QUANTITATIVE ISOPERIMETRIC INEQUALITIES, WITH APPLICATIONS TO THE STABILITY OF LIQUID DROPS AND CRYSTALS

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Abstract. Recently, in collaboration with Maggi and Pratelli, the author proved a sharp quantitative version of the anisotropic isoperimetric inequality using optimal transportation [10]. Subsequently, this result has been applied by the author and Maggi to study the stability of the shape of small liquid drops and crystals under the action of an exterior potential [9]. This note is a review of these results.

1. The anisotropic isoperimetric inequality

The anisotropic isoperimetric inequality arises in connection with a natural generalization of the Euclidean notion of perimeter. In dimension $n \geq 2$, we consider an open, bounded, convex set K of $\mathbb{R}^n$ containing the origin. Starting from K, we define a weight function on directions through the Euclidean scalar product

$$\|\nu\|_* := \sup\{x \cdot \nu : x \in K\}, \quad \nu \in \mathbb{S}^{n-1}, \tag{1.1}$$

where $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$. Given E a open smooth set in $\mathbb{R}^n$, its *anisotropic perimeter* is defined as

$$P_K(E) := \int_{\partial E} \|\nu_E(x)\|_* d\mathcal{H}^{n-1}(x), \tag{1.2}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$. This notion of perimeter obeys the scaling law $P_K(\lambda E) = \lambda^{n-1} P_K(E)$, $\lambda > 0$, and it is invariant under translations. However, at variance with the Euclidean perimeter, P_K is not invariant by the action of $O(n)$, or even of $SO(n)$, and in fact it may even happen that $P_K(E) \neq P_K(\mathbb{R}^n \setminus E)$, provided K is not symmetric with respect to the origin. When K is the Euclidean unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ of $\mathbb{R}^n$ then $\|\nu\|_* = 1$ for every $\nu \in \mathbb{S}^{n-1}$, and therefore $P_K(E)$ coincides with the Euclidean perimeter of E.

Apart from its intrinsic geometric interest, the anisotropic perimeter P_K arises as a model for surface tension in the study of equilibrium configurations of solid crystals with sufficiently small grains [24, 22], and constitutes the basic model for surface energies in phase transitions [15]. In both settings, one is naturally led to minimize $P_K(E)$ under a volume constraint. This is, of course, equivalent to study the isoperimetric problem

$$\inf\left\{\frac{P_K(E)}{|E|^{(n-1)/n}} : 0 < |E| < +\infty\right\}, \tag{1.3}$$

where $|E|$ is the Lebesgue measure of E. As conjectured by Wulff [24] back to 1901, the unique minimizer (modulo the invariance group of the functional, which

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consists of translations and scalings) is the set K itself. In particular the anisotropic isoperimetric inequality holds:

$$P_K(E) \geq n|K|^{1/n}|E|^{(n-1)/n}\,, \quad \text{if } |E| < +\infty\,. \tag{1.4}$$

(We refer to [10, 19] for more details and references.)

1.1. **Stability of isoperimetric problems.** Here we will show how an optimal transportation variant of Gromov's proof of (1.4) [20] can be used to get a stronger version. Let us introduce first some notation.

Whenever $0 < |E| < +\infty$, we introduce the *isoperimetric deficit of* E,

$$\delta(E) := \frac{P_K(E)}{n|K|^{1/n}|E|^{(n-1)/n}} - 1\,.$$

We observe that, due to (1.4), $\delta(E) \geq 0$. Moreover $\delta(E) = 0$ if and only if, modulo translations and dilations, E is equal to K up to modifications on a set of measure zero (this is a consequence of the characterization of equality cases of (1.4)). Thus $\delta(E)$ measures, in terms of the relative size of the perimeter and of the measure of E, the deviation of E from being optimal in (1.4). The stability problem consists in quantitatively relating this deviation to a more direct notion of distance from the family of optimal sets. To this end we introduce the *asymmetry index* of E,

$$A(E) := \inf\left\{\frac{|E\Delta(x_0 + rK)|}{|E|} : x_0 \in \mathbb{R}^n\,, r^n|K| = |E|\right\}$$

