A MASS TRANSPORTATION APPROACH TO QUANTITATIVE ISOPERIMETRIC INEQUALITIES

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Abstract. A sharp quantitative version of the anisotropic isoperimetric inequality is established, corresponding to a stability estimate for the Wulff shape of a given surface tension energy. This is achieved by exploiting mass transportation theory, especially Gromov's proof of the isoperimetric inequality and the Brenier-McCann Theorem. A sharp quantitative version of the Brunn-Minkowski inequality for convex sets is proved as a corollary.

1. Introduction

1.1. **Overview.** One dimensional parametrization arguments have been used for many years in the study of sharp inequalities of geometric-functional type. A major example is the proof of the Brunn-Minkowski inequality by Hadwiger and Ohmann [HO, Fe, Ga], where one dimensional monotone rearrangement plays a key role. The more direct generalization of this construction to higher dimension is that of the Knothe map [Kn], but alternative arguments, leading to maps with a more rigid structure, are also known. Starting from the Brenier map [Br], the theory of (optimal) mass transportation provides several results in this direction. All these maps can be used with success in establishing sharp inequalities of various kind [Vi, Chapter 6]. Here we shall be concerned with Gromov's striking proof of the anisotropic isoperimetric inequality [MS]. Our main result is a sharp estimate about the stability of optimal sets in this inequality, established via a quantitative study of transportation maps.

1.2. **Anisotropic perimeter.** The anisotropic isoperimetric inequality arises in connection with a natural generalization of the Euclidean notion of perimeter. In dimension $n \geq 2$, we consider an open, bounded, convex set K of $\mathbb{R}^n$ containing the origin. Starting from K, we define a weight function on directions through the Euclidean scalar product

$$\|\nu\|_* := \sup\{x \cdot \nu : x \in K\}\,, \quad \nu \in S^{n-1}\,, \tag{1.1}$$

where $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$. Let E be an open subset of $\mathbb{R}^n$, with smooth or polyhedral boundary ∂E oriented by its outer unit normal vector ν_E, and let $\mathcal{H}^{n-1}$ stand for the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$. The *anisotropic perimeter* of E is defined as

$$P_K(E) := \int_{\partial E} \|\nu_E(x)\|_* d\mathcal{H}^{n-1}(x)\,. \tag{1.2}$$

This notion of perimeter obeys the scaling law $P_K(\lambda E) = \lambda^{n-1} P_K(E)$, $\lambda > 0$, and it is invariant under translations. However, at variance with the Euclidean perimeter, P_K is not invariant by the action of $O(n)$, or even of $SO(n)$, and in fact it may even happen that $P_K(E) \neq P_K(\mathbb{R}^n \setminus E)$, provided K is not symmetric with respect to the origin. When K is the Euclidean unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ of $\mathbb{R}^n$ then $\|\nu\|_* = 1$ for every $\nu \in S^{n-1}$, and therefore $P_K(E)$ coincides with the Euclidean perimeter of E.

Apart from its intrinsic geometric interest, the anisotropic perimeter P_K arises as a model for surface tension in the study of equilibrium configurations of solid crystals with sufficiently small grains [Wu, He, Ty], and constitutes the basic model for surface energies in phase transitions [Gu]. In both settings, one is naturally led to minimize $P_K(E)$ under a volume constraint. This is, of course, equivalent to study the isoperimetric problem

$$\inf\left\{\frac{P_K(E)}{|E|^{1/n'}} : 0 < |E| < \infty\right\}, \tag{1.3}$$

where $|E|$ is the Lebesgue measure of E and $n' = n/(n-1)$. As conjectured by Wulff [Wu] back to 1901, the unique minimizer (modulo the invariance group of the functional, which consists of translations and scalings) is the set K itself. In particular the anisotropic isoperimetric inequality holds,

$$P_K(E) \geq n|K|^{1/n}|E|^{1/n'}, \quad \text{if } |E| < \infty\,. \tag{1.4}$$

Dinghas [Di] showed how to derive (1.4) from the Brunn-Minkowski inequality

$$|E+F|^{1/n} \geq |E|^{1/n} + |F|^{1/n}, \quad \forall E, F \subseteq \mathbb{R}^n\,. \tag{1.5}$$

The formal argument is well known. Indeed, (1.5) implies that

$$\frac{|E+\varepsilon K| - |E|}{\varepsilon} \geq \frac{(|E|^{1/n} + \varepsilon|K|^{1/n})^n - |E|}{\varepsilon}, \quad \forall \varepsilon > 0\,.$$

As $\varepsilon \to 0^+$, the right hand side converges to $n|K|^{1/n}|E|^{1/n'}$, while, if E is regular enough, the left hand side has $P_K(E)$ as its limit.

From a modern viewpoint, the natural framework for studying the isoperimetric inequality (1.4) is the theory of sets of finite perimeter. If E is a set of finite perimeter in $\mathbb{R}^n$ [AFP] then its anisotropic perimeter is defined as

$$P_K(E) := \int_{\mathcal{F}E} \|\nu_E(x)\|_* d\mathcal{H}^{n-1}(x)\,, \tag{1.6}$$

where $\mathcal{F}E$ denotes the *reduced boundary* of E and $\nu_E : \mathcal{F}E \to S^{n-1}$ is the measure-theoretic *outer* unit normal vector field to E (see Section 2.1). Whenever E has smooth or polyhedral boundary the above definition coincides with (1.2). Existence and uniqueness of minimizers for (1.3) in the class of sets of finite perimeter were first shown by Taylor [Ty], and later, with an alternative proof, by Fonseca and Müller [FM]. In [MS], Gromov deals with the functional version of (1.4), proving the anisotropic Sobolev inequality

$$\int_{\mathbb{R}^n} \|-\nabla f(x)\|_* dx \geq n|K|^{1/n}\|f\|_{L^{n'}(\mathbb{R}^n)}\,, \tag{1.7}$$

for every $f \in C^1_c(\mathbb{R}^n)$. Inequality (1.7) is equivalent to (1.4). Moreover, despite the fact that (1.7) is never saturated for $f \in C^1_c(\mathbb{R}^n)$, it turns out that, with the suitable technical tools from Geometric Measure Theory at hand, Gromov's argument can be adapted to obtain the characterization of the equality cases in (1.4) in the framework of sets of finite perimeter. This was done by Brothers and Morgan in [BM].

Alternative proofs of (1.4), that shall not be considered here, are also known. In particular, we mention the recent paper on anisotropic symmetrization by Van

Schaftingen [VS], and the proof by Dacorogna and Pfister [DP] (limited to the two dimensional case).

1.3. **Stability of isoperimetric problems.** Whenever $0 < |E| < \infty$, we introduce the *isoperimetric deficit of* E,

$$\delta(E) := \frac{P_K(E)}{n|K|^{1/n}|E|^{1/n'}} - 1\,.$$

This functional is invariant under translations, dilations and modifications on a set of measure zero of E. Moreover, $\delta(E) = 0$ if and only if, modulo these operations, E is equal to K (this is a consequence of the characterization of equality cases of (1.4), cf. Theorem A.1). Thus $\delta(E)$ measures, in terms of the relative size of the perimeter and of the measure of E, the deviation of E from being optimal in (1.4). The stability problem consists in quantitatively relating this deviation to a more direct notion of distance from the family of optimal sets. To this end we introduce the *asymmetry index*[1] of E,

$$A(E) := \inf\left\{\frac{|E\Delta(x_0 + rK)|}{|E|} : x_0 \in \mathbb{R}^n\,, r^n|K| = |E|\right\}, \tag{1.8}$$

where $E\Delta F$ denotes the symmetric difference between the sets E and F. The asymmetry is invariant under the same operations that leave the deficit unchanged. We look for constants C and α, depending on n and K only, such that the following quantitative form of (1.4) holds true:

$$P_K(E) \geq n|K|^{1/n}|E|^{1/n'}\left\{1 + \left(\frac{A(E)}{C}\right)^\alpha\right\}, \tag{1.9}$$

i.e., $A(E) \leq C\,\delta(E)^{1/\alpha}$. This problem has been thoroughly studied in the Euclidean case $K = B$, starting from the two dimensional case, considered by Bernstein [Be] and Bonnesen [Bo]. They prove (1.9) with the exponent $\alpha = 2$, that is optimal concerning the decay rate at zero of the asymmetry in terms of the deficit. The first general results in higher dimension are due to Fuglede [Fu], dealing with the case of convex sets. Concerning the unconstrained case, the main contributions are due to Hall, Hayman and Weitsman [HHW, Ha]. They prove (1.9) with a constant $C = C(n)$ and exponent $\alpha = 4$. It was, however, conjectured by Hall that (1.9) should hold with the sharp exponent $\alpha = 2$. This was recently shown in [FMP1] (see also the survey [Ma]).

A common feature of all these contributions is the use of *quantitative* symmetrization inequalities, that is clearly specific to the isotropic case. If K is a generic convex set, then the study of uniqueness and stability for the corresponding isoperimetric inequality requires the employment of entirely new ideas. Indeed, the methods developed in [HHW, FMP1] are of no use as soon as K is not a ball. Under the assumption of convexity on E, the problem has been studied by Groemer [Gr2], while the first stability result for (1.4) on generic sets is due to Esposito, Fusco, and Trombetti in [EFT]. Starting from the uniqueness proof of Fonseca and Müller [FM], they show the validity of (1.9) with some constant $C = C(n, K)$ and for the exponent

$$\alpha(2) = \frac{9}{2}\,, \quad \alpha(n) = \frac{n(n+1)}{2}\,, \quad n \geq 3\,.$$

[1] Also known as the *Fraenkel asymmetry* of E in the Euclidean case $K = B$.

This remarkable result leaves, however, the space for a substantial improvement concerning the decay rate at zero of the asymmetry index in terms of the isoperimetric deficit. Our main theorem provides the sharp decay rate.

Theorem 1.1. *Let E be a set of finite perimeter with $|E| < \infty$, then*

$$P_K(E) \geq n|K|^{1/n}|E|^{1/n'} \left\{ 1 + \left(\frac{A(E)}{C(n)} \right)^2 \right\}, \tag{1.10}$$

or, equivalently,

$$A(E) \leq C(n)\sqrt{\delta(E)}. \tag{1.11}$$

Here and in the following the symbols $C(n)$ and $C(n, K)$ denote positive constants depending on n, or on n and K, whose value is (generally) not specified. Concerning Theorem 1.1, we show that we may consider the value $C(n) = C_0(n)$ defined as

$$C_0(n) = \frac{181\, n^7}{(2 - 2^{1/n'})^{3/2}}. \tag{1.12}$$

Therefore $C_0(n)$ has polynomial growth in n as $n \to \infty$.

Our proof of Theorem 1.1 is based on a quantitative study of certain transportation maps between E and K, through the bounds that can be derived from Gromov's proof of the isoperimetric inequality. These estimates provide control, in terms of the isoperimetric deficit, and modulo scalings and translations, on the distance between such a transportation map and the identity. There are several directions in which one may develop this idea, and the strategy we have chosen requires to settle various purely technical issues that could obscure the overall simplicity of the proof. For these reasons we spend the next three sections of this introduction motivating our choices and describing our argument, adopting for the sake of clarity a quite informal style of presentation.

1.4. **Gromov's proof of the isoperimetric inequality.** Although Gromov's proof [MS] was originally based on the use of the Knothe map M between E and K, his argument works with any other transport map having suitable structure properties, such as the Brenier map. This is a well-known, common feature of all the proofs of geometric-functional inequalities based on mass transportation [CNV, Vi]. It seems however that, in the study of stability, the Brenier map is more efficient. We now give some informal explanations on this point, which could also be of interest in the study of related questions.

The Knothe construction, see Figure 1.1, depends on the choice of an ordered orthonormal basis of $\mathbb{R}^n$. Let us use, for example, the canonical basis of $\mathbb{R}^n$, with coordinates $x = (x_1, x_2, ..., x_n)$, and for every $x \in E$, $y \in K$ and $1 \leq k \leq n-1$, let us define the corresponding $(n-k)$-dimensional sections of E and K as

$$E_{(x_1,...,x_k)} = \{z \in E : z_1 = x_1\,, ..., z_k = x_k\}\,,$$
$$K_{(y_1,...,y_k)} = \{z \in K : z_1 = y_1\,, ..., z_k = y_k\}\,.$$

We define $M(x) = (M_1(x_1), M_2(x_1, x_2), ..., M_n(x))$ by setting

$$\frac{|\{z \in E : z_1 < x_1\}|}{|E|} = \frac{|\{z \in K : z_1 < M_1\}|}{|K|},$$

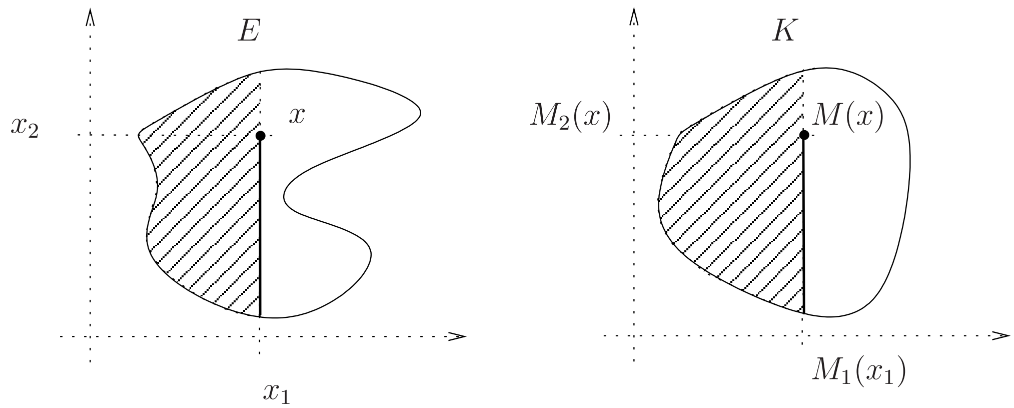

FIGURE 1.1. The construction of the Knothe map. The vertical section E_{x_1} of E is sent into the vertical section $K_{M_1(x_1)}$ of K, where $M_1(x_1)$ is chosen so that the relative measure of $\{z \in E : z_1 < x_1\}$ in E equals the relative measure of $\{z \in K : z_1 < M_1(x_1)\}$ in K. The same idea is used to displace E_{x_1} along $K_{M_1(x_1)}$: the point $x = (x_1, x_2)$ is placed in $K_{M_1(x_1)}$ at the height $M_2(x)$ such that the relative $\mathcal{H}^1$-measure of $\{z \in E_{x_1} : z_2 < x_2\}$ in E_{x_1} equals the relative $\mathcal{H}^1$-measure of $\{z \in K_{M_1(x_1)} : z_2 < M_2(x)\}$ in $K_{M_1(x_1)}$.

and, if $1 \le k \le n-1$,

$$\frac{\mathcal{H}^{n-k}(\{z \in E_{(x_1,\dots,x_k)} : z_{k+1} < x_{k+1}\})}{\mathcal{H}^{n-k}(E_{(x_1,\dots,x_k)})} = \frac{\mathcal{H}^{n-k}(\{z \in K_{(M_1,\dots,M_k)} : z_{k+1} < M_{k+1}\})}{\mathcal{H}^{n-k}(K_{(M_1,\dots,M_k)})}\,.$$

The resulting map has several interesting properties, that are easily checked at a formal level. Its gradient ∇M is upper triangular, its diagonal entries (the partial derivatives $\partial M_k/\partial x_k$) are positive on E, and their product, the Jacobian of M, is constantly equal to $|K|/|E|$, *i.e.*,

$$\det \nabla M = \prod_{k=1}^{n} \frac{\partial M_k}{\partial x_k} = \frac{|K|}{|E|}\,. \tag{1.13}$$

By the arithmetic-geometric mean inequality (which in turn implies the Brunn-Minkowski inequality (1.5) on n-dimensional boxes), we find

$$n(\det \nabla M)^{1/n} \le \operatorname{div} M \quad \text{on } E\,. \tag{1.14}$$

By (1.13), (1.14) and a formal application of the Divergence Theorem,

$$n|K|^{1/n}|E|^{1/n'} = \int_E n(\det \nabla M)^{1/n} \le \int_E \operatorname{div} M = \int_{\partial E} M \cdot \nu_E \, d\mathcal{H}^{n-1}\,. \tag{1.15}$$

Let us now define, for every $x \in \mathbb{R}^n$,

$$\|x\| = \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in K\right\}\,.$$

Note that this quantity fails to define a norm only because, in general, $\|x\| \ne \|-x\|$ (indeed, K is not necessarily symmetric with respect to the origin). The set K can be characterized as

$$K = \{x \in \mathbb{R}^n : \|x\| < 1\}\,. \tag{1.16}$$

Hence, $\|M\| \le 1$ on ∂E as $M(x) \in K$ for $x \in E$. Moreover,

$$\|\nu\|_* = \sup\{x \cdot \nu : \|x\| = 1\}\,,$$

which gives the following Cauchy-Schwarz type inequality

$$x \cdot y \le \|x\| \|y\|_*\,, \quad \forall x, y \in \mathbb{R}^n\,. \tag{1.17}$$

From (1.15), (1.17) and (1.16),

$$n|K|^{1/n}|E|^{1/n'} \le \int_{\partial E} \|M\| \|\nu_E\|_*\, d\mathcal{H}^{n-1} \le P_K(E)\,,$$

and the isoperimetric inequality is proved.

