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Variational models for the incompressible Euler equations

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1.1 Introduction

In these notes we consider different models to describe the motion of homogeneous incompressible fluids inside a bounded Lipschitz domain $D \subseteq \mathbb{R}^d$ without the action of external forces.

A classical model is given by the Euler equations, which describe the evolution of the *velocity field* of the fluid $v : [0,T] \times D \to \mathbb{R}^d$,

$$\begin{cases} \partial_t v + \big(v \cdot \nabla\big) v + \nabla p = 0 & \text{in } [0,T] \times D \\ \operatorname{div} v = 0 & \text{in } [0,T] \times D, \end{cases} \tag{1.1}$$

coupled with the boundary condition

$$v \cdot \nu = 0 \quad \text{on } [0,T] \times \partial D, \tag{1.2}$$

where ν is the unit exterior normal to ∂D. If $v = (v^1, \dots, v^d) : [0,T] \times D \to \mathbb{R}^d$, then (adopting the summation convention) $\operatorname{div} v = \partial_j v^j$ is the spatial divergence of v, ∇v is the spatial gradient, and $\big(v \cdot \nabla\big) v$ is the vector in $\mathbb{R}^d$ whose i-th component is given by $v^j \partial_j v^i$. Hence, (1.1) is a system of $(d+1)$ equations for the $(d+1)$ unknowns $(v^1, \dots, v^d, p)$, where $p : [0,T] \times D \to \mathbb{R}$ physically represents the *pressure field*.

The motion of an incompressible fluid inside D can be described also from a Lagrangian viewpoint, namely through the motion of its particles with respect to their initial position. To pass from the Eulerian to the Lagrangian formulation, let us assume that v is a smooth solution of (1.1), and let $g : [0,T] \times D \to \mathbb{R}^d$ be the flow map

$$\begin{cases} \dot g(t,a) = v(t, g(t,a)) & (t,a) \in [0,T] \times D \\ g(0,a) = a & a \in D. \end{cases} \tag{1.3}$$

Due to the boundary condition (1.2), we get $g(t,D) = D$ for all $t \in [0,T]$. Moreover, differentiating (1.3) and using the classical identity

$$\frac{d}{d\epsilon}_{|\epsilon=0} \det(A+\epsilon BA) = \mathrm{tr}(B)\det(A),$$

one obtains

$$\partial_t \big(\det \nabla g(t,a)\big) = \mathrm{div}\, v(t, g(t,a)) \det \nabla g(t,a), \tag{1.4}$$

which because of the incompressibility constraint $\mathrm{div}\, v = 0$ implies

$$\det \nabla g(t,a) \equiv 1, \quad \forall\, t \in [0,T].$$

Hence $g(t) := g(t,\cdot)$ belongs to the space $SDiff(D)$ of *orientation and measure-preserving diffeomorphisms* of D. Moreover, differentiating (1.3) with respect to t and using the Euler equations (1.1), we obtain that the map $t \mapsto g(t)$ satisfies the ODE

$$\ddot{g}(t,a) = -\nabla p(t, g(t,a)) \quad \text{in } [0,T] \times D \tag{1.5}$$

with the constraint

$$g(t) \in SDiff(D), \qquad \forall\, t \in [0,T]. \tag{1.6}$$

On the other hand it is not difficult to show that the converse is also true: in the smooth case $v : [0,T] \times D \to \mathbb{R}^d$ solves the Euler equations (1.1)-(1.2) with initial condition $v(0) = v_0$ if and only if its flow map g satisfies the ODE (1.5)-(1.6) with the initial conditions $\dot{g}(0) = v_0$ and $g(0) = i_D$, being $i_D : D \to D$ the identity map.

In the first part of these notes (Section 1.2) we review some results concerning the existence and uniqueness of solutions of (1.1), both in the classical and in the weak (distributional) setting. In Section 1.2.1 we will focus on the proof of the global (in time) existence and uniqueness of weak solutions with bounded vorticity in dimension $d = 2$ ([Yud63]). If on the one hand local existence and uniqueness of classical solutions of (1.1) can be obtained under suitable smoothness assumptions on the initial data (see e.g. [BM02]), on the other hand no global existence result is available in dimension $d \geq 3$, not even of weak (distributional) solutions. In Section 1.2.2 we present the notion of *generalized measure-valued solutions* introduced by DiPerna and Majda in [DiPM87]. The measure-valued solutions of DiPerna and Majda are globally defined and include the vanishing viscosity limits of Leray solutions of Navier-Stokes equations. Despite the fact that they are not unique, a weak-strong uniqueness result holds ([BDS11]).

The second and more substantial part of these notes (Sections 1.3, 1.4 and 1.5) is devoted to the study of some variational formulations of the problem (1.5).

The starting point of these variational models is Arnold's interpretation of the ODE (1.5) as the geodesic equation on $SDiff(D)$, seen (formally) as an infinite-dimensional submanifold of $L^2(D;\mathbb{R}^d)$ with respect to the induced metric ([Arn66]). Therefore, in analogy with the finite dimensional Riemannian setting, one is led to study the following:

Problem 1.1. Given g_0, $g_T \in SDiff(D)$, find a smooth curve $[0,T] \ni t \mapsto \bar{g}(t) \in SDiff(D)$ minimizing the energy

$$\mathcal{E}(g) = T \int_0^T \frac{1}{2} \|\dot{g}(t)\|^2_{L^2(D;\mathbb{R}^d)} \, dt, \tag{1.7}$$

among all curves in $[0,T] \ni t \mapsto g(t) \in SDiff(D)$ satisfying $g(0) = g_0$, $g(T) = g_T$.

Notice that Problem 1.1 is essentially different from (1.5). Indeed, instead of prescribing the initial velocity of the curves $\dot{g}(0) = v_0$, we assign their final position $g(T)$. Moreover, we look only for minimizers of (1.7) instead of considering all its critical points, which again formally correspond to solutions of (1.5).

As we will see in Section 1.3.1, the existence of energy minimizing curves for (1.7) is guaranteed only if g_0 and g_T are close in a very strong topology ([EM70]). In general, as shown by Shnirelman ([Shn87], [Shn94]) there could be no curves of finite energy if $d = 2$, or the infimum in (1.7) could not be attained.

From the classical variational viewpoint, the main difficulties lie in the fact that the topology induced by the energy (1.7) does not permit to preserve the constraint of being diffeomorphisms, or even maps, for all $t \in [0,T]$.

Even trying to attack the minimization problem (1.7) with the simpler strategy of projecting on $SDiff(D)$ time-discretized geodesics in $L^2(D;\mathbb{R}^d)$ (see Section 1.3.2), the need for relaxation in space is evident.

These preliminary considerations led Brenier [Bre89] to introduce a weaker variational formulation of (1.7), allowing both for non-injective flow maps $t \mapsto g(t)$ taking values in the space of (Lebesgue) measure-preserving maps, and also for the splitting/crossing of fluid particles (see Section 1.4).

Assuming g_0 to be equal to i_D (as explained in Section 1.3.1, this can be done without loss of generality), the dynamics of the possible trajectories followed by the particles is now described by probability measures concentrated on the space of continuous paths $\Omega(D) := C([0,T];D)$.

More precisely, one considers the following minimization problem (here e_t denotes the evaluation map at time t, that is $\mathrm{e}_t(\omega) = \omega(t)$, and $\mathcal{L}_D$ denotes the Lebesgue measure on D renormalized in such a way that $\mathcal{L}_D(D) = 1$, see Sections 1.3 and 1.4 for more details):

Problem 1.2. Given $h \in SDiff(D)$, find a minimizer of the action

$$\mathcal{A}(\boldsymbol{\eta}) = T \int_{\Omega(D)} \int_0^T \frac{1}{2} |\dot{\omega}(t)|^2 \, dt \, d\boldsymbol{\eta}(\omega) \tag{1.8}$$

among all $\boldsymbol{\eta} \in \mathcal{P}(\Omega(D))$ satisfying

$$(\mathrm{e}_t)_\sharp \boldsymbol{\eta} = \mathcal{L}_D, \quad \forall\, t \in [0,T], \tag{1.9}$$

and

$$(\mathrm{e}_0 \times \mathrm{e}_T)_\sharp \boldsymbol{\eta} = (i_D \times h)_\sharp \mathcal{L}_D.$$

Following Brenier [Bre89], probability measures $\boldsymbol{\eta} \in \mathcal{P}(\Omega(D))$ such that (1.9) holds are called *generalized incompressible flows*.

Notice that any flow of measure-preserving diffeomorphisms $t \mapsto g(t) \in SDiff(D)$ with $g(0) = i_D$ induces a generalized incompressible flow $\boldsymbol{\eta}_g := (\Phi_g)_\sharp \mathcal{L}_D$, where $\Phi_g : D \to \Omega(D)$, $\Phi_g(a) = g(\cdot, a)$, and $\mathcal{E}(g) = \mathcal{A}(\boldsymbol{\eta}_g)$.

However, the converse is not always true: by definition, for any intermediate time $t \in (0, T)$, $(\mathrm{e}_0, \mathrm{e}_t)_\sharp \boldsymbol{\eta} \in \Gamma(D)$ –where $\Gamma(D) = \{\gamma \in \mathcal{P}(D \times D) : (\pi_1)_\sharp \gamma = \mathcal{L}_D, (\pi_2)_\sharp \gamma = \mathcal{L}_D\}$ is the set of *measure-preserving transport plans* of D– but $(\mathrm{e}_0, \mathrm{e}_t)_\sharp \boldsymbol{\eta}$ is *deterministic* only if there exists a (measure-preserving) map $h_t : D \to D$ such that $(\mathrm{e}_0, \mathrm{e}_t)_\sharp \boldsymbol{\eta} = (i_D \times h_t)_\sharp \mathcal{L}_D$. However, there are generalized incompressible flows connecting the identity to some $h \in SDiff(D)$ for which $(\mathrm{e}_0, \mathrm{e}_t)_\sharp \boldsymbol{\eta}$ is not deterministic at some intermediate time (see Section 1.4.4). Hence, in Section 1.4.1 we will describe an extension of Brenier's formulation, introduced by Ambrosio and Figalli in [AF09], which allows to connect any $\eta, \gamma \in \Gamma(D)$ and then to study the behaviour of generalized flows also for intermediate times.

In Section 1.4.2 we prove the existence of minimizers of Problem 1.2 connecting i_D to any $\eta \in \Gamma(D)$ when D is the d-dimensional torus $D = \mathbb{T}^d$, $d \geq 2$ ([Bre89]). This result can be extended also to other Lipschitz domains D and flows between any $\eta, \gamma \in \Gamma(D)$ ([Shn94],[AF09]).

In Section 1.4.3 we deal with the consistency of minimizers of the action (1.8) with classical solutions of (1.5). As proved by Brenier in [Bre89], under appropriate assumptions on the pressure vector field (second spatial derivatives uniformly bounded in time) and for T sufficiently small, the generalized flows are unique and induced by classical solutions. However, for times larger than a fixed T (depending on the L^∞-norm of the second spatial derivatives of p) the uniqueness property may be lost, even among classical solutions. An interesting non-uniqueness example on the 2-dimensional disc is presented in Section 1.4.4 (see [Bre89], [BFS08]).

However, despite the lack of uniqueness of action minimizing flows, in [Bre99] (see Section 1.5.1) it has been shown that, given $h \in S(D)$ for which the infimum of the action among generalized incompressible flows between i_D and h is finite, there exists a distribution p which acts as a Lagrange multiplier for the incompressibility constraint. More precisely, a generalized incompressible flow is a minimizer of the action between i_D and h if and only if it minimizes an augmented action functional (including a distribution p) among a broader class of flows, called *almost-incompressible generalized flows* (see Section 1.5.1).

A surprising feature of this variational model is that this distribution p is unique, up to trivial modifications (see Section 1.5.2), and is called the *pressure field* since it coincides with the usual one in the smooth case.

In Section 1.5.3 we present an equivalent formulation of the minimization Problem 1.2 of mixed Eulerian-Lagrangian type. This model, introduced by Brenier in [Bre99], provides a finer description of the trajectories followed by the optimal generalized flows, and allows to show that p is indeed a function, and not merely a distribution ([Bre99], [AF08]). In particular, $p \in L^2_{\text{loc}}((0,T); L^{d/(d-1)}_{\text{loc}}(D))$.

In Section 1.6 we restrict for simplicity to the d-dimensional torus $\mathbb{T}^d$ and we report the analysis made in [AF09] on the necessary and sufficient conditions for optimality of generalized flows. In particular, as in the smooth case, we will see (Theorem 1.54) that almost every trajectory followed by an optimal flow is a local minimizer of the action

$$\int_0^T \frac{1}{2}|\dot\omega(t)|^2 - p(t,\omega(t))\,dt$$

among all curves of finite action connecting $\omega(0)$ and $\omega(T)$. The main difficulty here is to define the value of p along curves in D and near $t = 0$ and $t = T$, due to the local integrability assumption. A second necessary condition will be related to the local optimal transportation properties of the intermediate plans determined by the optimal generalized flows (see Theorem 1.56). In the end, we will see that the validity of both these necessary conditions is also sufficient for optimality (Theorem 1.57).

1.2 Incompressible Euler equations

In this section we consider the Cauchy problem for the Euler equations (1.1) with boundary condition (1.2). Here D is a bounded and simply connected domain of $\mathbb{R}^d$ with C^2 boundary, and we denote by $v_0 : D \to \mathbb{R}^d$ the initial condition.

The existence of *classical* (i.e., sufficiently smooth) *solutions* was already investigated by Gunther and Lichtenstein at the end of the '20s. In [Gun27] and [Lic25] they proved local (in time) existence and uniqueness of velocity fields $v \in C^{1,\lambda}$, $0 < \lambda < 1$, as soon as v_0 is sufficiently smooth. As for classical solutions, an existence result stated in modern form is the following (see e.g. [Tay96], Chapter 17, Theorem 2.1 and Proposition 2.2, or [BM02], Theorem 3.4):

Theorem 1.3. *If $v_0 \in H^s$ for some $s > \left[\frac{n}{2}\right] + 1$, then $\exists\, T = T(\|v_0\|_{H^s}) > 0$ such that there exists a unique classical solution of (1.1) on $[0,T] \times D$.*

However, no global existence theorem valid in all dimensions is presently known.

A property satisfied by classical solutions is *energy conservation*. Indeed,

$$\begin{aligned}
\frac{d}{dt}\int_D \frac{1}{2}|v(t,x)|^2\,dx &= \int_D v^i\partial_t v^i \overset{(1.1)}{=} -\int_D v^i v^j \partial_j v^i - \int_D v^i\partial_i p \\
&= -\int_D v^j \partial_j \frac{|v|^2}{2} + \int_D \partial_i v^i p \\
&= \int_D \partial_j v^j \left(\frac{|v|^2}{2} + p\right) \overset{(\operatorname{div} v=0)}{=} 0.
\end{aligned}$$

Henceforth, the natural energy space where to look for weak distributional solutions is $L^\infty([0,T];L^2(D;\mathbb{R}^d))$. As observed in [DLS09], the search for weak distributional solutions of the Euler equations, other than being natural from the PDEs viewpoint, comes also from Kolmogorov's theory of turbulence.

Definition 1.4 (Distributional solutions). *We say that a vector field $v \in L^\infty([0,T];L^2(D;\mathbb{R}^d))$ is a weak solution to the Cauchy problem for (1.1) with initial datum $v_0 \in L^2(D;\mathbb{R}^d)$ if, for all $\varphi \in C_c^\infty([0,T)\times D;\mathbb{R}^d)$ with $\operatorname{div}\varphi = 0$ and for all $\xi \in C^\infty(D)$,*

$$\int_0^T \int_D v \cdot \partial_t \varphi + (v \otimes v) : \nabla \varphi \, dx \, dt + \int_D v_0 \varphi \, dx = 0 \tag{1.10}$$

and

$$\int_D v(t,\cdot) \cdot \nabla \xi = 0, \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T].$$

We recall that any weak distributional solution $v \in L^\infty([0,T];L^2(D;\mathbb{R}^d))$ of (1.1) belongs (up to redefining it on a $\mathcal{L}^1$-negligible set of times) to the space $C([0,T];L^2_w(D;\mathbb{R}^d))$ (see e.g. [DLS10], Lemma 7.1).

Let us mention that, with the exception of the case $d = 2$ in which the existence and uniqueness of global weak solutions can be proved under additional regularity assumptions on the vorticity of the initial data (see Section 1.2.1), in higher dimensions no general theorem providing global solutions is known.

Moreover, as shown first by Scheffer in [Sch93] and Shnirelman [Shn97], weak solutions may not be unique. In [DLS09], De Lellis and Székelyhidi introduced a new framework in which to study oscillatory non-uniqueness phenomena for the Euler equations and proved the following theorem (implying the results of [Sch93] and [Shn97] with a much simpler proof).

Theorem 1.5. *For all $d \geq 2$, there exist $v \in L^\infty(\mathbb{R}\times\mathbb{R}^d;\mathbb{R}^d)$ and $p \in L^\infty(\mathbb{R}\times\mathbb{R}^d)$, solving (1.1) in the distributional sense, such that v is not identically zero and $\operatorname{supp} v$, $\operatorname{supp} p$ are compact subsets of $\mathbb{R}\times\mathbb{R}^d$.*

We note that the fact that Theorem 1.5 holds also in dimension two does not clash with the uniqueness result of Section 1.2.1 (see Theorem 1.6), since De Lellis and Székelyhidi's solutions do not satisfy the required regularity assumptions. Theorem 1.5 shows that there are very pathological examples of solutions to Euler: since the support of v is compact in space-time, it means that at some initial time the fluid is at rest ($v \equiv 0$) but then at some moment, without the action of any external force, it starts suddenly to move, and finally it comes back to rest. In particular, for such solutions the conservation of the L^2 norm is violated.

1.2.1 Weak solutions in the two dimensional case

In two dimensions much more is known about the existence of global solutions of (1.1). Indeed, as we will see, in the two dimensional case the equation satisfied by the vorticity of v has a nice structure, and the results obtained for this equation translate, under suitable regularity assumptions on the initial data, into the global (in time) existence and uniqueness of weak solutions of the Euler equations.

Recall that, for any $w \in L^1(D;\mathbb{R}^2)$, the *vorticity* of w is the distribution defined by

$$\operatorname{curl} w := \partial_2 w^1 - \partial_1 w^2.$$

Moreover, for time dependent vector fields $u \in L^1([0,T] \times D;\mathbb{R}^2)$ we will use the same notation $\operatorname{curl} u$ to denote the time-dependent distribution

$$t \mapsto \operatorname{curl} u(t) := \operatorname{curl}(u(t,\cdot)). \tag{1.11}$$

The global existence of classical solutions for sufficiently smooth initial data was proved by Wolibner [Wol33]; Kato presented the result in a modern form in [Kat68].

