A REFINED BRUNN-MINKOWSKI INEQUALITY FOR CONVEX SETS

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Abstract. Starting from a mass transportation proof of the Brunn-Minkowski inequality on convex sets, we improve the inequality showing a sharp estimate about the stability property of optimal sets. This is based on a Poincaré-type trace inequality on convex sets that is also proved in sharp form.

1. Introduction

We deal with the *Brunn-Minkowski inequality*: given E and F non-empty subsets of $\mathbb{R}^n$, we have

$$|E+F|^{1/n} \geq |E|^{1/n} + |F|^{1/n}\,, \tag{1}$$

where $E+F = \{x+y : x \in E\,, y \in F\}$ is the *Minkowski sum of E and F*, and where $|\cdot|$ stands for the (outer) Lebesgue measure on $\mathbb{R}^n$. The central role of this inequality in many branches of Analysis and Geometry, and especially in the theory of convex bodies, is well explained in the excellent survey [Ga] by R. Gardner. Concerning the case E and F are *open bounded convex sets* (shortly: *convex bodies*), it may be proved (see [BZ, HM]) that equality holds in (1) if and only if E and F are homothetic, i.e.

$$\exists\, \lambda > 0\,, x_0 \in \mathbb{R}^n : \quad E = x_0 + \lambda F\,. \tag{2}$$

Theorem 1 provides a refined Brunn-Minkowski inequality on convex bodies, in the spirit of [Dk, Gr, Sc, Ru]. We define the *relative asymmetry of E and F* as

$$A(E,F) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E\Delta(x_0+\lambda F)|}{|E|} : \lambda = \left(\frac{|E|}{|F|}\right)^{1/n} \right\}\,, \tag{3}$$

and the *relative size of E and F* as

$$\sigma(E,F) := \max\left\{\frac{|F|}{|E|}, \frac{|E|}{|F|}\right\}\,. \tag{4}$$

We note that $A(E,F) = A(F,E)$ and $\sigma(E,F) = \sigma(F,E)$.

Theorem 1. *If E and F are convex bodies, then*

$$|E+F|^{1/n} \geq \left(|E|^{1/n} + |F|^{1/n}\right)\left\{1 + \frac{A(E,F)^2}{C_0(n)\sigma(E,F)^{1/n}}\right\}\,. \tag{5}$$

In [FMP], inequality (5) was derived as a corollary of the sharp quantitative Wulff inequality, with a constant $C_0(n) \approx n^7$ and with explicit examples proving the sharpness of decay rate of $A(E,F)$ and $\sigma(E,F)$ in the regime $\beta(E,F) \to 0$. Here, we introduce the *Brunn-Minkowski deficit of the pair (E,F)* by setting

$$\beta(E,F) := \frac{|E+F|^{1/n}}{|E|^{1/n}+|F|^{1/n}} - 1\,,$$

so that (5) becomes equivalent to

$$C_0(n)\sqrt{\beta(E,F)\sigma(E,F)^{1/n}} \geq A(E,F)\,. \tag{6}$$

As in [FMP], our approach to (5) is based on the theory of mass transportation. A one dimensional mass transportation argument is at the basis of the beautiful proof of (1) by Hadwiger and Ohmann [HO], see [Fe, 3.2.41] and [Ga, Proof of Theorem 4.1]. The impact of mass transportation theory in the field of sharp functional-geometric inequalities is now widely recognized, with many old and new inequalities treated from a unified and elegant viewpoint (see [Vi, Chapter 6] for an introduction). A proof of the Brunn-Minkowski inequality in this framework is already contained in the seminal paper by McCann [McC], see also Step two in the proof of Theorem 1.

