Stability in geometric & functional inequalities

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Abstract. The aim of this note is to review recent stability results for some geometric and functional inequalities, and to describe applications to the long-time asymptotic of evolution equations.

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1. Introduction

Geometric and functional inequalities play a crucial role in several problems arising in the calculus of variations, partial differential equations, geometry, etc. More recently, there has been a growing interest in studying the stability for such inequalities. The basic question one wants to address is the following:

Suppose we are given a functional inequality for which minimizers are known. Can we prove, in some quantitative way, that if a function "almost attains the equality" then it is close (in some suitable sense) to one of the minimizers?

In recent years several results have been obtained in this direction, showing stability for isoperimetric inequalities [22, 18, 11, 16, 12], the Brunn-Minkowski inequality on convex sets [19], Sobolev [10, 20, 14] and Gagliardo-Nirenberg inequalities [7, 14], etc.

In this note we will describe two different ways to attack this kind of problems. More precisely, in Section 2 we will focus on the anisotropic isoperimetric inequality, proving a sharp stability result using optimal transport. Then in Section 3 we will address the stability issue for Sobolev and Gagliardo-Nirenberg inequalities. Finally, as an application, in Section 4 we will use results from Section 3 to obtain a quantitative rate of convergence for the critical mass Keller-Segel equation.

2. Stability for isoperimetric inequalities

2.1. The Euclidean case. The classical Euclidean isoperimetric inequality states that, for any bounded open smooth set $E \subset \mathbb{R}^n$, the perimeter $P(E)$ controls the volume $|E|$: more precisely,

$$P(E) \geq n|B_1|^{1/n}|E|^{(n-1)/n}, \qquad (1)$$

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B_1 being the unit ball in $\mathbb{R}^n$. Moreover equality holds if and only if E is a ball.

The stability question we want to ask is the following: if E is "almost a minimizer" does this imply that E is close to a ball, if possible in some quantitative way? In order to properly formulate the problem, we introduce some notation.

Whenever E is a smooth open set with $0 < |E| < +\infty$, we define its isoperimetric deficit as

$$\delta(E) := \frac{P(E)}{n|B_1|^{1/n}|E|^{(n-1)/n}} - 1 .$$

We observe that (1) implies that $\delta(E) \geq 0$, and by the characterization of the equality cases $\delta(E) = 0$ if and only if E is a ball. Thus $\delta(E)$ measures the deviation of E from being optimal in (1), and the stability problem consists in quantitatively relating this deviation to a more direct notion of distance from the family of optimal sets. To this end we introduce the asymmetry index of E,

$$A(E) := \inf\left\{\frac{|E\Delta(B_r(x))|}{|E|} : x \in \mathbb{R}^n\,, r^r|B_1| = |E|\right\}$$

(here $E\Delta F$ denotes the symmetric difference between the sets E and F, i.e., $E\Delta F := (E \setminus F) \cup (F \setminus E)$), and we look for positive constants C and α, depending only on n, such that the following stability version of (1) holds:

$$A(E) \leq C\,\delta(E)^{1/\alpha} . \tag{2}$$

This problem has been thoroughly studied (we refer to the survey [25, Section 3] for an extended list of references and the history of the problem). In particular, the first main contributions to the general problem in arbitrary dimension are due to Hall, Hayman, and Weitsman [24, 23], where they prove (2) with a constant $C = C(n)$ and the exponent $\alpha = 4$. It was however conjectured by Hall that (2) should hold with the sharp exponent $\alpha = 2$ (the sharpness of the exponent $\alpha = 2$ can be checked by looking at a sequence of ellipsoids converging to B_1). This has been recently shown by Fusco, Maggi and Pratelli in [22] making use of symmetrization techniques (see also [18, 12] for alternative proofs of this result).

2.2. The anisotropic case. The anisotropic isoperimetric inequality arises in connection with a natural generalization of the Euclidean notion of perimeter. In dimension $n \geq 2$, we consider an open bounded convex set K of $\mathbb{R}^n$ containing the origin. Starting from K, we define a weight function on directions through the Euclidean scalar product

$$\|\nu\|_* := \sup\{x \cdot \nu : x \in K\}\,, \quad \nu \in \mathbb{S}^{n-1}\,, \tag{3}$$

where $\mathbb{S}^{n-1}$ is the unit sphere, and $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$. Given E an open smooth set in $\mathbb{R}^n$, its anisotropic perimeter is defined as

$$P_K(E) := \int_{\partial E} \|\nu_E(x)\|_* d\mathcal{H}^{n-1}(x)\,, \tag{4}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. This notion of perimeter obeys the scaling law $P_K(\rho E) = \rho^{n-1} P_K(E)$ for $\rho > 0$, and it is invariant under translations. However, in contrast with the Euclidean perimeter, P_K is in general not invariant by the action of $O(n)$, or even of $SO(n)$, and in fact it may even happen that $P_K(E) \neq P_K(\mathbb{R}^n \setminus E)$ if K is not symmetric with respect to the origin. When K is the Euclidean unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ of $\mathbb{R}^n$ then $\|\nu\|_* = 1$ for every $\nu \in \mathbb{S}^{n-1}$, and therefore $P_K(E)$ coincides with the Euclidean perimeter $P(E)$.

