# How to recognize convexity of a set from its marginals

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**Abstract**

We investigate the regularity of the marginals onto hyperplanes for sets of finite perimeter. We prove, in particular, that if a set of finite perimeter has log-concave marginals onto a.e. hyperplane then the set is convex. Our proof relies on measuring the perimeter of a set through a Hilbertian fractional Sobolev norm, a fact that we believe has its own interest.

## 1 Introduction

Let $\varphi : \mathbb{R}^n \to [0, +\infty)$ be a log-concave function, that is, $\varphi$ is of the form $e^{-V}$ for some convex function $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. A well-known consequence of the Prékopa-Leindler inequality states that the marginals of $\varphi$ onto any hyperplane are log-concave (see for instance [6, Sections 10-11]): more precisely, for any direction $\mathbf{e} \in \mathbb{S}^{n-1}$ let $\pi_{\mathbf{e}} : \mathbb{R}^n \to \mathbf{e}^\perp$ denote the orthogonal projection onto the hyperplane $\mathbf{e}^\perp := \{x \in \mathbb{R}^n : \mathbf{e} \cdot x = 0\}$, and define

$$\varphi_{\mathbf{e}} : \mathbf{e}^\perp \to \mathbb{R}, \qquad \varphi_{\mathbf{e}}(x) := \int_{\mathbb{R}} \varphi(x + t\mathbf{e})\, dt.$$

Then $\varphi_{\mathbf{e}}$ is log-concave, i.e., $\varphi_{\mathbf{e}} = e^{-W_{\mathbf{e}}}$ for some convex function $W_{\mathbf{e}} : \mathbf{e}^\perp \simeq \mathbb{R}^{n-1} \to \mathbb{R} \cup \{+\infty\}$. In particular, if $E$ is a convex set and we denote by $\mathbf{1}_E$ the characteristic function of $E$ (that is $\mathbf{1}_E(x) = 1$ if $x \in E$, $\mathbf{1}_E(x) = 0$ if $x \notin E$), then $\mathbf{1}_E$ is log-concave, which implies that, for any $\mathbf{e} \in \mathbb{S}^{n-1}$,

$$w_{\mathbf{e}} : \mathbf{e}^\perp \to \mathbb{R}, \qquad w_{\mathbf{e}}(x) := \int_{\mathbb{R}} \mathbf{1}_E(x + t\mathbf{e})\, dt \tag{1.1}$$

is log-concave. Actually, by the Brunn-Minkowski inequality, an even stronger result is true, namely $w_{\mathbf{e}}$ is concave. (We refer to [6] for more details.)

In [5], Falconer proved that a converse of the previous statements is true: if a compact set $E$ has concave marginals onto a.e. hyperplane $\mathbf{e}^\perp$, then it is convex. The aim of this paper is to show that (under rather weak regularity assumptions on $E$) this converse statement is still true under weaker assumptions on the marginals: namely, we prove that if the marginals have convex support and are uniformly Lipschitz strictly inside their support, then the set is convex. In particular, this implies that if a set has log-concave marginals onto a.e. hyperplane $\mathbf{e}^\perp$ then it is convex. As we will also discuss below, this fact is false if we do not restrict to characteristic functions of sets:

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it is possible to construct examples of functions whose marginals are log-concave (actually, even concave) but the functions themselves are not.

To state our result, let us introduce some notation. Given a Borel set $E$, let $w_{\mathbf{e}}$ be as in (1.1), and define the set $A_{\mathbf{e}} := \{w_{\mathbf{e}} > 0\} \subset \mathbf{e}^{\perp}$. (Notice that if $w_{\mathbf{e}}$ is log-concave then $A_{\mathbf{e}}$ is convex.) Also, for any $\delta > 0$ we set $A_{\mathbf{e}}^{\delta} := \{x \in A_{\mathbf{e}} : \mathrm{dist}(x, \partial A_{\mathbf{e}}) \geq \delta\}$. We recall that a set $E$ is of *finite perimeter* if the distributional derivative $\nabla \mathbf{1}_E$ of $\mathbf{1}_E$ is a finite measure, that is

$$\int_{\mathbb{R}^n} |\nabla \mathbf{1}_E| < \infty.$$

Also, we use $\mathcal{H}^k$ to denote the $k$-dimensional Hausdorff measure. Here is our main result:

**Theorem 1.1.** *Let $E \subset \mathbb{R}^n$ be a bounded set of finite perimeter and assume that $A_{\mathbf{e}}$ is convex for $\mathcal{H}^{n-1}$-a.e. $\mathbf{e} \in \mathbb{S}^{n-1}$. Suppose further that $w_{\mathbf{e}}$ is locally Lipschitz inside $A_{\mathbf{e}}$ for $\mathcal{H}^{n-1}$-a.e. $\mathbf{e} \in \mathbb{S}^{n-1}$ and the following uniform bound holds: for any $\delta > 0$ there exists a constant $C_\delta$ such that*

$$|\nabla w_{\mathbf{e}}| \leq C_\delta \quad \text{a.e. inside } A_{\mathbf{e}}^{\delta}, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \mathbf{e} \in \mathbb{S}^{n-1}.$$

*Then $E$ is convex (up to a set of measure zero).*

Notice that, since log-concave functions are Lipschitz in the interior of their support, our assumption is weaker than asking that $w_{\mathbf{e}}$ is log-concave for $\mathcal{H}^{n-1}$-a.e. $\mathbf{e} \in \mathbb{S}^{n-1}$. Hence our theorem implies the following:[1]

**Corollary 1.2.** *Let $E \subset \mathbb{R}^n$ be a bounded set of finite perimeter and assume that, for $\mathcal{H}^{n-1}$-a.e. $\mathbf{e} \in \mathbb{S}^{n-1}$, $w_{\mathbf{e}}$ is log-concave. Then $E$ is convex (up to a set of measure zero).*

In light of the fact that marginals of log-concave functions are log-concave, one may wonder if the corollary above may be generalized to functions, that is, whether the fact that $\varphi : \mathbb{R}^n \to [0, +\infty)$ has log-concave marginals implies that $\varphi$ is log-concave. Unfortunately this stronger result is false. To see this, consider $\varphi := \mathbf{1}_{B_1} - \varepsilon\psi$, where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a smooth radial non-negative function supported in a small neighborhood of the origin. Since the marginals of $\mathbf{1}_{B_1}$ are positive and uniformly concave near the origin, it is easy to see that the marginals of $\varphi$ are concave for $\varepsilon > 0$ sufficiently small (hence, in particular, they are log-concave), but of course $\varphi$ is not log-concave.

