GENERIC REGULARITY OF EQUILIBRIUM MEASURES FOR THE LOGARITHMIC POTENTIAL WITH EXTERNAL FIELDS

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Abstract. It is a well-known conjecture in β-models and in their discrete counterpart that, generically, external potentials should be "off-critical" (or, equivalently, "regular"). Exploiting the connection between minimizing measures and thin obstacle problems, we give a positive answer to this conjecture.

1. Introduction

Given a potential $V : \mathbb{R} \to \mathbb{R}$, a central question arising both from the theory of Coulomb gases and the mean-field limit of β-ensembles is to understand the behavior of probability measures μ_V minimizing the energy

$$\mathcal{E}_V(\mu) := \int_{\mathbb{R}} \int_{\mathbb{R}} \big(V(x) + V(y) - \log|x-y|\big)\, d\mu(x)\, d\mu(y) \tag{1.1}$$

(see, for instance, [25, 27]). The classical case $V(x) = x^2$, which comes from Gaussian Orthogonal Ensembles, was first studied in [28], and the minimizing probability is known as the "semicircle law", since its density $\frac{d\mu_V}{dx} = \frac{2}{\pi}\sqrt{(x+1)(1-x)}$ is, up to a constant, the graph of the (upper) unit circle.

β-models. Most results on β-models (see, for instance, [2, 7, 14, 8, 4, 18, 6] and the references therein) rely on the assumption that the semicircle represents the general behavior of such minimizing measures. More precisely, given a potential V, the following assumption (usually called "off-criticality" or "regularity" assumption) is made:
(A) *The minimizing probability μ_V is supported over finitely many disjoint compact intervals and, inside each such interval $[a, b]$, it has the form*

$$\frac{d\mu_V}{dx} = Q_V(x)\sqrt{(x-a)(b-x)}, \tag{1.2}$$

for some function $Q_V : \mathbb{R} \to \mathbb{R}$ satisfying $0 < c \le Q_V(x) \le C$.

Unfortunately, the property above is known to be false for arbitrary potentials. Still, it is conjectured to be "generically" true. In the case of analytic potentials, this has indeed been shown in [20], but up to now, nothing is known in the non-analytic setting (except for some very special cases).

In this article, we give a positive answer to this conjecture for $C^{2,\alpha}$ potentials. Also, we show that higher regularity assumptions on V yield higher regularity for the functions Q. Because it is well-known that the space of regular potential is open, the challenge is to prove that they are dense.

Here and in the following, given $j \in \mathbb{N}$ and $\beta \in [0,1]$, we denoted by $C^{j,\beta}_{\mathrm{loc}}(\mathbb{R})$ the space of functions $g : \mathbb{R} \to \mathbb{R}$ that belong to $C^j(\mathbb{R})$ and whose j-th derivative is locally β-Hölder continuous, that is

$$\sup_{x \neq y \in [-R,R]} \frac{|D^j g(x) - D^j g(y)|}{|x-y|^\beta} < \infty \qquad \forall R > 0.$$

Our first main result is the following (see Theorem 2.3 below for a more precise statement):

Theorem 1.1. *Given $\alpha \in (0,1)$, let $V \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R})$ satisfy $\lim_{|x|\to+\infty} \frac{V(x)}{\log|x|} = +\infty$. Given $\gamma \in \mathbb{R}$, consider the family of potentials $V_{s,\gamma}(x) := \frac{V(s^\gamma x)}{s}$, $s > 0$. Then $V_{s,\gamma}$ is regular for a.e. $s > 0$.*

In particular, the set of potentials for which **(A)** *holds is open and dense in the class of* $C^{2,\alpha}_{\mathrm{loc}}$ *potentials that diverge at infinity faster than a logarithm.*

Furthermore, if **(A)** *holds and* $V \in C^{k+1/2+\beta}_{\mathrm{loc}}(\mathbb{R})$ *for some* $k \geq 2$ *and* $\beta \in (0,1)$,[1] *then* $Q_V \in C^{k-1,\beta}_{\mathrm{loc}}(\mathbb{R})$.

Remark 1.2. In the case of analytic potentials, this result with $\gamma = 1$ recovers the one of [20].

Discrete β-models. A discrete version of the previous problem naturally arises, for instance, from asymptotics of distributions of discrete β-ensembles or in the study of orthogonal polynomials of a discrete variable, see [5, 13].

In this case, given $N \in \mathbb{N}$ and $\theta > 0$, the energy (1.1) is minimized among probability measures with density bounded above by θ and supported in $\bigcup_{h=1}^N [a_h, b_h]$, where the previous intervals are disjoint. Also, one prescribes the mass of the admissible measures μ inside each interval:

$$\mu([a_h, b_h]) = \hat{n}_h \geq 0, \qquad \sum_{h=1}^N \hat{n}_h = 1. \tag{1.3}$$

Furthermore, the potential V is differentiable inside $\bigcup_{h=1}^N [a_h, b_h]$ and is assumed to satisfy

$$|V'(x)| \leq C\left(1 + \sum_{h=1}^N |\log(x - a_h)| + |\log(x - b_h)|\right) \tag{1.4}$$

for some constant $C > 0$.

In analogy to the continuous case, an important assumption on the minimizing measure is the following (see, for instance, [5]):

(B) *The set* $\{0 < \frac{d\mu_{V,\theta}}{dx} < \theta\}$ *is a finite union of intervals compactly contained inside* $\bigcup_{h=1}^N (a_h, b_h)$*. In addition, the density* $\frac{d\mu_{V,\theta}}{dx}$ *vanishes like a square root near* $\partial\{\psi > 0\}$ *and converges to* θ *like a square root near* $\partial\{\psi < \theta\}$*. More precisely, given* $p_- \in \partial\{\psi > 0\}$ *(resp.* $p_+ \in \partial\{\psi < \theta\}$*) there exists a uniformly positive bounded function* Q_- *(resp.* Q_+*), defined in neighborhood of* p_- *(resp.* p_+*), such that*

$$\begin{aligned} &\frac{d\mu_{V,\theta}}{dx}(x) = Q_-\sqrt{|x - p_-|} \text{ for } x \in \left\{0 < \tfrac{d\mu_{V,\theta}}{dx} < \theta\right\},\ |x - p_-| \ll 1, \\ &\text{(resp. } \frac{d\mu_{V,\theta}}{dx} = \theta - Q_+\sqrt{|x - p_+|} \text{ for } x \in \left\{0 < \tfrac{d\mu_{V,\theta}}{dx} < \theta\right\},\ |x - p_+| \ll 1\text{).} \end{aligned} \tag{1.5}$$

Given N disjoint intervals $[a_h, b_h]$, $j \in \mathbb{N}$, and $\beta \in [0,1]$, we denote by $C^{j,\beta}_{\mathrm{loc}}\left(\bigcup_h (a_h, b_h)\right)$ the space of functions $g\colon \bigcup_h (a_h, b_h) \to \mathbb{R}$ that belong to $C^j\left(\bigcup_h (a_h, b_h)\right)$ and whose j-th derivative is locally β-Hölder continuous, that is

$$\sup_{x \neq y \in \bigcup_h [-a_h + r, b_h - r]} \frac{|D^j g(x) - D^j g(y)|}{|x - y|^\beta} < \infty \qquad \forall\, r > 0 \text{ sufficiently small.}$$

Here is our main result (see Theorem 2.6 below for a more precise statement, in particular concerning the topology considered for our openness/denseness statement):

Theorem 1.3. *Given* $\alpha \in (0,1)$*, let* $V \in C^{2,\alpha}_{\mathrm{loc}}\left(\bigcup_h (a_h, b_h)\right)$ *satisfy* (1.4)*. Consider the family of potentials* $V_s(x) := \frac{V(sx)}{s}$*,* $s > 0$*. Then* V_s *is regular for a.e.* $s > 0$*.*

In particular, the set of potentials for which **(B)** *holds is open and dense in the class of* $C^{2,\alpha}_{\mathrm{loc}}$ *potentials satisfying* (1.4)*.*

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[1] Here and in the sequel, for convenience of notation, we denote

$$C^{k+1/2+\beta}_{\mathrm{loc}}(\mathbb{R}) = \begin{cases} C^{k,\beta+1/2}_{\mathrm{loc}}(\mathbb{R}) & \text{if } \beta \leq 1/2, \\ C^{k+1,\beta-1/2}_{\mathrm{loc}}(\mathbb{R}) & \text{if } \beta > 1/2. \end{cases}$$

Furthermore, if **(B)** *holds and* $V \in C^{k+1/2+\beta}_{\mathrm{loc}}(\bigcup_h (a_h, b_h))$ *for some* $k \geq 2$ *and* $\beta \in (0,1)$,[2] *then the functions* $Q_\pm$ *are of class* $C^{k-1,\beta}_{\mathrm{loc}}$.

Remark 1.4. Consider the potential

$$V(x) = \sum_{h=1}^{N} \big[(x-a_h)_+ \log(x-a_h) - (b_h - x)_+ \log(b_h - x)\big].$$

Then

$$\frac{V(sx)}{s} = \sum_{h=1}^{N} \left[\left(x - \frac{a_h}{s}\right)_+ \log\left(x - \frac{a_h}{s}\right) - \left(\frac{b_h}{s} - x\right)_+ \log\left(\frac{b_h}{s} - x\right) + \frac{b_h - a_h}{s} \log s \, \mathbf{1}_{[a_h, b_h]}(x)\right].$$

Since the last term in each addend corresponds to adding an additive constant to the potential inside each interval, it can be neglected due to the mass constraint. So, Theorem 1.3 implies that the family of potentials

$$x \mapsto \sum_{h=1}^{N} \left[\left(x - \frac{a_h}{s}\right)_+ \log\left(x - \frac{a_h}{s}\right) - \left(\frac{b_h}{s} - x\right)_+ \log\left(\frac{b_h}{s} - x\right)\right]$$

are regular for a.e. $s > 0$.

In particular, consider the class of potentials appearing in [5, Equation (122)]. These potentials depend on 6 parameters (A, B, C, t, D, H). Then, considering as in [5, Section 9.2] the case when these parameters are very large, our result implies that for a.e. choice of these parameters, the corresponding potentials are regular.

Riesz potentials. One can also consider more general energies of the form

$$\mathcal{E}(\mu) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \big(\mathfrak{g}(x-y) + V(x) + V(y)\big)\, d\mu(x)\, d\mu(y),$$

where

$$\mathfrak{g}(x) = \begin{cases} \frac{1}{\sigma}|x|^{-\sigma} & \text{if } \sigma \neq 0, \\ -\log|x| & \text{if } \sigma = 0, \end{cases} \qquad \text{with} \quad \sigma \in (d-2, d). \tag{1.6}$$

These energies arise from the study of Riesz gases (see for instance [27] and references therein). In this context, the analogue of assumption **(A)** is the following:

($\mathbf{A}_\sigma$) *The minimizing probability* μ_V *is supported over finitely many disjoint compact sets* $\{K_j\}_{1 \leq j \leq M} \subset \mathbb{R}^d$*, with* ∂K_j *a* $(d-1)$*-dimensional manifold of class* C^1*. Also, inside* K_j*, it has the form*

$$\frac{d\mu_V}{dx} = Q_V(x)\, \mathrm{dist}(x, \partial K_j)^{1 - \frac{d-\sigma}{2}}, \tag{1.7}$$

for some function $Q_V : \mathbb{R}^d \to \mathbb{R}$ *satisfying* $0 < c \leq Q_V(x) \leq C$.