Apart from its intrinsic geometric interest, the anisotropic perimeter P_K arises as a model for surface tension in the study of equilibrium configurations of solid crystals with sufficiently small grains, and constitutes the basic model for surface energies in phase transitions. In both settings one is naturally led to minimize $P_K(E)$ under a volume constraint, which leads to the anisotropic isoperimetric inequality:

$$P_K(E) \geq n|K|^{1/n}|E|^{(n-1)/n}\,, \tag{5}$$

with equality if and only if $E = x + rK$ for some $x \in \mathbb{R}^n$ and $r > 0$ [21].

Also in this case, we are interested in the stability of such inequality. Hence we introduce the anisotropic deficit of E,

$$\delta_K(E) := \frac{P_K(E)}{n|K|^{1/n}|E|^{(n-1)/n}} - 1\,,$$

and the anisotropic asymmetry of E,

$$A_K(E) := \inf\left\{ \frac{|E\Delta(x_0 + rK)|}{|E|} : x_0 \in \mathbb{R}^n\,, r^n|K| = |E| \right\}\,,$$

and we look for positive constants C and α, depending on n and K only, such that

$$A_K(E) \leq C\,\delta_K(E)^{1/\alpha}\,. \tag{6}$$

As mentioned before, the proof in [22] of the sharp version of (2) relies in quantitative symmetrization inequalities, that is clearly specific to the isotropic case. Hence, when K is a generic convex set, the study of (6) requires completely new ideas.

The first stability result for (5) on generic sets is due to Esposito, Fusco, and Trombetti in [15]: starting from the uniqueness proof in [21], they show the validity of (6) with some constant $C = C(n, K)$ and for the exponent

$$\alpha(2) = \frac{9}{2}\,, \qquad \alpha(n) = \frac{n(n+1)}{2} \quad \text{for } n \geq 3\,.$$

This non-trivial result still leaved the space for a substantial improvement concerning the decay rate at zero of the asymmetry index in terms of the isoperimetric deficit. Theorem 1.1 in [18] provides the sharp decay rate:

Theorem 2.1. *Let E be a set of finite perimeter with $|E| < +\infty$. Then*

$$A_K(E) \leq \frac{181\,n^7}{(2 - 2^{(n-1)/n})^{3/2}}\sqrt{\delta_K(E)}\,.$$

The proof of the above theorem is based on a quantitative study of the optimal transport map between E and K, through the bounds that can be derived from Gromov's proof of the isoperimetric inequality (see also [9] for a proof of Sobolev inequalities via optimal transportation). These estimates provide control, in terms of the isoperimetric deficit, and modulo scalings and translations, on the distance between such a transportation map and the identity, which can then be used to deduce a bound on the asymmetry index. In the next sections we sketch the proof of this result.

2.2.1. Gromov's proof of the anisotropic isoperimetric inequality. Here we describe a variant of Gromov's argument to prove the anisotropic isoperimetric inequality (in his original argument, Gromov did not use the optimal transport map but instead the Knothe map, see [18, Section 1.4] for more details).

Given a (smooth) bounded set $E \subset \mathbb{R}^n$, Brenier's Theorem [5] ensures the existence of a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$, whose gradient $T = \nabla\varphi$ pushes forward the probability density $|E|^{-1}1_E(x)dx$ into the probability density $|K|^{-1}1_K(y)dy$. In particular T takes E into K, and

$$\det \nabla T = \frac{|K|}{|E|} \quad \text{on } E\,. \tag{7}$$

Since T is the gradient of a convex function, ∇T is a symmetric positive definite $n \times n$ matrix. In particular, at any point x we can find an orthonormal basis $\{e_k(x)\}_{k=1}^n$ of $\mathbb{R}^n$ and n non-negative numbers $0 \le \lambda_1(x) \le \ldots \le \lambda_n(x)$, such that

$$\nabla T(x) = \sum_{k=1}^n \lambda_k(x) e_k(x) \otimes e_k(x)\,.$$

Then, by the arithmetic-geometric inequality we find

$$n(\det \nabla T)^{1/n} = n\left(\prod_{k=1}^n \lambda_k\right)^{1/n} \le \sum_{k=1}^n \lambda_k = \operatorname{div} T \quad \text{on } E\,. \tag{8}$$

Hence, by (7), (8), and the Divergence Theorem, we get

$$n|K|^{1/n}|E|^{(n-1)/n} = \int_E n(\det \nabla T)^{1/n} \le \int_E \operatorname{div} T = \int_{\partial E} T \cdot \nu_E \, d\mathcal{H}^{n-1}\,. \tag{9}$$