(where $E\Delta F$ denotes the symmetric difference between the sets E and F), and we look for positive constants C and α, depending on n and K only, such that the following quantitative form of (1.4) holds true:

$$P_K(E) \geq n|K|^{1/n}|E|^{(n-1)/n}\left\{1 + \left(\frac{A(E)}{C}\right)^{\alpha}\right\}\,,$$

i.e.,

$$A(E) \leq C\,\delta(E)^{1/\alpha}\,. \tag{1.5}$$

This problem has been thoroughly studied in the Euclidean case $K = B$, starting from the two dimensional case, considered by Bernstein [2] and Bonnesen [3]. They prove (1.5) with the exponent $\alpha = 2$, that is optimal concerning the decay rate at zero of the asymmetry in terms of the deficit. The first general results in higher dimension are due to Fuglede [12], dealing with the case of convex sets. Concerning the unconstrained case, the main contributions are due to Hall, Hayman and Weitsman [17, 16]. They prove (1.5) with a constant $C = C(n)$ and exponent $\alpha = 4$. It was, however, conjectured by Hall that (1.5) should hold with the sharp exponent $\alpha = 2$. This was recently shown in [13] (see also the survey [19]).

A common feature of all these contributions is the use of *quantitative* symmetrization inequalities, that is clearly specific to the isotropic case. If K is a generic convex set, then the study of uniqueness and stability for the corresponding isoperimetric inequality requires the employment of entirely new ideas. Indeed, the methods developed in [17, 13] are of no use as soon as K is not a ball. Under the assumption of convexity on E, the problem has been studied by Groemer [14], while the first stability result for (1.4) on generic sets is due to Esposito, Fusco, and Trombetti in [8]. Starting from the uniqueness proof of Fonseca and Müller [11], they show the validity of (1.5) with some constant $C = C(n, K)$ and for the exponent

$$\alpha(2) = \frac{9}{2}, \quad \alpha(n) = \frac{n(n+1)}{2}, \quad n \geq 3.$$

This remarkable result leaves, however, the space for a substantial improvement concerning the decay rate at zero of the asymmetry index in terms of the isoperimetric deficit. The main result in [10] provides the sharp decay rate, see [10, Theorem 1.1]:

Theorem 1.1. *Let E be a set of finite perimeter with $|E| < +\infty$. Then*

$$A(E) \leq \frac{181\, n^7}{(2 - 2^{(n-1)/n})^{3/2}} \sqrt{\delta(E)}\,. \tag{1.6}$$

The proof of the above theorem is based on a quantitative study of the optimal transport map between E and K, through the bounds that can be derived from Gromov's proof of the isoperimetric inequality (see also [7] for a proof of Sobolev inequalities via optimal transportation). These estimates provide control, in terms of the isoperimetric deficit, and modulo scalings and translations, on the distance between such a transportation map and the identity, which can then be used to deduce a bound on the asymmetry index.

In the next section, we sketch the proof of this result.

1.2. **Sketch of the proof of the Theorem 1.1.**

1.2.1. *Gromov's proof of the anisotropic isoperimetric inequality.* Given a (smooth) bounded set $E \subset \mathbb{R}^n$, Brenier's Theorem [4] ensures the existence of a convex, continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$, whose gradient $T = \nabla\varphi$ pushes forward the probability density $|E|^{-1}1_E(x)dx$ into the probability density $|K|^{-1}1_K(y)dx$. In particular, T takes E into K and

$$\det \nabla T = \frac{|K|}{|E|} \quad \text{on } E\,. \tag{1.7}$$

Since T is the gradient of a convex function and has positive Jacobian, $\nabla T(x)$ is a symmetric and positive definite $n\times n$ tensor, with n-positive eigenvalues $0 < \lambda_k(x) \leq \lambda_{k+1}(x)$, $1 \leq k \leq n-1$, such that