As discussed in Appendix C, our strategy also applies in this case and proves the generic validity of **($\mathbf{A}_\sigma$)** in low dimensions (see Theorem C.1).

1.1. **Comments on the proofs and the structure of the paper.** To prove Theorem 1.1, we follow Serfaty's strategy to associate to μ_V a solution to the thin obstacle problem (see [26, 21, 27, 22] and Section 2.1 below). In this way, we may hope to apply the regularity theory available for generic solutions of the thin obstacle problem [16]. Note that the latter theory applies only to a strictly monotone family of solutions, so one would need to find perturbations of V that induce monotone perturbations of the solutions. While this seems difficult to achieve, we can follow the strategy in [20] where, instead of changing the potential, we vary the mass constraint. In this way, the associated family of solutions is monotone (see Proposition 3.2) and the generic regularity theory for the thin

[2] As before, for convenience of notation, given $U \subset \mathbb{R}$ open we denote

$$C^{k+1/2+\beta}_{\mathrm{loc}}(U) = \begin{cases} C^{k,\beta+1/2}_{\mathrm{loc}}(U) & \text{if } \beta \leq 1/2, \\ C^{k+1,\beta-1/2}_{\mathrm{loc}}(U) & \text{if } \beta > 1/2. \end{cases}$$

obstacle problem applies. Then, using scaling properties for minimizers of $\mathcal{E}_V$, we obtain our result (see Remark 3.1).

While in the case of β-models we can rely on the available theory for the thin obstacle problem, the discrete case (Theorem 1.3) is much more challenging. More precisely, also in this case we can relate the regularity of the minimizing measure to the regularity of a solution of a PDE. However, the additional constraint on the upper bound of the density of the measure reflects in the fact that the solution will solve a "two-phase" thin obstacle problem (see Theorem 2.4), for which no generic regularity theory was available. In addition, now the potential V is not smooth at the boundary of its support, and this may create singularities in our solutions. These two facts create several challenges that are addressed in Appendix B.

Since this paper uses and develops nontrivial PDE tools but has applications beyond the PDE community, we structure the paper as follows: In Section 2, following Serfaty's argument, we show the connection between minimizing measures and obstacle problems. Then, in Sections 3 and 4 we prove our main results, taking for granted the generic regularity properties of solutions to obstacle problems. Finally, all PDE materials (both the known theory for the classical thin obstacle problem and the new theory needed for the case of discrete models) are postponed to Appendices A and B. In Appendix C, we briefly discuss the case of Riesz potentials.

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2. From minimizing measures to obstacle problems

The goal of this section is to discuss some preliminary facts about measures minimizing the energy (1.1) and show the connection to the thin obstacle problem. This will allow us to reformulate our main theorems in a slightly different form, see Theorems 2.3 and 2.6 below. The proofs of these two theorems will then be given in the next two sections.

We begin this section by discussing some preliminary facts about measures minimizing the energy (1.1).

2.1. From minimizing measures to the thin obstacle problem.

Here and in the sequel, we denote by $\mathcal{M}_1(\mathbb{R})$ the space of nonnegative probability measures on $\mathbb{R}$. We begin with the non-discrete case.

Theorem 2.1 ([2, Lemma 2.6.2])**.** *Let $V : \mathbb{R} \to \mathbb{R}$ be a continuous potential satisfying*

$$\lim_{|x|\to\infty} \frac{V(x)}{\log|x|} = +\infty.$$

Then there is a unique probability measure μ_V minimizing

$$\inf\{\mathcal{E}_V(\nu) \,:\, \nu \in \mathcal{M}_1(\mathbb{R})\}.$$

In addition, the measure μ_V is compactly supported and it is uniquely determined by the existence of a constant C_V such that

$$-\int_{\mathbb{R}} \log|x-t|\, d\mu_V(t) \geq C_V - V(x) \quad \textit{for every } x \in \mathbb{R}, \textit{ with equality } \mu_V\textit{-a.e.} \tag{2.1}$$

To connect minimizing measures to the thin obstacle problem, given the measure μ_V minimizing $\mathcal{E}_V$ in $\mathcal{M}_1(\mathbb{R})$, consider the function

$$u_V(x) = -\big(\log|\cdot| * \mu_V\big)(x) - C_V, \quad x \in \mathbb{R}^2. \tag{2.2}$$

Then, as discussed for instance in [27, Chapter 2.4], (2.1) is equivalent to asking that u_V solves the thin obstacle problem with obstacle $-V$ (cf. Appendix A):

$$\begin{cases} -\Delta u_V \geq 0 & \text{in } \mathbb{R}^2, \\ \Delta u_V = 0 & \text{in } \mathbb{R}^2 \setminus (\{u_V = -V\} \cap \{x_2 = 0\}), \\ u_V \geq -V & \text{in } \{x_2 = 0\}, \\ \frac{u_V(x)}{\log|x|} \to -1 & \text{as } |x| \to +\infty. \end{cases} \tag{2.3}$$

Furthermore, if a solution u of (2.3) is regular in the sense of Definition A.1, then the associated measure satisfies property **(A)** in the introduction, namely its support consists of finitely many intervals and its density satisfies (1.2). This justifies the following definition.

Definition 2.2 (Regular potential)**.** *We say that a potential V is* regular *if:*

i) $\{u_V = -V\} = \operatorname{supp}\mu_V$*;*
ii) $\operatorname{supp}\mu_V$ *is a finite union of disjoint compact intervals;*
iii) $\mu_V \ll \mathcal{L}^1$ *and its density is of the form* (1.2) *in each interval* $[a, b]$ *of* $\operatorname{supp}\mu_V$*.*

With this terminology, we can now state a slightly refined version of Theorem 1.1. However, we first precisely define the topology considered here when talking about an open and dense set. Define

$$X := \left\{ V \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}) \,:\, \lim_{|x|\to+\infty} \frac{V(x)}{\log|x|} = +\infty \right\}$$

endowed with the distance[3]

$$\rho(V, W) := \sum_k 2^{-k} \frac{\|V - W\|_{C^{2,\alpha}(-k,k)}}{1 + \|V - W\|_{C^{2,\alpha}(-k,k)}} + \sum_k 2^{-k} \frac{|G_k(V) - G_k(W)|}{1 + |G_k(V) - G_k(W)|},$$

where

$$G_k \colon X \to \mathbb{R}, \qquad G_k(V) := \inf_{|x|\geq k} \frac{V(x)}{\log|x|}.$$

Notice that $\rho(V_k, V) \to 0$ if and only if $V_k \to V$ in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R})$ and $\lim_{|x|\to+\infty} \frac{V_k(x)}{\log|x|} = +\infty$ uniformly in k.

We can now state our result.

Theorem 2.3. *Given $\alpha \in (0, 1)$, let $V \in C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R})$ satisfy $\lim_{|x|\to+\infty} \frac{V(x)}{\log|x|} = +\infty$. Given $\gamma \in \mathbb{R}$, consider the family of potentials $V_{s,\gamma}(x) := \frac{V(s^\gamma x)}{s}$, $s > 0$. Then $V_{s,\gamma}$ is regular for a.e. $s > 0$ in the sense of Definition 2.2. In particular, the set of regular potentials is an open and dense subset of (X, ρ).*

Furthermore, if $V : \mathbb{R} \to \mathbb{R}$ is regular and belongs to $C^{k+1/2+\beta}_{\mathrm{loc}}(\mathbb{R})$ for some $k \geq 2$ and $\beta \in (0, 1)$, then the function Q_V in (1.2) is of class $C^{k-1,\beta}_{\mathrm{loc}}(\mathbb{R})$.

2.2. **From minimizing measures to the two-phase thin obstacle problem: the discrete case.** Given $K = \bigcup_{h=1}^N [a_h, b_h]$ a finite union of intervals and nonnegative numbers $\hat{n}_h$ such that $\sum_{h=1}^N \hat{n}_h = 1$, define

$$\mathcal{M}_{1,\theta}(K) := \left\{ \nu = \eta\mathcal{L}^1 \in \mathcal{M}_1(\mathbb{R}) \,:\, 0 \leq \eta \leq \theta, \quad \operatorname{supp}\nu \subseteq K, \quad \nu([a_h, b_h]) = \hat{n}_h \right\},$$

where $\mathcal{L}^1$ denoted the Lebesgue measure on $\mathbb{R}$. We will say that the problem is saturated in one interval $[a_h, b_h]$ if $\theta(b_h - a_h) = \hat{n}_h$, namely if for every measure $\nu \in \mathcal{M}_{1,\theta}(K)$ we have that $\nu \llcorner [a_h, b_h] = \theta\mathcal{L}^1 \llcorner [a_h, b_h]$.[4]

Theorem 2.4. *For every $\theta > 0$ such that $\mathcal{M}_{1,\theta}(K) \neq \emptyset$, there exists a unique measure minimizer of*

$$\inf\{\mathcal{E}_V(\nu) \,:\, \nu \in \mathcal{M}_{1,\theta}(K)\}.$$

[3]In [20], the authors consider an analogous distance but in the space of C^3_{loc} potentials.

[4]Given a measure ν and a set $K \subset \mathbb{R}$, $\nu \llcorner K$ denotes the restriction of the measure ν to K.

Moreover, if the problem is not saturated in an interval $[a_h, b_h]$*, then a measure* μ *is minimizing if and only if there is a constant* C_h *such that the function*

$$u_h(x) = -\big(\log|\cdot| * (\mu \llcorner [a_h, b_h])\big)(x) - C_h, \quad x \in \mathbb{R}^2, \tag{2.4}$$

solves

$$\begin{cases} \Delta u_h = 0 & \text{in } \mathbb{R}^2 \setminus \{x_2 = 0, a_h \le x_1 \le b_h\}, \\ -2\pi\theta \le \partial_2 u_h \le 0 & \text{in } \{x_2 = 0, a_h \le x_1 \le b_h\}, \\ \partial_2 u_h = 0 & \text{in } \{x_2 = 0, a_h \le x_1 \le b_h\} \cap \{u_h > -V\}, \\ -\partial_2 u_h = 2\pi\theta & \text{in } \{x_2 = 0, a_h \le x_1 \le b_h\} \cap \{u_h < -V\}, \\ \frac{u_h(x)}{\log|x|} \to -\hat{n}_h & \text{as } |x| \to +\infty, \end{cases} \tag{2.5}$$

where u_h *is even in* x_2 *and the value* $\partial_2 u_h$ *at* $\{x_2 = 0\}$ *is intended as the limit from the right, namely*

$$\partial_2 u_h(x_1, 0) = \lim_{t \to 0^+} \frac{u_h(x_1, t) - u_h(x_1, 0)}{t}.$$

Proof. The existence and uniqueness of a minimising probability μ is shown in [5, Lemma 5.1]. The proof of the necessary and sufficient condition (2.5) follows as in [5, Lemma 5.5] or [13, Theorem 2.1]. $\square$

In analogy to the previous subsection, we now define regular potentials.

