REGULARITY THEORY FOR LOCAL AND NONLOCAL MINIMAL SURFACES: AN OVERVIEW

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Abstract. These notes record the lectures for the CIME Summer Course held by the second author in Cetraro during the week of July 4-8, 2016. The goal is to give an overview of some classical results for minimal surfaces, and describe recent developments in the nonlocal setting.

1. Introduction

Let $1 \leqslant k \leqslant n-1$ be two integers and $\Gamma \subset \mathbb{R}^n$ be a $(k-1)$-dimensional, smooth manifold without boundary. The classical *Plateau problem* consists in finding a k-dimensional set Σ with $\partial\Sigma = \Gamma$ such that

$$\text{Area}(\Sigma) = \min \Big\{ \text{Area}(\Sigma') : \partial\Sigma' = \Gamma \Big\}. \tag{1.1}$$

Here, with the notation $\text{Area}(\cdot)$ we denote a general "area-type functional" that we shall specify later. We will consider two main examples: one where the area is the standard Hausdorff k-dimensional measure, and one in which it represents a recently introduced notion of nonlocal (or fractional) perimeter.

We stress that we do not have a well-defined nonlocal perimeter for k-dimensional manifolds with $k \leqslant n-2$. Moreover, even in the codimension 1 case, we need Σ to be the boundary of some set in order to be able to define its fractional perimeter. Therefore, to make the parallel between the local and the nonlocal theories more evident, we shall always focus on the setting

$$k = n-1 \qquad \text{and} \qquad \Sigma = \partial E, \text{ with } E \subset \mathbb{R}^n \text{ } n\text{-dimensional.}$$

In the forthcoming sections, we will outline several issues and solutions relevant to this minimization problem.

Most of these notes will be devoted to presenting the main ideas involved in the case of the traditional area functional. Then, in the last section, we will briefly touch on the main challenges that arise in the nonlocal setting.

Remark 1.1. In these notes, we shall say that a surface is "minimal" if it *minimizes* the area functional. This notation is not universal: some authors call a surface "minimal" if it is a critical point of the area functional, and call it "area minimizing" when it is a minimizer.

1.1. **The minimization problem.** Given a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary and a $(n-2)$-dimensional smooth manifold Γ without boundary, we want to find a set $E \subset \mathbb{R}^n$ satisfying the boundary constraint

$$\partial E \cap \partial\Omega = \Gamma,$$

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and minimizing

$$\mathrm{Area}(\partial E \cap \Omega),$$

among all sets $E' \subset \mathbb{R}^n$ such that $\partial E' \cap \partial\Omega = \Gamma$.

Note that it is very difficult to give a precise sense to the intersection $\partial E \cap \partial\Omega$ when E has a rough boundary. In order to avoid unnecessary technical complications related to this issue, we argue as follows. For simplicity, we shall assume from now on that Ω is equal to the unit ball B_1, but of course this discussion can be easily extended to the general case.

Fix a smooth n-dimensional set $F \subset \mathbb{R}^n$ such that $\partial F \cap \partial B_1 = \Gamma$. Instead of prescribing the boundary of our set E on ∂B_1, we will require it to coincide with F on the complement of B_1. That is, we study the equivalent minimization problem

$$\min \Big\{ \mathrm{Area}(\partial E \cap B_1) : E \setminus B_1 = F \setminus B_1 \Big\}. \tag{1.2}$$

Still, there is another issue. It may happen that a non-negligible part of ∂E is not inside B_1, but on the boundary of B_1 (see Figure 1). As a consequence, this part would either contribute or not contribute to $\mathrm{Area}(\partial E \cap B_1)$, depending on our understanding of B_1 as open or closed.

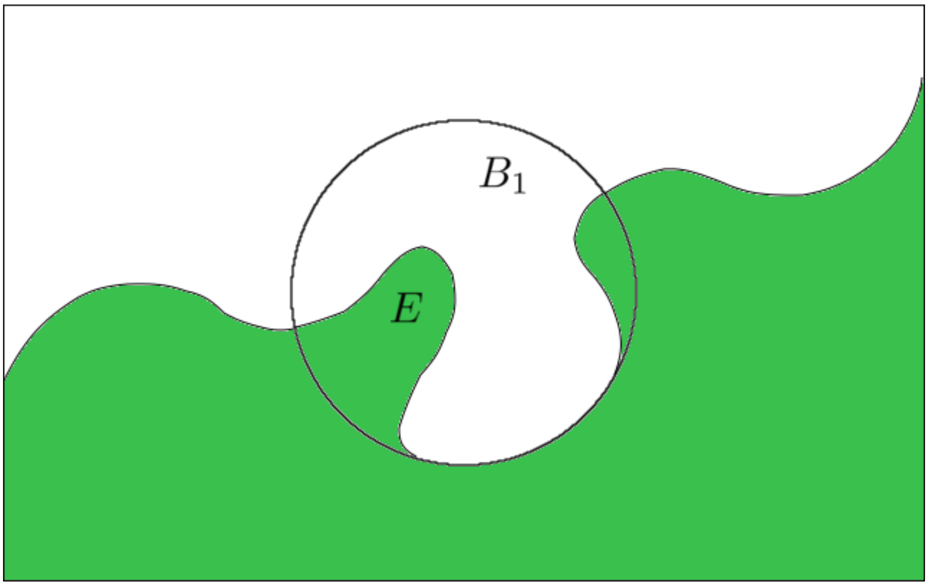

Figure 1. An example of a boundary ∂E that *sticks* to the sphere ∂B_1 on a non-negligible portion of it.

In order to overcome this ambiguity, we consider the slightly different minimization problem

$$\min \Big\{ \mathrm{Area}(\partial E \cap B_2) : E \setminus B_1 = F \setminus B_1 \Big\}. \tag{1.3}$$

Observe that, in contrast to (1.2), we are now minimizing the area inside the larger open ball B_2. In this way, we do not have anymore troubles with sticking boundaries. On the other hand, we prescribe the constraint outside of the smaller ball B_1. Hence, in (1.3) we are just adding terms which are the same for all competitors, namely the area of ∂F inside $B_2 \setminus B_1$. Notice that (1.3) is equivalent to

$$\min \Big\{ \mathrm{Area}(\partial E \cap \overline{B}_1) : E \setminus B_1 = F \setminus B_1 \Big\}.$$

However, as we shall see later, (1.3) is "analytically" better because the area inside an open set will be shown to be lower-semicontinuous under L^1_{loc}-convergence (see Proposition 2.3).

Now, the main question becomes: what is the area? For smooth boundaries, this is not an issue, since there is a classical notion of surface area. On the other hand, if in (1.3) we are only allowed to minimize among smooth sets, then it is not clear whether a minimizer exists in such class of sets. Actually, as we shall see later, minimizers are not necessarily smooth! Thus, we need a good definition of area for non-smooth sets.

2. Sets of finite perimeter

The main idea is the following: If E has smooth boundary, then it is not hard to verify that

$$\mathrm{Area}(\partial E) = \sup\left\{\int_{\partial E} X\cdot\nu_E : X\in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ |X|\leqslant 1\right\}, \tag{2.1}$$

where ν_E denotes the unit normal vector field of ∂E, pointing outward of E. Indeed, if ∂E is smooth, one can extend ν_E to a smooth vector field N defined on the whole $\mathbb{R}^n$ and satisfying $|N|\leqslant 1$. By setting $X=\eta_R N$, with η_R a cutoff function supported inside the ball B_R, and letting $R\to\infty$, one is easily led to (2.1).

Since, by the divergence theorem,

$$\int_{\partial E} X\cdot\nu_E = \int_E \mathrm{div}X,$$

we can rewrite (2.1) as

$$\mathrm{Area}(\partial E) = \sup\left\{\int_E \mathrm{div}X : X\in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ |X|\leqslant 1\right\}.$$

Notice that we do not need any regularity assumption on ∂E for the right-hand side of the formula above to be well-defined. Hence, one can use the right-hand side as the *definition* of perimeter for a non-smooth set.

More generally, given any open set $\Omega\subseteq\mathbb{R}^n$, the same considerations as above show that

$$\mathrm{Area}(\partial E\cap\Omega) = \sup\left\{\int_E \mathrm{div}X : X\in C^1_c(\Omega;\mathbb{R}^n),\ |X|\leqslant 1\right\}. \tag{2.2}$$

Again, this fact holds true when E has smooth boundary. Conversely, for a general set E, we can use (2.2) as a definition.

Definition 2.1. *Let $\Omega\subseteq\mathbb{R}^n$ be open and $E\subset\mathbb{R}^n$ be a Borel set. The* perimeter *of E inside Ω is given by*

$$\mathrm{Per}(E;\Omega) := \sup\left\{\int_E \mathrm{div}X : X\in C^1_c(\Omega;\mathbb{R}^n),\ |X|\leqslant 1\right\}.$$

When $\Omega=\mathbb{R}^n$, we write simply $\mathrm{Per}(E)$ *to indicate* $\mathrm{Per}(E;\mathbb{R}^n)$.

Note that $\mathrm{Per}(E;\Omega)$ is well-defined for any Borel set, but it might be infinite. For this reason, we will restrict ourselves to a smaller class of sets.

Definition 2.2. *Let $\Omega\subseteq\mathbb{R}^n$ be open and $E\subset\mathbb{R}^n$ be a Borel set. The set E is said to have* finite perimeter *inside Ω if $\mathrm{Per}(E;\Omega)<+\infty$. When $\Omega=\mathbb{R}^n$, we simply say that E has finite perimeter.*

With these definitions, the minimization problem becomes

$$\min \Big\{ \operatorname{Per}(E; B_2) : E \setminus B_1 = F \setminus B_1 \Big\}. \tag{2.3}$$

Of course, since F is a competitor and $\operatorname{Per}(F, B_2) < +\infty$, in the above minimization problem it is enough to consider only sets of finite perimeter.

In the remaining part of this section we examine two fundamental properties of the perimeter that will turn out to be crucial for the existence of minimizers: lower semicontinuity and compactness.

2.1. Lower semicontinuity.

In this subsection, we show that the perimeter is lower semicontinuous with respect to the L^1_{loc} topology. We recall that a sequence of measurable sets $\{E_k\}$ is said to converge in $L^1(\Omega)$ to a measurable set E if

$$\chi_{E_k} \longrightarrow \chi_E \text{ in } L^1(\Omega),$$

as $k \to +\infty$. Similarly, the convergence in L^1_{loc} is understood in the above sense.

The statement concerning the semicontinuity of Per is as follows.

Proposition 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\{E_k\}$ be a sequence of Borel sets, converging in $L^1_{\mathrm{loc}}(\Omega)$ to a set E. Then,*

$$\operatorname{Per}(E; \Omega) \leqslant \liminf_{k \to +\infty} \operatorname{Per}(E_k; \Omega).$$

Proof. Clearly, we can assume that each E_k has finite perimeter inside Ω. Fix any vector field $X \in C^1_c(\Omega; \mathbb{R}^n)$ such that $|X| \leqslant 1$. Then,

$$\int_E \operatorname{div} X = \int_\Omega \chi_E \operatorname{div} X = \lim_{k \to +\infty} \int_\Omega \chi_{E_k} \operatorname{div} X = \lim_{k \to +\infty} \int_{E_k} \operatorname{div} X.$$

Since by definition

$$\int_{E_k} \operatorname{div} X \leqslant \operatorname{Per}(E_k; \Omega) \qquad \text{for any } k \in \mathbb{N},$$

this yields

$$\int_E \operatorname{div} X \leqslant \liminf_{k \to +\infty} \operatorname{Per}(E_k; \Omega).$$

The conclusion follows by taking the supremum over all the admissible vector fields X on the left-hand side of the above inequality. □

Lower semicontinuity is the first fundamental property that one needs in order to prove the existence of minimal surfaces. However, alone it is not enough. We need another key ingredient.

2.2. Compactness.

Here we focus on a second important property enjoyed by the perimeter. We prove that a sequence of sets having perimeters uniformly bounded is precompact in the L^1_{loc} topology. That is, the next result holds true.

Proposition 2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\{E_k\}_{k \in \mathbb{N}}$ be a sequence of Borel subsets of Ω such that*

$$\operatorname{Per}(E_k; \Omega) \leqslant C \tag{2.4}$$

for some constant $C > 0$ independent of k. Then, up to a subsequence, E_k converges in $L^1_{\mathrm{loc}}(\Omega)$ to a Borel set $E \subseteq \Omega$.

The proof of the compactness result is more involved than that of the semicontinuity. We split it in several steps.

First, we recall the following version of the Poincaré's inequality. We denote by $(u)_A$ the integral mean of u over a set A with finite measure, that is

$$(u)_A := \fint_A u = \frac{1}{|A|}\int_A u.$$

Also, Q_r denotes a given (closed) cube of sides of length $r > 0$.

Lemma 2.5. *Let $r > 0$ and $u \in C^1(Q_r)$. Then,*

$$\int_{Q_r} |u - (u)_{Q_r}| \leqslant C_n r \int_{Q_r} |\nabla u|,$$

for some dimensional constant $C_n > 0$.

Proof. Up to a translation, we may assume that $Q_r = [0, r]^n$. Moreover, we initially suppose that $r = 1$.

We first prove the result with $n = 1$. In this case, note that for any $x, y \in [0, 1]$, we have

$$|u(x) - u(y)| \leqslant \int_x^y |\nabla u(z)|\, dz \leqslant \int_0^1 |\nabla u(z)| dz.$$

Choosing $y \in [0, 1]$ such that $u(y) = (u)_{[0,1]}$ (note that such a point exists thanks to the mean value theorem) and integrating the inequality above with respect to $x \in [0, 1]$, we conclude that

$$\int_0^1 |u - (u)_{[0,1]}| \leqslant \int_0^1 |\nabla u|,$$

which proves the result with $C_1 = 1$.

