

CONFINEMENT IN NONLOCAL INTERACTION EQUATIONS

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ABSTRACT. We investigate some dynamical properties of nonlocal interaction equations. We consider sets of particles interacting pairwise via a potential W , as well as continuum descriptions of such systems. The typical potentials we consider are repulsive at short distances, but attractive at long distances. The main question we consider is whether an initially localized configuration remains localized for all times, regardless of the number of particles or their arrangement. In particular we find sufficient conditions on the potential W for the above “confinement” property to hold. We use the framework of weak measure solutions developed in [8] to provide unified treatment of both particle and continuum systems.

Keywords: nonlocal interactions, confinement, gradient flows, particle approximation.

AMS Classification: 35B40, 45K05, 49K20, 92DXX.

1. INTRODUCTION

We consider a mass distribution of particles, represented by a measure $\mu \geq 0$, under the effect of an even *interaction potential*, W . The case of finitely many particles corresponds to purely atomic measures: $\mu = \sum_{i=1}^N m_i \delta_{x_i}$. A measure μ which is absolutely continuous with respect to the Lebesgue measure represents a continuum distribution of particles. The velocity field is given by $v = -\nabla W * \mu$, which represents the combined contributions, at a given particle, of the interaction with all other particles. More precisely, we consider the continuity equation

$$\frac{\partial \mu}{\partial t} = \operatorname{div} [(\nabla W * \mu)\mu] \quad x \in \mathbb{R}^d, t > 0. \quad (1.1)$$

The equation is typically coupled with an initial datum $\mu(0) = \mu_0$.

It is known (cf. [1, 8]) that the equation (1.1) has the structure of a gradient flow of the *interaction energy* functional

$$\mathcal{W}[\mu] := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y) d\mu(x) d\mu(y)$$

with respect to Wasserstein metric. The gradient flow structure is usually displayed via the formula

$$\frac{\partial \mu}{\partial t} = \operatorname{div} \left[\left(\nabla \frac{\delta \mathcal{W}}{\delta \mu} \right) \mu \right]$$

in which the symbol $\frac{\delta \mathcal{W}}{\delta \mu}$ represents the formal functional derivative of \mathcal{W} .

We shall be working with solutions for (1.1) in the sense introduced in [8] and briefly recapped in Subsection 3.1 below. Such a concept of solution allows μ to be an atomic

Date: August 28, 2011.

measure $\mu = \sum_{i=1}^N m_i \delta_{x_i(t)}$, representing N particles with masses $m_i > 0$ at location $x_i(t)$. In this case, the PDE is equivalent to the ODE system

$$\dot{x}_i = - \sum_{j \neq i} m_j \nabla W(x_i - x_j), \quad i = 1, \dots, N, \quad (1.2)$$

on any time interval for which C^1 solutions of the ODE exist. Particles collision in finite time is not ruled out for potentials W which are attractive and ‘singular’ at the origin. In this case the ODE system needs to be considered in a generalized sense as discussed in Remark 2.1.

Equations of the form (1.1) arise in several physical and biological contexts, with the choice of W depending on the phenomenon studied [2, 3, 4, 5, 6, 7, 9, 10, 16, 18, 19, 21, 25, 26, 27]. For instance, in population dynamics or collective motion of animals one is interested in the description of the evolution of a density of individuals. Very often the social interaction between two individuals only depends on the distance between them, which suggests a choice of W as a *radial* function, i.e.

$$W(x) = w(|x|).$$

The potential W is also often chosen to be attractive in the long range (modeling the fact that individuals want to remain in a cohesive group) and repulsive in the short range (modeling the fact that individuals repel each other when they are too close in order to avoid collision) [20, 21]. In the simplest situation, this means that

$$\begin{aligned} w'(r) &\leq 0 \text{ if } r < R_a, \\ w'(r) &\geq 0 \text{ if } r \geq R_a, \end{aligned}$$

for some threshold distance R_a . Recent numerical studies of equation (1.1) in \mathbb{R}^2 and \mathbb{R}^3 has shown that such repulsive-attractive potentials lead to the emergence of surprisingly rich geometric structures [17, 29] in which the *confinement* of particles plays a role. Some of these patterns are reminiscent from patterns observed in experiments with bacterial colonies growing on agar plates. Many swarming systems with repulsive-attractive potentials have been studied. Some of these models are discrete, some other are continuous. Specific phase transitions, as well as, the shape of the patterns and the geometry of the steady states have been studied [20, 19, 11, 12, 9, 17, 23, 13, 14].