Definition 2.5 (Regular potentials, discrete case)**.** *Given* $\theta > 0$ *such that* $\mathcal{M}_{1,\theta}(K) \neq \emptyset$ *and the problem is not saturated in any interval, we say that a potential* V *is regular if, denoting by* ψ *the density of the minimising measure, for every* $1 \le h \le N$ *it holds:*

i) $\{u_h = -V\} = \{0 < \psi < \theta\} \cap (a_h, b_h)$*;*

ii) $\{0 < \psi < \theta\}$ *is a finite union of open intervals and is compactly contained in the interior of* $\bigcup_{h=1}^N (a_h, b_h)$*;*

iii) ψ *(resp.,* $\theta - \psi$*) is of the form* (1.5) *at each point* $p_- \in \partial\{\psi > 0\}$ *(resp.* $p_+ \in \partial\{\psi < \theta\}$*).*

To state our refined version of Theorem 1.3, we introduce a topology on the set

$$\widetilde{X} := \Big\{(V, K) \mid K \subset \mathbb{R} \text{ is of the form } \cup_{k=1}^N [a_h, b_h] \text{ and } V \in C^{2,\alpha}_{\mathrm{loc}}(K) \text{ satisfies (1.4)}\Big\}$$

induced by the distance

$$\widetilde{\rho}((V,K),(W,C)) := \mathrm{dist_H}\big(\mathrm{graph}(V_{|K}), \mathrm{graph}(W_{|C})\big) + \sum_{n \in \mathbb{N}} \sum_{h=1}^N \frac{\|V - W\|_{C^{2,\alpha}((K \cap C)_{1/n})}}{1 + \|V - W\|_{C^{2,\alpha}((K \cap C)_{1/n})}},$$

where $\mathrm{dist_H}$ denotes the Hausdorff distance between two sets, and given $E \subset \mathbb{R}$ and $\rho > 0$, we denote $E_\rho = \{x \in E : \mathrm{dist}(x, E^c) > \rho\}$.

Note that $\widetilde{\rho}((V_k, K_k), (V, K)) \to 0$ if and only if $V_k \to V$ in $C^{2,\alpha}_{\mathrm{loc}}(K)$ and the graphs of V_k on K_k converge in the Hausdorff distance to the graph of V on K.

Theorem 2.6. *Given* $\alpha \in (0,1)$*, let* $V \in C^{2+\alpha}_{\mathrm{loc}}(\bigcup_h (a_h, b_h))$ *satisfy* (1.4)*. Consider the family of potentials* $V_s(x) := \frac{V(sx)}{s}$*,* $s > 0$*. Then* V_s *is regular in the sense of Definition 2.5 for a.e.* $s > 0$*. In particular, the set of regular potentials is an open and dense subset of* $(\widetilde{X}, \widetilde{\rho})$*.*

Furthermore, if $V : \bigcup_h [a_h, b_h] \to \mathbb{R}$ *is regular and belongs to* $C^{k+1/2+\beta}_{\mathrm{loc}}(\bigcup_h (a_h, b_h))$ *for some* $k \ge 2$ *and* $\beta \in (0,1)$*, then the functions* $Q_\pm$ *are of class* $C^{k-1,\beta}_{\mathrm{loc}}$*.*

3. Proof of Theorem 2.3

As mentioned in the introduction, the main difficulty is to prove the density of regular potentials. We will achieve this by showing that, given $\gamma \in \mathbb{R}$ and a potential $V \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R})$ satisfying $\lim_{|x| \to +\infty} \frac{V(x)}{\log|x|} = +\infty$ and $\gamma \in \mathbb{R}$, the potential $V_{s,\gamma} := \frac{V(s^\gamma \cdot)}{s}$ is regular for a.e. $s > 0$.

Here and in the sequel, we denote by $\mathcal{M}_s(\mathbb{R})$ the space of nonnegative measures on $\mathbb{R}$ with total mass $s > 0$. The following observation, already used in the proof of generic regularity for analytic potentials in [20] for $\gamma = 0$, relates scaling the potential $V_{s,\gamma}$ to rescaling the mass.

Remark 3.1. Given $s > 0$, denote $\rho_s(x) := sx$. Then, for every $\gamma \in \mathbb{R}$ and $s > 0$, a measure μ minimise $\{\mathcal{E}_V(\nu) : \nu \in \mathcal{M}_s(\mathbb{R})\}$ if and only if $\frac{1}{s}(\rho_{s^{-\gamma}})_{\#}\mu$ minimises $\{\mathcal{E}_{V_{s,\gamma}}(\nu) : \nu \in \mathcal{M}_1(\mathbb{R})\}$[5].

Given a potential V and $\gamma \in \mathbb{R}$, for every $s > 0$ let μ_s be the measure minimizing $\{\mathcal{E}_V(\nu) : \nu(\mathbb{R}) = s\}$,[6] and consider the function

$$u_s(x) = -\log|\cdot| * \mu_s(x) - \gamma \log s - sC_{V_{s,\gamma}}, \quad x \in \mathbb{R}^2, \tag{3.1}$$

where $C_{V_{s,\gamma}}$ is the constant associated to the probability measure minimising $\mathcal{E}_{V_{s,\gamma}}$ in $\mathcal{M}_1(\mathbb{R})$. Then, applying Theorem 2.1 with $V_{s,\gamma}$ and Remark 3.1, it follows that u_s solves the following thin obstacle problem with obstacle $-V$:

$$\begin{cases} -\Delta u_s \geq 0 & \text{in } \mathbb{R}^2, \\ \Delta u_s = 0 & \text{in } \mathbb{R}^2 \setminus \{u_V = -V\} \cap \{x_2 = 0\}, \\ u_s \geq -V & \text{in } \{x_2 = 0\}, \\ \frac{u_s(x)}{\log|x|} \to -s & \text{as } |x| \to +\infty. \end{cases} \tag{3.2}$$

The following result is crucial for us.

Proposition 3.2. *The functions u_s are decreasing in s, namely $u_s \geq u_{s'}$ for any $s < s'$. More precisely, they are strictly decreasing in the following sense: for every $M > 0$ there is R_0 such that for any $R \geq R_0$ there is $a = a(R) > 0$ such that*

$$u_{s-\delta} - u_s > a\delta \quad \text{in } \partial B_R \cap \{|x_2| > R/2\} \tag{3.3}$$

for every $0 < s - \delta \leq s \leq M$ and $R \geq R_0$.

A proof of the monotonicity is essentially contained in [9, Theorem 2], using potential theory. Here we use the comparison principle for (A.1), based on the following remark.

Remark 3.3. We will state the comparison principle for (A.1) allowing for different obstacles, since we will need it later. Let u, v be two solutions of (A.1) in B_1 with obstacle φ_u, φ_v. Assume that $\varphi_u \leq \varphi_v$ in $\operatorname{supp}\Delta u$. Then $(u - v)_+$ is subharmonic in B_1. Indeed, $\Delta(u - v) = -\Delta v \geq 0$ in $\mathbb{R}^2 \setminus \operatorname{supp}(\Delta u)$, while $u \leq v$ in $\operatorname{supp}(\Delta u)$, since for $x \in \operatorname{supp}(\Delta u)$ we have $u(x) = \varphi_u(x) \leq \varphi_v(x) \leq v(x)$. Thus $\Delta(u - v)_+ \geq 0$ in B_1.

For convenience of notation, we write $u \sim s\log|\cdot|$ to denote that $\lim_{|x|\to\infty} \frac{u(x)}{s\log|x|} \to 1$.

Proof of Proposition 3.2. Given $s < s'$, we note that

$$u_s(x) \sim -s\log|x| \gg u_{s'}(x) \sim -s'\log|x| \quad \text{as } x \to +\infty.$$

Hence, for any large enough ball B_R, we have $u_{s'} \leq u_s$ in ∂B_R. Thus $(u_{s'} - u_s)_+$ vanishes on ∂B_R, is nonnegative, and (by Remark 3.3) it is subharmonic in B_R. By the maximum principle, this implies $u_{s'} \leq u_s$ in B_R. Since R can be taken arbitrarily large, we conclude that $u_s \geq u_{s'}$ in $\mathbb{R}^2$.

Let us now show (3.3). Since $\operatorname{supp}\Delta u_s \subset \{u_s = -V\}$ and the functions are monotone, for every M there is R_0 such that $\{u_s = -V\} \subset B_{R_0/2}$ for every $s \leq M$. Thus we have

$$\Delta(u_{s-\delta} - u_s) = 0 \quad \text{in } \mathbb{R}^2 \setminus \{x_2 = 0, |x_1| \leq R_0/2\},$$

[5] Given measurable spaces X, Y, a measure μ on X, and a measurable function $f\colon X \to Y$, we denote by $f_{\#}\mu$ the push-forward of μ through f, namely the measure on Y defined by $f_{\#}\mu(A) = \mu(f^{-1}(A))$ for every measurable set $A \subset Y$.

[6] Note that Theorem 2.1 applies, with the same proof, to any value of $s > 0$.

$u_{s-\delta} - u_s \geq 0$ (by the previous step), and $u_{s-\delta} - u_s \sim \delta \log|\cdot|$ as $|x| \sim +\infty$. Hence, if η is a solution of the problem

$$\begin{cases} \Delta\eta = 0 & \text{in } \mathbb{R}^2 \setminus [-R_0/2, R_0/2] \times \{0\}, \\ \eta = 0 & \text{in } [-R_0/2, R_0/2] \times \{0\}, \\ \eta \sim \log|\cdot| & \text{as } |x| \to +\infty, \end{cases}$$

the maximum principle implies $u_{s-\delta} - u_s \geq \delta\eta$ in $\mathbb{R}^2$. To conclude, we can take $a(R) = \inf_{\partial B_R} \eta$, which is positive by the strong maximum principle. □

We can now prove our main result.

Proof of Theorem 2.3. We split the proof into 3 steps.
Step 1: Regular potentials are dense. Given $V \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R})$ satisfying $\lim_{|x|\to+\infty} \frac{V(x)}{\log|x|} = +\infty$ and $\gamma \in \mathbb{R}$, we show that the potentials $V_{s,\gamma}$ are regular for almost every $s > 0$, where we recall that $V_{s,\gamma}(x) = s^{-1}V(s^\gamma x)$.

For every $s > 0$ let μ_s be the measure minimizing E_V in $\mathcal{M}_s(\mathbb{R})$ given by Theorem 2.1, and let u_s be given by (3.1). Then, recalling Remark 3.1, to show that $V_{\gamma,s}$ is regular, we need to prove that u_s is regular.

By Proposition 3.2 and Theorem 2.1, the functions u_s defined in (3.1) are a decreasing family of solutions of (A.1) in $\mathbb{R}^2$ with obstacle $-V$ satisfying the strict monotonicity condition (3.3). Also, as shown in the proof of Proposition 3.2, the contact sets $\{u_s = -V\}$ are contained inside $B_{R/2}$ for every $s \leq M$. Hence we can apply Theorem A.3 to the family $\{u_{1/t}\}_{t>0}$ in B_R to find that u_s is regular for almost every $s > 0$, as desired.
Step 2: Stability of regular potentials. Let $V_k \to V$ in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R})$, with $\lim_{|x|\to+\infty} \frac{V_k(x)}{\log|x|} = +\infty$ uniformly in k. We prove that if the potentials V_k are not regular, then also V is not regular.