The generalized solutions in the two dimensional case were first introduced by Yudovich in [Yud63], providing the following global existence and uniqueness theorem for initial data having bounded vorticity.

Theorem 1.6 ([Yud63]). *For any $v_0 \in L^2(D;\mathbb{R}^2)$ such that $\operatorname{curl} v_0 \in L^\infty(D)$, there exists a unique weak distributional solution of (1.1) $v \in L^\infty([0,+\infty);L^2(D))$ with initial condition v_0 and satisfying*

$$\operatorname{curl} v \in L^\infty([0,+\infty) \times D)$$

More precisely, Yudovich proved that the global existence of solutions holds whenever $\operatorname{curl} v_0 \in L^p(D)$ for any given $p > 1$. However, the uniqueness property was shown only in the case of bounded vorticity. In [Yud95], the uniqueness result was improved allowing for initial vorticities which belong to $\cap_{p\in[1,+\infty)} L^p$, with some restriction on the growth of the L^p-norms as $p \to +\infty$. The existence part of Theorem 1.6 was improved by Delort [Del91] to initial vorticities of the form $\omega_0 = \omega_0' + \omega_0''$, where ω_0' is a non-negative compactly supported Radon measure in $H^{-1}(\mathbb{R}^2)$ and ω_0'' an L^p compactly supported function ($p > 1$). Delort's solutions belong as well to the space $L^\infty_{\mathrm{loc}}([0,+\infty);L^2_{\mathrm{loc}}(D))$ and their vorticities $\omega(t)$ are, for all $t \in [0,+\infty)$, of the form $\omega = \omega' + \omega''$, where ω' is a positive measure with mass uniformly bounded in t by the mass of ω_0' and $\omega'' \in L^\infty([0,+\infty);L^q(D))$, for all $q \in [1,p]$. The technique used in [Del91] is however rather different w.r.t. the one used by Yudovich to prove Theorem 1.6, and we will not describe it here.

The aim of this Section is to give a proof of Theorem 1.6. As anticipated, the first idea to obtain weak solutions of (1.1) is to study the PDE satisfied by their vorticity (Propositions 1.9 and 1.10).

In Proposition 1.8 we will see that it is possible to reconstruct a vector field from its vorticity, under suitable integrability assumptions. Before giving the precise statement, we need to define some elliptic and distributional operators.

For any $\rho \in L^\infty(D)$, we denote by $\triangle^{-1}\rho$ the weak solution of the Dirichlet problem

$$\begin{cases} \triangle\psi = \rho & \text{in } D \\ \psi = 0 & \text{on } \partial D \end{cases} \tag{1.12}$$

By standard elliptic theory, $\psi = \triangle^{-1}\rho \in W^{2,p}(D) \cap W^{1,p}_0(D)$ for all $p \in [2, +\infty)$. Moreover, one has the bounds

$$\|\triangle^{-1}\rho\|_{W^{2,p}(D)} \leq C(p,D)\|\rho\|_{L^\infty(D)}, \quad p \in [2,+\infty). \tag{1.13}$$

Denoting by $\nabla^\perp$ the distributional operator

$$\nabla^\perp\phi := (\partial_2\phi,\, -\partial_1\phi), \tag{1.14}$$

for any $\rho \in L^\infty$ define

$$K(\rho) := \nabla^\perp\triangle^{-1}\rho, \tag{1.15}$$

where $\nabla^\perp\triangle^{-1}$ is the composition of (1.12) and (1.14).

Notice that, by (1.13) and (1.14), $K(\rho) \in W^{1,p}(D;\mathbb{R}^2)$ for all $p \in [2,+\infty)$. Moreover, one has the following quantitative estimate on the growth with respect to p of its $W^{1,p}$ norms, which will be used in the proof of Theorem 1.6 (see e.g. [Ste70]):

Lemma 1.7. *For any* $\rho \in L^\infty(D)$*,*

$$\|\nabla K(\rho)\|_{L^p(D;\mathbb{R}^2)} \leq C(D)\frac{p^2}{p-1}\|\rho\|_{L^\infty(D)}, \tag{1.16}$$

where $C(D)$ *is a geometric constant.*

Proposition 1.8. *Let* $\rho \in L^\infty(D)$*. Then*

$$\operatorname{curl} K(\rho) = \rho, \tag{1.17}$$

and $K(\rho)$ *is the unique function satisfying* (1.17) *and lying in the space*

$$H := \left\{u \in L^2(D;\mathbb{R}^2) :\, \operatorname{div} u = 0,\, u\cdot\nu = 0 \text{ on } \partial D\right\}.$$

Proof (Proposition 1.8). Set $\psi := \triangle^{-1}\rho$. Then $K(\rho) = \nabla^\perp\psi = (\partial_2\psi, -\partial_1\psi)$ and $\operatorname{curl} K(\rho) = \triangle\psi = \rho$. Moreover, since $\operatorname{div}\nabla^\perp = 0$, $\operatorname{div} K(\rho) = 0$. Finally, since $K(\rho) = \nabla^\perp\psi$ and $\psi = 0$ on ∂D, it follows that $K(\rho)\cdot\nu = \frac{\partial\psi}{\partial\tau} = 0$ on ∂D. Hence $K(\rho) \in H$.

In order to prove that $K(\rho)$ is unique in H, let us assume that $\exists\, w \in H$ such that $\operatorname{curl} w = \rho$ and prove that $u := K(\rho) - w$ is identically zero.

Since both $K(\rho)$ and w have the same curl and are divergence free, $u \in H$ and $\operatorname{curl} u = 0$. Then, since by assumption D is simply connected, there exists $\phi : D \to \mathbb{R}$ satisfying

$$\begin{cases} \nabla\phi = u & \text{in } D, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial D, \end{cases}$$

which implies (since $\operatorname{div} u = 0$)

$$\begin{cases} \triangle\phi = 0 & \text{in } D, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial D. \end{cases}$$

Hence ϕ is constant, and so $u = 0$ as required.

From now on, as done in (1.11) for the vorticity of time dependent vector fields, for any function $\omega \in L^2([0,T] \times D)$ we define $K(\omega)$ as the time dependent distribution

$$K(\omega) : t \mapsto K(\omega(t,\cdot)).$$

Proposition 1.9. *Any smooth vector field $v : [0,T] \times D \to \mathbb{R}^2$ is a solution of* (1.1) *satisfying the boundary condition* (1.2) *if and only if the smooth scalar field $\omega : [0,T] \times D \to \mathbb{R}$ defined as $\omega(t,\cdot) := \operatorname{curl} v(t,\cdot)$ satisfies the transport equation*

$$\partial_t\omega + (K(\omega) \cdot \nabla)\,\omega = 0. \tag{1.18}$$

Proof (Proposition 1.9). Let $v : [0,T] \times D \to \mathbb{R}^2$ be a smooth solution of (1.1). Then, by simple computations, it is possible to show that the smooth scalar field ω satisfies (1.18).

Indeed, taking the curl in (1.1) one obtains

$$\partial_t\omega + (v \cdot \nabla)\,\omega + \operatorname{div} v\omega = 0,$$

that coincides with (1.18) since v is divergence free. (This is a peculiarity of the two dimensional case, since in higher dimensions the right-hand side of this equation contains an additional term.)

Viceversa, if a smooth scalar field ω satisfies

$$\partial_t\omega + (K(\omega) \cdot \nabla)\,\omega = 0 \text{ in } (0,T) \times D, \tag{1.19}$$

then the vector field $v(t,\cdot) := K(\omega(t,\cdot))$ is a solution of the Euler equations.

Indeed, the vorticity of the vector field $w = (w^1, w^2)$ whose components are given by

$$w^i = \partial_t v^i + v^j \partial_j v^i$$

coincides with the left-hand side of (1.19). Hence it is identically 0, and since D is simply connected there exists a scalar field $p : [0,T] \times D \to \mathbb{R}$ whose spatial gradient satisfies (1.1).

In the distributional setting, an analogous correspondence between weak solutions of (1.18) and weak solutions of the Euler equations having bounded vorticity holds.

Proposition 1.10. *Let $v_0 \in L^2(D;\mathbb{R}^2)$ with $\operatorname{curl} v_0 \in L^\infty(D)$. Then, $v \in L^\infty([0,T];L^2(D;\mathbb{R}^2))$ is a weak solution of (1.1) with initial datum $v_0 \in L^2(D)$ and $\operatorname{curl} v \in L^\infty([0,T]\times D)$ if and only if $\omega := \operatorname{curl} v \in L^\infty([0,T]\times D)$ is a weak solution of (1.18) with initial datum $\omega_0 := \operatorname{curl} v_0$, namely*

$$\int_0^T \int_D \omega \partial_t \phi + \omega K(\omega)\cdot \nabla \phi \, dx\, dt + \int_D \omega_0 \phi(0)\, dx = 0, \qquad (1.20)$$

for all $\phi \in C_c^\infty([0,T)\times D)$.

The proof follows by standard integration by parts arguments.

Thanks to Proposition 1.10, Theorem 1.6 is an immediate consequence of the following result.

Theorem 1.11. *For any $\omega_0 \in L^\infty(D)$, there exists a unique weak solution $\omega \in L^\infty([0,+\infty)\times D)$ of the Cauchy problem (1.18) with initial datum ω_0.*

Proof (Theorem 1.11).

Existence

The existence of a solution can be obtained via an explicit Euler scheme combined with a regularization argument. Let $\{\rho^\epsilon\}_{\epsilon>0} \subset C_c^\infty(D)$ be a family of smooth mollifiers with $\operatorname{supp}(\rho^\epsilon) \subset B_\epsilon$, and consider the following scheme: for any $n \in \mathbb{N}$ we define ω_n^ϵ as the solution of the linear transport equation

$$\begin{cases} \partial_t \omega_n^\epsilon + K(\bar\omega_n^\epsilon)\cdot \nabla \omega_n^\epsilon = 0, & \text{in } (0,+\infty)\times D_\epsilon, \\ \omega_n^\epsilon(0) = \omega_0 * \rho^\epsilon, \end{cases} \qquad (1.21)$$

where we have extended ω_0 to be identically zero outside D, and

$$\bar\omega_n^\epsilon(t) := \omega_n^\epsilon(k/n) \qquad \text{if } t \in [k/n,(k+1)/n),\ k \in \mathbb{N}.$$

More precisely, once ω_n^ϵ has been constructed up to a time k/n, we use $\omega_n^\epsilon(k/n)$ to define the vector field $K(\bar\omega_n^\epsilon)$ on the time interval $[k/n,(k+1)/n]$, and then we can define ω_n^ϵ on the interval $[k/n,(k+1)/n]$ by using the method of characteristics: ω_n^ϵ is given by the representation formula

$$\omega_n^\epsilon(t,x) = \omega_n^\epsilon\big(k/n,(X_{n,k}^\epsilon)^{-1}(t-k/n,x)\big), \qquad t \in [k/n,(k+1)/n], \quad (1.22)$$

where

$$\begin{cases} \dot X_{n,k}^\epsilon(s,x) = K(\omega_n^\epsilon(k/n))\left(X_{n,k}^\epsilon(s,x)\right), \\ X_{n,k}^\epsilon(0,x) = x, \end{cases}$$

is the flow of $K(\omega_n^\epsilon(k/n))$ in D (recall that by definition $K\cdot\nu = 0$ on ∂D, so the flow preserves D).

In particular, by induction on k, one immediately checks that ω_n^ϵ is smooth for every n. Moreover, by (1.22),

$$\|\omega_n^\epsilon\|_{L^\infty([0,+\infty)\times D_\epsilon)} \le \|\omega_0 * \rho^\epsilon\|_{L^\infty(D_\epsilon)} \le \|\omega_0\|_{L^\infty(D)} \tag{1.23}$$

and, by elliptic regularity (see (1.13))

$$\begin{aligned} \|K(\bar\omega_n^\epsilon)\|_{L^\infty([0,+\infty);W^{1,p}(D_\epsilon))} &\le C(p,D)\|\bar\omega_n^\epsilon\|_{L^\infty([0,+\infty)\times D_\epsilon)} \\ &\le C(p,D)\|\omega_0\|_{L^\infty(D)} \end{aligned} \tag{1.24}$$

for all $p \in [1,+\infty)$ and for some geometric constant $C(p,D)$ depending only on p and on the domain D.

Let us consider sequences $\{\epsilon_j\}_{j\in\mathbb{N}}$, $\{n_j\}_{j\in\mathbb{N}}$ such that $\epsilon_j \to 0$ and $n_j \to \infty$ as $j \to \infty$. Up to subsequences, (1.23) guarantees that $\exists\, \omega \in L^\infty([0,+\infty)\times D)$ such that $\omega_{n_j}^{\epsilon_j} \overset{*}{\rightharpoonup} \omega$ in $L^\infty([0,+\infty)\times D)$. In particular, by linearity of K, it is immediate to check that $K(\omega_{n_j}^{\epsilon_j}) \overset{*}{\rightharpoonup} K(\omega)$ in $L^\infty([0,+\infty)\times D)$. We now want to show that actually the convergence of $K(\omega_{n_j}^{\epsilon_j})$ is strong, and that it is still valid if instead of $K(\omega_{n_j}^{\epsilon_j})$ we consider $K(\bar\omega_{n_j}^{\epsilon_j})$.

To this aim we observe that, since $\mathrm{div}\,(K(\bar\omega_n^\epsilon)) = 0$, (1.21) can be rewritten as

$$\partial_t \omega_n^\epsilon = -\mathrm{div}\,\left(K(\bar\omega_n^\epsilon)\,\omega_n^\epsilon\right), \tag{1.25}$$

and the linearity of K implies that

$$\partial_t K(\omega_n^\epsilon) = -\nabla^\perp \triangle^{-1}\left[\mathrm{div}\,\left(K(\bar\omega_n^\epsilon)\,\omega_n^\epsilon\right)\right]. \tag{1.26}$$

Hence, thanks to (1.24), the maps $t \mapsto \partial_t K(\omega_{n_j}^{\epsilon_j}(t,\cdot))$ are uniformly bounded in $L^p(D;\mathbb{R}^2)$, uniformly in time. So, using (1.24) again, we can apply Aubin-Lions' lemma to deduce that $K(\omega_{n_j}^{\epsilon_j}) \to K(\omega)$ strongly in $L^1_{\mathrm{loc}}([0,+\infty)\times D)$.

Finally, we use (1.25), (1.23), and (1.24) to estimate $K(\omega_n^\epsilon) - K(\bar\omega_n^\epsilon)$:

$$\begin{aligned} \|K(\omega_n^\epsilon) - K(\bar\omega_n^\epsilon)\|_{L^\infty([0,+\infty),L^2(D))} &\le \|\omega_n^\epsilon - \bar\omega_n^\epsilon\|_{L^\infty([0,+\infty),H^{-1}(D))} \\ &\le \sup_{k\in\mathbb{N}} \int_{k/n}^{(k+1)/n} \|\partial_t \omega_n^\epsilon(t)\|_{H^{-1}(D)}\,dt \\ &\le \sup_{k\in\mathbb{N}} \int_{k/n}^{(k+1)/n} \|K(\bar\omega_n^\epsilon(t))\,\omega_n^\epsilon(t)\|_{L^2(D)}\,dt \\ &\le \frac{C}{n}, \end{aligned}$$

which implies that also $K(\bar\omega_{n_j}^{\epsilon_j}) \to K(\omega)$ in $L^1_{\mathrm{loc}}([0,+\infty)\times D)$.

Thanks to these facts, by taking the limit in (1.21) we obtain that ω is a solution of (1.18) with initial datum ω_0.

Uniqueness

Let ω, δ be two weak solutions of (1.18) with initial condition $\omega_0 \in L^\infty(D)$, and let $v = K(\omega)$, $w = K(\delta)$ be the corresponding weak solutions of the

Euler equations (see Proposition 1.10). Then, it is sufficient to show that $u := v - w = 0$.

By (1.13), (1.26), and the boundedness of ω, one has that, for all $p \in [1, +\infty)$,

$$v \in L^\infty([0,+\infty); W^{1,p}(D;\mathbb{R}^2)), \quad \partial_t v \in L^\infty([0,+\infty); L^p(D;\mathbb{R}^2)), \tag{1.27}$$

which implies in particular $v \in C([0,+\infty); L^p(D;\mathbb{R}^2))$ for all $p \in [1,+\infty)$. Since the same bounds hold also for w, by a simple computation using (1.10) for both v and w (observe that, thanks to the above bounds, v and w are also admissible test functions) we get (recall that $u(0) = 0$)

$$\int_D |u|^2(t,x)\,dx = -\int_0^t \int_D \left(\frac{\partial v^i}{\partial x_k} + \frac{\partial v^k}{\partial x_i} \right) u^i u^k \,dx\,ds. \tag{1.28}$$

Let us now define the following functions:

$$f := \frac{|u|^2}{2}, \quad a_{ik} := \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}, \quad g := \sqrt{\sum_{i,k} a_{ik}^2}, \quad L(t) := \int_D f(t,x)\,dx.$$

Then (1.27) and (1.28) imply that $t \mapsto L(t)$ belongs to $W^{1,1}_{\mathrm{loc}}([0,+\infty))$, and

$$\frac{d}{dt} L(t) \le \int_D f(t,x) g(t,x)\,dx. \tag{1.29}$$

Notice that if we knew that $|g(s,x)| \le \alpha(s)$ for some $\alpha \in L^1([0,t])$, then by Gronwall's Lemma we would conclude that $L(t) = 0$ for all t, hence $u \equiv 0$. However for this kind of estimate we would need to assume a strong ("almost Lipschitz") regularity in space for the vector field v, which is not available in this situation.

