

# Existence and uniqueness of maximal regular flows for non-smooth vector fields

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## Abstract

In this paper we provide a complete analogy between the Cauchy-Lipschitz and the DiPerna-Lions theories for ODE's, by developing a local version of the DiPerna-Lions theory. More precisely, we prove existence and uniqueness of a maximal regular flow for the DiPerna-Lions theory using only local regularity and summability assumptions on the vector field, in analogy with the classical theory, which uses only local regularity assumptions. We also study the behaviour of the ODE trajectories before the maximal existence time. Unlike the Cauchy-Lipschitz theory, this behaviour crucially depends on the nature of the bounds imposed on the spatial divergence of the vector field. In particular, a global assumption on the divergence is needed to obtain a proper blow-up of the trajectories.

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# 1 Introduction

Given a vector field  $\mathbf{b}(t, x)$  in  $\mathbb{R}^d$ , the theory of DiPerna-Lions, introduced in the seminal paper [21], provides existence and uniqueness of the flow (in the almost everywhere sense, with respect to Lebesgue measure  $\mathcal{L}^d$ ) under weak regularity assumptions on  $\mathbf{b}$ , for instance when  $\mathbf{b}(t, \cdot)$  is Sobolev [21] or  $BV$  [2] and satisfies global bounds on the divergence. In this respect, this theory could be considered as a weak Cauchy-Lipschitz theory for ODE's. This analogy is confirmed by many global existence results, by a kind of Lusin type approximation of DiPerna-Lions flows by Lipschitz flows [6, 19], and even by differentiability properties of the flow [22]. However, this analogy is presently not perfect, and the main aim of this paper is to fill this gap.

Indeed, the Cauchy-Lipschitz theory is not only pointwise but also purely local, meaning that existence and uniqueness for small intervals of time depend *only* on local regularity properties of the vector fields  $\mathbf{b}(t, x)$ . On the other hand, not only the DiPerna-Lions theory is an almost everywhere theory (and this really seems to be unavoidable) but also the existence results for the flow depend on *global* in space growth estimates on  $|\mathbf{b}|$ , the most typical one being

$$\frac{|\mathbf{b}(t, x)|}{1 + |x|} \in L^1((0, T); L^1(\mathbb{R}^d)) + L^1((0, T); L^\infty(\mathbb{R}^d)). \quad (1.1)$$

This is in contrast with the fact that the so-called “renormalization property”, which plays a key role in the theory, seems to depend only on local properties of  $\mathbf{b}$ , because it deals with distributional solutions to a continuity/transport equation with a source term: as a matter of fact, it is proved using only local regularity properties of  $\mathbf{b}$ .

Given an open set  $\Omega \subset \mathbb{R}^d$ , in this paper we consider vector fields  $\mathbf{b} : (0, T) \times \Omega \rightarrow \mathbb{R}^d$  satisfying only the local integrability property  $\int_0^T \int_{\Omega'} |\mathbf{b}| dx dt < \infty$  for all  $\Omega' \Subset \Omega$ , a local one-sided bound on the distributional divergence, and the property that the continuity equation with velocity  $\mathbf{b}$  is well-posed in the class of nonnegative bounded and compactly supported functions in  $\Omega$ . As illustrated in Remark 3.1, this last assumption is fulfilled in many cases of interest and it is known to be deeply linked to the uniqueness of the flow; in addition, building on the superposition principle (Theorem 2.1), it is proved in the appendix that even this assumption is purely local, as well as the other two ones concerning integrability and bounds on divergence.

Under these three assumptions we prove existence of a unique *maximal regular* flow  $\mathbf{X}(t, x)$  in  $\Omega$ , defined up to a maximal time  $T_{\Omega, \mathbf{X}}(x)$  which is positive  $\mathcal{L}^d$ -a.e. in  $\Omega$ . Here “maximal” refers to

$$\limsup_{t \uparrow T_{\Omega, \mathbf{X}}(x)} V_{\Omega}(\mathbf{X}(t, x)) = \infty \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \{T_{\Omega, \mathbf{X}} < T\}, \quad (1.2)$$

where  $V_{\Omega} : \Omega \rightarrow [0, \infty)$  is a given continuous “confining potential”, namely with  $V(x) \rightarrow \infty$  as  $x \rightarrow \partial\Omega$ ; hence, (1.2) is a synthetic way to state that, for any  $\Omega' \Subset \Omega$ ,  $\mathbf{X}(t, x)$  is not contained in  $\Omega'$  for  $t$  close to  $T_{\Omega, \mathbf{X}}(x)$ . In our axiomatization, which parallels the one of [2] and slightly differs from the one of the DiPerna-Lions theory (being only based on one-sided bounds on divergence and independent of the semigroup property), “regular” means

the existence of constants  $C(\Omega', \mathbf{X})$  such that

$$\int_{\Omega' \cap \{h_{\Omega'} > t\}} \phi(\mathbf{X}(t, x)) dx \leq C(\Omega', \mathbf{X}) \int_{\mathbb{R}^d} \phi(y) dy \quad \text{for all } \phi \in C_c(\mathbb{R}^d) \text{ nonnegative} \quad (1.3)$$

for all  $t \in [0, T]$ ,  $\Omega' \Subset \Omega$ , where  $h_{\Omega'}(x) \in [0, T_{\Omega, \mathbf{X}}(x)]$  is the first time that  $\mathbf{X}(\cdot, x)$  hits  $\mathbb{R}^d \setminus \Omega'$ . Under global bounds on the divergence, (1.3) can be improved to

$$\int_{\Omega \cap \{T_{\Omega, \mathbf{X}} > t\}} \phi(\mathbf{X}(t, x)) dx \leq C_*(\mathbf{X}) \int_{\mathbb{R}^d} \phi(y) dy \quad \text{for all } \phi \in C_c(\mathbb{R}^d) \text{ nonnegative} \quad (1.4)$$

for all  $t \in [0, T]$  and for a constant  $C_*(\mathbf{X}) > 0$ , but many structural properties can be proved with (1.3) only.

Uniqueness of the maximal regular flow follows basically from the “probabilistic” techniques developed in [2], which allow one to transfer uniqueness results at the level of the PDE (the continuity equation), here axiomatized, into uniqueness results at the level of the ODE. Existence follows by analogous techniques; the main new difficulty here is that even if we truncate  $\mathbf{b}$  by multiplying it by a  $C_c^\infty(\Omega)$  cut-off function, the resulting vector field has not divergence in  $L^\infty$  (just  $L^1$ , actually, when  $|\mathbf{b}_t| \notin L_{\text{loc}}^\infty(\Omega)$ ), hence the standard theory is not applicable. Hence, several new ideas and techniques need to be introduced to handle this new situation. These results are achieved in Section 5.

Besides existence and uniqueness, in Section 6 we prove a natural semigroup property for  $\mathbf{X}$  and for  $T_{\Omega, \mathbf{X}}$  and some additional properties which depend on *global* bounds on the divergence, more precisely on (1.4). The first property, well known in the classical setting, is properness of the blow-up, namely this enforcement of (1.2):

$$\lim_{t \uparrow T_{\Omega, \mathbf{X}}(x)} V_\Omega(\mathbf{X}(t, x)) = \infty \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \{T_{\Omega, \mathbf{X}} < T\}. \quad (1.5)$$

In other terms, for any  $\Omega' \Subset \Omega$  we have that  $\mathbf{X}(t, x) \notin \Omega'$  for  $t$  sufficiently close to  $T_{\Omega, \mathbf{X}}(x)$ . In  $\Omega = \mathbb{R}^d$ ,  $d \geq 2$ , we also provide an example of an autonomous Sobolev vector field showing that (1.2) cannot be improved to (1.5) when only local bounds on divergence are present. We also discuss the 2-dimensional case for  $BV_{\text{loc}}$  vector fields; these examples are presented in Section 7.1. The second property is the continuity of  $\mathbf{X}(\cdot, x)$  up to  $T_{\Omega, \mathbf{X}}(x)$ , discussed in Theorem 7.5, and sufficient conditions for  $T_{\Omega, \mathbf{X}}(x) = T$ . Finally, we discuss the stability properties of  $\mathbf{X}$  before the blow-up time  $T_{\mathbf{X}}$  with respect to perturbations of  $\mathbf{b}$ .

As shown in [4] (see also [18]) existence and uniqueness of a maximal regular flow can be applied to describe the Lagrangian structure of weak solutions to the transport equations, and in particular it has interesting consequences on kinetic equations such as the Vlasov-Poisson system. Indeed, in [4] we show that Eulerian solutions correspond to a Lagrangian evolution of particles even in the context of weak solutions; for instance, bounded solutions are transported by a suitable notion of maximal regular flow in the phase space. Moreover, thanks to the stability of maximal regular flows we prove new global existence results for weak solutions. Since in the Vlasov-Poisson system the vector field  $\mathbf{b}$  is coupled with the solution of the continuity equation via the Poisson equation, having stability results in

this new setting plays a crucial role to obtain existence of solutions via an approximation argument.

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## 2 Notation and preliminary results

We mostly use standard notation, denoting by  $\mathcal{L}^d$  the Lebesgue measure in  $\mathbb{R}^d$ , and by  $f_{\#}\mu$  the push-forward of a Borel nonnegative measure  $\mu$  under the action of a Borel map  $f$ , namely  $f_{\#}\mu(B) = \mu(f^{-1}(B))$  for any Borel set  $B$  in the target space. We denote by  $\mathcal{B}(\mathbb{R}^d)$  the family of all Borel sets in  $\mathbb{R}^d$ . In the family of positive finite measures in an open set  $\Omega$ , we will consider both the weak topology induced by the duality with  $C_b(\Omega)$  that we will call *narrow* topology, and the *weak* topology induced by  $C_c(\Omega)$ .

If  $J \subset \mathbb{R}$  is an interval and  $t \in J$ , we denote by  $e_t : C(J; \mathbb{R}^d) \rightarrow \mathbb{R}^d$  the evaluation map at time  $t$ , namely  $e_t(\eta) := \eta(t)$  for any continuous curve  $\eta : J \rightarrow \mathbb{R}^d$ . The rest of the section is devoted to the discussion of preliminary results on solutions to the continuity equation, with statements and proofs adapted to our problem. Also,  $\mathcal{M}_+(\mathbb{R}^d)$  will denote the space of finite Borel measures on  $\mathbb{R}^d$ , while  $\mathcal{P}(\mathbb{R}^d)$  denotes the space of probability measures. Let us fix  $T \in (0, \infty)$  and consider a weakly continuous family  $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ ,  $t \in [0, T]$ , solving in the sense of distributions the continuity equation

$$\frac{d}{dt}\mu_t + \nabla \cdot (\mathbf{b}_t \mu_t) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d$$

for a Borel vector field  $\mathbf{b} : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ , locally integrable with respect to the space-time measure  $\mu_t dt$ . When we restrict ourselves to probability measures  $\mu_t$ , then weak and narrow continuity w.r.t.  $t$  are equivalent; analogously, we may equivalently consider compactly supported test functions  $\varphi(t, x)$  in the weak formulation of the continuity equation, or functions with bounded  $C^1$  norm whose support is contained in  $I \times \mathbb{R}^d$  with  $I \Subset (0, T)$ .

We now recall the so-called superposition principle. We prove it under the general assumption that  $\mu_t$  may a priori vanish for some  $t \in [0, T]$ , but satisfies (2.1); we see in the proof of Theorem 2.1 that this assumption implies that there is no mass loss, namely  $\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d)$  for every  $t \in [0, T]$ . The proof of the superposition principle, as stated below, reduces then to [5, Theorem 12], which presents the same result assuming that the family  $\mu_t$  is made of probability measures. We mention also [10, Theorem 8.2.1], where a proof is presented in the even more special case of  $L^p$  integrability on  $\mathbf{b}$  for some  $p > 1$

$$\int_0^T \int_{\mathbb{R}^d} |\mathbf{b}_t(x)|^p d\mu_t(x) dt < \infty.$$

The superposition principle will play a role in the proof of the comparison principle stated in Proposition 3.3 and in the blow-up criterion of Theorem 7.6.

**Theorem 2.1** (Superposition principle and approximation). *Let  $\mathbf{b} : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a Borel vector field. Let  $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ ,  $0 \leq t \leq T$ , with  $\mu_t$  weakly continuous in  $[0, T]$  solution to the equation  $\frac{d}{dt}\mu_t + \operatorname{div}(\mathbf{b}\mu_t) = 0$  in  $(0, T) \times \mathbb{R}^d$ , with*

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\mathbf{b}_t(x)|}{1+|x|} d\mu_t(x) dt < \infty. \quad (2.1)$$

Then there exists  $\boldsymbol{\eta} \in \mathcal{M}_+(C([0, T]; \mathbb{R}^d))$  satisfying:

- (i)  $\boldsymbol{\eta}$  is concentrated on absolutely continuous curves  $\eta$  in  $[0, T]$ , solving the ODE  $\dot{\eta} = \mathbf{b}_t(\eta)$   $\mathcal{L}^1$ -a.e. in  $(0, T)$ ;
- (ii)  $\mu_t = (e_t)_\# \boldsymbol{\eta}$  (so, in particular,  $\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d)$ ) for all  $t \in [0, T]$ .

Moreover, there exists a family of measures  $\mu_t^R \in \mathcal{M}_+(\mathbb{R}^d)$ , narrowly continuous in  $[0, T]$ , solving the continuity equation and supported on  $\overline{B}_R$ , such that  $\mu_t^R \uparrow \mu_t$  as  $R \rightarrow \infty$  for all  $t \in [0, T]$ .

*Proof.* We first show that  $\mu_t$  does not lose nor gain mass, namely

$$\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d) \quad \forall t \in [0, T]. \quad (2.2)$$

Indeed, let  $R \geq 1$  and  $\chi_R \in C_c^\infty(B_{3R})$  be a cut-off function with  $0 \leq \chi_R \leq 1$ ,  $\chi_R \equiv 1$  on a neighborhood of  $B_R$ , and  $|\nabla \chi_R| \leq \frac{1}{R} \chi_{B_{3R} \setminus B_R}$ . Since  $\mu_t$  solves the continuity equation and  $1/R \leq 4/(1+|x|)$  for  $x \in B_{3R} \setminus B_R$ , we have

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \chi_R d\mu_0 - \int_{\mathbb{R}^d} \chi_R d\mu_t \right| &\leq \int_0^T \left| \frac{d}{dt} \int_{\mathbb{R}^d} \chi_R d\mu_t \right| dt = \int_0^T \left| \int_{B_{3R} \setminus B_R} \mathbf{b}_t \cdot \nabla \chi_R d\mu_t \right| dt \\ &\leq \frac{1}{R} \int_0^T \int_{B_{3R} \setminus B_R} |\mathbf{b}_t| d\mu_t dt \leq 4 \int_0^T \int_{B_{3R} \setminus B_R} \frac{|\mathbf{b}_t(x)|}{1+|x|} d\mu_t(x) dt, \end{aligned}$$

which gives

$$\mu_0(B_R) - \mu_t(B_{3R}) \leq \int_{\mathbb{R}^d} \chi_R d\mu_0 - \int_{\mathbb{R}^d} \chi_R d\mu_t \leq 4 \int_0^T \int_{B_{3R} \setminus B_R} \frac{|\mathbf{b}_t(x)|}{1+|x|} d\mu_t(x) dt \quad (2.3)$$

and

$$\mu_t(B_R) - \mu_0(B_{3R}) \leq \int_{\mathbb{R}^d} \chi_R d\mu_t - \int_{\mathbb{R}^d} \chi_R d\mu_0 \leq 4 \int_0^T \int_{B_{3R} \setminus B_R} \frac{|\mathbf{b}_t(x)|}{1+|x|} d\mu_t(x) dt. \quad (2.4)$$

Letting  $R \rightarrow \infty$  in (2.3) and (2.4), the right-hand sides converge to 0 by (2.1) and we find (2.2).

Thanks to (2.2), the first part of the theorem follows directly from [5, Theorem 12], while the last statement is obtained by restricting  $\boldsymbol{\eta}$  to the class of curves contained in  $\overline{B}_R$  for all  $t \in [0, T]$  to obtain positive finite measures  $\boldsymbol{\eta}^R \leq \boldsymbol{\eta}$  which satisfy  $\boldsymbol{\eta}^R \uparrow \boldsymbol{\eta}$ , and then defining  $\mu_t^R := (e_t)_\# \boldsymbol{\eta}^R$ .

□

### 3 Integrability and uniqueness of bounded solutions of the continuity equation

Given a closed interval  $I \subset \mathbb{R}$  and an open set  $\Omega \subset \mathbb{R}^d$ , let us define the class  $\mathcal{L}_{I,\Omega}$  of all nonnegative functions which are essentially bounded, nonnegative, and compactly supported in  $\Omega$ :

$$\mathcal{L}_{I,\Omega} := L^\infty(I; L^{\infty}_+(\Omega)) \cap \{w : \text{supp } w \text{ is a compact subset of } I \times \Omega\}. \quad (3.1)$$

We say that  $\rho \in \mathcal{L}_{I,\Omega}$  is weakly\* continuous if there is a representative  $\rho_t$  with  $t \mapsto \rho_t$  continuous in  $I$  w.r.t. the weak\* topology of  $L^\infty(\Omega)$ . Notice that, in the class  $\mathcal{L}_{I,\Omega}$ , weak\* continuity of  $\rho$  is equivalent to the narrow continuity of the corresponding measures  $\mu_t := \rho_t \mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$ .

For  $T \in (0, \infty)$  we are given a Borel vector field  $\mathbf{b} : (0, T) \times \Omega \rightarrow \mathbb{R}^d$  satisfying:

(a- $\Omega$ )  $\int_0^T \int_{\Omega'} |\mathbf{b}(t, x)| \, dx dt < \infty$  for any  $\Omega' \Subset \Omega$ ;

(b- $\Omega$ ) for any nonnegative  $\bar{\rho} \in L^{\infty}_+(\Omega)$  with compact support in  $\Omega$  and any closed interval  $I = [a, b] \subset [0, T]$ , the continuity equation

$$\frac{d}{dt} \rho_t + \text{div}(\mathbf{b} \rho_t) = 0 \quad \text{in } (a, b) \times \Omega \quad (3.2)$$

has at most one weakly\* continuous solution  $I \ni t \mapsto \rho_t \in \mathcal{L}_{I,\Omega}$  with  $\rho_a = \bar{\rho}$ .

**Remark 3.1.** Assumption (b- $\Omega$ ) is known to be true in many cases. The following list does not pretend to be exhaustive:

- Sobolev vector fields [21],  $BV$  vector fields whose divergence is a locally integrable function in space [13, 2, 16, 17], some classes of vector fields of bounded deformation [6];
  - vector fields  $\mathbf{B}(x, y) = (\mathbf{b}_1(x, y), \mathbf{b}_2(x, y))$  with different regularity w.r.t.  $x$  and  $y$  [22, 23];
  - two-dimensional Hamiltonian vector fields [1] (within this class, property (b- $\Omega$ ) has been characterized in terms of the so-called weak Sard property);
  - vector fields arising from the convolution of  $L^1$  functions with singular integrals [14, 15].
- In this case, the authors proved uniqueness of the regular lagrangian flow associated to  $\mathbf{b}$ ; we outline in the next remark how to obtain the eulerian uniqueness property (b- $\Omega$ ) following their argument.