Let $u_k = u_{V_k}$ and $u = u_V$ be defined as in (2.2). We first show that $u_k \to u$ locally uniformly in $\mathbb{R}^2$. To do this, given $\delta > 0$, consider $v_\delta^\pm$ solutions of (A.1) with obstacle $-V \pm \delta$ and satisfying $v_\delta^\pm \sim -(1 \mp \delta)\log|\cdot|$ as $|x| \to +\infty$. Since $\operatorname{supp}(\Delta v_\delta^+)$ is compact, there exists C_0 sufficiently large such that v_δ^+ still solves (A.1) with obstacle

$$-\tilde{V}_\delta = \max\big\{ -V + \delta, -C_0\big(1 + \log(1 + |x|)\big)\big\}.$$

Since $V_k \to V$ locally and the potentials diverge at infinity faster than a logarithm, for k sufficiently large we have

$$-V_k \leq -\tilde{V}_\delta \text{ on } \mathbb{R} \quad \text{and} \quad -V_k \geq -V - \delta \text{ on } \operatorname{supp}(\Delta v_\delta^-).$$

We claim that $v_\delta^- \leq u_k \leq v_\delta^+$ for $k \gg 1$. Indeed, for $k \gg 1$ the functions u_k, v_δ^+ solve (A.1) in B_R with ordered obstacles. Moreover, since $u_k \sim -\log R \ll v_\delta^+$ in ∂B_R for R sufficiently large, we can apply Remark 3.3 to get $u_k \leq v_\delta^+$. Analogously, $v_\delta^- \leq u_k$.

Since $v_\delta^\pm \to u$ locally uniformly as $\delta \to 0$ we find that $u_k \to u$ locally uniformly. In particular, the contact sets $\{u_k = -V_k\}$ are equibounded.

Since V_k are not regular, there is a sequence of points $x_k \in \operatorname{Sing}(u_k) \subset \{u_k = -V_k\}$ (see Appendix A for the definition of singular points). Since the contact sets are equibounded, there is an accumulation point x_∞, namely $x_k \to x_\infty$ up to a subsequence. Then Lemma A.4 implies that $x_\infty \in \operatorname{Sing}(u)$, as wanted.
Step 3: Higher order expansion. Since $\frac{d\mu_V}{dx} = -\partial_2 u_V$, we simply need to show the result for $\partial_2 u_V$. Since $u_V = -V \in C^{k+1/2+\alpha}_{\mathrm{loc}}$ in the interior of $\operatorname{supp}\mu_V$, local $C^{k-1/2+\alpha}$ regularity in the interior of $\operatorname{supp}\mu_V$ follows from boundary regularity for the Dirichlet problem.

Let us show the regularity at the boundary of $\operatorname{supp}\psi$. Up to a translation and rescaling, we can assume that u solves in the unit ball B_1 the following problem:

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \setminus \{x_2 = 0, x_1 \leq 0\}, \\ u = -V & \text{in } \{x_2 = 0, -1 < x \leq 0\}, \\ u > -V & \text{in } \{x_2 = 0, 0 < x_1 < 1\}. \end{cases}$$

Let P be the k-th order ($k+1$-th for $\alpha > 1/2$) Taylor expansion of $-V$ at 0, and let $\tilde{P}$ be its unique harmonic extension to $\mathbb{R}^2$ which is even in the x_2 variable. Define $\tilde{u} := u - (-V + \tilde{P} - P)$. Then $\tilde{u}$ satisfies

$$\begin{cases} |\Delta \tilde{u}| \le C|x|^{k+\alpha-3/2} & \text{in } B_1 \setminus \{x_2 = 0, x_1 \le 0\}, \\ \tilde{u} = 0 & \text{in } \{x_2 = 0, -1 < x_1 \le 0\}. \end{cases}$$

Defining in polar coordinates

$$U_0(\rho, \theta) := \rho^{1/2} \cos\theta/2,$$

we can apply Theorem A.5 to find a polynomial Q of degree k such that

$$\left|\tilde{u} - QU_0\right| \le C|x|^{k+\alpha+1/2}, \quad \Delta(QU_0) = 0, \quad \text{and} \quad Q(0) = 0,$$

where we used $|u| \le Cr^{3/2}$ in B_r to deduce that $Q(0) = 0$. Recalling the definition of $\tilde{u}$, this implies

$$|u - \tilde{P} - U_0 Q| \le C r^{k+1/2+\alpha} \quad \text{in } B_r,$$

for some polynomials $\tilde{P}$ with $\deg \tilde{P} \in \{k, k+1\}$ and Q of degree k, with $\partial_2 \tilde{P} \equiv 0$ and $Q(0) = 0$.

Let us set $\mathcal{C} := B_1 \setminus (B_{1/4} \cup \{x_2 = 0, x_1 \le 0\})$. Then, for every $0 < r < 1$, the function

$$v_r := \frac{u(r\,\cdot) - \tilde{P}(r\,\cdot) - U_0(r\,\cdot)Q(r\,\cdot)}{r^{k+1/2+\alpha}},$$

satisfies

$$\Delta v_r = 0 \quad \text{in } \mathcal{C}, \quad |v_r| \le C \text{ in } \mathcal{C}, \quad \text{and} \quad \|v_r(\cdot, 0)\|_{C^{k+1/2+\alpha}(-1,-1/4)} \le C,$$

so elliptic regularity yields

$$\|v_r\|_{C^{k+1/2+\alpha}} \le C \quad \text{in } B_{3/4} \setminus B_{1/2}.$$

Thus, recalling that $\partial_2 \tilde{P} \equiv 0$,

$$|\partial_2 u - \partial_2(U_0 Q))| \le C r^{k-1/2+\alpha} \quad \text{in } B_{3r/4} \setminus B_{r/2}.$$

By direct computation and recalling that $Q(0) = 0$ we have $\partial_2(U_0 Q)(x_1, 0) = (x_1)_-^{1/2} \mathcal{P}$ for some polynomial $\mathcal{P}(x_1)$ of degree $k-1$. Thus we can rewrite the previous inequality as

$$\left| \frac{\partial_2 u(x_1, 0)}{(x_1)_-^{1/2}} - \mathcal{P}(x_1) \right| \le C r^{k-1+\alpha} \quad \text{for } x_1 \in (-3r/4, -r/2)$$

for a polynomial $\mathcal{P}(x_1)$ of degree $k-1$. This yields the desired $C^{k-1,\alpha}$ regularity for Q_V at the boundary of $\operatorname{supp} \mu_V$, from which the result follows. □

4. Proof of Theorem 2.6

Given $K = \bigcup_{h=1}^N [a_h, b_h]$ a finite union of intervals and nonnegative numbers $\hat{n}_h$ such that $\sum_{h=1}^N \hat{n}_h = 1$, define

$$\mathcal{M}_{s,\theta}(K) := \left\{ \nu = \eta \mathcal{L}^1 \,:\, 0 \le \eta \le \theta, \quad \nu(\mathbb{R}) = s, \quad \operatorname{supp} \nu \subseteq K, \quad \nu([a_h, b_h]) = s\hat{n}_h \right\}.$$

We say that the problem is saturated in an interval $[a_h, b_h]$ if $\theta(b_h - a_h) = s\hat{n}_h$, namely if for every measure $\nu \in \mathcal{M}_{s,\theta}(K)$ we have $\nu \llcorner [a_h, b_h] = \theta \mathcal{L}^1 \llcorner [a_h, b_h]$. The following result is the natural generalization of Theorem 2.4.

Theorem 4.1. *For every $s, \theta > 0$ such that $\mathcal{M}_{s,\theta}(K) \neq \emptyset$, there exists a unique measure minimizing*

$$\inf\{\mathcal{E}_V(\nu) \,:\, \nu \in \mathcal{M}_{s,\theta}(K)\}$$

supported in K. Moreover, if the problem is not saturated in an interval (a_h, b_h), then a measure μ is minimizing if and only if there is a constant C_h such that the function

$$u_h(x) = -\log|\cdot| * (\mu \llcorner (a_h, b_h))(x) - C_h, \quad x \in \mathbb{R}^2, \tag{4.1}$$

solves

$$\begin{cases} \Delta u_h = 0 & \text{in } \mathbb{R}^2 \setminus \{x_2 = 0, a_h \le x_1 \le b_h\}, \\ -2\pi\theta \le \partial_2 u_h \le 0 & \text{in } \{x_2 = 0, a_h \le x_1 \le b_h\}, \\ \partial_2 u_h = 0 & \text{in } \{x_2 = 0, a_h \le x_1 \le b_h\} \cap \{u_h > -V\}, \\ -\partial_2 u_h = 2\pi\theta & \text{in } \{x_2 = 0, a_h \le x_1 \le b_h\} \cap \{u_h < -V\}, \\ \frac{u_h(x)}{\log|x|} \to -s\hat{n}_h & \text{as } |x| \to +\infty. \end{cases} \tag{4.2}$$

The proof of the theorem above is a direct consequence of Theorem 2.4 and the following observation that relates scalings of the potential to rescalings of the mass.

Remark 4.2. Given a nonnegative bounded function ψ, let $\psi_s := \psi(s\,\cdot)$. Then $\psi\mathcal{L}^1$ minimizes $\mathcal{E}_V$ in $\mathcal{M}_{s,\theta}(K)$ if and only if $\psi_s\mathcal{L}^1$ minimises $\mathcal{E}_{V_s}$ in $\mathcal{M}_{1,\theta}(K/s)$, where $V_s = \frac{V(s\,\cdot)}{s}$.

Given a potential V, a parameter θ, and $s > 0$, denote by ψ_s the density minimizing $\{\mathcal{E}_V(\eta) : \eta \in \mathcal{M}_{s,\theta}(K)\}$. Moreover, for every $1 \le h \le N$, we denote by $u_{h,s}$ the associated solution of (4.2) given by (4.1).

Proposition 4.3. *For every $1 \le h \le N$ the functions $u_{h,s}$ are decreasing in s, namely $u_{h,s} \ge u_{h,s'}$ for any $s < s'$. More precisely, they are strictly decreasing in the following sense: for every $M > 0$ there is R_0 such that for any $R \ge R_0$ there is $a = a(R) > 0$ such that*

$$u_{h,s-\delta} - u_{h,s} > a\delta \quad \text{in } \partial B_R \cap \{|x_2| > R/2\} \tag{4.3}$$

for every $0 < s - \delta \le s \le M$ and $R \ge R_0$.

The proof is based on the following Remark, which plays the same role as Remark 3.3 in the previous section.

Remark 4.4. We will state the comparison principle for (4.2) allowing for different obstacles, since we will need it later. Let $u, v \colon B_2 \subset \mathbb{R}^2 \to \mathbb{R}$ be two functions, even in x_2, solving (4.2) (on possibly different intervals $[a_u, b_u], [a_v, b_v]$). Assume that the obstacles of u and v are ordered, namely, $\varphi_u \ge \varphi_v$, where we use the convention $\varphi = -\infty$ outside $[a, b]$. Then $(v - u)_+$ is subharmonic in B_2.

Indeed, calling ψ_u, ψ_v the normal derivatives of u, v respectively at $\{x_2 = 0\}$, we have that the distributional Laplacian of $v-u$ is equal to twice the jump of the normal derivaties across $\{x_2 = 0\}$, that is,

$$\Delta(v - u) = 2(\psi_v - \psi_u)\mathcal{H}^1 \llcorner \{x_2 = 0\},$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. So it suffices to check that $\psi_v \ge \psi_u$ in $\{x_2 = 0\} \cap \{v > u\}$.