When considering models where individuals both attract and repulse one another, it is fundamental to understand whether or not the group will remain in a fixed bounded domain for all time or if it will expand and occupy larger and larger domains. This is the question of *confinement*. A potential W is said to be *confining* if, for any compact domain, there exists a ball of radius $R > 0$ such that for all initial data supported in the domain the solution of (1.1) remains supported in the ball of radius R for all time. In this paper we derive sufficient condition for a potential W to be confining. Loosely stated, our main result is the following: if there exists $C_W > 0$ such that

$$w'(r) \geq -C_W \text{ for all } r \leq R_a \quad (1.3)$$

$$\text{and } \lim_{r \rightarrow +\infty} w'(r) \sqrt{r} = +\infty \quad (1.4)$$

then the potential $W(x) = w(|x|)$ is confining. The precise result is given in Theorem 3.5. Inequality (1.3) means that the ‘repulsion strength’ between two particles is always

bounded above by C_W . Inequality (1.4) means that the “attraction strength” between two particles does not decay too fast as these particles get further and further apart, $r^{-1/2}$ being the critical decay rate in our proof. This specific balance between the “attraction strength” in the long range and the “repulsion strength” in the short range allows us to prove confinement.

Note that (1.4) does not prevent the “attraction strength” to go to zero at infinity. In this case we say that the potential is *weakly attractive* at infinity. On the other hand, if the attraction strength $w'(r)$ is greater than $4C_W$ for r large enough we say that the potential is *strongly attractive at infinity*. For potential which are strongly attractive at infinity we are able to derive a better a priori bound on the final size of the support of the solution than the one obtain with potentials that are only weakly attractive at infinity. The precise result is given in Theorem 3.3.

In [8], we developed a theory of well-posedness for measure solutions to (1.1). This theory allows to treat particle and continuum solutions at the same level. Moreover, the explicit bounds on the continuous dependence with respect to the initial data allow to approximate continuum solutions by particle or atomic measure solutions. Therefore, the strategy that we follow in this work is the following: in Section 2 we derive confinement results for the particle system (1.2) independent of the number of particles, and then, in Section 3, we use the existence and stability theory of [8] to pass to the continuum limit these confinement results.

Let us emphasize that (1.3)-(1.4) are the key hypotheses to obtain confinement. On the other hand, to obtain well-posedness of measure solutions, the exact set of hypotheses used in [8] is the following:

$$W(x) = W(-x) \text{ for all } x \in \mathbb{R}^d \text{ with } W \in C^1(\mathbb{R}^d \setminus \{0\}), \quad (1.5)$$

$$W \text{ is } \lambda\text{-convex for some } \lambda \leq 0, \text{ i.e. } W(x) - \frac{\lambda}{2}|x|^2 \text{ is convex}, \quad (1.6)$$

$$\text{There exists a constant } C > 0 \text{ such that } W(z) \leq C(1 + |z|^2), \text{ for all } z \in \mathbb{R}^d. \quad (1.7)$$

Note that (1.6) implies that the potential is Lipschitz at the origin, which has to be a local minimum if the potential W is not C^1 .

2. CONFINEMENT FOR PARTICLES

In this section we derive sufficient conditions on the potential W so that the particle system (1.2) remains confined for all times. One should keep in mind that additional conditions, (1.5)-(1.7), on W will be needed in next section to extrapolate these confinement results to the continuum model. Consider the system of particles $x_1, \dots, x_N \in \mathbb{R}^d$ the dynamics of which are described by

$$\dot{x}_i = - \sum_{j=1}^N m_j \partial^0 W(x_i - x_j) \quad i = 1, \dots, N \quad (2.1)$$

with m_i 's positive and satisfying $\sum_{i=1}^N m_i = 1$. The notation $\partial^0 W$ stands for

$$\partial^0 W(x) := \begin{cases} \nabla W(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We will assume in this section that the potential W is radially symmetric and continuously differentiable away from the origin:

$$W(x) = w(|x|) \quad \text{and} \quad w \in C^1(0, \infty). \quad (2.2)$$

We also assume that there exists a ball of finite radius such that W is attractive outside of this ball. Inside this ball W can be both repulsive and attractive, but its repulsive “strength” is bounded. To be more precise we assume that there exists constants $R_a > 0$ and $C_W > 0$ such that

$$w'(r) > 0 \quad \text{for all } r > R_a, \quad (2.3)$$

$$w'(r) > -C_W \quad \text{for all } 0 < r < R_a. \quad (2.4)$$

Finally, since the center of mass $\sum_{i=1}^N m_i x_i(t)$ is preserved by (2.1), we assume without loss of generality that

$$\sum_{i=1}^N m_i x_i(t) = 0 \quad \text{for all } t \geq 0$$

and we define $r(t)$ to be the radius of the cloud of particles:

$$r(t) = \max\{|x_1(t)|, \dots, |x_N(t)|\}.$$

Remark 2.1. Under assumptions (2.2), (2.3), (2.4) on W , solutions of (2.1) can be constructed as follows: as long as particles do not collide, the velocity field is continuous and so solutions exist by Peano Theorem (even if they may not be unique). Then, even if it is possible for particles to collide in finite time, since there is a finite number of particles there can only be a finite number of collision times. Hence the system of ODE has a solution in the time intervals between these collision times, while if two particles collide at time t^* then we *assume* that they stick together for $t \geq t^*$ (this may not be the only possibility for extending the solution without some assumption on W near the origin). Then Proposition 2.2 stated below guarantees that, *for any* solution to the ODE as described above, the radius $r(t)$ of the cloud of particles grows at most linearly with respect to time. In particular particles cannot reach infinity in finite time, which gives global existence in time. Let us also observe that, whenever condition (1.6) is enforced, the velocity field is one-sided Lipschitz [15], and so the solution of (2.1) is unique forward in time (but not backward in time). We refer to [15] for a classical reference of discontinuous dynamical systems and [22] for an application to transport equations. We note that this remark also corrects [8, Remark 2.10], where the regularity assumptions on the velocity fields were stated incorrectly. More precisely, [8, Remark 2.10] is true for potentials as in (2.2) with $w \in C^2(0, \infty)$ satisfying (2.3) and (2.4).

We now state the main results of this section, postponing all the proofs to the end of the section.

Proposition 2.2. *Suppose W satisfies (2.2), (2.3) and (2.4). For $r > 2R_a$ define*

$$\sigma(r) := \inf_{2r \geq s \geq r/2} w'(s) > 0. \quad (2.5)$$

Then

$$\frac{d^+}{dt} r(t) := \limsup_{h \rightarrow 0^+} \frac{r(t+h) - r(t)}{h} \leq -\frac{\sigma(r(t))}{6} + \frac{2}{3} C_W$$

whenever $r(t) > 2R_a$.

As a corollary we get an explicit bound for the radius of a cloud of particles evolving under the influence of a strongly attractive potential at infinity (i.e. its attractive strength at infinity is at least four times larger than its maximum repulsive strength).

Corollary 2.3. *Suppose W satisfies (2.2), (2.3) and (2.4). If there exists \bar{R} such that*

$$w'(r) \geq 4C_W \quad \text{for all } r \geq \bar{R}, \quad (2.6)$$

then there exists $R^ > 0$ depending only $r(0)$, \bar{R} , R_a , and W such that $r(t) \leq R^*$ for all $t \geq 0$.*

The next result states that potentials which are weakly attractive at infinity (i.e. their attractive strength goes to zero as $r \rightarrow +\infty$) can also be confining, as long as the rate of decay at infinity of the attractive strength is not too rapid. Here $w'(r) \sim r^{-1/2}$ is the threshold decay rate at infinity. In our proof, which is based on energetic arguments, it is essential for the potential W to be bounded on compact sets.

Proposition 2.4. *Suppose that, in addition to (2.2), (2.3) and (2.4), W satisfies*

$$\liminf_{r \rightarrow 0} w(r) > -\infty \quad \text{and} \quad \lim_{r \rightarrow \infty} w'(r)\sqrt{r} = +\infty, \quad (2.7)$$

then there exists $\bar{R} > 0$ depending only $r(0)$ and W such that $r(t) \leq \bar{R}$ for all $t \geq 0$.

Remark 2.5. In Corollary 2.3 an explicit bound for the radius of the support of the cloud of particles is provided. In the proof of Proposition 2.4 we also derive an explicit bound for the radius of the support. However, for strongly attractive potential at infinity which also satisfies $\liminf_{r \rightarrow 0} w(r) > -\infty$, the bound provided by Corollary 2.3 is better than the one provided in the proof of Proposition 2.4.

Remark 2.6. Conditions (2.2), (2.3) and (2.4) alone are not enough for confinement, and counterexamples follow from the work by Theil [24]. On the other hand we do believe that the conclusion of the Proposition 2.4 holds under significantly weaker assumptions on the growth of w . In particular we conjecture that (in addition to $\liminf_{r \rightarrow 0} w(r) > -\infty$) it is enough to assume that w is increasing on $[R, \infty)$ for some R and

$$\lim_{r \rightarrow +\infty} w(r) = +\infty.$$

The rest of the section is devoted to the proofs of all the above results.

Proof of Proposition 2.2 and Corollary 2.3. Since for any $t \geq 0$ (even a collision time) there exists Δt such that on $[t, t + \Delta t)$ the ODE system has a C^1 solution, it suffices to provide the proof at $t = 0$, under the assumption that $r_0 := r(0) > 2R_a$. From the definition of $r(t)$ we easily get

$$\frac{d^+}{dt} r^2(0) = \max_{\{i: |x_i|=r_0\}} -2 \sum_{j \neq i} m_j \frac{(x_i - x_j) \cdot x_i}{|x_i - x_j|} w'(|x_i - x_j|). \quad (2.8)$$

We can assume that the maximum is achieved for $i = 1$ and that $x_1 = |x_1| e_1 = r_0 e_1$. To find a good upper bound on the right-hand side of (2.8) we compare the repulsive effects of the nearby particles and the attractive effects of appropriately selected far-enough particles.