$$\nabla T(x) = \sum_{k=1}^{n} \lambda_k(x) e_k(x) \otimes e_k(x)\,,$$

for a suitable orthonormal basis $\{e_k(x)\}_{k=1}^n$ of $\mathbb{R}^n$. Then, by the arithmetic-geometric inequality, we find

$$n(\det \nabla T)^{1/n} = n\left(\prod_k \lambda_k\right)^{1/n} \leq \sum_k \lambda_k = \operatorname{div} T \quad \text{on } E\,. \tag{1.8}$$

Hence, by (1.7), (1.8) and the Divergence Theorem, we get

$$n|K|^{1/n}|E|^{(n-1)/n} = \int_E n(\det \nabla T)^{1/n} \leq \int_E \operatorname{div} T = \int_{\partial E} T \cdot \nu_E \, d\mathcal{H}^{n-1}\,. \tag{1.9}$$

Let us now define, for every $x \in \mathbb{R}^n$,

$$\|x\| := \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in K\right\}\,. \tag{1.10}$$

Note that this quantity fails to define a norm only because, in general, $\|x\| \neq \|-x\|$ (indeed, K is not necessarily symmetric with respect to the origin). Then the set K can be characterized as

$$K = \{x \in \mathbb{R}^n : \|x\| < 1\}\,.$$

Hence, as $T(x) \in K$ for $x \in E$, we obtain

$$\|T\| \leq 1\,, \qquad \text{on } \partial E\,. \tag{1.11}$$

Moreover,

$$\|\nu\|_* = \sup\{x \cdot \nu : \|x\| = 1\}\,,$$

which gives the following Cauchy-Schwarz type inequality

$$x \cdot y \leq \|x\|\|y\|_*\,, \quad \forall\, x, y \in \mathbb{R}^n\,. \tag{1.12}$$

From (1.9), (1.12) and (1.11),

$$n|K|^{1/n}|E|^{(n-1)/n} \leq \int_{\partial E} \|T\|\|\nu_E\|_*\, d\mathcal{H}^{n-1} \leq P_K(E)\,,$$

and the anisotropic isoperimetric inequality is proved. (This argument is formal, since a priori the transport map is not smooth. However, the proof can be made rigorous by using either Caffarelli's regularity theory [5] or the theory of BV function, see [10].)

1.2.2. *The equality case.* To give an example of the robustness of the above proof, we show here how the characterization of equality cases follows (at least formally) almost immediately from the above argument. (Actually, the proof can be easily made rigorous using some fine results on set of finite perimeter, see [10, Appendix].)

Assume E to be (a smooth open connected set) optimal in the isoperimetric inequality, which with no loss of generality we can assume to have the same volume as K. Then, from Gromov's argument we derive the condition $n(\det \nabla T)^{1/n} = \operatorname{div} T$. Recalling that equality in the arithmetic-geometric inequality holds if and only if all the numbers are equal, we get $\lambda_1(x) = \ldots = \lambda_n(x)$ on E. Then, since $\det \nabla T = 1$ on E and ∇T is symmetric, the above condition implies immediately $\nabla T = \mathrm{Id}$. Thus $T(x) = x + c$ for some vector $c \in \mathbb{R}^n$, that is, $E = K - c$, as desired.

1.2.3. *The quantitative argument.* We now discuss how the bounds on the isoperimetric deficit contained in Gromov's proof can be used to prove Theorem 1.1. If we assume $|E| = |K|$ and let T be the Brenier map between E and K, then from Gromov's proof we immediately find

$$n|K|\delta(E) \geq \int_{\partial E} (1 - \|T\|)\|\nu_E\|_* d\mathcal{H}^{n-1}\,, \tag{1.13}$$

$$|K|\delta(E) \geq \int_E \left\{ \frac{\operatorname{div} T}{n} - (\det \nabla T)^{1/n} \right\}. \tag{1.14}$$

As seen before, $\delta(E) = 0$ forces $\nabla T = \mathrm{Id}$ a.e. on E, therefore it is not surprising to derive from (1.14) the estimate

$$C(n)|K|\sqrt{\delta(E)} \geq \int_E |\nabla T - \mathrm{Id}\,|\,, \tag{1.15}$$

see [10, Corollary 2.4].