Let us now define, for every $x \in \mathbb{R}^n$,

$$\|x\| := \inf\left\{\rho > 0 : \frac{x}{\rho} \in K\right\}.$$

Then the set K can be characterized as $K = \{x \in \mathbb{R}^n : \|x\| < 1\}$, so by the fact that T maps E into K we obtain the bound

$$\|T\| \le 1 \qquad \text{on } \partial E\,. \tag{10}$$

Moreover, recalling (3), we easily obtain the identity $\|\nu\|_* = \sup\{x \cdot \nu : \|x\| \le 1\}$, from which we immediately deduce

$$x \cdot y \le \|x\| \|y\|_* \qquad \forall\, x, y \in \mathbb{R}^n \,. \tag{11}$$

Combining (9), (11), and (10) we obtain

$$n|K|^{1/n}|E|^{(n-1)/n} \le \int_{\partial E} \|T\| \|\nu_E\|_* \, d\mathcal{H}^{n-1} \le P_K(E) \,,$$

which proves the anisotropic isoperimetric inequality. (This argument is formal since a priori the transport map is not smooth, but the proof can be made rigorous by using for instance some fine results on BV functions and sets of finite perimeter, see [18].)

2.2.2. The equality case. To give an example of the robustness of the above proof, we show here how the characterization of equality cases follows almost immediately from the above argument. (We sketch here the formal argument, but again the proof can be made rigorous using the theory of BV functions, see [18, Appendix].)

Assume E to be a smooth open connected set which is optimal in the isoperimetric inequality. Up to rescale E, with no loss of generality we can assume $|E| = |K|$. Then from Gromov's argument we deduce that $n(\det \nabla T)^{1/n} = \operatorname{div} T$ inside E. Recalling that equality in the arithmetic-geometric inequality holds if and only if all numbers are equal, we get $\lambda_1(x) = \ldots = \lambda_n(x)$ on E. Then, since $\det \nabla T = 1$ on E and ∇T is symmetric, the above condition implies immediately $\nabla T = \mathrm{Id}$. Thus $T(x) = x + c$ for some vector $c \in \mathbb{R}^n$, that is $E = K - c$, as desired.

2.2.3. The quantitative argument. In order to prove the stability result, one has to make quantitative the previous uniqueness argument. More precisely, by knowing that $\delta_K(E)$ is small, we can quantify the gap between the arithmetic and the geometric mean of the eigenvalues [18, Lemma 2.5], and a simple argument allows then to show that

$$C(n, K)\sqrt{\delta_K(E)} \ge \int_E |\nabla T - \mathrm{Id}\,| \,,$$

see [18, Corollary 2.4].

From this we would like to deduce that T is close to the identity map, and in order to achieve this we may think to use some Poincaré-type inequality. However, since the Poincaré constant of a domain depends on the regularity of its boundary, if we just applied the Poincaré inequality directly on E we would deduce a stability result with a constant depending on E itself! Instead, the key ingredient to go further is to show that, if $\delta_K(E)$ is sufficiently small, then there exists a "good" set $G \subset E$ such that $|E \setminus G| \le C(n, K)\delta_K(E)$, and on G the Poincaré inequality holds with a universal constant depending only on n and K. Then the idea is to apply Gromov's argument to G instead of E, show that G is close to K, and then use that $|E \setminus G| \lesssim \delta_K(E)$ to conclude that E is close to K.

Actually, as explained in the introduction of [18], the use of a Poincaré inequality does not seem to provide the sharp exponent, and in order to conclude one should rather use a trace inequality [18, Sections 3.1-3.4], which combined with the addition information (coming from Gromov's proof) that

$$n|K|\,\delta_K(E) \geq \int_{\partial E} (1 - \|T\|)\|\nu_E\|_* d\mathcal{H}^{n-1}\,.$$

allows to prove

$$C(n,K)\sqrt{\delta_K(E)} \geq \int_{\partial E} |1 - \|x\||\,\|\nu_E\|_*\, d\mathcal{H}^{n-1}(x)\,,. \tag{12}$$

It is then not difficult to show that this last integral controls $|E \setminus K| = |E\Delta K|/2$ [18, Lemma 3.5], so Theorem 2.1 is proved.

Let us point out that, although the constant $C(n,K)$ in (12) depends a priori on K, by using a renormalization argument for convex sets one can find a bound on $C(n,K)$ depending on the dimension only. We refer to [18] for more details, and to [17] for an application of this stability result.

3. Stability for Sobolev and Gagliardo-Nirenberg inequalities

Sobolev and Gagliardo-Nirenberg inequalities allow to control the L^r norm of a function in terms of some L^p norm of its gradient, and perhaps a L^q norm of the function itself.

While for Sobolev inequalities minimizers are well-known [1, 27] and some general stability results are available [3, 10, 20], for Gagliardo-Nirenberg inequalities there are very few cases for which minimizers are explicitly known [13, 9] and all the techniques used up to now to prove stability for Sobolev inequalities seem to fail in this context.