The assumption that $E$ is of finite perimeter is technical, and it is likely that our result could be true without this assumption. However, finite perimeter plays an important role in the proof, which

[1] To be precise, since the bound on the Lipschitz constant of a convex function depends on its $L^\infty$ norm in a slightly larger domain, the bound on the Lipschitz constant for a log-concave function depends on a lower bound on the function itself in a slightly larger domain. Although in general there is no universal bound for a general class of log-concave functions, in our case all the functions $w_{\mathbf{e}}$ arise as marginals of a bounded set, which implies that $A_{\mathbf{e}_k} \to A_{\mathbf{e}}$ whenever $\mathbf{e}_k \to \mathbf{e}$ (a way to see this is to observe that $A_{\mathbf{e}}$ coincides with the projection onto $\mathbf{e}^{\perp}$ of the convex hull of the set of the density one points of $E$, see the proof of Theorem 1.1 in Section 4). Thanks to this fact it is not difficult to show that, if all the functions $w_{\mathbf{e}}$ are log-concave, then for any $\delta > 0$ there exists a constant $c_\delta > 0$ such that $w_{\mathbf{e}} \geq c_\delta$ inside $A_{\mathbf{e}}^{\delta}$ for $\mathcal{H}^{n-1}$-a.e. $\mathbf{e} \in \mathbb{S}^{n-1}$. In particular this implies that the assumptions of Theorem 1.1 are satisfied.

is based on measuring the perimeter using a fractional Hilbertian Sobolev norm of a smoothing of $\mathbf{1}_E$.[2] To formulate that result, we introduce some further notation.

For notational convenience, we will say that $a \lesssim b$ if there exists a dimensional constant $C$ such that $a \leq Cb$, and $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

Consider the 1/2 Sobolev norm, defined by

$$\|u\|_{H^{1/2}}^2 := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+1}} \, dy \, dx. \tag{1.2}$$

This norm can be used to measure the perimeter as follows:

**Theorem 1.3.** *Let $E \subset B_1 \subset \mathbb{R}^n$ have finite perimeter. Let $\gamma_n$ be the standard Gaussian in $\mathbb{R}^n$, $\gamma_{n,\varepsilon}(x) := \frac{1}{\varepsilon^n}\gamma_n(x/\varepsilon)$, and $\varphi_\varepsilon := \mathbf{1}_E * \gamma_{n,\varepsilon}$. Then*

$$\limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\varphi_\varepsilon\|_{H^{1/2}}^2 \simeq \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\varphi_\varepsilon\|_{H^{1/2}}^2 \simeq \int_{\mathbb{R}^n} |\nabla \mathbf{1}_E|,$$

*i.e., the ratios of these quantities are bounded above and below by positive dimensional constants.*

We have not investigated the connection, if any, between our norm and the notion of fractional perimeter introduced in [3] and whose relationship to the classical perimeter can be found in [1, 4]. Our norm is somewhat different in the spirit. On the one hand, as our norm is quadratic, the analysis performed in [4, 1] does not apply in our situation. On the other hand, the Hilbert structure allows us to exploit Fourier transform techniques.

As we shall see, Theorem 1.1 in $n$ dimensions follows easily from the case $n = 2$. In two dimensions the result says that if almost every marginal is supported on an interval and is uniformly Lipschitz in its interior, then the set is convex up to set of measure zero. The strategy of the proof is the following. Consider $E \subset \mathbb{R}^2$. Given $\theta \in [0, \pi]$, set $w_\theta := w_{\mathbf{e}_\theta}$ where $\mathbf{e}_\theta = (\cos(\theta), \sin(\theta))$. If $E$ is smoothly bounded, but not convex, then one expects that for some direction $\theta$, the derivative of the marginal $w_\theta$ is infinite at some interior point of (the convex hull of) its support (see Figure 1.1). For domains that merely have finite perimeter, the quantity that diverges is, roughly speaking, a suitable localized version of $\int\int |w_\theta'(t)|^2 dt\, d\theta$. We make this quantitative by considering the mollification $\varphi_\varepsilon$ of $\mathbf{1}_E$.

More precisely, given a bounded set of finite perimeter $E$, and $\varphi_\varepsilon$ as in Theorem 1.3 above, we show that

$$\|\varphi_\varepsilon\|_{H^{1/2}}^2 \lesssim |\log(\varepsilon)| \qquad \text{as } \varepsilon \to 0,$$

and a localized version of Theorem 1.3 saying that

$$\frac{1}{|\log(\varepsilon)|} \int_{B_r(x_0)} \int_{B_r(x_0)} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{n+1}} \, dy \, dx \tag{1.3}$$

controls from above the perimeter of $E$ inside $B_{r/2}(x_0)$ as $\varepsilon \to 0$.

[2]To our knowledge, this is the first time that a Hilbertian norm is used to measure the perimeter of a set, and we believe that this fact has its own interest.

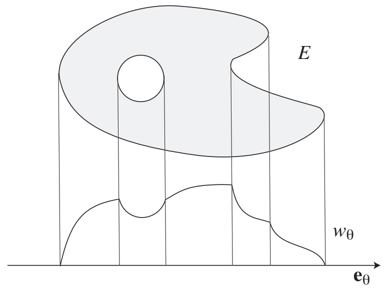


Figure 1.1: If $E$ is a smooth non-convex set, for a.e. $\theta$ the marginal $w_\theta$ has infinite derivative at some interior point. However, it is easy to check that this argument fails when $E$ is not smooth (consider for instance a disk with a small square removed from its interior). Still, we can show that some suitable integral quantity has to blow-up.