**Remark 3.2.** Under the assumptions on the vector field  $\mathbf{b}$  considered in [15], the authors proved in [15, Theorem 6.2] the uniqueness of the lagrangian flow. In their key estimate, the authors take two regular lagrangian flows  $\mathbf{X}$  and  $\mathbf{Y}$ , provide an upper and lower bound for the quantity

$$\Phi_\delta(t) := \int \log \left( 1 + \frac{|\mathbf{X}(t, x) - \mathbf{Y}(t, x)|}{\delta} \right) dx \quad t \in [0, T] \quad (3.3)$$

in terms of a parameter  $\delta > 0$ , and eventually let  $\delta \rightarrow 0$ . To show that property (b- $\Omega$ ) holds, we consider two nonnegative bounded solutions of the continuity equation with the same initial datum which are compactly supported in  $[a, b] \times \Omega$ . By Theorem 2.1 there

exist  $\boldsymbol{\eta}^1, \boldsymbol{\eta}^2 \in \mathcal{P}(C([a, b]; \mathbb{R}^d))$  which are concentrated on absolutely continuous solutions  $\eta \in AC([a, b]; \Omega)$  of the ODE  $\dot{\eta} = \mathbf{b}(t, \eta)$   $\mathcal{L}^1$ -a.e. in  $(a, b)$ , and satisfy  $(e_t)_\# \boldsymbol{\eta}^i \leq C \mathcal{L}^d$  for any  $t \in [a, b]$ ,  $i = 1, 2$ . Moreover, we have that  $(e_a)_\# \boldsymbol{\eta}^1 = (e_a)_\# \boldsymbol{\eta}^2$ . Given  $\delta > 0$ , we consider the quantity

$$\Psi_\delta(t) := \int_\Omega \int \int \log \left( 1 + \frac{|\gamma(t) - \eta(t)|}{\delta} \right) d\boldsymbol{\eta}_x^1(\gamma) d\boldsymbol{\eta}_x^2(\eta) d[(e_0)_\# \boldsymbol{\eta}^1](x) \quad t \in [a, b], \quad (3.4)$$

where  $\boldsymbol{\eta}_x^1, \boldsymbol{\eta}_x^2$  are the conditional measures of  $\boldsymbol{\eta}^1$  and  $\boldsymbol{\eta}^2$  with respect to the map  $e_a$ . Since  $\boldsymbol{\eta}^1$  and  $\boldsymbol{\eta}^2$  are concentrated on curves in  $C([a, b]; \Omega)$ , to show that  $\boldsymbol{\eta}^1 = \boldsymbol{\eta}^2$  we can neglect the behavior of  $\mathbf{b}$  outside  $\Omega$ . Following the same computations of [15] with the functional (3.4) instead of (3.3), we show that  $\boldsymbol{\eta}_x^1 = \boldsymbol{\eta}_x^2$  for  $(e_a)_\# \boldsymbol{\eta}^1$ -a.e.  $x \in \Omega$  and this implies the validity of property (b- $\Omega$ ).

More recently, these well-posedness results have also been extended to vector fields in infinite-dimensional spaces (see [8] and the bibliography therein). It is interesting to observe that the uniqueness assumption in (b- $\Omega$ ) actually implies the validity of a comparison principle.

**Proposition 3.3** (Comparison principle). *If (a- $\Omega$ ) and (b- $\Omega$ ) are satisfied, then the following implication holds:*

$$\rho_0^1 \leq \rho_0^2 \quad \implies \quad \rho_t^1 \leq \rho_t^2 \quad \forall t \in [0, T]$$

for all weakly\* continuous solutions of (3.2) in the class  $\mathcal{L}_{[0, T], \Omega}$ .

*Proof.* Let  $\boldsymbol{\eta}^i$  be representing  $\mu_t^i := \rho_t^i \mathcal{L}^d$  according to Theorem 2.1, and let  $\boldsymbol{\eta}_x^i$  be the conditional probability measures induced by  $e_0$ , that is

$$\int F(\eta) d\boldsymbol{\eta}^i = \int_{\mathbb{R}^d} \left( \int F(\eta) d\boldsymbol{\eta}_x^i \right) d\mu_0^i(x) \quad \forall F : C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R} \text{ bounded,}$$

or (in a compact form)  $\boldsymbol{\eta}^i(d\eta) = \int \boldsymbol{\eta}_x^i(d\eta) d\mu_0^i(x)$ . Defining

$$\tilde{\boldsymbol{\eta}}(d\eta) := \int \boldsymbol{\eta}_x^2 d\mu_0^1(x), \quad \tilde{\mu}_t := (e_t)_\# \tilde{\boldsymbol{\eta}},$$

because  $\mu_0^1 \leq \mu_0^2$ , we get  $\tilde{\boldsymbol{\eta}} \leq \boldsymbol{\eta}^2$ . Moreover, the densities of measures  $\tilde{\mu}_t$  and  $\mu_t^1$  provide two elements in  $\mathcal{L}_{[0, T], \Omega}$ , solving the continuity equation with the same initial condition  $\mu_0^1$ . Therefore assumption (b- $\Omega$ ) gives  $\tilde{\mu}_t = \mu_t^1$  for all  $t \in [0, T]$ , and  $\mu_t^1 = \tilde{\mu}_t = (e_t)_\# \tilde{\boldsymbol{\eta}} \leq (e_t)_\# \boldsymbol{\eta}^2 = \mu_t^2$  for all  $t \in [0, T]$ , as desired.  $\square$

**Theorem 3.4.** *Assume that  $\mathbf{b}$  satisfies (a- $\Omega$ ) and (b- $\Omega$ ), and let  $\boldsymbol{\lambda} \in \mathcal{P}(C([0, T]; \mathbb{R}^d))$  satisfy:*

(i)  $\boldsymbol{\lambda}$  is concentrated on

$$\{\eta \in AC([0, T]; \Omega) : \dot{\eta}(t) = \mathbf{b}(t, \eta(t)) \text{ for } \mathcal{L}^1\text{-a.e. } t \in (0, T)\};$$

(ii) there exists  $C_0 \in (0, \infty)$  such that

$$(e_t)_\# \lambda \leq C_0 \mathcal{L}^d \quad \forall t \in [0, T]. \quad (3.5)$$

Then the conditional probability measures  $\lambda_x$  induced by the map  $e_0$  are Dirac masses for  $(e_0)_\# \lambda$ -a.e.  $x$ ; equivalently, there exist curves  $\eta_x \in AC([0, T]; \Omega)$  solving the Cauchy problem  $\dot{\eta} = \mathbf{b}(t, \eta)$  with the initial condition  $\eta(0) = x$ , satisfying

$$\lambda = \int \delta_{\eta_x} d[(e_0)_\# \lambda](x).$$

*Proof.* Let  $\{A_n\}_{n \in \mathbb{N}}$  be an increasing family of open subsets of  $\Omega$  whose union is  $\Omega$ , with  $A_n \subseteq A_{n+1} \subseteq \Omega$  for every  $n$ . Possibly considering the restriction of  $\boldsymbol{\eta}$  to the sets

$$\{\eta \in C([0, T]; \mathbb{R}^d) : \eta(t) \in \overline{A_n} \text{ for every } t \in [0, T]\}$$

it is not restrictive to assume that  $\boldsymbol{\eta}$  is concentrated on a family  $\Gamma$  of curves satisfying  $\bigcup_{\eta \in \Gamma} \eta([0, T]) \subseteq \Omega$ . Then, using the uniqueness assumption for uniformly bounded and compactly supported solutions to the continuity equation, the result follows from the decomposition procedure of [5, Theorem 18] (notice that the latter slightly improves the original argument of [2, Theorem 5.4], where comparison principle for the continuity equation was assumed, see also Proposition 3.3 and its proof).  $\square$

**Remark 3.5.** The assumption (b- $\Omega$ ) is purely local, as it is proved in the Appendix. Moreover, it could be reformulated in terms of a local uniqueness property of regular lagrangian flows: for any  $t_0 \geq 0, x_0 \in \Omega$  there exists  $\varepsilon := \varepsilon(t_0, x_0) > 0$  such that for any Borel set  $B \subset B_\varepsilon(x_0) \subset \Omega$  and any closed interval  $I = [a, b] \subset [t_0 - \varepsilon, t_0 + \varepsilon] \cap [0, T]$ , there exists at most one regular lagrangian flow in  $B \times [a, b]$  with values in  $B_\varepsilon(x_0)$  (see Definition 4.1). Indeed, (b- $\Omega$ ) implies the local uniqueness of regular lagrangian flows by Theorem 3.4 applied to  $\lambda = \frac{1}{2} \int_B (\delta_{\mathbf{X}(\cdot, x)} + \delta_{\mathbf{Y}(\cdot, x)}) d\mathcal{L}^d(x)$ , where  $\mathbf{X}$  and  $\mathbf{Y}$  are regular lagrangian flows in  $B \times [a, b]$ ; on the other hand, we obtain the converse implication through the superposition principle. This approach has the advantage to state the assumptions and the results of this paper only in terms of the lagrangian point of view on the continuity equation. On the other hand, in concrete examples it is usually easier to verify assumption (b- $\Omega$ ) than the corresponding lagrangian formulation.

## 4 Regular flow, hitting time, maximal flow

**Definition 4.1** (Local regular flow). *Let  $B \in \mathcal{B}(\mathbb{R}^d)$ ,  $\tau > 0$ , and  $\mathbf{b} : (0, \tau) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  Borel. We say that  $\mathbf{X} : [0, \tau] \times B \rightarrow \mathbb{R}^d$  is a local regular flow starting from  $B$  (relative to  $\mathbf{b}$ ) up to  $\tau$  if the following two properties hold:*

(i) for  $\mathcal{L}^d$ -a.e.  $x \in B$ ,  $\mathbf{X}(\cdot, x) \in AC([0, \tau]; \mathbb{R}^d)$  and solves the ODE  $\dot{x}(t) = \mathbf{b}(t, x(t))$   $\mathcal{L}^1$ -a.e. in  $(0, \tau)$ , with the initial condition  $\mathbf{X}(0, x) = x$ ;

(ii) there exists a constant  $C = C(\mathbf{X})$  satisfying  $\mathbf{X}(t, \cdot)_\# (\mathcal{L}^d \llcorner B) \leq C \mathcal{L}^d$ .



In the previous definition, as long as the image of  $[0, \tau] \times B$  through  $\mathbf{X}$  is contained in an open set  $\Omega$ , it is not necessary to specify the vector field  $\mathbf{b}$  outside  $\Omega$ ; the flow is called “regular” because of the bounded compression condition (ii). By Theorem 3.4 we obtain a consistency result of the local regular flows with values in  $\Omega$  in the intersection of their domains.

**Lemma 4.2** (Consistency of local regular flows). *Assume that  $\mathbf{b}$  satisfies (a- $\Omega$ ) and (b- $\Omega$ ). Let  $\mathbf{X}_i$  be local regular flows starting from  $B_i$  up to  $\tau_i$ ,  $i = 1, 2$ , with  $\mathbf{X}_i([0, \tau_i] \times B_i) \subset \Omega$ . Then*

$$\mathbf{X}_1(\cdot, x) \equiv \mathbf{X}_2(\cdot, x) \quad \text{in } [0, \tau_1 \wedge \tau_2], \text{ for } \mathcal{L}^d\text{-a.e. } x \in B_1 \cap B_2. \quad (4.1)$$

*Proof.* Take  $B \subset B_1 \cap B_2$  Borel with  $\mathcal{L}^d(B)$  finite, and apply Theorem 3.4 with  $T = \tau_1 \wedge \tau_2$  and

$$\lambda := \frac{1}{2} \int (\delta_{\mathbf{X}_1(\cdot, x)} + \delta_{\mathbf{X}_2(\cdot, x)}) d\mathcal{L}_B^d(x),$$

where  $\mathcal{L}_B^d$  is the normalized Lebesgue measure on  $B$ . □

If we consider a smooth vector field  $\mathbf{b}$  in a domain  $\Omega$ , a maximal flow of  $\mathbf{b}$  in  $\Omega$  would be given by the trajectories of  $\mathbf{b}$  until they hit the boundary of  $\Omega$ . In order to deal at the same time with bounded and unbounded domains (including the case  $\Omega = \mathbb{R}^d$ ) we introduce a continuous potential function  $V_\Omega : \Omega \rightarrow [0, \infty)$  satisfying

$$\lim_{x \rightarrow \partial\Omega} V_\Omega(x) = \infty, \quad (4.2)$$

meaning that for any  $M > 0$  there exists  $K \Subset \Omega$  with  $V_\Omega > M$  on  $\Omega \setminus K$  (in particular, when  $\Omega = \mathbb{R}^d$ ,  $V_\Omega(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ ). For instance, an admissible potential is given by  $V_\Omega(x) = \max\{[\text{dist}(x, \mathbb{R}^d \setminus \Omega)]^{-1}, |x|\}$ .

**Definition 4.3** (Hitting time in  $\Omega$ ). *Let  $\tau > 0$ ,  $\Omega \subset \mathbb{R}^d$  open, and  $\eta : [0, \tau) \rightarrow \mathbb{R}^d$  continuous. We define the hitting time of  $\eta$  in  $\Omega$  as*

$$h_\Omega(\eta) := \sup\{t \in [0, \tau) : \max_{[0, t]} V_\Omega(\eta) < \infty\},$$

*with the convention  $h_\Omega(\eta) = 0$  if  $\eta(0) \notin \Omega$ .*

It is easily seen that this definition is independent of the choice of  $V_\Omega$ , that  $h_\Omega(\eta) > 0$  whenever  $\eta(0) \in \Omega$ , and that

$$h_\Omega(\eta) < \tau \implies \limsup_{t \uparrow h_\Omega(\eta)} V_\Omega(\eta(t)) = \infty. \quad (4.3)$$

Using  $V_\Omega$  we can also define the concept of maximal regular flow.

**Definition 4.4** (Maximal regular flow in an open set  $\Omega$ ). *Let  $\mathbf{b} : (0, T) \times \Omega \rightarrow \mathbb{R}^d$  be a Borel vector field. We say that a Borel map  $\mathbf{X}$  is a maximal regular flow relative to  $\mathbf{b}$  in  $\Omega$  if there exists a Borel map  $T_{\Omega, \mathbf{X}} : \Omega \rightarrow (0, T]$  such that  $\mathbf{X}(t, x)$  is defined in the set  $\{(t, x) : t < T_{\Omega, \mathbf{X}}(x)\}$  and the following properties hold:*

(i) for  $\mathcal{L}^d$ -a.e.  $x \in \Omega$ ,  $\mathbf{X}(\cdot, x) \in AC_{\text{loc}}([0, T_{\Omega, \mathbf{X}}(x)]; \mathbb{R}^d)$ , and solves the ODE  $\dot{x}(t) = \mathbf{b}(t, x(t))$   $\mathcal{L}^1$ -a.e. in  $(0, T_{\Omega, \mathbf{X}}(x))$ , with the initial condition  $\mathbf{X}(0, x) = x$ ;

(ii) for any  $\Omega' \Subset \Omega$  there exists a constant  $C(\Omega', \mathbf{X})$  such that

$$\mathbf{X}(t, \cdot)_{\#}(\mathcal{L}^d \llcorner \{T_{\Omega'} > t\}) \leq C(\Omega', \mathbf{X}) \mathcal{L}^d \llcorner \Omega' \quad \forall t \in [0, T], \quad (4.4)$$

where

$$T_{\Omega'}(x) := \begin{cases} h_{\Omega'}(\mathbf{X}(\cdot, x)) & \text{for } x \in \Omega', \\ 0 & \text{otherwise;} \end{cases}$$

(iii)  $\limsup_{t \uparrow T_{\Omega, \mathbf{X}}(x)} V_{\Omega}(\mathbf{X}(t, x)) = \infty$  for  $\mathcal{L}^d$ -a.e.  $x \in \Omega$  such that  $T_{\Omega, \mathbf{X}}(x) < T$ .

In the previous definition, “regular” refers to the local bounded compression condition in (ii) and “maximal” refers to (iii). For  $\mathcal{L}^d$ -a.e.  $x \in \Omega$ , the function  $T_{\Omega, \mathbf{X}}(x)$  represents the *maximal existence time* (often called *blow-up time* in the following) of the trajectory  $\mathbf{X}(\cdot, x)$ . Notice that (4.4) could be equivalently written as

$$\mathbf{X}(t, \cdot)_{\#}(\mathcal{L}^d \llcorner \{T_{\Omega'} > t\}) \leq C(\Omega', \mathbf{X}) \mathcal{L}^d \quad \forall t \in [0, T],$$

because the push-forward measure is concentrated on  $\Omega'$ ; so the real meaning of this requirement is that the push forward measure must have a bounded density w.r.t.  $\mathcal{L}^d$ , at least as long as the trajectories remain inside  $\Omega'$ .

**Remark 4.5** (Maximal regular flows induce regular flows). Given any maximal regular flow  $\mathbf{X}$  in  $\Omega$ ,  $\tau \in (0, T)$ , and a Borel set  $B \subset \Omega$  such that  $T_{\Omega, \mathbf{X}} > \tau$  on  $B$  and

$$\{\mathbf{X}(t, x) : x \in B, t \in [0, \tau]\} \Subset \Omega,$$

we have an induced local regular flow in the set  $B$  up to time  $\tau$ .

**Remark 4.6** (Invariance in the equivalence class of  $\mathbf{b}$ ). It is important and technically useful (see for instance [3]) to underline that the concepts of local regular flow and of maximal regular flow are invariant in the Lebesgue equivalent class, exactly as our constitutive assumptions (a- $\Omega$ ), (b- $\Omega$ ), and the global/local bounds on the divergence of  $\mathbf{b}$ . Indeed, for local regular flows, Definition 4.1(ii) in conjunction with Fubini’s theorem implies that for any  $\mathcal{L}^{1+d}$ -negligible set  $N \subset (0, T) \times \mathbb{R}^d$  the set

$$\{x \in B : \mathcal{L}^1(\{t \in (0, \tau) : (t, \mathbf{X}(t, x)) \in N\}) > 0\}$$

is  $\mathcal{L}^d$ -negligible. An analogous argument, based on (4.4), applies to maximal regular flows.

## 5 Existence and uniqueness of the maximal regular flow

In this section we consider a Borel vector field  $\mathbf{b} : (0, T) \times \Omega \rightarrow \mathbb{R}^d$  which satisfies the assumptions (a- $\Omega$ ), (b- $\Omega$ ) of Section 3, and such that the spatial divergence  $\text{div } \mathbf{b}(t, \cdot)$  in the sense of distributions satisfies

$$\forall \Omega' \Subset \Omega, \quad \text{div } \mathbf{b}(t, \cdot) \geq m(t) \quad \text{in } \Omega', \quad \text{with } L(\Omega', \mathbf{b}) := \int_0^T |m(t)| dt < \infty. \quad (5.1)$$

**Remark 5.1.** Assumption (5.1) could be weakened to  $m \in L^1(0, T_0)$  for all  $T_0 \in (0, T)$ , but we made it global in time to avoid time-dependent constants in our estimates (and, in any case, the maximal flow could be obtained in this latter case by gluing together the maximal flows in  $(0, T_0)$  for  $T_0 \uparrow T$ ).

The first step in the construction of the maximal regular flow will be the following local existence result.

**Theorem 5.2** (Local existence). *Let  $\mathbf{b} : (0, T) \times \Omega \rightarrow \mathbb{R}^d$  be a Borel vector field which satisfies (a- $\Omega$ ), (b- $\Omega$ ), (5.1), and let  $A \Subset \Omega$  be open. Then there exist a Borel map  $T_A : A \rightarrow (0, T]$  and a Borel map  $\mathbf{X}(t, x)$ , defined for  $x \in A$  and  $t \in [0, T_A(x)]$ , such that:*

- (a) *for  $\mathcal{L}^d$ -a.e.  $x \in A$ ,  $\mathbf{X}(\cdot, x) \in AC([0, T_A(x)]; \mathbb{R}^d)$ ,  $\mathbf{X}(0, x) = x$ ,  $\mathbf{X}(t, x) \in A$  for all  $t \in [0, T_A(x))$ , and  $\mathbf{X}(T_A(x), x) \in \partial A$  when  $T_A(x) < T$ ;*
- (b) *for  $\mathcal{L}^d$ -a.e.  $x \in A$ ,  $\mathbf{X}(\cdot, x)$  solves the ODE  $\dot{\gamma} = \mathbf{b}(t, \gamma)$   $\mathcal{L}^1$ -a.e. in  $(0, T_A(x))$ ;*
- (c)  *$\mathbf{X}(t, \cdot)_{\#}(\mathcal{L}^d \llcorner \{T_A > t\}) \leq e^{L(A, \mathbf{b})} \mathcal{L}^d \llcorner A$  for all  $t \in [0, T]$ , where  $L(A, \mathbf{b})$  is the constant in (5.1).*

Notice that since the statement of the theorem is local (see also the appendix, in connection with property (b- $\Omega$ )), we need only to prove it under the assumption  $|\mathbf{b}| \in L^1((0, T) \times \Omega)$ , which is stronger than (a- $\Omega$ ).