Given $x \in \{v > u\}$, we consider two cases: if $v(x) \le \varphi_v(x)$ then we have $u(x) < v(x) \le \varphi_v(x) \le \varphi_u(x)$, hence $\psi_u(x) = -2\pi\theta \le \psi_v(x)$. If $v(x) > \varphi_v(x)$ then $\psi_v(x) = 0 \ge \psi_u(x)$, as wanted.

Proof of Proposition 4.3. The monotonicity is proved as in the proof of Proposition 3.2, using Remark 4.4 instead of Remark 3.3. The proof of (4.3) follows exactly that of (3.3). □

Proof of Theorem 2.6. We split the proof in 3 steps.
Step 1: Regular potentials are dense. Given $V \in C^{2,\alpha}_{\text{loc}}(\mathbb{R})$, we want to show that the potentials V_s are regular for almost every $s > 0$, where $V_s(x) = s^{-1}V(sx)$.

For this, we can argue for each u_h as in the proof of Theorem 2.3, using Proposition 4.3, Theorem B.7, and Remark 4.2 instead of Proposition 3.2, Theorem A.3, and Remark 3.1, respectively.
Step 2: Stability of regular potentials. We show that if $V_k \to V$ in $C^{2,\alpha}_{\text{loc}}\left(\bigcup_h (a_h, b_h)\right)$ and V_k are not regular, then also V is not regular.

We first show that, for every h, $u_{h,k} \to u_h$ locally uniformly in $\mathbb{R}^2$. Indeed, given $\delta > 0$, consider $v_\delta^\pm$ solutions of (2.5) with obstacle $-V_\delta^\pm(x) = -V((1 \mp \delta)x) \pm \varepsilon(\delta)$ and satisfying[7]

$$v_\delta^\pm \sim -(\hat{n}_h \mp \delta) \log|\cdot| \qquad \text{as } |x| \to +\infty,$$

where $\varepsilon(\delta) \to 0$ as $\delta \to 0$ and is chosen so that

$$-V_\delta^+ \geq -V + \delta \geq -V - \delta \geq -V_\delta^-.$$

Since $-V_\delta^- \leq -V_k \leq -V_\delta^+$ for $k \gg 1$, Remark 4.4 implies that $v_\delta^- \leq u_{h,k} \leq v_\delta^+$. Since $v_\delta^\pm \to u_h$ as $\delta \to 0$, we find that also $u_{h,k} \to u_h$ locally uniformly, and in particular, for k large enough. The contact sets $\{u_{h,k}(\cdot,0) = -V_k\}$ are compactly contained in $(a_{h,k}, b_{h,k})$. We can then conclude arguing as in the proof of Theorem 2.3, Step 2, using Lemma B.8 instead of Lemma A.4.

Step 3: Higher order expansion. The argument is the same as in Step 3 in the proof of Theorem 2.3. □

Appendix A. Results from Thin Obstacle Problem

Here we collect useful results from the regularity of the thin obstacle problem, referring the interested reader to [23, Chapter 9] and the recent survey [17] for more details. To keep the notation consistent with the rest of the paper, we restrict ourselves to the two-dimensional setting.

Recall that, given a function $\varphi\colon [-1,1] \to \mathbb{R}$, a (two-dimensional) solution of the thin obstacle problem with obstacle φ is a function $u\colon B_1 \subset \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\text{(A.1)} \qquad \begin{cases} \Delta u = 0 & \text{in } B_1 \setminus \{x_2 = 0\}, \\ \Delta u \leq 0 & \text{in } \{x_2 = 0\} \cap B_1 \\ \partial_{x_2} u = 0 & \text{in } \{x_2 = 0\} \cap \{u > \varphi\} \cap B_1, \\ u \geq \varphi & \text{in } \{x_2 = 0\} \cap B_1. \end{cases}$$

We will denote the contact set of u by $\Lambda(u) := \{u = \varphi\} \cap \{x_2 = 0\}$ and the free boundary by $\Gamma(u) = \partial_{\mathbb{R}} \Lambda(u)$ (i.e., the topological boundary of $\Lambda(u)$ as a subset of $\{x_2 = 0\} \simeq \mathbb{R}$). A point $x_0 \in \Gamma(u)$ is regular ($x_0 \in \mathrm{Reg}(u)$) provided there exists $c > 0$ such that

$$|u(x_0 + r\cdot)| \geq c r^{3/2} \quad \text{in } B_1, \quad \forall\, r < 1,$$

and it is singular ($x_0 \in \mathrm{Sing}(u)$) if there exists $C > 0$ such that

$$\text{(A.2)} \qquad |u(x_0 + r\cdot) - \varphi(x_0 + r\cdot)| \leq C r^2 \quad \text{in } B_1, \quad \forall\, r < 1,$$

where we mean $\varphi(x_1, x_2) = \varphi(x_1)$.

Definition A.1. *A solution u of* (A.1) *is* regular *if there is no point in $\Gamma(u)$ where* (A.2) *holds.*

We recall that solutions to the thin obstacle problem are $C^{1,1/2}$ (see [3, 24]).

Lemma A.2. *Let u solve* (A.1) *in $B_1 \subset \mathbb{R}^2$ with obstacle $\varphi \in C^{1,\alpha}([-1,1])$ for some $\alpha > \frac{1}{2}$. Then $u(\cdot, 0) \in C^{1,1/2}_{\mathrm{loc}}(-1,1)$.*

While singular points may exist, it has been recently proved that, in low dimensions, the singular set is generically empty. More precisely, consider a continuous 1-parameter family family of solutions $\{u_t\}_{t \in [-1,1]}$ to (A.1) in B_1, strictly increasing in the following sense: there exists $\eta > 0$ such that

$$\text{(A.3)} \qquad \begin{cases} u_{t+\varepsilon} \geq u_t & \text{in } \partial B_1 \\ u_{t+\varepsilon} \geq u_t + \eta\varepsilon & \text{in } \partial B_1 \cap \{|x_2| > \frac{1}{2}\} \end{cases}$$

for every $\varepsilon > 0$.

[7]The existence of these functions can be proved, for instance, using Theorem 4.1 (namely, minimizing $\mathcal{E}_{V_\delta^\pm}$ with an upper bound θ on the density and suitable mass constraints).

Theorem A.3 (Generic regularity for the thin obstacle problem, see [15, 16]). *Let $\varphi \in C^{2,\alpha}([-1,1])$ for some $0 < \alpha \le 1$ and let $u_t : B_1 \to \mathbb{R}$ be a family of solutions of* (A.1) *satisfying* (A.3). *Then, for a.e. $t \in [-1,1]$,*

i) $\mathrm{Sing}(u_t) \cap [-1/2, 1/2] = \emptyset$*;*
ii) $\Gamma(u_t) \cap [-1/2, 1/2]$ *is a finite set.*

We also recall two useful properties: limits of singular points are singular, and solutions with smoother obstacles are more regular.

Lemma A.4 (Convergence of singular points, see [15]). *Let u_k, u solve* (A.1) *in B_1 with obstacles $\varphi_k, \varphi \in C^{1,1}([-1,1])$. Assume that*

i) $u_k \to u, \varphi_k \to \varphi$ *uniformly, and* $\sup_n \|\varphi_k\|_{C^{1,1}([-1,1])} < +\infty$*;*
ii) there are points $x_k \in \mathrm{Sing}(u_k)$ with $x_k \to x \in B_{1/2}$.

Then $x \in \mathrm{Sing}(u)$.

Theorem A.5 (Higher order boundary Harnack, see [12, Theorem 2.3]). *Let u solve*

$$\begin{cases} |\Delta u| \le C|x|^{k+\alpha-3/2} & \text{in } B_1 \setminus \{x_2 = 0, x_1 \le 0\}, \\ u = 0 & \text{in } \{x_2 = 0, x_1 \le 0\}. \end{cases}$$

for some $k \ge 1$ and $\alpha > 0$.

Then there exist a polynomial Q of degree k and a constant $C' > 0$ such that

$$\sup_{B_r} |u - QU_0| \le C' r^{k+\alpha+1/2} \qquad \text{for all } 0 < r < 1,$$

where $U_0(\rho, \theta) = \rho^{1/2} \cos\theta/2$.

Appendix B. Generic regularity for the Two-phase Thin Obstacle

Let $\theta > 0$ be a fixed parameter. We will consider differentiable obstacles φ on $[-1,1]$ satisfying the analog of (1.4), namely

$$|\varphi'(x)| \le C\big(|\log(x+1)| + |\log(x-1)|\big). \tag{B.1}$$

We want to show a generic regularity result for (4.2). To this aim, we will assume $K = [-1,1]$ and we consider solutions in B_2 of

$$\begin{cases} -\Delta u = 0 & \text{in } B_2 \setminus \{x_2 = 0\}, \\ \partial_2 u = \theta & \text{in } \{x_2 = 0, |x_1| > 1\}, \\ |\partial_2 u| \le \theta & \text{in } \{x_2 = 0, |x_1| \le 1\}, \\ \partial_2 u = \mathrm{sign}(u - \varphi)\theta & \text{in } \{u \ne \varphi, |x_1| \le 1, x_2 = 0\}, \\ u(x_1, -x_2) = u(x_1, x_2) & \text{for } (x_1, x_2) \in \mathbb{R}^2, \end{cases} \tag{B.2}$$

where where u is even in x_2 and the value $\partial_2 u$ at $\{x_2 = 0\}$ is intended as the limit from the right, namely

$$\partial_2 u(x_1, 0) = \lim_{t \to 0^+} \frac{u(x_1, t) - u(x_1, 0)}{t}.$$

We remark that u_h solves (2.5) with $a_h = -1$ and $b_h = 1$ if and only if $\frac{1}{\pi} u_h + \theta|x_2|$ solves (B.2). Moreover, note that if u solves (B.2) and $u \ge \varphi$ in $\{x_2 = 0\} \cap B_r(x_0)$ for some $r > 0$ and $x_0 \in \{x_2 = 0\}$, then $u - \theta|x_2|$ solves the classical thin obstacle problem (cf. Appendix A) in $B_r(x_0)$.

In this section we will show a generic regularity result for (B.2), see Theorem B.7 below. To achieve this, the main challenge is proving a phase separation result for solutions of (B.2); namely, we show that given a point $x_0 \in \partial\{u \ne \varphi\}$, either $u - \theta x_2$ or $-\theta x_2 - u$ solve the thin obstacle problem in a sufficiently small ball around x_0. This allows us to recover $C^{1,1/2}$ regularity of the solution and generic regularity for a monotone family of solutions by applying the results for the thin obstacle problem. We note that a similar problem was recently studied in [10], but the results and techniques used there do not apply to our case, so a new argument is needed.

Observe that the form of the obstacle (B.1) (namely, its low regularity at ± 1) allows for non-Lipschitz solutions at $(\pm 1, 0)$ (see the Remark below), while we will show that the solutions must be locally Lipschitz near the interior of K, see Lemma B.3 below.