Now, we use a reduction step. Namely, in [10] we show the following result: If E has small deficit, up to the removal of a "critical" subset whose measure is controlled by $\delta(E)$, there exists a positive constant $\tau(n,K)$, independent of E, such that the following trace inequality holds true:

$$\int_E \|-\nabla f(x)\|_* dx \geq \tau(n,K) \inf_{c\in\mathbb{R}} \int_{\partial E} |f(x)-c| \|\nu_E(x)\|_* d\mathcal{H}^{n-1}(x) \,, \quad \forall\, f \in C^1_c(\mathbb{R}^n)\,,$$

see [10, Theorem 3.4]. Hence we can apply the trace inequality together with (1.15) to deduce that

$$C(n,K)\sqrt{\delta(E)} \geq \int_{\partial E} \|T(x)-x\| \|\nu_E\|_* \, d\mathcal{H}^{n-1}(x) \tag{1.16}$$

up to a translation of E. Since $\|T(x)\| \leq 1$ on ∂E we have

$$|1-\|x\|| \leq |1-\|T(x)\|\,| + \|T(x)-x\| = (1-\|T(x)\|) + \|T(x)-x\| \,,$$

for every $x \in \partial E$. Thus, by adding (1.13) and (1.16) we find

$$C(n,K)\sqrt{\delta(E)} \geq \int_{\partial E} |1-\|x\|| \, \|\nu_E\|_* \, d\mathcal{H}^{n-1}(x) \,. \tag{1.17}$$

Then it is not difficult to show that this last integral controls $|E\backslash K| = |E\Delta K|/2$ (see [10, Figure 1.5 and Lemma 3.5]), which allows to achieve the proof of Theorem 1.1.

Let us point out that, although the constant $C(n,K)$ in (1.17) depends a priori on K, by using a renormalization argument for convex sets one can find a bound for $C(n,K)$ depending on the dimension only. We refer to [10] for more details.

2. On the shape of small liquid drops and crystals

2.1. **The variational problem.** Let us consider a liquid drop or a crystal of mass m subject to the action of a potential. At equilibrium, its shape minimizes (under a volume constraint) the free energy, that consists of a (possibly anisotropic) interfacial surface energy plus a bulk potential energy induced by an external force field [6, 18]. Therefore one is naturally led to consider the variational problem

$$\inf \left\{ \mathcal{E}(E) := P_K(E) + \mathcal{G}(E) : |E| = m \right\} \,. \tag{2.1}$$

Here, $P_K(E)$ and $\mathcal{G}(E)$ are, respectively, the surface energy and the potential energy of E, that are introduced as follows.

• *Surface energy:* Assume that we are given a *surface tension*, that is, a convex, positively 1-homogeneous function $f : \mathbb{R}^n \to [0,+\infty)$. Correspondingly we define the surface energy of a smooth set $E \subset \mathbb{R}^n$ as

$$\int_{\partial E} f(\nu_E) \, d\mathcal{H}^{n-1} \,. \tag{2.2}$$

Let us observe that the function $f(\nu)$ corresponds to the weight function $\|\nu\|_*$ defined in the previous section, for some suitable Wulff shape. More precisely, if we define the set K as

$$K := \bigcap_{\nu\in S^{n-1}} \{x \in \mathbb{R}^n : (x\cdot\nu) < f(\nu)\} \,, \tag{2.3}$$

then $f(\nu) = \|\nu\|_*$ (see (1.1)), and the surface energy coincides with P_K. In particular, if $\mathcal{G} \equiv 0$ then the minimization problem (2.1) becomes equivalent to the isoperimetric problem (1.3), and the unique minimizer is given by K (up to translations and dilations).