We will obtain Theorem 5.2 via an approximation procedure which involves the concept of regular generalized flow in closed domains, where now “regular” refers to the fact that the bounded compression condition is imposed only in the interior of the domain.

**Definition 5.3** (Regular generalized flow in  $\bar{A}$ ). *Let  $A \subset \mathbb{R}^d$  be an open set and let  $\mathbf{c} : (0, T) \times \bar{A} \rightarrow \mathbb{R}^d$  be a Borel vector field. A probability measure  $\boldsymbol{\eta}$  in  $C([0, T]; \mathbb{R}^d)$  is said to be a regular generalized flow on  $\bar{A}$  if the following two conditions hold:*

- (i)  *$\boldsymbol{\eta}$  is concentrated on*

$$\{\eta \in AC([0, T]; \bar{A}) : \dot{\eta}(t) = \mathbf{c}(t, \eta(t)) \text{ for } \mathcal{L}^1\text{-a.e. } t \in (0, T)\};$$

- (ii) *there exists a constant  $C = C(\boldsymbol{\eta}) \in (0, \infty)$  satisfying*

$$((e_t)_{\#} \boldsymbol{\eta}) \llcorner A \leq C \mathcal{L}^d \quad \forall t \in [0, T]. \quad (5.2)$$

Any constant  $C$  for which (5.2) holds is called a *compressibility constant* of  $\boldsymbol{\eta}$ .

The class of regular generalized flows enjoys good tightness and stability properties. We recall that a sequence  $\boldsymbol{\eta}^n \in \mathcal{P}(C([0, T]; \bar{A}))$  is said *tight* if for every  $\varepsilon > 0$  there exists a compact set  $\Gamma_\varepsilon \subseteq C([0, T]; \bar{A})$  such that  $\boldsymbol{\eta}^n(C([0, T]; \bar{A}) \setminus \Gamma_\varepsilon) \leq \varepsilon$  for every  $n \in \mathbb{N}$ . Equivalently, the sequence  $\boldsymbol{\eta}^n$  is said to be tight if there exists a coercive, lower semicontinuous functional  $\Sigma : C([0, T]; \bar{A}) \rightarrow [0, \infty]$  such that  $\sup_{n \in \mathbb{N}} \int \Sigma d\boldsymbol{\eta}^n < \infty$ .

We state the tightness and stability properties in the case of interest for us, namely when the velocity vanishes at the boundary.

**Theorem 5.4** (Tightness and stability of regular generalized flows in  $\overline{A}$ ). *Let  $A \subset \mathbb{R}^d$  be a bounded open set, let  $\mathbf{c}, \mathbf{c}^n : (0, T) \times \overline{A} \rightarrow \mathbb{R}^d$  be Borel vector fields such that  $\mathbf{c} = \mathbf{c}^n = 0$  on  $(0, T) \times \partial A$  and*

$$\lim_{n \rightarrow \infty} \mathbf{c}^n = \mathbf{c} \quad \text{in } L^1((0, T) \times A; \mathbb{R}^d). \quad (5.3)$$

*Let  $\boldsymbol{\eta}^n \in \mathcal{P}(C([0, T]; \overline{A}))$  be regular generalized flows of  $\mathbf{c}^n$  in  $\overline{A}$  and let us assume that the best compressibility constants  $C_n$  of  $\boldsymbol{\eta}^n$  satisfy  $\sup_n C_n < \infty$ . Then  $(\boldsymbol{\eta}^n)$  is tight, any limit point  $\boldsymbol{\eta}$  is a regular generalized flow of  $\mathbf{c}$  in  $\overline{A}$ , and the following implication holds:*

$$((e_t)_\#(\boldsymbol{\eta}^n \llcorner \Gamma)) \llcorner A' \leq c_n \mathcal{L}^d \text{ for some } c_n > 0 \implies ((e_t)_\#(\boldsymbol{\eta} \llcorner \Gamma)) \llcorner A' \leq (\liminf_n c_n) \mathcal{L}^d \quad (5.4)$$

*for any choice of open sets  $\Gamma \subset C([0, T]; \overline{A})$  and  $A' \subset A$ .*

In the previous theorem the assumption that all the vector fields vanish on the boundary of  $A$  allows us to say the following: if an integral curve of  $\mathbf{c}^n$  in  $A$  hits  $\partial A$  and stops there, then it is still an integral curve of  $\mathbf{c}^n$  on the whole  $\overline{A}$ . We remark that the previous theorem is invariant if the vector fields  $\mathbf{c}^n$  are modified on a set of Lebesgue measure zero in  $(0, T) \times A$ , thanks to the compressibility condition (5.2) required in  $A$ ; on the contrary, the value of  $\mathbf{c}^n$  on  $\partial A$  has to be understood in a pointwise sense.

*Proof of Theorem 5.4.* By Dunford-Pettis' theorem, since the family  $\{\mathbf{c}^n\}$  is compact in  $L^1(\overline{A}; \mathbb{R}^d)$  (recall that  $\mathbf{c}_n(t, \cdot)$  vanish outside of  $A$ ), there exists a modulus of integrability for  $\mathbf{c}^n$ , namely an increasing, convex, superlinear function  $F : [0, \infty) \rightarrow [0, \infty)$  such that  $F(0) = 0$  and

$$\sup_{n \in \mathbb{N}} \int_0^T \int_A F(|\mathbf{c}^n(t, x)|) dx dt < \infty. \quad (5.5)$$

Let us introduce the functional  $\Sigma : C([0, T]; \mathbb{R}^d) \rightarrow [0, \infty]$  as follows:

$$\Sigma(\eta) := \begin{cases} \int_0^T F(|\dot{\eta}(t)|) dt & \text{if } \eta \in AC([0, T]; \overline{A}), \\ \infty & \text{if } \eta \in C([0, T]; \mathbb{R}^d) \setminus AC([0, T]; \overline{A}). \end{cases}$$

Using Ascoli-Arzelà theorem, the compactness of  $\overline{A}$ , and a well-known lower semicontinuity result due to Ioffe (see for instance [9, Theorem 5.8]), it turns out that  $\Sigma$  is lower semicontinuous and coercive, namely its sublevels  $\{\Sigma \leq M\}$  are compact.

Since  $\boldsymbol{\eta}^n$  is concentrated on  $AC([0, T]; \overline{A})$  we get

$$\int \Sigma d\boldsymbol{\eta}^n = \int \int_0^T F(|\dot{\eta}|) dt d\boldsymbol{\eta}^n(\eta) = \int_0^T \int_A F(|\mathbf{c}^n|) d[(e_t)_\# \boldsymbol{\eta}^n] dt \leq C_n \int_0^T \int_A F(|\mathbf{c}^n|) dx dt,$$

so that that  $\int \Sigma d\boldsymbol{\eta}^n$  is uniformly bounded thanks to (5.5) and the assumption  $\sup_n C_n < \infty$ . Therefore, Prokhorov compactness theorem provides the existence of limit points. Since  $\Sigma$  is lower semicontinuous we obtain that any limit point  $\boldsymbol{\eta}$  satisfies  $\int \Sigma d\boldsymbol{\eta} < \infty$ , therefore  $\boldsymbol{\eta}$  is concentrated on  $AC([0, T]; \overline{A})$ .

Let  $C := \liminf_{n \in \mathbb{N}} C_n < \infty$ . Since  $(e_t)_\# \boldsymbol{\eta}^n$  narrowly converge to  $(e_t)_\# \boldsymbol{\eta}$ , we know that for any open set  $A' \subset A$  there holds

$$(e_t)_\# \boldsymbol{\eta}(A') \leq \liminf_{n \rightarrow \infty} (e_t)_\# \boldsymbol{\eta}^n(A') \leq C \mathcal{L}^d(A') \quad \forall t \in [0, T].$$

Since  $A'$  is arbitrary we deduce that  $\boldsymbol{\eta}$  satisfies (5.2). A similar argument provides its localized version (5.4). To show that  $\boldsymbol{\eta}$  is concentrated on integral curves of  $\mathbf{c}$ , it suffices to show that

$$\int \left| \eta(t) - \eta(0) - \int_0^t \mathbf{c}(s, \eta(s)) ds \right| d\boldsymbol{\eta}(\eta) = 0 \quad (5.6)$$

for any  $t \in [0, T]$ . The technical difficulty is that this test function, due to the lack of regularity of  $\mathbf{c}$ , is not continuous. To this aim, we prove that

$$\int \left| \eta(t) - \eta(0) - \int_0^t \mathbf{c}'(s, \eta(s)) ds \right| d\boldsymbol{\eta}(\eta) \leq C \int_{(0, T) \times A} |\mathbf{c} - \mathbf{c}'| dx dt \quad (5.7)$$

for any continuous vector field  $\mathbf{c}' : [0, T] \times \overline{A} \rightarrow \mathbb{R}^d$  with  $\mathbf{c}' = 0$  in  $[0, T] \times \partial A$ . Then, choosing a sequence  $(\mathbf{c}'_n)$  of such vector fields converging to  $\mathbf{c}$  in  $L^1(\overline{A}; \mathbb{R}^d)$  and noticing that

$$\int \int_0^T |\mathbf{c}(s, \eta(s)) - \mathbf{c}'_n(s, \eta(s))| ds d\boldsymbol{\eta}(\eta) = \int_0^T \int_A |\mathbf{c} - \mathbf{c}'_n| d(e_s)_\# \boldsymbol{\eta} ds \leq C_n \int_{(0, T) \times A} |\mathbf{c} - \mathbf{c}'_n| dx dt,$$

converges to 0 as  $n \rightarrow \infty$ , we can take the limit in (5.7) with  $\mathbf{c}' = \mathbf{c}'_n$  to obtain (5.6).

It remains to show (5.7). This is a limiting argument based on the fact that (5.6) holds for  $\mathbf{c}^n, \boldsymbol{\eta}^n$ :

$$\begin{aligned} \int \left| \eta(t) - \eta(0) - \int_0^t \mathbf{c}'(s, \eta(s)) ds \right| d\boldsymbol{\eta}^n(\eta) &= \int \left| \int_0^t (\mathbf{c}^n(s, \eta(s)) - \mathbf{c}'(s, \eta(s))) ds \right| d\boldsymbol{\eta}^n(\eta) \\ &\leq \int \int_0^t |\mathbf{c}^n - \mathbf{c}'|(s, \eta(s)) ds d\boldsymbol{\eta}^n(\eta) \\ &= \int_0^t \int_A |\mathbf{c}^n - \mathbf{c}'| d[(e_s)_\# \boldsymbol{\eta}^n] ds \\ &\leq C_n \int_0^t \int_A |\mathbf{c}^n - \mathbf{c}'| dx ds. \end{aligned}$$

Taking the limit in the chain of inequalities above we obtain (5.7).  $\square$

Now we show how Theorem 5.2 can be deduced from the existence of regular generalized flows in  $\overline{A}$ ; at the same time, we show that flows associated to sufficiently smooth vector fields induce regular generalized flows (actually even classical ones, but we will need them in generalized form to take limits).

**Proposition 5.5.** (i) Let  $\mathbf{b} : (0, T) \times \Omega \rightarrow \mathbb{R}^d$  be a Borel vector field which satisfies (a- $\Omega$ ) and (b- $\Omega$ ), let  $A \Subset \Omega$  be an open set, and let  $\boldsymbol{\eta}$  be a regular generalized flow in  $\overline{A}$  relative

to  $\mathbf{c} = \chi_A \mathbf{b}$  with compressibility constant  $C$  and that satisfies  $(e_0)_\# \boldsymbol{\eta} = \rho_0 \mathcal{L}^d$  with  $\rho_0 > 0$   $\mathcal{L}^d$ -a.e. in  $A$ . Then there exist  $\mathbf{X}$  and  $T_A$  as in Theorem 5.2(a)-(b) that satisfy

$$\mathbf{X}(t, \cdot)_\# (\rho_0 \mathbf{L}\{T_A > t\}) \leq C \mathcal{L}^d \llcorner A \quad (5.8)$$

for all  $t \in [0, T]$ .

(ii) Let  $\mathbf{b} \in C^\infty([0, T] \times \bar{A}; \mathbb{R}^d)$ . Then there exists a regular generalized flow  $\boldsymbol{\eta}$  associated to  $\mathbf{b}\chi_A$ , with  $(e_0)_\# \boldsymbol{\eta}$  equal to the normalized Lebesgue measure in  $A$  and satisfying

$$((e_t)_\# \boldsymbol{\eta} \llcorner \{h_{A'}(\cdot) > t\}) \llcorner A' \leq \frac{e^{L(A', \mathbf{b})}}{\mathcal{L}^d(A)} \mathcal{L}^d \quad \forall t \in [0, T] \quad (5.9)$$

for any open set  $A' \Subset A$ .

*Proof.* We first prove (i). Set  $\mu_0 = \rho_0 \mathcal{L}^d$  and consider a family  $\{\boldsymbol{\eta}_x\} \subset \mathcal{P}(C([0, T]; \bar{A}))$  of conditional probability measures, concentrated on

$$\{\boldsymbol{\eta} \in AC([0, T]; \bar{A}) : \dot{\boldsymbol{\eta}} = \mathbf{c}(t, \boldsymbol{\eta}) \text{ } \mathcal{L}^1\text{-a.e. in } (0, T), \boldsymbol{\eta}(0) = x\}$$

and representing  $\boldsymbol{\eta}$ , i.e.,  $\int \boldsymbol{\eta}_x d\mu_0(x) = \boldsymbol{\eta}$ . We claim that  $\mu_0$ -almost every  $x \in A$ :

- (1)  $h_A(\boldsymbol{\eta})$  is equal to a positive constant for  $\boldsymbol{\eta}_x$ -a.e.  $\boldsymbol{\eta}$ ;
- (2) if  $T_A(x)$  is the constant in (1),  $(e_t)_\# \boldsymbol{\eta}_x$  is a Dirac mass for all  $t \in [0, T_A(x)]$ .

By our assumption on  $\mu_0$ , the properties stated in the claim hold  $\mathcal{L}^d$ -a.e. in  $A$ . Hence, given the claim, if we define

$$\mathbf{X}(t, x) := \int \boldsymbol{\eta}(t) d\boldsymbol{\eta}_x(\boldsymbol{\eta})$$

then for  $\mathcal{L}^d$ -a.e.  $x \in A$  the integrand  $\boldsymbol{\eta}(t)$  is independent of  $\boldsymbol{\eta}$  as soon as  $t < T_A(x)$ , hence  $\mathbf{X}(t, x)$  satisfies (a) and (b) in the statement of Theorem 5.2. The compressibility property (5.8) follows immediately from (5.2).

Let us prove our claim. We notice that the hitting time is positive for  $\mu_0$ -a.e.  $x \in A$ . For  $q \in \mathbb{Q} \cap (0, T)$ , we shall denote by  $\Gamma_q$  the set  $\{\boldsymbol{\eta} : h_A(\boldsymbol{\eta}) > q\}$  and by  $\Sigma^q : \Gamma_q \rightarrow C([0, q]; A)$  the map induced by restriction to  $[0, q]$ , namely  $\Sigma^q(\boldsymbol{\eta}) = \boldsymbol{\eta}|_{[0, q]}$ .

In order to prove the claim it clearly suffices to show that, for all  $q \in \mathbb{Q} \cap (0, T)$ ,  $\Sigma^q_\#(\boldsymbol{\eta}_x \llcorner \Gamma_q)$  is either a Dirac mass or it is null. So, for  $q \in \mathbb{Q} \cap (0, T)$  and  $\delta \in (0, 1)$  fixed, it suffices to show that

$$\boldsymbol{\lambda}_x := \frac{1}{\boldsymbol{\eta}_x(\Gamma_q)} \Sigma^q_\#(\boldsymbol{\eta}_x \llcorner \Gamma_q) \in \mathcal{P}(C([0, q]; A))$$

is a Dirac mass for  $\mu_0$ -a.e.  $x$  satisfying  $\boldsymbol{\eta}_x(\Gamma_q) \geq \delta$ .

By construction the measures  $\boldsymbol{\lambda}_x$  satisfy  $\boldsymbol{\lambda}_x \leq \Sigma^q_\#(\boldsymbol{\eta}_x \llcorner \Gamma_q)/\delta$  and they are concentrated on curves  $[0, q] \ni t \mapsto \boldsymbol{\eta}(t)$  starting at  $x$  and solving the ODE  $\dot{\boldsymbol{\eta}} = \mathbf{b}(t, \boldsymbol{\eta})$   $\mathcal{L}^1$ -a.e. in  $(0, q)$ . Therefore

$$\boldsymbol{\lambda} := \int_{\{x \in A : \boldsymbol{\eta}_x(\Gamma_q) \geq \delta\}} \boldsymbol{\lambda}_x d\mu_0(x) \in \mathcal{P}(C([0, q]; A))$$

satisfies all the assumptions of Theorem 3.4 with  $T = q$  and  $\Omega = A$ , provided we check (3.5). To check this property with  $C_0 = C/\delta$ , for  $t \in [0, q]$  and  $\varphi \in C_c(A)$  nonnegative we use the fact that  $\lambda_y \leq \Sigma_{\#}^q(\eta_y \llcorner \Gamma_q)/\delta$  and the fact that  $C$  is a compressibility constant of  $\eta$  to estimate

$$\int_{\mathbb{R}^d} \varphi d(e_t)_{\#} \lambda \leq \frac{1}{\delta} \int_{\mathbb{R}^d} \varphi d(e_t)_{\#} (\eta \llcorner \Gamma_q) \leq \frac{1}{\delta} \int_{\mathbb{R}^d} \varphi d(e_t)_{\#} \eta \leq \frac{C}{\delta} \int_A \varphi dx.$$

Therefore Theorem 3.4 can be invoked:  $\lambda_x$  is a Dirac mass for  $\mu_0$ -a.e.  $x$ , and this gives that  $\eta_x$  is a Dirac mass  $\mu_0$ -a.e. in  $\{x : \eta_x(\Gamma_q) \geq \delta\}$ . This concludes the proof of (i).

For (ii), we begin by defining  $\eta$  with the standard Cauchy-Lipschitz theory. More precisely, for  $x \in A$  we let  $\mathbf{X}(t, x)$  be the unique solution to the ODE  $\dot{\eta} = \mathbf{b}(t, \eta)$  with  $\eta(0) = x$  until the first time  $T_A(x)$  that  $\mathbf{X}(t, x)$  hits  $\partial A$ , and then we define  $\mathbf{X}(t, x) = \mathbf{X}(t, T_A(x))$  for all  $t \in [T_A(x), T]$ . Finally, denoting by  $\mathcal{L}_A^d$  the normalized Lebesgue measure in  $A$ , we define  $\eta$  as the law under  $\mathcal{L}_A^d$  of the map  $x \mapsto \mathbf{X}(\cdot, x)$ . With this construction it is clear that condition (i) in Definition 5.3 holds.