Here we describe the approach introduced in [7] to obtain (sharp) stability estimates for some Gagliardo-Nirenberg inequalities starting from the ones for Sobolev inequalities. We mention that, after the work in [7] was completed, Dolbeault and Toscani [14] were able to obtain stability results for some Sobolev and Gagliardo-Nirenberg inequalities using "entropy-entropy dissipation" techniques.

3.1. Equality cases for Gagliardo-Nirenberg inequalities. Although in general optimal constants in Gagliardo-Nirenberg inequalities are not known, there are some important special cases for which minimizers (and so also the optimal constants) have been found.

Let $W^{1,2}(\mathbb{R}^n)$ denote the space of measurable functions on $\mathbb{R}^n$ that have a square integrable distributional gradient. The Gagliardo-Nirenberg (GN) inequality states that, for $n \geq 2$ and all $1 \leq p \leq q < r(n)$ (with $r(2) := +\infty$, and

$r(n) := 2n/(n-2)$ if $n \geq 3$), there is a finite constant $C = C(n,p,q)$ such that for all $u \in W^{1,2}(\mathbb{R}^n)$,

$$\|u\|_q \leq C\|u\|_p^{1-\theta}\|\nabla u\|_2^\theta \qquad \text{with} \quad \frac{1}{q} = \frac{\theta}{r(n)} + \frac{1-\theta}{p} \ , \tag{13}$$

where $\|\cdot\|_s$ denotes the L^s norm of a function over $\mathbb{R}^n$.

For $n \geq 3$ (so that $r(n) < \infty$) the above inequality is valid also for $q = r(n)$, in which case $\theta = 1$ and we get the Sobolev inequality

$$\|u\|_{2n/(n-2)}^2 \leq S_n\|\nabla u\|_2^2 \ , \tag{14}$$

for which the sharp constant S_n is known.

There are a few other choices of the p and q for which sharp constants are known. In particular, in [13] Del Pino and Dolbeault found the sharp constant for a one-parameter family of GN inequalities for each $n \geq 2$: For $t > 1$, let $p = t+1$, and let $q = 2t$. Then

$$\|u\|_{2t} \leq A_{n,t}\|\nabla u\|_2^\theta\|u\|_{t+1}^{1-\theta} \ , \qquad \theta = \frac{n(t-1)}{t[2n-(1+t)(n-2)]} \ . \tag{15}$$

It turns out that there is a close relation between the sharp Sobolev inequality (14) and the family of GN inequalities (15). One aspect of this is that the functions u that saturate these inequalities are simply powers of one another: The optimal constant S_n in (14) is given by

$$S_n = \frac{\|v\|_{2n/(n-2)}^2}{\|\nabla v\|_2^2} \qquad \text{where} \quad v(x) = (1+|x|^2)^{-(n-2)/2} \tag{16}$$

(see [1, 27]), and moreover, with this value of S_n, there is equality in (14) if and only if u is a multiple of $v(\mu(x-x_0))$ for some $\mu > 0$ and some $x_0 \in \mathbb{R}^n$.

Likewise, the optimal constant $A_{n,t}$ in (15) is given by

$$A_{n,t} = \frac{\|v\|_{2t}}{\|v\|_{t+1}^{1-\theta}\|\nabla v\|_2^\theta} \qquad \text{where} \quad v(x) = (1+|x|^2)^{-1/(t-1)}$$

(see [13]), and with this value of $A_{n,t}$ there is equality in (14) if and only if u is a multiple of $v(\mu(x-x_0))$ for some $\mu > 0$ and some $x_0 \in \mathbb{R}^n$. However, this is a very particular feature of this family.

Another aspect of this close relation between (14) and (15) is that both inequalities can be proved using ideas coming from the theory of optimal mass transportation [9].

Our goal here is to prove some stability properties for the GN inequalities (15), and show applications of this stability to certain partial differential equations.

3.2. Stability for a GN inequality. Although one could generalize many of the arguments here to the whole family in (15), because of its application to the

Keller-Segel equation (that we will describe in the Section 4) we shall focus only on the special case $n = 2$ and $t = 3$.

This case may be written explicitly as

$$\pi \int_{\mathbb{R}^2} u^6(x)\, dx \leq \left(\int_{\mathbb{R}^2} |\nabla u(x)|^2\, dx\right) \left(\int_{\mathbb{R}^2} u^4(x)\, dx\right) . \tag{17}$$

Now, given a non-negative function u in $W^{1,2}(\mathbb{R}^2)$, we define $\delta_{\mathrm{GN}}[u]$ by

$$\delta_{\mathrm{GN}}[u] := \left(\int_{\mathbb{R}^2} |\nabla u|^2\, dy\right)^{1/2} \left(\int_{\mathbb{R}^2} u^4\, dy\right)^{1/2} - \left(\pi \int_{\mathbb{R}^2} u^6\, dy\right)^{1/2} . \tag{18}$$

Also, for $\lambda > 0$ and $x_0 \in \mathbb{R}^2$, define

$$v_{\lambda,x_0} := (1 + \lambda^2 |x - x_0|^2)^{-1/2} ,$$

and use v to denote the function $v_{1,0}$. By [13, Theorem 1] we have that $\delta_{\mathrm{GN}}[u] > 0$ unless u is a multiple of v_{λ,x_0} for some $\lambda > 0$ and some $x_0 \in \mathbb{R}^2$. The question addressed in [7] is:

When $\delta_{\mathrm{GN}}[u]$ is small, is u close (in some sense) to a multiple of v_{λ,x_0}?