As mentioned earlier, this argument could be repeated *verbatim* if the Knothe map is replaced by the Brenier map. The Brenier-McCann Theorem furnishes a transport map between E and K, which is analogous to the Knothe map, but enjoys a much more rigid structure. Postponing a rigorous discussion to the proof of Theorem 2.3, we recall that Brenier-McCann Theorem [Br, McC1, McC2] ensures the existence of a convex, continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$, whose gradient $T = \nabla\varphi$ *pushes forward* the probability density $|E|^{-1}1_E(x)dx$ into the probability density $|K|^{-1}1_K(y)dx$. In particular, T takes E into K and

$$\det \nabla T = \frac{|K|}{|E|} \quad \text{on } E\,.$$

Since T is the gradient of a convex function and has positive Jacobian, then $\nabla T(x)$ is a symmetric and positive definite $n \times n$ tensor, with n-positive eigenvalues $0 < \lambda_k(x) \le \lambda_{k+1}(x)$, $1 \le k \le n-1$, such that

$$\nabla T(x) = \sum_{k=1}^{n} \lambda_k(x) e_k(x) \otimes e_k(x)\,,$$

for a suitable orthonormal basis $\{e_k(x)\}_{k=1}^n$ of $\mathbb{R}^n$. The inequality $n(\det \nabla T)^{1/n} \le \operatorname{div} T$ is once again implied by the arithmetic-geometric mean inequality for the λ_k's, and the formal version of Gromov's argument presented above can be repeated with T in place of M.

1.5. **Uniqueness: a comparison between Knothe and Brenier map.** Concerning the determination of equality cases, and still arguing at a formal level, one can readily see some differences in the use of the two constructions. Let us consider, for example, a connected open set E having the same barycenter and measure as K, so that $\nabla M = \mathrm{Id}$ or $\nabla T = \mathrm{Id}$ would imply $E = K$. If we assume E to be optimal in the isoperimetric inequality, then we derive from Gromov's argument the conditions $n(\det \nabla M)^{1/n} = \operatorname{div} M$, and $n(\det \nabla T)^{1/n} = \operatorname{div} T$, respectively.

From $n(\det \nabla M)^{1/n} = \operatorname{div} M$ we find that the partial derivatives $\partial M_k/\partial x_k$ are all equal on E. Since $\det \nabla M = 1$, it must be

$$\frac{\partial M_k}{\partial x_k} = 1 \quad \text{on } E. \tag{1.18}$$

As ∇M is upper triangular, this is not sufficient to conclude $\nabla M = \mathrm{Id}$. However, we can still prove that $E = K$ starting from (1.18) by means of the following argument: let $v(t) = \mathcal{H}^{n-1}(\{x \in E : x_1 = t\})$ and $u(t) = \mathcal{H}^{n-1}(\{x \in K : x_1 = t\})$. As $\partial M_1/\partial x_1 = 1$ on E, and having assumed that E and K have the same barycenter,

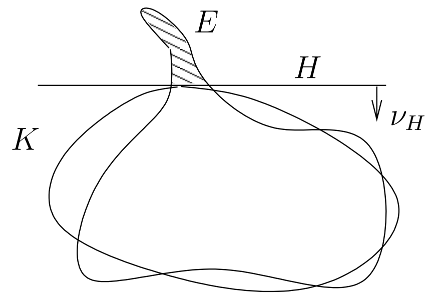

FIGURE 1.2. A quantitative analysis based on any Knothe map constructed on starting from the direction ν_H, leads to control the measure of the dashed zone $E \cap H$ in terms of $\sqrt{\delta(E)}$, see (1.19). Unless K has polyhedral boundary, this argument has to be repeated for infinitely many directions in order to control all of $|E \setminus K|$, finally leading to a non-sharp estimate.

it follows that $u = v$. In particular $\{u > 0\} = \{v > 0\}$ is an open interval (α, β), and we find

$$|E \cap \{x \in \mathbb{R}^n : x_1 \in \mathbb{R} \setminus (\alpha, \beta)\}| = 0\,.$$

If we now fix a direction $\nu \in S^{n-1}$, complete it into an orthonormal basis, apply the above argument to the corresponding Knothe map, and repeat this procedure for *every* direction ν, we find that $|E \setminus K| = 0$, and therefore $E = K$ (since $|E| = |K|$).

Though the use of infinitely many Knothe maps is harmless when proving uniqueness (and presents in fact an interesting analogy with the use of infinitely many Steiner symmetrizations in the uniqueness proof for the Euclidean case [DG]), it unavoidably leads to lose optimality in the decay rate of the asymmetry index in terms of the isoperimetric deficit when trying to prove (1.11). Indeed, it can be shown that if H is an half space disjoint from K such that ∂H is a supporting hyperplane to K, then, by looking at a Knothe map M constructed starting from the direction ν_H and on exploiting the bounds on $\nabla M - \mathrm{Id}$ that can be derived from Gromov's proof, we have

$$|E \cap H| \le C(n, K)\sqrt{\delta(E)}\,, \tag{1.19}$$

see Figure 1.2. To control all of $|E \setminus K|$ one has to repeat this argument for every normal direction to K, a process that, in general, takes infinitely many steps, thus leading to a loss of optimality in the decay rate. This remark also gives a reasonable explanation for the non-optimal exponent $\alpha(n)$ found in [EFT], where arguments related to the Knothe construction are implicitly used.

The Brenier map allows us to avoid all these difficulties: since $\det \nabla T = 1$ on E and ∇T is symmetric, the optimality condition $n(\det \nabla T)^{1/n} = \operatorname{div} T$ immediately implies $\nabla T = \mathrm{Id}$, thus $E = K$.

1.6. **Trace and Sobolev-Poincaré inequalities on almost optimal sets.** We now discuss how the bounds on the isoperimetric deficit contained in Gromov's proof adapted to the Brenier map may be used in proving Theorem 1.1. If we assume $|E| = |K|$ and let T be the Brenier map between E and K, then from (1.15), with

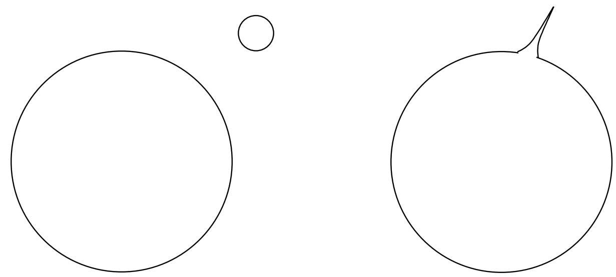

FIGURE 1.3. A set can have arbitrarily small isoperimetric deficit but degenerate Sobolev-Poincaré constant, either because it is not connected or because ∂E contains outward cusps (the picture is relative to the Euclidean case $K = B$).

M replaced by T, we find

$$n|K|\delta(E) \geq \int_{\partial E} (1 - \|T\|)\|\nu_E\|_* d\mathcal{H}^{n-1}\,, \tag{1.20}$$

$$|K|\delta(E) \geq \int_E \left\{ \frac{\operatorname{div} T}{n} - (\det \nabla T)^{1/n} \right\}. \tag{1.21}$$

As seen before, $\delta(E) = 0$ forces $\nabla T = \mathrm{Id}$ a.e. on E, therefore it is not surprising to derive from (1.21) the estimate

$$C(n)|K|\sqrt{\delta(E)} \geq \int_E |\nabla T - \mathrm{Id}\,|\,, \tag{1.22}$$

where we have endowed the space of $n \times n$ tensors with the trace norm $|A| = \sqrt{\mathrm{trace}(A^t A)}$. If we could apply the Sobolev-Poincaré inequality on E, we may control, up to a translation of E, the $L^{n'}$ norm of $T(x) - x$ over E, and therefore, in some form, the size of $|E\Delta K|$. But, of course, there is no reason for the set E to be connected, let alone to have the necessary boundary regularity for a Sobolev-Poincaré inequality to hold true! It turns out that, provided the set E is almost optimal, *i.e.*, that $\delta(E) \leq \delta(n)$ for some suitably small $\delta(n)$, one can identify a maximal "critical subset" of E for the validity of the Sobolev-Poincaré inequality, where the measure of this region is controlled by the isoperimetric deficit. So, up to a simple reduction argument one could directly assume that the Sobolev-Poincaré inequality holds true on E, *i.e.*, that

$$\int_E \|-\nabla f(x)\|_* dx \geq \gamma(n) \inf_{c \in \mathbb{R}} \left(\int_E |f(x) - c|^{n'} dx \right)^{1/n'}, \quad \forall f \in C^1_c(\mathbb{R}^n)\,, \tag{1.23}$$

for a positive constant $\gamma(n)$ that is *independent* of E. Therefore, modulo a translation of E we find

$$C(n, K)\sqrt{\delta(E)} \geq \left(\int_E \|T(x) - x\|^{n'} dx \right)^{1/n'}. \tag{1.24}$$

It remains to control $|E\Delta K|$ by the right hand side of (1.24). As a first step in this direction it is not difficult to find a non-sharp estimate like

$$A(E) \leq C(n, K)\delta(E)^{1/4}. \tag{1.25}$$

Indeed, as T takes values in K we have

$$\|T(x)-x\| \geq \inf\{\|z-x\| : z \in K\}\,, \quad \text{for } x \in E. \tag{1.26}$$

Thus, for every $\varepsilon \in (0,1)$, we find

$$\begin{aligned} |K|A(E) &\leq |E\Delta K| = 2|E\setminus K| \\ &\leq 2\{|E\setminus(1+\varepsilon)K| + |(1+\varepsilon)K\setminus K|\} \\ &\leq C(n)\left\{\frac{1}{\varepsilon}\int_E \|T(x)-x\|dx + \varepsilon|K|\right\} \\ &\leq C(n,K)\left\{\frac{1}{\varepsilon}\sqrt{\delta(E)}+\varepsilon\right\}, \end{aligned}$$

and a simple optimization over ε leads to (1.25).

The reasoning leading from (1.24) to (1.25) is clearly non-optimal. Indeed, it only uses the information that the Brenier map T moves points of E that have distance ε from K at least by a distance of order $\varepsilon \approx \delta(E)^{1/4}$. In fact, by monotonicity, the Brenier map has to move points also into a suitably larger zone *inside* E. This consequence of monotonicity can be clearly visualized on the Knothe map (see Figure 1.4): however, it does not seem easy to translate this intuition into an explicit estimate for the asymmetry.

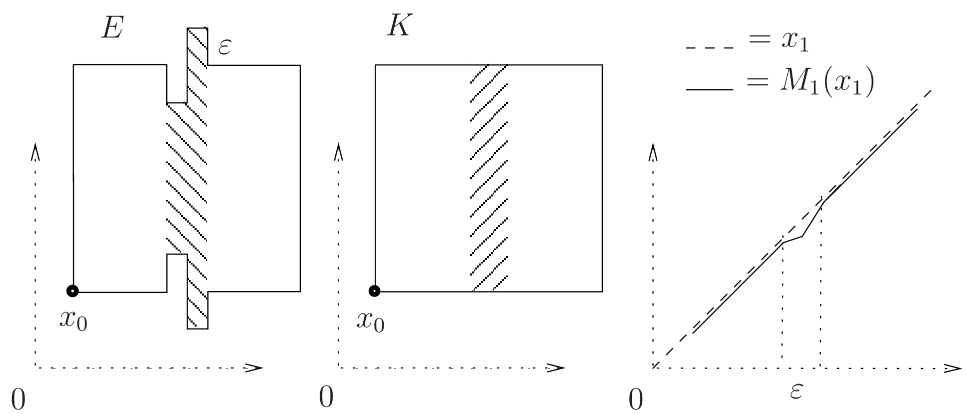

FIGURE 1.4. The set E is such that $A(E) = \varepsilon^2$. Knothe map (with respect to the canonical basis of $\mathbb{R}^2$) differs from the identity only in the dashed zone. In this zone, of measure ε, we have that $\|M(x)-x\|$ is of order ε. In particular $\int_E \|M(x)-x\|dx$ and $A(E)$ have the same size.

The argument that allows us to prove Theorem 1.1 is based on a stronger reduction step. Namely, we show the following. If E has small deficit, up to the removal of a maximal critical subset, there exists a positive constant $\tau(n)$ independent of E such that the following trace inequality holds true:

$$\int_E \|-\nabla f(x)\|_* dx \geq \tau(n)\inf_{c\in\mathbb{R}}\int_{\partial E}|f(x)-c|\|\nu_E(x)\|_* d\mathcal{H}^{n-1}(x)\,, \quad \forall f \in C^1_c(\mathbb{R}^n)\,,$$

see Theorem 3.4. Hence we can apply the trace inequality together with (1.22) to deduce that

$$C(n,K)\sqrt{\delta(E)} \geq \int_{\partial E}\|T(x)-x\|\|\nu_E\|_*\, d\mathcal{H}^{n-1}(x) \tag{1.27}$$

up to a translation of E. Since $\|T(x)\| \le 1$ on ∂E we have

$$|1 - \|x\|| \le |1 - \|T(x)\|\,| + \|T(x) - x\| = (1 - \|T(x)\|) + \|T(x) - x\|\,,$$

for every $x \in \partial E$. Thus, by adding (1.20) and (1.27) we find

$$C(n,K)\sqrt{\delta(E)} \ge \int_{\partial E} |1 - \|x\||\,\|\nu_E\|_*\,d\mathcal{H}^{n-1}(x)\,. \tag{1.28}$$

As shown in Lemma 3.5, this last integral controls $|E \setminus K| = |E\Delta K|/2$ (see Figure 1.5), and thus we achieve the proof of Theorem 1.1 (indeed, although the constant C in (1.28) depends on K, we will see that C can be bounded independently of K).