Let us check condition (ii) as well, in the stronger form (5.9). Recall that  $\mathbf{X}$  is smooth before the hitting time and that the map  $t \mapsto J(t) := \det \nabla_x \mathbf{X}(t, x)$  is nonnegative and solves the ODE

$$\begin{cases} \dot{J}(t) = J(t) \operatorname{div} \mathbf{b}(t, \mathbf{X}(t, x)), \\ J(0) = 1. \end{cases} \quad (5.10)$$

Now, fix an open set  $A' \Subset A$ , and observe that (5.9) is equivalent to prove that for every  $t \in [0, T]$

$$\int_{A' \cap \{x : \mathfrak{h}_{A'}(\mathbf{X}(\cdot, x)) > t\}} \varphi(\mathbf{X}(t, x)) dx \leq e^{L(A', \mathbf{b})} \int_{A'} \varphi(x) dx \quad \text{for every } \varphi \in C_c(A').$$

Fix  $\varphi \in C_c(A')$  nonnegative and notice that  $\varphi(\mathbf{X}(t, x)) = 0$  if  $t \geq \mathfrak{h}_{A'}(\mathbf{X}(\cdot, x))$ , hence  $\operatorname{supp} \varphi \circ \mathbf{X}(t, \cdot)$  is a compact subset of the open set  $G_t := \{x : \mathfrak{h}_{A'}(\mathbf{X}(\cdot, x)) > t\}$ . By the change of variables formula

$$\int_{\mathbb{R}^d} \varphi(\mathbf{X}(t, x)) \det \nabla_x \mathbf{X}(t, x) dx = \int_{\mathbb{R}^d} \varphi(x) dx,$$

and in order to estimate from below the left-hand side it suffices to bound from below  $\det \nabla_x \mathbf{X}(t, x)$  in  $G_t$ ; using (5.10) and Gronwall's lemma, this estimate is provided by  $e^{-L(A', \mathbf{b})}$ .  $\square$

**Remark 5.6.** For the proof of Theorem 6.2 we record the following facts, proved but not stated in Proposition 5.5: if  $\eta$  is as in the statement of the proposition, then for  $(e_0)_{\#} \eta$ -a.e.  $x$  the hitting time  $\mathfrak{h}_A(\eta)$  is equal to a positive constant  $T_A(x)$  for  $\eta_x$ -a.e.  $\eta$ ; furthermore,  $(e_t)_{\#} \eta_x$  is a Dirac mass for all  $t \in [0, T_A(x)]$ .

*Proof of Theorem 5.2.* By the first part of Proposition 5.5, it suffices to build a regular generalized flow  $\eta$  in  $\bar{A}$  relative to  $\mathbf{c} = \chi_A \mathbf{b}$  with compressibility constant  $e^{L(A, \mathbf{b})}/\mathcal{L}^d(A)$  such that  $(e_0)_{\#} \eta = \rho_0 \mathcal{L}^d$  with  $\rho_0 > 0$   $\mathcal{L}^d$ -a.e. in  $A$ . By the second part of the proposition,

we have existence of  $\boldsymbol{\eta}$  with  $(e_0)_\# \boldsymbol{\eta}$  equal to the normalized Lebesgue measure  $\mathcal{L}_A^d$  and satisfying (5.9) whenever  $\mathbf{b} \in C^\infty([0, T] \times \overline{A}; \mathbb{R}^d)$ .

Hence, to use this fact, extend  $\mathbf{b}$  to be identically zero on  $\mathbb{R} \times \mathbb{R}^d \setminus [0, T] \times \Omega$ , and let  $\mathbf{b}_\varepsilon$  be mollified vector fields. We have that  $L(A, \mathbf{b}_\varepsilon)$  are uniformly bounded (because  $A \Subset \Omega$ ) and, in addition, the properties of convolution immediately yield

$$\limsup_{\varepsilon \downarrow 0} L(A', \mathbf{b}_\varepsilon) \leq L(A, \mathbf{b}) \quad \text{for any } A' \Subset A \text{ open.} \quad (5.11)$$

If  $\boldsymbol{\eta}_\varepsilon$  are regular generalized flows associated to  $\mathbf{c}_\varepsilon = \chi_A \mathbf{b}_\varepsilon$ , we can apply Theorem 5.4 to get that any limit point  $\boldsymbol{\eta}$  is a regular generalized flow associated to  $\mathbf{c}$  and it satisfies  $(e_0)_\# \boldsymbol{\eta} = \mathcal{L}_A^d$ . In addition, given  $A' \Subset A$  open we have

$$((e_t)_\# \boldsymbol{\eta}_\varepsilon \llcorner \{\mathbf{h}_{A'}(\cdot) > t\}) \llcorner A' \leq \frac{e^{L(A', \mathbf{b}_\varepsilon)}}{\mathcal{L}^d(A')} \mathcal{L}^d \quad \forall t \in [0, T],$$

thus (5.4) and (5.11) yield

$$((e_t)_\# \boldsymbol{\eta} \llcorner \{\mathbf{h}_{A'}(\cdot) > t\}) \llcorner A' \leq \frac{e^{L(A, \mathbf{b})}}{\mathcal{L}^d(A)} \mathcal{L}^d \quad \forall t \in [0, T].$$

Letting  $A' \uparrow A$  gives that  $e^{L(A, \mathbf{b})} / \mathcal{L}^d(A)$  is a compressibility constant for  $\boldsymbol{\eta}$ .  $\square$

Using a gluing procedure in space, we can now build the maximal regular flow in  $\Omega$  using the flows provided by Theorem 5.2 in domains  $\Omega_n \Subset \Omega_{n+1}$  with  $\Omega_n \uparrow \Omega$ .

**Theorem 5.7.** *Let  $\mathbf{b} : (0, T) \times \Omega \rightarrow \mathbb{R}^d$  be a Borel vector field which satisfies (a- $\Omega$ ) and (b- $\Omega$ ). Then the maximal regular flow is unique, and existence is ensured under the additional assumption (5.1). In addition,*

- (a) *for any  $\Omega' \Subset \Omega$  the compressibility constant  $C(\Omega', \mathbf{X})$  in Definition 4.4 can be taken to be  $e^{L(\Omega', \mathbf{b})}$ , where  $L(\Omega', \mathbf{b})$  is the constant in (5.1);*
- (b) *if  $\mathbf{Y}$  is a regular flow in  $B$  up to  $\tau$  with values in  $\Omega$ , then  $T_{\Omega, \mathbf{X}} > \tau$   $\mathcal{L}^d$ -a.e. in  $B$  and*

$$\mathbf{X}(\cdot, x) = \mathbf{Y}(\cdot, x) \quad \text{in } [0, \tau], \text{ for } \mathcal{L}^d\text{-a.e. } x \in B. \quad (5.12)$$

*Proof.* Let us prove first the uniqueness of the maximal regular flow in  $\Omega$ . Given regular maximal flows  $\mathbf{X}^i$  in  $\Omega$ ,  $i = 1, 2$ , by Lemma 4.2 and Remark 4.5 we easily obtain

$$\mathbf{X}^1(\cdot, x) = \mathbf{X}^2(\cdot, x) \quad \text{in } [0, T_{\Omega, \mathbf{X}^1}(x) \wedge T_{\Omega, \mathbf{X}^2}(x)], \text{ for } \mathcal{L}^d\text{-a.e. } x \in \Omega.$$

On the other hand, for  $\mathcal{L}^d$ -a.e.  $x \in \{T_{\Omega, \mathbf{X}^1} > T_{\Omega, \mathbf{X}^2}\}$ , the image of  $[0, T_{\Omega, \mathbf{X}^2}(x)]$  through  $V_\Omega(\mathbf{X}^1(\cdot, x))$  is bounded in  $\mathbb{R}$ , whereas the image of  $[0, T_{\Omega, \mathbf{X}^2}(x))$  through  $V_\Omega(\mathbf{X}^2(\cdot, x))$  is not. It follows that the set  $\{T_{\mathbf{X}^1} > T_{\mathbf{X}^2}\}$  is  $\mathcal{L}^d$ -negligible. Reversing the roles of  $\mathbf{X}^1$  and  $\mathbf{X}^2$  we obtain that  $T_{\Omega, \mathbf{X}^1} = T_{\Omega, \mathbf{X}^2}$   $\mathcal{L}^d$ -a.e. in  $\Omega$ .

In order to show existence we are going to use auxiliary flows  $\mathbf{X}_n$  in  $\Omega_n$  with hitting times  $T_n : \Omega_n \rightarrow (0, T]$ , i.e.,



- (1) for  $\mathcal{L}^d$ -a.e.  $x \in \Omega_n$ ,  $\mathbf{X}_n(\cdot, x) \in AC([0, T_n(x)]; \mathbb{R}^d)$ ,  $\mathbf{X}_n(0, x) = x$ ,  $\mathbf{X}_n(t, x) \in \Omega_n$  for all  $t \in [0, T_n(x))$ , and  $\mathbf{X}_n(T_n(x), x) \in \partial\Omega_n$  when  $T_n(x) < T$ , so that  $\mathbf{h}_{\Omega_n}(\mathbf{X}_n(\cdot, x)) = T_n(x)$ ;
- (2) for  $\mathcal{L}^d$ -a.e.  $x \in \Omega_n$ ,  $\mathbf{X}_n(\cdot, x)$  solves the ODE  $\dot{\gamma} = \mathbf{b}(t, \gamma)$   $\mathcal{L}^1$ -a.e. in  $(0, T_n(x))$ ;
- (3)  $\mathbf{X}_n(t, \cdot)_{\#}(\mathcal{L}^d \llcorner \{T_n > t\}) \leq e^{L(\Omega_n, \mathbf{b})} \mathcal{L}^d \llcorner \Omega_n$  for all  $t \in [0, T]$ , where  $L(\Omega_n, \mathbf{b})$  is given as in (5.1).

The existence of  $\mathbf{X}_n$ ,  $T_n$  as in (1), (2), (3) has been achieved in Theorem 5.2.

If  $n \leq m$ , the uniqueness argument outlined at the beginning of this proof gives immediately that  $T_n(x) \leq T_m(x)$ , and that  $\mathbf{X}_n(\cdot, x) \equiv \mathbf{X}_m(\cdot, x)$  in  $[0, T_n(x)]$  for  $\mathcal{L}^d$ -a.e.  $x \in \Omega_n$ . Hence the limits

$$T_{\Omega, \mathbf{X}}(x) := \lim_{n \rightarrow \infty} T_n(x), \quad \mathbf{X}(t, x) = \lim_{n \rightarrow \infty} \mathbf{X}_n(t, x) \quad t \in [0, T_{\Omega, \mathbf{X}}(x)) \quad (5.13)$$

are well defined for  $\mathcal{L}^d$ -a.e.  $x \in \Omega$ . By construction

$$\mathbf{X}(\cdot, x) = \mathbf{X}_n(\cdot, x) \quad \text{in } [0, T_n(x)), \text{ for } \mathcal{L}^d\text{-a.e. } x \in \Omega_n. \quad (5.14)$$

We now check that  $\mathbf{X}$  and  $T_{\Omega, \mathbf{X}}$  satisfy the conditions (i), (ii), (iii) of Definition 4.4. Property (i) is a direct consequence of property (2) of  $\mathbf{X}_n$ , (5.13), and (5.14).

In connection with property (ii) of Definition 4.4, in the more specific form stated in (a) for any open set  $\Omega' \Subset \Omega$ , it suffices to check it for all open sets  $\Omega_n$ : indeed, it is clear that in the uniqueness proof we need it only for a family of sets that invade  $\Omega$  and, as soon as uniqueness is established, we can always assume in our construction that  $\Omega'$  is one of the sets  $\Omega_n$ . Now, given  $n$ , we first remark that property (1) of  $\mathbf{X}_n$  yields  $T_n(x) = \mathbf{h}_{\Omega_n}(\mathbf{X}(\cdot, x))$  for  $\mathcal{L}^d$ -a.e.  $x \in \Omega_n$ ; moreover (5.14) gives

$$\mathbf{X}(t, \cdot)_{\#}(\mathcal{L}^d \llcorner \{T_n > t\}) = \mathbf{X}_n(t, \cdot)_{\#}(\mathcal{L}^d \llcorner \{T_n > t\})$$

for all  $t \in [0, T]$ . Hence, we can now use property (3) of  $\mathbf{X}_n$  to get

$$\mathbf{X}(t, \cdot)_{\#}(\mathcal{L}^d \llcorner \{T_n > t\}) \leq e^{L(\Omega_n, \mathbf{b})} \mathcal{L}^d \llcorner \Omega_n \quad \text{for every } t \in [0, T], \quad (5.15)$$

which together with the identity  $T_n(x) = \mathbf{h}_{\Omega_n}(\mathbf{X}(\cdot, x))$  for  $\mathcal{L}^d$ -a.e.  $x \in \Omega_n$  concludes the verification of Definition 4.4(ii).

Now we check Definition 4.4(iii): we obtain that  $\limsup V_{\Omega}(\mathbf{X}(t, x)) = \infty$  as  $t \uparrow T_{\Omega, \mathbf{X}}(x)$  for  $\mathcal{L}^d$ -a.e.  $x \in \Omega$  such that  $T_{\Omega, \mathbf{X}}(x) < T$  from the fact that  $\mathbf{X}(t, T_n(x)) \in \partial\Omega_n$ , and the sets  $\Omega_n$  contain eventually any set  $K \Subset \Omega$ . This completes the existence proof and the verification of the more specific property (a).

The proof of property (b) in the statement of the theorem follows at once from Lemma 4.2 and Remark 4.5.  $\square$

## 6 Main properties of the maximal regular flow

### 6.1 Semigroup property

In order to discuss the semigroup property, we double the time variable and denote by

$$\mathbf{X}(t, s, x), \quad t \geq s,$$

the maximal flow with  $s$  as initial time, so that  $\mathbf{X}(t, 0, x) = \mathbf{X}(t, x)$  and  $\mathbf{X}(s, s, x) = x$ . The maximal time of  $\mathbf{X}(\cdot, s, x)$  will be denoted by  $T_{\Omega, \mathbf{X}, s}(x)$ .

The proof of the semigroup property and of the identity  $T_{\Omega, \mathbf{X}, s}(\mathbf{X}(s, x)) = T_{\Omega, \mathbf{X}}(x)$  satisfied by the maximal existence time follows the classical scheme. It is however a bit more involved than usual because we are assuming only one-sided bounds on the divergence of  $\mathbf{b}$ , therefore the inverse of the map  $\mathbf{X}(s, \cdot)$  (which corresponds to a flow with reversed time) is a priori not defined. For this reason, using conditional measures, we define in the proof a kind of multi-valued inverse of  $\mathbf{X}(s, \cdot)$ .

**Theorem 6.1** (Semigroup property). *Under assumptions (a- $\Omega$ ), (b- $\Omega$ ), and (5.1) on  $\mathbf{b}$ , for all  $s \in [0, T]$  the maximal regular flow  $\mathbf{X}$  satisfies*

$$T_{\Omega, \mathbf{X}, s}(\mathbf{X}(s, x)) = T_{\Omega, \mathbf{X}}(x) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \{T_{\Omega, \mathbf{X}} > s\}, \quad (6.1)$$

$$\mathbf{X}(\cdot, s, \mathbf{X}(s, x)) = \mathbf{X}(\cdot, x) \text{ in } [s, T_{\Omega, \mathbf{X}}(x)), \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \{T_{\Omega, \mathbf{X}} > s\}. \quad (6.2)$$

*Proof.* Let us fix  $s \geq 0$  and assume without loss of generality that  $\mathcal{L}^d(\{T_{\Omega, \mathbf{X}} > s\}) > 0$ . Let us fix a Borel  $B_s \subset \{T_{\Omega, \mathbf{X}} > s\}$  with positive and finite measure, and let  $\mathcal{L}_s^d$  denote the renormalized Lebesgue measure on  $B_s$ , namely  $\mathcal{L}_s^d := \mathcal{L}^d \llcorner B_s / \mathcal{L}^d(B_s)$ . We denote by  $\rho_s$  the bounded density of the probability measure  $\mathbf{X}(s, \cdot)_{\#} \mathcal{L}_s^d$  with respect to  $\mathcal{L}^d$ . We can disintegrate the probability measure  $\pi := (Id \times \mathbf{X}(s, \cdot))_{\#} \mathcal{L}_s^d$  with respect to  $\rho_s$ , getting a family  $\{\pi_y\}$  of probability measures in  $\mathbb{R}^d$  such that  $\pi = \int \pi_y \otimes \delta_y \rho_s(y) dy$ . Notice that in the case when  $\mathbf{X}(s, \cdot)$  is (essentially) injective,  $\pi_y$  is the Dirac mass at  $(\mathbf{X}(s, \cdot))^{-1}(y)$  for  $\mathbf{X}(s, \cdot)_{\#} \mathcal{L}_s^d$ -a.e.  $y$ .

For  $\varepsilon > 0$ , let us set

$$\pi_\varepsilon := \int_{\{\rho_s \geq \varepsilon\}} \pi_y \otimes \delta_y dy \in \mathcal{P}(\mathbb{R}^{2d})$$

Since  $\varepsilon \pi_\varepsilon \leq \pi$ , the first marginal  $\tilde{\rho}_\varepsilon$  of  $\pi_\varepsilon$  is bounded from above by  $\mathcal{L}_s^d / \varepsilon$ , therefore it has a bounded density  $\tilde{\rho}_\varepsilon$  with respect to  $\mathcal{L}^d$ . Moreover, since  $\pi \leq \|\rho_s\|_{L^\infty(\mathbb{R}^d)} \sup_{\varepsilon > 0} \pi_\varepsilon$  and the first marginal of  $\pi$  is  $\mathcal{L}_s^d$ , we obtain

$$\sup_{\varepsilon > 0} \tilde{\rho}_\varepsilon(x) > 0 \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in B_s. \quad (6.3)$$

Now, for  $\tau > s$  and  $\varepsilon > 0$  fixed, let  $B_s^\tau := \{T_{\Omega, \mathbf{X}} > \tau\}$  and define a generalized flow  $\eta_{\tau, \varepsilon} \in \mathcal{P}(C([s, \tau]; \mathbb{R}^d))$  by

$$\eta_{\tau, \varepsilon} := \int_{(x, y) \in B_s^\tau \times \{\rho_s \geq \varepsilon\}} \delta_{\mathbf{X}(\cdot, x)} d\pi_y(x) dy = \int_{B_s^\tau} \delta_{\mathbf{X}(\cdot, x)} \tilde{\rho}_\varepsilon(x) dx. \quad (6.4)$$

For any  $r \in [s, \tau]$  and any  $\phi \in C_b(\mathbb{R}^d)$  nonnegative there holds

$$\int_{\mathbb{R}^d} \phi d[(e_r)_\# \boldsymbol{\eta}_{\tau, \varepsilon}] = \int_{B_s^\tau} \phi(\mathbf{X}(r, x)) \tilde{\rho}_\varepsilon(x) dx \leq L \|\tilde{\rho}_\varepsilon\|_\infty \int_{\mathbb{R}^d} \phi(z) dz.$$

Evaluating at  $r = s$ , a similar computation gives

$$(e_s)_\# \boldsymbol{\eta}_{\tau, \varepsilon} = \mathbf{X}(s, \cdot)_\# (\chi_{B_s^\tau} \tilde{\rho}_\varepsilon).$$

By Theorem 3.4 (applied in the time interval  $[s, \tau]$  instead of  $[0, T]$ ) it follows that

$$\boldsymbol{\eta}_{\tau, \varepsilon} = \int \delta_{\eta_z} d[(e_s)_\# \boldsymbol{\eta}_{\tau, \varepsilon}](z). \quad (6.5)$$

Now, it is clear that  $\mathbf{W}(\cdot, z) := \eta_z(\cdot)$  is a regular flow in  $[s, \tau]$ , hence (by uniqueness)  $\eta_z = \mathbf{X}(\cdot, s, z)$  for  $(e_s)_\# \boldsymbol{\eta}_{\tau, \varepsilon}$ -a.e.  $z$ . Returning to (6.5) we get

$$\boldsymbol{\eta}_{\tau, \varepsilon} = \int \delta_{\mathbf{X}(\cdot, s, z)} d[(e_s)_\# \boldsymbol{\eta}_{\tau, \varepsilon}](z) = \int_{B_s^\tau} \delta_{\mathbf{X}(\cdot, s, \mathbf{X}(s, x))} \tilde{\rho}_\varepsilon(x) dx, \quad (6.6)$$

where in the second equality we used the formula for  $(e_s)_\# \boldsymbol{\eta}_{\tau, \varepsilon}$ . Comparing formulas (6.4) and (6.6), and taking (6.3) into account, we find that  $T_{\Omega, \mathbf{X}, s}(\mathbf{X}(s, x)) \geq \tau$  and that  $\mathbf{X}(\cdot, s, \mathbf{X}(s, x)) \equiv \mathbf{X}(\cdot, x)$  in  $[s, \tau]$ , for  $\mathcal{L}^d$ -a.e.  $x \in B_s^\tau$ . Since  $\tau > s$  is arbitrary, it follows that  $T_{\Omega, \mathbf{X}, s}(\mathbf{X}(s, x)) \geq T_{\Omega, \mathbf{X}}(x)$  and that  $\mathbf{X}(t, s, \mathbf{X}(s, x)) = \mathbf{X}(t, x)$   $\mathcal{L}^d$ -a.e. in  $B_s$ . If  $T_{\Omega, \mathbf{X}}(x) < T$ , by the semigroup identity it follows that

$$\limsup_{t \uparrow T_{\Omega, \mathbf{X}}(x)} V_\Omega(\mathbf{X}(t, s, \mathbf{X}(s, x))) = \limsup_{t \uparrow T_{\Omega, \mathbf{X}}(x)} V_\Omega(\mathbf{X}(t, x)) = \infty,$$

and hence

$$T_{\Omega, \mathbf{X}, s}(\mathbf{X}(s, x)) = T_{\Omega, \mathbf{X}}(x) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in B_s. \quad (6.7)$$

Eventually we use the arbitrariness of  $B_s$  to conclude (6.1) and (6.2).  $\square$

## 6.2 Stability

The following theorem provides a stability result for maximal regular flows in  $\Omega$  when the vector fields converge strongly in space and weakly in time, in analogy with the classical theory (see also Remark 6.3 below).