Now, assume by induction that the result is true up to dimension $n - 1$. Then, given a C^1 function $u : [0, 1]^n \to \mathbb{R}$, we can define the function $\bar{u} : [0, 1]^{n-1} \to \mathbb{R}$ given by

$$\bar{u}(x') = \int_0^1 u(x', x_n) dx_n.$$

With this definition, the 1-dimensional argument above applied to the family of functions $\{u(x', \cdot)\}_{x' \in [0,1]^{n-1}}$ shows that

$$\int_0^1 |u(x', x_n) - \bar{u}(x')| dx_n \leqslant \int_0^1 |\partial_n u(x', x_n)| dx_n \quad \text{for any } x' \in [0, 1]^{n-1}.$$

Hence, integrating with respect to x', we get

$$\int_{[0,1]^n} |u - \bar{u}| \leqslant \int_{[0,1]^n} |\partial_n u|. \tag{2.5}$$

We now observe that, by the inductive hypothesis,

$$\int_{[0,1]^{n-1}} |\bar{u} - (\bar{u})_{[0,1]^{n-1}}| \leqslant C_{n-1} \int_{[0,1]^{n-1}} |\nabla_{x'} \bar{u}|. \tag{2.6}$$

Noticing that

$$(\bar{u})_{[0,1]^{n-1}} = (u)_{[0,1]^n} \quad \text{and} \quad \int_{[0,1]^{n-1}} |\nabla_{x'} \bar{u}| \leqslant \int_{[0,1]^n} |\nabla_{x'} u|,$$

combining (2.5) and (2.6) we get

$$\int_{[0,1]^n} |u - (u)_{[0,1]^n}| \leqslant \int_{[0,1]^n} |\partial_n u| + C_{n-1} \int_{[0,1]^n} |\nabla_{x'} u|,$$

which proves the result with $C_n = 1 + C_{n-1}$.

Finally, the general case follows by a simple scaling argument. Indeed, if $u \in C^1(Q_r)$, then the rescaled function $u_r(x) := u(rx)$ belongs to $C^1(Q_1)$. Moreover, we have that

$$\int_{Q_r} |u - (u)_{Q_r}| = r^n \int_{Q_1} |u_r - (u_r)_{Q_1}|,$$

and

$$\int_{Q_r} |\nabla u| = r^{n-1} \int_{Q_1} |\nabla u_r|.$$

The conclusion then follows from the case $r = 1$ applied to u_r. □

We now plan to deduce a Poincaré-type inequality for the characteristic function χ_E of a bounded set E having finite perimeter. Of course, $\chi_E \notin C^1$ and Lemma 2.5 cannot be applied directly to it. Instead, we need to work with suitable approximations.

Given $r > 0$, consider a countable family of disjoint open cubes $\{Q^j\}$ of sides r such that $\cup_j \overline{Q^j} = \mathbb{R}^n$. We order this family so that

$$\begin{aligned} |Q^j \cap E| &\geqslant \frac{|Q^j|}{2} \text{ for any integer } j = 1, \ldots, N, \\ |Q^j \cap E| &< \frac{|Q^j|}{2} \text{ for any integer } j > N, \end{aligned} \tag{2.7}$$

for some uniquely determined $N \in \mathbb{N}$. Notice that such N exists since E is bounded. We then write

$$T_{E,r} := \bigcup_{j=1}^{N} Q^j, \tag{2.8}$$

see Figure 2.

Lemma 2.6. *Let $r > 0$ and $E \subset \mathbb{R}^n$ be a bounded set with finite perimeter. Then,*

$$\|\chi_E - \chi_{T_{E,r}}\|_{L^1(\mathbb{R}^n)} \leqslant C_n r \operatorname{Per}(E),$$

with C_n as in Lemma 2.5.

Proof. Consider a family $\{\rho_\varepsilon\}$ of radially symmetric smooth convolution kernels, and define $u_\varepsilon := \chi_E * \rho_\varepsilon$. Clearly, $u_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and $u_\varepsilon \to \chi_E$ in $L^1(\mathbb{R}^n)$, as $\varepsilon \to 0^+$. Furthermore, by considerations analogous to the ones at the beginning of Section 2, it is not hard to see that

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon| = \sup \left\{ - \int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot X : X \in C_c^1(\mathbb{R}^n; \mathbb{R}^n), \, |X| \leqslant 1 \right\}.$$

Integrating by parts and exploiting well-known properties of the convolution operator, we find that

$$\begin{aligned} - \int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot X &= \int_{\mathbb{R}^n} u_\varepsilon \operatorname{div} X = \int_{\mathbb{R}^n} (\chi_E * \rho_\varepsilon) \operatorname{div} X \\ &= \int_{\mathbb{R}^n} \chi_E (\rho_\varepsilon * \operatorname{div} X) = \int_E \operatorname{div}(X * \rho_\varepsilon). \end{aligned}$$

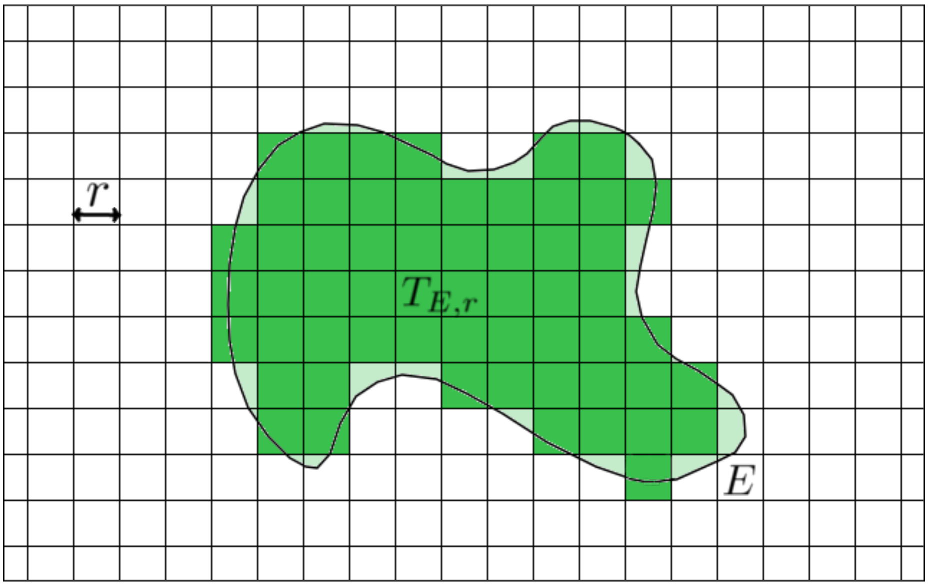

FIGURE 2. The grid made up of cubes of sides r, the set E (in light green) and the resulting set $T_{E,r}$ (in dark green).

Since $|X| \leqslant 1$, it follows that $|X * \rho_\varepsilon| \leqslant 1$. Therefore, by taking into account Definition 2.1, we obtain

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon| \leqslant \operatorname{Per}(E). \tag{2.9}$$

Recall now the partition (up to a set of measure zero) of $\mathbb{R}^n$ into the family of cubes $\{Q^j\}$ introduced earlier. Applying the Poincaré's inequality of Lemma 2.5 to u_ε in each cube Q^j, we get

$$C_n r \int_{\mathbb{R}^n} |\nabla u_\varepsilon| = C_n r \sum_{j\in\mathbb{N}} \int_{Q^j} |\nabla u_\varepsilon| \geqslant \sum_{j\in\mathbb{N}} \int_{Q^j} |u_\varepsilon - (u_\varepsilon)_{Q^j}|. \tag{2.10}$$

On the other hand, for any $j \in \mathbb{N}$,

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \int_{Q^j} |u_\varepsilon - (u_\varepsilon)_{Q^j}| &= \int_{Q^j} |\chi_E - (\chi_E)_{Q^j}| = \int_{Q^j} \left|\chi_E - \frac{|Q^j \cap E|}{|Q^j|}\right| \\
&= |Q^j \cap E| \frac{|Q^j| - |Q^j \cap E|}{|Q^j|} + |Q^j \setminus E| \frac{|Q^j \cap E|}{|Q^j|} \\
&= 2\frac{|Q^j \cap E||Q^j \setminus E|}{|Q^j|}.
\end{aligned}$$

Using this in combination with (2.9) and (2.10), we obtain that

$$C_n r \operatorname{Per}(E) \geqslant 2 \sum_{j\in\mathbb{N}} \frac{|Q^j \cap E||Q^j \setminus E|}{|Q^j|}.$$

But then, recalling (2.7) and (2.8), we conclude that

$$\begin{aligned} C_n r \operatorname{Per}(E) &\geqslant \sum_{j=1}^{N} \left(2 \frac{|Q^j \cap E|}{|Q^j|} \right) |Q^j \setminus E| + \sum_{j=N+1}^{+\infty} \left(2 \frac{|Q^j \setminus E|}{|Q^j|} \right) |Q^j \cap E| \\ &\geqslant \sum_{j=1}^{N} |Q^j \setminus E| + \sum_{j=N+1}^{+\infty} |Q^j \cap E| = |T_{E,r} \setminus E| + |E \setminus T_{E,r}| \\ &= \|\chi_E - \chi_{T_{E,r}}\|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

which concludes the proof. □

By virtue of Lemma 2.6, we see that $T_{E,r}$ converges to E in $L^1(\mathbb{R}^n)$, as r goes to 0, with a rate that is controlled by $\operatorname{Per}(E)$. Knowing this fact, we are now in position to deal with the proof of Proposition 2.4. The main step is represented by the next:

Lemma 2.7. *Let $C, R > 0$ be fixed. Let $\{E_k\}$ be a sequence of sets such that*

$$E_k \subseteq B_R, \tag{2.11}$$

and

$$\operatorname{Per}(E_k) \leqslant C,$$

for any $k \in \mathbb{N}$. Then, up to a subsequence, E_k converges in $L^1(\mathbb{R}^n)$ to a set E.

Notice that this result is slightly weaker than the one claimed by Proposition 2.4 (with $\Omega = \mathbb{R}^n$), since the E_k's are supposed to be uniformly bounded sets.

Proof of Lemma 2.7. Consider the following class of sets

$$\mathcal{X}_{R,C} := \Big\{ F \subseteq B_R : F \text{ is Borel}, \operatorname{Per}(F) \leqslant C \Big\},$$

and endow it with the metric defined by

$$d(E,F) := \|\chi_E - \chi_F\|_{L^1(\mathbb{R}^n)}, \qquad \text{for any } E, F \in \mathcal{X}_{R,C}.$$

Observe that the lemma will be proved if we show that the metric space $(\mathcal{X}_{R,C}, d)$ is compact.

We first claim that

$$(\mathcal{X}_{R,C}, d) \text{ is complete.} \tag{2.12}$$

Note that $(\mathcal{X}_{R,C}, d)$ may be seen as a subspace of $L^1(\mathbb{R}^n)$, via the identification of a set E with its characteristic function χ_E. Therefore, it suffices to prove that $\mathcal{X}$ is closed in $L^1(\mathbb{R}^n)$. To see this, let $\{F_k\} \subset \mathcal{X}_{R,C}$ be a sequence such that χ_{F_k} converges to some function f in $L^1(\mathbb{R}^n)$. Clearly $f = \chi_F$ for some set $F \subseteq B_R$, since a subsequence of $\{\chi_{F_k}\}$ converges to f a.e. in $\mathbb{R}^n$. In addition, Proposition 2.3 implies that

$$\operatorname{Per}(F) \leqslant \liminf_{k \to +\infty} \operatorname{Per}(F_k) \leqslant C.$$

This proves that $F \in \mathcal{X}_{R,C}$, hence $\mathcal{X}_{R,C}$ is closed in $L^1(\mathbb{R}^n)$ and (2.12) follows.

We now claim that

$$(\mathcal{X}_{R,C}, d) \text{ is totally bounded.} \tag{2.13}$$

To check (2.13), we need to show the existence of a finite ε-net. That is, for any $\varepsilon > 0$, we need to find a finite number of sets $F_1, \dots, F_{N_\varepsilon}$, for some $N_\varepsilon \in \mathbb{N}$, such that, for any $F \in \mathcal{X}_{R,C}$,

$$d(F, F_i) < \varepsilon, \text{ for some } i \in \{1, \dots, N_\varepsilon\}.$$

Fix $\varepsilon > 0$ and set

$$r_\varepsilon := \frac{\varepsilon}{2CC_n},$$

with C_n as in Lemma 2.5. Given any $F \in \mathcal{X}_{R,C}$, we consider the set T_{F,r_ε} introduced in (2.8). By Lemma 2.6, we have that

$$d(F, T_{F,r_\varepsilon}) = \|\chi_F - \chi_{T_{F,r_\varepsilon}}\|_{L^1(\mathbb{R}^n)} \leqslant C_n r_\varepsilon \operatorname{Per}(F) \leqslant CC_n r_\varepsilon < \varepsilon.$$

Since the cardinality of

$$\Big\{T_{F,r_\varepsilon} : F \in \mathcal{X}_{\mathcal{R},\mathcal{C}}\Big\},$$

is finite (as a quick inspection of definition (2.8) reveals), we have found the desired ε-net and (2.13) follows.