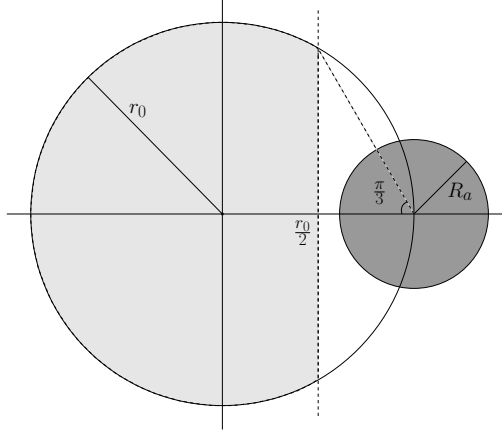


FIGURE 1. Sketch of the geometrical distribution of the location of particles.

Let J_R be the set of indices of particles that are within the radius R_a of x_1 , and thus may be pushing x_1 away. Let $J_A = \{j : x_j \cdot e_1 < \frac{r_0}{2}\}$ be the indices of particles that are “strongly attracting” x_1 . Let $J_{rest} = \{1, \dots, N\} \setminus (J_R \cup J_A)$.

We now show that the mass of “strongly attractive” particles is greater than $1/3$, which implies in particular that the mass of repulsive particles is less than $2/3$. Indeed, since the center of mass is 0, we have that $\sum_{j=1}^N m_j x_j \cdot e_1 = 0$, and we deduce that

$$r_0 \sum_{j \in J_A} m_j \geq - \sum_{j \in J_A} m_j x_j \cdot e_1 = \sum_{j \in J_R \cup J_{rest}} m_j x_j \cdot e_1 \geq \frac{r_0}{2} \sum_{j \in J_R \cup J_{rest}} m_j.$$

Let $m_A = \sum_{j \in J_A} m_j$ and $m_R = \sum_{j \in J_R} m_j$. It follows that

$$m_A \geq \frac{1}{2} \sum_{j \in J_R \cup J_{rest}} m_j = \frac{1}{2} (1 - m_R). \quad (2.9)$$

Hence $m_A \geq \frac{1}{3}$, and so $m_R \leq \frac{2}{3}$.

We now conclude: since x_1 is the particle the furthest away from the origin, it follows that $(x_1 - x_j) \cdot x_1 \geq 0$, and combined with (2.8), (2.4), (2.3) and (2.5) we obtain that at $t = 0$

$$\begin{aligned} \frac{1}{2} \frac{d^+}{dt} r^2(0) &\leq - \sum_{j \in J_R \cup J_A} m_j \frac{(x_1 - x_j) \cdot x_1}{|x_1 - x_j|} w'(|x_1 - x_j|) \\ &\leq \sum_{j \in J_R} m_j r_0 C_W - \sum_{j \in J_A} m_j \cos\left(\frac{\pi}{3}\right) r_0 \sigma(r_0) \\ &\leq \frac{2}{3} r_0 C_W - \frac{1}{6} r_0 \sigma(r_0), \end{aligned}$$

where in the last two steps we have used that the maximum angle between $x_1 - x_j$ for $j \in J_A$ and x_1 is $\pi/3$ as depicted in Figure 1. Dividing by r_0 establishes the claim of Proposition 2.2 at $t = 0$, which, as we remarked before, implies the claim for arbitrary $t \geq 0$.

Now, let us show Corollary 2.3. Let \bar{r} be a solution of

$$\begin{cases} \frac{d\bar{r}}{dt} = -\frac{\sigma(\bar{r})}{6} + \frac{2}{3}C_W \\ \bar{r}(0) = r_0. \end{cases}$$

Since σ is a continuous function a solution exists. If the solution is nonunique, we choose the maximal solution.

Consequently for all $t \geq 0$, $r(t) \leq \max\{\bar{r}(t), 2R_a\}$. Now, since by assumption $\sigma(r) \geq 4C_W$ for $r \geq \bar{R}$, we get $\frac{d\bar{r}}{dt} \leq 0$ whenever $\bar{r} \geq \bar{R}$. This implies $\bar{r}(t) \leq \max\{\bar{r}(0), \bar{R}\}$, so $r(t) \leq \max\{r(0), \bar{R}, 2R_a\}$ for $t \geq 0$, as desired. \square

We now turn to the proof of Proposition 2.4. Compared to the previous proof, here we will make use of the fact that the system of ODE (2.1) is a gradient flow of the interaction energy

$$\mathcal{W}[x_1, \dots, x_N] = \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N m_j m_k W(x_j - x_k). \quad (2.10)$$

The assumption that $W(x)$ remain bounded from below as $x \rightarrow 0$ guarantees that the interaction energy is finite for all time, even if collisions take place. The idea of the proof is as follow: Note that there are no direct energetic obstacles to prevent the support of the solution from becoming large. That is the boundedness of the interaction energy does not prevent a particle from traveling far from the origin, as long as its mass is small. However it turns out that for even a small particle to go far from the center of mass, there must exist significant mass nearby. That is for the small particle to go far, there must be particles of relatively large total mass which are “pushing” it out. However the existence of a “large” mass far from the center of mass does violate the fact that the energy is bounded.