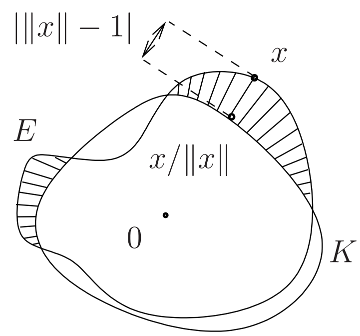

FIGURE 1.5. The term $\int_{\partial E} |\|x\|-1|\,\|\nu_E\|_*\,d\mathcal{H}^{n-1}(x)$ is sufficient to bound $|E \setminus K|$.

1.7. **The Brunn-Minkowski inequality on convex sets.** Whenever E and F are open bounded convex sets, equality holds in the Brunn-Minkowski inequality (1.5)

$$|E+F|^{1/n} \ge |E|^{1/n} + |F|^{1/n}\,,$$

if and only if there exist $r > 0$ and $x_0 \in \mathbb{R}^n$ such that $E = x_0 + rF$. Theorem 1.1 implies an optimal result concerning the stability problem with respect to the *relative asymmetry index of E and F*, defined as

$$A(E,F) = \inf\left\{\frac{|E\Delta(x_0+rF)|}{|E|} : x_0 \in \mathbb{R}^n\,, r^n|F| = |E|\right\}. \tag{1.29}$$

To state this result it is convenient to introduce the *Brunn-Minkowski deficit of E and F*,

$$\beta(E,F) := \frac{|E+F|^{1/n}}{|E|^{1/n} + |F|^{1/n}} - 1\,,$$

and the *relative size factor of E and F*, defined as

$$\sigma(E,F) := \max\left\{\frac{|F|}{|E|}, \frac{|E|}{|F|}\right\}. \tag{1.30}$$

Theorem 1.2. *If E and F are open bounded convex sets, then*

$$|E+F|^{1/n} \ge (|E|^{1/n} + |F|^{1/n})\left\{1 + \frac{1}{\sigma(E,F)^{1/n}}\left(\frac{A(E,F)}{C(n)}\right)^2\right\}, \tag{1.31}$$

or, equivalently,

$$C(n)\sqrt{\beta(E,F)\sigma(E,F)^{1/n}} \ge A(E,F)\,. \tag{1.32}$$

An admissible value for $C(n)$ in (1.32) is $C(n) = 2C_0(n)$, where $C_0(n)$ is the constant defined in (1.12). Moreover, we will show by suitable examples that the decay rate of A in terms of β and σ provided in (1.32) is sharp.

Refinements of the Brunn-Minkowski inequality such as (1.31) were already known in the literature. The role of the relative asymmetry $A(E, F)$ is there played by its counterpart based on the Hausdorff distance in the works of Diskant [Dk] and Groemer [Gr1], and on the Sobolev distance between support functions in the work of Schneider [Sc]. Although these results allow to derive controls on the relative asymmetry $A(E, F)$, they do not seem sufficient to derive the sharp lower bound expressed in (1.31). When the convexity assumption on E and F is dropped the problem becomes significantly more difficult. Some results were, however, obtained by Rusza [Ru].

In [FiMP] we prove Theorem 1.2 by a direct mass transportation argument that avoids the use of Theorem 1.1 and of any sophisticated tool from Geometric Measure Theory. However, that simpler approach has the drawback of producing a value of $C(n)$ in (1.31) that diverges exponentially as $n \to \infty$.

1.8. **Further links with Sobolev inequalities.** As previously mentioned, Gromov's argument was originally developed to prove the anisotropic Sobolev inequality (1.7). A sharp quantitative version of this inequality has been proved in [FMP2], where the suitable notion of *Sobolev deficit* of a function $f \in BV(\mathbb{R}^n)$ is shown to control the $L^{n'}$-distance of f from the set of optimal functions in (1.7) (this set amounts to the non-zero multiples, scalings and translations of 1_K). In a companion paper, we combine Gromov's argument with the theory of symmetric decreasing rearrangements to *improve* this stability result. More precisely, we show that the Sobolev deficit of f actually controls the total variation of f on a suitable super-level set. This kind of gradient estimate is the analogous in the BV-setting to the striking result obtained by Bianchi and Egnell [BE] for the (Euclidean) L^2-Sobolev inequality.

Concerning L^p-Sobolev inequalities, Gromov's proof has inspired a recent important contribution by Cordero-Erausquin, Nazaret and Villani. Indeed, in [CNV] they present a mass transportation proof of the (anisotropic) L^p-Sobolev inequalities, which provides a new (and more direct) way to deduce the classical characterization of equality cases [Au, Ta]. In [CFMP], the proof from [CNV] has been exploited in combination with the theory of symmetric decreasing rearrangements to extend to the (Euclidean) L^p-Sobolev inequalities the above mentioned result from [FMP2]. This has been done with *non-sharp* decay rates. The methods developed in this paper may help to employ, in a more efficient way, the argument from [CNV] in order to obtain the sharp decay rates that are missing in [CFMP], and, possibly, to prove a Bianchi-Egnell-type result for the L^p-Sobolev inequalities.

1.9. **Organization of the paper.** Section 2 contains a rigorous justification of Gromov's argument (applied to the Brenier map) in the framework of sets of finite perimeter, together with some bounds on the isoperimetric deficit in terms of the Brenier map. Section 3 is devoted to the proof of Theorem 1.1, and in particular to the reduction step to sets with a good trace inequality. In Section 4 we consider the Brunn-Minkowski inequality, proving Theorem 1.2 and showing two examples concerning its sharpness. Finally, in the appendix we briefly discuss how the characterization of equality cases for the isoperimetric inequality can be derived from

Gromov's proof in connection with the notion of indecomposability for sets of finite perimeter.

2. Brenier map and the isoperimetric inequality

2.1. **Some preliminaries on functions of bounded variation.** We will use some tools from the theory of sets of finite perimeter and of functions of bounded variation. We gather here some results that are particularly useful in our analysis, referring the reader to the book [AFP] for a detailed exposition.

2.1.1. *Reduced boundary, density points, traces and the Divergence Theorem.* If μ is a $\mathbb{R}^n$-valued Borel measure on $\mathbb{R}^n$ we shall define its (Euclidean) *total variation* as the non-negative Borel measure $|\mu|$ defined on the Borel set E by the formula

$$|\mu|(E) = \sup\left\{\sum_{h\in\mathbb{N}}|\mu(E_h)| : E_h\cap E_k=\emptyset\,,\ \bigcup_{h\in\mathbb{N}}E_h\subseteq E\right\}.$$

Given a measurable set E, we say that E has *finite perimeter* if the distributional gradient $D1_E$ of its characteristic function 1_E is a $\mathbb{R}^n$-valued Borel measure on $\mathbb{R}^n$ with finite total variation on $\mathbb{R}^n$, *i.e.*, with $|D1_E|(\mathbb{R}^n)<\infty$. If, for example, E is a bounded open set with smooth boundary ∂E and outer unit normal vector field ν_E, one can prove, starting from the Divergence Theorem, that E is a set of finite perimeter, with $D1_E=-\nu_E\,d\mathcal{H}^{n-1}\lfloor\partial E$ and $|D1_E|(\mathbb{R}^n)=\mathcal{H}^{n-1}(\partial E)$.

In general, we define the *reduced boundary* $\mathcal{F}E$ of the set of finiter perimeter E as follows: $\mathcal{F}E$ consists of those points $x\in\mathbb{R}^n$ such that $|D1_E|(B(x,r))>0$ for every $r>0$ and

$$\lim_{r\to 0^+}\frac{D1_E(B_r(x))}{|D1_E|(B_r(x))}\quad\text{exists and belongs to } S^{n-1}\,, \tag{2.1}$$

where we have defined $B_r(x)=x+rB$. For every $x\in\mathcal{F}E$ we denote by $-\nu_E(x)$ the limit in (2.1), and call the Borel vector field $\nu_E:\mathcal{F}E\to S^{n-1}$ the *measure theoretic outer unit normal to* E (the minus sign is due to obtaining the outer, instead of the inner, unit normal). The importance of the reduced boundary is clarified by the following result (cf. [AFP, Theorem 3.59]). Here we use the $L^1_{\rm loc}$ convergence of sets, defined by saying that $E_h\to E$ if 1_{E_h} converges to 1_E in $L^1_{\rm loc}$.

Theorem 2.1 (De Giorgi Rectifiability Theorem)**.** *Let E be a set of finite perimeter and let $x\in\mathcal{F}E$. Then*

$$\frac{(E-x)}{r}\longrightarrow\{y\in\mathbb{R}^n:\nu_E(x)\cdot(y-x)<0\}\,, \tag{2.2}$$

as $r\to 0^+$. Moreover, the following representation formulas hold true:

$$D1_E=-\nu_E\,d\mathcal{H}^{n-1}\lfloor\mathcal{F}E\,,\quad |D1_E|(\mathbb{R}^n)=\mathcal{H}^{n-1}(\mathcal{F}E)\,. \tag{2.3}$$

Starting from (2.3) and the distributional Divergence Theorem (see [AFP, Theorem 3.36 and (3.47)]), one finds that, if E is a set of finite perimeter, then

$$\int_E \operatorname{div}T(x)dx=\int_{\mathcal{F}E}T(x)\cdot\nu_E(x)d\mathcal{H}^{n-1}(x)\,, \tag{2.4}$$

for every vector field $T\in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$. We shall need a refinement of this result, relative to the case of a vector field $T\in BV(\mathbb{R}^n;\mathbb{R}^n)$, and stated in (2.18) below.

If E is a Borel set and $\lambda \in [0,1]$, we denote by $E^{(\lambda)}$ the set of points x of $\mathbb{R}^n$ having density λ with respect to E, *i.e.*, $x \in E^{(\lambda)}$ if

$$\lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \lambda\,.$$

We use the notation $\partial_{1/2}E$ for $E^{(1/2)}$, and introduce the *essential boundary* $\partial^* E$ of E by setting $\partial^* E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$. A theorem by Federer [AFP, Theorem 3.61] relates the reduced boundary $\mathcal{F}E$ to the set of points of density $1/2$ and to the essential boundary, ensuring that, if E is a set of finite perimeter, then

$$\mathcal{F}E \subseteq \partial_{1/2}E \subseteq \partial^* E\,,$$

and that, in fact, these three sets are $\mathcal{H}^{n-1}$-equivalent. In particular

$$\mathcal{H}^{n-1}(\mathbb{R}^n \setminus (E^{(1)} \cup E^{(0)} \cup \mathcal{F}E)) = 0\,, \tag{2.5}$$

$$\mathcal{H}^{n-1}(\mathcal{F}E \Delta \partial_{1/2}E) = 0\,. \tag{2.6}$$

Let now E and F be sets of finite perimeter. By [AFP, Proposition 3.38, Example 3.68, Example 3.97], $E \cap F$ is a set of finite perimeter and, if we let

$$J_{E,F} = \{x \in \mathcal{F}E \cap \mathcal{F}F : \nu_E(x) = \nu_F(x)\}\ , \tag{2.7}$$

then, up to $\mathcal{H}^{n-1}$-null sets,

$$\mathcal{F}(E \cap F) = J_{E,F} \cup [\mathcal{F}E \cap F^{(1)}] \cup [\mathcal{F}F \cap E^{(1)}]\,. \tag{2.8}$$

Moreover, at $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}(E \cap F)$ we find

$$\nu_{E\cap F}(x) = \begin{cases} \nu_E(x)\,, & \text{if } x \in \mathcal{F}E \cap F^{(1)}\,,\\ \nu_F(x)\,, & \text{if } x \in \mathcal{F}F \cap E^{(1)}\,,\\ \nu_E(x) = \nu_F(x)\,, & \text{if } x \in J_{E,F}\,. \end{cases} \tag{2.9}$$

In the particular case that $F \subseteq E$, (2.8) and (2.9) reduce to

$$\mathcal{F}F = [\mathcal{F}F \cap \mathcal{F}E] \cup [\mathcal{F}F \cap E^{(1)}]\,, \tag{2.10}$$

$$\nu_F(x) = \nu_E(x)\,, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{F}F \cap \mathcal{F}E\ , \tag{2.11}$$

where (2.10) is valid up to $\mathcal{H}^{n-1}$-null sets. We shall also use the following lemma concerning the union of two sets of finite perimeter:

Lemma 2.2. *Let E and F be sets of finite perimeter with $|E \cap F| = 0$. Then*

$$\nu_{E\cup F}\, d\mathcal{H}^{n-1}\lfloor \mathcal{F}(E \cup F) = \nu_E\, d\mathcal{H}^{n-1}\lfloor(\mathcal{F}E \setminus \mathcal{F}F) + \nu_F\, d\mathcal{H}^{n-1}\lfloor(\mathcal{F}F \setminus \mathcal{F}E)\,, \tag{2.12}$$

and $\nu_E(x) = -\nu_F(x)$ at $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$.

Proof. As $|E \cap F| = 0$, we have $1_{E\cup F} = 1_E + 1_F$. Therefore, by (2.3),

$$\begin{aligned} \nu_{E\cup F}\, d\mathcal{H}^{n-1}\lfloor \mathcal{F}(E \cup F) &= D1_{E\cup F} = D1_E + D1_F \\ &= \nu_E\, d\mathcal{H}^{n-1}\lfloor \mathcal{F}E + \nu_F\, d\mathcal{H}^{n-1}\lfloor \mathcal{F}F\,. \end{aligned} \tag{2.13}$$

Since $\partial_{1/2}E \cap \partial_{1/2}F \subseteq (E \cup F)^{(1)}$, we have $\mathcal{H}^{n-1}(\mathcal{F}(E\cup F) \cap \mathcal{F}E \cap \mathcal{F}F) = 0$ by (2.6). In particular, (2.12) follows from (2.13). Moreover,

$$0 = \int_C \nu_E + \nu_F\, d\mathcal{H}^{n-1}\,, \quad \text{for every Borel set } C \subseteq \mathcal{F}E \cap \mathcal{F}F\,,$$

i.e., $\nu_E = -\nu_F$ at $\mathcal{H}^{n-1}$-a.e. point in $\mathcal{F}E \cap \mathcal{F}F$. □

Let us now recall that we have endowed the space of $n \times n$ tensors $\mathbb{R}^{n\times n}$ with the metric $|A| = \sqrt{\mathrm{trace}(A^t A)}$. In particular, if $T \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ and DT is its $\mathbb{R}^{n\times n}$-valued distributional derivative, then we denote by $|DT|(C)$ the total variation of DT on the Borel set C defined with respect to this metric. We let $BV(\mathbb{R}^n;\mathbb{R}^n)$ be the space of $L^1(\mathbb{R}^n;\mathbb{R}^n)$ vector fields T such that $|DT|(\mathbb{R}^n) < \infty$. In this case we denote by ∇T the density of DT with respect to Lebesgue measure, and by $D_s T$ the corresponding singular part, so that $DT = \nabla T\, dx + D_s T$.