In view of (2.12) and (2.13), we know that $(\mathcal{X}_{R,C}, d)$ is closed and totally bounded. It is a standard fact in topology that this is in turn equivalent to the compactness of $(\mathcal{X}_{R,C}, d)$. Hence, Lemma 2.7 holds true. □

With the help of Lemma 2.7, we can now conclude this subsection by proving the validity of our compactness statement in its full generality.

Proof of Proposition 2.4. We plan to obtain the result combining Lemma 2.7 with a suitable diagonal argument. To do this, consider first $\{\Omega_\ell\}$ an exhaustion of Ω made of open bounded sets with smooth boundaries, so that, in particular, the perimeter of each set Ω_ℓ is finite. Moreover, we may assume without loss of generality that $\Omega_\ell \subset B_\ell$.

For any $\ell \in \mathbb{N}$, we define

$$E_k^\ell := E_k \cap \Omega_\ell.$$

For any fixed ℓ, it holds $E_k^\ell \subseteq \Omega_\ell \subset B_\ell$ for any $k \in \mathbb{N}$. In particular, E_k^ℓ satisfies (2.11) with $R = \ell$. Moreover, using (2.4), it is not hard to check that

$$\operatorname{Per}(E_k^\ell) \leqslant \operatorname{Per}(E_k; \Omega_\ell) + \operatorname{Per}(\Omega_\ell) \leqslant \operatorname{Per}(E_k; \Omega) + \operatorname{Per}(\Omega_\ell) \leqslant C_\ell,$$

for some constant $C_\ell > 0$ independent of k.

In light of these facts, the sequence $\{E_k^\ell\}_{k\in\mathbb{N}}$ satisfies the hypotheses of Lemma 2.7. Hence, we infer that, for any fixed ℓ, there exists a diverging sequence $\mathcal{K}^\ell = \{\varphi^\ell(j)\}_{j\in\mathbb{N}}$ of natural numbers such that $E^\ell_{\varphi^\ell(j)}$ converges in $L^1(\mathbb{R}^n)$ to a set $E^\ell \subseteq \Omega_\ell$, as $j \to +\infty$. By a diagonal argument we can suppose that $\mathcal{K}^m \subseteq \mathcal{K}^\ell$ if $\ell \leqslant m$. Furthermore, it is easy to see that $E^m \cap \Omega_\ell = E^\ell$, if $\ell \leqslant m$. We then define

$$E := \bigcup_{\ell\in\mathbb{N}} E^\ell,$$

and notice that $E \cap \Omega_\ell = E^\ell$ for any ℓ. Set $k_\ell := \varphi^\ell(\ell)$, for any $\ell \in \mathbb{N}$. Clearly, $\{k_\ell\}$ is a subsequence of each $\mathcal{K}^m$, up to a finite number of indices ℓ. Hence, for any fixed $m \in \mathbb{N}$, we have

$$\begin{aligned}\lim_{\ell\to+\infty} \|\chi_{E_{k_\ell}} - \chi_E\|_{L^1(\Omega_m)} &= \lim_{\ell\to+\infty} \|\chi_{E^m_{k_\ell}} - \chi_{E^m}\|_{L^1(\mathbb{R}^n)} \\ &= \lim_{j\to+\infty} \|\chi_{E^m_{\varphi^m(j)}} - \chi_{E^m}\|_{L^1(\mathbb{R}^n)} \\ &= 0.\end{aligned}$$

This proves that $E_{k_\ell} \to E$ in $L^1_{\mathrm{loc}}(\Omega)$ as $\ell \to +\infty$, completing the proof. □

3. Existence of minimal surfaces

With the help of the lower semicontinuity of the perimeter and the compactness property established in the previous section, we can now easily prove the existence of a solution to the minimization problem (2.3).

Theorem 3.1. *Let F be a set with finite perimeter inside B_2. Then, there exists a set E of finite perimeter inside B_2 such that $E \setminus B_1 = F \setminus B_1$ and*

$$\operatorname{Per}(E; B_2) \leqslant \operatorname{Per}(E'; B_2)$$

for any set E' such that $E' \setminus B_1 = F \setminus B_1$.

Proof. Our argument is based on the direct method of the calculus of variations. Set

$$\alpha := \inf \Big\{ P(E'; B_2) : E' \setminus B_1 = F \setminus B_1 \Big\}. \tag{3.1}$$

Note that α is finite since $\alpha \leqslant P(F; B_2)$.

Take a sequence $\{E_k\}$ of sets of finite perimeter such that $E_k \setminus B_1 = F \setminus B_1$ for any $k \in \mathbb{N}$ and

$$\lim_{k \to +\infty} \operatorname{Per}(E_k; B_2) = \alpha.$$

Clearly, we can assume without loss of generality that

$$\operatorname{Per}(E_k; B_2) \leqslant \alpha + 1 \quad \text{ for any } k \in \mathbb{N}.$$

Therefore, by Proposition 2.4, we conclude that there exists a subsequence E_{k_j} converging to a set E in $L^1_{\mathrm{loc}}(B_2)$, as $j \to +\infty$. Consequently, Proposition 2.3 yields

$$\operatorname{Per}(E; B_2) \leqslant \lim_{j \to +\infty} \operatorname{Per}(E_{k_j}; B_2) = \alpha.$$

Since $E \setminus B_1 = \lim_{j \to +\infty} E_{k_j} \setminus B_1 = F \setminus B_1$, the set E is admissible in (3.1) and we conclude that

$$\operatorname{Per}(E; B_2) = \alpha.$$

The set E is thus the desired minimizer. □

In the following sections, our goal will be to show that the minimizers just obtained are more than just sets with finite perimeter. That is, we will develop an appropriate regularity theory for minimal surfaces. However, to do that, we first need to describe some important facts about sets of finite perimeter.

4. Fine properties of sets of finite perimeter

In this section, we introduce a different concept of boundary for sets of finite perimeter: the *reduced boundary*. As we shall see, up to a "small" component, this new boundary is always contained in a collection of $(n-1)$-dimensional hypersurfaces of class C^1. Moreover, through this definition, one can compute the perimeter of a set in a more direct way via the Hausdorff measure.

We begin by recalling the definition of Hausdorff measure.

4.1. **Hausdorff measure.** The aim is to define a σ-dimensional surface measure for general non-smooth subsets of the space $\mathbb{R}^n$.

Fix $\sigma \geqslant 0$ and $\delta > 0$. Given a set E, we cover it with a countable family of sets $\{E_k\}$ having diameter smaller or equal than δ. Then, the quantity

$$\sum_{k\in\mathbb{N}} (\mathrm{diam}(E_k))^\sigma$$

represents more or less a notion of σ-dimensional measure of E, provided we take δ sufficiently small. Of course, if δ is not chosen small enough, we might lose the geometry of the set E (see Figure 3).

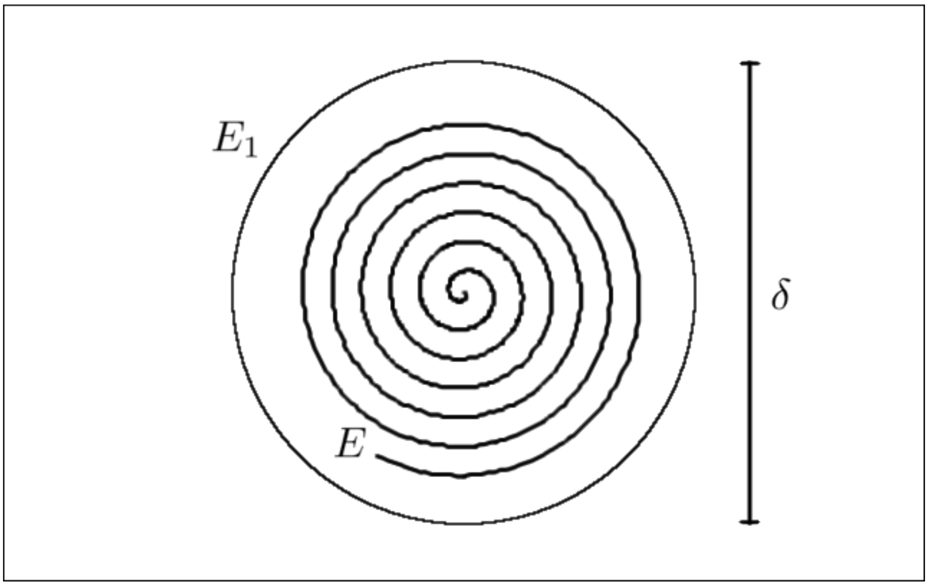

FIGURE 3. The spiral-like set E is covered by the ball $E_1 = B_{\delta/2}$. If δ is comparable to the diameter of E, the covering consisting only of the set E_1 cannot capture the geometry of E.

We give the following definition.

Definition 4.1. *Let $\sigma \geqslant 0$ and $\delta > 0$. Given any $E \subseteq \mathbb{R}^n$, we set*

$$\mathcal{H}^\sigma_\delta(E) := \inf\left\{\omega_\sigma \sum_{k\in\mathbb{N}} \left(\frac{\mathrm{diam}(E_k)}{2}\right)^\sigma : E \subseteq \bigcup_{k\in\mathbb{N}} E_k,\ \mathrm{diam}(E_k) \leqslant \delta\right\},$$

where

$$\omega_\sigma := \frac{\pi^{\frac{\sigma}{2}}}{\Gamma(\frac{\sigma}{2}+1)},$$

*and Γ is Euler's Gamma function. Then, we define the s-*dimensional Hausdorff measure of *E by*

$$\mathcal{H}^\sigma(E) := \lim_{\delta\to 0^+} \mathcal{H}^\sigma_\delta(E).$$

The factor ω_σ is a normalization constant that makes the Hausdorff measure consistent with the standard Lebesgue measure of $\mathbb{R}^n$. In particular, ω_n is precisely the volume of the n-dimensional unit ball.

It is immediate to check that the limit defining the Hausdorff measure $\mathcal{H}^\sigma$ always exists. Indeed, since $\mathcal{H}^\sigma_\delta$ is non-increasing in δ,

$$\mathcal{H}^\sigma(E) = \sup_{\delta>0} \mathcal{H}^\sigma_\delta(E).$$

Finally, it can be proved that, when $k \geqslant 0$ is an integer, $\mathcal{H}^k$ coincides with the classical k-dimensional measure on smooth k-dimensional surfaces of $\mathbb{R}^n$ (see for instance [20, Section 3.3.2 and 3.3.4.C] or [25, Chapter 11]).

4.2. **De Giorgi's rectifiability theorem.** Having recalled the definition of Hausdorff measure, we may now present the main result of this section, referring to [25, Chapter 15] for a proof.

Theorem 4.2 (De Giorgi's rectifiability theorem)**.** *Let E be a set of finite perimeter. Then, there exists a set $\partial^* E \subseteq \partial E$, such that:*

(i) we have

$$\partial^* E \subseteq \bigcup_{i \in \mathbb{N}} \Sigma_i \cup N,$$

for a countable collection $\{\Sigma_i\}$ of $(n-1)$-dimensional C^1 hypersurfaces and some set N with $\mathcal{H}^{n-1}(N) = 0$;

(ii) for any open set A, it holds

$$\operatorname{Per}(E; A) = \mathcal{H}^{n-1}(\partial^* E \cap A).$$

Notice that, thanks to (ii), we have now an easier way to compute the perimeter of any set.

The object $\partial^* E$ introduced in the above theorem is usually called *reduced boundary*. Typically, it differs from the usual topological boundary, which may be very rough for general Borel sets.

Example 4.3. Let $\{x_k\}$ be a sequence of points dense in $\mathbb{R}^n$. For $N \in \mathbb{N}$, define

$$E_N := \bigcup_{k=1}^{N} B_{2^{-k}}(x_k).$$

Then,

$$\operatorname{Per}(E_N) \leqslant \sum_{k=1}^{N} \operatorname{Per}(B_{2^{-k}}) = c_n \sum_{k=1}^{N} 2^{-k(n-1)} \leqslant \tilde{c}_n,$$

for some dimensional constants $c_n, \tilde{c}_n > 0$. Since

$$E_N \to E_\infty := \bigcup_{k=1}^{+\infty} B_{2^{-k}}(x_k) \qquad \text{in } L^1(\mathbb{R}^n),$$

Proposition 2.3 implies that

$$\operatorname{Per}(E_\infty) \leqslant \tilde{c}_n,$$

that is E_∞ is a set of finite perimeter. On the other hand, the topological boundary of E_∞ is very large: indeed, while

$$|E_\infty| \leqslant \sum_{k=1}^{+\infty} |B_{2^{-k}}| < +\infty,$$

since E_∞ is dense in $\mathbb{R}^n$ we have $\bar{E}_\infty = \mathbb{R}^n$, thus $|\partial E_\infty| = +\infty$. Also, although it does not follow immediately from the definition, it is possible to prove that

$$\partial^* E_\infty \subseteq \bigcup_{N=1}^{+\infty} \partial^* E_N \subseteq \bigcup_{k=1}^{+\infty} \partial B_{2^{-k}}(x_k).$$

This example shows that the topological boundary may be a very *bad* notion in the context of perimeters.

Luckily, this is not always the case for minimizers of the perimeter. In fact, we will shortly prove partial regularity results (i.e., smoothness outside a lower dimensional set) for the topological boundary of minimizers of the perimeter.

5. Regularity of minimal graphs

After the brief parenthesis of Section 4, we now focus on the regularity properties enjoyed by the minimizers of problem (2.3), whose existence has been established in Theorem 3.1.

We first restrict ourselves to minimal surfaces which can be written as graphs with respect to one fixed direction.