In Section 3 of this note we present a direct proof of (5), independent from the structure theory for sets of finite perimeter that was heavily used in [FMP]. As a technical drawback, this approach does not provide a polynomial bound on $C_0(n)$, but only an exponential behavior in n. However, we believe this proof is more broadly accessible and substantially simpler. A technical element of this proof that we believe of independent interest is the Poincaré-type trace inequality on convex sets proved in Section 2, with a constant having sharp dependence on the dimension n and on the ratio between the in-radius and the out-radius of the set (see Remark 3).

2. A Poincaré-type trace inequality on convex sets

In this section we aim to prove the following Poincaré-type trace inequality for a convex body:

Lemma 2. *Let E be a convex body such that $B_r \subset E \subset B_R$, for $0 < r < R$. Then*

$$\frac{n\sqrt{2}}{\log(2)}\,\frac{R}{r}\int_E |\nabla f| \geq \inf_{c\in\mathbb{R}}\int_{\partial E}|f-c|\,d\mathcal{H}^{n-1}\,, \tag{7}$$

for every $f \in C^\infty(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$.

It is quite easy to prove (7) by a contradiction argument, if we allow to replace $n(R/r)$ by a constant generically depending on E. However, in order to prove Theorem 1, we need to express this dependence just in terms of n and R/r, and thus require a more careful approach. Let us also note that, by a standard density argument, (7) holds true for every $f \in BV(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ (see [AFP, EG]), in the form

$$\frac{n\sqrt{2}}{\log(2)}\,\frac{R}{r}\,|Df|(E) \geq \inf_{c\in\mathbb{R}}\int_{\partial E}|\mathrm{tr}\,_E(f)-c|\,d\mathcal{H}^{n-1}\,,$$

where $|Df|$ denotes the total variation measure of Df and where $\mathrm{tr}\,_E(f)$ is the trace of f on ∂E, defined as an element of $L^1(\mathcal{H}^{n-1}\lfloor\partial E)$ (see [AFP, Theorem 3.87]). However, we shall not need this stronger form of the inequality.

Given a convex body E containing the origin in its interior, we introduce a weight function on directions defined for $\nu \in S^{n-1}$ as

$$\|\nu\|_E := \sup\{x\cdot\nu : x \in E\}\,.$$

When F is a set with Lipschitz boundary and outer unit normal ν_F, we define the anisotropic perimeter of F with respect to E as

$$P_E(F) := \int_{\partial F}\|\nu_F(x)\|_E\,d\mathcal{H}^{n-1}(x)\,,$$

and recall that $P_E(E) = n|E|$. Then, the anisotropic isoperimetric inequality, or Wulff inequality,

$$P_E(F) \geq n|E|^{1/n}|F|^{(n-1)/n}\,, \tag{8}$$

holds true, as it can be shown starting from (1) (see [Ga, Section 3]).

Proof of Lemma 2. Let us set

$$\tau(E) := \inf_F \frac{\mathcal{H}^{n-1}(E \cap \partial F)}{\mathcal{H}^{n-1}(F \cap \partial E)}$$

where F ranges over the class of open sets of $\mathbb{R}^n$ with smooth boundary such that $|E \cap F| \leq |E|/2$. Then, fixed $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we set $F_t = \{x \in \mathbb{R}^n : f(x) > t\}$ for every $t \in \mathbb{R}$. The proof of the lemma is then achieved on combining the following two statements.

Step one: We have that

$$\int_E |\nabla f| \geq \tau(E) \int_{\partial E} |f - m|\, d\mathcal{H}^{n-1}\,,$$

where m is a median of f in E, i.e.

$$|F_t \cap E| \leq \frac{|E|}{2}\,, \quad \forall t \geq m\,,$$
$$|F_t \cap E| > \frac{|E|}{2}\,, \quad \forall t < m\,.$$