The geometric properties of a Wulff shape are closely related to the analytic properties of the corresponding surface tension. Two relevant (and somehow complementary) situations are the following ones:

Uniformly elliptic case: The surface tension f is λ-*elliptic*, $\lambda > 0$, if $f \in C^2(\mathbb{R}^n \setminus \{0\})$ and

$$(\nabla^2 f(v)\tau) \cdot \tau \geq \frac{\lambda}{|v|} \left| \tau - \left(\tau \cdot \frac{v}{|v|} \right) \frac{v}{|v|} \right|^2 , \tag{2.4}$$

whenever $v, \tau \in \mathbb{R}^n$, $v \neq 0$. Under these assumptions, the boundary of the Wulff shape K is of class C^2 and uniformly convex. Moreover, the second fundamental form $\nabla \nu_K$ of K satisfies the identity

$$\nabla^2 f(\nu_K(x)) \nabla \nu_K(x) = \mathrm{Id}_{T_x \partial K} , \qquad \forall\, x \in \partial K . \tag{2.5}$$

This situation includes of course the *isotropic* case $f(\nu) = \lambda|\nu|$ ($\lambda > 0$). Evidently, in the isotropic case the Wulff shape is the Euclidean ball $B_\lambda = \lambda B$, and the Wulff inequality reduces to the Euclidean isoperimetric inequality. Isotropic (or smooth, nearly isotropic) surface energies are used to model liquid drops. Furthermore, they appear in phase transition problems, where the mean curvature of the interface is related to the pressure or the temperature on it, represented by g (this is the so-called Gibbs-Thompson relation).

Crystalline case: A surface tension f is *crystalline* if it is the maximum of finitely many linear functions, i.e., if there exists a finite set $\{x_j\}_{j=1}^N \subset \mathbb{R}^n \setminus \{0\}$, $N \in \mathbb{N}$, such that

$$f(\nu) = \max_{1 \leq j \leq N} (x_j \cdot \nu) , \qquad \forall\, \nu \in S^{n-1} . \tag{2.6}$$

The corresponding Wulff shape is a convex polyhedron. These are the surface tensions used in studying crystals [22].

- *Potential energy:* The *potential* is a locally bounded Borel function $g : \mathbb{R}^n \to [0, +\infty)$ that is coercive on $\mathbb{R}^n$, i.e., we have

$$g(x) \to +\infty \quad \text{as } |x| \to +\infty . \tag{2.7}$$

We also assume that

$$\inf_{\mathbb{R}^n} g = g(0) = 0 . \tag{2.8}$$

This is done without loss of generality in the study of (2.1), as it amounts to subtract to the free energy a suitable constant and to translate the origin in the system of coordinates. The potential energy of $E \subset \mathbb{R}^n$ is then defined as

$$\mathcal{G}(E) = \int_E g(x)\, dx . \tag{2.9}$$

Actually one could also allow g to take the value $+\infty$ in order to include a confinement constraint, since (whenever possible) a minimizer will always avoid the region $\{g = +\infty\}$. Observe that, when g is differentiable on the (open) set $\{g < +\infty\}$, then the energy term $\mathcal{G}(E)$ corresponds to the presence of the force field $-\nabla g$ acting on E.

2.2. **Geometric properties of minimizers.** In the presence of the potential term, the geometric properties of minimizers are not well understood. (A noticeable exception to this claim is the case of sessile/pendant liquid drops under the action of gravity.)

More precisely, if we look to (2.1) in its full generality, then the validity of various natural properties of minimizers is at present unknown. In particular, the following question was raised by Almgren (personal communication of Morgan):

(Q) If the surface energy dominates over the potential energy (e.g., if the potential g is almost constant or if the mass m is sufficiently small), to which extent are minimizers "close" to Wulff shapes?