Consider the cylinder

$$C_1 := B_1^{n-1} \times \mathbb{R},$$

with

$$B_1^{n-1} := \Big\{ (x', 0) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 1 \Big\}.$$

Given $g : \partial B_1^{n-1} \to \mathbb{R}$, we denote by $\Gamma \subset \partial C_1$ the graph of g.

The following result shows that minimizing the area among graphs is the same as minimizing the area among all sets.

Lemma 5.1. *Let* $\Sigma = \mathrm{graph}(u)$ *for some* $u : \bar{B}_1^{n-1} \to \mathbb{R}$ *such that* $u = g$ *on* ∂B_1^{n-1}. *Then,* Σ *is a minimal surface if and only if it satisfies*

$$\mathcal{H}^{n-1}(\Sigma) \leqslant \mathcal{H}^{n-1}(\mathrm{graph}(v)),$$

for any $v : \bar{B}_1^{n-1} \to \mathbb{R}$ *such that* $v = g$ *on* ∂B_1^{n-1}.

Sketch of the proof. Clearly, we just need to show that if Σ is a minimizer among graphs, then it also solves problem (1.1).

Let K be a convex set and denote with $\pi_K : \mathbb{R}^n \to \mathbb{R}^n$ the projection from $\mathbb{R}^n$ onto K. It is well-known that π_K is 1-Lipschitz (see for instance [21, Lemma A.3.8]). Hence, since distances (and therefore also areas) decrease under 1-Lipschitz maps,

$$\mathcal{H}^{n-1}(\pi_K(\Sigma')) \leqslant \mathcal{H}^{n-1}(\Sigma')$$

(see for instance [21, Lemma A.7] applied with $L = 1$). By applying this with $K = C_1$, it follows that we can restrict ourselves to consider only competitors Σ' which are contained in C_1.

We now show that the area decreases under vertical rearrangements. To explain this concept, we describe it in a simple example. So, we suppose for simplicity that $n = 2$ and Σ' is as in Figure 4, so that

$$\Sigma' \cap C_1 = \mathrm{graph}(f_1) \cup \mathrm{graph}(f_2) \cup \mathrm{graph}(f_3),$$

for some smooth functions $f_i : [-1, 1] \to \mathbb{R}$, $i = 1, 2, 3$. Then it holds.

$$\mathcal{H}^1(\Sigma') = \sum_{i=1}^{3} \int_{-1}^{1} \sqrt{1 + (f_i')^2}.$$

Consider now the function $h := f_1 - f_2 + f_3$. Note that h is geometrically obtained as follows: given $x \in [-1, 1]$, consider the vertical segment $I_x := \{x\} \times [f_2(x), f_3(x)]$ and shift it vertically unit it touches $\{x\} \times (-\infty, f_1(x)]$. Then the set constructed in this way coincides with the epigraph of h.

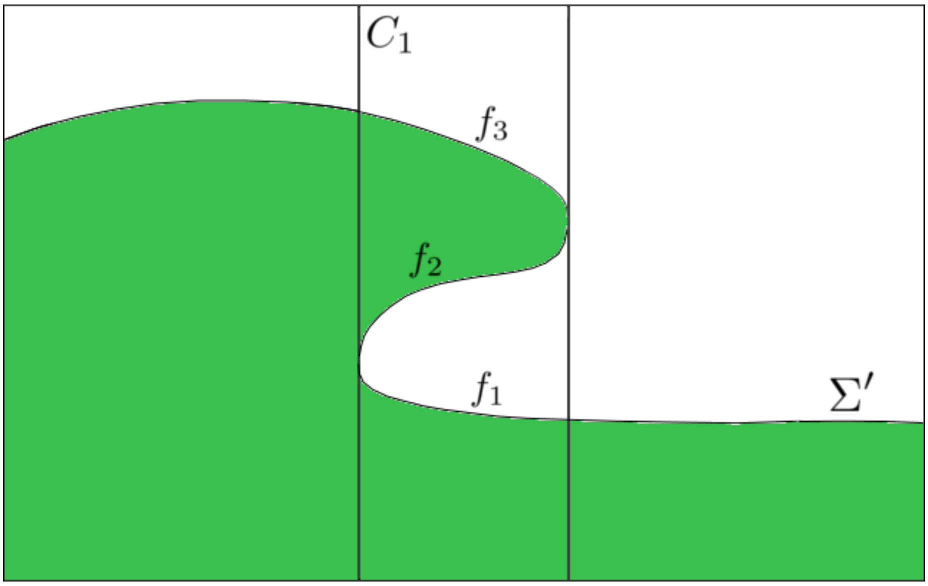

FIGURE 4. The curve Σ', given by the union of the graphs of f_1, f_2 and f_3.

We note that, thanks to the numerical inequality

$$\sqrt{1+(a+b+c)^2} \leqslant \sqrt{1+a^2}+\sqrt{1+b^2}+\sqrt{1+c^2} \quad \text{for any } a,b,c>0,$$

it follows that

$$\begin{aligned}\mathcal{H}^1(\mathrm{graph}(h)\cap C_1) &= \int_{-1}^{1} \sqrt{1+(f_1'-f_2'+f_3')^2} \\ &\leqslant \int_{-1}^{1} \sqrt{1+(|f_1'|+|f_2'|+|f_3'|)^2} \\ &\leqslant \sum_{i=1}^{3}\int_{-1}^{1} \sqrt{1+(f_i')^2} \\ &= \mathcal{H}^1(\Sigma').\end{aligned}$$

In other words, the area decreases under vertical rearrangement.

We note that this procedure can be generalized to arbitrary dimension and to any set $E \subset C_1$, allowing us to construct a function $h_E : B_1^{n-1} \to \mathbb{R}$ whose epigraph has boundary with less area than ∂E. However, to make this argument rigorous one should notice that the function h_E may jump at some points (see Figure 5). Hence, one needs to introduce the concept of BV functions and discuss the area of the graph of such a function. Since this would be rather long and technical, we refer the interested reader to [24, Chapters 14-16]. □

In view of the above result, we may limit ourselves to minimize area among graphs, i.e., we may restrict to the problem

$$\min\left\{\int_{B_1^{n-1}} \sqrt{1+|\nabla u|^2} : u = g \text{ on } \partial B_1^{n-1}\right\}. \tag{5.1}$$

Note that the existence of a solution to such problem is not trivial, as the functional has linear growth at infinity, which may determine a lack of compactness since the Sobolev space $W^{1,1}$ is not weakly compact. We shall not discuss the existence problem here and we refer to [24] for more details.

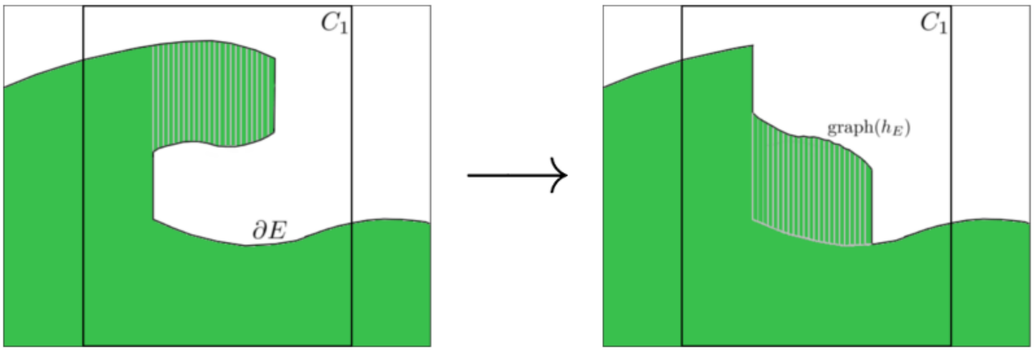

FIGURE 5. On the left is the original set E, while on the right is its vertical rearrangement, given as the epigraph of the function h_E. In gray are depicted the segments I_x (on the left), and their vertical translations (on the right). As it is clear from the picture, the graph of h_E may have jumps.

The following comparison principle is easily established.

Lemma 5.2. *Suppose that g is bounded. Then, the solution u to the minimizing problem (5.1) is bounded as well, and it holds*

$$\|u\|_{L^\infty(B_1^{n-1})} \leqslant \|g\|_{L^\infty(\partial B_1^{n-1})}.$$

Proof. Let $M := \|g\|_{L^\infty(\partial B_1^{n-1})}$. Then, $u_M := (u \wedge M) \vee -M = g$ on ∂B_1^{n-1}. Also, since $1 = \sqrt{1 + |\nabla u_M|} \leqslant \sqrt{1 + |\nabla u|}$ inside $\{|u| \geqslant M\}$,

$$\begin{aligned} \mathcal{H}^{n-1}(\mathrm{graph}(u_M)) &= \int_{\{-M<u<M\}} \sqrt{1 + |\nabla u|^2} + |\{|u| \geqslant M\}| \\ &\leqslant \int_{B_1^{n-1}} \sqrt{1 + |\nabla u|^2} \\ &= \mathcal{H}^{n-1}(\mathrm{graph}(u)). \end{aligned}$$

By the minimality of u, it follows that the above inequality is in fact an identity. Hence, $|u| \leqslant M$. □

Starting from this, the regularity theory for minimal graphs can be briefly described as follows. First of all, the well-known gradient estimate of Bombieri, De Giorgi, and Miranda [6] ensures that minimizers are locally Lipschitz functions.

Theorem 5.3. *Let u be a bounded solution to the minimizing problem (5.1). Then, u is locally Lipschitz inside B_1^{n-1}.*

Knowing that u is locally Lipschitz, we may differentiate the area functional to infer more information on the smoothness of u. Fix $\varphi \in C_c^\infty(B_1^{n-1})$. By the minimality of u, we have that

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{B_1^{n-1}} \sqrt{1 + |\nabla u + \varepsilon \nabla \varphi|^2}.$$

From this, we deduce that

$$\int_{B_1^{n-1}} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla \varphi = 0 \ \text{ for any } \varphi \in C_c^\infty(B_1^{n-1}),$$

which is the weak formulation of the Euler-Lagrange equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{5.2}$$

Write now $F(p) := \sqrt{1+|p|^2}$ for any $p \in \mathbb{R}^{n-1}$. Since $DF(q) = q/\sqrt{1+|q|^2}$, we see that (5.2) may be read as

$$\operatorname{div}\left(DF(\nabla u)\right) = 0.$$

By differentiating this equation with respect to the direction e_ℓ, we get[1]

$$\operatorname{div}\left(D^2F(\nabla u) \cdot \nabla(\partial_\ell u)\right) = 0,$$

for any $\ell = 1, \dots n-1$. Setting now $A(x) := D^2F(\nabla u(x))$ and $v := \partial_\ell u$, the above equation becomes

$$\operatorname{div}\left(A(x)\nabla v\right) = 0.$$

Note that, because u is locally Lipschitz, given any ball $B_r(x) \subset B_1^{n-1}$, there exists a constant $L_{x,r}$ such that $|\nabla u| \leqslant L_{x,r}$ inside $B_r(x)$. Hence, since

$$0 < \lambda_{x,r}\mathrm{Id}_{n-1} \leqslant D^2F(q) \leqslant \Lambda_{x,r}\mathrm{Id}_{n-1} \quad \text{for any } |q| \leqslant L_{x,r},$$

we deduce that

$$\lambda_{x,r}\mathrm{Id}_{n-1} \leqslant A(y) = D^2F(\nabla u(y)) \leqslant \Lambda_{x,r}\mathrm{Id}_{n-1} \quad \text{for any } y \in B_r(x).$$

This proves that A is measurable and uniformly elliptic, therefore we may apply the De Giorgi-Nash-Moser theory [13, 27, 26] and conclude that $\partial_\ell u = v \in C^{0,\alpha}_{\mathrm{loc}}$, for some $\alpha \in (0,1)$. Hence, $u \in C^{1,\alpha}_{\mathrm{loc}}$ and consequently $A = D^2F(\nabla u) \in C^{0,\alpha}_{\mathrm{loc}}$. Then, by Schauder theory (see e.g. [23]), we get that $v \in C^{1,\alpha}_{\mathrm{loc}}$, i.e. $u \in C^{2,\alpha}_{\mathrm{loc}}$. Accordingly, $A \in C^{1,\alpha}_{\mathrm{loc}}$ and we can keep iterating this procedure to show that u is of class C^∞. Actually, by elliptic regularity, one can even prove that u is analytic. Hence, we obtain the following result.

Theorem 5.4. *Let $u : \bar{B}_1^{n-1} \to \mathbb{R}$ be a bounded solution to problem (5.1). Then u is analytic inside B_1^{n-1}.*

We have therefore proved that minimal graphs are smooth. This is no longer true for general minimal sets, as we will see in the next section.

6. Regularity of general minimal surfaces

We deal here with the regularity of minimal sets which are not necessarily graphs.

Let E be a minimal surface. By De Giorgi's rectifiability theorem (Theorem 4.2), we have the tools to work as if ∂E were already smooth (of course, there are technicalities involved, but the philosophy is the same). Thus, for simplicity we shall make computations are if ∂E were smooth, and we will prove estimates that are independent of the smoothness of ∂E.

6.1. Density estimates.

In this subsection we show that, nearby boundary points, minimal sets occupy *fat* portions of the space, at any scale. That is, we rule out the behavior displayed in Figure 6.

[1] Of course, to make this rigorous one should first check that $u \in W^{2,2}$. This can be done in a standard way, starting from equation (5.2) and exploiting the Lipschitz character of u to prove a Caccioppoli inequality on the incremental quotients of ∇u.