The following sharp stability result for (17) is proved in [7, Theorem 1.2]:

Theorem 3.1. *Let $u \in W^{1,2}(\mathbb{R}^2)$ be a non-negative function such that $\|u\|_6 = \|v\|_6$. Then there exists a universal constant $K_1 > 0$ such that*

$$\inf_{\lambda>0, x_0 \in \mathbb{R}^2} \|u^6 - \lambda^2 v^6_{\lambda,x_0}\|_1 \leq K_1\, \delta_{\mathrm{GN}}[u]^{1/2}. \tag{19}$$

We now explain the main ingredients in the proof.

3.2.1. Stability for the Sobolev inequality. We begin by recalling that a stability result for the sharp Sobolev inequality (14) has been obtained some time ago in [3] by Bianchi and Egnell: It states that there is a constant C_n, $n \geq 3$, such that for all $f \in W^{1,2}(\mathbb{R}^n)$,

$$C_n \left(\|\nabla f\|_2^2 - S_n \|f\|^2_{2n/(n-2)}\right) \geq \inf_{c\in\mathbb{R},\, \mu>0,\, x_0\in\mathbb{R}^n} \|\nabla f - c\nabla h_{\mu,x_0}\|_2^2 , \tag{20}$$

where

$$h_{\mu,x_0}(x) := (1 + \mu^2 |x - x_0|^2)^{-(n-2)/2} .$$

The proof is based on a spectral analysis argument which strongly exploits the Hilbertian structure of $W^{1,2}$: let us endow $W^{1,2}(\mathbb{R}^n)$ with the Hilbertian norm $\|f\|_* := \|\nabla f\|_2$. Then we can write any function $f \in W^{1,2}(\mathbb{R}^n)$ with $\|f\|_* = 1$ as $f = h + \alpha g$, where $\alpha > 0$, $\|g\|_* = 1$, and $h \in \{c h_{\mu,x_0}\}_{c,\mu,x_0}$ is the closest minimizers (with respect to the $\|\cdot\|_*$ norm) to f. One then distinguish between two cases, depending on the size of α.

If α is very small then one can expand (20) in powers of α, and showing some spectral gap property one proves that the inequality is true at the dominant order.

On the other hand, if α is not small then a concentration-compactness argument shows that the deficit (i.e., the left hand side in (20)) has to be bounded away from zero, so the inequality is trivial by choosing C_n sufficiently large.

Observe that since the proof sketched above uses a compactness argument, there is no information on the value of C_n. On the other hand, the metric used on the right hand side in (20) is as strong as one could hope for, and in this sense this result is remarkably strong.

3.2.2. From Sobolev to Gagliardo-Nirenberg. It has recently been shown in [2] that one may deduce the sharp forms of the GN inequalities in (15) from the sharp Sobolev inequality (14). Of course, using Hölder inequality it is quite easy to deduce (15) with a non-optimal constant from (14). The argument in [2], which we learned from Bakry, is more subtle: In particular, as we show below, one deduces the particular two-dimensional GN inequality (17) from the four-dimensional Sobolev inequality.

The four-dimensional version of the sharp Sobolev inequality (14) has the explicit form

$$\|f\|_4^2 \le \frac{1}{4\pi}\sqrt{\frac{3}{2}}\|\nabla f\|_2^2 \,, \tag{21}$$

and equality holds if $f = g$, where

$$g(x,y) := \frac{1}{1+|y|^2+|x|^2} \qquad x, y \in \mathbb{R}^2 \,. \tag{22}$$

The key observation is that g can be written as

$$g(x,y) = \frac{1}{G(y)+|x|^2} \qquad \text{with} \quad G(y) := v^{-2}(y) = 1+|y|^2.$$

Hence, if $u \in W^{1,2}(\mathbb{R}^2)$ is a non-negative function satisfying

$$\|u\|_6 = \|v\|_6 = \frac{\pi}{2} \,, \qquad \sqrt{2}\|\nabla u\|_2 = \|u\|_4^2 \,, \tag{23}$$

and we define $f : \mathbb{R}^4 \to \mathbb{R}$ as

$$f(x,y) := \frac{1}{F(y)+|x|^2} \,, \qquad F(y) := u^{-2}(y) \,, \qquad x, y \in \mathbb{R}^2 \,,$$

then by a direct computation one gets

$$\begin{aligned}\delta_{\mathrm{GN}}[u] &= \left(\int_{\mathbb{R}^2} |\nabla u|^2\,dy\right)^{1/2} \left(\int_{\mathbb{R}^2} u^4\,dy\right)^{1/2} - \left(\pi\int_{\mathbb{R}^2} u^6\,dy\right)^{1/2} \\ &= \sqrt{3}\left(\frac{1}{4\pi}\sqrt{\frac{3}{2}}\|\nabla f\|_2^2 - \|f\|_4^2\right) ,\end{aligned}$$

see [2, Chapter 7] or [7, Proposition 2.1].