**Theorem 6.2** (Stability of maximal regular flows in  $\Omega$ ). *Let  $\Omega \subset \mathbb{R}^d$  be an open set. Let  $\mathbf{X}^n$  be maximal regular flows in  $\Omega$  relative to locally integrable Borel vector fields  $\mathbf{b}^n : (0, T) \times \Omega \rightarrow \mathbb{R}^d$ . Assume that:*

- (a) *for any  $A \Subset \Omega$  open the compressibility constants  $C(A, \mathbf{X}^n)$  in Definition 4.4 are uniformly bounded;*

(b) for any  $A \Subset \Omega$  open, setting  $A^\varepsilon := \{x \in A : \text{dist}(x, \mathbb{R}^d \setminus A) \geq \varepsilon\}$  for  $\varepsilon > 0$ , there holds, uniformly w.r.t.  $n$ ,

$$\lim_{h \rightarrow 0} |\chi_{A|h}|(x+h)\mathbf{b}^n(t, x+h) - \chi_A(x)\mathbf{b}^n(t, x)| = 0 \quad \text{in } L^1((0, T) \times A); \quad (6.8)$$

(c) there exists a Borel vector field  $\mathbf{b} : (0, T) \times \Omega \rightarrow \mathbb{R}^d$  satisfying (a- $\Omega$ ) and (b- $\Omega$ ) such that

$$\mathbf{b}^n \rightharpoonup \mathbf{b} \quad \text{weakly in } L^1((0, T) \times A; \mathbb{R}^d) \quad \text{for all } A \Subset \Omega \text{ open.} \quad (6.9)$$

Then there exists a unique maximal regular flow  $\mathbf{X}$  for  $\mathbf{b}$  and, for every  $t \in [0, T]$  and any open set  $A \Subset \Omega$ , we have

$$\lim_{n \rightarrow \infty} \left\| \max_{s \in [0, t]} |\mathbf{X}_A^n(s, \cdot) - \mathbf{X}(s, \cdot)| \wedge 1 \right\|_{L^1(\{x : h_A(\mathbf{X}(\cdot, x)) > t\})} = 0, \quad (6.10)$$

where

$$\mathbf{X}_A^n(t, x) := \begin{cases} \mathbf{X}^n(t, x) & \text{for } t \in [0, h_A(\mathbf{X}^n(\cdot, x))], \\ \mathbf{X}^n(h_A(\mathbf{X}^n(\cdot, x)), x) & \text{for } t \in [h_A(\mathbf{X}^n(\cdot, x)), T]. \end{cases}$$

**Remark 6.3.** The convergence (6.9) and (6.8) of  $\mathbf{b}^n$  to  $\mathbf{b}$  is implied by the strong convergence of  $\mathbf{b}^n$  to  $\mathbf{b}$  in space-time. It is however quite natural to state the convergence in these terms in view of some applications. For example, the weak convergence of (6.9) and the boundedness in a fractional Sobolev space  $\mathbf{b}^n \in L^1((0, T); W^{m,p}(\mathbb{R}^d))$ ,  $p > 1, m > 0$ , is enough to guarantee that (6.8) holds. The same kind of convergence appears in [21, Theorem II.7] to prove convergence of distributional solutions of the continuity equation, and in [19, Remark 2.11] in the context of quantitative estimates on the flows of Sobolev vector fields.

The convergence of the flows in (6.10) is localised to the trajectories of  $\mathbf{b}$  which are inside  $A$  in  $[0, t]$ . This is indeed natural: even with smooth vector fields one can construct examples where the existence time of  $\mathbf{X}(\cdot, x)$  is strictly smaller than the existence time of  $\mathbf{X}^n(\cdot, x)$  and the convergence of  $\mathbf{X}^n(\cdot, x)$  to  $\mathbf{X}(\cdot, x)$ , or to its constant extension beyond the existence time  $T_{\Omega, \mathbf{X}}(x)$ , fails after  $T_{\Omega, \mathbf{X}}(x)$  (see Figure 1).

The stability of maximal flows in Theorem 6.2 implies a lower semicontinuity property of hitting times.

**Corollary 6.4** (Semicontinuity of hitting times). *With the same notation and assumptions of Theorem 6.2, for every  $t \in [0, T]$  we have that*

$$\lim_{n \rightarrow \infty} \mathcal{L}^d(\{x : h_A(\mathbf{X}^n(\cdot, x)) \leq t < h_A(\mathbf{X}(\cdot, x))\}) = 0. \quad (6.11)$$

In particular, there exists a subsequence  $n(k) \rightarrow \infty$  (which depends, in particular, on  $A$ ) such that

$$h_A(\mathbf{X}(\cdot, x)) \leq \liminf_{k \rightarrow \infty} h_A(\mathbf{X}^{n(k)}(\cdot, x)) \quad \mathcal{L}^d\text{-a.e. in } A. \quad (6.12)$$

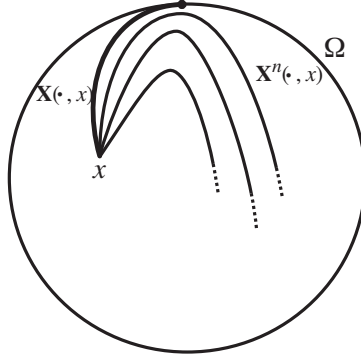


Figure 1: One can build a sequence of smooth vector fields  $\mathbf{b}^n$  whose trajectories  $\mathbf{X}^n(\cdot, x)$  starting from a point  $x$  is drawn in the figure. These trajectories fail to converge to the constant extension of  $\mathbf{X}(\cdot, x)$  after  $T_{\Omega, \mathbf{X}}(x)$ .

*Proof.* For every  $x$  such that  $\mathfrak{h}_A(\mathbf{X}^n(\cdot, x)) \leq t < \mathfrak{h}_A(\mathbf{X}(\cdot, x))$  we have that

$$\max_{s \in [0, t]} |\mathbf{X}_A^n(s, x) - \mathbf{X}(s, x)| \geq \text{dist}(\partial A, \mathbf{X}([0, t], x)) > 0.$$

It implies, together with (6.10), that (6.11) holds.

Up to a subsequence and with a diagonal argument, by (6.11) we deduce that for every  $t \in \mathbb{Q} \cap [0, T]$  the functions  $1_{\{\mathfrak{h}_A(\mathbf{X}^{n(k)}(\cdot, x)) \leq t\}}$  converge pointwise a.e. to 0 in  $\{\mathfrak{h}_A(\mathbf{X}(\cdot, x)) > t\}$  and therefore for  $\mathcal{L}^d$ -a.e.  $x$  such that  $t < \mathfrak{h}_A(\mathbf{X}(\cdot, x))$  we have  $\mathfrak{h}_A(\mathbf{X}^n(\cdot, x)) > t$  for  $n$  large enough. This implies that for every  $t \in \mathbb{Q} \cap [0, T]$ , for  $\mathcal{L}^d$ -a.e.  $x$  such that  $t < \mathfrak{h}_A(\mathbf{X}(\cdot, x))$  we have

$$t \leq \liminf_{k \rightarrow \infty} \mathfrak{h}_A(\mathbf{X}^{n(k)}(\cdot, x)) \quad \mathcal{L}^d\text{-a.e. in } A,$$

which implies (6.12). □

The proof of the stability of maximal regular flows in  $\Omega$  is based on a tightness and stability result for regular generalized flows in  $\overline{A}$  (according to Definition 5.3), as the one presented in Theorem 5.4 under the assumption of the strong space-time convergence of the vector fields.

**Proposition 6.5** (Tightness and stability of generalized regular flows). *Let  $A \subset \mathbb{R}^d$  be a bounded open set. The result of Theorem 5.4 holds true also if we replace the strong convergence of the vector fields (5.3) with the assumptions*

$$\lim_{h \rightarrow 0} \chi_{A|h|}(x+h) \mathbf{c}^n(t, x+h) = \chi_A(x) \mathbf{c}^n(t, x) \quad \text{in } L^1((0, T) \times A), \text{ uniformly w.r.t. } n, \quad (6.13)$$

$$\mathbf{c}^n \rightharpoonup \mathbf{c} \quad \text{weakly in } L^1((0, T) \times A), \quad (6.14)$$

where  $A^\varepsilon := \{x \in A : \text{dist}(x, \mathbb{R}^d \setminus A) \geq \varepsilon\}$  for  $\varepsilon > 0$  (compare with (6.8) and (6.9)).

*Proof.* The tightness was based on Dunford-Pettis' theorem and it can be repeated in this context thanks to (6.14): in particular, there exists a modulus of integrability  $F$  such that

$$\sup_{n \in \mathbb{N}} \int \int_0^T F(|\dot{\eta}(t)|) dt d\boldsymbol{\eta}^n < \infty. \quad (6.15)$$

We show that  $\boldsymbol{\eta}$  is concentrated on integral curves of  $\mathbf{c}$ , namely

$$\int \left| \eta(t) - \eta(0) - \int_0^t \mathbf{c}(s, \eta(s)) ds \right| d\boldsymbol{\eta}(\eta) = 0 \quad (6.16)$$

for any  $t \in [0, T]$ . To this end we consider  $\mathbf{c}^\varepsilon := (\mathbf{c}\chi_{A^\varepsilon}) * \rho_\varepsilon$ , where  $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(x/\varepsilon)$ ,  $\rho \in C_c^\infty(\mathbb{R}^d)$  nonnegative, is a standard convolution kernel in the space variable with compact support in the unit ball. Notice that  $\mathbf{c}^\varepsilon \in L^1((0, T); C_c^\infty(A; \mathbb{R}^d))$  and that  $|\mathbf{c}^\varepsilon - \mathbf{c}| \rightarrow 0$  in  $L^1((0, T) \times A)$  as  $\varepsilon \rightarrow 0$ . Similarly, for every  $n \in \mathbb{N}$  we set  $\mathbf{c}^{n, \varepsilon} := (\mathbf{c}^n \chi_{A^\varepsilon}) * \rho_\varepsilon$ . We first prove that, for every  $\varepsilon > 0$ ,

$$\int \left| \eta(t) - \eta(0) - \int_0^t \mathbf{c}^\varepsilon(s, \eta(s)) ds \right| d\boldsymbol{\eta}(\eta) \leq \omega(\varepsilon), \quad (6.17)$$

where  $\omega : (0, \infty) \rightarrow (0, \infty)$  is a nondecreasing function which goes to 0 as  $\varepsilon \rightarrow 0$  to be chosen later.

Since the integrand is a continuous (possibly unbounded) function of  $\eta \in C([0, T]; \mathbb{R}^d)$  and  $\boldsymbol{\eta}^n$  is concentrated on integral curves of  $\mathbf{c}^n$ , by the triangular inequality we have the estimate

$$\begin{aligned} & \int \left| \eta(t) - \eta(0) - \int_0^t \mathbf{c}^\varepsilon(s, \eta(s)) ds \right| d\boldsymbol{\eta}(\eta) \\ & \leq \liminf_{n \rightarrow \infty} \int \left| \eta(t) - \eta(0) - \int_0^t \mathbf{c}^\varepsilon(s, \eta(s)) ds \right| d\boldsymbol{\eta}^n(\eta) \\ & \leq \liminf_{n \rightarrow \infty} \left[ \int \left| \int_0^t [\mathbf{c}^n - \mathbf{c}^{n, \varepsilon}](s, \eta(s)) ds \right| d\boldsymbol{\eta}^n(\eta) + \int \left| \int_0^t [\mathbf{c}^{n, \varepsilon} - \mathbf{c}^\varepsilon](s, \eta(s)) ds \right| d\boldsymbol{\eta}^n(\eta) \right] \end{aligned} \quad (6.18)$$

To estimate the first term in the right-hand side of (6.18), we notice that

$$\sup_{n \in \mathbb{N}} \|\mathbf{c}^{n, \varepsilon} - \mathbf{c}^n\|_{L^1((0, T) \times A)} \leq \omega(\varepsilon)$$

and  $\omega(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Indeed, consider a nondecreasing function  $\omega_0 : (0, \infty) \rightarrow (0, \infty)$  which goes to 0 as  $\varepsilon \rightarrow 0$  and such that

$$\|\chi_{A|h|}(x-h)\mathbf{c}^n(t, x-h) - \chi_A(x)\mathbf{c}^n(t, x)\|_{L^1((0, T) \times A)} \leq \omega_0(|h|) \quad (6.19)$$

for every  $n \in \mathbb{N}$ , which exists thanks to (6.13). We notice that

$$\begin{aligned}
\int_0^T \int_A |\mathbf{c}^{n,\varepsilon} - \mathbf{c}^n| dx dt &\leq \int_{\mathbb{R}^d} \rho_\varepsilon(z) \int_0^T \int_A |\chi_{A^\varepsilon}(x-z) \mathbf{c}^n(t, x-z) - \mathbf{c}^n(t, x)| dx dt dz \\
&\leq \int_{\mathbb{R}^d} \rho_\varepsilon(z) \int_0^T \int_A [|\chi_{A^{|z|}}(x-z) - \chi_{A^\varepsilon}(x-z)|] |\mathbf{c}^n(t, x-z)| dx dt dz \\
&\quad + \int_{\mathbb{R}^d} \rho_\varepsilon(z) \int_0^T \int_A |\chi_{A^{|z|}}(x-z) \mathbf{c}^n(t, x-z) - \mathbf{c}^n(t, x)| dx dt dz \\
&\leq \int_{\mathbb{R}^d} \rho_\varepsilon(z) \int_0^T \int_{\mathbb{R}^d} [|\chi_A(x) - \chi_{A^\varepsilon}(x)|] |\mathbf{c}^n(t, x)| dx dt dz + \omega_0(\varepsilon)
\end{aligned}$$

and the first term converges to 0 uniformly in  $n$  thanks to (6.14), Dunford-Pettis' theorem and since  $A^\varepsilon \uparrow A$  as  $\varepsilon \rightarrow 0$ .

Hence, using the fact that  $\mathbf{c}^n = 0$  on  $\partial A$  and the definition (5.2) of compressibility constant  $C_n$  for  $\boldsymbol{\eta}^n$  we get

$$\int \left| \int_0^t [\mathbf{c}^n - \mathbf{c}^{n,\varepsilon}](s, \eta(s)) ds \right| d\boldsymbol{\eta}^n(\eta) \leq C_n \int_\Omega \int_0^t |\mathbf{c}^n - \mathbf{c}^{n,\varepsilon}| ds dx \leq \sup_n C_n \omega(\varepsilon). \quad (6.20)$$

We now estimate the second term in the right-hand side of (6.18). To this end, for every  $k > 0$  we consider the set of curves

$$\Gamma_k := \left\{ \eta \in AC([0, T]; \bar{A}) : \int_0^T F(|\dot{\eta}(t)|) dt \leq k \right\}.$$

We notice that all curves in  $\Gamma_k$  have a uniform modulus of continuity that we denote by  $\tilde{\omega}_k$ . By Chebyshev's inequality and (6.15) we deduce that

$$\boldsymbol{\eta}^n(C([0, T]; \bar{A}) \setminus \Gamma_k) \leq \frac{C}{k}$$

for some constant  $C > 0$ , hence in the complement of  $\Gamma_k$  we estimate the integrand with its  $L^\infty$  norm:

$$\begin{aligned}
\int_{\Gamma_k^c} \left| \int_0^t [\mathbf{c}^{n,\varepsilon} - \mathbf{c}^\varepsilon](s, \eta(s)) ds \right| d\boldsymbol{\eta}^n(\eta) &\leq \boldsymbol{\eta}^n(\Gamma_k^c) \int_0^T \|[\mathbf{c}^{n,\varepsilon} - \mathbf{c}^\varepsilon](s, \cdot)\|_{L^\infty(A)} ds \\
&\leq \frac{C}{k} \|\mathbf{c}^n - \mathbf{c}\|_{L^1((0, T) \times A)} \|\rho_\varepsilon\|_{L^\infty(A)}.
\end{aligned} \quad (6.21)$$

Hence, choosing  $k$  large enough we can make this term as small as we wish uniformly with respect to  $n$ , since  $\|\mathbf{c}^n - \mathbf{c}\|_{L^1((0, T) \times A)} \leq \|\mathbf{c}^n\|_{L^1((0, T) \times A)} + \|\mathbf{c}\|_{L^1((0, T) \times A)}$  is bounded.