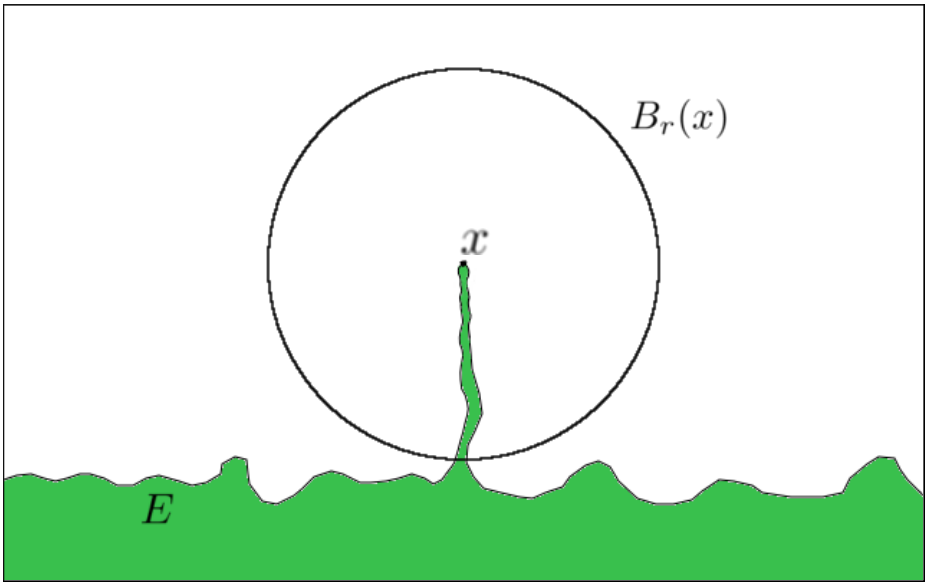

FIGURE 6. An example of a set E which cannot be area minimizing. In fact, the measure of $E \cap B_r(x)$ is too small.

Lemma 6.1. *There exists a dimensional constant $c_\star > 0$ such that*

$$|B_r(x) \cap E| \geqslant c_\star r^n \quad \text{and} \quad |B_r(x) \setminus E| \geqslant c_\star r^n, \tag{6.1}$$

for any $x \in \partial E$ and any $r > 0$.

Proof. First, recall the isoperimetric inequality: there is a dimensional constant $c_n > 0$ such that

$$c_n \operatorname{Per}(F) \geqslant |F|^{\frac{n-1}{n}}, \tag{6.2}$$

for any bounded set $F \subset \mathbb{R}^n$. One can show (6.2) via Sobolev inequality. Indeed, let $\{\varphi_\varepsilon\}$ be a family of smooth convolution kernels and apply e.g. [19, Section 5.6.1, Theorem 1] to the function $\chi_F * \rho_\varepsilon$, for any $\varepsilon > 0$. Recalling also (2.9), we get

$$\|\chi_F * \rho_\varepsilon\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leqslant c_n \|\nabla(\chi_F * \rho_\varepsilon)\|_{L^1(\mathbb{R}^n)} \leqslant c_n \operatorname{Per}(F).$$

Inequality (6.2) follows by letting $\varepsilon \to 0^+$.

Let $V(r) := |B_r(x) \cap E|$. By the minimality of E, we have that

$$\mathcal{H}^{n-1}(B_r(x) \cap \partial E) \leqslant \mathcal{H}^{n-1}(\partial B_r(x) \cap E).$$

Therefore, by this and (6.2), we obtain

$$\begin{aligned} V(r)^{\frac{n-1}{n}} &\leqslant c_n \left[\mathcal{H}^{n-1}(B_r(x) \cap \partial E) + \mathcal{H}^{n-1}(\partial B_r(x) \cap E)\right] \\ &\leqslant 2c_n \mathcal{H}^{n-1}(\partial B_r(x) \cap E). \end{aligned} \tag{6.3}$$

Using polar coordinates, we write

$$V(r) = \int_0^r \mathcal{H}^{n-1}(\partial B_s(x) \cap E)\, ds.$$

Accordingly,

$$V'(r) = \mathcal{H}^{n-1}(\partial B_r(x) \cap E),$$

and hence, by (6.3), we are led to the differential inequality

$$V(r)^{\frac{n-1}{n}} \leqslant 2c_n V'(r).$$

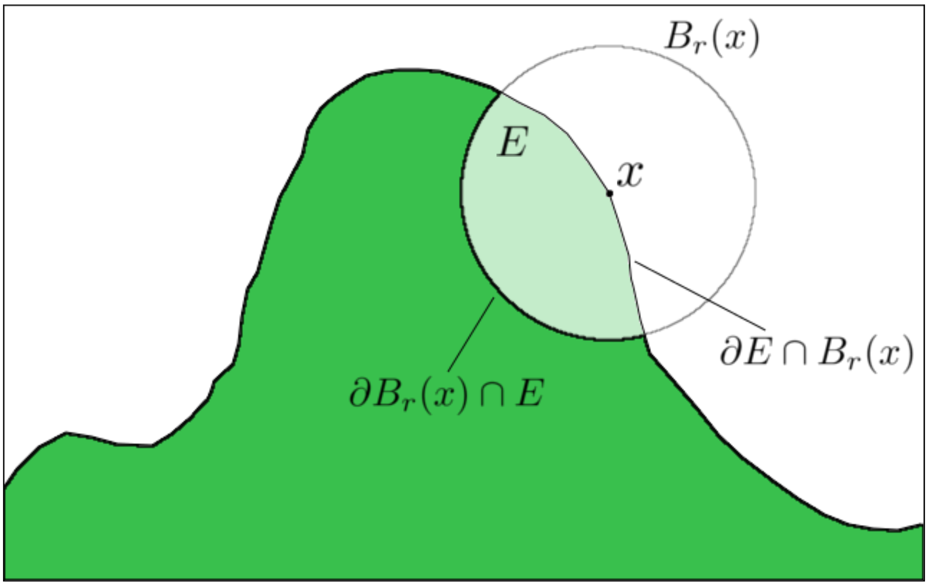

FIGURE 7. The set $E \cap B_r(x)$ (in light green) and $E \setminus B_r(x)$ (in dark green). Then, one uses $E \setminus B_r(x)$ as competitor in the minimality of E. The boundary of $B_r(x) \cap E$ is the union of the two sets $\partial B_r(x) \cap E$ and $\partial E \cap B_r(x)$.

By this, we find that

$$\left(V^{\frac{1}{n}}(r)\right)' = \frac{1}{n}\frac{V'(r)}{V^{\frac{n-1}{n}}(r)} \geqslant \frac{1}{2nc_n},$$

and thus, since $V(0) = 0$, we conclude that

$$V^{\frac{1}{n}}(r) \geqslant \frac{r}{2nc_n}.$$

This is equivalent to the first estimate in (6.1). The second one is readily obtained by applying the former to $\mathbb{R}^n \setminus E$ (note that if E is minimal, so is $\mathbb{R}^n \setminus E$). □

An immediate corollary of the density estimates of Lemma 6.1 is given by the following result.

Corollary 6.2. *Let $\{E_k\}$ be a sequence of minimal surfaces, converging in L^1_{loc} to another minimal surface E. Then E_k converges to E in L^∞_{loc}.*

Notice that convergence in L^∞_{loc} means that the boundaries of E_k and E are (locally) uniformly close.

Proof of Corollary 6.2. Fix a compact set $K \subset \mathbb{R}^n$. We need to prove that, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$K \cap \partial E_k \subset \Big\{x \in K : \operatorname{dist}(x, \partial E) < \varepsilon\Big\}$$

and

$$K \cap \partial E \subset \Big\{x \in K : \operatorname{dist}(x, \partial E_k) < \varepsilon\Big\}$$

for any $k \geqslant N$. We just prove the first inclusion, the proof of the second one being analogous.

We argue by contradiction, and suppose that there exist a diverging sequence $\{k_j\}$ of integers and a sequence of points $\{x_j\} \subset K$ such that $x_j \in \partial E_{k_j}$ and $\operatorname{dist}(x_j, \partial E) \geqslant$

ε, for any $j \in \mathbb{N}$. Up to a subsequence, $\{x_j\}$ converges to a point $\bar{x} \in K$. Clearly, $\operatorname{dist}(x, \partial E) \geqslant \varepsilon$ and thus, in particular,

$$\text{either} \quad B_{\varepsilon/2}(x) \subset \mathring{E} \quad \text{or} \quad B_{\varepsilon/2}(x) \subset \mathbb{R}^n \setminus \bar{E}.$$

Suppose without loss of generality that the latter possibility occurs, i.e., that

$$B_{\varepsilon/2}(x) \subset \mathbb{R}^n \setminus \bar{E}.$$

By this, Lemma 6.1, and the L^1_{loc} convergence of the E_k's, we get

$$c_\star \left(\frac{\varepsilon}{2}\right)^n \leqslant \lim_{j\to+\infty} |B_{\varepsilon/2}(x_j) \cap E_{k_j}| = \lim_{j\to+\infty} \int_{B_{\varepsilon/2}(x_j)} \chi_{E_{k_j}} = \int_{B_{\varepsilon/2}(x)} \chi_E = 0,$$

which is a contradiction. The proof is therefore complete. □

The regularity theory in this case does not proceed as the one for minimal graphs (see Section 5). In fact, we need a more refined strategy.

6.2. **ε-regularity theory.** The aim of this subsection is to prove the following deep result, due to De Giorgi [14].

Theorem 6.3. *There exists a dimensional constant $\varepsilon > 0$ such that, if*

$$\partial E \cap B_r \subseteq \{|x_n| \leqslant \varepsilon r\}$$

for some radius $r > 0$ and $0 \in \partial E$, then

$$\partial E \cap B_{r/2} \text{ is a } C^{1,\alpha} \text{graph}$$

for some $\alpha \in (0,1)$.

Theorem 6.3 ensures that, if a minimal surface is sufficiently flat in one given direction, then it is a $C^{1,\alpha}$ graph. The proof presented here is based on several ideas contained in the work [28] by Savin. The key step is represented by the following lemma.

Lemma 6.4. *Let $\rho \in (0,1)$. There exist $\eta, \rho \in (0,1)$ and $\varepsilon_0 > 0$ such that, if*

$$\partial E \cap B_1 \subseteq \{|x_n| \leqslant \varepsilon\},$$

for some $\varepsilon \in (0, \varepsilon_0)$, and $0 \in \partial E$, then

$$\partial E \cap B_\rho \subseteq \{|x \cdot e| \leqslant \eta\rho\varepsilon\},$$

for some unit vector $e \in \mathbb{S}^{n-1}$.

Lemma 6.4 yields a so-called *improvement of flatness* for the minimal surface ∂E. Indeed, it tells that, shrinking from the ball B_1 to the smaller B_ρ, the oscillation of ∂E around some hyperplane is dumped by a factor η smaller than 1, possibly changing the direction of the hyperplane under consideration. Of course, even if ∂E is a smooth surface, its normal at the origin may not be e_n. Hence, we really need to tilt our reference frame in some new direction $e \in \mathbb{S}^{n-1}$ in order to capture the $C^{1,\alpha}$ behavior of ∂E at the origin.

We now suppose the validity of Lemma 6.4 and show how Theorem 6.3 can be deduced from it.

Sketch of the proof of Theorem 6.3. First of all, we only consider the case of $r = 1$, as one can replace E with $r^{-1}E$. To this regard, observe that the minimality of a set is preserved under dilations.

We then suppose for simplicity that the rotation that sends e_n to e may be avoided in Lemma 6.4. That is, we assume that we can prove that

$$\partial E \cap B_1 \subseteq \{|x_n| \leqslant \varepsilon\} \text{ implies } \partial E \cap B_\rho \subseteq \{|x_n| \leqslant \eta\rho\varepsilon\}, \tag{6.4}$$

provided that $\varepsilon \leqslant \varepsilon_0$. As pointed out before, this clearly cannot be true. Nevertheless, we argue supposing the validity of (6.4), since the general case may be obtained using the same ideas and only slightly more care.

Thanks to the hypothesis of the theorem, we may apply (6.4) and deduce that

$$\partial E \cap B_\rho \subseteq \{|x_n| \leqslant \eta\rho\varepsilon\}.$$

Consider now the rescaled set $E_1 := \rho^{-1}E$. The previous inclusion can be read as

$$\partial E_1 \cap B_1 \subseteq \{|x_n| \leqslant \eta\varepsilon\}.$$

Since $\rho\varepsilon \leqslant \varepsilon \leqslant \varepsilon_0$, we can apply (6.4) to $E_1 := \rho^{-1}E$, and we get

$$\partial E_1 \cap B_\rho \subseteq \left\{|x_n| \leqslant \eta^2\rho\varepsilon\right\}.$$

Getting back to E, this becomes

$$\partial E \cap B_{\rho^2} \subseteq \left\{|x_n| \leqslant \eta^2\rho^2\varepsilon\right\}.$$

By iterating this procedure, we find that, for any $k \in \mathbb{N}$,

$$\partial E \cap B_{\rho^k} \subseteq \left\{|x_n| \leqslant \eta^k\rho^k\varepsilon\right\} = \left\{|x_n| \leqslant \rho^{(1+\alpha)k}\varepsilon\right\}, \tag{6.5}$$

where $\alpha > 0$ is chosen so that $\rho^\alpha = \eta$. Now, given $s \in (0,1)$, there exists $k \in \mathbb{N}$ such that $\rho^k \leqslant s \leqslant \rho^{k-1}$. Hence

$$\begin{aligned}\partial E \cap B_s \subseteq \partial E \cap B_{\rho^{k-1}} &\subseteq \left\{|x_n| \leqslant \rho^{(1+\alpha)(k-1)}\varepsilon\right\}\\ &= \left\{|x_n| \leqslant \rho^{-(1+\alpha)}\rho^{(1+\alpha)k}\varepsilon\right\} \subseteq \left\{|x_n| \leqslant \rho^{-(1+\alpha)}s^{1+\alpha}\varepsilon\right\}\end{aligned}$$

Thus, we deduce that

$$\partial E \cap B_s \subseteq \left\{|x_n| \leqslant C\varepsilon s^{1+\alpha}\right\},$$

for any $s \in (0,1]$, where $C := \rho^{-(1+\alpha)}$.