We then focus on the case $n = 2$ and, by use of the Fourier transform, show that (1.3) is majorized by

$$\frac{1}{|\log(\varepsilon)|}\int_0^\pi \int_{\mathbb{R}} \psi(t)^2 |w'_\theta * \gamma_{1,\varepsilon}|^2(t)\, dt\, d\theta, \tag{1.4}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a suitable smooth cut-off function. Now, the fact that $|\log \varepsilon| \to \infty$ as $\varepsilon \to 0$ shows that the measures

$$\frac{1}{|\log(\varepsilon)|}|w'_\theta * \gamma_{1,\varepsilon}|^2(t)\, dt\, d\theta$$

concentrate near the points where $w'_\theta$ is infinite, which (under our assumption on $w_\theta$) is contained inside the union over $\theta$ of the boundaries of the support of $w_\theta$, and this latter set corresponds to the boundary of the support of the projections of the convex hull of $E$. Then, if $E$ is not convex, it is not difficult to see that this information is incompatible with the fact that the expression in (1.3) controls from above the perimeter of $E$ at every point (in particular, at points on the reduced boundary of $E$ which are inside the support of the convex hull), proving the result. Once the theorem is proved in two dimensions, the higher dimensional case follows by a slicing argument.

The paper is organized as follows. In Section 2 we prove Theorem 1.3 and a local version, valid in all dimensions, showing that (1.3) controls the perimeter. Then, in Section 3 we majorize (1.3) by (1.4) in dimension 2. Finally, in Section 4 we prove Theorem 1.1.

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## 2 A Hilbert norm for sets of finite perimeter

In this section we prove Theorem 1.3. This argument is valid in any dimension. Let us recall that $u \in BV(\mathbb{R}^n)$ if its distributional derivative $\nabla u$ is a finite measure, and

$$\|u\|_{BV} := \int_{\mathbb{R}^n} |\nabla u| = |\nabla u|(\mathbb{R}^n).$$

Given a set $E$ of finite perimeter, there is a suitable notion of boundary, called the *reduced boundary* and denoted by $\partial^* E$, such that the following is true:

$$\int_{\mathbb{R}^n} |\nabla \mathbf{1}_E| = \mathcal{H}^{n-1}(\partial^* E),$$

and for any $x \in \partial^* E$ the following hold:

$$\mathcal{H}^{n-1}(\partial^* E \cap B_r(x)) \simeq r^{n-1} \qquad \text{as } r \to 0 \tag{2.1}$$

and there exists a unit vector $\nu(x)$ (called *inner measure theoretical normal to $E$ at $x$*) such that

$$\frac{1}{|B_r|} \int_{B_r} |\mathbf{1}_E(x+y) - \mathbf{1}_{\mathbb{R}^+}(\nu(x)\cdot y)|\, dy \to 0. \qquad \text{as } r \to 0 \tag{2.2}$$

Furthermore, $\nu$ is a $\mathcal{H}^{n-1}$-measurable function of $x \in \partial^* E$. (We refer to [2, Sections 3.3 and 3.5] or [7, Chapters 12 and 15] for more details.)

**Lemma 2.1.** *Let $u \in BV(\mathbb{R}^n)$ be supported in the unit ball $B_1$. Assume $|u| \le 1$, and set $u_\varepsilon := u * \gamma_{n,\varepsilon}$. Then there is a dimensional constant $C$ such that*

$$\|u_\varepsilon\|^2_{H^{1/2}} \le C|\log(\varepsilon)| \int_{\mathbb{R}^n} |\nabla u|, \qquad 0 < \varepsilon < 1/2.$$

*Proof.* We begin by showing that

$$\|u_\varepsilon\|^2_{H^{1/2}} \le C \int_\varepsilon^\infty \int_{\mathbb{R}^n} |\partial_t u_t|^2\, dx\, dt, \qquad u_t = u * \gamma_{n,t}. \tag{2.3}$$

Indeed recall that, in Fourier space, the $H^{1/2}$ norm can be expressed as

$$\|u_\varepsilon\|^2_{H^{1/2}} = c \int_{\mathbb{R}^n} |\xi| |\hat{u}_\varepsilon(\xi)|^2\, d\xi$$

for some positive dimensional constant $c$.[3] Furthermore,

$$\partial_t \hat{u}_{\varepsilon+t}(\xi) = \partial_t \big(e^{-(t+\varepsilon)^2|\xi|^2}\big)\hat{u}(\xi) = -2(t+\varepsilon)|\xi|^2 e^{-(t^2+2\varepsilon t)|\xi|^2} \hat{u}_\varepsilon(\xi),$$

[3] This follows from the fact that the Fourier transform of $|x|^{-(n+1)}$ is given by $c|\xi|$ $(c>0)$, which can be easily proven by homogeneity.

thus, using the fact that the Fourier transform is an isometry in $L^2$, up to a multiplicative constant we get

$$\int_\varepsilon^\infty \int_{\mathbb{R}^n} |\partial_t u_t|^2 \, dx \, dt = \int_0^\infty \int_{\mathbb{R}^n} (t+\varepsilon)^2 |\xi|^4 e^{-2(t^2+2\varepsilon t)|\xi|^2} |\hat{u}_\varepsilon(\xi)|^2 \, d\xi \, dt.$$

Moreover,

$$\begin{aligned}\int_0^\infty (t+\varepsilon)^2 |\xi|^4 e^{-2(t^2+2\varepsilon t)|\xi|^2} \, dt &\geq \int_\varepsilon^\infty t^2 |\xi|^4 e^{-8t^2|\xi|^2} \, dt + \int_0^\varepsilon \varepsilon^2 |\xi|^4 e^{-8\varepsilon t|\xi|^2} \, dt \\ &= \int_{\varepsilon|\xi|}^\infty s^2 |\xi|^2 e^{-8s^2} \frac{ds}{|\xi|} + \int_0^{\varepsilon^2|\xi|^2} \varepsilon^2 |\xi|^4 e^{-8s^2} \frac{ds}{\varepsilon|\xi|^2} \\ &\geq c|\xi|.\end{aligned}$$

(If $\varepsilon|\xi| \leq 1$, the first integral majorizes $|\xi|$, and if $\varepsilon|\xi| \geq 1$, then the second integral majorizes $|\xi|$.) Therefore, (2.3) follows.