Proof of Proposition 2.4. Since $w(r)$ is increasing for r large enough, and since it does not diverge as $r \rightarrow 0$, it is bounded from below and we can assume without loss of generality that $w(r) \geq 0$ for all $r > 0$ by adding a suitable constant to w . Let

$$\theta(r) := \inf_{s \geq r} w'(s) \sqrt{s}.$$

Let $r_0 := r(0)$. We now introduce \bar{R} , which we show to be an upper bound on the radius of the support of the solution. Let \bar{R} be any number such that

$$\bar{R} \geq 6R_a, \bar{R} \geq r_0, \text{ and } \theta\left(\frac{\bar{R}}{6}\right) > 6^{5/4} \sqrt{C_W \|W\|_{L^\infty(B(0, 2r_0))}}. \quad (2.11)$$

Note that it is possible to choose such an \bar{R} because of (2.7). With such a choice for \bar{R} we will be able to show that if $r(t)$ were to reach \bar{R} , this would lead to the energy at time t being larger than at time 0, violating the fact that the energy is non-increasing. The choice of the constants involved in (2.11) will become clearer in the course of the proof.

Let us first observe that for any $r > 2R_a$

$$w(r) \geq \frac{\sqrt{r}}{2} \theta\left(\frac{r}{2}\right). \quad (2.12)$$

This follows by noting that $w'(s) \geq \theta(r/2)/\sqrt{s} \geq \theta(r/2)/\sqrt{r}$ for all $s \in (r/2, r)$ (here we have used the fact that since $r > 2R_a$, $\theta(r/2) \geq 0$). Then integrating from $r/2$ to r , and using that $w(r/2) \geq 0$ leads to (2.12).

Assume that the statement of the proposition does not hold. Let t_1 be the first time at which a particle reaches the distance \bar{R} from the origin. Consider the ODE system (2.1) in which this particle is identified as $x_1(t_1)$ and assume without loss of generality that $x_1(t_1) = |x_1(t_1)|e_1$.

We can also assume without loss of generality that there are no collisions at time t_1 , that is that the ODE system has C^1 solutions on the time interval $(t_1 - \Delta t, t_1 + \Delta t)$, for Δt small enough. Indeed if there is a collision at time t_1 , we can always replace \bar{R} by $\bar{R} + \Delta R$. Since we have assumed that the claim of the lemma does not hold the radius of the support will eventually reach $\bar{R} + \Delta R$, and since there are finitely many collisions (if any) one can choose ΔR so that there are no collisions at the first time the support reaches $\bar{R} + \Delta R$. By the choice of t_1 , we have that $|x_1(t_1)| = \bar{R}$ and

$$\frac{1}{2} \frac{d^+}{dt} r^2(t_1) = \dot{x}_1(t_1) \cdot x_1(t_1) \geq 0.$$

Therefore, we deduce

$$-\sum_{j \geq 2} m_j \nabla W(x_1(t_1) - x_j(t_1)) \cdot x_1(t_1) \geq 0. \quad (2.13)$$

Let J_R , J_A , and J_{rest} be as in the proof of Proposition 2.2: $J_R = \{j : x_j(t_1) \in B(x_1(t_1), R_a)\}$, $J_A = \{j : x_j(t_1) \cdot e_1 < \frac{\bar{R}}{2}\}$, and $J_{rest} = \{1, \dots, N\} \setminus (J_R \cup J_A)$ with the geometrical interpretation of Figure 1. Arguing as for (2.9) one obtains $m_A \geq \frac{1}{3}$. We are now going to derive a lower bound for m_R : on one hand because of (2.4) we clearly have

$$\sum_{j \in J_R} -m_j \nabla W(x_1 - x_j) \cdot e_1 \leq m_R C_W. \quad (2.14)$$

On the other hand, since $\sum_{j \in J_{rest}} -m_j \nabla W(x_1 - x_j) \cdot e_1 \leq 0$ as can be seen on Figure 1, we have from (2.13) that

$$\sum_{j \in J_A} m_j \nabla W(x_1(t_1) - x_j(t_1)) \cdot e_1 \leq \sum_{j \in J_R} -m_j \nabla W(x_1(t_1) - x_j(t_1)) \cdot e_1. \quad (2.15)$$

Thus combining (2.14) and (2.15), and using again the fact that the maximum angle between $x_1 - x_j$ for $j \in J_A$ and x_1 is $\pi/3$, we obtain:

$$m_R C_W \geq \sum_{j \in J_A} m_j \nabla W(x_1 - x_j) \cdot e_1 \geq \frac{1}{2} \sum_{j \in J_A} m_j w'(|x_1 - x_j|) \geq \frac{1}{6\sqrt{2\bar{R}}} \theta\left(\frac{\bar{R}}{2}\right). \quad (2.16)$$

To obtain the last inequality we have used the fact that for all $\bar{R}/2 \leq s \leq 2\bar{R}$, $w'(s) \geq \theta(\bar{R}/2)/\sqrt{s} \geq \theta(\bar{R}/2)/\sqrt{2\bar{R}}$. The above computation gives a lower bound on the mass m_R of particles repulsing the particle the furthest away. It shows that, in order for the particle the furthest away to be pushed even further, there must be significant mass nearby.