As mentioned above, this estimate is obtained forgetting about the fact that one needs to tilt the system of coordinates. If one takes into account such tilting, instead of (6.5) one would obtain an inclusion of the type

$$\partial E \cap B_{\rho^k} \subseteq \left\{|x \cdot e^k| \leqslant \rho^{(1+\alpha)k}\varepsilon\right\},$$

for some sequence $\{e^k\} \subset \mathbb{S}^{n-1}$. However, at each step the inner product $e^{k+1} \cdot e^k$ cannot be too far from 1 (see Figure 8). By obtaining a quantification of this defect, one can show that the tiltings $\{e^k\}$ converge at some geometric rate to some unit vector e_0. Hence, the correct bound is

$$\partial E \cap B_s \subseteq \left\{|x \cdot e_0| \leqslant C\varepsilon s^{1+\alpha}\right\} \qquad \text{for any } s \in (0,1], \text{ for some } e_0 \in \mathbb{S}^{n-1}. \tag{6.6}$$

Let now z be any point in $B_{1/2} \cap \partial E$. As $B_{1/2}(z) \subset B_1$, we clearly have that

$$\partial E \cap B_{1/2}(z) \subseteq \{|x_n| \leqslant \varepsilon\}.$$

Assume that $\varepsilon \leqslant \varepsilon_0/2$. Then the set $E_z := 2(E - z)$ satisfies

$$\partial E_z \cap B_1 \subseteq \{|x_n| \leqslant 2\varepsilon\}.$$

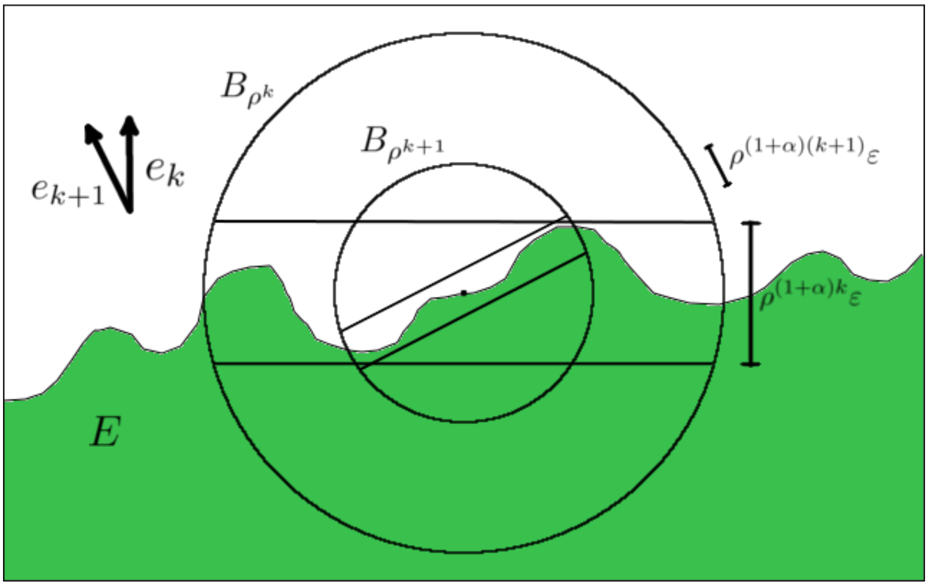

FIGURE 8. The boundary of a minimal set E may be trapped in slabs of different orientations inside balls of different radii. However, the discrepancy between these orientations cannot be too large.

This allows us to repeat the argument above with E_z in place of E and conclude that

$$\partial E \cap B_s(z) \subseteq \left\{ |(x-z)\cdot e_z| \leqslant 2C\varepsilon s^{1+\alpha} \right\} \qquad \text{for some } e_z \in \mathbb{S}^{n-1},$$

for any $z \in B_{1/2} \cap \partial E$ and any $s \in (0, 1/2)$. With this in hand, one can then conclude that ∂E is a $C^{1,\alpha}$ graph. □

In order to finish the proof of the ε-regularity theorem, we are therefore only left to show the validity of Lemma 6.4. We do this in the remaining part of the subsection.

To prove Lemma 6.4 we argue by contradiction and suppose that, given two real numbers $\rho, \eta \in (0,1)$ to be fixed later, there exist an infinitesimal sequence $\{\varepsilon_k\}$ of positive real numbers and a sequence of minimizers $\{E_k\}$ for which $0 \in \partial E_k$,

$$\partial E_k \cap B_1 \subseteq \{|x_n| \leqslant \varepsilon_k\},$$

but

$$\partial E_k \cap B_\rho \not\subseteq \{|x \cdot e| \leqslant \eta\rho\varepsilon_k\} \qquad \text{for any } e \in \mathbb{S}^{n-1}. \tag{6.7}$$

Consider the changes of coordinates $\Psi_k : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Psi_k(x', x_n) := \left(x', \frac{x_n}{\varepsilon_k} \right),$$

and define $\tilde{E}_k := \Psi_k(E_k)$. Observe that the new sets $\tilde{E}_k$ are not minimizers, as stretching in one variable does not preserve minimality. However, thanks to the following result, the surfaces $\tilde{E}_k$ are precompact:

Lemma 6.5 (Savin [28]). *Up to a subsequence, the surfaces $\{\partial \tilde{E}_k\}$ converge in $L^\infty_{\rm loc}$ to the graph of some function u.*

What can we say about u? Let us deal with the easier case in which the original boundaries ∂E_k are already the graphs of some functions u_k. This is of course not always the case, but the Lipschitz approximation theorem for minimal surfaces (see for instance [25, Theorem 23.7]) tells that a flat minimal surface is a Lipschitz graph at many points (the measure of the points being more and more as the surface gets flatter and flatter).

Under this assumption, we have that $\partial \tilde{E}_k = \text{graph}(\tilde{u}_k)$, with $\tilde{u}_k := \varepsilon_k^{-1} u_k$. Observe that $|\tilde{u}_k| \leqslant 1$, since $\partial \tilde{E}_k \cap B_1 \subseteq \{|x_n| \leqslant 1\}$. In view of the minimality of ∂E_k, we compute

$$0 = \frac{1}{\varepsilon_k} \operatorname{div} \left(\frac{\nabla u_k}{\sqrt{1 + |\nabla u_k|^2}} \right) = \operatorname{div} \left(\frac{\nabla \tilde{u}_k}{\sqrt{1 + \varepsilon_k^2 |\nabla \tilde{u}_k|^2}} \right).$$

Assuming that $|\nabla \tilde{u}_k|$ is bounded, by taking the limit as $k \to +\infty$ in the above expression, we find that u solves

$$\begin{cases} \Delta u = 0 \text{ in } B_1^{n-1} \\ \|u\|_{L^\infty(B_1^{n-1})} \leqslant 1. \end{cases} \tag{6.8}$$

(In order to rigorously obtain the claimed equation for u, one needs to use the concept of viscosity solutions that we shall not discuss here. We refer to [28, 7] for more details.)

From (6.8), it follows by regularity theory for harmonic functions that $\|u\|_{C^2(B_{3/4})} \leqslant \bar{C}_n$, for some dimensional constant $\bar{C}_n \geqslant 0$. Therefore,

$$|u(x') - u(0) - \nabla u(0) \cdot x'| \leqslant \bar{C}_n \rho^2 \quad \text{for any } x' \in B_{2\rho}^{n-1}.$$

Taking $\rho \leqslant \eta/(4\bar{C}_n)$ and observing that $u(0) = 0$, this becomes

$$|u(x') - \nabla u(0) \cdot x'| \leqslant \frac{\eta \rho}{2} \quad \text{for any } x' \in B_{2\rho}^{n-1}.$$

As $\partial \tilde{E}_k$ converges uniformly to graph(u) in L^∞_{loc}, the above estimate implies that

$$\partial \tilde{E}_k \cap \left(B_\rho^{n-1} \times \mathbb{R} \right) \subseteq \{|x \cdot \tilde{v}| \leqslant \eta \rho\}, \quad \text{with } \tilde{v} := (-\nabla u(0), 1),$$

for $k \gg 1$. Dilating back we easily obtain

$$\partial E_k \cap B_\rho \subseteq \{|x \cdot \tilde{e}_k| \leqslant \eta \rho \varepsilon_k\} \quad \text{with } \tilde{e}_k := \frac{(-\varepsilon_k \nabla u(0), 1)}{\sqrt{1 + \varepsilon_k^2 |\nabla u(0)|^2}} \in \mathbb{S}^{n-1},$$

in contradiction with (6.7).

We have therefore proved Theorem 6.3 in its entirety (up to the compactness result in Lemma 6.5, and some small technical details). By this result, we know that if a minimal surface is sufficiently flat around a point, then it is locally the graph of a $C^{1,\alpha}$ function. Note that, by the regularity theory discussed in Section 5, such a function will actually be analytic.

In order to proceed further in the understanding of the regularity theory, the next question becomes: at how many points minimal surfaces are flat?

An answer to this question is provided via the so-called *blow-up procedure*.

6.3. **Blow-up technique.** The idea is to look at points of ∂E from closer and closer. More precisely, for a fixed $x \in \partial E$, we define the family of minimal surfaces $\{E_{x,r}\}$ as

$$E_{x,r} := \frac{E - x}{r}, \tag{6.9}$$

for any $r > 0$. By taking the limit as $r \to 0^+$ of such close-ups, one reduces to problem of counting flat points to that of classifying limits of blow-ups.

In order to rigorously describe the above anticipated blow-up procedure, we first need some preliminary results.

We recall that a set C is said to be a cone with respect to a point x if

$$y \in C \text{ implies that } \lambda(y-x) \in C - x \text{ for any } \lambda > 0.$$

Theorem 6.6 (Monotonicity formula)**.** *The function*

$$\Psi_E(r) := \frac{\mathcal{H}^{n-1}(\partial E \cap B_r(x))}{r^{n-1}},$$

is monotone non-decreasing in r.

Proof. Let Σ_r be the cone centered at x and such that

$$\Sigma_r \cap \partial B_r(x) = \partial E \cap \partial B_r(x). \tag{6.10}$$

Set $f(r) := \mathcal{H}^{n-1}(\partial E \cap B_r(x))$. By minimality,

$$f(r) \leqslant \mathcal{H}^{n-1}(\Sigma_r \cap B_r(x)).$$

Using polar coordinates, the fact that Σ_r is a cone, and again (6.10), we compute

$$\begin{aligned} \mathcal{H}^{n-1}(\Sigma_r \cap B_r(x)) &= \int_0^r \mathcal{H}^{n-2}(\Sigma_r \cap \partial B_s(x))\, ds \\ &= \frac{\mathcal{H}^{n-2}(\Sigma_r \cap \partial B_r(x))}{r^{n-2}} \int_0^r s^{n-2}\, ds \\ &= \frac{r\mathcal{H}^{n-2}(\partial E \cap \partial B_r(x))}{n-1}. \end{aligned}$$

As $\mathcal{H}^{n-2}(\partial E \cap \partial B_r(x)) = f'(r)$, we conclude that

$$f(r) \leqslant \frac{r}{n-1} f'(r).$$

This in turn implies that

$$\Psi_E'(r) = \left(\frac{f(r)}{r^{n-1}}\right)' = \frac{rf'(r) - (n-1)f(r)}{r^n} \geqslant 0,$$

and the monotonicity follows. □

By a more careful inspection of the proof, one can show that Ψ_E is constant if and only if E is a cone with respect to the point x.

Proposition 6.7. *There exists an infinitesimal sequence $\{r_j\}$ of positive real numbers such that $\{E_{x,r_j}\}$ converges to a set F in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, as $j \to +\infty$. Furthermore,*

(i) ∂F is a minimal surface;
(ii) F is a cone.

Sketch of the proof. By scaling and Theorem 6.6, given $R > 0$, for any $r \in (0, 1/R]$ we estimate

$$\mathrm{Per}(E_{x,r}; B_R) = \mathcal{H}^{n-1}(\partial E_{x,r} \cap B_R) = \frac{\mathcal{H}^{n-1}(\partial E \cap B_{rR}(x))}{r^{n-1}} \leqslant R^{n-1}\mathcal{H}^{n-1}(\partial E \cap B_1(x)).$$

This proves that the perimeter of $\partial E_{x,r}$ in B_R is uniformly bounded for all $r \in (0, 1/R]$. Accordingly, by Proposition 2.4 the family $\{E_{x,r}\}$ is compact in $L^1_{\mathrm{loc}}(B_R)$. Since this is true for any $R > 0$, a diagonal argument yields the existence of an infinitesimal sequence $\{r_j\}$ such that

$$E_{x,r_j} \longrightarrow F \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

for some set $F \subseteq \mathbb{R}^n$. Since the sets $E_{x,r}$ are minimal, exploiting the lower semicontinuity of the perimeter (see Proposition 2.3) it is not difficult to show that ∂F is a minimal surface and that $\mathcal{H}^{n-1}(\partial E_{x,r_j} \cap B_s) \to \mathcal{H}^{n-1}(\partial F \cap B_s)$ for a.e. s (see, for instance, [24, Lemma 9.1]).