Remark 3.2. Observe that, given $u \in W^{1,2}(\mathbb{R}^2)$ with $u \not\equiv 0$, we can always multiply it by a constant so that $\|u\|_6 = \|v\|_6$, and then scale it as $\mu^{1/3}u(\mu y)$ choosing μ to ensure that $\sqrt{2}\|\nabla u\|_2 = \|u\|_4^2$. Since (17) is invariant under this scaling, the above inequality proves in particular (17).

3.2.3. Sketch of the proof of Theorem 3.1. The stability estimate (20) combined with the Sobolev inequality (21) asserts the existence of a universal constant C_0 such that

$$C_0\sqrt{3}\left(\frac{1}{4\pi}\sqrt{\frac{3}{2}}\|\nabla f\|_2^2 - \|f\|_4^2\right) \geq \inf_{c\in\mathbb{R},\, \mu>0,\, z_0\in\mathbb{R}^4} \|f - g_{c,\mu,z_0}\|_4^2 .$$

Hence, by the result discussed in the previous Section 3.2.2, whenever u satisfies conditions (23) we have

$$C_0\,\delta_{\rm GN}[u] \geq \inf_{c\in\mathbb{R},\, \mu>0,\, x_0,y_0\in\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\frac{1}{u^{-2}(y)+|x|^2} - \frac{c\mu}{1+\mu^2|x+x_0|^2+\mu^2|y+y_0|^2}\right|^4 dx\,dy\right)^{1/2}. \tag{24}$$

Now, to prove the theorem we need to show that the right hand side in (24) controls $\|u^6 - v_{1,y_0}^6\|_1^{1/2}$. This is obtained in [7, Lemmas 2.3 and 2.4] using some elementary (though non-trivial) arguments.

4. A quantitative convergence result for the critical mass Keller-Segel equation

In this last section we describe the results from [7] on the long-time asymptotic for the critical mass Keller-Segel (KS) equation in $\mathbb{R}^2$:

$$\frac{\partial\rho}{\partial t}(t,x) = \operatorname{div}\big[\nabla\rho(t,x) - \rho(t,x)\nabla c(t,x)\big] , \tag{25}$$

where $\rho(0,x) \geq 0$ belongs to $L^1(\mathbb{R}^2)$, and c satisfies $-\Delta c = \rho$, that is

$$c(t,x) = -\frac{1}{2\,\pi}\int_{\mathbb{R}^2}\log|x-y|\rho(t,y)\,dy\,.$$

It is immediate to check that, at least formally, the total mass $\int_{\mathbb{R}^2}\rho(t,x)\,dx$ is constant in time. Moreover, again by a formal argument, the time evolution for the second moments gives

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2\rho(t,x)\,dx = 4M - \frac{1}{2\pi}M^2, \qquad \text{where} \quad M := \int_{\mathbb{R}^2}\rho(0,x)\,dx\,,$$

see for instance [4, Section 1.1]. Since the second moments cannot become negative, the above computation suggests that for $M > 8\pi$ the formal argument has to fail, and in particular the solution cannot be smooth.

Indeed, it is by now well-known (see for instance [4] and the references therein) that if the initial datum has a mass less than 8π, diffusion dominates and the solution diffuses away to infinity; if the initial datum has a mass greater than 8π, the restoring drift dominates and the solution collapses in finite time; if the initial datum has a mass equal to 8π (the critical mass case), then the solution exists globally in time and there are infinitely many steady-states, which (up to a translation) are given by

$$\sigma_\kappa(x) := \frac{8\kappa}{(\kappa + |x|^2)^2}\,, \qquad \kappa > 0\,. \tag{26}$$

Note that $\int_{\mathbb{R}^2} \sigma_\kappa(x)\,dx = 8\pi$ for all κ.

The existence of infinitely many steady states raises the question of knowing to which one of them a solution of KS would converge. A natural answer is provided by the following energy functionals: for any $\kappa > 0$ we define

$$\mathcal{H}_\kappa[\rho] := \int_{\mathbb{R}^2} \frac{|\sqrt{\rho}(y) - \sqrt{\sigma_\kappa}(y)|^2}{\sqrt{\sigma_\kappa}(y)}\,dy\,. \tag{27}$$

It is evident that $\mathcal{H}_\kappa[\rho]$ is uniquely minimized at $\rho = \sigma_\kappa$, and as shown in [6, 4] this functional is decreasing in time along a solution of KS. Moreover it is not difficult to check that if $\mathcal{H}_\kappa[\rho] < \infty$ for some κ, then $\mathcal{H}_{\kappa'}[\rho] = +\infty$ for any $\kappa' \neq \kappa$. Hence, if we consider an initial datum such that $\mathcal{H}_{\kappa_0}[\rho(0,\cdot)] := E_0 < \infty$ for some $\kappa_0 > 0$, then $\mathcal{H}_{\kappa_0}[\rho(t,\cdot)] \le E_0$ for all $t > 0$ and we may expect that $\rho(t,\cdot)$ should converge to σ_{κ_0} as $t \to \infty$. This has been proved in [4] using a compactness argument. Our goal here is to show how to obtain an explicit rate of convergence.