Let us denote by $\mathrm{Div}\, T$ the distributional divergence of T, and consider the case when DT takes values in the set of $n \times n$ tensors that are symmetric and positive definite. Then $\mathrm{Div}\, T$ is a non-negative Radon measure on $\mathbb{R}^n$, which is bounded above and below by the total variation of T: for every Borel set C in $\mathbb{R}^n$,

$$\frac{1}{\sqrt{n}}\mathrm{Div}\, T(C) \le |DT|(C) \le \mathrm{Div}\, T(C)\,, \tag{2.14}$$

as a consequence of $n^{-1/2}\sum_{i=1}^n \lambda_i \le (\sum_{i=1}^n \lambda_i^2)^{1/2} \le \sum_{i=1}^n \lambda_i$ whenever $\lambda_i \ge 0$. Moreover, if we set $\mathrm{div}\, T(x) = \mathrm{trace}(\nabla T(x))$, then

$$\mathrm{Div}\, T = \mathrm{div}\, T\, dx + (\mathrm{Div}\, T)_s\,, \quad (\mathrm{Div}\, T)_s = \mathrm{trace}(D_s T) \ge |D_s T|\,. \tag{2.15}$$

Note that, as a consequence of (2.15), $\mathrm{Div}\, T - \mathrm{div}\, T\, dx$ is a non-negative Radon measure.

Whenever $T \in BV(\mathbb{R}^n;\mathbb{R}^n)$ and E is a set of finite perimeter, for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E$ there exists a vector $\mathrm{tr}_E(T)(x) \in \mathbb{R}^n$ such that

$$\lim_{r\to 0}\frac{1}{r^n}\int_{B_r(x)\cap\{y:(y-x)\cdot\nu_E(x)<0\}} |T(y) - \mathrm{tr}_E(T)(x)|dy = 0\,, \tag{2.16}$$

called the *inner trace of T on E*, see [AFP, Theorem 3.77]. Note that, as a by-product of (2.2) we have in fact

$$\lim_{r\to 0}\frac{1}{r^n}\int_{B_r(x)\cap E} |T(y) - \mathrm{tr}_E(T)(x)|dy = 0\,. \tag{2.17}$$

Moreover, as a consequence of [AFP, Example 3.97] (applied to the pair of functions T and 1_E) the Divergence Theorem holds true in the form

$$\mathrm{Div}\, T(E^{(1)}) = \int_{\mathcal{F}E} \mathrm{tr}_E(T)\cdot\nu_E\, d\mathcal{H}^{n-1}\,, \tag{2.18}$$

whenever $T \in BV(\mathbb{R}^n;\mathbb{R}^n)$ and E is a set of finite perimeter.

2.1.2. *Anisotropic perimeter.* If μ is a $\mathbb{R}^n$-valued Borel measure, its *anisotropic total variation* $\|\mu\|_*$ is the non-negative Borel measure defined on the Borel set E as

$$\|\mu\|_*(E) = \sup\left\{\sum_{h\in\mathbb{N}}\|\mu(E_h)\|_* : E_h\cap E_k = \emptyset\,, \bigcup_{h\in\mathbb{N}} E_h \subseteq E\right\}.$$

If Ω is an open set in $\mathbb{R}^n$ then we have

$$\|\mu\|_*(\Omega) = \sup\left\{\int_{\mathbb{R}^n} T\cdot d\mu : T \in C^1_c(\Omega;K)\right\}.$$

The anisotropic total variation of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined as

$$TV_K(f) := \sup\left\{\int_{\mathbb{R}^n}\mathrm{div}\, T(x) f(x) dx : T \in C^1_c(\mathbb{R}^n;K)\right\}. \tag{2.19}$$

If $f \in BV(\mathbb{R}^n)$ then $TV_K(f) = \|-Df\|_*(\mathbb{R}^n)$. Note that, when $K = B$, then $TV_K(f) = |Df|(\mathbb{R}^n)$ is the total variation over $\mathbb{R}^n$ of the distributional gradient Df of f. In particular, since K is a bounded open set containing the origin, $TV_K(f) < \infty$ if and only if $|Df|(\mathbb{R}^n) < \infty$. If E is a set of finite perimeter and 1_E denotes its characteristic function, then $TV_K(1_E) = P_K(E)$, while, if $f \in C^1_c(\mathbb{R}^n)$,

$$TV_K(f) = \int_{\mathbb{R}^n} \|-\nabla f(x)\|_* dx \,. \tag{2.20}$$

The reason why $-\nabla f(x)$ appears in (2.20) is that it is parallel to the *outer* normal direction to $\{f > f(x)\}$. In this way,

$$P_K(E) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \|-\nabla f_\varepsilon(x)\|_* dx \,,$$

where $f_\varepsilon = 1_E * \rho_\varepsilon$ and ρ_ε is an ε-scale convolution kernel (*i.e.*, $\rho_\varepsilon(z) = \varepsilon^{-n}\rho(z/\varepsilon)$ for $\rho \in C^\infty_c(B;[0,\infty))$, $\int_{\mathbb{R}^n} \rho = 1$). If E is a Borel set and Ω is open, the *anisotropic perimeter of E relative to* Ω is defined by

$$P_K(E|\,\Omega) = \|-D1_E\|_*(\Omega) = \sup\left\{\int_E \operatorname{div} T(x) dx : T \in C^1_c(\Omega;K)\right\}.$$

Therefore, $E \mapsto P_K(E|\Omega)$ is lower semicontinuous with respect to the local convergence of sets. Moreover, this definition agrees with (1.6) when $\Omega = \mathbb{R}^n$, and in general

$$P_K(E|\,\Omega) = \int_{\Omega \cap \mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} \,.$$

Relative perimeters appear in the Fleming-Rishel Coarea Formula for the anisotropic total variation on Ω of a function $f \in C^1(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$. Namely, under these assumptions we have that

$$\int_\Omega \|-\nabla f(x)\|_* dx = \int_{\mathbb{R}} P_K(\{f > t\}|\Omega) dt \,. \tag{2.21}$$

More generally, if $f \in BV(\mathbb{R}^n)$ and $\psi : \mathbb{R}^n \to [0,\infty]$ is a Borel function, then

$$\int_{\mathbb{R}^n} \psi \, d\|-Df\|_* = \int_{\mathbb{R}} dt \int_{\mathcal{F}\{f>t\}} \psi \|\nu_{\{f>t\}}\|_* d\mathcal{H}^{n-1} \,. \tag{2.22}$$

Starting from (2.21), and arguing as in the analogous proof for the Euclidean perimeter, it can be shown that for every set of finite perimeter E one can find a sequence E_h of open bounded set, with polyhedral or smooth boundary, such that

$$|E_h \Delta E| \to 0 \,, \qquad P_K(E_h) \to P_K(E) \,. \tag{2.23}$$

In particular, $A(E_h) \to A(E)$ and $\delta(E_h) \to \delta(E)$.

2.1.3. *A technical remark.* In the proof of Theorem 1.1 we use some non-trivial results from the theory of sets of finite perimeter, as the generalized form of the Divergence Theorem stated in (2.18). This can be avoided, but only up to a certain extent, if one relies on the regularity theory for the Monge-Ampere equation by Caffarelli and Urbas. Indeed, when proving Theorem 1.1, one may assume without loss of generality that E is a bounded open set with smooth boundary (thanks to the approximation given in (2.23)), and derive from [Ca] that the Brenier map T belongs to $C^\infty(E,K)$. However, in the proof of Theorem 1.1 we will need to apply Gromov's proof not to E but to the set G provided by Theorem 3.4 (see Section 3.5).

Since there is a priori no (simple) reason for the set G provided by Theorem 3.4 to be open, the use of (2.18) seems unavoidable.

2.2. **The isoperimetric inequality.** We come now to a rigorous justification of Gromov's argument.

Theorem 2.3. *Whenever* $|E| < \infty$*, we have*

$$P_K(E) \geq n|K|^{1/n}|E|^{1/n'} .$$

Proof. We may assume E has finite perimeter and, by a simple scaling argument, that $|E| = |K|$. The Brenier-McCann Theorem [Br, McC1], suitably modified by taking into account that K is bounded (see, for example, [MV, Section 2.1]), ensures the existence of a convex, continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that, if we set $T = \nabla\varphi$, then $T(x)$ belongs to K for a.e. $x \in \mathbb{R}^n$ and $T_\#(1_E(x)dx) = 1_K(y)dy$, *i.e.*,

$$\int_K h(y)dy = \int_E h(T(x))dx \,, \tag{2.24}$$

for every Borel function $h : \mathbb{R}^n \to [0,\infty]$. As T is the gradient of convex function, its distributional derivative DT takes values in the set of symmetric and non-negative definite $n \times n$-tensors. Therefore, (see e.g. [AA, Proposition 5.1]) $T \in BV(\mathbb{R}^n; K)$, and (2.14) and (2.15) are in force. Moreover, a localization argument starting from (2.24) proves that

$$\det \nabla T(x) = 1 \,, \quad \text{for a.e. } x \in E \,,$$

see [McC2]. Since $\nabla T(x)$ is a positive semi-definite symmetric tensor for a.e. $x \in \mathbb{R}^n$, we can define measurable functions $\lambda_k : \mathbb{R}^n \to [0,\infty)$ and $e_k : \mathbb{R}^n \to S^{n-1}$, $k = 1, ..., n$, such that

$$0 < \lambda_k \leq \lambda_{k+1} \,, \quad e_i \cdot e_j = \delta_{i,j} \,, \quad \nabla T = \sum_{k=1}^n \lambda_k e_k \otimes e_k \,.$$

The arithmetic-geometric mean inequality implies that, for a.e. $x \in E$,

$$n = n(\det \nabla T(x))^{1/n} = n\left(\prod_{k=1}^n \lambda_k(x)\right)^{1/n} \leq \sum_{k=1}^n \lambda_k(x) = \operatorname{div} T(x) \,. \tag{2.25}$$

By (2.25), (2.15) and by the general version of the Divergence Theorem (2.18)

$$\begin{aligned} n|K|^{1/n}|E|^{1/n'} = n|E| &= \int_E n(\det \nabla T(x))^{1/n}\, dx \\ &\leq \int_E \operatorname{div} T(x)dx = \int_{E^{(1)}} \operatorname{div} T(x)\, dx \qquad (2.26) \\ &\leq \operatorname{Div} T(E^{(1)}) = \int_{\mathcal{F}E} \operatorname{tr}_E(T) \cdot \nu_E \, d\mathcal{H}^{n-1} \,, \qquad (2.27) \end{aligned}$$

By (2.16), since T takes values in K, we find $\|\operatorname{tr}_E(T)(x)\| \leq 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E$. The Cauchy-Schwarz inequality (1.17) allows therefore to conclude that

$$n|K|^{1/n}|E|^{1/n'} \leq \int_{\mathcal{F}E} \|\operatorname{tr}_E(T)\|\, \|\nu_E\|_* d\mathcal{H}^{n-1} \leq \int_{\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} = P_K(E) \,, \tag{2.28}$$

as desired. □

A characterization of the equality cases for the isoperimetric inequality could be directly derived from the above proof. However, the argument is slightly technical, and we are going to prove the stronger stability result of Theorem 1.1 without relying on this characterization. Thus, we postpone the details of the equality case to the appendix.

We now exploit Gromov's proof to deduce some bounds on the distance of the Brenier map from a translation, in terms of the size of the isoperimetric deficit.

Corollary 2.4. *Let E be a set of finite perimeter with $|E| = |K|$, and let T be the Brenier map of E into K. If $\delta(E) \le 1$, then*

$$n|K|\delta(E) \ge \int_{\mathcal{F}E} (1 - \|\mathrm{tr}_E(T)\|)\|\nu_E\|_* \, d\mathcal{H}^{n-1}\,, \tag{2.29}$$

$$9n^2|K|\sqrt{\delta(E)} \ge \int_E |\nabla T(x) - \mathrm{Id}\,|dx + |D_sT|(E^{(1)}) = |DS|(E^{(1)})\,, \tag{2.30}$$

where $S(x) = T(x) - x$.

The proof of the corollary is based on the following elementary lemma.

Lemma 2.5. *Let $0 < \lambda_1 \le \ldots \le \lambda_n$ be positive real numbers, and set*

$$\lambda_A := \frac{1}{n}\sum_{k=1}^n \lambda_k\,, \qquad \lambda_G := \left(\prod_{k=1}^n \lambda_k\right)^{1/n}. \tag{2.31}$$

Then

$$7n^2(\lambda_A - \lambda_G) \ge \frac{1}{\lambda_n}\sum_{k=1}^n (\lambda_k - \lambda_G)^2\,. \tag{2.32}$$

Proof of Lemma 2.5. By the inequality

$$\log(s) \le \log(t) + \frac{s-t}{t} - \frac{(s-t)^2}{2\max\{s,t\}^2}\,, \qquad s,t \in (0,\infty)\,,$$

we find that

$$\begin{aligned}\log(\lambda_G) &= \frac{1}{n}\sum_{k=1}^n \log(\lambda_k) \le \frac{1}{n}\sum_{k=1}^n \left\{\log(\lambda_A) + \frac{\lambda_k - \lambda_A}{\lambda_A} - \frac{(\lambda_k-\lambda_A)^2}{2\lambda_n^2}\right\}\\ &= \log(\lambda_A) - \frac{1}{2n\lambda_n^2}\sum_{k=1}^n (\lambda_k - \lambda_A)^2 = \log(\lambda_A) - z\,,\end{aligned}$$

i.e., $\lambda_G \le \lambda_A e^{-z}$. Clearly $z \in [0,1/2]$, and $1 - e^{-t} \ge 3t/4$ for every $t \in [0,1/2]$. Thus

$$\lambda_A - \lambda_G \ge \lambda_A(1 - e^{-z}) \ge \frac{3}{8\,n^2\lambda_n}\sum_{k=1}^n (\lambda_k - \lambda_A)^2$$

where we have also kept into account that $n\lambda_A \ge \lambda_n$. We conclude by noticing that, since $2n \le n^2$,

$$\begin{aligned}\sum_{k=1}^n (\lambda_k - \lambda_G)^2 &\le 2\sum_{k=1}^n (\lambda_k - \lambda_A)^2 + 2n(\lambda_A - \lambda_G)^2\\ &\le \frac{16}{3}n^2\lambda_n(\lambda_A - \lambda_G) + 2n\lambda_n(\lambda_A - \lambda_G) \le 7n^2\lambda_n(\lambda_A - \lambda_G)\,.\end{aligned}$$

□

We now come to the proof of the corollary.