Next, using the formula above for $\partial_t \hat{u}_{1+t}(\xi)$ (i.e., $\varepsilon = 1$) and integrating over $t \in [1, \infty)$, we have, up to a multiplicative constant,

$$\int_2^\infty \int_{\mathbb{R}^n} |\partial_t u_t|^2 \, dx \, dt = \int_{\mathbb{R}^n} \int_1^\infty (t+1)^2 |\xi|^4 e^{-2(t^2+2t)|\xi|^2} \, dt \, |\hat{u}_1(\xi)|^2 \, d\xi$$

and

$$\int_1^\infty (t+1)^2 |\xi|^4 e^{-2(t^2+2t)|\xi|^2} \, dt \leq \int_0^\infty 4t^2 |\xi|^4 e^{-t^2|\xi|^2} \, dt = c|\xi|.$$

Hence, since $\gamma_{n,1}$ is a smooth rapidly decaying function and $u$ is bounded by 1 and supported in the unit ball, we have[4]

$$\int_2^\infty \int_{\mathbb{R}^n} |\partial_t u_t|^2 \, dx \, dt \leq C \|u * \gamma_{n,1}\|_{H^{1/2}}^2 \leq C \|u\|_{L^2}^2 \leq C \|u\|_{L^1}$$

for some dimensional constant $C$. Finally, again using that $u$ is supported in the unit ball, the Sobolev inequality for BV functions [2, Chapter 3.4] implies that the $L^1$ norm of $u$ is controlled by the total variation of its derivative, thus giving

$$\int_2^\infty \int_{\mathbb{R}^n} |\partial_t u_t|^2 \, dx \, dt \leq C \int_{\mathbb{R}^n} |\nabla u|. \tag{2.4}$$

We turn next to the integral from $\varepsilon$ to 2. Recall that since $\gamma_{n,t}(x) := \frac{1}{t^n}\gamma_n(x/t)$, the time derivative of $u_t$ can be written as

$$\partial_t u_t(x) = \partial_t \int_{\mathbb{R}^n} u(x - tz) \gamma_n(z) \, dz = -\int_{\mathbb{R}^n} \nabla u(x - tz) \cdot z \gamma_n(z) \, dz.$$

[4] A possible way to show that $\|u * \gamma_{n,1}\|_{H^{1/2}}^2 \leq C\|u\|_{L^2}^2$ is to use that $\|f\|_{H^{1/2}}^2 \leq \|f\|_{L^2} \|\nabla f\|_{L^2}$ for all smooth rapidly decaying functions $f$ (this can be easily proved in Fourier space), and to observe that

$$\|u * \gamma_{n,1}\|_{L^2} \leq \|u\|_{L^2}, \qquad \|u * \nabla \gamma_{n,1}\|_{L^2} \leq C\|u\|_{L^2}.$$

It follows from Fubini's theorem that

$$\int_{\mathbb{R}^n} |\partial_t u_t| \, dx \leq C \int_{\mathbb{R}^n} |\nabla u|.$$

We can also write

$$\begin{aligned}\partial_t u_t(x) &= \partial_t \int_{\mathbb{R}^n} u(x-z) \frac{1}{t^n} \gamma_n\left(\frac{z}{t}\right) dz = -\frac{1}{t} \int_{\mathbb{R}^n} u(x-z) \left[ n\gamma_n\left(\frac{z}{t}\right) + \nabla \gamma_n\left(\frac{z}{t}\right) \cdot \frac{z}{t} \right] \frac{dz}{t^n} \\ &= -\frac{1}{t} \int_{\mathbb{R}^n} u(x-tz) \big[ n\gamma_n(z) + \nabla \gamma_n(z) \cdot z \big] \, dz\end{aligned}$$

from which, along with $|u| \leq 1$, we get

$$\|\partial_t u_t\|_\infty \leq \frac{C}{t}.$$

Thus

$$\int_{\mathbb{R}^n} |\partial_t u_t|^2 \, dx \leq \frac{C}{t} \int_{\mathbb{R}^n} |\nabla u|. \tag{2.5}$$

Combining (2.3), (2.4), and (2.5), we conclude that

$$\begin{aligned}\|u_\varepsilon\|_{H^{1/2}}^2 &\lesssim \int_\varepsilon^\infty \int_{\mathbb{R}^n} |\partial_t u_t|^2 \, dx \, dt \\ &\leq \left( C + \int_\varepsilon^2 \frac{C}{t} \, dt \right) \int_{\mathbb{R}^n} |\nabla u| \ \leq C |\log \varepsilon| \int_{\mathbb{R}^n} |\nabla u|, \qquad 0 < \varepsilon < 1/2,\end{aligned}$$

as desired. □

We now show that the norm (1.2) controls the perimeter locally.