Indeed, let $g = \max\{f - m\,, 0\}$ and let $G_t = \{x \in \mathbb{R}^n : g(x) > t\}$. Then by the Coarea Formula, the choice of m and the definition of $\tau(E)$ (note that F_t is admissible in $\tau(E)$ for a.e. $t \geq m$ by Morse-Sard Lemma)

$$\begin{aligned}\int_{E\cap F_m} |\nabla f| &= \int_E |\nabla g| = \int_0^\infty \mathcal{H}^{n-1}(E \cap \partial G_t)\, dt \\ &\geq \tau(E) \int_0^\infty \mathcal{H}^{n-1}(G_t \cap \partial E)\, dt = \tau(E) \int_{\partial E} g\, d\mathcal{H}^{n-1} \\ &= \tau(E) \int_{\partial E} \max\{f - m\,, 0\}\, d\mathcal{H}^{n-1}\,.\end{aligned}$$

The choice of m allows to argue similarly with $\max\{m - f\,, 0\}$ in place of g and to eventually achieve the proof of step one.

Step two: We have that

$$\tau(E) \geq \frac{r}{R}\left(1 - \frac{1}{2^{1/n}}\right).$$

To prove this, let us consider an admissible set F for $\tau(E)$ and set for simplicity

$$\lambda := \frac{\mathcal{H}^{n-1}(E \cap \partial F)}{\mathcal{H}^{n-1}(F \cap \partial E)}\,. \tag{9}$$

On denoting $F_1 = F \cap E$ and $F_2 = E \setminus \overline{F}$, we have that

$$E \cap \partial F_1 = E \cap \partial F_2 = E \cap \partial F\,, \qquad \text{with } \nu_F = \nu_{F_1} = -\nu_{F_2} \text{ on } E \cap \partial F.$$

Therefore

$$
\begin{aligned}
P_E(E) &\geq P_E(F_1) + P_E(F_2) - \int_{E\cap\partial F_1} \|\nu_{F_1}\|_E \, d\mathcal{H}^{n-1} - \int_{E\cap\partial F_2} \|\nu_{F_2}\|_E \, d\mathcal{H}^{n-1} \\
&\geq P_E(F_1) + P_E(F_2) - 2\,R\,\mathcal{H}^{n-1}\big(E\cap\partial F\big) \\
&= P_E(F_1) + P_E(F_2) - 2\,R\lambda\,\mathcal{H}^{n-1}\big(F\cap\partial E\big) \\
&\geq P_E(F_1) + P_E(F_2) - 2\,R\lambda\,\mathcal{H}^{n-1}\big(\partial F_1\big) \\
&\geq \left(1 - 2\lambda\,\frac{R}{r}\right) P_E(F_1) + P_E(F_2)\,,
\end{aligned} \tag{10}
$$

where we have used (9) and the elementary inequality

$$
r \leq \|\nu\|_E \leq R\,,
$$

for every $\nu \in S^{n-1}$. On combining (10), the anisotropic isoperimetric inequality (8) and the fact that $P_E(E) = n|E|$, we come to

$$
n|E| \geq n|E|^{1/n}\left\{\left(1 - 2\lambda\,\frac{R}{r}\right)|F_1|^{1/n'} + |F_2|^{1/n'}\right\},
$$

i.e. we have proved that

$$
\lambda\, t^{1/n'} \geq \frac{r}{2R}\left(t^{1/n'} + (1-t)^{1/n'} - 1\right)\,,
$$

where $t = |F_1|/|E|$. As $t \in (0, 1/2]$ by construction and

$$
s^{1/n'} + (1-s)^{1/n'} - 1 \geq (2 - 2^{1/n'})s^{1/n'}\,, \quad \forall s \in (0, 1/2]\,,
$$

the proof of step two is easily concluded. □

Remark 3. Let us point out that the dependence on n and R/r given in the above result, that is $n(R/r)$, is sharp. In $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, it suffices to consider the box E defined as

$$
E = Q \times [-R_0, R_0]\,, \quad Q = \left[-\frac{r}{2}, \frac{r}{2}\right]^{n-1}.
$$