Proof of Corollary 2.4. Inequality (2.29) follows immediately from (2.28). We deduce similarly from (2.26), (2.27) and (2.15) that

$$
\begin{aligned}
|K|\delta(E) &\geq \int_E \left\{ \frac{\operatorname{div} T(x)}{n} - (\det \nabla T(x))^{1/n} \right\} dx + \frac{|D_s T|(E^{(1)})}{n} \\
&= \int_E (\lambda_A - \lambda_G) dx + \frac{|D_s T|(E^{(1)})}{n} .
\end{aligned} \tag{2.33}
$$

With the same notation as in the proof of Theorem 2.3, as $\lambda_G = 1$, we have

$$
\begin{aligned}
\int_E |\nabla T(x) - \operatorname{Id}| dx &= \int_E \sqrt{\sum_{k=1}^n (\lambda_k - 1)^2} \leq \sqrt{\|\lambda_n\|_{L^1(E)}} \sqrt{\int_E \sum_{k=1}^n \frac{(\lambda_k - \lambda_G)^2}{\lambda_n}} \\
&\leq \sqrt{7n^2 \|\lambda_n\|_{L^1(E)} |K| \delta(E)} ,
\end{aligned} \tag{2.34}
$$

where we have applied Lemma 2.5, Hölder inequality and (2.33). By (2.34) we can derive the required upper bound on $\|\lambda_n\|_{L^1(E)}$. Indeed, we have $|\lambda_n - 1| \leq |\nabla T - \operatorname{Id}|$, and moreover, $\delta(E) \leq 1$. Thus, by (2.34),

$$
\|\lambda_n\|_{L^1(E)} \leq |K| + \sqrt{7}\, n \sqrt{\|\lambda_n\|_{L^1(E)} |K|} \leq |K| + \sqrt{7}\, n \left\{ \frac{\varepsilon}{2} \|\lambda_n\|_{L^1(E)} + \frac{|K|}{2\varepsilon} \right\} .
$$

Choosing $\varepsilon = 1/\sqrt{7}n$ we easily deduce that $\|\lambda_n\|_{L^1(E)} \leq 8n^2|K|$, so that

$$
\int_E |\nabla T(x) - \operatorname{Id}| dx \leq \sqrt{56} n^2 |K| \sqrt{\delta(E)} , \tag{2.35}
$$

thanks to (2.34). As $\sqrt{56} + 1 \leq 9$, (2.33) and (2.35) imply (2.30). □

3. Stability for the isoperimetric inequality

In this section we prove Theorem 1.1. The proof is split into several lemmas. We shall often refer to the constants m_K and M_K, defined as

$$
m_K := \inf\{\|\nu\|_* : \nu \in S^{n-1}\} , \quad M_K := \sup\{\|\nu\|_* : \nu \in S^{n-1}\} . \tag{3.1}
$$

We note that, for every $x \in \mathbb{R}^n$,

$$
\frac{|x|}{M_K} \leq \|x\| \leq \frac{|x|}{m_K} . \tag{3.2}
$$

3.1. **Trace inequalities.** We start with a brief review on trace and Sobolev-Poincaré type inequalities on domains of $\mathbb{R}^n$. This topic is developed in great detail in the book of Maz'ja [Mz], especially in relation with the notion of relative perimeter. For technical reasons we propose here a slightly different discussion. Given a set of finite perimeter E with $0 < |E| < \infty$, we consider the constant

$$
\tau(E) = \inf \left\{ \frac{P_K(F)}{\int_{\mathcal{F}F \cap \mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1}} : F \subseteq E \,, 0 < |F| \leq \frac{|E|}{2} \right\} ,
$$

see Figure 3.1. By (2.11), $\tau(E) \geq 1$. When $\tau(E) > 1$ a non-trivial trace inequality holds on E, as shown in the following result.

FIGURE 3.1. The trace constant $\tau(E)$. When $\tau(E)=1$ we have a trivial Sobolev-Poincaré trace inequality (3.3). This happens, for example, if E has multiple connected components (choose F to be any of these components with $|F|\le|E|/2$) or if E contains an outward cusp (consider a sequence F_h converging towards the tip of the cusp).

Lemma 3.1. *For every function $f\in BV(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and for every set of finite perimeter E with $|E|<\infty$ we have*

$$\|-Df\|_*(E^{(1)})\ge\frac{m_K}{M_K}(\tau(E)-1)\inf_{c\in\mathbb{R}}\int_{\mathcal{F}E}\mathrm{tr}_E(|f-c|)\|\nu_E\|_*d\mathcal{H}^{n-1}\,. \tag{3.3}$$

Proof. For every $t\in\mathbb{R}$, let $F_t=E\cap\{f>t\}$. There exists $c\in\mathbb{R}$ such that

$$|F_t|\le\frac{|E|}{2}\,,\quad\forall t\ge c\,,\qquad |E\setminus F_t|\le\frac{|E|}{2}\,,\quad\forall t<c\,.$$

It is convenient to introduce the following notation:

$$u_1(t)=\max\{t-c,0\}\,,\quad u_2(t)=\max\{c-t,0\}\,,\quad\forall t\in\mathbb{R}\,.$$

We start by considering $g=u_1\circ f=\max\{f-c,0\}$ and set $G_s=E\cap\{g>s\}$. By the Coarea Formula (2.22) we have that

$$\|-Dg\|_*(E^{(1)})=\int_0^\infty ds\int_{E^{(1)}\cap\mathcal{F}\{g>s\}}\|\nu_{\{g>s\}}\|_*d\mathcal{H}^{n-1}\,. \tag{3.4}$$

Moreover, by (2.8) and (2.9) we find that $E^{(1)}\cap\mathcal{F}\{g>s\}$ is $\mathcal{H}^{n-1}$-equivalent to $E^{(1)}\cap\mathcal{F}G_s$, and that $\nu_{G_s}=\nu_{\{g>s\}}$ at $\mathcal{H}^{n-1}$-a.e. point in $E^{(1)}\cap\mathcal{F}G_s$. Hence, thanks to (2.10), (2.11) and the definition of $\tau(E)$, we find

$$\begin{aligned}\int_{E^{(1)}\cap\mathcal{F}\{g>s\}}\|\nu_{\{g>s\}}\|_*d\mathcal{H}^{n-1}&=\int_{E^{(1)}\cap\mathcal{F}G_s}\|\nu_{G_s}\|_*d\mathcal{H}^{n-1}\\&=\int_{\mathcal{F}G_s}\|\nu_{G_s}\|_*d\mathcal{H}^{n-1}-\int_{\mathcal{F}E\cap\mathcal{F}G_s}\|\nu_{G_s}\|_*d\mathcal{H}^{n-1}\\&\ge(\tau(E)-1)\int_{\mathcal{F}E\cap\mathcal{F}G_s}\|\nu_{G_s}\|_*d\mathcal{H}^{n-1}\\&=(\tau(E)-1)\int_{\mathcal{F}E\cap\mathcal{F}G_s}\|\nu_E\|_*d\mathcal{H}^{n-1}\,.\end{aligned} \tag{3.5}$$

We now remark that, by Fubini Theorem,

$$\int_{\mathcal{F}E}\mathrm{tr}_E(g)\|\nu_E\|_*d\mathcal{H}^{n-1}=\int_0^\infty ds\int_{\mathcal{F}E\cap\{\mathrm{tr}_E(g)>s\}}\|\nu_E\|_*d\mathcal{H}^{n-1}\,.$$

We claim that, up to $\mathcal{H}^{n-1}$-null sets,

$$\mathcal{F}E\cap\{\mathrm{tr}_E(g)>s\}\subseteq\mathcal{F}E\cap\partial_{1/2}G_s\,.$$

Indeed, by (2.5) it suffices to show that $\mathcal{H}^{n-1}(\mathcal{F}E\cap\{\mathrm{tr}_E(g)>s\}\cap[G_s^{(1)}\cup G_s^{(0)}])=0$. As $G_s^{(1)}\cap\partial_{1/2}E=\emptyset$, by (2.6) we find $\mathcal{H}^{n-1}(\mathcal{F}E\cap G_s^{(1)})=0$. Moreover, if $x\in G_s^{(0)}$,

$$\lim_{r\to 0}\frac{1}{|B_r(x)|}\int_{B_r(x)\cap G_s}g(y)dy\le\|g\|_{L^\infty(\mathbb{R}^n)}\lim_{r\to 0}\frac{|B_r(x)\cap G_s|}{|B_r(x)|}=0\,.$$

Therefore, there is no $x\in\mathcal{F}E\cap\{\mathrm{tr}_E(g)>s\}\cap G_s^{(0)}$, as otherwise, by (2.17),

$$\begin{aligned}s<\mathrm{tr}_E(g)(x)&=\lim_{r\to 0}\frac{1}{|B_r(x)|}\int_{B_r(x)\cap E}g(y)dy\\&=\lim_{r\to 0}\left(\frac{1}{|B_r(x)|}\int_{B_r(x)\cap E\cap\{g\le s\}}g(y)dy+\frac{1}{|B_r(x)|}\int_{B_r(x)\cap G_s}g(y)dy\right)\le s\,,\end{aligned}$$

a contradiction. Thanks to (2.5) and (2.6) our claim is proved. In particular we find that

$$\int_{\mathcal{F}E}\mathrm{tr}_E(g)\|\nu_E\|_*d\mathcal{H}^{n-1}\le\int_0^\infty ds\int_{\mathcal{F}E\cap\mathcal{F}G_s}\|\nu_E\|_*d\mathcal{H}^{n-1}\,,\tag{3.6}$$

and the combination of (3.4), (3.5) and (3.6) leads to

$$\|-D(u_1\circ f)\|_*(E^{(1)})\ge(\tau(E)-1)\int_{\mathcal{F}E}\mathrm{tr}_E(u_1\circ f)\|\nu_E\|_*d\mathcal{H}^{n-1}\,.\tag{3.7}$$

Now, the choice of c allows to repeat the above argument with $\max\{c-f,0\}$ in place of $\max\{f-c,0\}$, thus finding

$$\|D(u_2\circ f)\|_*(E^{(1)})\ge(\tau(E)-1)\int_{\mathcal{F}E}\mathrm{tr}_E(u_2\circ f)\|\nu_E\|_*d\mathcal{H}^{n-1}\,.\tag{3.8}$$

Observing now that

$$\|y\|_*\le\frac{M_K}{m_K}\|-y\|_*\,,\quad\forall y\in\mathbb{R}^n\,,\tag{3.9}$$

on gathering (3.7), (3.8) and (3.9), and by taking into account the linearity of the trace operator as well as that $u_1(t)+u_2(t)=|t-c|$ for every $t\in\mathbb{R}$, we have proved that

$$\sum_{k=1}^2\|-D(u_k\circ f)\|_*(E^{(1)})\ge(\tau(E)-1)\frac{m_K}{M_K}\int_{\mathcal{F}E}\mathrm{tr}_E(|f-c|)\|\nu_E\|_*d\mathcal{H}^{n-1}\,.$$

We will conclude the proof by showing that, for every open set Ω in $\mathbb{R}^n$

$$\sum_{k=1}^2\|-D(u_k\circ f)\|_*(\Omega)\le\|-Df\|_*(\Omega)\,.\tag{3.10}$$

Indeed, let Ω be fixed. Then we can find a sequence $\{f_h\}_{h\in\mathbb{N}}\subseteq C^\infty(\Omega)$ such that $f_h\to f$ in $L^1(\Omega)$ and $\int_\Omega\|-\nabla f_h(x)\|_*dx\to\|-Df\|_*(\Omega)$ as $h\to\infty$. As $\nabla f_h=0$ at a.e. $x\in f_h^{-1}(\{c\})$ and $\sum_{k=1}^2|u_k'(t)|=1$ for every $t\ne c$, we clearly have that

$$\sum_{k=1}^2\int_\Omega\|-\nabla(u_k\circ f_h)\|_*dx\le\int_\Omega\sum_{k=1}^2|u_k'(f_h)|\|-\nabla f_h\|_*dx=\int_\Omega\|-\nabla f_h\|_*dx\,.$$

Letting $h\to\infty$, since $u_k\circ f_h\to u_k\circ f$ in $L^1(\Omega)$, by lower semicontinuity of the anisotropic total variation on open sets we come to (3.10), and achieve the proof of the lemma. □

3.2. **Maximal critical sets.** Here we show the existence of a maximal critical set for the trace inequality.

Lemma 3.2 (Existence of a maximal critical set). *Let E be a set of finite perimeter with $0 < |E| < \infty$, and let $\lambda > 1$. If the family of sets*

$$\Gamma_\lambda = \left\{ F \subseteq E : 0 < |F| \le \frac{|E|}{2}, \quad P_K(F) \le \lambda \int_{\mathcal{F}F \cap \mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} \right\}$$

is non-empty, then it admits a maximal element with respect to the order relation defined by set inclusion up to sets of measure zero.

Proof. We define by induction a sequence of sets F_h in Γ_λ. We let F_1 be any element of Γ_λ and, once F_h has been defined for $h \ge 1$, we consider

$$\Gamma_\lambda(h) = \{F \in \Gamma_\lambda : F_h \subseteq F\} .$$

We let F_{h+1} be any element of $\Gamma_\lambda(h)$ such that

$$|F_{h+1}| \ge \frac{|F_h| + s_h}{2}, \quad \text{where } s_h = \sup_{F \in \Gamma_\lambda(h)} |F| .$$

It is clear that $\{F_h\}_{h\in\mathbb{N}}$ is an increasing sequence of sets, and we denote by F_∞ its limit. We claim that $F_\infty \in \Gamma_\lambda$ and that F_∞ is a maximal element in Γ_λ.

Clearly, $|F_\infty| = \sup_{h\in\mathbb{N}} |F_h| \le |E|/2$. Moreover, by lower semicontinuity of the perimeter we have

$$P_K(F_\infty) \le \liminf_{h\to\infty} P_K(F_h) \le \lambda \liminf_{h\to\infty} \int_{\mathcal{F}F_h \cap \mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} . \tag{3.11}$$

Since $F_h \subseteq F_{h+1} \subseteq F_\infty \subseteq E$, we find

$$(\partial_{1/2} F_h \cap \partial_{1/2} E) \subseteq (\partial_{1/2} F_{h+1} \cap \partial_{1/2} E) \subseteq (\partial_{1/2} F_\infty \cap \partial_{1/2} E) , \tag{3.12}$$

therefore, by (3.11) and (2.6), $F_\infty \in \Gamma_\lambda$. We are left to show that F_∞ is maximal. Indeed, let H be a subset of E, disjoint from F_∞, such that $F_\infty \cup H \in \Gamma_\lambda$. By construction $F_\infty \cup H \in \Gamma_\lambda(h)$, so that

$$s_h \ge |F_\infty \cup H| \ge |F_{h+1}| + |H| \ge \frac{|F_h| + s_h}{2} + |H| ,$$

i.e., $|H| \le (s_h - |F_h|)/2$. Since $s_h - |F_h| \le 2|F_{h+1} \setminus F_h| \to 0$ as $h \to \infty$, we have found $|H| = 0$, thus proving the maximality of F_∞. □

3.3. **Critical sets in almost optimal sets.** Here we show that, provided E is almost optimal, every set $F \subseteq E$ that makes $\tau(E) - 1$ small enough has small volume (in terms of the isoperimetric deficit) with respect to E.