Remark B.1. Considering $u = -\frac{\theta}{2\pi}\log|\cdot| * \chi_{(-1,1)}$, then u is a non-Lipschitz solution of (B.2) with $\varphi = u$ in $[-1,1]$. Note that φ satisfies (B.1) in $(-1,1)$.

We start by showing Lipschitz regularity in the normal direction.

Lemma B.2. *Let u solve* (B.2)*. Then there exists a constant $C > 0$ such that*

$$|\partial_2 u| \le \theta + C|x_2| \quad \text{in } B_1. \tag{B.3}$$

Proof. Since $u - \theta|x_2|$ is harmonic in $B_2 \setminus B_1$, elliptic estimates yield

$$|\nabla u| \le \theta + |\nabla(u - \theta|x_2|)| \le C' \quad \text{in } \partial B_{4/3},$$

while from (B.2) we have $|\partial_2 u| \le \theta$ in $\{x_2 = 0\}$. If we denote by w the solution of

$$\begin{cases} \Delta w = 0 & \text{in } B_{4/3} \cap \{x_2 > 0\}, \\ w = \theta & \text{in } \{x_2 = 0\}, \\ w = C' & \text{in } \partial B_{4/3} \cap \{x_2 > 0\}, \end{cases}$$

by boundary regularity there exists $C > 0$ such that $w \le \theta + C|x_2|$ in $B_1 \cap \{x_2 > 0\}$. Thus, since $|\partial_2 u|$ is subharmonic in $B_{4/3} \cap \{x_2 > 0\}$, the maximum principle yields $|\partial_2 u| \le w \le \theta + C|x_2|$ in $B_1 \cap \{x_2 \ge 0\}$. Recalling that u is even in x_2, (B.3) follows. □

B.1. **Local regularity.** In this subsection we show the separation of phases for (B.2), i.e., we exclude the possibility of having a point $z \in \partial\{u(\cdot,0) > \varphi\} \cap \partial\{u(\cdot,0) < \varphi\}$. In this way, the regularity results for (A.1) will imply local $C^{1,1/2}$ regularity.

We localise the problem in the interior of $[-1,1]$, considering solutions in B_1 of

$$\begin{cases} -\Delta u = 0 & \text{in } B_1 \setminus \{x_2 = 0\}, \\ |\partial_2 u| \le \theta & \text{in } \{x_2 = 0\}, \\ \partial_2 u = \operatorname{sign}(u - \varphi)\theta & \text{in } \{x_2 = 0\} \cap \{u \ne \varphi\}, \end{cases} \tag{B.4}$$

and we assume that $\varphi \in \operatorname{Lip}([-1,1])$. We will also always assume that (B.3) is satisfied. We say that $z \in (-1/2, 1/2)$ is a two-phase point if $u(z,0) = \varphi(z)$ and

$$\{u(\cdot,0) > \varphi\} \cap (z - r, z + r) \quad \text{and} \{u(\cdot,0) < \varphi\} \cap \quad (z - r, z + r) \ne \emptyset \quad \text{for all } r > 0.$$

Lemma B.3. *Let u be a solution of* (B.4) *with $\varphi \in \operatorname{Lip}([-1,1];\mathbb{R})$. Then*

$$\sup_{B_{1/2}} |\partial_1 u| \le C\|\nabla u\|_{L^2(B_1)} + \operatorname{Lip}(\varphi). \tag{B.5}$$

Proof. If u solves (B.4) with obstacle φ, then $u_h \coloneqq u(\cdot + h\mathbf{e}_1) + \operatorname{Lip}(\varphi)|h|$ solves (B.4) with obstacle $\varphi(\cdot + h\mathbf{e}_1) + |h|\operatorname{Lip}(\varphi) \ge \varphi$. Applying Remark 4.4 to u_h and u gives

$$\Delta\left(\frac{u(x) - u(x + h\mathbf{e}_1)}{|h|} - \operatorname{Lip}(\varphi)\right)_+ \ge 0,$$

and the local maximum principle implies

$$\frac{u(x) - u(x + h\mathbf{e}_1)}{|h|} - \operatorname{Lip}(\varphi) \le C\|\nabla u\|_{L^2(B_1)} \qquad \text{for } x \in B_{1/2}$$

By the same argument applied to $u(\cdot + h\mathbf{e}_1) - \operatorname{Lip}(\varphi)|h|$ we also get

$$\frac{u(x + h\mathbf{e}_1) - u(x)}{|h|} - \operatorname{Lip}(\varphi) \le C\|\nabla u\|_{L^2(B_1)} \qquad \text{for } x \in B_{1/2}$$

So (B.5) follows. □

This is the main result of this section.

Proposition B.4 (Phase separation)**.** *Let u be a solution of* (B.4) *with $\varphi \in C^{1,1}$. Then there are no two-phase points.*

To prove phase separation, we will make use of the following result, which is a scale-invariant formulation of [10, Proof of Proposition 4.1, Step 2]. This result will also be useful later.

Lemma B.5. *For every $c < 1$ there is $\delta > 0$ such that the following holds:*
Let u solve (B.4) *with obstacle φ, and assume that:*

i) $\varphi(0) = \varphi'(0) = 0$ *and* $|\varphi''| \le \delta$*;*
ii) $u \le \delta$ *in* $\{x_2 = 0\} \cap B_1$*, and* $|\partial_2 u| \le \theta + \delta|x_2|$ *in* B_1*;*
iii) there is $\hat{z} \in (-\frac{1}{100}, \frac{1}{100})$ *satisfying* $u(\hat{z}, 0) > 0$*.*

Then

$$\partial_2 u \ge c\theta \quad \textit{in} \quad \left\{x_2 = \tfrac{1}{100}, |x_1| \le 1/2\right\}.$$

Proof. Up to considering $u(x) - \varphi(x_1)$ instead of u, we can assume that u solves

$$\begin{cases} \Delta u = f(x_1) & \mathbb{R}^2 \setminus \{x_2 = 0\}, \\ |\partial_2 u| \le \theta & \{x_2 = 0\}, \\ \partial_2 u = \theta \operatorname{sign} u & \{u \neq 0\} \cap \{x_2 = 0\}, \end{cases}$$

where $f(x_1) = -\varphi''(x_1)$ satisfies $|f(x_1)| \le \delta$, $u(0) = 0$, $u \le 2\delta$ in $\{x_2 = 0\} \cap B_1$, and

$$(-r, r) \cap \{u > 0\} \neq \emptyset \quad \text{and} \quad (-r, r) \cap \{u < 0\} \neq \emptyset \quad \text{for all } r > 0. \tag{B.6}$$

Given $M \ge 1$ we define

$$V(x_1, x_2) := u(x_1, x_2) - M\delta(x_1 - \hat{z})^2 + (M + 1/2)\delta x_2^2 - (\theta - \delta)x_2,$$

and let $\Gamma := [-1, 1] \times [0, \eta]$, where $\eta > 0$ is small enough (for instance, $\eta = \frac{1}{100}$ will work).

Since

$$\Delta V = \Delta u + \delta \ge 0 \quad \text{in } \Gamma,$$

by the maximum principle there is $(\bar{x}_1, \bar{x}_2) \in \partial\Gamma$ such that

$$V(\bar{x}_1, \bar{x}_2) = \max_{\partial\Gamma} V = \max_{\Gamma} V \ge V(\hat{z}, 0) = u(\hat{z}, 0) > 0. \tag{B.7}$$

Step 1. Let us show that $\bar{x}_2 = \eta$ and $|\bar{x}_1| \le 1/2$, provided M is (universally) large enough. We argue by contradiction, splitting the proof into several cases.

• $\bar{x}_2 = 0$. In this case we have

$$0 \ge \partial_2 V(\bar{x}_1, 0) = \partial_2 u(\bar{x}_1, 0) - \theta + \delta,$$

hence $\partial_2 u(\bar{x}_1, 0) \le \theta - \delta < \theta$. This implies $u(\bar{x}_1, 0) \le 0$, which yields $V(\bar{x}_1, 0) = u(\bar{x}_1, 0) - M\delta(\bar{x}_1 - \hat{z})^2 \le 0$, contradicting (B.7).

• $|\bar{x}_1| \ge 1/2$. Since

$$0 < V(\bar{x}_1, \bar{x}_2) = u(\bar{x}_1, \bar{x}_2) - M\delta|\bar{x}_1 - \hat{z}|^2 + (M + 1/2)\delta\bar{x}_2^2 - (\theta - \delta)\bar{x}_2,$$

recalling that $M \ge 1$ and $\bar{x}_2 \le \eta = \frac{1}{100}$ we have

$$u(\bar{x}_1, \bar{x}_2) > \frac{M}{5}\delta - (M + 1/2)\delta\bar{x}_2^2 + (\theta - \delta)\bar{x}_2 \ge \frac{M}{10}\delta + \theta\bar{x}_2.$$

On the other hand, since $|\partial_2 u| \le \theta + \delta|x_2|$ and $u \le 2\delta$ in $\{x_2 = 0\} \cap B_1$, we get

$$u(\bar{x}_1, \bar{x}_2) \le u(\bar{x}_1, 0) + \theta\bar{x}_2 + \delta\frac{\bar{x}_2^2}{2} \le (2 + \eta^2/2)\delta + \theta\bar{x}_2,$$

which is impossible for M large enough.

Step 2. Let us conclude the proof. From Step 1 we have $\bar{x}_2 = \eta$ and $|\bar{x}_1| \le 1/2$. Moreover, (B.7) gives

$$0 \le \partial_2 V(\bar{x}_1, \eta) = \partial_2 u(\bar{x}_1, \eta) + 2\eta(M + 1/2)\delta - (\theta - \delta),$$

hence

$$\theta - \partial_2 u(\bar{x}_1, \eta) \le C\delta. \tag{B.8}$$

By assumption the function $\theta + 2\delta\eta - \partial_2 u$ is harmonic and positive in $(-1, 1) \times (0, 2\eta)$. Thus, by Harnack inequality, there exists $C > 0$ such that, setting $\Gamma' = [-1/2, 1/2] \times [\eta/2, 3\eta/2]$, we have

$$\sup_{\Gamma'} \big(\theta + 2\delta\eta - \partial_2 u\big) \le C \inf_{\Gamma'} \big(\theta + 2\delta\eta - \partial_2 u\big).$$

Since $(\bar{x}_1, \eta) \in \Gamma'$, by the equation above and (B.8) we have

$$\sup_{\Gamma'} \big(\theta + 2\delta\eta - \partial_2 u\big) \le C\delta,$$

therefore

$$\partial_2 u \ge \theta - C\delta > c\theta \quad \text{in } \big\{x_2 = \tfrac{1}{100}, |x_1| \le 1/2\big\}$$

for δ small enough, as we wanted. □

Proof of Proposition B.4. We argue by contradiction, assuming that $x_0 = 0$ is a two-phase point. Up to subtracting $\varphi(0) + \varphi'(0)x_1$, we can assume that $\varphi(0) = \varphi'(0) = 0$.
Step 1. Let $\delta > 0$ be given by Lemma B.5 with $c = 1/2$. Let us show that $|u(\cdot, 0)| \le r\delta$ in $(-r, r)$ for all $r > 0$ small enough.