In our case, by (1.27) we have the following: there exist constants $M, \Theta > 0$, and a function $\sigma : \mathbb{R} \to \mathbb{R}$, such that, for all $t \in [0,T]$,

$$\begin{aligned} &\|f(t)\|_{L^\infty(D)} \le M, \\ &\|g(t)\|_{L^p(D)} \le \Theta\,\sigma(p), \quad \forall\, p \in [1,+\infty). \end{aligned}$$

(The first inequality follows from the embedding $W^{1,p} \hookrightarrow L^\infty$ for $p > 2$). Hence, for all $\epsilon \in (0,1)$

$$\begin{aligned} \int_D f(t,x) g(t,x)\,dx &\le M^\epsilon \int_D f(t,x)^{1-\epsilon} g(t,x)\,dx \\ &\le M^\epsilon \left(\int_D f(t,x)\,dx \right)^{1-\epsilon} \left(\int_D g(t,x)^{1/\epsilon}\,dx \right)^\epsilon \\ &\le \Theta \left(\frac{M}{L(t)} \right)^\epsilon L(t)\,\sigma\!\left(\frac{1}{\epsilon}\right). \end{aligned}$$

Define

$$\tau(a) := \inf_{0<\epsilon\leq 1} a^\epsilon \sigma\left(\frac{1}{\epsilon}\right).$$

Then

$$\frac{d}{dt}L(t) \leq \Theta\, L(t)\, \tau\left(\frac{M}{L(t)}\right)$$

By Lemma 1.7 we see that we can take $\sigma(p) = p$. Moreover, up to enlarge M we can assume that $L(t) \leq M$. Then it is easy to see that

$$\tau\left(\frac{M}{L(t)}\right) = \inf_{0<\epsilon\leq 1}\left(\frac{M}{L(t)}\right)^\epsilon \frac{1}{\epsilon} = \log\left(\frac{M}{L(t)}\right),$$

and by (1.29) we get

$$\frac{d}{dt}L(t) \leq \Theta\, L(t) \log\left(\frac{M}{L(t)}\right).$$

Since $\int_0^\delta \frac{1}{s\log s}\, ds = +\infty$ for any $\delta > 0$, by the classical Osgood condition for ODEs we conclude that $L(t) = 0$ for all $t \in [0, +\infty)$, as desired.

1.2.2 DiPerna-Majda measure-valued solutions

As already mentioned in Section 1.2.1, the existence of global (in time) weak solutions to the Euler equations in dimension $d \geq 3$ is an open problem.

Fix an initial datum $v_0 \in L^2(D;\mathbb{R}^d)$, and consider instead the Navier-Stokes equations with viscosity parameter $\epsilon > 0$:

$$\begin{cases} \partial_t v^\epsilon + \operatorname{div}(v^\epsilon \otimes v^\epsilon) = -\nabla p^\epsilon + \epsilon \triangle v^\epsilon & \text{in } (0,+\infty)\times D \\ \operatorname{div} v^\epsilon = 0 & \text{in } (0,+\infty)\times D \\ v^\epsilon(0,\cdot) = v_0 & \text{in } D. \end{cases} \tag{1.30}$$

Notice that, for $\epsilon = 0$, the first equation of the system (1.30) corresponds to the Euler equation expressed in the equivalent form

$$\partial_t v + \operatorname{div}(v \otimes v) = -\nabla p. \tag{1.31}$$

In 1934 ([Ler34]), Leray proved global existence of weak distributional solutions of (1.30) for any $v_0 \in L^2(D;\mathbb{R}^2)$ with $\operatorname{div} v_0 = 0$, satisfying the uniform energy bound

$$\sup_{\epsilon\leq\epsilon_0,\, t\geq 0} \int_D |v^\epsilon|^2(t,x)\, dx \leq \int_D |v_0|^2(x)\, dx \tag{1.32}$$

Henceforth, viewing ϵ as a regularization parameter, an interesting problem is to explore the connections between the structure of solutions to the Euler equations and the behaviour of solutions of the Navier-Stokes system as $\epsilon \to 0$. As noticed by DiPerna and Majda in [DiPM87], for smooth initial data there exists a time interval $[0,T]$, $T = T(v_0)$, on which Leray's solutions converge strongly in L^2 as ϵ vanishes. Then, the limit vector fields on $[0,T]$ must be solutions of the Euler equations (1.31). However, numerical computations show that the behaviour of the limit flow is much more complex as time evolves. Indeed, after some time, due to the persistence of oscillations and concentrations phenomena, the Navier-Stokes solutions converge only weakly (and not strongly) in L^2 –the weak convergence being guaranteed by the energy bound (1.32). In particular, the nonlinear term $\operatorname{div}(v^\epsilon \otimes v^\epsilon)$ is not stable under weak limit and then the limit vector field may not necessarily satisfy, after the critical time $T(v_0)$, the Euler equations.

In order to describe the wilder behaviour of the vanishing viscosity limits of (1.30), DiPerna and Majda [DiPM87] introduced a new framework in which to incorporate the possible oscillation and concentration phenomena of the Euler flows. This was done defining a new notion of solution of (1.1), the so called *measure-valued solutions*, which is based on an extension of the concept of Young measure. We recall that the first to recognize the importance of Young measures to represent the oscillations of solutions of PDEs was Tartar [Tar79], [Tar83] (see also [DiP85]), who dealt with weak limits of L^∞-bounded sequences of solutions to 1-dimensional conservation laws.

In the context of the Navier-Stokes approximation of the Euler equations and the existence of globally defined solutions of the latter, the main theorem proved by DiPerna and Majda is the following:

Theorem 1.12 ([DiPM87]). *Let $v_0 \in L^2(\mathbb{R}^3)$ be a divergence free vector field, and let v^ϵ be any (Leray) weak solution of the Navier-Stokes equations* (1.30) *with initial data v_0. Then, as $\epsilon \to 0$, there exists a subsequence of v^ϵ which converges to a measure-valued solution of the Euler equations* (1.1) *on* $[0,+\infty)$.

Notice that Theorem 1.12 provides globally defined solutions to the Euler equations, even though in a "generalized" sense.

The aim of this section is to give a rigorous definition of measure-valued solution of the Euler equations as presented in [DiPM87]. At the end we report also a weak-strong uniqueness result for measure-valued solutions obtained as limits of Leray solutions of (1.30) proved in [BDS11] (see Theorem 1.20).

First we recall the concept of Young measure. In the following, if X is a locally compact Hausdorff space, we denote by $\mathcal{M}(X)$ the space of Radon measures on X of finite total mass, by $\mathcal{M}^+(X) \subset \mathcal{M}(X)$ the subspace of non-negative measures, and by $\mathrm{Prob}(X)$ the subset of non-negative measures with unit mass. The notation $w_\epsilon \rightharpoonup w$ denotes weak convergence in $\mathcal{M}(X)$ or $L^p(X)$, while $w_\epsilon \to w$ stands for strong convergence. By $\langle \cdot, \cdot \rangle$ we denote the duality product between measures and continuous functions on X. We denote by Ω any bounded domain of $\mathbb{R} \times \mathbb{R}^d$, e.g. $\Omega = [0,T] \times D$.

Theorem 1.13. *Let $\{u_j\}_{j\in\mathbb{N}}$ be any sequence of vector fields $u_j : \Omega \to \mathbb{R}^d$ satisfying*

$$\sup_j \|u_j\|_{L^\infty(\Omega;\mathbb{R}^d)} \leq C$$

for some real constant C, *and*

$$u_j \rightharpoonup u \quad \text{in } L^1(\Omega;\mathbb{R}^d)$$

for some function $u \in L^\infty(\Omega;\mathbb{R}^d)$. *Then, there exists a Lebesgue measurable mapping*

$$(x,t) \mapsto \nu_{(x,t)} \in \mathrm{Prob}(\mathbb{R}^d) \tag{1.33}$$

with $\mathrm{supp}\,\nu_{(x,t)} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq C\}$, *such that*

$$g \circ u_j \rightharpoonup \langle \nu_{(x,t)}, g\rangle$$

for all $g \in C(\mathbb{R}^d)$, *i.e.*

$$\lim_{j\to\infty} \int_\Omega \phi g(u_j)\,dx\,dt = \int_\Omega \phi \langle \nu_{(x,t)}, g\rangle\,dx\,dt \quad \text{for all } \phi \in C_0(\Omega).$$

Furthermore,

$$u_j \to u \text{ in } L^1(\Omega) \quad \Leftrightarrow \quad \nu_{(x,t)} = \delta_{u(x,t)} \text{ for a.e. } (t,x).$$

Definition 1.14 (Young measure). *The mapping* (1.33) *is called* Young measure *associated to the sequence* $\{u_j\}_{j\in\mathbb{N}}$.

Roughly speaking, a Young measure $\nu \equiv \{\nu_{(x,t)}\}$ represents *all* the composite weak limits of an L^∞-bounded and weakly convergent sequence, which are not Dirac deltas in case of persistence of oscillations.

It is easy to see that, if $v_j : [0,T] \times D \to \mathbb{R}^d$ is a sequence of weak solutions of the Euler equation (1.1) such that

$$\sup_j \|v_j\|_{L^\infty} \leq C,$$

then the Young measure ν constructed from this sequence satisfies

$$\int_0^T \int_D \langle \nu_{(x,t)}, \xi\rangle \partial_t \phi + \langle \nu_{(x,t)}, \xi \otimes \xi\rangle : \nabla\phi\,dx\,dt = 0 \tag{1.34}$$

for all divergence free $\phi \in C_c^\infty((0,T)\times D;\mathbb{R}^d)$, and

$$\int_0^T \int_D \langle \nu_{(x,t)}, \xi\rangle \cdot \nabla\psi\,dx\,dt = 0 \tag{1.35}$$

for all $\psi \in C_c^\infty((0,T)\times D)$. Indeed, to obtain (1.34) and (1.35) it is sufficient to take $g(\xi) = \xi$ and $g(\xi) = \xi \otimes \xi$, and apply Theorem 1.13 to the weak formulation of (1.1).

However, as discussed at the beginning, it is not natural to impose a uniform L^∞ bound on solutions of (1.1), the natural energy space being L^2. For this reason, DiPerna and Majda generalized the concept of Young measure to deal with the composite weak limits of sequences satisfying the uniform L^2-bound (1.24) with continuous functions of the form

$$g(\xi) = g_0(\xi)(1+|\xi|^2) + g_H\left(\frac{\xi}{|\xi|}\right)|\xi|^2, \tag{1.36}$$

where g_0 lies in the space $C_0(\mathbb{R}^d)$ of continuous functions vanishing at infinity, and g_H lies in the space $C(\mathbb{S}^{d-1})$ of continuous functions on the unit sphere. Indeed, the defining functions ξ and $\xi \otimes \xi$ of the Euler equations belong to this class with $g_0(\xi) = \xi/(1+|\xi|^2)$ and $g_H(\xi/|\xi|) = (\xi/|\xi|) \otimes (\xi/|\xi|)$.

Theorem 1.15 ([DiPM87], Theorem 1). *If $\{u_j\}_{j\in\mathbb{N}} \subset L^2(\Omega;\mathbb{R}^d)$ is a family of functions such that*

$$\sup_j \|u_j\|_{L^2} \le C,$$

then, up to subsequences, $\{u_j\}_{j\in\mathbb{N}}$ satisfies the following properties: there exist a measure $\sigma \in \mathcal{M}^+(\Omega)$ such that

$$|u_j|^2 dx\,dt \rightharpoonup \sigma \qquad \text{in } \mathcal{M}^+(\Omega)$$

and a σ-measurable map

$$\Omega \ni (x,t) \mapsto \left(\nu^1_{(x,t)}, \nu^2_{(x,t)}\right) \in \mathcal{M}^+(\mathbb{R}^d) \times \mathrm{Prob}(\mathbb{S}^{d-1})$$

such that, for all g in (1.36),

$$g \circ u_j \rightharpoonup \langle \nu^1, g_0\rangle (1+f)\,dx\,dt + \langle \nu^2, g_H\rangle\, d\mu, \tag{1.37}$$

where f denotes the Radon-Nikodym derivative of σ with respect to $dx\,dt$, namely

$$\lim_{j\to+\infty} \int_\Omega \phi g(u_j)\,dx\,dt = \int_\Omega \phi\langle \nu^1_{(x,t)}, g_0\rangle(1+f)\,dx\,dt + \int_\Omega \phi \langle \nu^2_{(x,t)}, g_H\rangle\, d\sigma$$

for all $\phi \in C_c^\infty(\Omega)$.

Definition 1.16 (Generalized Young measure). *A triple $\nu = \left(\sigma, \nu^1_{(x,t)}, \nu^2_{(x,t)}\right)$ as in Theorem 1.15 is called* generalized Young measure.

We also use the notation

$$\nu = \nu^1(\,dx\,dt + \,d\sigma) + \nu^2\,d\sigma.$$

Remark 1.17. The oscillations of weak-L^2 limits on functions of the form (1.36) with $g_H = 0$ are represented by the family of non-negative measures $\{\nu^1_{(x,t)}\} \subset \mathcal{M}^+(\mathbb{R}^d)$. The reason why they may not have unit mass is that, if the functions u_j are not uniformly bounded, then some of their mass can escape to infinity. Whenever this happens, it can be encoded in the composite limits with homogeneous functions g_H on the support of the singular part of σ with respect to $dx\,dt$.

Definition 1.18. *A generalized Young measure* $\nu = \nu^1(\,dx\,dt + \,d\sigma) + \nu^2\,d\sigma$ *is called a* measure-valued solution *of the Euler equations* (1.1) *if*

$$\int_0^T \int_D \langle \nu^1_{(x,t)}, \xi\rangle \cdot \partial_t \phi\,(1+f)dx\,dt + \langle \nu^2_{(x,t)}, \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}\rangle : \nabla\phi\,d\sigma = 0$$

for all divergence free $\phi \in C_c^\infty([0,T]\times D;\mathbb{R}^d)$, *and*

$$\int_0^T \int_D \langle \nu^1_{(x,t)}, \xi\rangle \cdot \nabla\psi(1+f)\,dx\,dt = 0$$

for all $\psi \in C_c^\infty([0,T]\times D)$.

Measure-valued solutions in the sense of Definition 1.18 may not be unique. However, as noticed e.g. in [Li96] and [BDS11], any reasonable notion of solution should satisfy the so-called weak-strong uniqueness property: in case the Cauchy problem admits a classical solution, then the generalized ones should coincide with it. In [Li96], Lions introduced a notion of *dissipative solution* for which he could prove existence and weak-strong uniqueness. Despite its applications to the analysis of various singular perturbations of the Euler equations, Lions' notion of solution does not obviously include the weak solutions of the Euler equations –because of the existence of compactly supported (in time) solutions (see [Sch93], [Shn97],[DLS09])– and seemed also too restrictive to include weak limits of Leray's solutions of (1.30).

In [BDS11] Brenier, De Lellis and Székelyhidi were finally able to find a suitable class of measure-valued solutions, called *admissible measure-valued solutions*, for which existence and weak-strong uniqueness hold and which include the limits of Leray's solutions of the Navier-Stokes system (see Proposition 1.19 and Theorem 1.20 below). These kinds of solutions are actually closer to the original DiPerna-Majda's solutions, with the addition of an entropy condition on the energy. We remark that in the proof of the weak-strong uniqueness property the authors of [BDS11] exploit the fact that their barycenter is a dissipative solution in the sense of Lions.

Proposition 1.19 ([BDS11], Proposition 1). *For any initial data* $v_0 \in L^2(D)$, *any sequence of Leray's solutions of* (1.30) *with vanishing viscosity has a subsequence converging to an admissible measure-valued solution of* (1.1) *with initial datum* v_0.

Theorem 1.20 ([BDS11], Theorem 2). *Let $v \in C([0,T];L^2(\mathbb{R}^d;\mathbb{R}^d))$ be a solution of* (1.1) *satisfying*

$$\int_0^T \|\nabla v(t) + (\nabla v(t))^T\|_{L^\infty}\, dt < +\infty,$$

and let $(\sigma, \nu^1_{(x,t)}, \nu^2_{(x,t)})$ be any measure-valued solution with initial datum $v(0)$. Then $\sigma = 0$ and $\nu^1_{(x,t)} = \delta_{v(x,t)}$ for a.e. (x,t).

1.3 Geodesics of measure-preserving diffeomorphisms

Let us now consider the Lagrangian description of the motion of an incompressible fluid through the ODE

$$\ddot{g}(t,a) = -\nabla p(t, g(t,a)), \quad (t,a) \in (0,T) \times D, \tag{1.38}$$

with the constraint

$$g(t,\cdot) \in SDiff(D), \quad \forall\, t \in [0,T].$$

First we derive Arnold's interpretation [Arn66] of the ODE (1.38) as the geodesics equation on $SDiff(D)$ with respect to the $L^2(D;\mathbb{R}^d)$ metric. Subsequently, we deal with Problem 1.1 of finding energy minimizing geodesics in $SDiff(D)$ and resume both "positive" and "negative" results.

Let us set some preliminary notation. In the following, D will be either a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary, or a d-dimensional smooth submanifold of $\mathbb{R}^m$ with no boundary, such as the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. By $\mathcal{L}_D$ we denote either the Lebesgue measure on D renormalized in such a way that $\mathcal{L}_D(D) = 1$ or, in the second case, the unitary volume measure on D. Sometimes $\mathcal{L}_D$ is denoted also by dx, when we want to make the independent variable x explicit.

Viewing formally $SDiff(D)$ as an infinite-dimensional submanifold of $L^2(D;\mathbb{R}^d)$, in analogy with the definition of geodesic on a submanifold of $\mathbb{R}^d$, we say that $t \mapsto g(t) \in SDiff(D)$ is a geodesic if

$$\ddot{g}(t) \in \big(T_{g(t)} SDiff(D)\big)^{\perp}, \tag{1.39}$$

where $T_{g(t)}SDiff(D)$ is the tangent space to $SDiff(D)$ at the point $g(t)$ and $\big(T_{g(t)}SDiff(D)\big)^{\perp}$ is the orthogonal in $L^2(D;\mathbb{R}^d)$ to the tangent space with respect to the L^2 scalar product $<,>_{L^2(D;\mathbb{R}^d)}$.