We consider the strictly concave function $\Psi : [0,1] \to [0, 2^{1/n} - 1]$ defined by

$$\Psi(s) := s^{1/n'} + (1-s)^{1/n'} - 1 , \quad s \in [0,1] ,$$

and notice that

$$\Psi(s) \ge (2 - 2^{1/n'}) s^{1/n'} , \quad s \in [0, 1/2] . \tag{3.13}$$

Set

$$k(n) = \frac{2 - 2^{1/n'}}{3} , \tag{3.14}$$

see Figure 3.2. Then we have the following lemma.

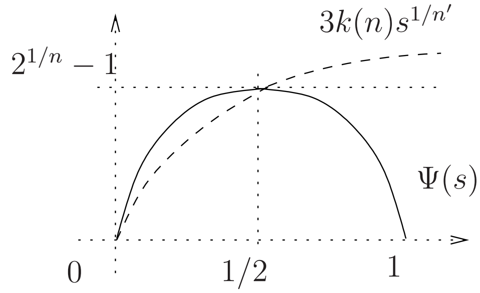

FIGURE 3.2. The constant $k(n)$ is defined so that $3k(n)s^{1/n'} \le \Psi(s)$ for every $s \in [0,1/2]$, with equality for $s \in \{0,1/2\}$.

Lemma 3.3 (Removal of a critical set)**.** *Let E and F be two sets of finite perimeter, with $F \subseteq E$ such that*

$$0 < |F| \le \frac{|E|}{2} < \infty\,,\quad P_K(F) \le \left(1 + \frac{m_K}{M_K}k(n)\right)\int_{\mathcal{F}F\cap\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1}\,.$$

Then

$$|F| \le \left(\frac{\delta(E)}{k(n)}\right)^{n'}|E|\,,\quad P_K(E\setminus F) \le P_K(E)\,,$$

and in particular, provided $\delta(E) \le k(n)$,

$$\delta(E\setminus F) \le \frac{3}{k(n)}\delta(E)\,.$$

Proof. Let us set for the sake for brevity $\lambda = 1 + (m_K/M_K)k(n)$ and $G = E \setminus F$. Thanks to (2.10) and (2.11),

$$P_K(F) = \int_{\mathcal{F}F\cap\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{E^{(1)}\cap\mathcal{F}F} \|\nu_F\|_* d\mathcal{H}^{n-1}\,,$$
$$P_K(G) = \int_{\mathcal{F}G\cap\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{E^{(1)}\cap\mathcal{F}G} \|\nu_G\|_* d\mathcal{H}^{n-1}\,.$$

It is easily seen that $\partial_{1/2}F\cap\partial_{1/2}G\cap\partial_{1/2}E = \emptyset$. Moreover, $\partial_{1/2}F\cap E^{(1)} = \partial_{1/2}G\cap E^{(1)}$, and, by Lemma 2.2, $\nu_G = -\nu_F$ at $\mathcal{H}^{n-1}$-a.e. point of $\partial_{1/2}F\cap E^{(1)}$. Gathering these remarks, and taking into account (3.9), we find that

$$\begin{aligned} P_K(E) &= \int_{\mathcal{F}G\cap\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{\mathcal{F}F\cap\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} \\ &\ge P_K(G) + P_K(F) - \left(1 + \frac{M_K}{m_K}\right)\int_{E^{(1)}\cap\mathcal{F}F} \|\nu_F\|_* d\mathcal{H}^{n-1}\,. \end{aligned} \tag{3.15}$$

By our assumptions on F,

$$\int_{E^{(1)}\cap\mathcal{F}F} \|\nu_F\|_* d\mathcal{H}^{n-1} \le (\lambda-1)\int_{\mathcal{F}F\cap\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} \le (\lambda-1)P_K(F)\,.$$

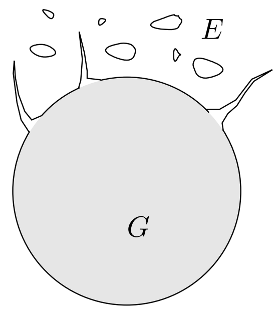

FIGURE 3.3. The set G is obtained by cutting away from E a maximal critical subset F_∞ for the Sobolev-Poincaré trace inequality (see also Figure 3.1). If $G = E \setminus F_\infty$ and $\delta(E)$ is small enough, then $|E \setminus G|$ and $\delta(G)$ are bounded from above by $\delta(E)$, while $\tau(G) - 1$ is bounded from below in terms of n and K only.

As $(1 + (M_K/m_K))(\lambda - 1) \le 2k(n)$, by (3.15) and thanks to the isoperimetric inequality (1.4) we derive that

$$\begin{aligned} P_K(E) &\ge P_K(G) + (1 - 2k(n))P_K(F) \\ &\ge n|K|^{1/n}\{|G|^{1/n'} + (1 - 2k(n))|F|^{1/n'}\}\,. \end{aligned} \tag{3.16}$$

Let us consider $t = |F|/|E|$, so that $t \in (0, 1/2]$. By definition of Ψ and of $k(n)$,

$$\delta(E) \ge \Psi(t) - 2k(n)t^{1/n'} \ge k(n)t^{1/n'}\,,$$

and it follows that $|F| \le (\delta(E)/k(n))^{n'}|E|$. Since $k(n) \le 1/2$ we also deduce from (3.16) that $P_K(G) \le P_K(E)$. Finally, if $\delta(E) \le k(n)$, as

$$t \le \min\{1/2, \delta(E)/k(n)\}$$

we find

$$\begin{aligned} \delta(G) &= \frac{P_K(G)}{n|K|^{1/n}|G|^{1/n'}} - 1 \le \frac{P_K(E)}{n|K|^{1/n}|E|^{1/n'}(1-t)^{1/n'}} - 1 \\ &\le \frac{P_K(E)}{n|K|^{1/n}|E|^{1/n'}}\,(1 + 2t) - 1 = \delta(E) + 2t(\delta(E) + 1) \le \frac{3}{k(n)}\,\delta(E)\,. \end{aligned}$$

This completes the proof of the lemma. □

3.4. **Reduction to a better set.** We next show that an almost optimal set can be replaced (to the end of proving Theorem 1.1) by a set that satisfies the trace inequality with a constant bounded from below in terms of n and m_K/M_K only.

Theorem 3.4. *Let E be a set of finite perimeter, with $0 < |E| < \infty$ and $\delta(E) \le k(n)^2/8$. Then there exists $G \subseteq E$, having finite perimeter, such that*

$$|E \setminus G| \le \frac{\delta(E)}{k(n)}\,|E|\,, \quad \delta(G) \le \frac{3}{k(n)}\,\delta(E)\,, \tag{3.17}$$

and

$$\tau(G) \ge 1 + \frac{m_K}{M_K}\,k(n)\,. \tag{3.18}$$

Proof. We consider the family Γ_λ introduced in Lemma 3.2, and take

$$\lambda = 1 + \frac{m_K}{M_K}\, k(n)\,.$$

Let F_∞ be the maximal set constructed in Lemma 3.2, and let $G = E \setminus F_\infty$. Since $F_\infty \in \Gamma_\lambda$, by Lemma 3.3 we deduce the validity of (3.17). It remains to show that $\tau(G) \ge \lambda$. Let otherwise H be a subset of G such that

$$0 < |H| \le \frac{|G|}{2}\,, \quad P_K(H) < \lambda \int_{\mathcal{F}H\cap\mathcal{F}G} \|\nu_G\|_* d\mathcal{H}^{n-1}\,. \tag{3.19}$$

We will prove that $F_\infty \cup H \in \Gamma_\lambda$, thus violating the maximality of F_∞. By Lemma 3.3 we find

$$|H| \le \frac{\delta(G)}{k(n)}|G| \le 3\frac{\delta(E)}{k(n)^2}|E|\,,$$

and likewise, since $\delta(E) \le k(n)^2/8$,

$$|F_\infty \cup H| = |F_\infty| + |H| \le \frac{\delta(E)}{k(n)}|E| + 3\frac{\delta(E)}{k(n)^2}|E| \le 4\frac{\delta(E)}{k(n)^2}|E| \le \frac{|E|}{2}\,.$$

We are thus left to show that

$$P_K(F_\infty \cup H) \le \lambda \int_{\mathcal{F}(F_\infty\cup H)\cap\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1}\,,$$

or equivalently that

$$\int_{\mathcal{F}(F_\infty\cup H)\cap E^{(1)}} \|\nu_{F_\infty\cup H}\|_* d\mathcal{H}^{n-1} \le \frac{m_K}{M_K} k(n) \int_{\mathcal{F}(F_\infty\cup H)\cap\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1}\,. \tag{3.20}$$

To this end, we remark that by Lemma 2.2

$$\begin{aligned}\int_{\mathcal{F}(F_\infty\cup H)\cap E^{(1)}} &\|\nu_{F_\infty\cup H}\|_* d\mathcal{H}^{n-1} \\ &= \int_{[\mathcal{F}F_\infty\setminus\mathcal{F}H]\cap E^{(1)}} \|\nu_{F_\infty}\|_* d\mathcal{H}^{n-1} + \int_{[\mathcal{F}H\setminus\mathcal{F}F_\infty]\cap E^{(1)}} \|\nu_H\|_* d\mathcal{H}^{n-1}\,.\end{aligned} \tag{3.21}$$

Since $(\partial_{1/2}H \setminus \partial_{1/2}F_\infty) \cap E^{(1)} \subseteq G^{(1)}$, by (2.5) and (2.6),

$$\begin{aligned}\int_{[\mathcal{F}H\setminus\mathcal{F}F_\infty]\cap E^{(1)}} \|\nu_H\|_* d\mathcal{H}^{n-1} &\le \int_{\mathcal{F}H\cap G^{(1)}} \|\nu_H\|_* d\mathcal{H}^{n-1} \\ &\le \frac{m_K}{M_K} k(n) \int_{\mathcal{F}H\cap\mathcal{F}G} \|\nu_H\|_* d\mathcal{H}^{n-1}\,,\end{aligned} \tag{3.22}$$

where we have also used (3.19). By Lemma 2.2, $\nu_{F_\infty} = -\nu_H$ at $\mathcal{H}^{n-1}$-a.e. point of $\mathcal{F}F_\infty \cap \mathcal{F}H$. Therefore, due to (3.9),

$$\int_{\mathcal{F}H\cap\mathcal{F}G\cap E^{(1)}} \|\nu_H\|_* d\mathcal{H}^{n-1} \le \frac{M_K}{m_K} \int_{\mathcal{F}H\cap\mathcal{F}F_\infty\cap E^{(1)}} \|\nu_{F_\infty}\|_* d\mathcal{H}^{n-1}\,. \tag{3.23}$$

Combining (3.21), (3.22) and (3.23) we find that

$$
\begin{aligned}
\int_{\mathcal{F}(F_\infty\cup H)\cap E^{(1)}} & \|\nu_{F_\infty\cup H}\|_* d\mathcal{H}^{n-1} \\
&\le \int_{[\mathcal{F}F_\infty\setminus\mathcal{F}H]\cap E^{(1)}} \|\nu_{F_\infty}\|_* d\mathcal{H}^{n-1} \\
&\qquad + \frac{m_K}{M_K}k(n)\left\{\int_{\mathcal{F}H\cap\mathcal{F}G\cap E^{(1)}} \|\nu_H\|_* d\mathcal{H}^{n-1} + \int_{\mathcal{F}H\cap\mathcal{F}G\cap\mathcal{F}E} \|\nu_H\|_* d\mathcal{H}^{n-1}\right\} \\
&\le \int_{\mathcal{F}F_\infty\cap E^{(1)}} \|\nu_{F_\infty}\|_* d\mathcal{H}^{n-1} + \frac{m_K}{M_K}k(n)\int_{\mathcal{F}H\cap\mathcal{F}G\cap\mathcal{F}E} \|\nu_H\|_* d\mathcal{H}^{n-1} \\
&\le \frac{m_K}{M_K}k(n)\left\{\int_{\mathcal{F}F_\infty\cap\mathcal{F}E} \|\nu_{F_\infty}\|_* d\mathcal{H}^{n-1} + \int_{\mathcal{F}H\cap\mathcal{F}G\cap\mathcal{F}E} \|\nu_H\|_* d\mathcal{H}^{n-1}\right\} \\
&\le \frac{m_K}{M_K}k(n)\int_{\mathcal{F}(F_\infty\cup H)\cap\mathcal{F}E} \|\nu_{F_\infty\cup H}\|_* d\mathcal{H}^{n-1}\,,
\end{aligned}
$$

where in the last step we have applied again Lemma 2.2. This proves (3.17) and concludes the proof of the theorem. □

3.5. **Proof of Theorem 1.1.** Before coming to the proof of the theorem, we are left to show a last estimate, the one explained in Figure 1.5.

Lemma 3.5. *If E is a set of finite perimeter in $\mathbb{R}^n$ with $|E|<\infty$, then*

$$
\int_{\mathcal{F}E} \big|\|x\|-1\big|\,\|\nu_E(x)\|_* d\mathcal{H}^{n-1}(x) \ge \frac{m_K}{M_K}|E\setminus K|\,. \tag{3.24}
$$

Proof. Let us set for simplicity of notation $K_t = tK$ for every $t>0$, and consider the convex function $u:\mathbb{R}^n\to[0,\infty)$ defined by $u(x)=\|x\|$, so that $K_t=\{u<t\}$ and $u^{-1}\{t\}=\partial K_t$. Since u is sub-additive it is easily seen that (3.2) implies

$$
\frac{1}{M_K}\le|\nabla u(x)|\le\frac{1}{m_K}\,, \tag{3.25}
$$

for a.e. $x\in\mathbb{R}^n$. By a simple approximation argument [Ty, Theorem 3.1] it suffices to prove (3.24) in the case that K is uniformly convex and has smooth boundary. Correspondingly, we have that $u\in C^\infty(\mathbb{R}^n\setminus\{0\})$. We notice that if $x\in\partial K_t$ for some $t>0$ then

$$
\nu_{K_t}(x)=\frac{\nabla u(x)}{|\nabla u(x)|}\,,
$$

and due to the convexity of K the mean curvature $H_t(x)$ of ∂K_t at x satisfies

$$
H_t(x)=\operatorname{div}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)\ge 0\,. \tag{3.26}
$$

Finally, we notice that $\mathcal{H}^{n-1}(\mathcal{F}E\cap\partial K_t)=0$ for a.e. $t>0$. Therefore, again by an approximation argument we can directly assume that

$$
\mathcal{H}^{n-1}(\mathcal{F}E\cap\partial K)=0\,. \tag{3.27}
$$

We are now in the position to prove (3.24). By (3.25) we have

$$\begin{aligned}|E\setminus K| &\le M_K\int_{E\setminus K}|\nabla u| = M_K\int_{E\setminus K}\nabla(u-1)\cdot\frac{\nabla u}{|\nabla u|}\\ &= M_K\int_{E\setminus K}\mathrm{div}\left((u-1)\frac{\nabla u}{|\nabla u|}\right)-(u-1)\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right).\end{aligned}\tag{3.28}$$

By the Coarea Formula and by (3.26) we find that

$$\int_{E\setminus K}(u-1)\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)=\int_1^\infty(t-1)\,dt\int_{E\cap\partial K_t}H_t\,\frac{d\mathcal{H}^{n-1}}{|\nabla u|}\ge 0\,.$$

Hence (3.28) and the Divergence Theorem imply

$$|E\setminus K|\le M_K\int_{\mathcal{F}(E\setminus K)}(u-1)\frac{\nabla u}{|\nabla u|}\cdot\nu_{E\setminus K}\,d\mathcal{H}^{n-1}\,.\tag{3.29}$$

By a suitable variant of Lemma 2.2 and by (3.27) we readily see that

$$\nu_{E\setminus K}\mathcal{H}^{n-1}\llcorner\mathcal{F}(E\setminus K)=\nu_E\mathcal{H}^{n-1}\llcorner[(\mathcal{F}E)\setminus K]-\nu_K\,\mathcal{H}^{n-1}\llcorner(E^1\cap\partial K)\,.$$

Since $\partial K=\{u=1\}$, we conclude from (3.29) that

$$\begin{aligned}|E\setminus K| &\le M_K\int_{(\mathcal{F}E)\setminus K}(u-1)\,\frac{\nabla u}{|\nabla u|}\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &\le M_K\int_{(\mathcal{F}E)\setminus K}(u-1)\,d\mathcal{H}^{n-1}\le\frac{M_K}{m_K}\int_{(\mathcal{F}E)\setminus K}|u-1|\,\|\nu_E\|_*\,d\mathcal{H}^{n-1}\,,\end{aligned}$$

that is (3.24). □

We are finally ready for the proof of the main theorem.