We now turn toward energetic arguments. Note that at time 0 the interaction energy (2.10) satisfies

$$\|W\|_{L^\infty(B(0, 2r_0))} \geq 2W[x_1, \dots, x_N].$$

On the other hand at time t_1 , using the positivity of W together with the fact that $\bar{R} > 6R_a$, we get that

$$2\mathcal{W}[x_1, \dots, x_N] \geq \sum_{j \in J_R} \sum_{k \in J_A} m_j m_k W(x_j - x_k) \geq \frac{m_R}{3} \inf_{r \geq \bar{R}/3} w(r) = \frac{m_R}{3} w(\bar{R}/3)$$

where we have used the fact that if $\bar{R} > 6R_a$, then particles in J_A are at least at a distance $\bar{R}/3$ from particles in J_R . Since the interaction energy is a decreasing function of time, we conclude that

$$\|W\|_{L^\infty(B(0, 2r_0))} \geq \frac{m_R}{3} w(\bar{R}/3).$$

We now use the lower bound on m_R from (2.16), and the lower bound on $w(\bar{R}/3)$ from (2.12) (where we used $r = \bar{R}/3$, which is possible due to the assumption that $\bar{R} \geq 6R_a$):

$$\|W\|_{L^\infty(B(0, 2r_0))} \geq \frac{1}{3} \left[\frac{1}{6\sqrt{2}\bar{R}C_W} \theta\left(\frac{\bar{R}}{2}\right) \right] \left[\frac{\sqrt{\bar{R}/3}}{2} \theta\left(\frac{\bar{R}}{6}\right) \right] \geq \frac{1}{6^{5/2}C_W} \theta\left(\frac{\bar{R}}{6}\right)^2,$$

where for the last inequality we used that $\theta(r)$ is increasing. This contradicts (2.11) and concludes the proof. \square

3. CONFINEMENT FOR GENERAL MEASURE SOLUTIONS

In this section we use the theory developed in [8] in order to pass to the limit the confinement results derived in the previous section for particles. We start with a short summary of the results of [8].

3.1. Weak measure solutions. We shall briefly resume here the weak measure solution theory for the equation (1.1) developed in [1, 8]. We shall work in the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures on \mathbb{R}^d , thus normalizing the total mass to 1. This is not restrictive in view of the following invariance property: if $\mu(t)$ is a solution, so is $M\mu(Mt)$ for all $M > 0$. We additionally require our measure solution to belong to the metric space

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 d\mu(x) < +\infty \right\}$$

of probability measures with finite second moment, endowed with the 2-Wasserstein distance d_W (see [1, 28] for further details).

Definition 3.1 (Weak measure solutions). *A locally absolutely continuous curve*

$$\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$$

*is a weak measure solution to (1.1) with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 W * \mu$ belongs to $L^1_{loc}([0, +\infty); L^2(\mu(t)))$ and*

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}^d} \frac{\partial \varphi}{\partial t}(x, t) d\mu(t)(x) dt + \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) = \\ \int_0^{+\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla \varphi(x, t) \cdot \partial^0 W(x - y) d\mu(t)(x) d\mu(t)(y) dt, \end{aligned} \quad (3.1)$$

for all test functions $\varphi \in C_c^\infty([0, +\infty) \times \mathbb{R}^d)$.

The case of a measure $\mu(t)$ given by a finite combination of Dirac deltas centered at $x_i(t)$, $i = 1, \dots, N$ solving (2.1) is included in the notion of solution provided in Definition 3.1 (see [8, Remark 2.10]).

We remark that the assumption that the velocity is in $L^1_{loc}([0, +\infty), L^2(\mu(t)))$ is needed for the theory of gradient flows in spaces of probability measures to apply. In particular under this assumption the solutions are curves of locally finite length and the right-hand side of (3.1) is well-defined by Hölder's inequality.

The following result is a combination of [8, Theorems 2.12 and 2.13]:

Theorem 3.2 (Existence and d_W -Stability). *Let W satisfy the assumptions (1.5), (1.6) and (1.7). Then, there exists a unique weak measure solution to (1.1) in the sense of Definition 3.1. Moreover, given two weak measure solutions $\mu^1(t)$ and $\mu^2(t)$, we have*

$$d_W(\mu^1(t), \mu^2(t)) \leq e^{-\lambda t} d_W(\mu_0^1, \mu_0^2) \quad (3.2)$$

for all $t \geq 0$.

3.2. Confinement. We are now ready to state and prove the two main theorems of this paper. As in the proof of Corollary 2.3, let us consider $\bar{r}(t)$ to be the maximal solution of

$$\begin{cases} \frac{d\bar{r}}{dt} = -\frac{\sigma(\bar{r})}{6} + \frac{2}{3}C_W, \\ \bar{r}(0) = r_0 \end{cases},$$

with the considerations done there.