**Lemma 2.2.** *Let $E$ be a set of finite perimeter, and let $\varphi_\varepsilon$ be as in Theorem 1.3. Fix $x_0 \in \partial^* E$. Then, for any $r_0 > 0$,*

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log(\varepsilon)|} \int_{B_{r_0}(x_0)} \int_{B_{r_0}(x_0)} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{n+1}} \, dx \, dy \gtrsim \mathcal{H}^{n-1}(B_{r_0/2}(x_0) \cap \partial^* E).$$

*Proof.* For $x \in \partial^* E$ define

$$D_k(x) := \sup_{j \geq k} \frac{1}{|B_{2^{-j}}|} \int_{B_{2^{-j}}} |\mathbf{1}_E(x+y) - \mathbf{1}_{\mathbb{R}^+}(\nu(x) \cdot y)| \, dy,$$

Let $\delta > 0$ be a small dimensional constant (to be fixed later). Let

$$F_m := \{x \in \partial^* E : D_m(x) \leq \delta\}.$$

Then

$$F_m \subset F_{m+1} \quad \text{and} \quad \bigcup_m F_m = \partial^* E$$

by (2.2). Furthermore, $F_m$ is $\mathcal{H}^{n-1}$-measurable because $\nu$ is $\mathcal{H}^{n-1}$-measurable. Therefore, we can choose $m$ sufficiently large that

$$\mathcal{H}^{n-1}(B_{r_0/2}(x_0) \cap F_m) \geq \frac{1}{2}\mathcal{H}^{n-1}(B_{r_0/2}(x_0) \cap \partial^* E).$$

Fix $m$ as above, set $\rho = 2^{-m}$, and suppose that $\sqrt{\varepsilon} < \rho/100$. Fix $r \in (100\varepsilon, \sqrt{\varepsilon})$. By a standard covering argument[5] there are $N = N_r$ disjoint balls $B_r(x_j)$, $j = 1, \ldots, N_r$, such that $x_j \in F_m \cap B_{r_0/2}(x_0)$, and

$$N_r \gtrsim \mathcal{H}^{n-1}(B_{r_0/2}(x_0) \cap \partial^* E)/r^{n-1}. \tag{2.6}$$

We now want to estimate from below

$$\int_{x,y \in B_{r_0}(x_0),\, r/4 \leq |x-y| \leq r/2} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{n+1}}\, dx\, dy.$$

Since the balls $B_r(x_j)$ are disjoint we have

$$\begin{aligned}
&\int_{x,y \in B_{r_0}(x_0),\, r/4 \leq |x-y| \leq r/2} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{n+1}}\, dx\, dy \\
&\geq \sum_{j=1}^{N} \int_{B_{r/2}(x_j) \cap \{\nu(x_j)\cdot(x-x_j) \geq r/10\}} dx \int_{B_{r/2}(x_j) \cap \{\nu(x_j)\cdot(y-x_j) \leq -r/10\} \cap \{r/4 \leq |x-y| \leq r/2\}} dy\, \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{n+1}}.
\end{aligned}$$

Because inside $B_{r/2}(x_j)$ the set $E$ is very close in $L^1$ to the hyperplane $\{\nu(x_j)\cdot(x-x_j) \geq 0\}$ (since $D_m(x_j) \leq \delta$) and $r \geq 100\varepsilon$, we deduce that, provided $\delta$ is chosen sufficiently small (the smallness depending only on the dimension), $|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| \gtrsim 1$ on a substantial fraction of the latter integrals (since $\varphi_\varepsilon$ is close to the characteristic function of $E$, and thus to the characteristic function of the hyperplane). Thus, since $|x-y| \leq r/2$ and both $x$ and $y$ vary inside some sets whose measure is of order $r^n$, we get

$$\begin{aligned}
\sum_{j=1}^{N} \int_{B_{r/2}(x_j) \cap \{\nu(x_j)\cap(x-x_j) \geq r/10\}} dx \int_{B_{r/2}(x_j) \cap \{\nu(x_j)\cap(y-x_j) \leq -r/10\}} dy\, \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{n+1}} \\
\gtrsim N_r r^{n-1} \gtrsim \mathcal{H}^{n-1}(B_{r_0/2}(x_0) \cap \partial^* E),
\end{aligned}$$

where the last inequality follows by (2.6). Thus, we proved that for any $r \in (100\varepsilon, \sqrt{\varepsilon})$

$$\int_{x,y \in B_{r_0}(x_0),\, r/4 \leq |x-y| \leq r/2} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{n+1}}\, dx\, dy \gtrsim \mathcal{H}^{n-1}(B_{r_0/2}(x_0) \cap \partial^* E).$$

[5] If $\{B_r(x_j)\}_{1 \leq j \leq N_r}$ is a maximal disjoint family of balls with $x_j \in F_m \cap B_{r_0/2}(x_0)$, then $\bigcup_{1 \leq j \leq N_r} B_{3r}(x_j)$ covers $F_m \cap B_{r_0/2}(x_0)$, and by the definition of $\mathcal{H}^{n-1}$ (see [2, Section 2.8] or [7, Chapter 3]) we get

$$\mathcal{H}^{n-1}(F_m \cap B_{r_0/2}(x_0)) \leq C_n \sum_{j=1}^{N_r} (3r)^{n-1} = C_n 3^{n-1} N_r\, r^{n-1},$$

where $C_n > 0$ is a dimensional constant.

Hence, choosing $r = 4^{-k}$ and letting $k$ vary between $\ell_1 := \lfloor -\log_4(\sqrt{\varepsilon})\rfloor + 1$ and $\ell_2 := \lfloor -\log_4(100\varepsilon)\rfloor$, for $\varepsilon$ sufficiently small we get

$$\begin{aligned}\int_{B_{r_0}(x_0)}\int_{B_{r_0}(x_0)} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{n+1}}\,dx\,dy &\geq \sum_{k=\ell_1}^{\ell_2} \int_{x,y\in B_{r_0}(x_0),\,4^{-k}/4\leq|x-y|\leq 4^{-k}/2} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{n+1}}\,dx\,dy\\ &\gtrsim (\ell_2-\ell_1)\,\mathcal{H}^{n-1}(B_{r_0/2}(x_0)\cap\partial^* E)\\ &\gtrsim |\log(\varepsilon)|\,\mathcal{H}^{n-1}(B_{r_0/2}(x_0)\cap\partial^* E)\end{aligned}$$

as desired. □

*Proof of Theorem 1.3.* The upper bound in Theorem 1.3 follows from Lemma 2.1, and the lower bound follows from Lemma 2.2 letting $x_0 = 0$, $r_0 = 4$. □

# 3 The $H^{1/2}$ norm expressed in terms of the marginals

Here we show that the $H^{1/2}(\mathbb{R}^2)$ norm of a function is equal (up to constants) to the average of the $H^1(\mathbb{R}^2)$ norm of its marginals, and then we prove a localized version of this identity.[6] The arguments in this section are specific to the case $n = 2$.