Proof of Theorem 1.1. **Step one:** We prove that

$$A(E)\le C_0(n,K)\sqrt{\delta(E)}\,,$$

where

$$C_0(n,K)=\frac{181\,n^3}{(2-2^{1/n'})^{3/2}}\left(\frac{M_K}{m_K}\right)^4\,.$$

Without loss of generality, due to (2.23), we can assume that E is bounded. Since $A(E)\le 2$, if $\delta(E)\ge k(n)^2/8$, then

$$A(E)\le 2\le\frac{4\sqrt{2}}{k(n)}\sqrt{\delta(E)}\le C_0(n,K)\sqrt{\delta(E)}\,.$$

Therefore we assume that $\delta(E)\le k(n)^2/8$ and apply Theorem 3.4 to find $G\subseteq E$ such that (3.17) and (3.18) hold true. We dilate E and G by the same factor, so that $|G|=|K|$. Of course, this operation leaves unchanged the validity of (3.17) and (3.18). We let T be the Brenier map between G and K, let $S(x):=T(x)-x$, and denote by $S^{(i)}$ its i-th component of S, for $1\le i\le n$. By Corollary 2.4 and by Lemma 3.1, up to a translation, we have

$$9n^2|K|\sqrt{\delta(G)}\ge\frac{1}{M_K}\|-DS^{(i)}\|_*(G^{(1)})\ge\frac{m_K^2}{M_K^3}k(n)\int_{\mathcal{F}G}\left|\mathrm{tr}\,_G(S^{(i)})\right|\,\|\nu_G\|_*d\mathcal{H}^{n-1}\,.$$

Adding up over $i = 1, ..., n$, as $\sum_{i=1}^{n} |y_i| \geq |y| \geq m_K \|y\|$, we find that

$$\frac{9\,n^3}{k(n)} \left(\frac{M_K}{m_K}\right)^3 |K| \sqrt{\delta(G)} \geq \int_{\mathcal{F}G} \|\mathrm{tr}\,_G(S)\| \|\nu_G\|_* d\mathcal{H}^{n-1} \,. \tag{3.30}$$

Once again by Corollary 2.4 we have that

$$n|K|\delta(G) \geq \int_{\mathcal{F}G} (1 - \|\mathrm{tr}\,_G(T)\|) \|\nu_G\|_* \, d\mathcal{H}^{n-1} \,,$$

so that (3.30) and Lemma 3.5 give

$$\frac{10\,n^3}{k(n)} \left(\frac{M_K}{m_K}\right)^3 |K| \sqrt{\delta(G)} \geq \int_{\mathcal{F}G} |1 - \|x\|| \; \|\nu_G\|_* d\mathcal{H}^{n-1} \geq \frac{m_K}{M_K} |G \setminus K| \,.$$

As $|K| A(G) \leq |G \Delta K| = 2|G \setminus K|$, we come to

$$A(G) \leq \frac{20\,n^3}{k(n)} \left(\frac{M_K}{m_K}\right)^4 \sqrt{\delta(G)} \,.$$

Let us now set $r_E = (|E|/|K|)^{1/n}$ and let $x_G \in \mathbb{R}^n$ be such that

$$|K| A(G) = |G \Delta (x_G + K)| \,.$$

Since $|K \Delta (r_E K)| = |E| - |K| = |E \setminus G| = |E \Delta G|$, we obtain

$$\begin{aligned} |E| A(E) &\leq |E \Delta (x_G + r_E K)| \leq |E \Delta G| + |G \Delta (x_G + K)| + |K \Delta (r_E K)| \\ &= 2|E \setminus G| + |G| A(G) \,. \end{aligned}$$

We divide by $|E|$ and take into account (3.17), (3.14) and the fact that $|G| \leq |E|$, to find that

$$\begin{aligned} A(E) &\leq \frac{6\,\delta(E)}{2 - 2^{1/n'}} + \frac{60\,n^3}{2 - 2^{1/n'}} \left(\frac{M_K}{m_K}\right)^4 \sqrt{\frac{9\,\delta(E)}{2 - 2^{1/n'}}} \\ &\leq \frac{181\,n^3}{(2 - 2^{1/n'})^{3/2}} \left(\frac{M_K}{m_K}\right)^4 \sqrt{\delta(E)} \,. \end{aligned}$$

This concludes the proof of this step.

Step two: We complete the proof of the theorem by showing that

$$A(E) \leq C_0(n) \sqrt{\delta(E)} \,. \tag{3.31}$$

Here $C_0(n)$ is the constant defined in (1.12), *i.e.*,

$$C_0(n) = \frac{181\,n^7}{(2 - 2^{1/n'})^{3/2}} \,,$$

and we can assume once again by (2.23) that E has smooth boundary. By John's Lemma [J, Theorem III] there exists an affine map L_0 on $\mathbb{R}^n$ such that

$$B_1 \subset L_0(K) \subset B_n \,, \quad \det L_0 > 0 \,.$$

Thus we can find $r > 0$ and an affine map L on $\mathbb{R}^n$ such that

$$B_r \subset L(K) \subset B_{rn} \,, \quad \det L = 1 \,.$$

By step one we have that

$$A(L(E), L(K)) \leq C_0(n, L(K)) \sqrt{\frac{P_{L(K)}(L(E))}{n |L(K)|^{1/n} |L(E)|^{1/n'}} - 1} \,,$$

where $A(L(E),L(K))$ is the relative asymmetry between $L(E)$ and $L(K)$ as introduced in (1.29). By construction

$$\frac{M_{L(K)}}{m_{L(K)}} \leq n\,,$$

and moreover, as $\det L = 1$, we have $A(L(E),L(K)) = A(E,K)$, therefore

$$A(E,K) \leq C_0(n)\sqrt{\frac{P_{L(K)}(L(E))}{n|K|^{1/n}|E|^{1/n'}} - 1}\,.$$

As E has smooth boundary

$$\begin{aligned} P_K(E) &= \lim_{\varepsilon\to 0^+}\frac{|E+\varepsilon\, K|-|E|}{\varepsilon} = \lim_{\varepsilon\to 0^+}\frac{|L(E+\varepsilon\, K)|-|L(E)|}{\varepsilon} \\ &= \lim_{\varepsilon\to 0^+}\frac{|L(E)+\varepsilon\, L(K)|-|L(E)|}{\varepsilon} = P_{L(K)}(L(E))\,, \end{aligned}$$

and we have therefore achieved the proof of the theorem. □

4. Stability for the Brunn-Minkowski inequality on convex sets

In this section we prove Theorem 1.2 and discuss its sharpness. To this end we follow the standard derivation of the Brunn-Minkowski inequality for convex sets from the anisotropic isoperimetric inequality [HO]. Note first that whenever E and F are open bounded convex sets containing the origin and G is a set of finite perimeter, then

$$P_E(G) + P_F(G) = P_{E+F}(G)\,. \tag{4.1}$$

This is easily verified by starting from the definition of $E+F$ and of $\|\cdot\|_*$, see (1.1). As $E+F$ is an open bounded convex set containing the origin, by (4.1) and the anisotropic isoperimetric inequality we infer

$$\begin{aligned} n|E+F| = P_{E+F}(E+F) &= P_E(E+F) + P_F(E+F) \\ &\geq n|E|^{1/n}|E+F|^{1/n'} + n|F|^{1/n}|E+F|^{1/n'}\,, \end{aligned}$$

that is the Brunn-Minkowski inequality $|E+F|^{1/n} \geq |E|^{1/n} + |F|^{1/n}$ for E and F.

Before coming to the details of the proof, let us recall that whenever E, F and G are sets of finite measure, then

$$A(E,F) \leq A(E,G) + A(G,F)\,. \tag{4.2}$$

Indeed, by scaling and translation invariance of the relative asymmetry, it may be assumed that $|E| = |F| = |G| = 1$ and $A(E,G) = |E\Delta G|$, $A(G,F) = |G\Delta F|$. Therefore,

$$A(E,F) \leq |E\Delta F| \leq |E\Delta G| + |G\Delta F| = A(E,G) + A(G,F)\,,$$

by the triangle inequality.

Proof of Theorem 1.2. Let E and F be open bounded convex sets. By translation invariance of β, σ and A, we may assume that both E and F contain the origin. By

Theorem 1.1 we have that

$$P_E(E+F) \geq n|E|^{1/n}|E+F|^{1/n'}\left\{1+\left(\frac{A(E+F,E)}{C_0(n)}\right)^2\right\},$$

$$P_F(E+F) \geq n|F|^{1/n}|E+F|^{1/n'}\left\{1+\left(\frac{A(E+F,F)}{C_0(n)}\right)^2\right\}.$$

Adding up the two inequalities, thanks to (4.1) and the fact that $P_{E+F}(E+F) = n|E+F|$, we find that

$$\begin{aligned}\beta(E,F) &\geq \frac{|E|^{1/n}}{|E|^{1/n}+|F|^{1/n}}\left(\frac{A(E+F,E)}{C_0(n)}\right)^2 + \frac{|F|^{1/n}}{|E|^{1/n}+|F|^{1/n}}\left(\frac{A(E+F,F)}{C_0(n)}\right)^2 \\ &\geq \frac{A(E+F,E)^2+A(E+F,F)^2}{2\,C_0(n)^2\sigma(E,F)^{1/n}} \geq \frac{[A(E+F,E)+A(E+F,F)]^2}{4\,C_0(n)^2\sigma(E,F)^{1/n}}.\end{aligned}$$

Thus we have $2\,C_0(n)\sqrt{\beta(E,F)\sigma(E,F)^{1/n}} \geq A(E,F)$, due to (4.2). □

We now discuss the sharpness of

$$C(n)\sqrt{\beta(E,F)\sigma(E,F)^{1/n}} \geq A(E,F) \tag{4.3}$$

in the regimes $\beta(E,F) \to 0^+$ or $\sigma(E,F) \to +\infty$. More precisely, we exhibit two sequences of open, bounded, convex sets $\{E_h^{(1)}\}_{h\in\mathbb{N}}$ and $\{E_h^{(2)}\}_{h\in\mathbb{N}}$ such that

$$\begin{cases}\lim_{h\to\infty}\beta(E_h^{(1)},B)=0\,, \\ \lim_{h\to\infty}\sigma(E_h^{(1)},B)=1\,,\end{cases} \qquad \limsup_{h\to\infty}\frac{\sqrt{\beta(E_h^{(1)},B)\sigma(E_h^{(1)},B)^{1/n}}}{A(E_h^{(1)},B)} < \infty\,, \tag{4.4}$$

and

$$\begin{cases}\lim_{h\to\infty}\beta(E_h^{(2)},Q)=0\,, \\ \lim_{h\to\infty}\sigma(E_h^{(2)},Q)=+\infty\,,\end{cases} \qquad \limsup_{h\to\infty}\frac{\sqrt{\beta(E_h^{(2)},Q)\sigma(E_h^{(2)},Q)^{1/n}}}{A(E_h^{(2)},Q)} < \infty\,, \tag{4.5}$$

where $Q=\{x\in\mathbb{R}^n : 0<x_k<1\,,\text{for } 1\leq k\leq n\}$. We note that (4.4) implies that the factor $\beta^{1/2}$ in (4.3) can not be replaced by β^α for any exponent $\alpha>1/2$. In the same way, (4.5) implies that in (4.3) the factor $\sigma^{1/2n}$ can not be replaced by σ^α for any $\alpha<1/2n$.

Proof of (4.4)**.** Let us consider, for $\lambda\in(1,2)$, the deformations $T_\lambda:\mathbb{R}^n\to\mathbb{R}^n$ defined by $T_\lambda(x)=(\lambda x_1,\hat{x})$, where we have decomposed $x\in\mathbb{R}^n$ as $x=(x_1,\hat{x})\in\mathbb{R}\times\mathbb{R}^{n-1}$. Let $E_\lambda=T_\lambda(B)$, *i.e.*, E_λ is the ellipsoid characterized as

$$E_\lambda=\{x\in\mathbb{R}^n : f_\lambda(x)<1\}\,,\quad f_\lambda(x)=\frac{x_1^2}{\lambda^2}+\hat{x}^2\,.$$

We will show that

$$A(E_\lambda,B)\geq c(n)(\lambda-1)\,, \tag{4.6}$$

$$\beta(E_\lambda,B)\sigma(E_\lambda,B)^{1/n}\leq C(n)(\lambda-1)^2\,, \tag{4.7}$$

for $c(n),C(n)\in(0,\infty)$. Then (4.4) will follow from (4.6) and (4.7) on taking $E_h^{(1)}=E_{\lambda_h}$ for any sequence $\{\lambda_h\}_{h\in\mathbb{N}}$ such that $\lambda_h\to 1^+$.