Assume that the claim does not hold. Then, up to changing the sign of u, there exists a sequence $x_k \to 0$ such that $u(x_k, 0) \ge \delta|x_k|$. We can also assume that $x_k > 0$ infinitely many times.
Step 1a. We show that there is $\bar{x} > 0$ such that

$$u(x, 0) \ge \frac{1}{2}\delta|x| > \varphi(x) \quad \text{for } 0 < x < \bar{x}. \tag{B.9}$$

To show this, first of all notice that

$$\partial_{11}u(\cdot, 0) \ge -C \text{ in } \{u(\cdot, 0) > \varphi\} \quad \text{and} \quad \partial_{11}u(\cdot, 0) \le C \text{ in } \{u(\cdot, 0) < \varphi\}. \tag{B.10}$$

Indeed, by (B.3) we have $|\partial_2 u| \le \theta + Cx_2$ with equality in $\{x_2 = 0\} \cap \{u > \varphi\}$, hence $\partial_{22}u \le C$ in $\{x_2 = 0\} \cap \{u > \varphi\}$ and by harmonicity we find $\partial_{11}u(\cdot, 0) \ge -C$ on this set. Similarly, $\partial_{11}u(\cdot, 0) \le C$ on $\{u < \varphi\} \cap \{x_2 = 0\}$, thus proving (B.10).

Thus, the function $v(x) := u(x, 0) + \frac{C}{2}x^2$ is convex in $\{u > \varphi\}$ for some $C > 0$. Let $0 \le y_k < x_k$ be such that $u(y_k, 0) = \varphi(y_k)$ but $u(\cdot, 0) > \varphi$ in (y_k, x_k). Since $\varphi'(0) = \varphi(0) = 0$ we have $\varphi(x) \le \delta|x|/4$ for $|x|$ small enough, and in particular it is true at y_k for $k \gg 1$. Hence, the convexity of v in $(y_k, x_k + h)$ implies

$$\begin{aligned} u(x_k + h, 0) + \frac{C}{2}(x_k + h)^2 = v(x_k + h) &\ge v(x_k) + h\frac{v(x_k) - v(y_k)}{x_k - y_k} \\ &= u(x_k, 0) + \frac{C}{2}x_k^2 + h\frac{u(x_k, 0) + \frac{C}{2}x_k^2 - \varphi(y_k) - \frac{C}{2}y_k^2}{x_k - y_k} \\ &\ge \delta x_k + \frac{C}{2}x_k^2 + h\frac{\delta x_k + \frac{C}{2}x_k^2 - \frac{\delta}{4}x_k - \frac{C}{2}y_k^2}{x_k} \\ &\ge \delta x_k + \frac{3}{4}h\delta \ge \frac{3}{4}\delta(x_k + h) \end{aligned}$$

for any $h > 0$. This implies that, for $k \gg 1$ and $h > 0$ universally small, as long as $u(\cdot, 0) > \varphi$ we have

$$u(x_k + h, 0) \ge \frac{3}{4}\delta(x_k + h) - \frac{C}{2}(x_k + h)^2 \ge \frac{1}{2}\delta(x_k + h) > \varphi(x_k + h).$$

This implies that actually u cannot touch φ again on an interval $(x_k, x_k + h_0)$. Thus, letting $k \to \infty$ we obtain (B.9) with $\bar{x} = h_0$.
Step 1b. We show that there are $\varepsilon, \bar{x}' > 0$ such that

$$u(x, 0) \le -\varepsilon|x| < \varphi(x) \quad \text{for } -\bar{x}' < x < 0. \tag{B.11}$$

Notice that it is enough to show that there is a sequence $y_k \to 0$ ($y_k < 0$ by Step 1) such that $u(y_k, 0) < -\varepsilon'|y_k|$ for some $\varepsilon' > 0$, since if this holds then we can argue as in Step 1 to prove (B.11), using the second inequality in (B.10).

Assume by contradiction that $u(\cdot, 0) \ge -o(x)$ as $x \to 0$. Let us show that, up to passing to a sequence, $u_r := \frac{1}{r}u(r\cdot) \to q$ locally uniformly in $\mathbb{R}^2$, where $q \in \mathrm{Lip}(\mathbb{R}^2)$ satisfies:

i) $q(0) = 0$ and $q(x, 0) \ge \frac{\delta}{2}x$ for $x > 0$;
ii) $q - \theta x_2$ solves (A.1) in $\mathbb{R}^2$ with obstacle $\varphi \equiv 0$.

Notice that if this holds then we have proven Step 1b, since this contradicts the C^1 regularity of solutions of the thin obstacle problem (see Lemma A.2).

Since u is Lipschitz in $B_{1/2}$ (as follows from (B.3) and (B.5)) there is $q \in \mathrm{Lip}(\mathbb{R}^2)$ such that, up to passing to a sequence, $u_r \to q$ locally uniformly in $\mathbb{R}^2$; by uniform convergence we have $q(0) = 0$, and since $u(x, 0) \geq -o(x)$ as $x \to 0$ we obtain $q(\cdot, 0) \geq 0$ in $\mathbb{R}$. Moreover, u_r solve (B.4) in $B_{1/r}$ with obstacle $\varphi_r := \frac{1}{r}\varphi(r\cdot)$. Since $\varphi(0) = \varphi'(0) = 0$ we have that $\varphi_r \to 0$ locally uniformly in $\mathbb{R}^2$.

Let us check that $q - \theta x_2$ solves (A.1) with obstacle $\varphi \equiv 0$. Since $\Delta(u_r - \theta x_2) = 0$ in $\mathbb{R}^2 \setminus \{x_2 = 0\}$, the same is true for $q - \theta x_2$. Moreover, since $|\partial_2 u_r| \leq \theta + Cr|x_2|$ in $B_{1/2r}$, letting $r \to 0$ we find $\partial_2(q - \theta x_2) \leq 0$ in $\mathbb{R}^2$. Finally, if $q(x) - \theta x_2 > 0$ for some $x \in \{x_2 = 0\}$ then $u_r > \varphi_r$ in $B_\delta(x) \cap \{x_2 = 0\}$ for r small enough (recall that $\varphi_r \to 0$ locally uniformly in $\mathbb{R}^2$). Hence $\partial_2 u_r = \theta$ in $B_\delta(x) \cap \{x_2 = 0\}$ for r small enough, so that $\partial_2(q - \theta x_2) = 0$, as we wanted. This concludes the proof of Step 1b.

Step 1c. We show that eqs. (B.9) and (B.11) contradict the Lipschitz regularity of u.

From the previous steps there exists $\bar{r} > 0$ small such that $u(\cdot, 0) > \varphi$ in $\{0 < x < \bar{r}\}$ and $u(\cdot, 0) < \varphi$ in $\{-\bar{r} < x < 0\}$. Thus, setting

$$\mu := \mathcal{H}^1 \llcorner \{x_2 = 0, x_1 > 0\} - \mathcal{H}^1 \llcorner \{x_2 = 0, x_1 < 0\},$$

the function $\bar{u} := \frac{1}{\bar{r}} u(\bar{r}\cdot) \in \mathrm{Lip}(B_1)$ solves, in the distributional sense,

$$\Delta \bar{u} = \theta\mu \quad \text{in } B_1.$$

Defining the function

$$w := \frac{\theta}{2\pi} \log|\cdot| * \mu,$$

we have that, for $x \in \mathbb{R}$,

$$w(x, 0) = \frac{\theta}{2\pi} \log|\cdot| * (\chi_{(0,1)} - \chi_{(-1,0)})(x),$$

therefore

$$\partial_1 w(x, 0) = \frac{\theta}{2\pi}\big(\log|\cdot| * (2\delta_0 - \delta_1 - \delta_{-1})\big)(x) = \frac{\theta}{2\pi} \log \frac{x^2}{|x-1||x+1|}. \tag{B.12}$$

In particular w is not Lipschitz at 0, and since $\bar{u} \in \mathrm{Lip}(B_1)$, also $w - \bar{u}$ is not Lipschitz at 0. This is however a contradiction, since $w - \bar{u}$ is harmonic and hence smooth in B_1.

Step 2. We reach a contradiction, thus concluding the proof.

By Step 1 there is $\bar{r} > 0$ small enough such that $u_r := \frac{1}{r} u(r\cdot)$ satisfies $|u_r| \leq \delta$ in $B_1 \cap \{x_2 = 0\}$ for all $0 < r < \bar{r}$. We can also assume that r is small enough so that $|\varphi_r''| \leq \delta$ and $|\partial_2 u_r| \leq \theta + \delta|x_2|$. Indeed, since u_r solves (B.4) with obstacle $\varphi_r := \frac{1}{r}\varphi(r\cdot)$, we can choose r sufficiently small so that $|\varphi_r''| = |r\varphi''(r\cdot)| \leq \delta$; also, by (B.3), $|\partial_2 u_r| \leq \theta + Cr|x_2| \leq \theta + \delta|x_2|$ for r small enough.

Since we are assuming that 0 is a two phase point for u, there are points $\hat{z}_+, \hat{z}_-$ with $|\hat{z}_+|, |\hat{z}_-| \leq \frac{1}{100}$ such that $u_r(\hat{z}_+, 0) > 0$ and $u_r(\hat{z}_-, 0) < 0$. Thus we can apply Lemma B.5 to u and $-u$ with $c = 1/2$ to find

$$\partial_2 u \geq \tfrac{\theta}{2} \quad \text{and} \quad \partial_2 u \leq -\tfrac{\theta}{2} \quad \text{in } \{x_2 = \tfrac{1}{100}, |x_1| \leq 1/2\},$$

reaching a contradiction. □

Corollary B.6. *Let u be a solution of* (B.2) *with $\varphi \in C^{1,1}_{\mathrm{loc}}(-1, 1)$. Then $u(\cdot, 0) \in C^{1,1/2}_{\mathrm{loc}}(-1, 1)$.*

Proof. Given a point x_0 in the interior of K, by Proposition B.4 there is $r > 0$ such that either $u(\cdot, 0) \geq \varphi$ in $(x_0 - r, x_0 + r)$ or $u(\cdot, 0) \leq \varphi$ in $(x_0 - r, x_0 + r)$.

With no loss of generality, we can assume that we are in the first case. Then $u - \theta|x_2|$ solves (A.1) in $B_r(x_0)$ with a $C^{1,1}$ obstacle. This implies $u(\cdot, 0) \in C^{1,1/2}(x_0 - r, x_0 + r)$ (see Lemma A.2). If $u(\cdot, 0) \leq \varphi$ then similarly we find that $u(\cdot, 0) \in C^{1,1/2}(x_0 - r, x_0 + r)$. This concludes the proof. □

B.2. **Generic regularity.** We show generic regularity for a strictly monotone family of solutions of (B.2). More precisely, we consider a continuous family of solutions $\{u_t\}_{t\in[-1,1]}$ of (B.2) in B_2, strictly increasing in the following sense:

$$\begin{cases} u_{t+\varepsilon} \geq u_t & \text{in } \partial B_2, \\ u_{t+\varepsilon} \geq u_t + \eta\varepsilon & \text{in } \partial B_2 \cap \{|x_2| > 1\}, \end{cases} \tag{B.13}$$

for some $\eta > 0$ and every $-1 \leq t < t+\varepsilon \leq 1$.