In [9] we investigated this question, providing some optimal results, both in the planar case and in general dimension. All the estimates are quantitative, in the sense that we shall present explicit bounds on the proximity to a Wulff shape in terms of the small mass m. Moreover, the value of the "critical" mass below which these estimates hold could be made completely explicit from the proof. The first main result establishes the connectedness and the uniform L^∞-closeness of minimizers to Wulff shapes below a critical mass [9, Theorem 1.1]:

Theorem 2.1. *There exist positive constants $m_c = m_c(n, f, g)$ and $C = C(n, f, g)$ with the following property: If E is a minimizer in the variational problem* (2.1) *with mass $|E| = m \leq m_c$, then E is connected and uniformly close to a Wulff shape, i.e., there exist $x_0 \in \mathbb{R}^n$ and $r_0 > 0$, with*

$$r_0 \leq C\, m^{1/n^2} ,$$

such that

$$x_0 + K_{s(m)(1-r_0)} \subset E \subset x_0 + K_{s(m)(1+r_0)} ,$$

where

$$K_{s(m)(1\pm r_0)} = s(m)(1 \pm r_0)K \quad \textit{and} \quad s(m) := \left(\frac{m}{|K|}\right)^{1/n} .$$

If $n = 2$ then E is a convex set. Moreover, if f is crystalline (or, equivalently, if the Wulff shape K is a convex polygon), then E is a convex polygon with sides parallel to that of K.

The above theorem shows that in the planar crystalline case minimizers possess a particularly rigid structure, and this raises the question whether or not an analogous property should hold in higher dimension (see [9, Remark 1.2 and Figure 2]).

The main question left open by Theorem 2.1 concerns the convexity of minimizers at small mass in dimension $n \geq 3$. In [9], this problem is addressed in the case of smooth λ-elliptic surface tensions, and potentials of class C^1. In this situation the Wulff shape turns out to be a uniformly convex set with smooth boundary. Correspondingly, one can prove that minimizers at small mass are not merely convex, but that they are in fact uniformly convex sets with smooth boundary and with principal curvatures uniformly close to that of a (properly rescaled) Wulff shape. To express this last property, we made use of the second order characterization (2.5) of Wulff shapes [9, Theorem 1.2]:

Theorem 2.2. *If $g \in C^1_{\mathrm{loc}}(\mathbb{R}^n)$, $f \in C^{2,\alpha}(\mathbb{R}^n \setminus \{0\})$ for some $\alpha \in (0,1)$, and f is λ-elliptic, then there exist a critical mass $m_0 = m_0(n, g, f)$ and a constant*

$C = C(n,g,f,\alpha)$ with the following property: If E is a minimizer in (2.1) *with $|E| = m \le m_0$ and if we set*

$$F = \left(\frac{|K|}{m}\right)^{1/n} E\,,$$

then ∂F is of class $C^{2,\alpha}$ and

$$\max_{\partial F} |\nabla^2 f(\nu_F)\nabla\nu_F - \mathrm{Id}_{T_x\partial F}| \le C\, m^{2\alpha/(n+2\alpha)}\,. \tag{2.10}$$

In particular, if m is small enough (the smallness depending on n, f, and g only) then F (and so E) is a convex set.

Let us remark that the above result differs from similar results (stating for instance the asymptotic convexity of isoperimetric regions with small mass on Riemannian manifolds) in the fact of being "quantitative": not only we can find an explicit rate of convergence in terms of m, but also the constant C appearing in (2.10) is obtained by a constructive method, and so it is a priori computable.

In the sequel, we explain the strategy of the proof of the above theorems.