We now prove that F is a cone. For any $s > 0$, we have

$$\Psi_F(s) = \frac{\mathcal{H}^{n-1}(\partial F \cap B_s)}{s^{n-1}} = \lim_{j\to+\infty} \frac{\mathcal{H}^{n-1}(\partial E_{x,r_j} \cap B_s)}{s^{n-1}}$$
$$= \lim_{j\to+\infty} \frac{\mathcal{H}^{n-1}(\partial E \cap B_{r_j s}(x))}{(r_j s)^{n-1}} = \lim_{\rho\to 0^+} \frac{\mathcal{H}^{n-1}(\partial E \cap B_\rho(x))}{\rho^{n-1}}.$$

Thus, $\Psi_F(s)$ is constant, which implies that F is a cone. □

We have thus established that the blow-up sequence (6.9) converges to a minimal cone. Notice now that halfspaces are particular examples of cones. Also, if F is a halfspace and $E_{x,r}$ is close to F in L^1_{loc} (and hence in L^∞_{loc}, see Corollary 6.2), then $\partial E_{x,r}$ becomes flatter and flatter as $r \to 0^+$. In particular, we may apply Theorem 6.3 to $E_{x,r}$ for some r sufficiently small to deduce the smoothness of ∂E around x. Hence, the goal now is to understand whether minimal cones are always halfplanes or not. The desired classification result is given by the following theorem.

Theorem 6.8. *If $n \leqslant 7$, all minimal cones are halfplanes. If $n \geqslant 8$, then there exist minimal cones which are not halfplanes.*

Theorem 6.8 has been obtained by De Giorgi [15] for $n = 3$, by Almgren [3] for $n = 4$ and, finally, by Simons [30] in any dimension $n \leqslant 7$. The counterexample in dimension $n = 8$ is given by the so-called Simons cone

$$C := \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| < |y|\}.$$

Simons conjectured in [30] that the above set was a minimal cone in dimension 8, and this was proved by Bombieri, De Giorgi, and Giusti [5].

Notice that the case $n = 2$ of Theorem 6.8 is trivial. Indeed, the cone on the left of Figure 9 cannot be minimal, as the competitor showed on the right has less perimeter (by the triangle inequality).

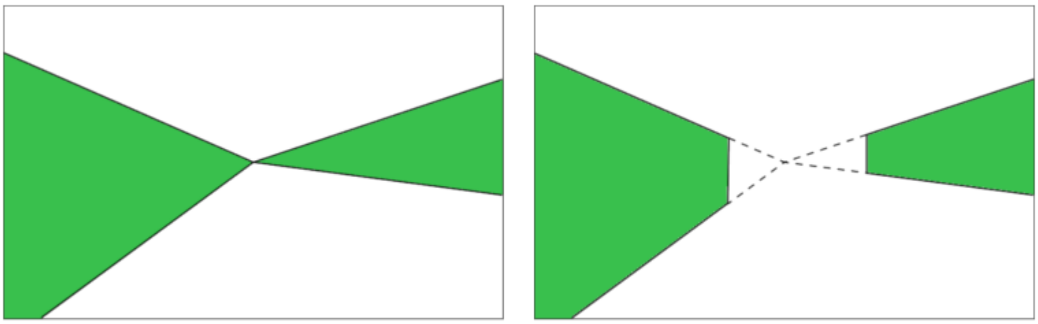

FIGURE 9. The cone on the left is not minimal, since the perturbation on the right has lower perimeter.

In conclusion, the discussion above shows that minimal surfaces are smooth up to dimension 7. Although in higher dimension minimal surfaces may develop a singular set $\mathcal{S}$, an argument due to Federer (called "Federer reduction argument") allows one to exploit the absence of singular minimal cones in dimension 7 to give a bound on dimension of $\mathcal{S}$. We can summarize this in the following result (see [24, Chapter 11] or [25, Chapter 28] for more details):

Corollary 6.9. *Let $E \subset \mathbb{R}^n$ be minimal. We have:*

(i) if $n \leqslant 7$, then ∂E is analytic;
(ii) if $n \geqslant 8$, then there exists $\mathcal{S} \subset \partial E$ such that $\mathcal{S}$ is closed, $\partial E \setminus \mathcal{S}$ is analytic, and $\mathcal{H}^\sigma(\mathcal{S}) = 0$ for any $\sigma > n-8$.

7. Nonlocal minimal surfaces

In this last section, we consider a different *nonlocal* notion of area, introduced by Caffarelli, Roquejoffre, and Savin in [7]. After briefly motivating its definition, we discuss which of the results and approaches described up to now can be carried over to this new setting.

To begin with, we should ask ourselves why we study perimeters. Of course, perimeters model surface tension, as for example in soap bubbles. Moreover, perimeters naturally arise in phase transition problems. Suppose that we have two different media (e.g. water and ice, or water and oil) that are put together in the same container. Of course, the system pays an energy for having an interface between them. Since nature tends to minimize such an energy, interfaces must be (almost) minimal surfaces (e.g. spheres of oil in water, planar regions, etc.).

Hence, perimeters are useful for interpreting in simple ways several complex events that take place in our world. In general, perimeters give good *local* descriptions of intrinsically *nonlocal* phenomena. We now address the problem of establishing a truly nonlocal energy that may hopefully better model the physical situation.

Let E be a subset of $\mathbb{R}^n$, representing the region occupied by some substance. In order to obtain an energy that incorporates the full interplay between E and its complement—that we think to be filled with a different composite—we suppose that each point x of E interacts with each point y of $\mathbb{R}^n \setminus E$. Of course, we need to weigh this interaction, so that closer points interact more strongly than farther ones. Moreover, E must not interact with itself, and similarly for its complement. Finally, because the regularity theory only depends on the interaction for extremely close-by points, it is natural to consider energies that have some scaling invariance. In the end, one comes up with the following notion of a *fractional perimeter* of E:

$$\mathrm{Per}_s(E) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy = 2\int_E\int_{\mathbb{R}^n\setminus E}\frac{dx\,dy}{|x-y|^{n+s}},$$

for any fixed $s \in (0,1)$. Notice that, since we chose a homogeneous weight, rescalings of minimal surfaces are still minimal, as for the standard perimeter. But why did we restrict to the above range for the power s?

To answer this question, we first need to define a preliminary restricted version of the fractional perimeter. Consider the quantity

$$\mathrm{Per}_s^{B_1}(E) := \int_{B_1}\int_{B_1} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy. \tag{7.1}$$

Observe that $\mathrm{Per}_s^{B_1}(E)$ sums up all the interactions between E and its complement that occur inside B_1. If we take $s<0$, then

$$\mathrm{Per}_s^{B_1}(E) \leqslant 2\int_{B_1}\int_{B_1}\frac{dx\,dy}{|x-y|^{n+s}} \leqslant C_n\int_0^2 \frac{d\rho}{\rho^{1+s}} < +\infty,$$

that is, $\mathrm{Per}_s^{B_1}(E)$ is always finite, no matter how rough the boundary of E is. Hence, this would lead to a too weak notion of perimeter.

On the other hand, suppose that $s \geqslant 1$. Then, if we take as E the upper halfspace $\{x_n > 0\}$, a simple computation reveals that

$$
\begin{aligned}
\mathrm{Per}_s^{B_1}(E) &= 2 \int_{B_1 \cap \{x_n > 0\}} \int_{B_1 \cap \{x_n < 0\}} \frac{dx\, dy}{|x-y|^{n+s}} \\
&\geqslant 2 \int_0^{\frac{\sqrt{2}}{2}} \int_{-\frac{\sqrt{2}}{2}}^0 \int_{B^{n-1}_{\frac{\sqrt{2}}{2}}} \int_{B^{n-1}_{\frac{\sqrt{2}}{2}}} \frac{dx'\, dy'\, dx_n\, dy_n}{\left[|x'-y'|^2 + (x_n - y_n)^2\right]^{\frac{n+s}{2}}} \\
&\geqslant c_n \int_0^{\frac{\sqrt{2}}{4}} \frac{dt}{t^{1+s}} = +\infty.
\end{aligned}
$$

Thus, halfspaces have infinite s-perimeter in the ball B_1 if $s \geqslant 1$. As halfspaces represents the simplest examples of surfaces, this is clearly something we do not want to allow for. Consequently, we restrict ourselves to consider weights corresponding to $s \in (0,1)$.[2]

It can be easily seen that $\mathrm{Per}_s(E) = +\infty$ if E is a halfspace, even for $s \in (0,1)$. This is due to the fact that Per_s takes into account also interactions coming from infinity (actually this happens also in the case of classical perimeters, as the perimeter of a halfspace in the whole $\mathbb{R}^n$ is not finite). Therefore, we need to restrict our definition (7.1) to bounded containers.

Fix an open set Ω, and prescribe E outside Ω, i.e., suppose that $E \setminus \Omega = F \setminus \Omega$ for some given set F. Then,

$$
\begin{aligned}
\mathrm{Per}_s(E) = & \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy + 2 \int_\Omega \left(\int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\, dy \right) dx \\
& + \int_{\mathbb{R}^n \setminus \Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy \\
= & \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy + 2 \int_\Omega \left(\int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\, dy \right) dx \\
& + \int_{\mathbb{R}^n \setminus \Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_F(x) - \chi_F(y)|}{|x-y|^{n+s}}\, dx\, dy.
\end{aligned}
$$

Notice that the last integral only sees outside of B_1 and is hence independent of E once the boundary datum F is fixed. Thus, when minimizing $\mathrm{Per}_s(E)$, it is enough to restrict ourselves to the two other terms. Thus, given a bounded open set Ω, we define

(7.2)
$$
\mathrm{Per}_s(E; \Omega) := \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy.
$$

One can check that, with this definition, halfspaces have finite s-perimeters inside any bounded set Ω.

Accordingly, we have the following notion of minimal surface for Per_s.

Definition 7.1 (Caffarelli-Roquejoffre-Savin [7]). *Given a bounded open set Ω, a measurable set $E \subseteq \mathbb{R}^n$ is said to be a* nonlocal s-minimal surface *inside Ω if*

$$
\mathrm{Per}_s(E; \Omega) \leqslant \mathrm{Per}_s(E'; \Omega)
$$

[2] Although the choice $s = 0$ is in principle admissible for the restricted perimeter $\mathrm{Per}_s^{B_1}$, we discard it anyway. In fact, it determines a weight with too fat tails at infinity, which would not be suitable for the full fractional perimeter Per_s.

for any measurable E' such that $E' \setminus \Omega = E \setminus \Omega$.

In the following subsections, we proceed to investigate some important properties shared by nonlocal minimal surfaces.

7.1. **Existence of s-minimal surfaces.** We begin by showing the existence of s-minimal surfaces. Assuming for simplicity that $\Omega = B_1$, we have the following result.

Theorem 7.2. *Let F be a set with locally finite s-perimeter. Then, there exists a s-minimal surface E in B_1 with $E \setminus B_1 = F \setminus B_1$.*

As in Section 3, the proof of the existence of minimal surfaces is based on the semicontinuity of Per_s and on a compactness result similar to Proposition 2.4. The lower semicontinuity of Per_s in L^1_{loc} can be easily established right from definition (7.2), using for instance Fatou's lemma. On the other hand, the needed compactness statement amounts to show that

$$\mathrm{Per}_s(E_k; B_1) \leqslant C \text{ implies that } \{E_k\} \text{ is precompact in } L^1(B_1). \tag{7.3}$$

To check this fact, we first notice that

$$\begin{aligned}\mathrm{Per}_s(F; B_1) &\geqslant \int_{B_1}\int_{B_1} \frac{|\chi_F(x) - \chi_F(y)|}{|x-y|^{n+s}}\,dx\,dy \\ &= \int_{B_1}\int_{B_1} \frac{|\chi_F(x) - \chi_F(y)|^2}{|x-y|^{n+s}}\,dx\,dy \\ &= [\chi_F]^2_{H^{s/2}(B_1)},\end{aligned}$$

where $[\,\cdot\,]_{H^{s/2}}$ denotes the Gagliardo seminorm of the fractional Sobolev space $H^{s/2}$. By the compact fractional Sobolev embedding (see e.g. [16, Theorem 7.1]), the uniform boundedness of $\{\chi_{E_k}\}$ in $H^{s/2}(B_1)$ implies that, up to a subsequence, it converges in $L^1(B_1)$ to χ_F, for some measurable set F. Hence, (7.3) is true.