We clearly have that $B_r \subset E \subset B_R$, with $R = \sqrt{R_0^2 + (n-1)r^2}$. Now, let us consider as a test set for the trace constant the half-space $F = \mathbb{R}^{n-1} \times (0, \infty)$, so that

$$
\partial F \cap E = Q \times \{0\}\,, \quad \partial E \cap F = (\partial Q \times (0, R_0)) \cup (Q \times \{R_0\})\,.
$$

The boundary ∂Q is the union of $2(n-1)$ cubes of dimension $(n-2)$ and size r. Thus,

$$
\mathcal{H}^{n-1}(\partial F \cap E) = r^{n-1}, \qquad \mathcal{H}^{n-1}(\partial E \cap F) = 2(n-1)R_0 r^{n-2} + r^{n-1}.
$$

For $R_0 \gg \sqrt{n-1}\,r$ we have $R \approx R_0$, and therefore

$$
\frac{n\sqrt{2}}{\log(2)}\,\frac{R}{r} \leq \tau(E) \leq \frac{2(n-1)R_0 r^{n-2} + r^{n-1}}{r^{n-1}} \approx n\,\frac{R_0}{r} \approx n\,\frac{R}{r}\,.
$$

This shows the sharpness of our trace constant, up to a numeric factor.

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We consider two convex bodies E and F, and we aim to prove (6). Without loss of generality, we may assume that $|E| \geq |F|$. By approximation, we can also assume that E and F are smooth and uniformly convex. Eventually, we can directly consider the case

$$\beta(E,F)\sigma(E,F)^{1/n} \leq 1\,. \tag{11}$$

Indeed, as we always have $A(E,F) \leq 2$, if $\beta(E,F)\sigma(E,F)^{1/n} > 1$ then (6) holds trivially with $C_0(n) = 2$. Observe further that, since $\sigma(E,F) \geq 1$, (11) implies

$$\beta(E,F) \leq 1\,. \tag{12}$$

We divide the proof in several steps.

Step one: John's normalization. A classical result in the theory of convex bodies by F. John [J] ensures the existence of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$B_1 \subset L(E) \subset B_n\,.$$

We note that

$$\beta(E,F) = \beta\big(L(E), L(F)\big)\,, \quad A(E,F) = A\big(L(E), L(F)\big)\,, \quad |L(E)| \geq |L(F)|\,.$$

Therefore in the proof of Theorem 1 we may also assume that

$$B_1 \subset E \subset B_n\,. \tag{13}$$

In particular, under this assumption one has $1 \leq r \leq R \leq n$, so that by Lemma 2 we can write

$$\frac{n^2\sqrt{2}}{\log(2)} \int_E |\nabla f| \geq \inf_{c\in\mathbb{R}} \int_{\partial E} |f - c| d\mathcal{H}^{n-1} \tag{14}$$

for every $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Step two: Mass transportation proof of Brunn-Minkowski. We prove the Brunn-Minkowski inequality by mass transportation. By the Brenier Theorem [Br1, Br2], there exists a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that its gradient $T = \nabla\varphi$ defines a map $T \in BV(\mathbb{R}^n, \overline{F})$ pushing forward $|E|^{-1}1_E(x)dx$ to $|F|^{-1}1_F(x)dx$, i.e.

$$\frac{1}{|F|}\int_F h(y)\,dy = \frac{1}{|E|}\int_E h(T(x))\,dx\,, \tag{15}$$

for every Borel function $h : \mathbb{R}^n \to [0,\infty)$. As shown by Caffarelli [Ca1, Ca2], under our assumptions the Brenier map is smooth up to the boundary, i.e. $T \in C^\infty(\overline{E}, \overline{F})$. Moreover, the push-forward condition (15) takes the form

$$\det \nabla T(x) = \frac{|F|}{|E|}\,, \quad \forall x \in E\,. \tag{16}$$

We are going to consider the eigenvalues $\{\lambda_k(x)\}_{k=1,\ldots,n}$ of $\nabla T(x) = \nabla^2\varphi(x)$, ordered so that $\lambda_k \leq \lambda_{k+1}$ for $1 \leq k \leq n-1$. We also define, for every $x \in E$,

$$\lambda_A(x) = \frac{\sum_{k=1}^n \lambda_k(x)}{n}\,, \qquad \lambda_G(x) = \Big(\prod_{k=1}^n \lambda_k(x)\Big)^{1/n}\,.$$