2.3. **Strategy of the proof of Theorems 2.1 and 2.2.** . The proofs of the above results rely on the notion and some important properties of (ε, R)-minimizers: Given $\varepsilon, R > 0$, a set (of finite perimeter) $E \subset \mathbb{R}^n$ is a (volume constrained) (ε, R)-minimizer of P_K provided

$$P_K(E) \le P_K(F) + \varepsilon\,|K|^{1/n}|E|^{(n-1)/n}\frac{|E\Delta F|}{|E|}\,, \tag{2.11}$$

for every set (of finite perimeter) $F \subset \mathbb{R}^n$ with

$$|F| = |E| \quad \text{and} \quad F \subset I_R(E)\,,$$

where $I_R(E)$ is the R-neighborhood of E with respect to K, i.e.,

$$I_R(E) = \{x \in \mathbb{R}^n : \mathrm{dist}_K(x,E) < R\}\,, \qquad \mathrm{dist}_K(x,E) = \inf_{y\in E}\|x-y\|\,, \tag{2.12}$$

with $\|\cdot\|$ defined as in (1.10).

2.3.1. *Proof of Theorem 2.1. Step 1: boundedness.* The first step consists in showing that optimal shapes for (2.1) are uniformly bounded in terms of their mass, the dimension n and the way g grows at infinity. This result is proved by some cut-and-paste operations followed by a mass adjustment. This kind of arguments are used many times in our proofs. We usually adjust mass in two ways: either by a first variation argument, where the surface energy variation depends on the set itself in a quite involved way (this lemma is sometimes referred to as "Almgren's Lemma", see [21, Lemma 13.5]); or by a dilation, in which case the surface energy variation is trivial but the variation of the potential energy requires an estimate.

As a consequence of the uniform boundedness and the local boundedness of g, it is easy to show that any minimizer E for (2.1) with $|E| = m$ is an $(\varepsilon, n+1)$-minimizer for $\varepsilon \le C\,m^{1/n}$, where C is an (explicitly computable) constant depending on n, f, and g only.

Step 2: L^1-closeness. The L^1-proximity (in terms of the smallness of ε) of every $(\varepsilon, n+1)$-minimizer to a properly rescaled and translated Wulff shape is an almost direct consequence of Theorem 1.1.

Step 3: L^∞-closeness. Once the L^1-closeness is established, the fact that $(\varepsilon, n+1)$-minimizers are connected and uniformly L^∞-close to Wulff shapes may appear to the specialists as a classical application of standard density estimates combined with the above mentioned L^1-estimate. However, at least to our knowledge, for a general integrand f (which we do not assume either smooth or uniformly elliptic) there are no universal density estimates available for (ε, R)-minimizers (i.e., density estimate independent of the minimizer). For this reason this closeness result, although it follows the lines of many other proofs of the same kind (again by using cut-and-paste arguments), presents some subtle points. Anyhow, a careful approach allows to show that the uniform proximity result holds for every $(\varepsilon, n+1)$-minimizer with $\varepsilon \le \varepsilon(n)$, where $\varepsilon(n)$ depends on the dimension n only, and not on f. This proves the first part of Theorem 2.1.

Step 4: the case $n = 2$. In the planar case $n = 2$ one can take advantage of the fact that the surface energy P_K decreases under convexification to show that $(\varepsilon, 3)$-minimizers are convex. Moreover, a (local) perturbation argument allows to prove that, in the crystalline case, minimizers are convex polygons with sides parallel to that of K provided $\varepsilon \le \varepsilon_0$, where ε_0 is a universal constant independent of f. This concludes the proof of Theorem 2.1.

2.3.2. *Proof of Theorem 2.2. Step 1: $C^{1,\alpha}$-regularity.* As a preparatory step towards the proof of Theorem 2.2, we consider λ-elliptic surface tensions and apply the regularity theory for almost minimizing rectifiable currents to show that the boundaries of $(\varepsilon, n+1)$-minimizers of P_K satisfy uniform $C^{1,\alpha}$-estimates for every $\alpha \in (0, 1)$.