Given a smooth rapidly decaying function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, for any $\theta \in [0, \pi]$ we define the marginal

$$w_\theta(t) := \int_{\mathbb{R}} \varphi\big(R_\theta(t,s)\big)\,ds, \tag{3.1}$$

where $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counterclockwise rotation by an angle $\theta$ around the origin.

## 3.1 A global identity

We claim that the norm

$$\int_0^\pi \int_{\mathbb{R}} |w_\theta'|^2(t)\,dt\,d\theta$$

is equivalent to the $H^{1/2}$ norm of $\varphi$.

To prove this, we first compute the Fourier transform of $w_\theta$. We denote by $\mathbf{e}_\theta := R_\theta e_1 = (\cos\theta, \sin\theta)$. Then

$$\begin{aligned}\hat{w}_\theta(\tau) &:= \int_{\mathbb{R}} w_\theta(t)e^{it\tau}\,dt = \int_{\mathbb{R}^2} \varphi\big(R_\theta(t,s)\big)e^{it\tau}\,dt\,ds\\ &= \int_{\mathbb{R}^2} \varphi\big(R_\theta x\big)e^{i\tau e_1\cdot x}\,dx = \int_{\mathbb{R}^2}\varphi(x)e^{i(\tau\mathbf{e}_\theta)\cdot x}\,dx = \hat{\varphi}\big(\tau\mathbf{e}_\theta\big),\end{aligned} \tag{3.2}$$

where (by abuse of notation) we used $\hat{w}$ and $\hat{\varphi}$ to denote respectively the Fourier transform on $\mathbb{R}$ and on $\mathbb{R}^2$.

[6] The global identity (3.3) is a well-known fact in the theory of Radon transform, and can be found, for instance, in the proof of Theorem 5.1 on page 43 of [8]. We thank Sigurdur Helgason for providing us with this reference.

Thanks to the formula above and the fact that the Fourier transform is an isometry in $L^2$, we get (up to a multiplicative constant)

$$\int_0^\pi \int_{\mathbb{R}} |w_\theta'|^2(t)\,dt\,d\theta = \int_0^\pi \int_{\mathbb{R}} |\tau|^2|\hat{w}_\theta|^2(\tau)\,d\tau\,d\theta = \int_0^\pi \int_{\mathbb{R}} |\tau|^2|\hat{\varphi}|^2\big(\tau\mathbf{e}_\theta\big)\,d\tau\,d\theta.$$

It is now easy to check that the last integral is simply an integration in polar coordinates, so by setting $\xi := \tau e^{i\theta}$ (so that $|\tau| = |\xi|$ and $d\xi = |\tau|\,d\tau\,d\theta$) we get

$$\int_0^\pi \int_{\mathbb{R}} |w_\theta'|^2(t)\,dt\,d\theta = c\int_{\mathbb{R}^2} |\xi|\,|\hat{\varphi}|^2(\xi)\,d\xi = \bar{c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^3}\,dy\,dx \tag{3.3}$$

for some dimensional constant $\bar{c} > 0$, which proves the claim.

### 3.2 A localized identity

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth compactly supported function. By (3.2) and the properties of the Fourier transform, we have

$$\int_0^\pi \int_{\mathbb{R}} \psi(t)^2|w_\theta'|^2(t)\,dt\,d\theta = \int_0^\pi \int_{\mathbb{R}} \left|\int_{\mathbb{R}} \hat{\psi}(\tau-\sigma)\sigma\hat{\varphi}(\sigma\mathbf{e}_\theta)\,d\sigma\right|^2 d\tau\,d\theta.$$

We now notice that, since $\hat{\varphi}(s\mathbf{e}_\theta) = \int_{\mathbb{R}^2} \varphi(x)e^{-is\mathbf{e}_\theta\cdot x}\,dx$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \hat{\psi}(\tau-\sigma)\sigma\hat{\varphi}(\sigma\mathbf{e}_\theta)\,d\sigma\right|^2 \\
&= \int\int\int\int \hat{\psi}(\tau-\sigma)\sigma\varphi(x)e^{-i\sigma\mathbf{e}_\theta\cdot x}\overline{\hat{\psi}(\tau-\upsilon)}\upsilon\varphi(y)e^{i\upsilon\mathbf{e}_\theta\cdot y}\,d\sigma\,d\upsilon\,dx\,dy \\
&= \int\int \varphi(x)\varphi(y)\int \hat{\psi}(\tau-\sigma)\sigma e^{-i\sigma\mathbf{e}_\theta\cdot x}\,d\sigma\overline{\int \hat{\psi}(\tau-\upsilon)\upsilon e^{-i\upsilon\mathbf{e}_\theta\cdot y}\,d\upsilon}\,dx\,dy \\
&= \int\int \varphi(x)\varphi(y)[\mathbf{e}_\theta\cdot\nabla_x]\int \hat{\psi}(\tau-\sigma)e^{-i\sigma\mathbf{e}_\theta\cdot x}\,d\sigma[\mathbf{e}_\theta\cdot\nabla_y]\overline{\int \hat{\psi}(\tau-\upsilon)e^{-i\upsilon\mathbf{e}_\theta\cdot y}\,d\upsilon}\,dx\,dy \\
&= \int\int \varphi(x)\varphi(y)[\mathbf{e}_\theta\cdot\nabla_x][\mathbf{e}_\theta\cdot\nabla_y]\Big(\psi(\mathbf{e}_\theta\cdot x)\psi(\mathbf{e}_\theta\cdot y)e^{-i\tau\mathbf{e}_\theta\cdot(x-y)}\Big)\,dx\,dy.
\end{aligned}$$