In  $\Gamma_k$ , for any  $N \in \mathbb{N}$  we can use the triangular inequality, the fact that  $\mathbf{c}^{n,\varepsilon}$  and  $\mathbf{c}^\varepsilon$  are null on  $(0, T) \times \partial A$ , and the bounded compression condition  $(e_{i/N})_\# \boldsymbol{\eta}^n \llcorner A \leq C_n \mathcal{L}^d$  for every

$i = 1, \dots, N$ , to get

$$\begin{aligned}
\int_{\Gamma_k} \left| \int_0^t [\mathbf{c}^{n,\varepsilon} - \mathbf{c}^\varepsilon](s, \eta(s)) ds \right| d\boldsymbol{\eta}^n(\eta) &\leq \sum_{i=1}^N \int_{\Gamma_k} \left| \int_{t_{i-1}^N}^{t_i^N} [\mathbf{c}^{n,\varepsilon} - \mathbf{c}^\varepsilon](s, \eta(s)) ds \right| d\boldsymbol{\eta}^n(\eta) \quad (6.22) \\
&\leq \sum_{i=1}^N \int_{\Gamma_k} \left| \int_{t_{i-1}^N}^{t_i^N} [\mathbf{c}^{n,\varepsilon} - \mathbf{c}^\varepsilon](s, \eta(t_i^N)) ds \right| d\boldsymbol{\eta}^n(\eta) \\
&\quad + \tilde{\omega}_k \left( \frac{t}{N} \right) \sum_{i=1}^N \int_{t_{i-1}^N}^{t_i^N} \|\nabla[\mathbf{c}^{n,\varepsilon} - \mathbf{c}^\varepsilon](s, \cdot)\|_{L^\infty(A)} ds \\
&\leq C_n \sum_{i=1}^N \int_A \left| \int_{t_{i-1}^N}^{t_i^N} [\mathbf{c}^{n,\varepsilon} - \mathbf{c}^\varepsilon] ds \right| dx \\
&\quad + \tilde{\omega}_k \left( \frac{t}{N} \right) \|\mathbf{c}^n - \mathbf{c}\|_{L^1((0,T) \times A)} \|\nabla \rho_\varepsilon\|_{L^\infty(\mathbb{R}^d)},
\end{aligned}$$

where  $t_i^N = it/N$ . Choosing  $N$  large enough we can make the second term in the right-hand side as small as we want, uniformly in  $n$ . Letting  $n \rightarrow \infty$  in (6.22), each term in the first sum in the right-hand side converges to 0 pointwise in  $x$  by the weak convergence (6.9) tested with the function  $\varphi_x(s, y) = \mathbf{1}_{[t_{i-1}^N, t_i^N]}(s) \rho_\varepsilon(x - y)$ , namely, for every  $x \in A$ ,

$$\lim_{n \rightarrow \infty} \int_{t_{i-1}^N}^{t_i^N} [\mathbf{c}^{n,\varepsilon} - \mathbf{c}^\varepsilon](s, x) ds = \lim_{n \rightarrow \infty} \int_{t_{i-1}^N}^{t_i^N} [\mathbf{c}^n - \mathbf{c}](s, y) \rho_\varepsilon(x - y) ds = 0.$$

These functions are bounded by  $\|\mathbf{c}^n - \mathbf{c}\|_{L^1((0,T) \times A)} \|\rho_\varepsilon\|_{L^\infty(\mathbb{R}^d)}$ , thus by dominated convergence the first sum in the right-hand side of (6.22) converges to 0. It follows that, given  $\varepsilon$  and  $k$ , by choosing  $N$  sufficiently large we can make also this term as small as we wish, hence (6.17) follows from (6.18). We now let  $\varepsilon \rightarrow 0$  in (6.17) and notice that, since  $\boldsymbol{\eta}$  satisfies (5.2) with  $C = \liminf_n C_n$  and  $\mathbf{c}^\varepsilon \rightarrow \mathbf{c}$  in  $L^1((0, T) \times A)$ ,

$$\lim_{\varepsilon \rightarrow 0} \int \left| \int_0^t [\mathbf{c} - \mathbf{c}^\varepsilon](s, \eta(s)) ds \right| d\boldsymbol{\eta}(\eta) \leq C \lim_{\varepsilon \rightarrow 0} \int_A \int_0^t |\mathbf{c} - \mathbf{c}^\varepsilon| ds dx = 0,$$

proving the validity of (6.16).  $\square$

The following lemma is a standard tool in optimal transport theory (for a proof, see for instance [3, Lemma 22], or [24, Corollary 5.23]).

**Lemma 6.6.** *Let  $X_1, X_2$  be Polish metric spaces, let  $\mu \in \mathcal{P}(X_1)$ , and let  $F_n : X_1 \rightarrow X_2$  be a sequence of Borel functions. If*

$$(\text{Id}, F_n)_\# \mu \rightharpoonup (\text{Id}, F)_\# \mu \quad \text{narrowly in } \mathcal{P}(X_1 \times X_2), \quad (6.23)$$

then  $F_n$  converge to  $F$  in  $\mu$ -measure, namely

$$\lim_{n \rightarrow \infty} \mu(\{d_{X_2}(F_n, F) > \varepsilon\}) = 0 \quad \forall \varepsilon > 0.$$



*Proof of Theorem 6.2.* Fix  $A \Subset \Omega$  open, denote by  $\mathcal{L}_A^d$  the normalized Lebesgue measure on  $A$ , and define  $\mathbf{X}_A^n$  as in the statement of the theorem. Then the laws  $\eta^n$  of  $x \mapsto \mathbf{X}_A^n(\cdot, x)$  under  $\mathcal{L}_A^d$  define regular generalized flows in  $\bar{A}$  relative to  $\mathbf{c}^n = \chi_A \mathbf{b}^n$ , according to Definition 5.3, with compressibility constants  $C_n = C(A, \mathbf{X}^n)$ .

Hence we can apply Proposition 6.5 to obtain that, up to a subsequence,  $\eta^n$  weakly converge to a generalized flow  $\eta$  in  $\bar{A}$  relative to the vector field  $\mathbf{c} = \chi_A \mathbf{b}$ , with compressibility constant  $C = \liminf_n C_n$ . Let  $\eta_x$  be the conditional probability measures induced by the map  $e_0$ , and let  $\mathbf{X}_A$  and  $T_A$  be given by Proposition 5.5; recall that  $\mathbf{X}_A(\cdot, x)$  is an integral curve of  $\mathbf{b}$  in  $[0, T_A(x)]$ , that  $\mathbf{X}_A([0, T_A(x)), x) \subset A$ , and that  $\mathbf{X}_A(T_A(x), x) \in \partial A$  if  $T_A(x) < T$ ; as explained in Remark 5.6, for  $\mathcal{L}_A^d$ -almost every  $x$  the hitting time  $\mathfrak{h}_A(\eta)$  is equal to  $T_A(x)$  for  $\eta_x$ -a.e.  $\eta$ , and  $(e_t)_\# \eta_x = \delta_{\mathbf{X}_A(t, x)}$  for all  $t \in [0, T_A(x)]$ . For every  $t \in [0, T]$  we set  $E_{t, A} := \{T_A(x) > t\}$ ; since

$$\mathbf{X}_A(s, \cdot)_\# (\mathcal{L}^d \llcorner E_{t, A}) = (e_s)_\# \int_{E_{t, A}} \delta_{\mathbf{X}_A(\cdot, x)} d\mathcal{L}^d \leq (e_s)_\# \tilde{\eta} \leq C \mathcal{L}^d \quad \forall s \in [0, t],$$

we obtain that  $\mathbf{X}_A$  is a regular flow for  $\mathbf{b}$  on  $[0, t] \times E_t$ . Applying Theorem 5.7(b) to  $\mathbf{X}_{A_1}$  and  $\mathbf{X}_{A_2}$  with  $A_1 \subset A_2$  we deduce that  $\mathbf{X}_{A_1} = \mathbf{X}_{A_2}$  on  $E_{t, A_1}$ , and this allows us (by a gluing argument) to obtain a maximal regular flow for  $\mathbf{b}$ .

To prove the last statement, we apply Lemma 6.6 with  $X_1 = \mathbb{R}^d$ ,  $\mu = (\mathcal{L}^d \llcorner \{T_A > t\}) / \mathcal{L}^d(\{T_A > t\})$ ,  $X_2 = C([0, t]; \bar{A})$ ,  $F_n(x) = \mathbf{X}_A^n(\cdot, x)$ ,  $F(x) = \mathbf{X}_A(\cdot, x)$ . More precisely, we consider the laws  $\tilde{\eta}^n \in \mathcal{P}(C([0, t]; \mathbb{R}^d))$  of  $x \mapsto \mathbf{X}_A^n(\cdot, x)$  under  $\mu$ ; with the same argument as above, we know that  $\tilde{\eta}^n$  weakly converge to  $\tilde{\eta}$  and that the disintegration  $\tilde{\eta}_x$  coincides with  $\delta_{\mathbf{X}_A(\cdot, x)}$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$  (notice that  $\mathbf{X}_A(\cdot, x)$  is defined in  $[0, t]$  for  $\mu$ -a.e.  $x$ ). The assumption (6.23) is satisfied, since for every bounded continuous function  $\varphi : \mathbb{R}^d \times C([0, T]; \bar{A}) \rightarrow \mathbb{R}$  we have

$$\int \varphi(x, \gamma) d(\text{Id}, \mathbf{X}_A^n(\cdot, x))_\# \mu(x, \gamma) = \int \varphi(\gamma(0), \gamma) d\tilde{\eta}^n(\gamma)$$

(and similarly with  $\tilde{\eta}$ ) and the weak convergence of  $\tilde{\eta}^n$  to  $\tilde{\eta}$  shows that

$$\lim_{n \rightarrow \infty} \int \varphi(x, \gamma) d(\text{Id}, \mathbf{X}_A^n(\cdot, x))_\# \mu(x, \gamma) = \int \varphi(x, \gamma) d(\text{Id}, \mathbf{X}_A(\cdot, x))_\# \mu(x, \gamma).$$

We deduce the convergence in  $\mu$ -measure of  $\mathbf{X}_A^n$  to  $\mathbf{X}_A$  in  $C([0, t]; \bar{A})$ , i.e.,

$$\lim_{n \rightarrow \infty} \mathcal{L}^d \left( \left\{ x \in \{T_A > t\} : \sup_{s \in [0, t]} |\mathbf{X}_A^n(s, x) - \mathbf{X}_A(s, x)| > \varepsilon \right\} \right) = 0 \quad \forall \varepsilon > 0,$$

from which (6.10) follows easily.  $\square$

## 7 Further properties implied by global bounds on divergence

### 7.1 Proper blow-up of trajectories

Recall that the blow-up time  $T_{\Omega, \mathbf{X}}(x)$  for maximal regular flows is characterized by the property  $\limsup_{t \uparrow T_{\Omega, \mathbf{X}}(x)} V_{\Omega}(\mathbf{X}(t, x)) = \infty$  when  $T_{\Omega, \mathbf{X}}(x) < T$ . We say that  $\mathbf{X}(\cdot, x)$  blows

up *properly* (i.e. with no oscillations) if the stronger condition  $\lim_{t \uparrow T_{\Omega, \mathbf{X}}(x)} V_{\Omega}(\mathbf{X}(t, x)) = \infty$  holds. In the following theorem we prove this property when a global bounded compression condition on  $\mathbf{X}$  is available, see (7.2) below. Thanks to the properties of the maximal regular flow the global bounded compression condition is fulfilled, for instance, in all cases when the divergence bounds  $L(\Omega')$  in (5.1) are uniformly bounded. More precisely

$$\operatorname{div} \mathbf{b}(t, \cdot) \geq m(t) \quad \text{in } \Omega, \text{ with } L(\Omega) := \int_0^T |m(t)| dt < \infty \quad (7.1)$$

implies (7.2) with  $C_* \leq e^{L(\Omega)}$ .

**Theorem 7.1.** *Let  $\mathbf{X}$  be a maximal regular flow relative to a Borel vector field  $\mathbf{b}$  satisfying (a- $\Omega$ ) and (b- $\Omega$ ), and assume that the bounded compression condition is global, namely there exists a constant  $C_* \geq 0$  satisfying*

$$\mathbf{X}(t, \cdot)_{\#}(\mathcal{L}^d \llcorner \{T_{\Omega, \mathbf{X}} > t\}) \leq C_* \mathcal{L}^d \quad \forall t \in [0, T]. \quad (7.2)$$

Then

$$\liminf_{t \uparrow T_{\Omega, \mathbf{X}}(x)} |\mathbf{X}(t, x)| = \infty \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \text{ such that } \limsup_{t \uparrow T_{\Omega, \mathbf{X}}(x)} |\mathbf{X}(t, x)| = \infty,$$

and in particular  $\lim_{t \uparrow T_{\Omega, \mathbf{X}}(x)} V_{\Omega}(\mathbf{X}(t, x)) = \infty$  for  $\mathcal{L}^d$ -a.e.  $x$  with  $T_{\Omega, \mathbf{X}}(x) < T$ .

*Proof.* Let  $\Omega_n$  be open sets with  $\Omega_n \Subset \Omega_{n+1} \Subset \Omega$ , with  $\cup_n \Omega_n = \Omega$ . We consider cut-off functions  $\psi_n \in C_c^\infty(\Omega_{n+1})$  with  $0 \leq \psi_n \leq 1$  and  $\psi_n \equiv 1$  on a neighborhood of  $\overline{\Omega}_n$ .

Since  $\mathbf{X}(\cdot, x)$  is an integral curve of  $\mathbf{b}$  for  $\mathcal{L}^d$ -a.e.  $x \in \Omega$  we can use (7.2) to estimate

$$\begin{aligned} \int_{\Omega} \int_0^{T_{\Omega, \mathbf{X}}(x)} \left| \frac{d}{dt} \psi_n(\mathbf{X}(t, x)) \right| dt dx &\leq \int_{\Omega} \int_0^{T_{\Omega, \mathbf{X}}(x)} |\nabla \psi_n(\mathbf{X}(t, x))| |\mathbf{b}(t, \mathbf{X}(t, x))| dt dx \\ &= \int_0^T \int_{\{T_{\Omega, \mathbf{X}} > t\}} |\nabla \psi_n(\mathbf{X}(t, x))| |\mathbf{b}(t, \mathbf{X}(t, x))| dx dt \\ &\leq C_* \int_0^T \int_{\mathbb{R}^d} |\nabla \psi_n(y)| |\mathbf{b}(t, y)| dy dt \\ &\leq C \|\nabla \psi_n\|_{L^\infty(\Omega)} \int_0^T \int_{\Omega_{n+1}} |\mathbf{b}(t, x)| dx dt. \end{aligned} \quad (7.3)$$

Hence  $\psi_n(\mathbf{X}(\cdot, x))$  is the restriction of an absolutely continuous map in  $[0, T_{\Omega, \mathbf{X}}(x)]$  (and therefore uniformly continuous in  $[0, T_{\Omega, \mathbf{X}}(x))$ ) for  $\mathcal{L}^d$ -a.e.  $x \in \Omega$ .

Let us fix  $x \in \Omega$  such that  $\limsup_{t \uparrow T_{\Omega, \mathbf{X}}(x)} V_{\Omega}(\mathbf{X}(t, x)) = \infty$  and  $\psi_n(\mathbf{X}(\cdot, x))$  is uniformly continuous in  $[0, T_{\Omega, \mathbf{X}}(x))$  for every  $n \in \mathbb{N}$ . The lim sup condition yields that the limit of all  $\psi_n(\mathbf{X}(t, x))$  as  $t \uparrow T_{\Omega, \mathbf{X}}(x)$  must be 0. On the other hand, if the lim inf of  $V_{\Omega}(\mathbf{X}(t, x))$  as  $t \uparrow T_{\Omega, \mathbf{X}}(x)$  were finite, we could find an integer  $n$  and  $t_k \uparrow T_{\Omega, \mathbf{X}}(x)$  with  $\mathbf{X}(t_k, x) \in \Omega_n$  for all  $k$ . Since  $\psi_{n+1}(\mathbf{X}(t_k, x)) = 1$  we obtain a contradiction.  $\square$

**Remark 7.2.** Under the assumptions of the previous theorem with  $\Omega = \mathbb{R}^d$ , given any probability measure  $\mu_0 \leq C\mathcal{L}^d$  for some  $C > 0$ , it can be easily shown that the measure

$$\mu_t := \mathbf{X}(t, \cdot)_{\#}(\mu_0 \llcorner \{T_{\mathbf{X}} > t\}), \quad t \in [0, T] \quad (7.4)$$

is a bounded (by (7.2)), weakly\* continuous, distributional solution to the continuity equation. We notice that the same statement is not true if we assume only a local bound on  $\operatorname{div} \mathbf{b}$ , since the measure (7.4) can be locally unbounded, as in the example of Proposition 7.3, and therefore we cannot write the distributional formulation of the continuity equation.

To see that (7.4) is a distributional solution of the continuity equation, we consider  $\varphi \in C_c^\infty(\mathbb{R}^d)$  and we define the function  $g(t, x)$  as  $\varphi(\mathbf{X}(t, x))$  if  $t < T_{\mathbf{X}}(x)$  or  $t = T_{\mathbf{X}}(x) = T$ , and  $g(t, x) = 0$  otherwise. By Theorem 7.1 we notice that  $g(t, x)$  is absolutely continuous with respect to  $t$  for  $\mathcal{L}^d$ -a.e.  $x \in \mathbb{R}^d$  and that  $\frac{d}{dt}g(t, x) = 1_{\{T_{\mathbf{X}}(x) > t\}} \nabla \varphi(\mathbf{X}(t, x)) \mathbf{b}(t, \mathbf{X}(t, x))$  for  $\mathcal{L}^1$ -a.e.  $t \in (0, T)$ , for  $\mathcal{L}^d$ -a.e.  $x \in \mathbb{R}^d$ . We deduce that  $t \mapsto \int_{\{T_{\mathbf{X}} > t\}} \varphi(\mathbf{X}(t, x)) d\mu_0(x)$  is absolutely continuous and that its derivative is given by

$$\begin{aligned} \frac{d}{dt} \int_{\{T_{\mathbf{X}} > t\}} \varphi(\mathbf{X}(t, x)) d\mu_0(x) &= \frac{d}{dt} \int_{\mathbb{R}^d} g(t, x) d\mu_0(x) \\ &= \int_{\{T_{\mathbf{X}} > t\}} \nabla \varphi(\mathbf{X}(t, x)) \mathbf{b}(t, \mathbf{X}(t, x)) d\mu_0(x). \end{aligned}$$

The proper blow up may fail for the maximal regular flow due only to the lack of a global bound on the divergence of  $\mathbf{b}$ , as shown in the next example.

In the following we denote by  $\mathbf{e}_1, \dots, \mathbf{e}_d$  the canonical basis of  $\mathbb{R}^d$  and  $B_r^{(d-1)}(x') \subset \mathbb{R}^{d-1}$  the ball of center  $x' \in \mathbb{R}^{d-1}$  and radius  $r$ . We denote each point  $x \in \mathbb{R}^d$  as  $x = (x', x_d)$ , where  $x'$  are the first  $d - 1$  coordinates of  $x$ . For simplicity we write  $T_{\mathbf{X}}$  for  $T_{\mathbb{R}^d, \mathbf{X}}$ .

**Proposition 7.3.** *Let  $d \geq 3$ . There exist an autonomous vector field  $\mathbf{b} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and a Borel set of positive measure  $\Sigma \subset \mathbb{R}^d$  such that  $\mathbf{b} \in W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$  for some  $p > 1$ ,  $\operatorname{div} \mathbf{b} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ , and*

$$T_{\mathbf{X}}(x) \leq 2, \quad \liminf_{t \uparrow T_{\mathbf{X}}(x)} |\mathbf{X}(t, x)| = 0, \quad \limsup_{t \uparrow T_{\mathbf{X}}(x)} |\mathbf{X}(t, x)| = \infty \quad (7.5)$$

for every  $x \in \Sigma$ .

*Proof.* We build a vector field whose trajectories are represented in Figure 2.

Let  $\{a_k\}_{k \in \mathbb{N}}$  be a fastly decaying sequence to be chosen later. For every  $k = 1, 2, \dots$  we define the cylinders

$$E_k = \begin{cases} B_{a_k}^{(d-1)}(2^{-k} \mathbf{e}_1) \times [-2^{k-1}, 2^k] & \text{if } k \text{ is odd} \\ B_{a_k}^{(d-1)}(2^{-k} \mathbf{e}_1) \times [-2^k, 2^{k-1}] & \text{if } k \text{ is even.} \end{cases}$$

We also define

$$E_0 = B_{a_1}^{(d-1)}(2^{-1} \mathbf{e}_1) \times (-\infty, -1].$$

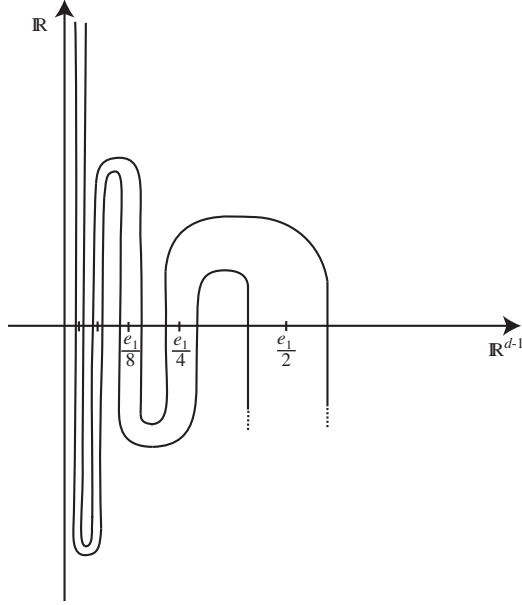


Figure 2: The trajectories of  $\mathbf{b}$  oscillate between 0 and  $\infty$ .