Step 2: $C^{2,\alpha}$-regularity. If we come back to minimizers of (2.1), we can use the above $C^{1,\alpha}$-regularity to perform a first variation argument on the energy functional and obtain some Euler-Lagrange equations which locally looks like

$$\mathrm{div}'\left(\nabla f^{\#}(\nabla u(z))\right) = g(z, u(z)) - \mu\,,$$

where $f^{\#}$ is defined in terms of f, and $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is a local parameterization of ∂E (see [9, Appendix A] for more details). Then we can apply standard Schauder estimates for elliptic equations to get uniform $C^{2,\alpha}$-bounds on u, and by a partition argument we get a uniform bound of the form

$$\|\partial F\|_{C^{2,\alpha}} \le C(n, f, g, \alpha)$$

(recall that $F := (|K|/m)^{1/n} E$ is a dilation of E so that $|F| = |K|$).

Step 3: $W^{2,2}$-closeness. We now assume that $g \in C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then we can exploit the non-negativity of the second variation of the free energy with respect to normal variations to deduce the validity of the minimality condition

$$\int_{\partial E} \mathrm{grad}\,\zeta \cdot (\mathrm{Hess} f(\nu_E)\,\mathrm{grad}\,\zeta) - \zeta^2 \left[\mathrm{tr}\,(\mathrm{Hess} f(\nu_E) A_E^2) - (\nabla g \cdot \nu_E)\right] d\mathcal{H}^{n-1} \ge 0\,, \tag{2.13}$$

for every $\zeta \in C_c^\infty(\mathbb{R}^n)$ satisfying the constraint

$$\int_{\partial E} \zeta \, d\mathcal{H}^{n-1} = 0 \,. \tag{2.14}$$

(Here "grad" and "Hess" denote the first and second tangential derivatives with respect to ∂E.) The idea is to exploit (2.13) to prove a quantitative bound on the L^2-distance of the second fundamental form of ∂E from that of ∂K. More precisely, when $g \equiv 0$ and $f \equiv 1$, Barbosa and do Carmo [1] used the test function

$$\zeta(x) = 1 - \beta(x - x_0) \cdot \nu_E(x) \,, \qquad \forall\, x \in \partial E \,, \tag{2.15}$$

(where β is determined by (2.14) and x_0 is arbitrary) to show that the principal curvatures of ∂E have to be all equal to each other and constant. In particular, ∂E is forced to be an Euclidean sphere.

The above result has been generalized by Winklmann [23] to the case of a general integrand f, using the test function

$$\zeta(x) = f(\nu_E(x)) - \beta\,(x - x_0) \cdot \nu_E(x) \,, \qquad \forall\, x \in \partial E \,, \tag{2.16}$$

to show that E is a Wulff shape.

In our situation, due to the small mass regime, we can consider the term $\nabla g \cdot \nu_E$ in (2.13) as a small perturbation and try to gain some information by using (2.13) with the test function used by Winklmann. (One main difference comes from the fact that we have to replace the constant anisotropic mean curvature condition $H_f = \mu$ of Winklmann with the stationarity condition $H_f + g = \mu$.) In this way, we prove that the quantity

$$\frac{1}{P(E)} \int_{\partial E} \|\mathrm{Hess} f(\nu_E)\nabla\nu_E - \mu\,\mathrm{Id}_{\,T_x\partial E}\|^2 \, d\mathcal{H}^{n-1} \,,$$

is uniformly bounded in terms of n, f, and g only. After rescaling, this bound becomes

$$\frac{1}{P(F)} \int_{\partial F} \left\|\nabla^2 f(\nu_F)\nabla\nu_F - \mathrm{Id}_{\,T_x\partial F}\right\|^2 \, d\mathcal{H}^{n-1} \le C\, m^{2/n} \,.$$

which shows the L^2-proximity of the second fundamental form of ∂F to the one of ∂K, as desired.

Step 4: conclusion. Using standard interpolation inequalities, the uniform $C^{2,\alpha}$-bound on ∂F from Step 2 together with the L^2-closeness from Step 3 give the C^0-closeness of the second fundamental form of ∂E to that of ∂K, implying in particular the convexity of E at small mass.

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