Hence, integrating this expression with respect to $\tau$ and $\theta$ we get

$$\begin{aligned}
&\int_0^\pi \int_{\mathbb{R}} \psi(t)^2|w_\theta'|^2(t)\,dt\,d\theta \\
&= \int_0^\pi \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(x)\varphi(y)[\mathbf{e}_\theta\cdot\nabla_x][\mathbf{e}_\theta\cdot\nabla_y]\Big(\psi(\mathbf{e}_\theta\cdot x)\psi(\mathbf{e}_\theta\cdot y)\delta\big(\mathbf{e}_\theta\cdot(x-y)\big)\Big)\,dx\,dy\,d\theta,
\end{aligned} \tag{3.4}$$

where $\delta(\cdot)$ denotes the delta measure. We now claim that

$$\int_0^\pi \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Phi(x)[\mathbf{e}_\theta\cdot\nabla_x][\mathbf{e}_\theta\cdot\nabla_y]\Big(\psi(\mathbf{e}_\theta\cdot x)\psi(\mathbf{e}_\theta\cdot y)\delta\big(\mathbf{e}_\theta\cdot(x-y)\big)\Big)\,dx\,dy\,d\theta = 0 \tag{3.5}$$

for any smooth rapidly decaying function $\Phi$. Indeed, the integral above is equal to the limit of

$$\int_0^\pi \int_{B_R} \int_{\mathbb{R}^2} \Phi(x)[\mathbf{e}_\theta \cdot \nabla_x][\mathbf{e}_\theta \cdot \nabla_y]\Big(\psi(\mathbf{e}_\theta \cdot x)\psi(\mathbf{e}_\theta \cdot y)\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\Big)\, dx\, dy\, d\theta$$

as $R \to \infty$, and the latter integral is equal to

$$-\int_0^\pi \int_{\mathbb{R}^2} [\mathbf{e}_\theta \cdot \nabla_x \Phi(x)]\bigg(\psi(\mathbf{e}_\theta \cdot x)\int_{\partial B_R} \psi(\mathbf{e}_\theta \cdot y)\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\, \nu_{\partial B_R}(y)\cdot \mathbf{e}_\theta \, d\mathcal{H}^1(y)\bigg)\, dx\, d\theta.$$

Next, we have the majorization

$$\Big|\psi(\mathbf{e}_\theta \cdot y)\, \nu_{\partial B_R}(y)\cdot \mathbf{e}_\theta\Big| \le \frac{C}{R} \qquad \text{for } R \gg 1$$

(since $\psi$ is compactly supported, so $|\nu_{\partial B_R}(y)\cdot \mathbf{e}_\theta| \le C/R$ on the support of $\psi(\mathbf{e}_\theta \cdot y)$). Furthermore, for each $\theta$, $\psi(\mathbf{e}_\theta \cdot y)$ is supported on portion of the circle $\partial B_R$ of length $O(1)$. Thus the integral is $O(1/R)$, and the claim follows.

By exchanging the roles of $x$ and $y$, we deduce that (3.5) holds also if we replace $\Phi(x)$ by $\Phi(y)$. Hence, by (3.5) applied with $\Phi(x) = \varphi(x)^2$ and $\Phi(y) = \varphi(y)^2$ we deduce that the expression (3.4) is equal to

$$\begin{aligned}
&\int_0^\pi \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\varphi(x)-\varphi(y)|^2 [\mathbf{e}_\theta \cdot \nabla_x][\mathbf{e}_\theta \cdot \nabla_y]\Big(\psi(\mathbf{e}_\theta \cdot x)\psi(\mathbf{e}_\theta \cdot y)\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\Big)\, dx\, dy\, d\theta \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\varphi(x)-\varphi(y)|^2 \int_0^\pi \Big(\psi(\mathbf{e}_\theta \cdot x)\psi(\mathbf{e}_\theta \cdot y)\delta''\big(\mathbf{e}_\theta \cdot (x-y)\big)\Big)\, d\theta\, dx\, dy \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\varphi(x)-\varphi(y)|^2 \int_0^\pi \Big(\big[\psi'(\mathbf{e}_\theta \cdot x)\psi(\mathbf{e}_\theta \cdot y) + \psi(\mathbf{e}_\theta \cdot x)\psi'(\mathbf{e}_\theta \cdot y)\big]\, \delta'\big(\mathbf{e}_\theta \cdot (x-y)\big)\Big)\, d\theta\, dx\, dy \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\varphi(x)-\varphi(y)|^2 \int_0^\pi \Big(\psi'(\mathbf{e}_\theta \cdot x)\psi'(\mathbf{e}_\theta \cdot y)\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\Big)\, d\theta\, dx\, dy.
\end{aligned}$$

We now observe that, by the chain-rule,

$$\delta'\big(\mathbf{e}_\theta \cdot (x-y)\big) = \frac{\partial_\theta \delta\big(\mathbf{e}_\theta \cdot (x-y)\big)}{e_\theta^\perp \cdot (y-x)},$$

$$\delta''\big(\mathbf{e}_\theta \cdot (x-y)\big) = \frac{\partial_{\theta\theta} \delta\big(\mathbf{e}_\theta \cdot (x-y)\big)}{\big(e_\theta^\perp \cdot (y-x)\big)^2} + \frac{\partial_\theta \delta\big(\mathbf{e}_\theta \cdot (x-y)\big)}{\big(e_\theta^\perp \cdot (y-x)\big)\big(e_\theta \cdot (y-x)\big)}.$$