Let  $\varphi \in C_0^\infty(B_1^{(d-1)})$  be a nonnegative cutoff function which is equal to 1 in  $B_{1/2}$ . In every  $E_k$  the vector field  $\mathbf{b}$  points in the  $d$ -th direction and it depends only on the first  $d-1$  variables

$$\mathbf{b}(x) := \begin{cases} (-1)^{k+1} 4^k \varphi\left(\frac{x' - 2^{-k} \mathbf{e}_1}{a_k}\right) \mathbf{e}_d & \forall x \in E_k, k \geq 1 \\ 4\varphi\left(\frac{x' - 2^{-1} \mathbf{e}_1}{a_1}\right) \mathbf{e}_d & \forall x \in E_0. \end{cases} \quad (7.6)$$

Notice that  $\operatorname{div} \mathbf{b} = 0$  in every  $E_k$  and that  $\mathbf{b}$  is 0 on the lateral boundary of every cylinder  $E_k$  since  $\varphi$  is compactly supported.

For every  $k \geq 1$  we define the cylinders  $E'_k \subset \mathbb{R}^d$  as

$$E'_k = \begin{cases} B_{a_k/2}^{(d-1)}(2^{-k} \mathbf{e}_1) \times [-2^{k-1}, 2^k] & \text{if } k \text{ is odd} \\ B_{a_k/2}^{(d-1)}(2^{-k} \mathbf{e}_1) \times [-2^k, 2^{k-1}] & \text{if } k \text{ is even.} \end{cases}$$

For every  $k \in \mathbb{N}$  we define a handle  $F_k$  which connects  $E_k$  with  $E_{k+1}$  as in Figure 3. It is made of a family of smooth, nonintersecting curves of length less than 1 which connect the top of  $E_k$  to the top of  $E_{k+1}$  and  $E'_k$  with  $E'_{k+1}$ . We denote by  $F'_k$  the handle between  $E'_k$  and  $E'_{k+1}$ , as in Figure 3.

The vector field  $\mathbf{b}$  is extended to be 0 outside  $\cup_{k=0}^\infty (E_k \cup F_k)$ . It is extended inside every  $F_k$  by choosing a smooth extension in a neighborhood of each handle, whose trajectories are the ones described by the handle. The modulus of  $\mathbf{b}$  is chosen to be between  $4^k$  and  $4^{k+1}$  in  $F'_k$  (notice that  $|\mathbf{b}(x)| = 4^k$  on the top of  $E'_k$  thanks to (7.6)).

With this choice, every trajectory in  $F'_k$  is not longer than 1 and the vector field  $\mathbf{b}$  is of size  $4^k$ . We deduce that the handle is traversed in time less than  $4^{-k}$ .

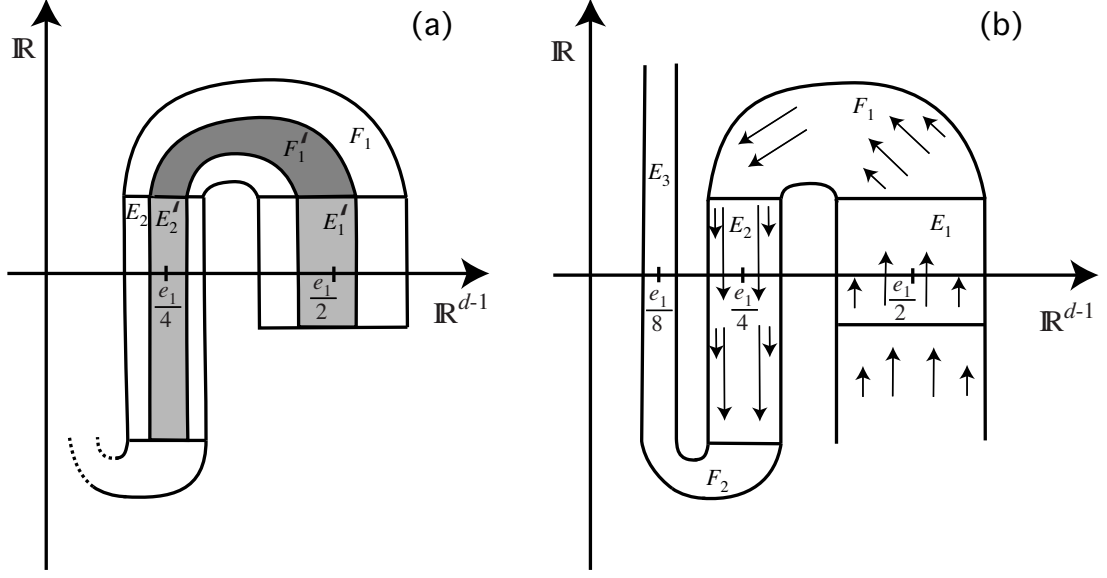


Figure 3: The sets  $E_k$ ,  $F_k$ ,  $E'_k$ , and  $F'_k$  and the vector field  $\mathbf{b}$ .

By the construction it is clear that  $\mathbf{b}$  is smooth in  $\mathbb{R}^d \setminus \mathbb{R}e_d$ . We show that  $\mathbf{b} \in W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$  for some  $p > 1$  by estimating the  $W^{1,p}$  norm of  $\mathbf{b}$  in every ball  $B_R$ . With this estimate, one can easily see that  $\mathbf{b}$  is the limit of smooth vector fields with bounded  $W^{1,p}$  norms on  $B_R$ ; it is enough to consider the restriction of  $\mathbf{b}$  to the first  $n$  sets  $E_k \cup F_k$ .

Fix  $R > 0$ . The  $W^{1,p}$  norm of  $\mathbf{b}$  in  $B_R$  is estimated by

$$\|\mathbf{b}\|_{W^{1,p}(B_R)} \leq \|\mathbf{b}\|_{W^{1,p}(E_0 \cap B_R)} + \sum_{k=1}^{\infty} \|\mathbf{b}\|_{W^{1,p}(F_k \cap B_R)} + \sum_{k=1}^{\infty} \|\mathbf{b}\|_{W^{1,p}(E_k)}. \quad (7.7)$$

The first term is obviously finite (depending on  $R$ ); since  $B_R$  intersects at most finitely many  $F_k$ , the second sum in the right-hand side of (7.7) has only finitely many nonzero terms. As regards the third sum, we compute the  $W^{1,p}$  norm of  $\mathbf{b}$  in each set  $E_k$ . For every  $k \in \mathbb{N}$

$$\|\mathbf{b}\|_{L^p(E_k)} \leq 4^k (2R)^{1/p} \left\| \varphi \left( \frac{x' - 2^{-k} \mathbf{e}_1}{a_k} \right) \right\|_{L^p(B_{a_k}^{(d-1)}(2^{-k} \mathbf{e}_1))} = 4^k (2R a_k^{d-1})^{1/p} \|\varphi\|_{L^p(B_1^{(d-1)})}$$

and similarly

$$\|\nabla \mathbf{b}\|_{L^p(E_k)} \leq \frac{4^k (2R)^{1/p}}{a_k} \left\| \nabla \varphi \left( \frac{x' - 2^{-k} \mathbf{e}_1}{a_k} \right) \right\|_{L^p(B_{a_k}^{(d-1)}(2^{-k} \mathbf{e}_1))} = \frac{4^k (2R a_k^{d-1})^{1/p}}{a_k} \|\nabla \varphi\|_{L^p(B_1^{(d-1)})}.$$

Since  $a_k \leq 1$ , the series in the right-hand side of (7.7) is estimated by

$$\sum_{k=1}^{\infty} \|\mathbf{b}\|_{W^{1,p}(E_k)} \leq C(R, \varphi) \sum_{k=1}^{\infty} 4^k a_k^{(d-1)/p-1}$$

and it is convergent for every  $p < d - 1$  provided that we take  $a_k \leq 8^{-pk/(d-1-p)}$ . Hence  $\mathbf{b} \in W^{1,p}(B_R; \mathbb{R}^d)$  for every  $R > 0$ .

To check that  $\operatorname{div} \mathbf{b} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ , we notice that  $\mathbf{b}$  is divergence free in  $\mathbb{R}^d \setminus \cup_{k=0}^\infty F_k$  and that for every  $R > 0$  the ball  $B_R$  intersects only finitely many handles  $F_k$ ; in particular  $\mathbf{b}$  is divergence free in  $B_1$ . Since  $\mathbf{b}$  is smooth in a neighbourhood of each handle, we deduce that  $\operatorname{div} \mathbf{b}$  is bounded in every  $B_R$ .

Finally we set  $\Sigma = B_{a_1/2}(\mathbf{e}_1/2) \times [0, 1]$  and we show that for every  $x \in \Sigma$  the smooth trajectory of  $\mathbf{b}$  starting from  $x$  satisfies (7.5). The trajectory of  $x$  lies by construction in  $\cup_{k=0}^\infty (E'_k \cup F'_k)$ . For every  $k \in \mathbb{N}$ , the time requested to cross the set  $E'_k$  is  $2^k/4^k$  and, as observed before, the time requested to cross  $F'_k$  is less than  $4^{-k}$ . Hence

$$T_{\mathbf{X}}(x) \leq \sum_{k=1}^{\infty} \frac{2^k + 1}{4^k} \leq 2 \quad \forall x \in \Sigma.$$

The other properties in (7.5) are satisfied by construction.  $\square$

In dimension  $d = 2$ , thanks to the smoothness of the vector field built in the previous example outside the  $x_2$ -axis, there exists only an integral curve of  $\mathbf{b}$  for every  $x \in \mathbb{R}^2 \setminus \{x_1 = 0\}$ . Hence, thanks to the superposition principle the previous example satisfies the assumption (b- $\Omega$ ) on  $\mathbf{b}$  and therefore provides a two-dimensional counterexample to the proper blow-up of trajectories. On the other hand, the vector field built in the previous example is not in  $BV_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ . We show indeed in the next proposition that for any autonomous  $BV_{\text{loc}}$  vector field in dimension  $d = 2$  the behavior of the previous example (see Figure 2) cannot happen and the trajectories must blow up properly. It looks likely that, with  $d = 2$  and a nonautonomous vector field, one can build an example following the lines of the example in Proposition 7.3.

**Proposition 7.4.** *Let  $\mathbf{b} \in BV_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ ,  $\operatorname{div} \mathbf{b} \in L_{\text{loc}}^\infty(\mathbb{R}^2)$ . Then*

$$\liminf_{t \uparrow T_{\mathbf{X}}(x)} |\mathbf{X}(t, x)| = \infty \quad \text{for } \mathcal{L}^2\text{-a.e. } x \in \mathbb{R}^2 \text{ such that } \limsup_{t \uparrow T_{\mathbf{X}}(x)} |\mathbf{X}(t, x)| = \infty. \quad (7.8)$$

*Proof.* **Step 1.** Let  $R > 0$ . We prove that for every vector field  $\mathbf{b} \in BV_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$

$$\int_R^{R+1} \operatorname{ess\,sup}_{x \in \partial B_r} |\mathbf{b}(x)| \, dr \leq \frac{1}{2\pi R} \int_{B_{R+1} \setminus B_R} |\mathbf{b}(x)| \, dx + |D\mathbf{b}|(B_{R+1} \setminus B_R). \quad (7.9)$$

For this, let  $\mathbf{b}_\varepsilon$  be a sequence of smooth vector fields which approximate  $\mathbf{b}$  in  $BV(B_{R+1} \setminus B_R)$ , namely

$$\lim_{\varepsilon \rightarrow 0} |\mathbf{b}_\varepsilon - \mathbf{b}| = 0 \quad \text{in } L^1(B_{R+1} \setminus B_R), \quad \lim_{\varepsilon \rightarrow 0} \int_{B_{R+1} \setminus B_R} |\nabla \mathbf{b}_\varepsilon(x)| \, dx = |D\mathbf{b}|(B_{R+1} \setminus B_R).$$

Up to a subsequence (not relabeled) we deduce that for  $\mathcal{L}^1$ -a.e.  $r \in (R, R + 1)$

$$\lim_{\varepsilon \rightarrow 0} \mathbf{b}_\varepsilon = \mathbf{b} \quad \text{in } L^1(\partial B_r; \mathbb{R}^2).$$

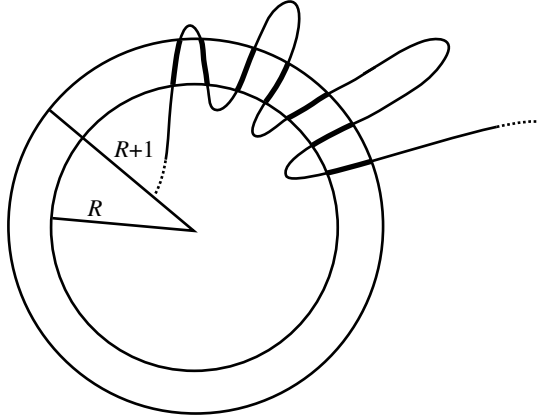


Figure 4: For an autonomous vector field  $\mathbf{b}$  in the plane, we consider an integral curve of a suitable representative of  $\mathbf{b}$ , namely a vector field which coincides  $\mathcal{L}^2$ -a.e. with  $\mathbf{b}$ . Given  $R > 0$ , the time needed for the integral curve to cross the annulus  $B_{R+1} \setminus B_R$  is greater or equal than the constant  $\|\text{ess sup}_{\partial B_r} |\mathbf{b}|\|_{L^1(R, R+1)}^{-1}$  (see (7.10) below). For this reason, every trajectory can cross only finitely many times the annulus in finite time and therefore every unbounded trajectory must blow up properly, as in (7.5).

Since we can control the supremum of the one dimensional restriction of  $\mathbf{b}_\varepsilon$  to  $\partial B_r$  through the  $L^1$  norm of  $\mathbf{b}_\varepsilon$  and the total variation we have that

$$\sup_{x \in \partial B_r} |\mathbf{b}_\varepsilon(x)| \leq \frac{1}{2\pi r} \int_{\partial B_r} |\mathbf{b}_\varepsilon(x)| dx + \int_{\partial B_r} |\nabla \mathbf{b}_\varepsilon(x)| dx.$$

Hence, integrating with respect to  $r$  in  $(R, R+1)$ , (7.9) holds for  $\mathbf{b}_\varepsilon$ :

$$\int_R^{R+1} \sup_{x \in \partial B_r} |\mathbf{b}_\varepsilon(x)| dr \leq \frac{1}{2\pi R} \int_{B_{R+1} \setminus B_R} |\mathbf{b}_\varepsilon(x)| dx + \int_{B_{R+1} \setminus B_R} |\nabla \mathbf{b}_\varepsilon(x)| dx.$$

Taking the  $\liminf$  in both sides as  $\varepsilon$  goes to 0, by Fatou lemma we deduce that

$$\begin{aligned} \int_R^{R+1} \text{ess sup}_{x \in \partial B_r} |\mathbf{b}(x)| dr &\leq \int_{B_{R+1} \setminus B_R} \liminf_{\varepsilon \rightarrow 0} \sup_{x \in \partial B_r} |\mathbf{b}_\varepsilon(x)| dr \\ &\leq \liminf_{\varepsilon \rightarrow 0} \int_R^{R+1} \sup_{x \in \partial B_r} |\mathbf{b}_\varepsilon(x)| dr \\ &\leq \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{2\pi R} \int_{B_{R+1} \setminus B_R} |\mathbf{b}_\varepsilon(x)| dx + \int_{B_{R+1} \setminus B_R} |D\mathbf{b}_\varepsilon(x)| dx \right) \\ &= \frac{1}{2\pi R} \int_{B_{R+1} \setminus B_R} |\mathbf{b}(x)| dx + |D\mathbf{b}|(B_{R+1} \setminus B_R). \end{aligned}$$

**Step 2.** Let  $R > 0$  and let  $\mathbf{c} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a Borel vector field such that

$$f(r) := \sup_{x \in \partial B_r} |\mathbf{c}(x)| \in L^1(R, R+1).$$

Let  $\gamma : [0, \tau] \rightarrow \overline{B_{R+1}} \setminus B_R$  be an absolutely continuous integral curve of  $\mathbf{c}$  (namely  $\dot{\gamma} = \mathbf{c}(\gamma)$ )  $\mathcal{L}^1$ -a.e. in  $(0, \tau)$  such that  $\gamma(0) \in \partial B_R$  and  $\gamma(\tau) \in \partial B_{R+1}$ . We claim that

$$\tau \geq \left( \int_R^{R+1} f(r) dr \right)^{-1}. \quad (7.10)$$

To prove this, we define the nondecreasing function  $\sigma : [0, \tau] \rightarrow \mathbb{R}$

$$\sigma(t) = \max_{s \in [0, t]} |\gamma(s)| \quad \forall t \in [0, \tau]; \quad (7.11)$$

we have that  $\sigma(0) = R$  and  $\sigma(\tau) = R + 1$ . For every  $s, t \in [0, \tau]$  with  $s < t$  there holds

$$0 \leq \sigma(t) - \sigma(s) \leq \sup_{r \in (s, t]} (|\gamma(r)| - |\gamma(s)|)^+ \leq \int_s^t \left| \frac{d}{dr} |\gamma(r)| \right| dr \leq \int_s^t |\dot{\gamma}(r)| dr.$$

Thus  $\sigma$  is absolutely continuous and  $\dot{\sigma} \leq |\dot{\gamma}|$   $\mathcal{L}^1$ -a.e. in  $(0, \tau)$ . In addition, for every  $t \in (0, \tau)$  such that  $\sigma(t) \neq |\gamma(t)|$  the function  $\sigma$  is constant in a neighborhood of  $t$ , hence  $\dot{\sigma} \leq \chi_{\{\sigma=|\gamma|\}} |\dot{\gamma}|$   $\mathcal{L}^1$ -a.e. in  $(0, \tau)$ . Therefore

$$\dot{\sigma}(t) \leq 1_{\{\sigma=|\gamma|\}}(t) |\dot{\gamma}(t)| = 1_{\{\sigma=|\gamma|\}}(t) |\mathbf{c}(\gamma(t))| \leq f(\sigma(t)) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, \tau).$$

By Hölder inequality and the change of variable formula we deduce that

$$\begin{aligned} 1 \leq [\sigma(\tau) - \sigma(0)]^2 &\leq \left( \int_0^\tau \dot{\sigma}(t) dt \right)^2 \leq \tau \int_0^\tau [\dot{\sigma}(t)]^2 dt \\ &\leq \tau \int_0^\tau \dot{\sigma}(t) f(\sigma(t)) dt = \tau \int_R^{R+1} f(\sigma) d\sigma, \end{aligned}$$

which proves (7.10).

**Step 3.** We conclude the proof. Using the invariance of the concept of maximal regular flow (see Remark 4.6) we can work with a well-chosen representative which allows us to apply the estimate in Step 2. For this specific representation of  $\mathbf{b}$ , we show that *every* integral unbounded trajectory blows up properly.

For  $\mathcal{L}^d$ -a.e.  $r > 0$  the restriction  $\mathbf{b}_r(x) = \mathbf{b}(rx)$ ,  $x \in \mathbb{S}^1$ , of the vector field  $\mathbf{b}$  to  $\partial B_r$  is  $BV$ . We remind that every 1-dimensional  $BV$  function has a precise representative given at every point by the average of the right approximate limit and of the left approximate limit, which exist everywhere. We define the Borel vector field  $\mathbf{c} : \mathbb{R}^2 \rightarrow \mathbb{R}$  as

$$\mathbf{c}(rx) = \text{the precise representative of } \mathbf{b}_r \text{ at } x \quad \forall x \in \mathbb{S}^1$$

for all  $r$  such that  $\mathbf{b}_r \in BV(\mathbb{S}^1)$ , and 0 otherwise. Notice that, by Fubini theorem,  $\mathbf{c}$  coincides  $\mathcal{L}^2$ -a.e. with  $\mathbf{b}$ , and that  $\sup |\mathbf{c}(r \cdot)| \leq \text{ess sup } |\mathbf{b}(r \cdot)|$  for all  $r > 0$ .