Thanks to (16) we have

$$\lambda_G(x) = \left(\frac{|F|}{|E|}\right)^{1/n}$$

for every $x \in E$. We are in the position to prove the Brunn-Minkowski inequality. Let $S(x) := x + T(x)$, then $S(E) \subset E + F$. As $\det \nabla S = \prod_{k=1}^{n}(1+\lambda_k) > 1$, we have $|\det \nabla S| = \det \nabla S$. Thus

$$|E+F|^{1/n} \geq |S(E)|^{1/n} = \left(\int_E \det \nabla S\right)^{1/n} = \left(\int_E \prod_{k=1}^{n}(1+\lambda_k)\right)^{1/n}. \tag{17}$$

We observe that

$$\prod_{k=1}^{n}(1+\lambda_k) = 1 + \sum_{m=1}^{n} \sum_{\{1\leq i_1<\cdots<i_m\leq n\}} \prod_{j=1}^{m} \lambda_{i_j}. \tag{18}$$

Note that the set of indexes $(i_1, \ldots, i_m)$ with $1 \leq i_j < i_{j+1} \leq n$ counts $\binom{n}{m}$ elements. For each fixed $m \geq 1$, the arithmetic-geometric mean inequality implies that

$$\sum_{\{1\leq i_1<\cdots<i_m\leq n\}} \prod_{j=1}^{m} \lambda_{i_j} \geq \binom{n}{m} \prod_{\{1\leq i_1<\cdots<i_m\leq n\}} \left(\prod_{j=1}^{m} \lambda_{i_j}\right)^{1/\binom{n}{m}}. \tag{19}$$

This last term is equal to

$$\binom{n}{m} \prod_{k=1}^{n} \lambda_k^{\binom{n-1}{m-1}/\binom{n}{m}} = \binom{n}{m} \lambda_G^m. \tag{20}$$

On putting (18), (19) and (20) together, and applying the binomial formula to $(1+\lambda_G)^n$ we come to

$$\prod_{k=1}^{n}(1+\lambda_k) - (1+\lambda_G)^n = \sum_{m=1}^{n} \Gamma_m, \tag{21}$$

where Γ_m denotes the difference between the left and the right hand side of (19). We observe that $\Gamma_m \geq 0$ whenever $1 \leq m \leq n$, and in particular $\Gamma_1 = n(\lambda_A - \lambda_G)$. On combining this with (17), (16), and $\lambda_G = (\det \nabla T)^{1/n}$, we find that

$$|E+F|^{1/n} \geq \left(\int_E (1+\lambda_G)^n\right)^{1/n} = |E|^{1/n}\left(1 + \left(\frac{|F|}{|E|}\right)^{1/n}\right) = |E|^{1/n} + |F|^{1/n},$$

i.e. we prove the Brunn-Minkowski inequality for E and F.

Step three. Lower bounds on the deficit. In this step we aim to prove

$$\frac{1}{|E|}\int_E |\nabla T(x) - \lambda_G \mathrm{Id}\,|\,dx \leq C(n)\sqrt{\beta(E,F)}\sqrt{\beta(E,F) + \sigma(E,F)^{-1/n}}. \tag{22}$$