Given $x_0 \in \{u(\cdot,0) = \varphi\}$ we say that $x_0 \in \mathrm{Sing}^+(u)$ (respectively $x_0 \in \mathrm{Sing}^-(u)$) if $u(\cdot,0) \geq 0$ (resp. $u(\cdot,0) \leq 0$) in a neighbourhood of x_0 and it is a singular point (in the sense of (A.2)) for the function $u - \theta|x_2|$ (resp. $-u - \theta|x_2|$). We say that $x_0 \in \mathrm{Sing}(u)$ if $x_0 \in \mathrm{Sing}^+(u) \cup \mathrm{Sing}^-(u)$, or if $x_0 \in \{\pm 1\}$ and $u(x_0, 0) = \varphi(x_0)$.

Theorem B.7. *Assume $\varphi \in C^{2,\alpha}$ and let u_t be a family of solutions of* (B.2) *satisfying* (B.13)*. Then, for a.e. $t \in [-1,1]$,*

i) $\mathrm{Sing}(u_t) \cap [-1,1] = \emptyset$*;*

ii) $\partial\{u_t(\cdot,0) \neq \varphi\} \cap [-1,1]$ *is a finite set.*

Proof. We divide the proof in 3 steps.

Step 1. We show that there are at most 2 values of t such that $\{u_t = \varphi\} \cap \{\pm 1\} \neq \emptyset$.

More precisely, we show that for each point $a \in \{\pm 1\}$ there is at most one value of t such that $u_t(a,0) = \varphi(a)$. Since ∂K is a finite set, this will prove Step 1.

We argue by contradiction, assuming that there are two values $t > t'$ such that $u_t(a,0) = u_{t'}(a,0) = \varphi(a)$.

We can translate the problem so that $a = 0$ and the interval $[-1,1]$ becomes $K = [-2,0]$. Since $u_t \geq u_{t'}$ and both are harmonic in $\mathbb{R}^2 \setminus K$, by the strong maximum principle the function $w := u_t - u_{t'}$ solves

$$\begin{cases} \Delta w = 0 & \text{in } B_\delta \setminus \{x_2 = 0, x_1 \leq 0\}, \\ w \geq 0 & \text{in } B_\delta, \\ w > 0 & \text{in } B_\delta \setminus \{x_2 = 0, x_1 \leq 0\} \end{cases}$$

and $w(0,0) = 0$. It follows that there is $\varepsilon > 0$ sufficiently small such that $w \geq \varepsilon U_0$ in $B_{\delta/4}$, where $U_0(\rho,\theta) = \rho^{1/2}\cos\theta/2$.

On the other hand, by (B.3) we have $w(0,x_2) \leq Cx_2$ for x_2 small enough. This implies $|x_2|^{1/2} \sim U_0(0,x_2) \leq Cx_2$ for x_2 small enough, which is a contradiction.

Step 2. We show that if u_t is a continuous family of solutions of (B.4), there is $\bar{r} > 0$ such that for all $|t| \leq 1$ and all $x_0 \in \mathrm{Sing}^+(u_t)$ we have

$$u_t(x,0) \geq 0 \quad \text{for all } x \in (x_0 - \bar{r}, x_0 + \bar{r}).$$

We claim that there is $\hat{r} > 0$ such that for every x, y such that $u_t(x,0) > \varphi(x)$ and $u_t(y,0) < \varphi(y)$ for some $t \in [-1,1]$, then $|x - y| > \hat{r}$. Notice that, thanks to regularity for (A.1), this claim yields *a priori* estimates for $\|\partial_1 u\|_{C^{0,1/2}}$, from which Step 2 immediately follows.

To prove the claim, assume by contradiction there are $x_k \in \partial\{u_{t_k}(\cdot,0) > \varphi\}$ with $y_k \in \{u_{t_k}(\cdot,0) < \varphi\}$ such that $|y_k - x_k| < 1/k$. Also, up to a subsequence, $t_k \to t_0$ and $y_k, x_k \to x_0$.

Consider the rescaled functions $u_{t_k,r} := \frac{1}{r}u_{t_k}(x_0 + r\cdot)$ with r sufficiently small but fixed. Since this family is compact in the C^0_{loc} topology, for r small enough and $k \gg 1$ all the functions $u_{t_k,r}$ and $-u_{t_k,r}$ satisfy the assumptions of Lemma B.5 with $c = 1/2$. Hence, we deduce that

$$\partial_2 u_{t_k} \geq \frac{\theta}{2} \quad \text{and} \quad \partial_2 u_{t_k} \leq -\frac{\theta}{2} \quad \text{in } [-r/2, r/2] \times \left\{\tfrac{r}{100}\right\},$$

for $k \gg 1$, a contradiction.

Step 3. We show $\mathcal{H}^1(\{t : \mathrm{Sing}^+(u_t) \neq \emptyset\}) = 0$.

Firstly, by Step 1 we have $u_t(\cdot,0) \neq \varphi$ in $\{\pm 1\}$ for all but 2 values of t. Thus, it is sufficient to show the result assuming that $\bigcup_t \{u_t(\cdot,0) = \varphi\}$ is compactly supported inside $(-1,1)$.

Now, assume by contradiction that $\mathcal{H}^1(\{t : \mathrm{Sing}^+(u_t) \neq \emptyset\}) > 0$. Then there must be a point $t_0 \in \{t : \mathrm{Sing}^+(u_t) \neq \emptyset\}$ with density 1. Given $\bar r$ as in Step 2, Lemma A.4 implies that

$$\bigcup_{t_0-\delta<s<t_0+\delta} \mathrm{Sing}^+(u_s) \subset \{y \,:\, \mathrm{dist}(y, \mathrm{Sing}^+(u_{t_0})) < \bar r/2\}$$

for some $\delta > 0$. Since $\mathrm{Sing}^+(u_{t_0})$ is compact and t_0 has density 1, we can find $x_0 \in \mathrm{Sing}^+(u_{t_0})$ such that

$$\mathcal{H}^1\big(\{s \in (t_0-\delta, t_0+\delta) : \mathrm{Sing}^+(u_s) \cap (x_0 - \bar r/2, x_0 + \bar r/2) \neq \emptyset\}\big) > 0.$$

Since the functions $v_s := u_s - \theta|x_2|$ are a strictly monotone continuous family of solutions of (A.1) in $B_{\bar r}$ for $s > t_0 - \delta$, we can apply Theorem A.3 to find a contradiction. □

Lemma B.8 (Convergence of singular points). *Let u_k, u solve* (B.4) *in B_1 with obstacles $\varphi_k, \varphi \in C^{1,1}$. Assume that:*

i) $u_k \to u, \varphi_k \to \varphi$ uniformly, and $\sup_n \|\varphi_k\|_{C^{1,1}(-1,1)} < +\infty$;
ii) there are points $x_k \in \mathrm{Sing}(u_k)$ with $x_k \to x_\infty \in B_{1/2}$.

Then $x_\infty \in \mathrm{Sing}(u)$.

Proof. We can assume without loss of generality that $x_k \in \mathrm{Sing}^+(u_k)$. Arguing as in Step 2 of the proof of Theorem B.7 we find that for k large enough $u_k(\cdot, 0) \geq 0$ in $(x_\infty - \bar r, x_\infty + \bar r)$ for some $\bar r$ independent of k. By uniform convergence the same is true for u. Thus, for k large enough the functions $u_k - \theta|x_2|$ solve (A.1) in $B_{\bar r}(x)$, and Lemma A.4 implies $x_\infty \in \mathrm{Sing}^+(u)$. □

Appendix C. Riesz potentials

Given an integer $d \geq 1$ and a potential $V : \mathbb{R}^d \to [-\infty, +\infty)$, our strategy also applies to minimisers of the energies

$$\mathcal{E}(\mu) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathfrak{g}(x-y)\, d\mu(x)\, d\mu(y) + 2\int_{\mathbb{R}^d} V(x)\, d\mu(x),$$

where $\mathfrak{g}$ is as in (1.6).

Existence and uniqueness of minimising measures hold under mild assumptions on V, see [27, Chapter 2]. Similarly to Section 2.1, given μ_V the minimizing measure, following [27, Proposition 2.15 and Section 2.4] we associate the function

$$h^{\mu_V}(x) := \mathfrak{g} * \mu_V(x), \qquad x \in \mathbb{R}^d.$$

Then, there is a constant $c \in \mathbb{R}$ such that the function $u^{\mu_V} := h^{\mu_V} - c$ solves

$$\text{(C.1)} \qquad \begin{cases} u^{\mu_V} \geq -V & \text{in } \mathbb{R}^d, \\ u^{\mu_V} = -V & \text{in } \mathrm{supp}(\mu_V), \\ (-\Delta)^{\frac{d-\sigma}{2}} u^{\mu_V} = c_{d,\sigma}\mu_V & \text{in } \mathbb{R}^d, \end{cases}$$

for constants $c_{d,\sigma}$. The converse is also true, namely a probability measure μ is minimising if there is a solution u^μ of (C.1) with $(-\Delta)^{\frac{d-\sigma}{2}} u^\mu = c_{d,\sigma}\mu$.

Also in this context, if u^{μ_V} is a regular solution of (C.1) (in a sense analogous to Definition A.1) then μ_V satisfies condition $(\mathbf{A}_\sigma)$. Thus, thanks to [11, Theorem 1.1] (see also [15, 16]), the same strategy adopted to prove Theorem 1.1 yields the following:

Theorem C.1. *Given an integer $d \geq 1$, a real number $\sigma \in (d-2, d)$, a potential $V \in C^{4,\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\alpha \in (0,1)$, and $\gamma \in \mathbb{R}$, consider the family $V_{\gamma,s} := s^{\gamma\sigma-1}V(s^\gamma\,\cdot)$, $s > 0$. Let $\mu_{V_{\gamma,s}}$ denote the associated minimizing probabilities, and assume that one of these holds:*

(a) $d \leq 2$;
(b) $d \leq 3$ and $\sigma \geq d-1$.

Then $V_{\gamma,s}$ is regular for a.e. $s > 0$.

Remark C.2. Thanks to [19, 1], higher regularity of the potential V yields higher regularity on the density of $\mu_{V_{\gamma,s}}$ and the geometry of the support. More precisely, if V is a regular potential of class $C^{k+\frac{d-\sigma}{2}+\alpha}(\mathbb{R}^d)$ for $k > 2$ and $\alpha \in (0,1)$ with $\alpha \pm \frac{d-\sigma}{2} \notin \mathbb{N}$,[8] then μ_V is supported over finitely many disjoint compact sets $\{K_j\}_{1\le j\le M} \subset \mathbb{R}^d$, with ∂K_j a $(d-1)$-dimensional manifold of class $C^{k,\alpha}$. Also, the function Q_V in (1.7) is of class $C^{k-1,\alpha}(K_j)$ in a neighborhood of ∂K_j.

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[8] As in the introduction, we denote

$$C^{k+\frac{d-\sigma}{2}+\beta}_{\mathrm{loc}}(\mathbb{R}^d) = \begin{cases} C^{k,\beta+\frac{d-\sigma}{2}}_{\mathrm{loc}}(\mathbb{R}^d) & \text{if } \beta \le 1-\frac{d-\sigma}{2}, \\ C^{k+1,\beta-\frac{d-\sigma}{2}}_{\mathrm{loc}}(\mathbb{R}^d) & \text{if } \beta > 1-\frac{d-\sigma}{2}. \end{cases}$$

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