To make the geodesic equation (1.39) explicit, we have first to identify the tangent space at a point of $SDiff(D)$. As in the finite-dimensional setting, we consider tangent vector fields to curves $t \mapsto g(t)$:

$$w(t,\cdot) := \dot{g}(t,\cdot) \in T_{g(t)}SDiff(D).$$

Then, by definition of flow map (1.3) and by the identity (1.4), we get

$$\begin{aligned} T_{g(t)}SDiff(D) &= \big\{w : D \to \mathbb{R}^d :\ \mathrm{div}\,(w \circ g(t)^{-1}) = 0, \quad w \cdot \nu = 0\big\} \\ &= \big\{u \circ g(t) : D \to \mathbb{R}^d :\ \mathrm{div}\, u = 0, \quad u \cdot \nu = 0\big\}. \end{aligned}$$

Now we observe that, since $g(t)$ is a measure-preserving diffeomorphism, for all $h, f \in L^2(D;\mathbb{R}^d)$

$$\begin{aligned} < h \circ g(t), f \circ g(t) >_{L^2(D;\mathbb{R}^d)} &= \int_D \big(h \circ g(t)\big) \cdot \big(f \circ g(t)\big)\, dx \\ &= \int_D h \cdot f\, dx =< h, f >_{L^2(D;\mathbb{R}^d)} . \end{aligned} \tag{1.40}$$

Finally, by the Helmholtz decomposition,

$$\big\{u :\ \mathrm{div}\, u = 0, \quad u \cdot \nu = 0\big\}^{\perp} = \big\{\nabla p :\ p : D \to \mathbb{R}\big\},$$

so we find the characterization

$$\big(T_{g(t)}SDiff(D)\big)^{\perp} = \big\{\nabla p \circ g(t) :\ p : D \to \mathbb{R}\big\}$$

from which we deduce the equivalence between (1.38) and (1.39).

With this interpretation of the ODE (1.38) at hand, we transpose on $SDiff(D)$ the standard variational problem of finding geodesics on a finite-dimensional Riemannian manifold as minimizers of the "kinetic energy" functional. More precisely, our main issue becomes now to solve Problem 1.1 introduced in Section 1.1.

1.3.1 Existence and non-existence results

The first result concerning the existence of solutions for the minimization Problem 1.1 is due to Ebin and Marsden [EM70].

Observe that the energy functional $\mathcal{E}$ defined in (1.7) is invariant with respect to the right composition of maps on $SDiff(D)$. Thus, by right composing any curve connecting g_0 to g_T with the map g_0^{-1}, we see that Problem 1.1 is equivalent to connect the identity map i_D to $g_T \circ g_0^{-1}$, Hence, without loss of generality, we can always assume $g_0 = i_D$.

Theorem 1.21 ([EM70]). *If D is a smooth compact manifold with no boundary and $\|g_T - i_D\|_{H^s(D;\mathbb{R}^d)} << 1$ for some $s > \left[\frac{n}{2}\right] + 1$, then there exists a unique minimizer for the action* (1.7)*.*

However, no general existence result for arbitrary initial and final data is available. Indeed, as observed in the Introduction, the quantity to be minimized does not contain any spatial derivatives of the maps, while the constraint is expressed in terms of their Jacobian. Hence, the appropriate strong topology in order to have convergence of minimizing sequences in $SDiff(D)$ is unrelated to the one induced by the energy (1.7), and no classical variational method can be applied.

The existence of connecting curves of finite energy on the d-dimensional unit cube $D = [0,1]^d$, $d \geq 3$, was proved by Shnirelman in [Shn87].

Theorem 1.22 ([Shn87]). *If $D = [0,1]^d$ and $d \geq 3$, then for all $h \in SDiff(D)$ there exists a curve $t \mapsto g(t) \in SDiff(D)$, with $\mathcal{E}(g) < \infty$, connecting i_D to h.*

On the other hand, in [Shn87], [Shn94] Shnirelman found two fundamental counterexamples to the existence of minimizers.

Theorem 1.23 ([Shn87]). *If $D = [0,1]^d$ and $d \geq 3$, there exists $h \in SDiff(D)$ for which the infimum in* (1.7) *among the curves connecting i_D to h is not achieved.*

Theorem 1.24 ([Shn94], Corollary 2.5). *If $D = [0,1]^2$, there exists $h \in SDiff(D)$ for which there is no curve $t \mapsto g(t) \in SDiff(D)$ satisfying $g(0) = i_D$, $g(T) = h$, and $\mathcal{E}(g) < +\infty$.*

Here we report a simplified proof of Theorem 1.23, given in [Shn94] (see also [Bre99], Section 1.3).

Proof (Theorem 1.23). Up to rescaling time, without loss of generality we can assume $T = 1$.

Let us denote by $SDiff_2([0,1]^3)$ the subset of $SDiff([0,1]^3)$ given by diffeomorphisms of the form

$$h(x_1, x_2, x_3) = (H(x_1, x_2), x_3),$$

where $H \in SDiff([0,1]^2)$, and define

$$\begin{aligned} \mathcal{I}_2(h) &:= \inf\{\mathcal{E}(g) :\, g(t) \in SDiff_2([0,1]^3),\, g(0) = i_D,\, g(T) = h\}, \\ \mathcal{I}_3(h) &:= \inf\{\mathcal{E}(g) :\, g(t) \in SDiff([0,1]^3),\, g(0) = i_D,\, g(T) = h\} \end{aligned}$$

Obviously, $\mathcal{I}_3(h) \leq \mathcal{I}_2(h)$. Moreover, by Theorems 1.22 and 1.24 it is possible to choose $h \in SDiff_2([0,1]^3)$ such that $\mathcal{I}_3(h) < \mathcal{I}_2(h)$ (for instance, one can take H as in Theorem 1.24). For such h, choose $t \mapsto g(t) \in SDiff([0,1]^3)$ such that $\mathcal{E}(g) < \mathcal{I}_2(h)$. In particular $g(t) \notin SDiff_2([0,1]^3)$ for some $t \in (0,T)$, which implies that $g_3(t, x_1, x_2, x_3) \neq x_3$. Set $\eta(x_3) := \min\{2x_3, 2 - 2x_3\}$, and let $u := \dot{g} \circ g^{-1}$ be the vector field associated to g. Then, it is fairly easy to check that the rescaled vector field

$$\tilde{u}(t, x_1, x_2, x_3) := \begin{cases} u_1(t, x_1, x_2, \eta(x_3)) \\ u_2(t, x_1, x_2, \eta(x_3)) \\ \eta'(x_3)^{-1} u_3(t, x_1, x_2, \eta(x_3)) \end{cases}$$

induces a path $t \mapsto \tilde{g}(t) \in SDiff([0,1]^3)$ which still connects i_D to h but with $\mathcal{E}(\tilde{g}) < \mathcal{E}(g)$ (actually, to be precise, the maps $\tilde{g}(t)$ are just bi-Lipschitz measure-preserving maps, but a regularization argument allows to take care of this problem). In particular, if g was assumed to be minimal, this would lead to a contradiction.

1.3.2 L^2-relaxation: measure-preserving maps and $|\cdot|^2$-optimal transportation

Before introducing Brenier's relaxed model for geodesics of measure-preserving diffeomorphisms, we try first to implement on $SDiff(D)$ a standard discretization method which can be used to find energy minimizing geodesics on a finite-dimensional Riemannian submanifold of $\mathbb{R}^d$.

Actually we will not carry this technique to its end, but it will be interesting to see how its basic step naturally leads to some relaxations of the space $SDiff(D)$ which will appear in Brenier's model.

Given two points x, y on a closed compact submanifold $M^n \subset \mathbb{R}^d$, $n < d$, suppose that we want to find an energy minimizing geodesic between x and y, namely a curve $\bar{\gamma} : [0,1] \to M^n$ s.t. $\bar{\gamma}(0) = x$, $\bar{\gamma}(1) = y$ which minimizes the energy

$$\frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2\,dt \tag{1.41}$$

among all curves $\gamma \subset M^n$ connecting x to y.

Then, one can try to perform the following iterative procedure. At each step j, $j \in \mathbb{N}$, we look for $2^j + 1$ points $\{z_i^j\}_{i=0}^{2^j}$ on M^n, with $z_0^j = x$ and $z_{2^j}^j = y$, minimizing the discrete energy

$$2^{j-1}\sum_{i=0}^{2^j-1} |z_{i+1}^j - z_i^j|^2.$$

Hence, the piecewise affine curves $\{\gamma_j\}_{j\in\mathbb{N}} \subset L^2([0,1];\mathbb{R}^d)$

$$\gamma_j(t) = z_i^j + (2^j t - i)(z_{i+1}^j - z_i^j) \quad \text{ if } t \in \left[\frac{i}{2^j}, \frac{i+1}{2^j}\right],\ i = 0,\ldots,2^j - 1$$

may be suitable (in the energy sense) approximations of an energy minimizing geodesic.

At the first step $j = 1$, the problem reduces to the one-point minimization problem

$$\min_{z\in M^n} |x - z|^2 + |z - y|^2,$$

which can be rewritten as

$$\frac{1}{2}|x - y|^2 + 2\min_{z\in M^n}\left|z - \frac{x+y}{2}\right|^2$$

that is, z_1^1 must be the projection on M^n of the midpoint of the segment $[x, y]$.

Analogously, on $SDiff(D)$ we look for

$$\min_{g \in SDiff(D)} \left\| g - \frac{g_0 + g_1}{2} \right\|_{L^2(D;\mathbb{R}^d)}. \tag{1.42}$$

However, since $SDiff(D)$ is infinite-dimensional but neither closed nor convex, no classical theory is available to infer the existence of such a projection.

The first thing one is led to do is then to relax (1.42) looking for minimizers in the L^2-closure of $SDiff(D)$. Hence, one needs first to characterize such a closure.

To this aim, we first introduce the space of measure-preserving maps: recalling that $\mathcal{L}_D$ is the renormalized Lebesgue/volume measure on D, and denoting by $f_\sharp \mathcal{L}_D$ the *push-forward measure* defined by $f_\sharp \mathcal{L}_D(B) = \mathcal{L}_D(f^{-1}(B))$ for all Borel sets $B \subset D$, we define the space of *measure-preserving maps* of D as

$$S(D) := \{f : D \to D : \ f_\sharp \mathcal{L}_D = \mathcal{L}_D\} \tag{1.43}$$

A proof of the following theorem was first given in [Shn87], and then, with different techniques, by various authors (see [BG03] for complete references on this topic). In [BG03], Brenier and Gangbo gave a new proof, based on Birkhoff's Theorem on doubly stochastic matrices and on the polar factorization theorem proved in [Bre91].

Theorem 1.25. *If* $D = [0, 1]^d$, $d \geq 2$,

$$\overline{SDiff(D)}^{L^2} = S(D). \tag{1.44}$$

We note that the proof of the above result given in [BG03] could be extended to bounded Lipschitz domains of $\mathbb{R}^d$. Hence, problem (1.42) turns into the following: given $h \in L^2(D; \mathbb{R}^d)$, solve

$$\min_{s \in S(D)} \int_D |h - s|^2 \, d\mathcal{L}_D. \tag{1.45}$$

The existence and uniqueness of a minimizer can be obtained using the polar factorization theorem proved by Brenier in [Bre91].

Theorem 1.26. *Let* $D \subset \mathbb{R}^d$ *be a bounded Lipschitz domain. Then, for all* $h : D \to D$ *such that* $h_\sharp(\mathcal{L}_D) \ll \mathcal{L}_D$*, there exists a unique solution of* (1.45).

We present the proof of this result since it reveals some strong links between the variational theory of incompressible fluids and optimal transportation. In Section 1.6 we will actually see that the presence of optimal transportation arises also when dealing with the problem of finding necessary and sufficient conditions for optimality of *generalized incompressible flows*.

Proof (Theorem 1.26).

Existence

Set $\mu := h_\sharp(\mathcal{L}_D)$, and denote by $\pi_i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, 2$, the canonical projections, that is $\pi_1(x, y) = x$, $\pi_2(x, y) = y$. Then

$$\begin{aligned} \inf_{s\in S(D)} \int_D |h - s|^2 \, d\mathcal{L}_D &= \inf_{s\in S(D)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^2 \, d(h \times s)_\sharp \, d\mathcal{L}_D \\ &\geq \inf_{\substack{(\pi_1)_\sharp\gamma=\mu \\ (\pi_2)_\sharp\gamma=\mathcal{L}_D}} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^2 \, d\gamma(x, y), \end{aligned} \tag{1.46}$$

where the last inequality follows from the fact that $(h \times s)_\sharp \mathcal{L}_D$ satisfies both the marginal constraints in (1.46).

Since $\mu \ll \mathcal{L}_D$, by optimal transportation theory we know that there exists a unique optimal transport plan $\bar{\gamma}$ solving the variational problem (1.46) (see [Bre91]). Moreover

$$\bar{\gamma} = (i_D \times \nabla\phi)_\sharp \mathcal{L}_D = (\nabla\psi \times i_D)_\sharp \mu,$$

where $\phi, \psi : \mathbb{R}^d \to \mathbb{R}$ are convex conjugate functions satisfying

$$\nabla\phi \circ \nabla\psi = i_D \quad \mu\text{-a.e.}, \qquad \nabla\psi \circ \nabla\phi = i_D \quad \mathcal{L}_D\text{-a.e.}$$

Define $s := \nabla\psi \circ h$. Then it is immediate to check that $s \in S(D)$. Furthermore, since $h = \nabla\phi \circ s$, we have

$$\begin{aligned} \int_D |h - s|^2 \, d\mathcal{L}_D &= \int_D |\nabla\phi \circ s - s|^2 \, d\mathcal{L}_D \\ &= \int_D |\nabla\phi - i_D|^2 \, d\mathcal{L}_D \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^2 \, d\bar{\gamma} \\ &= \min_{\substack{(\pi_1)_\sharp\gamma=\mu \\ (\pi_2)_\sharp\gamma=\mathcal{L}_D}} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^2 \, d\gamma(x, y). \end{aligned}$$

Hence, by (1.46), s is a minimizer of (1.45).

Uniqueness

Let $s, s' \in S(D)$ be two solutions of (1.45). Then, as seen in the Existence part, both $(h\times s)_\sharp \mathcal{L}_D$ and $(h\times s')_\sharp \mathcal{L}_D$ solve the optimal transportation problem

$$\min_{\substack{(\pi_1)_\sharp\gamma=\mu \\ (\pi_2)_\sharp\gamma=\mathcal{L}_D}} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^2 \, d\gamma(x, y).$$

Thus, by the uniqueness of the minimizing plan $\bar{\gamma}$ we get $(h \times s)_\sharp \mathcal{L}_D = (h \times s')_\sharp \mathcal{L}_D$, namely

$$\int_D F(h(x), s(x))\,dx = \int_D F(h(x), s'(x))\,dx, \quad \forall\, F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \text{ Borel.} \tag{1.47}$$

Choosing $F(x,y) = \nabla\psi(x) \cdot y$ and $F(x,y) = |y|^2$, we obtain respectively

$$\int_D s^2 = \int_D \nabla\psi \circ h \cdot s \overset{(1.47)}{=} \int_D \nabla\psi \circ h \cdot s' = \int_D s \cdot s',$$

$$\int_D |s|^2 \overset{(1.47)}{=} \int_D |s'|^2.$$

This implies

$$\int_D |s - s'|^2 = \int_D |s|^2 + \int_D |s'|^2 - 2\int_D s \cdot s' = 0,$$

which proves the desired uniqueness result.

1.4 Generalized incompressible flows

The aim of this section is to introduce the first of Brenier's variational models for incompressible fluids ([Bre89]), and consider the basic issues of existence, uniqueness, and consistency with classical solutions. Here we partly follow the presentation in [AF09].

We consider the space of continuous paths

$$\Omega(D) := C([0,T]; D),$$

whose typical element will be denoted by ω, $[0,T] \ni t \mapsto \omega(t) \in D$.

$\Omega(D)$ is a separable Banach space with respect to the supremum norm. We denote by $\mathcal{P}(\Omega(D))$ the space of probability measures on $\Omega(D)$.

For each finite subset of times $\{t_1, \ldots, t_k\} \subset [0,T]$, define the evaluation map

$$(\mathrm{e}_{t_1}, \ldots, \mathrm{e}_{t_k}) : D^{[0,T]} \to D^k, \quad (\mathrm{e}_{t_1}, \ldots, \mathrm{e}_{t_k})(\omega) := (\omega(t_1), \ldots, \omega(t_k))$$

and the marginal of $\boldsymbol{\mu} \in \mathcal{P}(\Omega(D))$ at times $(t_1, \ldots, t_k)$ as

$$\boldsymbol{\mu}^{t_1,\ldots,t_k} := (\mathrm{e}_{t_1}, \ldots, \mathrm{e}_{t_k})_\sharp \boldsymbol{\mu}.$$

In particular, each $\boldsymbol{\mu} \in \mathcal{P}(\Omega(D))$ induces a curve of *plans*

$$[0,T] \ni t \mapsto \boldsymbol{\mu}^{0,t} \in \mathcal{P}(D \times D). \tag{1.48}$$

It is easy to see that every smooth family of diffeomorphisms $[0,T] \ni t \mapsto g(t)$ induces a generalized flow $\boldsymbol{\mu}_g$ setting

$$\boldsymbol{\mu}_g := \Phi_{g\sharp}\mathcal{L}_D, \qquad \Phi_g : D \to \Omega(D), \quad \Phi_g(a) := g(\cdot, a). \tag{1.49}$$

Viceversa, if $g(0) = i_D$, one can reconstruct g from $\boldsymbol{\mu}_g$ via the disintegration formula

$$(\mathrm{e}_0, \mathrm{e}_t)_\sharp \boldsymbol{\mu}_g = \delta_{g(t,a)} \otimes da. \tag{1.50}$$

(Here we are using the "more expressive" formula $\delta_{g(t,a)} \otimes da$ to denote the measure $(i_D \times g(t,\cdot))_\sharp \mathcal{L}_D$. In the sequel, we shall use both notations.)

Given two diffeomorphisms g_0, $g_T : D \to D$, we say that $\boldsymbol{\mu}$ *connects* g_0 to g_T if

$$\boldsymbol{\mu}^{0,T} = (g_0, g_T)_\sharp \mathcal{L}_D. \tag{1.51}$$

If $g_0 \in SDiff(D)$, (1.51) is equivalent to $\boldsymbol{\mu}^{0,T} = (i_D \times g_T \circ g_0^{-1})_\sharp \mathcal{L}_D$.

A simple but essential remark is that, in general, even if $\boldsymbol{\mu}^{0,T}$ is *deterministic* –i.e. it is concentrated on the graph of a function– the curve of plans (1.48) may not be deterministic in between.

Now we introduce the class of generalized flows that will replace the curves of orientation and measure-preserving diffeomorphisms.

Definition 1.27. *A generalized flow* $\boldsymbol{\eta} \in \mathcal{P}(\Omega(D))$ *is* incompressible *if*

$$\boldsymbol{\eta}^t = \mathcal{L}_D, \quad \forall\, t \in [0,T]. \tag{1.52}$$

Indeed, notice that $\boldsymbol{\eta}_g$ is incompressible if and only if $\boldsymbol{\eta}_g^t = g(t)_\sharp \mathcal{L}_D = \mathcal{L}_D$ for all $t \in [0,T]$, that is, if and only if $g(t) \in S(D)$ (see (1.43)).