Let us set, for the sake of brevity,

$$s = \frac{1}{|E|}\int_E \det \nabla S, \qquad t = (1+\lambda_G)^n.$$

From Step two we deduce that

$$\frac{|E+F|^{1/n} - (|E|^{1/n} + |F|^{1/n})}{|E|^{1/n}} \geq s^{1/n} - t^{1/n} = \frac{s-t}{\sum_{h=1}^{n} s^{(n-h)/n} t^{(h-1)/n}}. \tag{23}$$

As $t \le s$ and $|E|s = |S(E)| \le |E+F|$,

$$\begin{aligned}\sum_{h=1}^{n} s^{(n-h)/n} t^{(h-1)/n} &\le n s^{(n-1)/n} \le n \left(\frac{|E+F|}{|E|}\right)^{(n-1)/n} \\ &= n\left(\Big(1+\beta(E,F)\Big)\frac{|E|^{1/n}+|F|^{1/n}}{|E|^{1/n}}\right)^{n-1} \le C(n)\,,\end{aligned} \tag{24}$$

where we have also made use of (12) and of the fact that $|F| \le |E|$. A similar argument shows that the left hand side of (23) is controlled by $2\beta(E,F)$, and therefore we conclude that

$$C(n)\beta(E,F) \ge s-t = \frac{1}{|E|}\int_E \left(\prod_{k=1}^{n}(1+\lambda_k) - (1+\lambda_G)^n\right) dx\,. \tag{25}$$

Then, by (25) and (21), as $\Gamma_m \ge 0$ whenever $1 \le m \le n$ and $\Gamma_1 = n(\lambda_A - \lambda_G)$, we get

$$C(n)\beta(E,F) \ge \frac{1}{|E|}\int_E \sum_{m=1}^{n}\Gamma_m(x)\,dx \ge \frac{1}{|E|}\int_E \Gamma_1(x)\,dx = \frac{n}{|E|}\int_E (\lambda_A - \lambda_G)\,. \tag{26}$$

An elementary quantitative version of the arithmetic-geometric mean inequality proved in [FMP, Lemma 2.5], ensures that

$$7n^2(\lambda_A - \lambda_G) \ge \frac{1}{\lambda_n}\sum_{k=1}^{n}(\lambda_k - \lambda_G)^2\,.$$

In particular, as $(\lambda_n - \lambda_1)^2 \le 2[(\lambda_n-\lambda_G)^2 + (\lambda_G - \lambda_1)^2]$ we obtain from (26)

$$C(n)\beta(E,F) \ge \frac{1}{|E|}\int_E \frac{(\lambda_n-\lambda_1)^2}{\lambda_n}\,dx\,. \tag{27}$$

By Hölder inequality

$$\frac{1}{|E|}\int_E (\lambda_n - \lambda_1)dx \le C(n)\sqrt{\beta(E,F)\frac{1}{|E|}\int_E \lambda_n}\,. \tag{28}$$

As $\lambda_1 \le (|F|/|E|)^{1/n} = \sigma(E,F)^{-1/n}$, from (28) we come to

$$\frac{1}{|E|}\int_E \lambda_n \le C(n)\sqrt{\beta(E,F)\frac{1}{|E|}\int_E \lambda_n} + \sigma(E,F)^{-1/n}\,,$$

which easily implies

$$\frac{1}{|E|}\int_E \lambda_n \le C(n)\left(\beta(E,F) + \sigma(E,F)^{-1/n}\right) \tag{29}$$

by Young's inequality. We eventually combine (29) with (28), and prove that

$$\frac{1}{|E|}\int_E (\lambda_n - \lambda_1)\,dx \le C(n)\sqrt{\beta(E,F)}\sqrt{\beta(E,F)+\sigma(E,F)^{-1/n}}\,. \tag{30}$$

Then (22) follows immediately.