By formally differentiating $\mathcal{H}_{\kappa_0}[\rho(t,\cdot)]$ in time one gets

$$\frac{d}{dt}\mathcal{H}_{\kappa_0}[\rho(t,\cdot)] = -\mathcal{D}[\rho(t,\cdot)]\,,$$

where the "dissipation functional" $\mathcal{D}$ is defined as

$$\mathcal{D}[\sigma] := \frac{1}{\pi}\left(\|\nabla u\|_2^2 \|u\|_4^4 - \pi\|u\|_6^6\right)\,, \qquad u := \sigma^{1/4}. \tag{28}$$

At a rigorous level, it is proved in [4] that if $\mathcal{H}_{\kappa_0}[\rho(0,\cdot)] < \infty$, then there exists a solution of KS (called "properly dissipative") such that

$$\mathcal{H}_{\kappa_0}[\rho(T,\cdot)] + \int_0^T \mathcal{D}[\rho(t,\cdot)]\,dt \le \mathcal{H}_{\kappa_0}[\rho(0,\cdot)] \tag{29}$$

for all $T > 0$. Hence, as an immediate consequence of (29) we get that, for any $T \ge 2$,

$$\inf_{t\in[1,T]} \mathcal{D}[\rho(t,\cdot)] \le \frac{1}{T-1}\int_1^T \mathcal{D}[\rho(t,\cdot)]\,dt \le \frac{1}{T-1}\mathcal{H}_{\kappa_0}[\rho(0,\cdot)]\,. \tag{30}$$

(The reason for considering $t \geq 1$ is to ensure that some time passes so that the solution enjoys some further regularity properties needed to apply our estimates.)

Observe now that for any density σ on $\mathbb{R}^2$ such that $\|\nabla\sigma^{1/4}\|_2 < \infty$,

$$\mathcal{D}[\sigma] = \Big(\|\nabla\sigma^{1/4}\|_2\|\sigma^{1/4}\|_4^2 + \sqrt{\pi}\|\sigma^{1/4}\|_6^3\Big)\,\delta_{\mathrm{GN}}(\sigma^{1/4})\ . \tag{31}$$

Hence, taking advantage of some uniform a priori bound on solutions to KS, we deduce the existence of some $\bar{t} \in [1, T]$ such that

$$\delta_{\mathrm{GN}}[\rho^{1/4}(\bar{t},\cdot)] \leq \frac{C}{T}\mathcal{H}_{\kappa_0}[\rho(0,\cdot)]$$

for some universal constant C, so by the stability Theorem 3.1 we conclude that

$$\|\rho(\bar{t},\cdot)^{3/2} - \sigma_\kappa(\cdot - x_0)^{3/2}\|_1 \leq C\left(\frac{1}{T}\mathcal{H}_{\kappa_0}[\rho]\right)^{1/2}$$

for some $x_0 \in \mathbb{R}^2$ and $\kappa > 0$ (recall that the density v_λ^4 is a multiple of $\sigma_{1/\lambda}$).

Now, using some uniform estimates on the pth moments of the solution and its L^q norms for all $p < 2$ and $q < \infty$, and exploiting that the KS evolution preserves the baricenter (in particular, without loss of generality we can assume that $\rho(t,\cdot)$ has baricenter at the origin for all t), we obtain

$$\|\rho(\bar{t},\cdot) - \sigma_\kappa\|_1 \leq C\left(\frac{1}{T}\mathcal{H}_{\kappa_0}[\rho]\right)^{(p-1)/4p}\ , \tag{32}$$

for all $p < 2$ (here C depends also on p).

Hence the above inequality bounds the time it takes a solution of the critical mass Keller-Segel equation to approach σ_κ for some κ. However, to get a quantitative convergence result, we must do two more things:

(A) Show that $\rho(t,\cdot)$ approaches σ_κ for $\kappa = \kappa_0$.

(B) Show that eventually it remains close.

While (A) is relatively easy since $H_{\kappa_0}[\sigma_\kappa] = +\infty$ if $\kappa \neq \kappa_0$, (B) is much more involved. To achieve it, we first recall that there is another functional which is decreasing along the KS evolution: this is the Logarithmic Hardy-Littlewood-Sobolev (Log-HLS) functional $\mathcal{F}$, defined by

$$\mathcal{F}[\rho] := \int_{\mathbb{R}^2}\rho(x)\log\rho(x)\,dx + 2\left(\int_{\mathbb{R}^2}\rho(x)\,dx\right)^{-1}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\log|x-y|\rho(y)\,dx\,dy.$$

(The fact that such a functional is decreasing along the KS equation is not really surprising, since the KS equation can be interpreted as the gradient flow of $\mathcal{F}$ with respect to the 2-Wasserstein distance.) This functional is invariant under scale changes $a \mapsto a^2\rho(ax)$. In particular, $\mathcal{F}[\sigma_\kappa]$ is independent of κ. Moreover the functions $\{\sigma_\kappa\}_{\kappa>0}$ uniquely minimize $\mathcal{F}$, see [**?**, 8].