Given $\boldsymbol{\eta} \in \mathcal{P}(\Omega(D))$ and t_1, $t_2 \in [0,1]$, the incompressibility constraint (1.52) implies that $\boldsymbol{\eta}^{t_1,t_2} = (\mathrm{e}_{t_1}, \mathrm{e}_{t_2})_\sharp \boldsymbol{\eta}$ belongs to the space of *doubly stochastic measures* (or *measure-preserving plans*) $\Gamma(D)$, where

$$\Gamma(D) := \big\{\eta \in \mathcal{P}(D \times D) :\ (\pi_1)_\sharp \eta = \mathcal{L}_D,\ (\pi_2)_\sharp \eta = \mathcal{L}_D\big\}. \tag{1.53}$$

(As in the previous sections, $\pi_1, \pi_2 : D \times D \to D$ denote the canonical projections.) In particular, $[0,T] \ni t \mapsto \boldsymbol{\eta}^{0,t}$ is a curve of measure-preserving plans. We remark that, if $\boldsymbol{\eta}^{0,t}$ is deterministic, namely $\boldsymbol{\eta}^{0,t} = (i_D \times h_t)_\sharp \mathcal{L}_D$ for some measurable function $h_t : D \to D$, then $h_t \in S(D)$.

Given two maps h_0, $h_T \in S(D)$, we say that $\boldsymbol{\eta} \in \mathcal{P}(\Omega(D))$ *connects* h_0 to h_T if $\boldsymbol{\eta}^{0,T} = (h_0 \times h_T)_\sharp \mathcal{L}_D$. More in general, given $\eta \in \Gamma(D)$, we say that $\boldsymbol{\eta} \in \mathcal{P}(\Omega(D))$ is *compatible with* η if $\boldsymbol{\eta}^{0,T} = \eta$.

After these preliminary definitions, we can define the variational model proposed by Brenier in [Bre89].

Problem 1.28. Given $\eta \in \Gamma(D)$, find $\bar{\boldsymbol{\eta}} \in \mathcal{P}(\Omega(D))$ which minimizes the *action*

$$\mathcal{A}(\boldsymbol{\eta}) := T \int_{\Omega(D)} \int_0^T \frac{1}{2} |\dot{\omega}(t)|^2 \, dt \, d\boldsymbol{\eta}(\omega), \tag{1.54}$$

among all generalized incompressible flows $\boldsymbol{\eta} \in \mathcal{P}(\Omega(D))$ which are compatible with η.

Notice that, if $\boldsymbol{\eta} = \boldsymbol{\eta}_g$ is induced by a smooth family of diffeomorphisms $t \mapsto g(t)$, then

$$\mathcal{A}(\boldsymbol{\eta}_g) = \mathcal{E}(g)$$

(see (1.7)).

1.4.1 An extended Lagrangian model

From the physical point of view, Problem 1.28 allows fluid particles to split/cross at intermediate times. (Although this may look unphysical, it is shown in [Bre08] that these generalized solutions are actually quite conventional, in the sense they obey, up to a suitable change of variable, a well-known variant of the Euler equations in $(d+1)$-dimensions for which the vertical acceleration is neglected, according to the so-called hydrostatic approximation.)

In Section 1.4.4 we will show an example [Bre89] in which two given diffeomorphisms can be connected by minimizing generalized incompressible flows which are not concentrated on a graph for all intermediate times $t \in (0,T)$.

For this reasons it seems natural to look for a variational model which permits also to connect couples of plans $\eta, \gamma \in \Gamma(D)$.

This is made possible by adding to the model a new Lagrangian variable $a \in D$ which tracks the initial position of the trajectories followed by the fluid particles (see [Bre99] and [AF09]).

More precisely, following the construction performed in [AF09], one considers the space

$$\tilde{\Omega}(D) := \Omega(D) \times D,$$

whose typical element will be denoted by (ω, a). In this setting, a *generalized flow* is a probability measure $\boldsymbol{\eta} \in \mathcal{P}(\tilde{\Omega}(D))$ such that $\pi_{D\sharp}\boldsymbol{\eta} = \mathcal{L}_D$, where $\pi_D : \tilde{\Omega}(D) \to D$ denotes the canonical projection onto the second factor, that is $\pi_D(\omega, a) = a$. By disintegration with respect to the map π_D, any generalized flow $\boldsymbol{\eta}$ can be represented as

$$\boldsymbol{\eta} = \boldsymbol{\eta}_a \otimes d\mathcal{L}_D(a),$$

where $\boldsymbol{\eta}_a \in \mathcal{P}(\Omega(D))$. Then the incompressibility constraint in this framework becomes

$$\int_D \boldsymbol{\eta}_a^t \, d\mathcal{L}_D(a) = \mathcal{L}_D, \quad \forall\, t \in [0,T],$$

or equivalently $(\mathrm{e}_t)_\sharp\boldsymbol{\eta} = \mathcal{L}_D$ if we let $\mathrm{e}_t : \tilde{\Omega}(D) \to D$, $\mathrm{e}_t(\omega, a) := \omega(t)$.

Given $\eta = \eta_a \otimes d\mathcal{L}_D(a)$ and $\gamma = \gamma_a \otimes d\mathcal{L}_D(a)$ in $\Gamma(D)$, we say that $\boldsymbol{\eta}$ *connects* η to γ if $\boldsymbol{\eta}_a^0 = \eta_a$ and $\boldsymbol{\eta}_a^T = \gamma_a$ for $\mathcal{L}_D$-a.e. $a \in D$. Setting $\boldsymbol{\eta}^t := \boldsymbol{\eta}_a^t \otimes d\mathcal{L}_D(a)$, this is equivalent to say that $\boldsymbol{\eta}^0 = \eta$, $\boldsymbol{\eta}^T = \gamma$. The measures η and γ are called respectively *initial* and *final configuration* of the generalized flow.

The action (1.54) becomes now

$$\mathcal{A}(\boldsymbol{\eta}) := T \int_{\tilde{\Omega}(D)} \int_0^T \frac{1}{2} |\dot{\omega}(t)|^2 \, dt \, d\boldsymbol{\eta}(\omega, a). \tag{1.55}$$

Notice that if $\eta_a = \delta_a$ (namely, $\eta = (i_D \times i_D)_\sharp \mathcal{L}_D$), then the condition $\boldsymbol{\eta}_a^0 = \eta_a$ tells us that almost all the trajectories on which $\boldsymbol{\eta}_a$ is concentrated start from a. Then $\int_D \boldsymbol{\eta}_a \, d\mathcal{L}_D(a)$ provides us with a generalized incompressible flow according to Brenier's original model, having the same action as $\boldsymbol{\eta}$.

Moreover, in the case in which $\boldsymbol{\eta}$ is induced by a flow of measure-preserving maps $[0,T] \ni t \mapsto h(t)$, this new formulation permits to reconstruct the maps from $\boldsymbol{\eta}$ even if $h(0)$ is not invertible –which was not possible before.

Remark 1.29. In the rest of these notes we will use this extended Lagrangian formulation only when needed (namely in Section 1.6, which is devoted to the necessary and sufficient conditions for optimality in Problem 1.28), while for the rest we will use Brenier's formulation. Indeed, while the extended formulation will permit to study the behaviour of the trajectories followed by minimizing generalized flows also between intermediate times $s, t \in (0,T)$, using as much as possible Brenier's formulation allows us to make the presentation of the results simpler and does not affect the generality of the following statements, since they could be all rephrased in the extended Lagrangian framework with no difficulties.

1.4.2 Existence

In this section we present the main results (given in [Bre89]) concerning the existence of minimizers of the action (1.54). All of them have been extended in [AF09] to the Lagrangian model defined in Section 1.4.1. The most complete statement is given in Theorem 1.33 below.

For a general domain D, and $\eta \in \Gamma(D)$, the existence of generalized incompressible flows of finite action compatible with η guarantees, by standard compactness and lower semicontinuity arguments, the existence of a minimizer.

Proposition 1.30 ([Bre89], Proposition 3.3). *Let $D \subset \mathbb{R}^d$ be a compact set. Then, for all $\eta \in \Gamma(D)$ for which there exists a generalized incompressible flow of finite action compatible with η, Problem 1.28 has a solution.*

In the proof of this proposition we will need the following compactness result, which is a simple consequence of the compact embedding of $H^1([0,T];D) \hookrightarrow C([0,T];D)$ (see [AF09], proof of Theorem 3.3, for more details).

Proposition 1.31. *For all $R > 0$, the set*

$$\mathcal{P}_R(\Omega(D)) := \left\{\boldsymbol{\eta} \in \mathcal{P}(\Omega(D)) : \mathcal{A}(\boldsymbol{\eta}) \le R\right\}$$

is sequentially weak-compact in $\mathcal{P}(\Omega(D))$.*

Proof (Proposition 1.30). For any $\omega \in \Omega(D)$, let us define

$$a(\omega) := \begin{cases} \displaystyle\int_0^T |\dot\omega(t)|^2 \, dt & \text{if } \omega \in H^1([0,T];D) \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that $a : \Omega(D) \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous with respect to the uniform convergence, so the functional

$$\mathcal{A}(\boldsymbol{\eta}) = \int_{\Omega(D)} a(\omega)\, d\boldsymbol{\eta}(\omega)$$

is weakly*-lower semicontinuous on $\mathcal{P}(\Omega(D))$.

Then, taking a minimizing sequence of generalized incompressible flows compatible with $\eta \in \Gamma(D)$ having uniformly bounded action (which exists by the assumptions), Proposition 1.31 immediately gives the desired result.

In the case in which $D = \mathbb{T}^d$, $d \geq 2$, the following theorem combined with Proposition 1.30 gives then a complete existence result.

Theorem 1.32 ([Bre89], Proposition 4.3). *Let $D = \mathbb{T}^d$, $d \geq 2$. Then, for any $\eta \in \Gamma(D)$ there exists a generalized incompressible flow $\boldsymbol{\eta} \in \mathcal{P}(\Omega(D))$ compatible with η and such that $\mathcal{A}(\boldsymbol{\eta}) \leq 2d$.*

Proof (Theorem 1.32).

We give an explicit construction of the generalized flow of finite energy. First, let us define the geodesic map

$$\gamma : [0,1] \times D \times D \to D, \quad (t,x,y) \mapsto \gamma(t,x,y) := x + t(y - x).$$

We claim that γ satisfies the following properties:

$$\gamma(0,x,y) = x, \quad \gamma(1,x,y) = y, \quad \forall\, x,y \in D \tag{1.56}$$

$$\int_0^1 |\partial_t \gamma(t,x,y)|^2\, dt \leq d, \quad \forall\, x,y \in D \tag{1.57}$$

$$\int_{D\times D} f(\gamma(t,x,y))\, dx\, dy = \int_D f(x)\, dx, \quad \forall\, f \in C(D),\ t \in [0,1]. \tag{1.58}$$

Indeed (1.56) is trivially true, and (1.57) follows from the fact that the diameter of $[0,1]^d$ is bounded by $\sqrt{d}$. Finally, since

$$\gamma(t,x,y) = x + \gamma(t,0,y-x),$$

for all $f \in C(D)$ we have

$$\begin{aligned}\int_D \int_D f(\gamma(t,x,y))\, dx\, dy &= \int_D \int_D f(x + \gamma(t,0,y-x))\, dx\, dy \\ &= \int_D \int_D f(x + \gamma(t,0,y'))\, dx\, dy' \\ &= \int_D \int_D f(x)\, dx\, dy' = \int_D f(x)\, dx,\end{aligned}$$

which proves (1.58).

Now, let $G : [0, T] \times D \times D \times D \to D$ be the map

$$G(t, x, y, x') := \begin{cases} \gamma\Big(\dfrac{2t}{T}, x, x'\Big) & \text{if } t \le \dfrac{T}{2} \\ \gamma\Big(\Big(\dfrac{2t}{T} - 1\Big), x', y\Big) & \text{if } t \ge \dfrac{T}{2} \end{cases} \tag{1.59}$$

and define $\boldsymbol{\eta} \in \mathcal{P}(\Omega(D))$ as

$$\boldsymbol{\eta} := \mathcal{H}^1 \llcorner G(\cdot, x, y, x') \otimes dx' \otimes d\eta(x, y), \tag{1.60}$$

where $\mathcal{H}^1 \llcorner G(\cdot, x, y, x')$ denotes the 1-dimensional Hausdorff measure restricted to the curve $t \mapsto G(t, x, y, x')$. Roughly speaking, according to $\boldsymbol{\eta}$ each particle starting from a point x spreads with uniform probability on the whole D at time $t = \frac{T}{2}$, and then reaches a point y with law $d\eta(x, y)$.

In order to prove the theorem, we have to check that (i) $\boldsymbol{\eta}$ satisfies the incompressibility constraint (1.52), (ii) that it is compatible with η, and (iii) that $\mathcal{A}(\boldsymbol{\eta}) \le 2d$.

(i) Given $f \in C(D)$ and $t \le \frac{T}{2}$,

$$\begin{aligned} \int_{\Omega(D)} f(\omega(t))\, d\boldsymbol{\eta}(\omega) &\overset{(1.60)}{=} \int_{D\times D} \int_D f(\gamma(2t/T, x, x'))\, dx'\, d\eta(x, y) \\ &\overset{\eta \in \Gamma(D)}{=} \int_D \int_D f(\gamma(2t/T, x, x'))\, dx'\, dx \\ &\overset{(1.58)}{=} \int_D f(x)\, dx. \end{aligned}$$

The case $t \ge \frac{T}{2}$ is analogous.

(ii) For any $f \in C(D \times D)$,

$$\begin{aligned} \int_{\Omega(D)} f(\omega(0), \omega(T))\, d\boldsymbol{\eta}(\omega) &= \int_D \int_{D\times D} f(G(0, x, y, x'), G(T, x, y, x'))\, d\eta(x, y)\, dx' \\ &\overset{(1.56)}{=} \int_D \int_{D\times D} f(x, y)\, d\eta(x, y)\, dx' \\ &= \int_{D\times D} f(x, y)\, d\eta(x, y). \end{aligned}$$

(iii)

$$
\begin{aligned}
\mathcal{A}(\boldsymbol{\eta}) &= \int_{D\times D}\int_D T\int_0^T \frac{1}{2}|\partial_t G(t,x,y,x')|^2\,dt\,dx'\,d\eta(x,y)\\
&\overset{(1.59)}{=} \int_{D\times D}\int_D \frac{T}{2}\Bigg[\int_0^{\frac{T}{2}}\left|\partial_t\gamma\Big(\frac{2t}{T},x,x'\Big)\right|^2 dt\\
&\qquad\qquad + \int_{\frac{T}{2}}^{T}\left|\partial_t\gamma\Big(\frac{2t}{T}-1,x',y\Big)\right|^2 dt\Bigg]\,dx'\,d\eta(x,y)\\
&= \int_{D\times D}\int_D\left[\int_0^1 |\partial_s\gamma(s,x,x')|^2\,ds + \int_0^1 |\partial_s\gamma(s,x',y)|^2\,ds\right]dx'\,d\eta(x,y)\\
&\overset{(1.57)}{\leq} 2d.
\end{aligned}
$$

This completes the proof.

In [Shn94], using a (non-injective) Lipschitz measure-preserving map from $\mathbb{T}^d$ to $[0,1]^d$, Shnirelman was able to produce finite action flows for all $\eta \in \Gamma(D)$ also in the case $D = [0,1]^d$. In [AF09], the authors extended Theorem 1.32 to any domain D for which there exists a Lipschitz measure preserving map Ψ from $[0,1]^d$ to D. Moreover, they deal with generalized flows according to the extended Lagrangian formulation of Section 1.4.1.

We end this part with a theorem ([AF09], Theorem 3.3) collecting all the results mentioned above in a concise form.

For all $\eta,\gamma \in \Gamma(D)$ we denote by $\bar{\delta}^2(\eta,\gamma) \in \mathbb{R}\cup\{+\infty\}$ the infimum of the action (1.55) among all generalized incompressible flows $\boldsymbol{\eta} \in \mathcal{P}(\tilde{\Omega}(D))$ between η and γ.

Theorem 1.33 ([AF09], Theorem 3.3). *Let $D \subset \mathbb{R}^d$ be a open bounded set. Then, the (possibly infinite) infimum in the definition of $\bar{\delta}(\eta,\gamma)$ is achieved. Furthermore, $\sup_{\eta,\gamma\in\Gamma(D)} \bar{\delta}(\eta,\gamma) \leq \sqrt{2d}$ whenever $D = [0,1]^d$ or $D = \mathbb{T}^d$. More generally, if $D' \subset \mathbb{R}^d$ is another open bounded set, and $\Psi : D' \to D$ is a Lipschitz measure-preserving map, then*

$$
\sup_{\eta,\gamma\in\Gamma(D)} \bar{\delta}(\eta,\gamma) \leq \mathrm{Lip}(\Psi) \sup_{\eta',\gamma'\in\Gamma(D')} \bar{\delta}(\eta',\gamma').
$$

In [AF09] it was also proved, through operations such as the *restriction*, *reparameterization* and *concatenation* of flows in $\mathcal{P}(\tilde{\Omega}(D))$, that $(\Gamma(D),\bar{\delta})$ is a complete metric space, whose convergence is stronger than the narrow convergence in $\mathcal{P}(D\times D)$.

For all $h \in SDiff(D)$, let us denote by $\delta(i_D,h)^2$ the infimum of the energy (1.7) among all smooth flows $t \mapsto g(t)$ connecting i_D to h, and let $\delta^*(i_D,h)$ be the $L^2(D;\mathbb{R}^d)$ relaxation

$$
\delta^*(i_D,h) := \inf\left\{\liminf_{n\to\infty}\delta(i_D,h_n) :\ h_n \in SDiff(D),\ \int_D |h_n-h|^2\,d\mathcal{L}_D \to 0\right\}.
$$

By (1.49) and the lower semicontinuity of $\bar{\bar{\delta}}$ with respect to narrow convergence, we have that

$$\bar{\bar{\delta}}\big((i_D \times i_D)_\sharp \mathcal{L}_D, (i_D \times h)_\sharp \mathcal{L}_D\big) \le \delta^*(i_D, h). \tag{1.61}$$

In the case $D = [0,1]^d$, $d \ge 3$, Shnirelman [Shn94] proved that the equality in (1.61) holds. In [AF09] it is shown that, if $d \ge 3$, the gap phenomenon still does not occur even when non-deterministic final data are considered (i.e., when considering as final configuration a plan $\gamma \in \Gamma(D)$ instead of a map $h \in S(D)$). Hence, the strict inequality may hold only if $d = 2$ ([Shn94]).