Theorem 3.3. *Assume W satisfies (1.5)–(1.7) as well as (2.2)–(2.4). Let μ_0 be a compactly supported probability measure with radius of support $r_0 > 0$. Let $\mu(t)$ be the solution to (1.1) and $r(t)$ its radius of support, then $r(t) \leq \max\{\bar{r}(t), 2R_a\}$. Moreover, if W also satisfies (2.6), then $r(t) \leq R^*$ for all $t \geq 0$.*

Remark 3.4. Of course (2.2) implies (1.5). Also, (1.6) and (2.3) imply (2.4). We choose to write it like this in order to separate the hypotheses necessary for well-posedness of measure solutions from the ones necessary for confinement.

Proof. We can assume without the loss of generality that μ_0 has center of mass 0. Since \mathcal{W} is translation invariant, $\mu(t)$ remains centered at 0 for all $t > 0$. Let us remark that the claims of the Theorem hold in the case of an initial data formed by a finite number of atoms due to Proposition 2.2 and Corollary 2.3.

To show the first claim for general initial data, let us consider a sequence of particle approximations $\mu_n(0)$ of $\mu(0)$ satisfying

$$\mu_n(0) = \sum_{i=1}^n m_{n,i} \delta_{x_{n,i}(0)}, \quad m_{n,i} > 0, \quad \sum_{i=1}^n m_{n,i} = 1 \quad (3.3)$$

$$|x_{n,i}(0)| < r_0 \text{ for all } n \text{ and } i \text{ with } \sum_{i=1}^n m_{n,i} x_{n,i} = 0 \quad (3.4)$$

$$\lim_{n \rightarrow \infty} d_w(\mu_n(0), \mu(0)) = 0 \quad (3.5)$$

Then, by stability of solutions given in Theorem 3.2, given any $t > 0$ we have

$$\lim_{n \rightarrow \infty} d_w(\mu_n(t), \mu(t)) = 0. \quad (3.6)$$

Reasoning as in the proof of corollary 2.3 we deduce that the support of $\mu_n(t)$ is contained in $\bar{B}(0, \max\{\bar{r}(t), 2R_a\})$ for all $t \geq 0$. Because of (3.6) this implies that the support of $\mu(t)$ must also be contained in $\bar{B}(0, \max\{\bar{r}(t), 2R_a\})$ for all $t \geq 0$. The second claim follows analogously using Corollary 2.3. \square

Theorem 3.5. *Assume W satisfies (1.5)–(1.7) as well as (2.2)–(2.4) together with (2.7). Then, given a compactly supported probability measure μ_0 with center of mass at x_0 such that $\text{supp } \mu_0 \subset \bar{B}(x_0, r_0)$, there exists $\bar{R} \geq r_0$, depending only on r_0 and W , such that the solution $\mu(t)$ to (1.1) satisfies*

$$\text{supp } \mu(t) \subset \bar{B}(x_0, \bar{R}) \quad \text{for all } t \geq 0.$$

Proof. As before we can assume that μ_0 has center of mass 0, which implies that for all times $\mu(t)$ has center of mass 0 as well. As in the proof of Theorem 3.3 we consider a sequence of particle approximations $\mu_n(0)$ of $\mu(0)$ satisfying (3.3)–(3.5). Because of Proposition 2.4 the claim of the theorem holds for each particle approximation. Therefore, due to (3.6), it also holds for the limit $\mu(t)$. \square

Acknowledgements. JAC and MdF acknowledge support from the project MTM2011-27739-C04-02 DGI-MCI (Spain), 2009-SGR-345 from AGAUR-Generalitat de Catalunya and the 2007 Azioni Integrate Italia-Spagna 25. AF acknowledges the support from NSF Grant DMS-0969962. TL acknowledges the support from NSF Grant DMS-1109805. DS is grateful to NSF for support via the grants DMS-0638481 and DMS-0908415, and is also thankful to the CNA (NSF grant DMS-0635983) and PIRE NSF grant OISE-0967140) for its support during the preparation of this paper. All authors acknowledge IPAM-UCLA where this work was started during the thematic program on “Optimal Transport”.