Keeping this in mind, for (B) we proceed as follows: first we show almost Lipschitz regularity of $\mathcal{F}$ in L^1 [7, Theorem 3.7], which combined with (32) and the fact that p can be chosen close to 2, allows us to deduce that for any $\epsilon > 0$ and $T \geq 2$ there exists $\bar{t} \in [1, T]$ such that

$$\mathcal{F}[\rho(\bar{t}, \cdot)] - \min \mathcal{F} \leq C\, T^{-(1-\epsilon)/8}.$$

Then, since $\bar{t} \leq T$ and $\mathcal{F}[\rho(t, \cdot)]$ is decreasing in time, we get

$$\mathcal{F}[\rho(T, \cdot)] - C(8\pi) \leq C\, T^{-(1-\epsilon)/8} \tag{33}$$

for all $T \geq 2$. Finally, in order to conclude that $\rho(T, \cdot)$ is close to some σ_κ we prove a stability result for the Log-HLS functional [7, Theorem 1.9]. (This is obtained exploiting Theorem 3.1 and some dissipation properties of the Log-HLS functional along a fast diffusion equation, see the proof of [7, Theorem 1.9] for more details.) Using this second stability result (combined with some additional time regularity estimates on the solution, see [7, Lemma 3.8]), one finally deduces for all $t \geq 2$ the existence of some constant $\kappa(t) > 0$ such that

$$\|\rho(t, \cdot) - \sigma_{\kappa(t)}\|_1 \leq C\, t^{-(1-\epsilon)/320}.$$

Finally, a simple argument using the sensitive dependence of $\mathcal{H}_{\kappa_0}$ on tails allows us to show that $\kappa(t)$ converges at a logarithmic rate to κ_0.

Thus, the final convergence result proved in [7, Theorem 3.5] becomes:

Theorem 4.1. *Let $\rho(t, x)$ be any properly dissipative solution of the Keller-Segel equation of critical mass $M = 8\pi$ such that $\mathcal{H}_{\kappa_0}[\rho(0, \cdot)] < \infty$ for some $\kappa_0 > 0$, and $\mathcal{F}[\rho(0, \cdot)] < \infty$. Assume that $\int_{\mathbb{R}^2} x\rho(x, 0)\, dx = 0$. Then, for all $\epsilon > 0$ there are constants C_1 and C_2, depending only on ϵ, κ, $\mathcal{H}_{\kappa,8\pi}[\rho(0, \cdot)]$ and $\mathcal{F}[\rho(0, \cdot)]$, such that, for all $t > 0$,*

$$\mathcal{F}[\rho(t, \cdot)] - C(8\pi) \leq C_1(1 + t)^{-(1-\epsilon)/8}$$

$$\inf_{\kappa > 0} \|\rho(t, \cdot) - \sigma_{\kappa,8\pi}\|_1 \leq C_2(1 + t)^{-(1-\epsilon)/320}.$$

Moreover, there is a positive number $a > 0$, depending only on $\mathcal{H}_{\kappa_0}[\rho(0, \cdot)]$ and $\mathcal{F}[\rho(0, \cdot)]$, so that for each $t > 0$,

$$\inf_{\kappa > 0} \|\rho(t, \cdot) - \sigma_\kappa\|_1 = \min_{a < \kappa < 1/a} \|\rho(t, \cdot) - \sigma_\kappa\|_1.$$

Finally, for each $t > 0$ the above minimum is achieved at some value $\kappa(t)$ satisfying

$$(\kappa(t) - \kappa_0)^2 \leq \frac{C}{\log(e + t)}\ .$$

In particular

$$\|\rho(t, \cdot) - \sigma_{\kappa_0}\|_1 \leq \frac{C}{\sqrt{\log(e + t)}}\ .$$

It is interesting that the approach to equilibrium described by these quantitative bounds takes place on two separate time scales: The solution approaches the one-parameter family of (centered) stationary states with at least a polynomial rate. Then, perhaps much more gradually, at only a logarithmic rate, the solution adjusts its spatial scale to finally converge to the unique stationary solution within its basis of attraction. It looks reasonable to expect such behavior: The initial data may, for example, be exactly equal to σ_{κ_0} on the complement of a ball of very large radius R, and yet may "look much more like" σ_κ on a ball of smaller radius for some $\kappa \neq \kappa_0$. One can then expect the solution to first approach σ_κ, and then only slowly begin to feel its distant tails and make the necessary adjustments to the spatial scale.

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