1.4.3 Consistency with classical solutions

A natural question is whether the notion of action minimizing generalized incompressible flow –whose existence has been proved in Section 1.4.2– is consistent with the one of classical solution of the ODE (1.38).

The answer, provided by Brenier in [Bre89], states that whenever a generalized flow satisfies the incompressibility constraint and is concentrated on solutions of the ODE (1.38) for a sufficiently regular pressure field p, then it is a minimizer of the action (1.54) for small times.

More precisely, we have the following result:

Theorem 1.34 ([Bre89], Theorem 5.1). *Let $\eta \in \Gamma(D)$, and let $\boldsymbol{\eta} \in \mathcal{P}(\Omega(D))$ be a generalized incompressible flow compatible with η such that*

$$\ddot{\omega}(t) = -\nabla p(t, \omega(t)), \quad \text{for } \boldsymbol{\eta}\text{-a.e. } \omega \in \Omega(D), \tag{1.62}$$

where $p : [0,T] \times D \to \mathbb{R}$ is continuously differentiable with respect to x and satisfies

$$T^2 \sup_{(t,x)\in(0,T)\times D} \nabla_x^2 p(t,x) \le \pi^2 I_d \tag{1.63}$$

in the sense of distributions and symmetric matrices.

Then $\boldsymbol{\eta}$ solves Problem 1.28 among all incompressible flows compatible with η.

Moreover, if the inequality (1.63) *is strict, then $\boldsymbol{\eta}$ is the unique minimizer and has the following* deterministic *property: $\boldsymbol{\eta}$-almost surely, two paths ω and ω' satisfying both $\omega(0) = \omega'(0)$ and $\omega(T) = \omega'(T)$ are equal.*

We notice that the above result holds also without the C^1 assumption on p: indeed (1.63) implies that p is Lipschitz continuous with respect to x, and using that Lipschitz functions are differentiable $\mathcal{L}_D$-a.e. one can easily generalize the above result to this more general situation. However, in order to avoid extra technicalities, we have decided to state the result in this simplified form.

The following corollary, which is a direct consequence of Theorem 1.34, deals with the particular case of deterministic generalized flows induced by orientation and measure-preserving diffeomorphisms.

Corollary 1.35. *Let* $[0,T] \ni t \mapsto g(t) \in SDiff(D)$ *be a smooth flow whose trajectories* $t \mapsto g(t,a)$ *satisfy the ODE* (1.38) *for* $\mathcal{L}_D$*-a.e* $a \in D$*, with* $p : [0,T]\times D \to \mathbb{R}$ *a pressure field which is continuously differentiable with respect to* x *and satisfies* (1.63)*. Then* $\boldsymbol{\eta}_g$ *is a minimizer of* (1.54) *among all the generalized incompressible flows compatible with* $(g(0)\times g(T))_\sharp \mathcal{L}_D$*. Moreover, if the inequality* (1.63) *is strict, then* $\boldsymbol{\eta}_g$ *is the unique minimizer.*

Before giving the proof of Theorem 1.34, we observe that for general final configurations the result is sharp. Indeed, in Section 1.4.4 we will see an example on the two dimensional disc showing that at the time T for which the equality in (1.63) holds there can be several (both classical and non-deterministic) action minimizers.

The proof of Theorem 1.34 is based on the disintegration of generalized flows with respect to initial/final coordinates, and on the following two propositions (whose proofs are postponed after the one of Theorem 1.34).

Proposition 1.36. *Given a vector function* $q \in L^1([0,T]\times D;\mathbb{R}^d)$ *such that*

$$\int_0^T \int_D q(t,x)\,dt\,dx = 0, \tag{1.64}$$

and a generalized flow $\boldsymbol{\eta}$*, let us define the action*

$$\mathcal{A}^q(\boldsymbol{\eta}) := T\int_{\Omega(D)}\int_0^T \frac{1}{2}|\dot\omega(t)|^2 - q(t,\omega(t))\,dt\,d\eta(\omega).$$

Then, for any $\eta \in \Gamma(D)$*,*

$$\min_{\substack{(\mathrm{e}_t)_\sharp\boldsymbol{\eta}=\mathcal{L}_D\\ (\mathrm{e}_0,\mathrm{e}_T)_\sharp\boldsymbol{\eta}=\eta}} \mathcal{A}^q(\boldsymbol{\eta}) = \min_{\substack{(\mathrm{e}_t)_\sharp\boldsymbol{\eta}=\mathcal{L}_D\\ (\mathrm{e}_0,\mathrm{e}_T)_\sharp\boldsymbol{\eta}=\eta}} \mathcal{A}(\boldsymbol{\eta}).$$

Proposition 1.37. *Let* $\gamma \in H^1((0,T);D)$ *be a solution of the ODE*

$$\ddot\gamma(t) = -\nabla p(t,\gamma(t)) \tag{1.65}$$

for some function $p : [0,T]\times D \to \mathbb{R}$ *which is* C^1 *is space and satisfies* (1.63)*. Then, for all* $\omega \in H^1((0,T);D)$ *with* $\omega(0)=\gamma(0)$ *and* $\omega(T)=\gamma(T)$*,*

$$\int_0^T \frac{1}{2}|\dot\gamma(t)|^2 - p(t,\gamma(t))\,dt \le \int_0^T \frac{1}{2}|\dot\omega(t)|^2 - p(t,\omega(t))\,dt. \tag{1.66}$$

Moreover, if the inequality in (1.63) *is strict, then* γ *is the unique minimizer of* (1.66)*.*

Proof (Theorem 1.34). Up to adding a constant to p, we can assume that p satisfies (1.64). Hence by Proposition 1.36 it suffices to show that $\boldsymbol{\eta}$ minimizes $\mathcal{A}^p$.

Now, for all $\boldsymbol{\nu} \in \mathcal{P}(\Omega(D))$ compatible with η we rewrite the action $\mathcal{A}^p(\boldsymbol{\nu})$ in terms of the disintegration of $\boldsymbol{\nu}$ with respect to the map $(\mathrm{e}_0, \mathrm{e}_T) : \Omega(D) \to D \times D$:

$$\begin{aligned}\mathcal{A}^p(\boldsymbol{\nu}) = T \int_{D\times D} \int_{\{\omega\in\Omega(D):\,\omega(0)=x,\omega(T)=y\}} \int_0^T |\dot\omega(t)|^2 \, dt \, d\boldsymbol{\nu}_{x,y}(\omega)\, d\eta(x,y) \\ - T \int_{D\times D} \int_{\{\omega\in\Omega(D):\,\omega(0)=x,\omega(T)=y\}} \int_0^T p(t,\omega(t)) \, dt \, d\boldsymbol{\nu}_{x,y}(\omega)\, d\eta(x,y).\end{aligned} \tag{1.67}$$

For all $x,\ y \in D$ define

$$a^p(x,y) := \inf_{\omega(0)=x,\,\omega(T)=y} T \int_0^T |\dot\omega(t)|^2 - p(t,\omega(t))\, dt.$$

In particular, since $\boldsymbol{\eta}$ is concentrated on trajectories satisfying the assumptions of Proposition 1.37,

$$\mathcal{A}^p(\boldsymbol{\eta}) = \int_{D\times D} a^p(x,y)\, d\eta(x,y) \tag{1.68}$$

By the pointwise inequality between (1.68) and the inner integrands of (1.67), this proves the result.

The fact that if the inequality (1.63) is strict then $\boldsymbol{\eta}$ is uniquely determined, is an immediate consequence of the final assertion in Proposition 1.37.

Proof (Proposition 1.36). By Fubini Theorem and the incompressibility constraint (1.52) we get

$$\int_{\Omega(D)} \int_0^T q(t,\omega(t))\, dt\, d\boldsymbol{\eta}(\omega) = \int_0^T \int_D q(t,x)\, dx = 0,$$

hence $\mathcal{A}^q(\boldsymbol{\eta}) = \mathcal{A}(\boldsymbol{\eta})$ for all $\boldsymbol{\eta}$.

Proof (Proposition 1.37). Let us write ω as a variation of γ:

$$\omega(t) = \gamma(t) + \delta(t), \quad \delta(0) = \delta(T) = 0.$$

Then

$$\begin{aligned}
\int_0^T \frac{1}{2}|\dot\omega|^2 - p(t,\omega)\,dt &= \int_0^T \frac{1}{2}|\dot\gamma+\dot\delta|^2 - p(t,\gamma+\delta)\,dt\\
&= \int_0^T \frac{1}{2}|\dot\gamma|^2 - p(t,\gamma)\,dt + \int_0^T \dot\delta\cdot\dot\gamma\,dt\\
&+ \int_0^T \frac{1}{2}|\dot\delta|^2\,dt + \int_0^T p(t,\gamma) - p(t,\delta+\gamma)\,dt\\
&= \int_0^T \frac{1}{2}|\dot\gamma|^2 - p(t,\gamma)\,dt + \int_0^T \frac{1}{2}|\dot\delta|^2\,dt\\
&- \int_0^T \big(p(t,\gamma+\delta) - p(t,\gamma) - \delta\cdot\nabla p(t,\gamma)\big)\,dt, \qquad (1.69)
\end{aligned}$$

where the last equality is obtained integrating by parts the term $\int \dot\delta\dot\gamma$ and using (1.65). Now, thanks to (1.63) we notice that, for any $t \in [0,T]$,

$$p(t,\gamma(t)+\delta(t)) - p(t,\gamma(t)) - \delta(t)\cdot\nabla p(t,\gamma(t)) \le \frac{\pi^2}{2T^2}|\delta(t)|^2. \qquad (1.70)$$

Moreover, by the Poincaré inequality,

$$\frac{T^2}{\pi^2}\int_0^T |\dot\delta|^2\,dt \ge \int_0^T |\delta|^2\,dt. \qquad (1.71)$$

(A simple way to prove such inequality is to use Fourier series.) Hence, substituting (1.70) and (1.71) in (1.69), we get that $t \mapsto \gamma(t)$ is a minimizer, and it is the unique one in case of strict inequality in (1.63).

1.4.4 A two dimensional non-uniqueness example

In this section we discuss a non-uniqueness example of minimal generalized flows in two dimensions, first put forward in [Bre89] and then further investigated in [BFS08].

Let $D = B_1(0) \subset \mathbb{R}^2$ be the two dimensional unit disc and consider the minimization Problem 1.28 among all generalized incompressible flows connecting i_D to $-i_D$ at time $T = \pi$.

It is easy to see that the generalized incompressible flows $\boldsymbol{\eta}_{g_\pm}$ induced respectively by the *clockwise* and *counterclockwise rotations* of angle $t \in [0,\pi]$,

$$g_\pm(t,a) := (a_1\cos t \mp a_2 \sin t, \pm a_1 \sin t + a_2 \cos t),$$

connect i_D to $-i_D$ at time $T = \pi$ and are concentrated on solutions of the ODE

$$\ddot x(t) = -x(t). \qquad (1.72)$$

Notice that (1.72) corresponds to the "geodesic equation" (1.62) for the smooth *pressure field* $p(t,x) = \frac{1}{2}|x|^2$. Hence, since for such p (1.63) is satisfied for any $T \leq \pi$, one can apply the consistency result of Theorem 1.34 and deduce that both $\boldsymbol{\eta}_{g_+}$ and $\boldsymbol{\eta}_{g_-}$ are minimizers of the action (1.54). Observe that, again by Theorem 1.34, for any time $T < \pi$ the flow $\boldsymbol{\eta}_{g_+}$ (resp. $\boldsymbol{\eta}_{g_-}$) is the unique minimizer of Problem 1.28 between i_D and $g_+(T, \cdot)$ (resp. $g_-(T, \cdot)$).

However, as noticed by Brenier in [Bre89], the loss of uniqueness at time $T = \pi$ is not limited to the previous examples but it is also due to the existence of *non-deterministic* optimal generalized flows. Thanks to the results of Section 1.4.3, to obtain such minimizers it is sufficient to find generalized incompressible flows on D which are concentrated – as $\boldsymbol{\eta}_{g_\pm}$ – on solutions of (1.72). The example provided by Brenier is obtained considering the family of minimizing curves $\omega_{x,\theta}$ connecting x to $-x$ defined by

$$\omega_{x,\theta}(t) := x \cos t + \sqrt{1 - |x|^2}(\cos\theta, \sin\theta) \sin t, \quad \theta \in (0, 2\pi)$$

and taking

$$\boldsymbol{\eta} := \frac{1}{2\pi} \omega_{x,\theta}{}_\sharp \big(\mathcal{L}_D(dx) \times \mathcal{L}^1 \llcorner [0, 2\pi](d\theta)\big).$$

Intuitively speaking, this non-deterministic flow spreads each particle of the fluid uniformly in all directions, along minimal geodesics for the augmented energy functional $\omega \mapsto \int_0^\pi \frac{1}{2}|\dot\omega(t)|^2 - p(t, \omega(t))\, dt$.

In [BFS08], Bernot, Figalli, and Santambrogio resumed Brenier's example and constructed a rich set of new solutions to Problem 1.28 in this setting. Moreover, they provided also with a quite general characterization of the possible minimizers.

With respect to Brenier's original work [Bre89], the main additional information on the problem which is available to the authors of [BFS08] is the fact that being concentrated on solutions of the ODE (1.72) is not only sufficient but also necessary for a generalized incompressible flow to be a minimizer. Indeed, by the analysis of the general Problem 1.28 carried out in [Bre93], [Bre99], [AF08], and [AF09] (see Section 1.5), it turns out that the pressure field $p(t, x) = \frac{1}{2}|x|^2$ is common to all the optimal generalized flows connecting i_D to $-i_D$. More precisely, as we will see in Theorem 1.40, the pressure arises as a (unique, up to time dependent distributions) Lagrange multiplier for the incompressibility constraint in the minimization Problem 1.28 with fixed initial and final configurations (given in this case by $\pm i_D$).

Therefore, the problem of finding and then characterizing the solutions of Problem 1.28 in this case can be reformulated in the following way: The basic observation is that, being solutions of the ODE (1.72), the trajectories of the flow are uniquely determined by their initial position and velocity. Hence (see Lemma 2.3 of [BFS08]), denoting by $t \mapsto \varPhi(t, x, v) \in \mathbb{R}^2$ the unique integral curve of (1.72) starting from $x \in B_1$ with velocity $v \in \mathbb{R}^2$, any generalized flow is optimal if and only if it is of the form

$$\boldsymbol{\eta}_\mu = \varPhi_\sharp \mu,$$

for some $\mu \in \mathrm{Prob}(TD)$ –being $TD := B_1 \times \mathbb{R}^2$– and satisfies the incompressibility constraint (1.52). Notice that $\Phi(t,\cdot,\cdot) = \phi_t$, where $t \mapsto \phi_t$ is the *Hamiltonian flow* given by the solutions of the system

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = -x(t) \\ x(0) = x \\ v(0) = v. \end{cases}$$

Therefore, our minimization problem is equivalent to find measures $\mu \in \mathrm{Prob}(TD)$ (called *minimal measures*) such that

$$\phi_{t\sharp}\mu \in \mathrm{Prob}(TD) \quad \text{and} \quad (\pi_D \circ \phi_t)_\sharp \mu = \mathcal{L}_D, \qquad \text{for all } t \in [0,\pi].$$

In this framework, Brenier's generalized optimal flow (see [Bre89], Section 6) is obtained taking

$$\mu(dx,dv) = \frac{1}{2\pi\sqrt{1-|x|^2}} \mathcal{H}^1 \llcorner \{v = \sqrt{1-|x|^2}\}(dv) \otimes \mathcal{L}_D(dx). \tag{1.73}$$

Starting from to the above considerations, the authors of [BFS08] showed that Brenier's flow can be decomposed into two minimal generalized flows, one concentrated on clockwise rotating geodesics, and the other on counterclockwise ones. More precisely, defining the sets

$$TD^+ := \{(x,v) : \, x^\perp \cdot v > 0\}, \quad TD^- := \{(x,v) : \, x^\perp \cdot v < 0\},$$

where $(x_1,x_2)^\perp = (x_2,-x_1)$, both the measures

$$\mu^+ := 2\mu \llcorner TD^+, \quad \mu^- := 2\mu \llcorner TD^-,$$

with μ given by (1.73), give rise to optimal generalized flows.

Actually, the analysis made in [BFS08] shows that –roughly speaking– the characterizing properties of a significantly large set of minimal measures can be encoded from the ones satisfied by μ, μ^+ and μ^-.

First of all, notice that the supports of all these measures are contained in a level set of the Hamiltonian $E(x,v) := |x|^2 + |v|^2$ (i.e., $\{E(x,v) = 1\}$). In particular, since the Hamiltonian flow ϕ_t preserves the value of E, it is natural to reduce the study to minimal measures which are concentrated on a single level set of the energy.

Secondly, μ, μ^+, and μ^- are *stationary*, namely

$$\phi_{t\sharp}\mu = \mu, \quad \forall\, t \in [0,\pi],$$

and *rotationally invariant*, i.e.

$$(R_\theta,\, R_\theta)_\sharp \mu = \mu, \quad \forall\, \theta \in [0,2\pi),$$

being $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation of angle θ.

Having these properties in mind, the authors of [BFS08] proved the following result, D being now either a disc or an annulus centered at the origin (note that solutions in the disc can always be constructed as convex combinations of solutions in disjoint annuli, so it is important to understand also the case when D is an annulus):

1. There is only one rotationally invariant clockwise (resp. counterclockwise) minimal measure μ that is concentrated on "the appropriate" energy level $\{E(x, v) = K\}$ (see Section 4.2 of [BFS08]);

2. There is only one stationary clockwise (resp. counterclockwise) minimal measure μ that is concentrated on "the appropriate" energy level $\{E(x, v) = K\}$, and in particular this measure is rotationally invariant (see Section 4.3 of [BFS08]).

Without entering into the details, we just observe that, from the dimensional point of view, the results are in accordance with the uniqueness of generalized incompressible flows in dimension 1 (i.e., $D = [-1, 1]$) proved in [BFS08]. Indeed, in the 1-dimensional setting, the phase space is 2-dimensional, and in this case the authors can prove uniqueness. Analogously, the "submanifold" of TB_1 (which is 4-dimensional) defined by the energy constraint and the stationarity (resp. the rotational invariance) of the measures is two dimensional, so it is natural to expect uniqueness there. However, as the results mentioned above show, with respect to 1-dimensional case, in order to ensure uniqueness one still has to fix the degree of freedom given by the orientation of the trajectories.