The set E_λ is symmetric with respect to the coordinate hyperplanes, with $|E_\lambda| = \lambda|B| = |B_{\lambda^{1/n}}|$. By [Ma, Lemma 5.2],

$$A(E, B) \geq \frac{|E_\lambda \Delta B_{\lambda^{1/n}}|}{3} \geq c(n)(\lambda - 1)\,,$$

and (4.6) is proved. We now prove (4.7). As $\sigma(E_\lambda, B) = \lambda \leq 2$, we only need to show that

$$|E_\lambda + B|^{1/n} - \big(|E_\lambda|^{1/n} + |B|^{1/n}\big) \leq C(n)(\lambda - 1)^2\,. \tag{4.8}$$

Let us consider the set $F_\lambda = (\mathrm{Id} + T_\lambda)(B)$. By construction $F_\lambda \subset (E_\lambda + B)$, and

$$|F_\lambda|^{1/n} - \big(|E_\lambda|^{1/n} + |B|^{1/n}\big) = |B|^{1/n}\left((1+\lambda)^{1/n}2^{(n-1)/n} - \lambda^{1/n} - 1\right) = |B|^{1/n}\varphi(\lambda)\,,$$

where $\varphi(\lambda) = (1+\lambda)^{1/n}2^{(n-1)/n} - \lambda^{1/n} - 1$. Since $\varphi(1) = \varphi'(1) = 0$ we have $\varphi(\lambda) \leq C(n)(\lambda - 1)^2$ for every $\lambda \in (1, 2)$. Therefore, in order to prove (4.8) we are left to show that

$$|E_\lambda + B|^{1/n} - |F_\lambda|^{1/n} \leq C(n)(\lambda - 1)^2\,. \tag{4.9}$$

Now, for every $y \in \partial(E_\lambda + B)$ there exists a unique $s(y) > 1$ such that $s(y)^{-1}y \in \partial F_\lambda$. Let us set $\sigma = \lambda - 1$ for the sake of brevity. By showing that

$$s(y) = 1 + O(\sigma^2)\,, \quad \forall y \in \partial(E_\lambda + B)\,, \tag{4.10}$$

we will infer the validity of (4.9). In order to prove (4.10), we note that F_λ can be characterized as

$$F_\lambda = \{x \in \mathbb{R}^n : g_\lambda(x) < 1\}\,, \quad g_\lambda(x) = \frac{x_1^2}{(1+\lambda)^2} + \frac{\hat{x}^2}{4}\,.$$

Thus, for every $y \in \partial(E_\lambda + B)$ we have

$$s(y)^2 = \frac{y_1^2}{(2+\sigma)^2} + \frac{\hat{y}^2}{4} = \frac{y^2}{4} - \frac{y_1^2}{4}\sigma + O(\sigma^2)\,. \tag{4.11}$$

On the other hand, for every $y \in \partial(E_\lambda + B)$ there exists a unique $x \in \partial E_\lambda$ such that

$$y = x + \nu_{E_\lambda}(x)\,, \tag{4.12}$$

see Figure 4.1. Let us note that if $x \in \partial E_\lambda$ then $1 = f_\lambda(x) = (1-2\sigma)x_1^2 + \hat{x}^2 + O(\sigma^2)$.

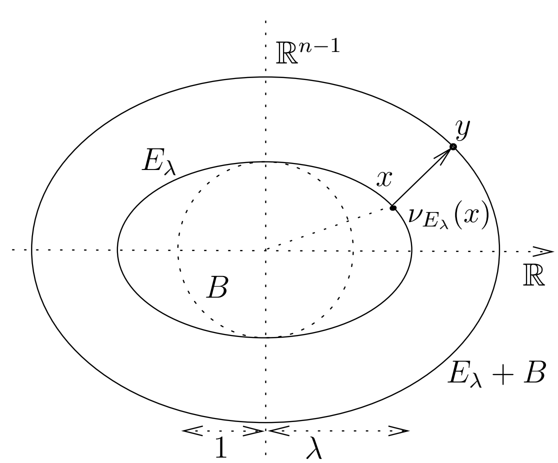

FIGURE 4.1. For every $y \in \partial(E_\lambda + B)$ there exists $x \in \partial E_\lambda$ such that $y = x + \nu_{E_\lambda}(x)$.

Therefore

$$x^2 = 1 + 2x_1^2\sigma + O(\sigma^2)\,. \tag{4.13}$$

Moreover, the outer unit normal vector $\nu_{E_\lambda}(x)$ to E_λ at x is parallel to $\nabla f_\lambda(x)$, and thus to

$$\Big(\frac{x_1}{\lambda^2},\hat{x}\Big) = \Big(x_1(1-2\sigma+O(\sigma^2)),\hat{x}\Big) = x - 2(x_1,0)\sigma + O(\sigma^2)\,. \tag{4.14}$$

By (4.14) and (4.13) we find

$$\Big|\Big(\frac{x_1}{\lambda^2},\hat{x}\Big)\Big|^2 = x^2 - 4x_1^2\sigma + O(\sigma^2) = 1 - 2x_1^2\sigma + O(\sigma^2)\,, \tag{4.15}$$

and so by (4.14) and (4.15) we get

$$\begin{aligned}\nu_{E_\lambda}(x) &= \frac{(\lambda^{-2}x_1,\hat{x})}{|(\lambda^{-2}x_1,\hat{x})|} = (1+x_1^2\sigma)\Big(x - 2(x_1,0)\sigma\Big) + O(\sigma^2)\\ &= x + m(x)\sigma + O(\sigma^2)\,,\end{aligned} \tag{4.16}$$

where $m(x) = x_1^2\,x - 2(x_1,0)$. Hence, combining (4.11) with (4.12) and (4.16) we find

$$\begin{aligned}s(y)^2 &= \frac{|2x+m(x)\sigma|^2}{4} - \frac{(2x_1+m_1(x)\sigma)^2}{4}\sigma + O(\sigma^2)\\ &= x^2 + (x\cdot m(x) - x_1^2)\sigma + O(\sigma^2) = 1 + (x\cdot m(x) + x_1^2)\sigma + O(\sigma^2)\,,\end{aligned}$$

where in the last equality we have used (4.13). By using the explicit formula for $m(x)$ in (4.16), and again due to (4.13), we conclude that

$$x\cdot m(x) + x_1^2 = x_1^2(x^2-1) = 2x_1^4\sigma + O(\sigma^2)\,.$$

This gives $s(y) = 1 + O(\sigma^2)$ and concludes the proof.

Proof of (4.5). It suffices to define $E_h^{(2)} = B_{\varepsilon_h}$ for any sequence $\{\varepsilon_h\}_{h\in\mathbb{N}}$ such that $\varepsilon_h \to 0^+$. Indeed, for every $\varepsilon > 0$ we have that $|B_\varepsilon + Q| = |Q| + \mathcal{H}^{n-1}(\partial Q)\varepsilon + o(\varepsilon) = 1 + 2n\varepsilon + o(\varepsilon)$, therefore

$$\beta(B_\varepsilon,Q)\sigma(B_\varepsilon,Q)^{1/n} = \left\{\frac{(1+2n\varepsilon+o(\varepsilon))^{1/n}}{1+\varepsilon|B|^{1/n}} - 1\right\}\frac{1}{\varepsilon|B|^{1/n}} = \left(\frac{2}{|B|^{1/n}} - 1\right) + \frac{o(\varepsilon)}{\varepsilon}\,.$$

As we have $A(B_\varepsilon,Q) = A(B,Q) = c(n)$ for some positive constant $c(n)$ depending on the dimension n only, we immediately deduce (4.5).

Appendix A. Characterization of isoperimetric sets

In this appendix we wish to discuss the following theorem, originally proved in [Ty, FM, BM]. As in the work of Brothers and Morgan, Gromov's original argument is developed in the framework of Geometric Measure Theory. In the present case we take further advantage from the use of the Brenier map. In this way we avoid the use of infinitely many Knothe maps, and present a more direct proof.

Theorem A.1. *Let E be a set of finite perimeter with $0 < |E| < \infty$. Then $P_K(E) = n|K|^{1/n}|E|^{1/n'}$ if and only if $|E\Delta(x_0 + rK)| = 0$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$.*

The stability result proved in Theorem 1.1 implies of course Theorem A.1 (note that, conversely, we have not used Theorem A.1 in proving Theorem 1.1!). Here we show how to directly derive Theorem A.1 from Gromov's argument.

A set of finite perimeter E is said *indecomposable* if for every $F \subseteq E$ having finite perimeter and such that

$$\mathcal{H}^{n-1}(\mathcal{F}E) = \mathcal{H}^{n-1}(\mathcal{F}F) + \mathcal{H}^{n-1}(\mathcal{F}(E \setminus F)), \tag{A.1}$$

we have that $\min\{|F|, |E \setminus F|\} = 0$. Indecomposability plays the role of connectedness in the theory of sets of finite perimeter, see [ACMM]. We shall need the following lemma, that is stated without proof in [DM, Proposition 2.12].

Lemma A.2. *Let E be an indecomposable set and let $f \in BV(\mathbb{R}^n)$. If $|Df|(E^{(1)}) = 0$, then there exists $c \in \mathbb{R}$ such that $f(x) = c$ for a.e. $x \in E$.*

Proof. Let $F_t = E \cap \{f > t\}$. As E is indecomposable, it suffices to show that (A.1) holds with $F = F_t$ for a.e. $t \in \mathbb{R}$. In fact, it is enough to prove that

$$\mathcal{H}^{n-1}(\mathcal{F}E) \geq \mathcal{H}^{n-1}(\mathcal{F}F_t) + \mathcal{H}^{n-1}(\mathcal{F}(E \setminus F_t)), \tag{A.2}$$

for a.e. $t \in \mathbb{R}$, as the converse inequality follows from the subadditivity of the distributional perimeter [AFP, Proposition 3.38 (d)]. To this end we start by noticing that

$$\{f \leq t\}^{(1)} = \{f > t\}^{(0)}, \quad \partial_{1/2}\{f > t\} = \partial_{1/2}\{f \leq t\}, \tag{A.3}$$

$$\mathcal{H}^{n-1}(\mathcal{F}\{f > t\}\Delta\mathcal{F}\{f \leq t\}) = 0, \quad \mathcal{H}^{n-1}(J_{E,\{f>t\}} \cap J_{E,\{f\leq t\}}) = 0, \tag{A.4}$$

where (A.3) is trivially checked, and where (A.4) follows from (A.3), Lemma 2.2 and (2.6). We now come to the proof of (A.2). By (2.8), as $E \setminus F_t = E \cap \{f \leq t\}$, we have that, up to $\mathcal{H}^{n-1}$-null sets,

$$\mathcal{F}F_t = J_{E,\{f>t\}} \cup [E^{(1)} \cap \mathcal{F}\{f > t\}] \cup [\mathcal{F}E \cap \{f > t\}^{(1)}], \tag{A.5}$$

$$\mathcal{F}(E \setminus F_t) = J_{E,\{f\leq t\}} \cup [E^{(1)} \cap \mathcal{F}\{f \leq t\}] \cup [\mathcal{F}E \cap \{f \leq t\}^{(1)}], \tag{A.6}$$

being $J_{E,F}$ as in (2.7). Hence, by the Coarea Formula (2.22) (applied to the Euclidean total variation) we get

$$0 = |Df|(E^{(1)}) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(E^{(1)} \cap \mathcal{F}\{f > t\})dt,$$

i.e., $\mathcal{H}^{n-1}(E^{(1)} \cap \mathcal{F}\{f > t\}) = 0$ for a.e. $t \in \mathbb{R}$. By (A.4), we also have

$$\mathcal{H}^{n-1}(E^{(1)} \cap \mathcal{F}\{f \leq t\}) = 0$$

for a.e. $t \in \mathbb{R}$. Therefore, thanks to (A.3), from (A.5) and (A.6) we deduce

$$\begin{aligned}\mathcal{H}^{n-1}(\mathcal{F}F_t) + \mathcal{H}^{n-1}(\mathcal{F}(E \setminus F_t)) =& \mathcal{H}^{n-1}(J_{E,\{f>t\}}) + \mathcal{H}^{n-1}(J_{E,\{f\leq t\}}) \\ &+ \mathcal{H}^{n-1}(\mathcal{F}E \cap [\{f > t\}^{(1)} \cup \{f > t\}^{(0)}]).\end{aligned} \tag{A.7}$$

Since (A.4) implies that

$$\mathcal{H}^{n-1}(J_{E,\{f>t\}}) + \mathcal{H}^{n-1}(J_{E,\{f\leq t\}}) \leq \mathcal{H}^{n-1}(\mathcal{F}E \cap \mathcal{F}\{f > t\}),$$

(A.2) follows from (2.5) and (A.7). □

We now derive Theorem A.1 from the proof of Theorem 2.3.

Proof of Theorem A.1. We only need to show that, if $P_K(E) = n|K|^{1/n}|E|^{1/n'}$, then $E = x_0 + rK$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$.

To this end, we start by proving that E is indecomposable. Indeed, let F be a set of finite perimeter contained in E and such that (A.1) holds true. The usual considerations based on (2.5), (2.6) and (2.8) serves to show that

$$\mathcal{H}^{n-1}(\mathcal{F}F) = \mathcal{H}^{n-1}(\mathcal{F}F \cap E^{(1)}) + \mathcal{H}^{n-1}(\mathcal{F}E \cap \mathcal{F}F)\,,$$
$$\mathcal{H}^{n-1}(\mathcal{F}(E \setminus F)) = \mathcal{H}^{n-1}(\mathcal{F}F \cap E^{(1)}) + \mathcal{H}^{n-1}(\mathcal{F}E \setminus \mathcal{F}F)\,.$$

Thus, by (A.1), we deduce $\mathcal{H}^{n-1}(\mathcal{F}F \cap E^{(1)}) = 0$. In particular

$$\begin{aligned} P_K(F) + P_K(E \setminus F) &= \int_{\mathcal{F}F\cap\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{\mathcal{F}F\cap E^{(1)}} \|\nu_F\|_* d\mathcal{H}^{n-1} \\ &\quad + \int_{\mathcal{F}E\setminus\mathcal{F}F} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{\mathcal{F}F\cap E^{(1)}} \|-\nu_F\|_* d\mathcal{H}^{n-1} \\ &= \int_{\mathcal{F}F\cap\mathcal{F}E} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{\mathcal{F}E\setminus\mathcal{F}F} \|\nu_E\|_* d\mathcal{H}^{n-1} = P_K(E)\,. \end{aligned}$$

By the isoperimetric inequality (1.4)

$$\begin{aligned} P_K(E) = P_K(F) + P_K(E \setminus F) &\geq n|K|^{1/n}(|F|^{1/n'} + |E \setminus F|^{1/n'}) \\ &\geq n|K|^{1/n}(|F| + |E \setminus F|)^{1/n'} = P_K(E)\,, \end{aligned}$$

and so by strict concavity $\min\{|F|, |E \setminus F|\} = 0$, that is E is indecomposable.

We now assume without loss of generality that $|E| = |K|$, and repeat the proof of Theorem 2.3. As $P_K(E) = n|K|^{1/n}|E|^{1/n'}$, we deduce in particular that the Brenier map T between E and K satisfies

$$0 = \int_E \left\{ \frac{\operatorname{div} T(x)}{n} - (\det \nabla T(x))^{1/n} \right\} dx + \frac{(\operatorname{Div} T)_s(E^{(1)})}{n}\,.$$

In particular $T \in W^{1,1}(\mathbb{R}^n; K)$. As $\det \nabla T(x) = 1$ a.e. on E, we find that $\nabla T(x) = \mathrm{Id}$ at a.e. $x \in E$, therefore $DT(C) = \int_C \mathrm{Id}\, dx$ for every Borel set $C \subseteq E^{(1)}$. If we let $S(x) = T(x) - x$, then $S \in W^{1,1}(\mathbb{R}^n; \mathbb{R}^n)$, with $|DS|(E^{(1)}) = 0$. Thus, applying Lemma A.2 to each component of the vector field S we deduce the existence of $x_0 \in \mathbb{R}^n$ such that $T(x) = x - x_0$ for a.e. $x \in E^{(1)}$. As $T(x) \in K$ for a.e. $x \in E$, we deduce that $E^{(1)}$ is a subset of $x_0 + K$, and since $|K| = |E| = |E^{(1)}|$ we conclude the proof. □

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