Let us assume by contradiction the existence of  $\bar{x} \in \mathbb{R}^d$  such that  $\mathbf{X}(\cdot, \bar{x})$  is an integral curve of the precise representative  $\mathbf{c}$  and

$$\liminf_{t \uparrow T_{\mathbf{X}}(\bar{x})} |\mathbf{X}(t, \bar{x})| < \infty, \quad \limsup_{t \uparrow T_{\mathbf{X}}(\bar{x})} |\mathbf{X}(t, \bar{x})| = \infty. \quad (7.12)$$



We fix  $R > 0$  greater than the liminf in (7.12), as in Figure 4 and we define  $f(r) := \sup_{x \in \partial B_r} |\mathbf{c}(x)|$ ,  $r \in [R, R+1]$ . Thanks to (7.9) applied to  $\mathbf{c}$ , we deduce that  $f \in L^1(R, R+1)$ . Therefore we can apply Step 2 to deduce that every transition from inside  $B_R$  to outside  $B_{R+1}$  requires at least time  $1/\|f\|_{L^1(R, R+1)} > 0$ . Hence the trajectory  $\mathbf{X}(\cdot, \bar{x})$  can cross the set  $B_{R+1} \setminus B_R$  only finitely many times in finite time, a contradiction.  $\square$

## 7.2 No blow-up criteria

If one is interested in estimating the blow-up time  $T_{\Omega, \mathbf{X}}$  of the maximal regular flow, or even if one wants to rule out the blow up, one may easily adapt to this framework the classical criterion based on the existence of a Lyapunov function  $\Psi : \mathbb{R}^d \rightarrow [0, \infty]$  satisfying  $\Psi(z) \rightarrow \infty$  as  $|z| \rightarrow \infty$  and

$$\frac{d}{dt} \Psi(x(t)) \leq C_{\Psi}(1 + \Psi(x(t)))$$

along absolutely continuous solutions to  $\dot{x} = \mathbf{b}(t, x)$ . On the other hand, in some cases, by a suitable approximation argument one can exhibit a solution  $\mu_t = \rho_t \mathcal{L}^d$  to the continuity equation with velocity field  $\mathbf{b}$  with  $|\mathbf{b}_t| \rho_t$  integrable. As in [10, Proposition 8.1.8] (where locally Lipschitz vector fields were considered) we can use the existence of this solution to rule out the blow-up.

In the next theorem we provide a sufficient condition for the continuity of  $\mathbf{X}$  at the blow-up time, using a global version of (a- $\Omega$ ) and the global bounded compression condition (7.2), implied by the global bound on divergence (7.1).

**Theorem 7.5.** *Let  $\mathbf{b} \in L^1((0, T) \times \Omega; \mathbb{R}^d)$  satisfy (b- $\Omega$ ) and assume that the maximal regular flow  $\mathbf{X}$  satisfies (7.2). Then  $\mathbf{X}(\cdot, x)$  is absolutely continuous in  $[0, T_{\Omega, \mathbf{X}}(x)]$  for  $\mathcal{L}^d$ -a.e.  $x \in \Omega$ , and the limit of  $\mathbf{X}(t, x)$  as  $t \uparrow T_{\Omega, \mathbf{X}}(x)$  belongs to  $\partial\Omega$  whenever  $T_{\Omega, \mathbf{X}}(x) < T$ .*

*Proof.* By (7.2) we have that

$$\begin{aligned} \int_{\Omega} \int_0^{T_{\Omega, \mathbf{X}}(x)} |\dot{\mathbf{X}}(t, x)| dt dx &= \int_{\Omega} \int_0^{T_{\Omega, \mathbf{X}}(x)} |\mathbf{b}(t, \mathbf{X}(t, x))| dt dx \\ &= \int_0^T \int_{\{T_{\Omega, \mathbf{X}} > t\}} |\mathbf{b}(t, \mathbf{X}(t, x))| dx dt \\ &\leq C_* \int_0^T \int_{\Omega} |\mathbf{b}(t, z)| dz dt < \infty, \end{aligned}$$

which proves that  $\mathbf{X}(\cdot, x)$  is absolutely continuous in  $[0, T_{\Omega, \mathbf{X}}(x)]$  for  $\mathcal{L}^d$ -a.e.  $x \in \Omega$ . Since the limsup  $V_{\Omega}(\mathbf{X}(t, x))$  as  $t \uparrow T_{\Omega, \mathbf{X}}$  is  $\infty$  whenever  $T_{\Omega, \mathbf{X}}(x) < T$ , we obtain that in this case the limit of  $\mathbf{X}(t, x)$  as  $t \rightarrow T_{\Omega, \mathbf{X}}(x)$  belongs to  $\partial\Omega$ .  $\square$

In the case  $\Omega = \mathbb{R}^d$  we now prove a simple criterion for global existence, which allows us to recover the classical result in the DiPerna-Lions theory on the existence of a global flow under the growth condition

$$\frac{|\mathbf{b}(t, x)|}{1 + |x|} \in L^1((0, T); L^1(\mathbb{R}^d)) + L^1((0, T); L^{\infty}(\mathbb{R}^d)). \quad (7.13)$$

As in the previous section, we will use in the next theorem the simplified notation  $T_{\mathbf{X}}$  for  $T_{\mathbb{R}^d, \mathbf{X}}$ .

**Theorem 7.6** (No blow-up criterion). *Let  $\mathbf{b} : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a Borel vector field which satisfies (a- $\mathbb{R}^d$ ) and (b- $\mathbb{R}^d$ ), and assume that the maximal regular flow  $\mathbf{X}$  satisfies (7.2). Assume that  $\rho_t \in L^\infty((0, T); L^1_+(\mathbb{R}^d))$  is a weakly\* continuous solution of the continuity equation satisfying the integrability condition*

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\mathbf{b}(t, x)|}{1 + |x|} \rho_t(x) dx dt < \infty. \quad (7.14)$$

*Then  $T_{\mathbf{X}}(x) = T$  and  $\mathbf{X}(\cdot, x) \in AC([0, T]; \mathbb{R}^d)$  for  $\rho_0 \mathcal{L}^d$ -a.e.  $x \in \mathbb{R}^d$ . In addition, if the growth condition (7.13) holds, then for any  $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$  nonnegative there exists a solution  $\rho_t$  satisfying (7.14), thus  $\mathbf{X}$  is defined in the whole  $[0, T] \times \mathbb{R}^d$ .*

*Proof.* For the first part of the statement we apply Theorem 2.1 to deduce that  $\rho_t$  is the marginal at time  $t$  of a measure  $\boldsymbol{\eta} \in \mathcal{M}_+(C([0, T]; \mathbb{R}^d))$  concentrated on absolutely continuous curves  $\eta$  in  $[0, T]$  solving the ODE  $\dot{\eta} = \mathbf{b}(t, \eta)$ . We then apply Theorem 3.4 to obtain that the conditional probability measures  $\boldsymbol{\eta}_x$  induced by the map  $e_0$  are Dirac masses for  $(e_0)_\# \boldsymbol{\eta}$ -a.e.  $x$ , hence (by uniqueness of the maximal regular flow)  $\rho_t$  is transported by  $\mathbf{X}$ . Notice that, as a consequence of the fact that  $\boldsymbol{\eta}$  is concentrated on absolutely continuous curves in  $[0, T]$ , the flow is globally defined on  $[0, T]$ , thus  $T_{\mathbf{X}}(x) = T$ .

For the second part, under assumption (7.13) the existence of a nonnegative and weakly\* continuous solution of the continuity equation  $\rho_t$  in  $L^\infty((0, T); L^1 \cap L^\infty(\mathbb{R}^d))$  can be achieved by a simple smoothing argument. So, the bound in  $L^1 \cap L^\infty$  on  $\rho_t$  can be combined with (7.13) to obtain (7.14).  $\square$

**Remark 7.7.** We remark that if only a local bound on the divergence is assumed as in Section 5, the growth assumption (7.13) is not enough to guarantee that the trajectories of the regular flow do not blow up. On the other hand, it can be easily seen that if we assume that  $\mathbf{b}$  satisfies (a- $\mathbb{R}^d$ ), (b- $\mathbb{R}^d$ ), (5.1) and  $|\mathbf{b}(t, x)|/(1 + |x|) \in L^1((0, T); L^\infty(\mathbb{R}^d))$ , every integral curve of  $\mathbf{b}$  cannot blow up in finite time and therefore the maximal regular flow satisfies  $T_{\mathbf{X}}(x) = T$  and  $\mathbf{X}(\cdot, x) \in AC([0, T]; \mathbb{R}^d)$  for  $\mathcal{L}^d$ -a.e.  $x \in \mathbb{R}^d$ .

Theorem 7.6 is useful in applications when one constructs solutions by approximation. For instance, for the Vlasov-Poisson system in dimension  $d \leq 4$ , a variant of this result can be used to show that trajectories which transport a bounded solution with finite energy do not explode in the phase space (see [4]).

## Appendix: On the local character of the assumption (b- $\Omega$ )

Here we prove that the property (b- $\Omega$ ) of Section 3 is local, in analogy with the other assumptions (a- $\Omega$ ) and the local bounds on distributional divergence made throughout this paper. More precisely, the following assumption is equivalent to (b- $\Omega$ ):

(b'- $\Omega$ ) for any  $t_0 \geq 0$  and  $x_0 \in \Omega$ , there exists  $\varepsilon := \varepsilon(t_0, x_0) > 0$  such that for any nonnegative  $\bar{\rho} \in L^\infty(\mathbb{R}^d)$  with compact support contained in  $B_\varepsilon(x_0) \subset \Omega$  and any closed interval  $I = [a, b] \subset [t_0 - \varepsilon, t_0 + \varepsilon] \cap [0, T]$ , the continuity equation

$$\frac{d}{dt}\rho_t + \operatorname{div}(\mathbf{b}\rho_t) = 0 \quad \text{in } (a, b) \times \mathbb{R}^d$$

has at most one weakly\* continuous solution  $I \ni t \mapsto \rho_t \in \mathcal{L}_{I, \Omega}$  with  $\rho_a = \bar{\rho}$  and  $\rho_t$  compactly supported in  $B_\varepsilon(x_0)$  for every  $t \in [a, b]$ .

**Lemma 7.8.** *If the assumptions (a- $\Omega$ ) and (b'- $\Omega$ ) on the vector field  $\mathbf{b}$  are satisfied, then (b- $\Omega$ ) is satisfied.*

*Proof. Step 1.* Let  $\boldsymbol{\eta} \in \mathcal{P}(C([a, b]; \mathbb{R}^d))$ ,  $0 \leq a < b \leq T$ , be concentrated on absolutely continuous curves  $\eta \in AC([a, b]; K)$  for some  $K \subset \Omega$  compact, solving the ODE  $\dot{\eta} = \mathbf{b}(t, \eta)$   $\mathcal{L}^1$ -a.e. in  $(a, b)$ , and such that  $(e_t)_\# \boldsymbol{\eta} \leq C \mathcal{L}^d$  for any  $t \in [0, T]$ . We claim that the conditional probability measures  $\boldsymbol{\eta}_x$  induced by the map  $e_a$  are Dirac masses for  $(e_a)_\# \boldsymbol{\eta}$ -a.e.  $x$ .

To this end, for  $s, t \in [a, b]$ ,  $s < t$ , we denote by  $\Sigma^{s,t} : C([a, b]; \mathbb{R}^d) \rightarrow C([s, t]; \mathbb{R}^d)$  the map induced by restriction to  $[s, t]$ , namely  $\Sigma^{s,t}(\eta) = \eta|_{[s, t]}$ . For  $(e_a)_\# \boldsymbol{\eta}$ -a.e.  $x \in \mathbb{R}^d$  we define  $\tau(x)$  the first splitting time of  $\boldsymbol{\eta}_x$ , namely the infimum of all  $t > a$  such that  $(\Sigma^{a,t})_\# \boldsymbol{\eta}_x$  is not a Dirac mass. We agree that  $\tau(x) = b$  if  $\boldsymbol{\eta}_x$  is a Dirac mass. We also define the splitting point  $B(x)$  as  $\eta(\tau(x))$  for any  $\eta \in \operatorname{supp} \boldsymbol{\eta}_x$ . By contradiction, we assume that the set  $\{x \in \mathbb{R}^d : \tau(x) < T\}$  has positive  $(e_a)_\# \boldsymbol{\eta}$  measure.

For every  $t_0 > 0$  and  $x_0 \in \mathbb{R}^d$  let  $\varepsilon(t_0, x_0) > 0$  be as in (b'- $\Omega$ ). By a covering argument, we can take a finite cover of  $[a, b] \times K$  with sets of the form

$$I_{t_0, x_0, \varepsilon(t_0, x_0)} = (t_0 - \varepsilon(t_0, x_0), t_0 + \varepsilon(t_0, x_0)) \times B_{\varepsilon(t_0, x_0)/2}(x_0).$$

We deduce that there exist  $t_0 > 0$  and  $x_0 \in \mathbb{R}^d$  such that the set

$$E_0 := \{x \in \mathbb{R}^d : \tau(x) < T, (\tau(x), B(x)) \in I_{t_0, x_0, \varepsilon(t_0, x_0)}\} \quad (7.15)$$

has positive  $(e_a)_\# \boldsymbol{\eta}$  measure.

For every  $p, q \in \mathbb{Q}$  with  $a \leq p < q \leq b$  we define the open set

$$E_{p,q} := \{\eta \in C([a, b]; \mathbb{R}^d) : \eta([p, q]) \subset B_{\varepsilon(t_0, x_0)/2}(x_0)\}.$$

We claim that there exist a set  $E_1 \subset E_0$  and  $p, q \in \mathbb{Q} \cap [a, b]$ ,  $p < q$  such that  $(e_a)_\# \boldsymbol{\eta}(E_1) > 0$  and for every  $x \in E_1$  the measure  $\Sigma_{\#}^{p,q}(1_{E_{p,q}} \boldsymbol{\eta}_x)$  is not a Dirac delta.

To this end, it is enough to show that for  $\mathcal{L}^d$ -a.e.  $x \in E_0$  there exist  $p_x, q_x \in \mathbb{Q} \cap [a, b]$ ,  $p_x < q_x$ , such that  $\Sigma_{\#}^{p_x, q_x}(1_{E_{p_x, q_x}} \boldsymbol{\eta}_x)$  is not a Dirac delta.

Let us consider  $\eta_1 \in \operatorname{supp} \boldsymbol{\eta}_x$ ; it satisfies  $\eta_1(\tau(x)) = B(x) \in B_{\varepsilon(t_0, x_0)/2}(x_0)$ . Let  $p_x, q_x$  be chosen such that  $\eta_1([p_x, q_x]) \subseteq B_{\varepsilon(t_0, x_0)/2}(x_0)$ . By definition of  $\tau(x)$  we know that  $\Sigma_{\#}^{p_x, q_x} \boldsymbol{\eta}_x$  is not a Dirac delta. Hence there exists  $\eta_2 \in C([a, b]; \mathbb{R}^d)$  such that  $\eta_2 \in \operatorname{supp}(\boldsymbol{\eta}_x)$ ,  $\eta_2(\tau(x)) = B(x)$ ,  $\eta_1(t) \neq \eta_2(t)$  for every  $t \in [a, \tau(x)]$ ,  $\eta_1(t) \neq \eta_2(t)$  for some  $t \in [\tau(x), q_x]$ .

Up to reducing  $q_x$ , we can assume that  $\Sigma^{p_x, q_x}(\eta_1), \Sigma^{p_x, q_x}(\eta_2)$  are curves whose image is contained in  $B_{\varepsilon(t_0, x_0)/2}(x_0)$ , so that  $\eta_1, \eta_2 \in E_{p_x, q_x}$ , and which do not coincide. Moreover, since  $\text{supp}(\Sigma_{\#}^{p_x, q_x} \eta_x) = \Sigma^{p_x, q_x}(\text{supp} \eta_x)$ , we deduce that both  $\Sigma^{p_x, q_x}(\eta_1)$  and  $\Sigma^{p_x, q_x}(\eta_2)$  belong to the support of  $\Sigma_{\#}^{p_x, q_x}(\eta_x)$  and hence  $\Sigma_{\#}^{p_x, q_x}(1_{E_{p_x, q_x}} \eta_x) = 1_{\Sigma^{p_x, q_x}(E_{p_x, q_x})} \Sigma_{\#}^{p_x, q_x} \eta_x$  is not a Dirac delta.

Let  $\delta > 0$  be small enough so that  $E_\delta = E_1 \cap \{x : \eta_x(E_{p, q}) \geq \delta\}$  has positive  $(e_a)_{\#} \eta$ -measure. We introduce the probability measure  $\tilde{\eta} \in \mathcal{P}(C([a, b]; \mathbb{R}^d))$

$$\tilde{\eta} := ((e_a)_{\#} \eta \llcorner E_\delta) \otimes \left( \frac{1_{E_{p, q}} \eta_x}{\eta_x(E_{p, q})} \right) = ((e_a)_{\#} \eta \llcorner E_\delta) \otimes \tilde{\eta}_x,$$

which is nonnegative, and less than or equal to  $\eta/\delta$ . Moreover  $\Sigma_{\#}^{p, q} \tilde{\eta} \in \mathcal{P}(C([p, q]; \mathbb{R}^d))$  is concentrated on curves in  $B_{\varepsilon(t_0, x_0)/2}(x_0)$ , and

$$\Sigma_{\#}^{p, q} \tilde{\eta}_x = \frac{\Sigma_{\#}^{p, q}(1_{E_{p, q}} \eta_x)}{\eta_x(E_{p, q})} \text{ is not a Dirac mass for } (e_a)_{\#} \eta\text{-a.e. } x \in E_\delta.$$

Applying Theorem 3.4 with  $\lambda = \Sigma_{\#}^{p, q} \tilde{\eta}$  and  $\Omega = B_{\varepsilon(t_0, x_0)}(x_0)$  in the time interval  $[p, q]$ , and thanks to the local uniqueness of bounded, nonnegative solutions of the continuity equation in  $I_{t_0, x_0, \varepsilon(t_0, x_0)}$ , which in turn follows from (b'- $\Omega$ ), we deduce that the disintegration  $\Sigma_{\#}^{p, q} \tilde{\eta}_x$  of  $\Sigma_{\#}^{p, q} \tilde{\eta}$  induced by  $e_a$  is a Dirac mass for  $(e_a)_{\#} \eta$ -a.e.  $x \in E_\delta$ . By the uniqueness of the disintegration, we obtain a contradiction.

**Step 2.** Let  $\mu^1$  and  $\mu^2$  be two solutions of the continuity equation as in (b) with the same initial datum. Let  $\eta^1, \eta^2 \in \mathcal{P}(C([a, b]; \mathbb{R}^d))$  be the representation of  $\mu^1$  and  $\mu^2$  obtained through the superposition principle; they are concentrated on absolutely continuous integral curves of  $\mathbf{b}$  and they satisfy  $\mu_t^i = (e_t)_{\#} \eta^i$  for any  $t \in [0, T]$ ,  $i = 1, 2$ . Since there exists a compact set  $K \subset \Omega$  such that  $\mu_t^i$  is concentrated on  $K$  for every  $t \in [0, T]$ ,  $\eta^i$  is concentrated on absolutely continuous curves contained in  $K$  for  $i = 1, 2$ . Then by the linearity of the continuity equation  $(e_t)_{\#}[(\eta_1 + \eta_2)/2] = (\mu_t^1 + \mu_t^2)/2$  is still a solution to the continuity equation; by Step 1 we obtain that  $(\eta_x^1 + \eta_x^2)/2$  are Dirac masses for  $\mu_0$ -a.e.  $x$ . This shows that  $\eta_x^1 = \eta_x^2$  for  $\mu_0$ -a.e.  $x$  and therefore that  $\mu_t^1 = \mu_t^2$  for every  $t \in [0, T]$ .  $\square$

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