(ii)

$$\int_0^T \left(\int_E \|(\nabla b_t)^{\mathrm{sym}}(x)\|_{HS}^q \, d\gamma(x) \right)^{1/q} dt < \infty, \tag{4.3.4}$$

and $\mathrm{div}_\gamma b_t \in L^1\big((0,T); L^q(\gamma)\big)$*;*

(iii) $\exp(c[\mathrm{div}_\gamma b_t]^-) \in L^\infty\big((0,T); L^1(\gamma)\big)$ *for some* $c > 0$*.*

If $r := \max\{p', q'\}$ *and* $c \geq rT$*, then the* L^r*-regular flow exists and is unique in the following sense: any two* L^r*-regular flows* X *and* $\tilde{X}$ *satisfy*

$$X(\cdot, x) = \tilde{X}(\cdot, x) \quad \text{in } [0,T], \text{ for } \gamma\text{-a.e. } x \in E.$$

Furthermore, X *is* L^s*-regular for all* $s \in [1, \frac{c}{T}]$ *and the density* u_t *of the law of* $X(t,\cdot)$ *under* γ *satisfies*

$$\int (u_t)^s \, d\gamma \leq \left\| \int_E \exp\big(Ts[\mathrm{div}_\gamma b_t]^-\big) \, d\gamma \right\|_{L^\infty(0,T)} \qquad \text{for all } s \in [1, \frac{c}{T}].$$

In particular, if $\exp(c[\mathrm{div}_\gamma b_t]^-) \in L^\infty\big((0,T); L^1(\gamma)\big)$ *for all* $c > 0$*, then the* L^r*-regular flow exists globally in time, and is* L^s*-regular for all* $s \in [1, \infty)$*.*

We remark that, in the previous results in this setting by Cruzeiro [46, 47, 48], Peters [69], and Bogachev and Wolf [34], the assumptions on the vector field were

$$\|b\|_{\mathcal{H}} \in \bigcap_{p \in [1,\infty)} L^p(\gamma),$$

$$\exp(c\|\nabla b\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}) \in L^1(\gamma) \qquad \text{for all } c > 0,$$

$$\exp(c|\mathrm{div}_\gamma b|) \in L^1(\gamma) \qquad \text{for some } c > 0.$$

Therefore the main difference between these results and our is that we replaced exponential integrability of b and the *operator* norm of ∇b by p-integrability of b and q-integrability of the *Hilbert-Schmidt* norm of (the symmetric part of) ∇b_t. These hypotheses are in some sense closer to the ones in the finite dimensional case, and so our result can really be seen as an extension of the finite dimensional theory to an infinite dimensional setting.

A natural problem, on which I would like to work in the future, is to try to understand how much this result is optimal, and whether it can be applied to prove "a.e. well-posedness" for PDEs, looking at them as infinite dimensional ODEs.

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