It is an open question whether there exist minimal measures which are not rotationally invariant. Finally, we refer to Remark 1.42 in Section 1.5.2 for the relation between the minimizing generalized flows constructed in [BFS08] and the weak distributional solutions of the Euler equations defined in Section 1.2.

1.5 The pressure field

In the examples of Section 1.4.4 we have seen that, even for smooth final configurations, action minimizing generalized incompressible flows may not be unique. However, in all the examples, the minimizing flows with the same final configuration shared the same pressure. The aim of this section is to show that this is not a coincidence, but a general fact.

Indeed, as shown by Brenier in [Bre93], for all the solutions of Problem 1.28 which are compatible with the same configuration $\eta \in \Gamma(D)$ there exists a *unique*, up to time dependent constants, *pressure field*. As will be shown below, such pressure field is a distribution arising as a Lagrange multiplier for the incompressibility constraint when considering the action over the larger class of *almost-incompressible generalized flows*.

1.5.1 The pressure as a Lagrange multiplier

By the invariance of the action under time reparameterization, from now on we assume without loss of generality that $T = 1$. Moreover, in order to avoid arguments aiming at controlling the behaviour of generalized flows near the boundary, we assume that $D = \mathbb{T}^d$. Henceforth, D satisfies the assumptions of Theorem 1.33 for the existence of minimizers of the action (1.54).

For all $\eta \in \Gamma(D)$, define

$$\mathcal{A}^*_\eta := \inf\bigl\{\mathcal{A}(\boldsymbol{\eta}) : \boldsymbol{\eta} \in \mathcal{P}(\Omega(D)),\, \boldsymbol{\eta}^{0,1} = \eta\bigr\}.$$

Definition 1.38 (Almost-incompressible flows). *A generalized flow $\boldsymbol{\nu} \in \mathcal{P}(\Omega(D))$ is* almost-incompressible *if there exists a smooth strictly positive function $\rho : [0,1] \times D \to \mathbb{R}^+$ such that*

$$\boldsymbol{\nu}_t = \rho(t,\cdot)\mathcal{L}_D, \qquad \forall\, t \in [0,1]$$

and

$$\|\rho - 1\|_{C^1([0,1]\times D)} \le \frac{1}{2}.$$

We call $\rho = \rho^{\boldsymbol{\nu}}$ the density field *associated to $\boldsymbol{\nu}$.*

Proposition 1.39 ([Bre93], Proposition 2.1). *There exist $\epsilon_0 \in \bigl[0, \frac{1}{2}\bigr)$ and $c' > 0$ such that, for all smooth positive functions $\rho : [0,1]\times D \to \mathbb{R}^+$ satisfying $\rho(0,\cdot) = \rho(1,\cdot) = 1$ and*

$$\int_D \rho(t,x)\, dx = 1 \quad \forall\, t \in [0,1], \qquad \|\rho - 1\|_{C^1([0,1]\times D)} \le \epsilon_0,$$

one can construct a Lipschitz continuous family of diffeomorphisms $[0,1] \ni t \mapsto \gamma(t,\cdot)$ on D, with $\gamma(0,\cdot) = \gamma(1,\cdot) = i_D$, and

$$\det \nabla_x \gamma^{-1}(t,x) = \rho(t,x), \quad \forall\, t \in [0,1],\, x \in D.$$

Moreover, for each generalized incompressible flow $\boldsymbol{\eta}$ such that $\mathcal{A}(\boldsymbol{\eta}) < +\infty$, the image measure of $\boldsymbol{\eta}$ through the mapping

$$\Omega(D) \ni \omega = \bigl(t \mapsto \omega(t)\bigr) \;\mapsto\; \omega_\gamma = \bigl(t \mapsto \gamma(t,\omega(t))\bigr) \in \Omega(D)$$

defines an almost-incompressible flow $\boldsymbol{\nu} := \boldsymbol{\eta}_\gamma$ satisfying $\rho^{\boldsymbol{\nu}} = \rho$, $\boldsymbol{\nu}^{0,1} = \boldsymbol{\eta}^{0,1}$, and

$$\mathcal{A}(\boldsymbol{\nu}) \le \mathcal{A}(\boldsymbol{\eta}) + c'\|\rho - 1\|_{C^1}(1 + \mathcal{A}(\boldsymbol{\eta})).$$

Theorem 1.40 ([Bre93], Theorem 1.1). *For all $\eta \in \Gamma(D)$ there exists $p \in [C^1_0([0,1]\times D)]^*$ such that*

$$\langle p, \rho^{\boldsymbol{\nu}} - 1\rangle \le \mathcal{A}(\boldsymbol{\nu}) - \mathcal{A}^*_{\boldsymbol{\eta}}, \tag{1.74}$$

for all almost-incompressible flows $\boldsymbol{\nu}$ compatible with η.

Observe that, if $\boldsymbol{\nu}$ satisfies the assumptions of Theorem 1.40, then $\rho^{\boldsymbol{\nu}}(0) = \rho^{\boldsymbol{\nu}}(1) = 1$. In particular, the duality bracket in the left-hand side of (1.74) is well defined. Here we report a simplified proof of Theorem 1.40 given by Ambrosio and Figalli in Theorem 6.2 of [AF09].

Proof (Theorem 1.40).
Let $\mathcal{C}$ be the closed convex set

$$\mathcal{C} := \left\{\rho \in C^1([0,1]\times D) : \|\rho - 1\|_{C^1([0,1]\times D)} \le \frac{1}{2},\ \rho(0,\cdot) = \rho(1,\cdot) = 1\right\}$$

and let us define the function $\phi : C^1([0,1]\times D) \to \mathbb{R}^+ \cup \{+\infty\}$,

$$\phi(\rho) := \begin{cases} \inf\{\mathcal{A}(\boldsymbol{\nu}) :\ \rho^{\boldsymbol{\nu}} = \rho \text{ and } \boldsymbol{\nu}^{0,1} = \eta\} & \text{if } \rho \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that $\phi(1) = \mathcal{A}^*_\eta$. It is easy to check that ϕ is convex and lower semicontinuous on $C^1([0,1]\times D)$. Now we prove that ϕ has *bounded slope* at 1, namely

$$\limsup_{\rho\to 1} \frac{(\phi(1) - \phi(\rho))^+}{\|\rho - 1\|_{C^1}} < +\infty. \tag{1.75}$$

Indeed, if (1.75) holds, then applying the Hahn-Banach Theorem it follows that the subdifferential of ϕ at 1 is nonempty, namely there exists $p \in [C^1_0([0,1]\times D)]^*$ such that

$$\langle p, \rho - 1\rangle \le \phi(\rho) - \phi(1),$$

which is equivalent to (1.74).

In order to prove (1.75) we observe that, given any smooth and positive ρ as in Proposition 1.39, for all incompressible flows $\boldsymbol{\eta}$ connecting i_D to η we can find an almost incompressible flow $\boldsymbol{\nu}$ with density field ρ, still connecting i_D to η and satisfying

$$\mathcal{A}(\boldsymbol{\nu}) \le \mathcal{A}(\boldsymbol{\eta}) + c'\|\rho - 1\|_{C^1}(1 + \mathcal{A}(\boldsymbol{\eta})).$$

In particular, if $\boldsymbol{\eta}$ is a minimizer of the action, for any ρ is a C^1-neighborhood of 1 we get

$$\phi(\rho) \le \phi(1) + c'\|\rho - 1\|_{C^1}(1 + \phi(1)),$$

which proves (1.75).

1.5.2 Uniqueness

Thanks to Theorem 1.40 one can now perform first variations of the action functional $\mathcal{A}$ among almost-incompressible flows. Exploiting this fact, Brenier proved in [Bre93] that the pressure field is uniquely determined, up to time dependent distributions:

Theorem 1.41 ([Bre93], Section 7). *For all generalized incompressible flows $\boldsymbol{\eta}$ such that $\mathcal{A}(\boldsymbol{\eta}) = \mathcal{A}^*_\eta$, define the vector-valued measures*

$$\bar{v}^{\boldsymbol{\eta}}_t \mathcal{L}_D = \mathrm{e}_{t\sharp}(\dot\omega\eta), \quad \overline{(v^{\boldsymbol{\eta}} \otimes v^{\boldsymbol{\eta}})}_t \mathcal{L}_D = \mathrm{e}_{t\sharp}(\dot\omega \otimes \dot\omega\eta). \tag{1.76}$$

Then, for all p satisfying (1.74),

$$\partial_t \bar{v}^{\boldsymbol{\eta}}_t + \operatorname{div} \overline{(v^{\boldsymbol{\eta}} \otimes v^{\boldsymbol{\eta}})}_t + \nabla p = 0 \tag{1.77}$$

in the weak distributional sense. As a consequence, p is uniquely determined up to time dependent distributions.

Proof. We perturb $\boldsymbol{\eta}$ as follows: let $w : (0,1) \times D \to \mathbb{R}$ be a smooth compactly supported vector field such that $w(0,\cdot) = w(1,\cdot) = 0$, and for $\epsilon \geq 0$ small define the flow map

$$\frac{d}{d\epsilon} X_t(\epsilon, x) = w(t, X_t(\epsilon, x)), \quad X_t(0,x) = x.$$

Let $\Phi_\epsilon : \Omega(D) \to \Omega(D)$ denote the family of diffeomorphisms

$$\Phi_\epsilon(\omega(t)) = X_t(\epsilon, \omega(t)).$$

Then the density fields $\rho_\epsilon = \rho^{\boldsymbol{\eta}_\epsilon}$ associated to $\boldsymbol{\eta}_\epsilon := \Phi_{\epsilon\sharp}\boldsymbol{\eta}$ satisfy

$$\frac{d}{d\epsilon}\rho_\epsilon(t,x) + \operatorname{div}(w(t,x)\rho_\epsilon(t,x)) = 0, \quad \forall\, t \in [0,1]. \tag{1.78}$$

Hence, if ϵ is sufficiently small, $\boldsymbol{\eta}_\epsilon$ is an almost-incompressible flow, and we have

$$\frac{\mathcal{A}(\boldsymbol{\eta}_\epsilon) - \mathcal{A}(\boldsymbol{\eta})}{\epsilon} = \frac{1}{2\epsilon}\int_{\Omega(D)}\int_0^1 \left|\frac{d}{dt}\Phi_\epsilon(\omega(t))\right|^2 - |\dot\omega(t)|^2\, dt\, d\boldsymbol{\eta}(\omega). \tag{1.79}$$

Now, applying (1.74) and (1.78) to the left-hand side of (1.79), and thanks to the fact that $\boldsymbol{\eta}$ is concentrated on $H^1((0,1);D)$, we can let $\epsilon \to 0^+$ in (1.79) and use (1.74) to obtain

$$\begin{aligned} -\langle p, \operatorname{div} w\rangle &\leq \int_0^1 \int_{\Omega(D)} \dot\omega(t)\cdot \frac{d}{dt}\big[w(t,\omega(t))\big]\, d\boldsymbol{\eta}(\omega)\, dt \\ &= \int_0^1 \int_{\Omega(D)} \dot\omega(t)\cdot \partial_t w(t,\omega(t)) + (\dot\omega(t)\cdot\nabla) w(t,\omega(t))\cdot\dot\omega(t)\, d\boldsymbol{\eta}(\omega)\, dt. \end{aligned} \tag{1.80}$$

Replacing w with $-w$, we deduce that equality in (1.80) holds.

Finally, defining $\bar{v}^{\boldsymbol{\eta}}$ and $\overline{(v^{\boldsymbol{\eta}} \otimes v^{\boldsymbol{\eta}})}$ as in (1.76), by the arbitrariness of w (1.80) is equivalent to (1.77).

Remark 1.42. Notice that $\bar{v}^{\boldsymbol{\eta}}$ is not a distributional solution of Euler, since in general

$$\overline{(v^{\boldsymbol{\eta}} \otimes v^{\boldsymbol{\eta}})} \neq \bar{v}^{\boldsymbol{\eta}} \otimes \bar{v}^{\boldsymbol{\eta}}. \tag{1.81}$$

However it is interesting to observe that, for the minimizing flows of the problem considered in Section 1.4.4 which have been constructed in [BFS08], the difference of the two terms in (1.81) is a gradient. Hence, in that case, one can find true distributional solutions of (1.1) replacing the "microscopic" pressure field p with a new "macroscopic" one.

1.5.3 An Eulerian-Lagrangian formulation

In this section we present a second variational relaxation of the least action principle (1.7). This formulation was introduced by Brenier in [Bre99] to describe the limits of certain approximate solutions to Problem 1.1. Compared to the Lagrangian model, the main advantage of this one consists in the fact that it allows to prove important regularity estimates on the pressure (1.74) ([Bre99], [AF08]) which are crucial to derive necessary and sufficient conditions for optimality of generalized incompressible flows ([AF09]).

Let us assume with no loss of generality that $T = 1$, and let $\eta = \eta_a \otimes d\mathcal{L}_D(a) \in \varGamma(D)$, $\gamma = \gamma_a \otimes d\mathcal{L}_D(a) \in \varGamma(D)$ be a pair of initial and final configurations. Then consider the distributional solutions of the *continuity equation*

$$\partial_t c_{t,a} + \nabla \cdot (v_{t,a} c_{t,a}) = 0, \qquad \text{in } \mathcal{D}'((0,1) \times D), \text{ for } \mathcal{L}_D\text{-a.e. } a \in D, \tag{1.82}$$

with the initial and final conditions

$$c_{0,a} = \eta_a, \quad c_{1,a} = \gamma_a, \qquad \text{for } \mathcal{L}_D\text{-a.e. } a \in D. \tag{1.83}$$

Defining the measures

$$c(dt, dx, da) := c_{t,a} \otimes (dt \times d\mathcal{L}_D(a)), \quad c_t(dx, da) := c_{t,a} \otimes d\mathcal{L}_D(a) \tag{1.84}$$

and the vector field

$$v(t, x, a) := v_{t,a}(x),$$

we observe that (1.82) is equivalent to

$$\frac{d}{dt} \int_{D \times D} \phi(x, a)\, dc_t(x, a) = \int_{D \times D} \langle \nabla_x \phi(x, a), v(t, x, a) \rangle\, dc_t(x, a) \tag{1.85}$$

for all $\phi \in C_b(D \times D)$ which are continuously differentiable with respect to x.

Then one studies the following:

Problem 1.43. Minimize the action

$$\mathcal{A}(c, v) := \int_0^1 \int_{D \times D} \frac{1}{2} |v(t, x, a)|^2\, dc_t(x, a)\, dt \tag{1.86}$$

among all couples (c, v) satisfying (1.82), (1.83), and the *incompressibility constraint*

$$\int_D c_{t,a}\, d\mathcal{L}_D(a) = \mathcal{L}_D, \qquad \text{for all } t \in [0,1]. \tag{1.87}$$

Remark 1.44. According to [BB00], the minimization of (1.86) without the global constraint (1.87) would give the optimal transportation problem (with quadratic cost) between η and γ.

The existence of minimizers was proved by standard compactness and semicontinuity arguments in [Bre99]. This variational model is said to be of *Eulerian-Lagrangian* type since it contains two state variables $(a, x) \in D \times D$, a representing the initial position of the fluid particles (Lagrangian coordinate) and x their actual position (Eulerian coordinate), and $v_t(x, a)$, which is the "average" velocity field of all particles which start from a and are at position x at time t.

Remark 1.45. As noticed in [Bre99], to each minimizer (c, v) of Problem 1.43 one can associate a measure-valued solution ν in the sense of DiPerna and Majda (see Definition 1.18) by setting

$$\int_{[0,1]\times D\times\mathbb{R}^d} f(t,x,\xi)\, d\nu(t,x,\xi) := \int_{[0,1]\times D\times D} f(t,x,v(t,x,a))\, dc(t,x,a),$$

for all $f \in C([0,1] \times D \times \mathbb{R}^d)$ with at most quadratic growth as $\xi \to \infty$.

The equivalence between Problem 1.43 and the extended Lagrangian formulation in Problem 1.28 was proved by Ambrosio and Figalli:

Theorem 1.46 ([AF09], Theorem 4.1). *For all* η, $\gamma \in \Gamma(D)$,

$$\min_{\boldsymbol{\eta}} \mathcal{A}(\boldsymbol{\eta}) = \min_{c,v} \mathcal{A}(c, v),$$

where $\boldsymbol{\eta}$ *are the generalized incompressible flows connecting* η *to* γ*, and* c, v *satisfy* (1.82)*,* (1.83)*, and* (1.87)*. Moreover, every action minimizing generalized incompressible flow* $\boldsymbol{\eta}$ *connecting* η *to* γ *induces a minimizer* (c, v) *of the Eulerian-Lagrangian model and satisfies, for* $\mathcal{L}^1$*-a.e.* $t \in [0,1]$*, the condition*

$$\dot\omega(t) = v_{t,a}(\omega(t)), \qquad \textit{for } \boldsymbol{\eta}\textit{-a.e. } (\omega, a).$$

Here we give only a sketch of the proof, mainly in order to understand the relationships between the configuration spaces of the two formulations.

Proof (Theorem 1.46).
On the one hand, one can check that the correspondence

$$\boldsymbol{\eta} \mapsto (c^{\boldsymbol{\eta}}_{t,a}, v^{\boldsymbol{\eta}}_{t,a}) \quad \text{with} \quad c^{\boldsymbol{\eta}}_{t,a} := (\mathrm{e}_t)_\sharp \boldsymbol{\eta}_a, \quad c^{\boldsymbol{\eta}}_{t,a} v^{\boldsymbol{\eta}}_{t,a} := (\mathrm{e}_t)_\sharp (\dot\omega \boldsymbol{\eta}_a)$$

maps generalized incompressible flows $\boldsymbol{\eta} \in \mathcal{P}(\tilde{\Omega}(D))$ connecting η to γ into couples $(c^{\boldsymbol{\eta}}, v^{\boldsymbol{\eta}})$ satisfying (1.82), (1.83), (1.87), and

$$\int_D |v^{\boldsymbol{\eta}}_{t,a}|^2 \, dc^{\boldsymbol{\eta}}_{t,a} \le \int_{\Omega(D)} |\dot\omega(t)|^2 \, d\boldsymbol{\eta}_a(\omega), \quad \forall\, t \in [0,1]. \tag{1.88}$$

Hence, integrating (1.88) with respect to $a \in D$ and $t \in [0,1]$ one has $\mathcal{A}(c^{\boldsymbol{\eta}}, v^{\boldsymbol{\eta}}) \le \mathcal{A}(\boldsymbol{\eta})$.