REFERENCES

- [1] L. AMBROSIO, N. GIGLI AND G. SAVARÉ, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics, Birkhäuser, (2005).
- [2] D. BENEDETTO, E. CAGLIOTI, M. PULVIRENTI, *A kinetic equation for granular media*, RAIRO Modél. Math. Anal. Numér., 31, (1997), 615–641.
- [3] S. BOI, V. CAPASSO AND D. MORALE, *Modeling the aggregative behavior of ants of the species *Polyergus rufescens**, Spatial heterogeneity in ecological models (Alcalá de Henares, 1998), Nonlinear Anal. Real World Appl., 1, (2000), 163-176.
- [4] M. BURGER AND M. DI FRANCESCO, *Large time behavior of nonlocal aggregation models with nonlinear diffusion*, Networks and Heterogeneous Media, 3, (2008), 749-785.
- [5] M. BURGER, V. CAPASSO AND D. MORALE, *On an aggregation model with long and short range interactions*, Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal, 8 (2007), 939-958.
- [6] M. BODNAR AND J.J.L. VELÁZQUEZ, *An integro-differential equation arising as a limit of individual cell-based models*, J. Differential Equations, 222, (2006), 341-380.
- [7] J.A. CARRILLO, R.J. MCCANN AND C. VILLANI, *Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates*, Rev. Matemática Iberoamericana, 19, (2003), 1-48.
- [8] J.A. CARRILLO, M. DI FRANCESCO, A. FIGALLI, T. LAURENT, AND D. SLEPČEV, *Global-in-time weak measure solutions, and finite-time aggregation for nonlocal interaction equations*, Duke Math. J., 156, No. 2 (2011), 229-271.
- [9] J.A. CARRILLO, M.R. D’ORSOGNA AND V. PANFEROV, *Double milling in self-propelled swarms from kinetic theory*, Kin. Rel. Mod., 2, (2009), 363-378.

- [10] Y.-L. CHUANG, Y.R. HUANG, M.R. D'ORSOGNA AND A.L. BERTOZZI, *Multi-vehicle flocking: scalability of cooperative control algorithms using pairwise potentials*, IEEE International Conference on Robotics and Automation, (2007), 2292-2299.
- [11] M. R. D'ORSOGNA, Y. CHUANG, A. BERTOZZI AND L. CHAYES, *Self-propelled particles with soft-core interactions: patterns, stability and collapse*, Phys. Rev. Lett., 96, (2006).
- [12] Y. CHUANG, M. R. D'ORSOGNA, D. MARTHALER, A. BERTOZZI AND L. CHAYES, *State transitions and the continuum limit for interacting, self-propelled particles*, Phys. D, 232, (2007), 33-47
- [13] K. FELLNER AND G. RAOUL, *Stable stationary states of non-local interaction equations*, to appear in Math. Models Methods Appl. Sci.
- [14] K. FELLNER AND G. RAOUL, to appear in Math. Comput. Modelling, *Stability of stationary states of non-local equations with singular interaction potentials*, to appear in Math. Comput. Modelling
- [15] A.F. FILIPPOV, *Differential equations with discontinuous righthand sides*, Mathematics and its Applications (Soviet Series) 18, Kluwer Academic Publishers Group, Dordrecht 1988.
- [16] F. GOLSE, *The mean-field limit for the dynamics of large particle systems*, Journées "Équations aux Dérivées Partielles", Exp. No. IX, 47, Univ. Nantes, Nantes, (2003).
- [17] T. KOLOKONIKOV, H. SUN, D. UMINSKY AND A. BERTOZZI, *Stability of ring patterns arising from 2D particle interactions*, Physical Review E, 84, (2011), 015203.
- [18] D. MORALE, V. CAPASSO AND K. OELSCHLÄGER, *An interacting particle system modelling aggregation behavior: from individuals to populations*, J. Math. Biol., 50, (2005), 49-66.
- [19] A. MOGILNER AND L. EDELSTEIN-KESHET, *A non-local model for a swarm*, J. Math. Bio., 38, (1999), 534-570.
- [20] A. MOGILNER, L. EDELSTEIN-KESHET, L. BENT, L. AND A. SPIROS *Mutual interactions, potentials, and individual distance in a social aggregation*, J. Math. Biol., 47, (2003), 353-389
- [21] A. OKUBO, S. LEVIN, *Diffusion and Ecological Problems: Modern Perspectives*, Springer, Berlin, (2002).
- [22] F. POUPAUD, M. RASCLE, *Measure solutions to the linear multi-dimensional transport equation with non-smooth coefficients*, Comm. Partial Differential Equations, 22, (1997), 337-358.
- [23] G. RAOUL, *Non-local interaction equations: Stationary states and stability analysis*, preprint.
- [24] F. THEIL, *A proof of crystallization in two dimensions*, Comm. Math. Phys. 262 No. 1 (2006), 209-236.
- [25] C.M. TOPAZ AND A.L. BERTOZZI, *Swarming patterns in a two-dimensional kinematic model for biological groups*, SIAM J. Appl. Math., 65, (2004), 152-174.
- [26] C.M. TOPAZ, A.L. BERTOZZI, AND M.A. LEWIS, *A nonlocal continuum model for biological aggregation*, Bulletin of Mathematical Biology, 68, No. 7 (2006), 1601-1623.
- [27] G. TOSCANI, *One-dimensional kinetic models of granular flows*, RAIRO Modél. Math. Anal. Numér., 34, No. 6 (2000), 1277-1291.
- [28] C. VILLANI, *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, (2003).
- [29] J. VON BRECHT, D. UMINSKY, T. KOLOKOLNIKOV, AND A. BERTOZZI, *Predicting pattern formation in particle interactions*, preprint.

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