Step four. Trace inequality. On combining (22) with (14), we conclude that, up to a translation of F,

$$C(n)\sqrt{\beta(E,F)}\sqrt{\beta(E,F)+\sigma(E,F)^{-1/n}}|E| \geq \int_{\partial E} |T(x)-\lambda_G x|\, d\mathcal{H}^{n-1}(x)\,.$$

If $F' = \lambda_G^{-1}F$ and $P : \mathbb{R}^n \setminus F' \to \partial F'$ denotes the projection of $\mathbb{R}^n \setminus F'$ over F', then, since by construction T takes value in $\overline{F}$, we get

$$C(n)\sqrt{\beta(E,F)}\sqrt{\beta(E,F)+\sigma(E,F)^{-1/n}} \geq \frac{\lambda_G}{|E|}\int_{\partial E\setminus F'} |P(x)-x|\, d\mathcal{H}^{n-1}(x)\,. \quad (31)$$

We now consider the map $\Phi : (\partial E \setminus F') \times (0,1) \to E \setminus F'$ defined by

$$\Phi(x,t) = tx + (1-t)P(x)\,.$$

Let $\{\varepsilon_k(x)\}_{k=1}^{n-1}$ be a basis of the tangent space to ∂E at x. Since Φ is a bijection, we find

$$\begin{aligned}|E\setminus F'| = \int_0^1 dt \int_{(\partial E\setminus F')}\Big|(x-P(x)) \wedge \Big(\bigwedge_{k=1}^{n-1}\Big(t\varepsilon_k(x) \\ + (1-t)dP_x(\varepsilon_k(x))\Big)\Big)\Big| d\mathcal{H}^{n-1}(x),\end{aligned} \quad (32)$$

where dP_x denotes the differential of the projection P at x. As P is the projection over a convex set, it decreases distances, i.e. $|dP_x(e)| \leq 1$ for every $e \in S^{n-1}$. Thus,

$$|t\varepsilon_k(x) + (1-t)dP_x(\varepsilon_k(x))| \leq 1\,, \quad \forall k \in \{1,\dots,n-1\}\,.$$

Recalling that $\lambda_G = \sigma(E,F)^{-1/n}$, we combine this last inequality with (31) and (32) to get

$$\begin{aligned}\frac{|E\setminus F'|}{|E|} &\leq \frac{1}{|E|}\int_{\partial E\setminus F'} |x-P(x)|\, d\mathcal{H}^{n-1}(x) \\ &\leq C(n)\sigma(E,F)^{1/n}\sqrt{\beta(E,F)}\sqrt{\beta(E,F)+\sigma(E,F)^{-1/n}} \\ &\leq C(n)\sigma(E,F)^{1/n}\sqrt{\beta(E,F)}\left(\sqrt{\beta(E,F)}+\sigma(E,F)^{-1/2n}\right) \\ &= C(n)\left(\sqrt{\beta(E,F)\sigma(E,F)^{1/n}}+\beta(E,F)\sigma(E,F)^{1/n}\right) \\ &\leq C(n)\sqrt{\beta(E,F)\sigma(E,F)^{1/n}}\,,\end{aligned}$$

where in the last inequality we have used (11). As

$$A(E,F) \leq \frac{|E\Delta F'|}{|E|} = 2\frac{|E\setminus F'|}{|E|}\,,$$

this proves (6) and we achieve the proof of the theorem.

We conclude noticing that the constant $C_0(n)$ in the above theorem can be taken to be

$$C_0(n) \approx p(n)\, c_0^n,$$

where $p(n)$ is a polynomial in n, and c_0 is any constant greater than $\sqrt{2}$. Indeed, a quick inspection of the proof shows that all the terms to be considered for $C(n)$ are polynomials, except for the estimate given in Step three –more precisely in (24)–

which gives a term like $n\,c^n$, with $c > 2$ (recall that, up to loosing a numeric factor in $C_0(n)$, we can assume from the beginning that $\beta(E, F)$ is smaller than an arbitrarily small constant). Eventually, when applying Hölder inequality in (28) we take a square root of the constant $C(n)$ appearing in (27), thus coming to the choice $c_0 > \sqrt{2}$.

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