7.2. **Euler-Lagrange equation.** Suppose that E is a nonlocal minimal surface in B_1 and let $\{E_\varepsilon\}$ be a continuous family of perturbations of E, with $E_\varepsilon \setminus B_1 = E \setminus B_1$ for any ε. From the minimality of E, we have that

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{Per}_s(E_\varepsilon; B_1).$$

Eventually, we are led to the equation

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathbb{R}^n\setminus E}(y)}{|x-y|^{n+s}}\,dy = 0 \text{ for any } x \in \partial E \cap B_1 \tag{7.4}$$

(see [7, Section 5]). Heuristically, this means that

$$\int_E \frac{dy}{|x-y|^{n+s}} = \int_{\mathbb{R}^n\setminus E} \frac{dy}{|x-y|^{n+s}},$$

at any point $x \in \partial E \cap B_1$. In other words, each point $x \in \partial E$ interacts in the same way both with E and with $\mathbb{R}^n \setminus E$. However, the above identity cannot be interpret in a rigorous way, as both integrals do not converge. Hence, (7.4) must be understood in the principal value sense, that is

$$0 = \mathrm{P.V.}\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathbb{R}^n\setminus E}(y)}{|x-y|^{n+s}}\,dy = \lim_{\delta\to 0^+}\int_{\mathbb{R}^n\setminus B_\delta(x)} \frac{\chi_E(y) - \chi_{\mathbb{R}^n\setminus E}(y)}{|x-y|^{n+s}}\,dy.$$

When E is the (global) subgraph of a function $u : \mathbb{R}^{n-1} \to \mathbb{R}$, this can be written as a nonlocal equation for u: more precisely, if we assume that u is small enough so that we neglect nonlinear terms, we find that

$$0 = I[u](x) \simeq (-\Delta)^{\frac{1+s}{2}} u(x) = \text{P.V.} \int_{\mathbb{R}^{n-1}} \frac{u(x) - u(y)}{|x-y|^{(n-1)+(1+s)}} \, dy,$$

where I denotes a suitable integral operator (cp. [7, Lemma 6.11] and [4, Section 3]). The fact that $I[u]$ is close to the fractional Laplacian of order $1+s$ when the Lipschitz norm of u is small should be compared with the classical mean curvature operator appearing in (5.2), which is close to the classical Laplacian when ∇u is small.

7.3. **s-minimal graphs.** As we did before, we begin by addressing the problem of obtaining regularity results for minimal surfaces in the case when they are (locally) the graph of a function u. So, we consider a s-minimal surface E in the infinite cylinder $C_1 = B_1^{n-1} \times \mathbb{R}$ such that

$$E \cap C_1 = \Big\{ (x', x_n) \in C_1 : x_n < u(x') \Big\}, \tag{7.5}$$

for some function $u : B_1^{n-1} \to \mathbb{R}$, with $u(0) = 0$.

In Section 5, we saw that bounded classical minimal graphs are smooth functions. The first step in the proof of this result was the gradient estimate of [6], which established their Lipschitz character. From this, additional regularity then followed by the De Giorgi-Nash-Moser and Schauder theories.

In the nonlocal setting, we are still missing the initial step of this argument. In fact, we can propose the following open problem.

Open Problem. *Suppose that u is bounded. What can be said of the regularity of u in the ball $B_{1/2}^{n-1}$? Is it locally Lipschitz?*

When u is already Lipschitz, then its smoothness follows. This is achieved in two essential steps. First, we have:

Theorem 7.3 (Figalli-Valdinoci [22])**.** *If u is Lipschitz, then u is $C^{1,\alpha}$ for any $\alpha < s$.*

Then, the following Schauder-type result allows one to conclude:

Theorem 7.4 (Barrios-Figalli-Valdinoci [4])**.** *If u is $C^{1,\alpha}$ for some $\alpha > s/2$, then u is C^∞.*

At the moment it is not known whether smooth s-minimal graphs are actually analytic. The results in [1] show that they enjoy some Gevrey regularity.

We conclude the subsection by observing that s-minimal surfaces with graph properties as (7.5) indeed exist, for instance when their boundary data are graphs too.

Theorem 7.5 (Dipierro-Savin-Valdinoci [18])**.** *Suppose that E is a s-minimal surface in C_1 such that*

$$E \setminus C_1 = \Big\{ (x', x_n) \in (\mathbb{R}^n \setminus B_1^{n-1}) \times \mathbb{R} : x_n < v(x') \Big\},$$

for some bounded, continuous function $v : \mathbb{R}^{n-1} \to \mathbb{R}$. Then, (7.5) holds true for some continuous function $u : \bar{B}_1^{n-1} \to \mathbb{R}$.

7.4. **Regularity of general s-minimal sets.** The regularity theory for nonlocal minimal surfaces established in [7] follows an analogous strategy to that outlined in Section 6.

The density estimates follow via the same argument of the proof of Lemma 6.1, using the fractional Sobolev inequality in place of the isoperimetric inequality (see [7, Section 4]).

The ε-regularity theory is also similar [7, Section 6], but we need to check what happens with the behavior at infinity of the s-minimal surface. The key step is represented by the following improvement of flatness result.

Lemma 7.6. *Let E be a s-minimal surface in B_1. For any fixed $\alpha \in (0,s)$, there exists $k_0 \in \mathbb{N}$ such that if*

$$\partial E \cap B_{2^{-k}} \subseteq \left\{ |x \cdot e_k| \leqslant 2^{-k(1+\alpha)} \right\},$$

for some unit vector $e_k \in \mathbb{S}^{n-1}$ and for any $k = 0, \dots, k_0$, then

$$\partial E \cap B_{2^{-k_0-1}} \subseteq \left\{ |x \cdot e_{k_0+1}| \leqslant 2^{-(k_0+1)(1+\alpha)} \right\},$$

for some $e_{k_0+1} \in \mathbb{S}^{n-1}$.

Lemma 7.6 tells that if ∂E is sufficiently flat for a sufficiently large number of geometric scales, then it is flatter and flatter at all smaller scales. Compare this with Lemma 6.4: in the local case it was sufficient to check the flatness of the boundary of E at only one scale to deduce its improvement at smaller scales.

We can rephrase the above statement by rescaling everything by a factor 2^{k_0} (in other words, replacing E by $2^{k_0}E$). Lemma 7.6 is then equivalent to prove that

$$\partial E \cap B_{2^j} \subseteq \left\{ |x \cdot e_j| \leqslant \varepsilon (2^j)^{1+\alpha} \right\},$$

for any $j = 0, \dots, k_0$ and with $\varepsilon = 2^{-k_0 \alpha}$, implies that

$$\partial E \cap B_{1/2} \subseteq \left\{ |x \cdot \bar{e}| \leqslant \varepsilon 2^{-1-\alpha} \right\}.$$

From this formulation, the role played by the nonlocality of Per_s is even more evident: to obtain information inside the ball $B_{1/2}$ we need to have it already in $B_{2^{k_0}}$, with k_0 sufficiently large.

To prove the improvement of flatness that we just stated, we argue by contradiction. As in Subsection 6.2, we pick two sequences of s-minimal surfaces E_m and positive real numbers ε_m, with $\varepsilon_m \to 0$. We suppose that each E_m violates the implication above, with $\varepsilon = \varepsilon_m$ and $k_0 = |\log \varepsilon_m| / (\alpha \log 2)$. It can be shown that suitable rescalings of the sets E_m (analogue to the rescaling in Subsection 6.2) converge to the graph of a function u that satisfies

$$\begin{cases} (-\Delta)^{\frac{1+s}{2}} u = 0 \text{ in } \mathbb{R}^{n-1} \\ |u(x)| \leqslant C \left(1 + |x|^{1+\alpha}\right) \text{ for any } x \in \mathbb{R}^{n-1}, \end{cases} \tag{7.6}$$

for some $C > 0$. The conclusive step of the proof of Lemma 7.6 is then provided by the next general Liouville-type result.

Lemma 7.7. *Suppose that u satisfies (7.6) for some $\alpha \in (0,s)$ and $s \in (0,1]$. Then, u is affine.*

Sketch of the proof. We include the proof of the lemma in the classical case $s = 1$. The argument for the fractional powers of the Laplacian is analogous (see [7, Proposition 6.7]).

Fix $R \geqslant 1$ and set $u_R(x) := R^{-1-\alpha}u(Rx)$. Clearly, $\Delta u_R = 0$ and $\|u_R\|_{L^\infty(B_1)} \leqslant C$. Consequently, by elliptic regularity, we have that $\|D^2 u_R\|_{L^\infty(B_{1/2})} \leqslant C_n C$. But

$$D^2 u_R(x) = R^{1-\alpha} D^2 u(Rx),$$

and therefore we get that

$$\|D^2 u\|_{L^\infty(B_{R/2})} \leqslant \frac{C_n C}{R^{1-\alpha}}.$$

The result follows by letting $R \to +\infty$. □

In view of the ε-regularity theory outlined above, we know that flat s-minimal surfaces are smooth.

The next step is then to use blow-ups in order to understand at how many points a nonlocal minimal surface is flat. To this aim, we first need an appropriate monotonicity formula, as in Theorem 6.6. Instead of working with the nonlocal perimeter Per_s as defined in 7.1, we consider a slightly different energy coming from the so-called *extension problem* (see [7, 8]).

Let $\mathbb{R}^{n+1}_+$ denote the upper halfspace $\{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y > 0\}$ and $u : \bar{\mathbb{R}}^{n+1}_+ \to \mathbb{R}$ be the unique solution to the problem

$$\begin{cases} \mathrm{div}_{\mathbb{R}^{n+1}} \left(y^{1-s} \nabla_{\mathbb{R}^{n+1}} u \right) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u|_{y=0} = \chi_E - \chi_{\mathbb{R}^n \setminus E} & \text{on } \mathbb{R}^n. \end{cases}$$

Then, define

$$\Phi_E(r) := \frac{1}{r^{n-s}} \int_{B_r^+} y^{1-s} \left| \nabla_{\mathbb{R}^{n+1}} u \right|^2,$$

for any $r > 0$. The notation B_r^+ is used here to indicate the upper half-ball of radius r, centered at the origin of $\mathbb{R}^{n+1}$, i.e. $B_r^+ := B_r^{n+1} \cap \mathbb{R}^{n+1}_+$. We have the following:

Theorem 7.8 (Caffarelli-Roquejoffre-Savin [7])**.** *The function Φ_E is monotone nondecreasing in r.*

With the help of this monotonicity result, we can successfully perform the standard blow-up procedure.

Proposition 7.9. *Let E be a s-minimal surface and let $x \in \partial E$. For small $r > 0$, set $E_{x,r} := r^{-1}(E - x)$. Then, up to a subsequence,*

$$E_{x,r} \longrightarrow F \text{ in } L^1_{\mathrm{loc}},$$

as $r \to 0^+$, with F a s-minimal cone.

As in the classical case, to complete our investigation on the regularity properties of minimal surfaces we are left with the problem of classifying minimal cones. This task turns out to be not trivial at all, even in the plane. In fact, here one cannot argue as easily as for the standard perimeter (recall Figure 9). However, a more refined approach can be developed to show that in $\mathbb{R}^2$ there are no non-trivial s-minimal cones.

Theorem 7.10 (Savin-Valdinoci [29])**.** *If E is a s-minimal cone in $\mathbb{R}^2$, then E is a halfspace. In particular, s-minimal surfaces in $\mathbb{R}^2$ are smooth.*

This result has been recently improved, via quantitative flatness estimates, in [10].

Another way to attack the problem of the regularity for s-minimal surfaces, when s is close to 1, is by taking advantage of the classical regularity theory. First, we recall the following result due to Davila [11] (see also [9, 2]).

Theorem 7.11. *There exists a dimensional constant $c_\star > 0$ such that*

$$(1-s)\operatorname{Per}_s(E;B_1) \longrightarrow c_\star \operatorname{Per}(E;B_1),$$

as $s \to 1^-$.

In view of the above theorem, (a suitable rescaling of) the nonlocal perimeter converges to the standard one as $s \to 1^-$. Similarly, nonlocal minimal surfaces approaches classical ones in the same limit. Hence, as we already know that classical minimal surfaces are smooth up to dimension $n = 7$, the same is true for s-minimal surfaces, provided s is sufficiently close to 1. More precisely, the following result holds as a consequence of Theorem 7.11:

Corollary 7.12. *Let $n \geqslant 2$, and let $E \subset \mathbb{R}^n$ be s-minimal. There exists $s_n \in (0,1)$ close to 1 such that, if $s \geqslant s_n$, then:*

(1) if $n \leqslant 7$, then $\partial E \in C^\infty$ (in particular, the only s-minimal cones are halfspaces);

(2) if $n \geqslant 8$, then there exists $\mathcal{S} \subset \partial E$ such that $\mathcal{S}$ is closed, $\partial E \setminus \mathcal{S}$ is smooth, and $\mathcal{H}^\sigma(\mathcal{S}) = 0$ for any $\sigma > n-8$.

On the contrary, as $s \to 0^+$, a suitable rescaling of Per_s converges to the volume [17]. In this respect, Per_s is a very natural way to interpolate between the volume and the perimeter.

We note that, if s is small, there is an example of a cone $F \subset \mathbb{R}^7$ such that, for any continuous family $\{F_\varepsilon\}$ of perturbations of F, it holds

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \operatorname{Per}_s(F_\varepsilon;B_1) = 0 \quad \text{and} \quad \left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \operatorname{Per}_s(F_\varepsilon;B_1) \geqslant 0.$$

That is, F is a stable solution of (7.4). If one could prove that F actually minimizes the s-perimeter, then one would have found a counterexample to the above corollary when s is far from 1. We refer the interested reader to [12] for more details on this construction.

In conclusion, the regularity theory for nonlocal minimal surfaces that we just described is often based on ideas that also work for classical ones. Often these methods are simpler and work better in the local scenario, but there are some tools and techniques that are naturally better suited for nonlocal objects.

For instance, as we saw in Section 5 the proof that classical Lipschitz minimal graphs are $C^{1,\alpha}$ is based on the De Giorgi-Nash-Moser theory for elliptic PDEs with bounded measurable coefficients. On the other hand, this strategy does not seem to work for nonlocal minimal surfaces. Conversely, a new geometric argument can be successfully applied and the same regularity result is true [22].

As one can see, several important questions in this theory are still open (the most fundamental one being the classification of minimal cones). We hope that new results will come in the next few years.

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