To show the reverse inequality one uses the following theorem with $\mu_t = c_{t,a}$ (see [AGS05], Theorem 8.2.1):

Theorem 1.47 (Superposition principle). *Assume that $D \subset \mathbb{R}^m$ is a compact set and that $[0,1] \ni t \mapsto \mu_t \in \mathcal{P}(D)$ is a narrowly continuous solution of the continuity equation*

$$\partial \mu_t + \nabla \cdot (v_t \mu_t) = 0$$

for some vector field $v \in L^1([0,1]; L^2(\mathbb{R}^d, \mu_t))$. Then there exists $\boldsymbol{\nu} \in \mathcal{P}(\Omega(D))$ such that

(i) $\mu_t = (\mathrm{e}_t)_\sharp \boldsymbol{\nu}, \quad \forall\, t \in [0,1],$

(ii) $\displaystyle\int_{\Omega(D)} \int_0^1 |\dot\omega(t)|^2 \, dt \, d\boldsymbol{\nu}(\omega) \le \int_0^1 \int_D |v(t,x)|^2 \, dx \, dt.$

Remark 1.48. In the Eulerian-Lagrangian model the pressure field p is uniquely determined as a distribution, up to time dependent constants, by

$$\nabla p(t,x) = -\partial_t \left(\int_D v(t,x,a) \, dc_{t,x}(a) \right) - \nabla \cdot \left(\int_D v(t,x,a) \otimes v(t,x,a) \, dc_{t,x}(a) \right), \tag{1.89}$$

where (c, v) is an optimal couple for Problem 1.43.

The use of the same letter to denote the pressure in both models is justified by the fact that the correspondence

$$\boldsymbol{\eta} \mapsto (c^{\boldsymbol{\eta}}_{t,a}, v^{\boldsymbol{\eta}}_{t,a}) \quad \text{with} \quad c^{\boldsymbol{\eta}}_{t,a} := (\mathrm{e}_t)_\sharp \boldsymbol{\eta}_a, \quad c^{\boldsymbol{\eta}}_{t,a} v^{\boldsymbol{\eta}}_{t,a} := (\mathrm{e}_t)_\sharp (\dot\omega \boldsymbol{\eta}_a)$$

maps optimal solutions of the Lagrangian problem into optimal solutions of the Eulerian-Lagrangian one. Hence, since under this correspondence (1.89) reduces to (1.77), the two pressure fields must coincide.

1.6 Necessary and sufficient conditions for optimality

In this Section we deal (essentially without proofs) with necessary and sufficient conditions for minimality of generalized incompressible flows. As in Section 1.5 we assume that $D = \mathbb{T}^d$ and we consider the extended minimization problem defined in Section 1.4.1 on the time interval $[0,1]$. The study carried in this section is due to Ambrosio and Figalli ([AF09]), with the exception of Theorem 1.49 ([Bre99]). The main results are Theorems 1.54 and 1.56, containing respectively the first and the second necessary condition, and finally Theorem 1.57, showing that their joint validity is also sufficient for optimality.

Theorem 1.40 (applied to the extended Lagrangian model) gives the following necessary condition for action minimizing incompressible flows: let $\boldsymbol{\eta} \in \mathcal{P}(\tilde{\Omega}(\mathbb{T}^d))$ be an optimal incompressible flow between $\eta, \gamma \in \Gamma(\mathbb{T}^d)$, and consider the augmented action

$$\mathcal{A}^p(\boldsymbol{\nu}) := \int_{\tilde{\Omega}(D)} \int_0^1 \frac{1}{2} |\dot{\omega}(t)|^2 \, dt \, d\boldsymbol{\nu}(\omega, a) - \langle p, \rho^{\boldsymbol{\nu}} - 1 \rangle, \tag{1.90}$$

where p is given by Theorem 1.40. Then $\boldsymbol{\eta}$ minimizes (1.90) among all almost-incompressible flows connecting η to γ.

As in Theorem 1.34, one would like to evince from the minimization of the augmented action (1.90) further necessary conditions for optimality which could be expressed in terms of the trajectories followed by the generalized flows. However, if p was merely a distribution and not a function, it would be impossible to compute its values along curves in $\mathbb{T}^d$. To reach this goal it has been necessary to improve the regularity properties of p.

The first result concerning the regularity of the pressure was obtained by Brenier [Bre99] by means of the Eulerian-Lagrangian formulation described in Section 1.5.3 (see Remark 1.48).

Theorem 1.49 ([Bre99], Theorem 1.2). *For all $\eta, \gamma \in \Gamma(\mathbb{T}^d)$, the distributional spatial derivatives $\{\partial_{x_i} p\}_{i=1}^d$ of the distribution p given by Theorem 1.40 belong to the space of locally finite measures $\mathcal{M}_{\text{loc}}((0,1) \times \mathbb{T}^d))$.*

However, the regularity obtained in Theorem 1.49 is still not sufficient to define path functionals of the type

$$H^1((0,T); \mathbb{T}^d) \ni \gamma \mapsto \int_0^T \frac{1}{2} |\dot{\gamma}(t)|^2 - p(t, \gamma(t)) \, dt.$$

In [AF08], Brenier's result was improved by Ambrosio and Figalli as follows:

Theorem 1.50 ([AF08], Theorem 3.1 and Corollary 3.3). *For all $\eta, \gamma \in \Gamma(\mathbb{T}^d)$, let p be the distribution given by Theorem 1.40. Then $\partial_{x_i} p \in L^2_{\text{loc}}((0,1) \times \mathcal{M}(\mathbb{T}^d))$. In particular,*

$$p \in L^2_{\text{loc}}((0,1); BV(\mathbb{T}^d)) \subset L^2_{\text{loc}}((0,1); L^{d/(d-1)}(\mathbb{T}^d)). \tag{1.91}$$

Thanks to (1.91) we have that p is a function, both in the time and space variables. Hence, one can try to derive from (1.90) some informations on the curves followed by a minimizing flow $\boldsymbol{\eta}$.

Up to adding a time dependent constant, without loss of generality we can assume that

$$\int_{\mathbb{T}^d} p(t,x)\,dx = 0, \qquad \text{for a.e. } t \in [0,1]. \tag{1.92}$$

Recall that, by Theorem 1.34 and Corollary 1.35, at least for short time intervals any smooth solution of the Euler equations induces an incompressible flow $\boldsymbol{\eta}$ such that $\boldsymbol{\eta}$-a.e. $\omega \in \Omega(\mathbb{T}^d)$ minimizes the Lagrangian action $\frac{1}{2}|v|^2 - p(t,x)$.

In order to generalize this kind of result to optimal generalized flows whose pressure field p satisfies the properties of Theorem 1.50, one has to deal with the following technical problems:

1. p is defined only $dt \times \mathcal{L}_{\mathbb{T}^d}$-a.e.;

2. p might not be integrable with respect to t in a neighborhood of 0 and 1.

To solve **1**, one chooses a particular representative of p in its equivalence class. A good choice is

$$\bar{p}(t,x) := \liminf_{\epsilon \to 0} p_\epsilon(t,x) \tag{1.93}$$

where, considering $p(t,\cdot)$ as a 1-periodic function on $\mathbb{R}^d$,

$$p_\epsilon(t,x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} p(t, x+\epsilon y) \mathrm{e}^{-|y|^2/2}\,dy.$$

Notice that $p_\epsilon(t,\cdot)$ is still 1-periodic and

$$(p_\epsilon)_{\epsilon'} = p_{\epsilon+\epsilon'}. \tag{1.94}$$

Observe that $\bar{p} = p$ at every Lebesgue point of $p(t,\cdot)$, therefore $\bar{p}$ is a representative of p.

To deal with **2**, one replaces the concept of minimizing curves of the Lagrangian action with the one of *locally minimizing curves*, namely curves minimizing the action on all time intervals $[s,t] \subset (0,1)$. Moreover, when defining which are the curves that are "admissible competitors" in the minimization problem, we need to avoid that such curves concentrate on regions where $p = +\infty$ for a set of times having positive measure. To prevent this fact, we first introduce the concept of *maximal function*.

Definition 1.51. *For all $f \in L^1(\mathbb{T}^d)$, the maximal function of f is defined as*

$$Mf(x) := \sup_{\epsilon > 0} (2\pi)^{-d/2} \int_{\mathbb{R}^d} |f|(t, x+\epsilon y) \mathrm{e}^{-|y|^2/2}\,dy.$$

In the rest of this section we will use the following facts: first, by (1.94)

$$Mf_\epsilon = \sup_{\epsilon' > 0} \big(|f|_\epsilon\big)_{\epsilon'} = \sup_{\epsilon'} |f|_{\epsilon+\epsilon'} \le \sup_{\epsilon'' > 0} |f|_{\epsilon''} = Mf.$$

Moreover, by standard maximal inequalities one has

$$\|Mf\|_{L^p(\mathbb{T}^d)} \le C_p \|f\|_{L^p(\mathbb{T}^d)}, \qquad \text{for all } p > 1.$$

Setting $Mp(t,x) := Mp(t,\cdot)$, by Theorem 1.50 one gets that $Mp \in L^2_{\mathrm{loc}}((0,1); L^{d/(d-1)}(\mathbb{T}^d))$, so in particular

$$Mp \in L^1_{\mathrm{loc}}((0,1) \times \mathbb{T}^d). \tag{1.95}$$

The regularity of p expressed by (1.95) is the one that is needed to formulate the first necessary condition in terms of minimal trajectories.

Definition 1.52 (q-minimizing paths). *Let $q : [0,1] \times \mathbb{T}^d \to \mathbb{R}$ satisfy (1.92). Fix $\omega \in H^1((0,1) \times \mathbb{T}^d)$, and assume that $Mq(\tau,\omega(\tau)) \in L^1((0,1))$. We say that ω is a* q-minimizing path *if*

$$\int_0^1 \frac{1}{2}|\dot\omega(\tau)|^2 - q(\tau,\omega(\tau))\, d\tau \le \int_0^1 \frac{1}{2}|\dot\omega(\tau) + \dot\delta(\tau)|^2 - q(\tau,\omega(\tau)+\delta(\tau))\, d\tau,$$

for all $\delta \in H^1_0((0,1);\mathbb{T}^d)$ with $Mq(\tau,\omega(\tau)+\delta(\tau)) \in L^1((0,1))$.

If we only have that $Mq(\tau,\omega(\tau)) \in L^1_{\mathrm{loc}}((0,1))$, we say that ω is a locally q-minimizing path *if*

$$\int_s^t \frac{1}{2}|\dot\omega(\tau)|^2 - q(\tau,\omega(\tau))\, d\tau \le \int_s^t \frac{1}{2}|\dot\omega(\tau) + \dot\delta(\tau)|^2 - q(\tau,\omega(\tau)+\delta(\tau))\, d\tau,$$

for all $[s,t] \subset (0,1)$ and for all $\delta \in H^1_0((s,t);\mathbb{T}^d)$ with $Mq(\tau,\omega(\tau)+\delta(\tau)) \in L^1((s,t))$.

We observe that, when taking $q = p$, the integrability condition on Mp along the curve $\omega + \delta$ is done exactly to avoid that the curve spend some "non-negligible time" in regions where $p = +\infty$.

Remark 1.53. Observe that if $\boldsymbol{\eta}$ is incompressible, $\mathcal{A}(\boldsymbol{\eta}) < +\infty$, and $Mq \in L^1_{(\mathrm{loc})}((0,1) \times \mathbb{T}^d)$, then for all $\delta \in H^1_0((0,1);\mathbb{T}^d)$ the following holds: for $\boldsymbol{\eta}$-a.e. (ω,a),

$$Mq(\tau,\omega(\tau)+\delta(\tau)) \in L^1((0,1)), \quad (\text{resp. } L^1((s,t)) \text{ for all } [s,t] \subset (0,1)).$$

Indeed, the incompressibility of $\boldsymbol{\eta}$, Fubini Theorem, and the translation invariance of the Lebesgue measure give

$$\begin{aligned}\int_{\tilde\Omega(D)} \int_J Mq(\tau,\omega+\delta)\, d\boldsymbol{\eta}(\omega,a) &= \int_J \int_{\mathbb{T}^d} Mq(\tau, x+\delta(\tau))\, dx\, d\tau \\ &= \int_J \int_{\mathbb{T}^d} Mq(\tau,x)\, dx\, d\tau < +\infty\end{aligned}$$

for all intervals $J \subset (0,1)$. This shows that the set of "admissible perturbations" is in some sense very large.

The first necessary condition for optimality is expressed by the following:

Theorem 1.54 ([AF09], Theorem 6.8). *Let $\boldsymbol{\eta}$ be an action-minimizing incompressible generalized flow. Then $\boldsymbol{\eta}$ is concentrated on locally $\bar{p}$-minimizing paths, where $\bar{p}$ is the precise representative of the pressure field p defined in* (1.93)*, and on $\bar{p}$-minimizing paths if $Mp \in L^1((0,1)\times\mathbb{T}^d)$.*

Remark 1.55. The fact that an optimal generalized flow $\boldsymbol{\eta}$ is concentrated on locally action-minimizing paths should imply some further regularity properties of these paths. In particular, exploiting that p is BV in the spatial variable (see (1.91)) one may expect that $\boldsymbol{\eta}$-a.e. path solves some weak form of the Euler-Lagrange equation. (In [FM] this question is addressed under the additional assumption that p is Sobolev in the space variable.)

In order to state the second necessary condition we need some preliminary definition.

Given $q \in L^1([s,t]\times\mathbb{T}^d)$ satisfying (1.92), define the cost $c_q^{s,t} : \mathbb{T}^d\times\mathbb{T}^d \to \mathbb{R}\cup\{+\infty\}$ of the q-minimal connection between x and y as

$$c_q^{s,t}(x,y) := \inf\left\{ \int_s^t \frac{1}{2}|\dot\omega(\tau)|^2 - q(\tau,\omega(\tau))\,d\tau : \omega(s)=x,\,\omega(t)=y,\right.$$
$$\left. Mq(\tau,\omega(\tau)) \in L^1((s,t)) \right\}, \qquad (1.96)$$

with the convention that $c_q^{s,t}(x,y) = +\infty$ if there are no admissible curves ω between x and y.

Using this cost function, one can consider the optimal transport problem

$$W_{c_q^{s,t}}(\mu_1,\mu_2) = \inf\left\{ \int_{\mathbb{T}^d\times\mathbb{T}^d} c_q^{s,t}(x,y)\,d\lambda(x,y) : (\pi_1)_\sharp\lambda = \mu_1,\,(\pi_2)_\sharp\lambda = \mu_2,\right.$$
$$\left. (c_q^{s,t})^+ \in L^1(\lambda) \right\}, \qquad (1.97)$$

where again $W_{c_q^{s,t}}(\mu_1,\mu_2) = +\infty$ if no admissible λ exists.

Notice that, in general, the existence of minimizers for (1.97) is not guaranteed because the functions $c_q^{s,t}$ may be not lower semicontinuous.

Choosing $q = \bar{p}_{|[s,t]}$, where $\bar{p} \in L^1_{\rm loc}((0,1)\times\mathbb{T}^d)$ is the representative (1.93) of the pressure field, we can state the second necessary condition for optimality.

Theorem 1.56 ([AF09], Theorem 6.11). *Let $\boldsymbol{\eta}$ be an optimal generalized incompressible flow between η and $\gamma \in \Gamma(\mathbb{T}^d)$. Then, for all $[s,t] \subset (0,1)$, $W_{c_{\bar{p}}^{s,t}}(\boldsymbol{\eta}_a^s, \boldsymbol{\eta}_a^t) < +\infty$ and the plan $(\mathrm{e}_s,\mathrm{e}_t)_\sharp\boldsymbol{\eta}_a$ is optimal, relative to the cost $c_{\bar{p}}^{s,t}$ defined in* (1.96)*, for $\mathcal{L}_{\mathbb{T}^d}$-a.e. $a \in \mathbb{T}^d$.*

Roughly speaking, the above result says that an optimal generalized flow not only chooses to follow locally action-minimizing paths, but actually more is true. In order to understand this second necessary condition, consider for instance the following simple example: a particle starts from a and splits into two particles with equal mass, which at times $s, t \in (0,1)$ are respectively at positions $a_1(s), a_2(s)$ and $a_1(t), a_2(t)$. Then the optimal flow not only will join $a_1(s)$ to $a_1(t)$ (resp. $a_2(s)$ to $a_2(t)$) using action-minimizing paths, but the total cost is also minimized, i.e.

$$c_p^{s,t}(a_1(s), a_1(t)) + c_p^{s,t}(a_2(s), a_2(t)) \leq c_p^{s,t}(a_1(s), a_2(t)) + c_p^{s,t}(a_2(s), a_1(t)).$$

We remark that this additional necessary condition is specific to the relaxed model, and it is empty if the flow is deterministic (that is, if the particles do not split).

Finally, the conditions given in Theorems 1.54 and 1.56 turns out to be also sufficient for optimality.

Theorem 1.57 ([AF09], Theorem 6.12). *Let $\boldsymbol{\eta} \in \mathcal{P}(\tilde{\Omega}(\mathbb{T}^d))$ be a generalized incompressible flow between η and $\gamma \in \Gamma(\mathbb{T}^d)$, and assume that for some function $q \in L^1_{\mathrm{loc}}((0,1) \times \mathbb{T}^d)$ satisfying (1.92) the following properties hold:*

(a) $Mq \in L^1((0,1) \times \mathbb{T}^d)$ and $\boldsymbol{\eta}$ is concentrated on q-minimizing paths;

(b) the plan $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \boldsymbol{\eta}_a$ is optimal, relative to the cost $c_q^{0,1}$ defined in (1.96), for $\mathcal{L}_{\mathbb{T}^d}$-a.e. $a \in \mathbb{T}^d$.

Then $\boldsymbol{\eta}$ is optimal and q is the unique pressure field. In addition, if (a) and (b) are replaced by:

(a′) $Mq \in L^1_{\mathrm{loc}}((0,1) \times \mathbb{T}^d)$ and $\boldsymbol{\eta}$ is concentrated on locally q-minimizing paths;

(b′) For all intervals $[s,t] \subset (0,1)$ the plan $(\mathrm{e}_s, \mathrm{e}_t)_\sharp \boldsymbol{\eta}_a$ is optimal, relative to the cost $c_q^{s,t}$ defined in (1.96), for $\mathcal{L}_{\mathbb{T}^d}$-a.e. $a \in \mathbb{T}^d$;

then the same conclusions hold.

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