Hence, if we integrate by parts in $\theta$ so that no derivatives fall onto $\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)$, we get that there is only one term with $\big(\mathbf{e}_\theta \cdot (x-y)\big)^{-2}$, and all the others are smooth functions of $\theta$ in a neighborhood of the support of $\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)$ multiplied at most by $\big(\mathbf{e}_\theta \cdot (x-y)\big)^{-1}$, namely

$$\begin{aligned}
\int_0^\pi [\mathbf{e}_\theta \cdot \nabla_x][\mathbf{e}_\theta \cdot \nabla_y]\Big(\psi(\mathbf{e}_\theta \cdot x)\psi(\mathbf{e}_\theta \cdot y)\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\Big)\, d\theta &= \int_0^\pi \frac{\psi(\mathbf{e}_\theta \cdot x)\psi(\mathbf{e}_\theta \cdot y)}{\big(\mathbf{e}_\theta \cdot (x-y)\big)^2}\, \delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\, d\theta \\
&\quad + \int_0^\pi \frac{\Psi(\theta, x, y)}{\big(\mathbf{e}_\theta \cdot (x-y)\big)} \delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\, d\theta,
\end{aligned}$$

where, for any $x \neq y$, $\Psi(\theta, x, y)$ is a smooth function of $\theta$ when $\mathbf{e}_\theta$ is almost orthogonal to $(x-y)$ (that is, near the support of $\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)$). Hence, since $\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)$ is a distribution which is homogeneous of degree $-1$, we deduce that

$$\left|\int_0^\pi \frac{\Psi(\theta,x,y)}{\big(\mathbf{e}_\theta \cdot (x-y)\big)}\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\,d\theta\right| \leq \frac{C}{|x-y|^2},$$

and we obtain

$$\begin{aligned}
&\int_0^\pi \int_{\mathbb{R}} \psi(t)^2 |w_\theta'|^2(t)\,dt\,d\theta \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\varphi(x)-\varphi(y)|^2 \int_0^\pi \frac{\psi(\mathbf{e}_\theta \cdot x)\psi(\mathbf{e}_\theta \cdot y)}{\big(\mathbf{e}_\theta \cdot (x-y)\big)^2}\,\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\,d\theta \\
&\qquad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\varphi(x)-\varphi(y)|^2 O(|x-y|^{-2})\,dx\,dy.
\end{aligned}$$

We now observe that, being the expression inside the first integral positive (because $\psi(\mathbf{e}_\theta \cdot x)\psi(\mathbf{e}_\theta \cdot y)\delta\big(\mathbf{e}_\theta \cdot (x-y)\big) = \psi(\mathbf{e}_\theta \cdot x)^2\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\big)$, it decreases if we localize it with a cut-off function $\chi(x)\chi(y)$. In particular, if the support of $\chi(x)\chi(y)$ is contained inside the one of $\psi(\mathbf{e}_\theta \cdot x)\psi(\mathbf{e}_\theta \cdot y)$ for any $\theta \in [0,\pi]$, since[7]

$$\int_0^\pi \frac{1}{\big(\mathbf{e}_\theta \cdot (x-y)\big)^2}\,\delta\big(\mathbf{e}_\theta \cdot (x-y)\big)\,d\theta = \frac{\hat{c}}{|x-y|^3}, \qquad \hat{c} > 0,$$

we get

$$\begin{aligned}
\int_0^\pi \int_{\mathbb{R}} \psi(t)^2 |w_\theta'|^2(t)\,dt\,d\theta \geq \hat{c} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^3}\chi(x)\chi(y)\,dx\,dy& \\
- C \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^2}\,dx\,dy&
\end{aligned} \tag{3.6}$$

for any $\chi : \mathbb{R}^n \to [0,1]$ whose support is contained inside the one of $\psi(\mathbf{e}_\theta \cdot x)$ for any $\theta \in [0,\pi]$.

# 4 Proof of Theorem 1.1

We first prove the result when $n = 2$.

Let $\varphi_\varepsilon := \mathbf{1}_E * \gamma_{2,\varepsilon}$, and let $w_\theta$ and $w_{\theta,\varepsilon}$ denote respectively the marginals of $\mathbf{1}_E$ and of $\varphi_\varepsilon$ (see (3.1)). Because Gaussian densities tensorize, it is immediate to check that they commute with marginals, so the following identity holds:

$$w_{\theta,\varepsilon} = w_\theta * \gamma_{1,\varepsilon}.$$

By (3.3) and Lemma 2.1 we see that

$$\frac{1}{|\log(\varepsilon)|}\int_0^\pi \int_{\mathbb{R}} |w_{\theta,\varepsilon}'|^2(t)\,dt\,d\theta = \frac{\bar{c}}{|\log(\varepsilon)|}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2}{|x-y|^3}\,dx\,dy \leq C,$$

[7]One way to prove this identity is to observe that $\int_0^\pi \frac{1}{(\mathbf{e}_\theta \cdot (x-y))^2}\,\delta\left(\mathbf{e}_\theta \cdot (x-y)\right)d\theta$ is homogeneous of degree $-3$ in $x-y$, it is invariant under rotations, and it is positive on positive functions.

so the measures

$$\nu_\varepsilon(dt,d\theta) := \frac{|w'_{\theta,\varepsilon}|^2(t)}{|\log(\varepsilon)|}\,dt\,d\theta, \qquad \mu_\varepsilon(dx,dy) := \frac{1}{|\log(\varepsilon)|}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2}{|x-y|^3}\,dx\,dy$$

are equibounded and they weakly converge (up to a subsequence) to measures $\nu(dt,d\theta)$ and $\mu(dx,dy)$.

Since $\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2}{|\log(\varepsilon)|} \to 0$ as $\varepsilon \to 0$, it follows that the measure $\mu$ is concentrated on the diagonal $\{x=y\}$.

Concerning $\nu$, let us denote by $(a_\theta,b_\theta)$ the interval $\{w_\theta>0\}$, and observe that, thanks to our assumption, there exists a constant $\bar C_\delta>0$ such that $|w'_{\theta,\varepsilon}| \le \bar C_\delta$ inside $[a_\theta+\delta,b_\theta-\delta]$ for all $\delta>0$, uniformly in $\varepsilon$ and $\theta$. Hence, by integrating $\nu_\varepsilon$ against a test function $f(\theta,t)$ which is zero in a neighborhood of

$$S := \bigcup_{\theta\in[0,\pi]} \{a_\theta,b_\theta\}\times\{\theta\} \subset \mathbb{R}\times[0,\pi]$$

and letting $\varepsilon\to 0$ we get

$$\int f(t,\theta)\,\nu(dt,d\theta)=0,$$

and by the arbitrariness of $f$ we deduce that $\nu$ is concentrated on the closure of $S$.

We now make the following observation: in (3.6) we have related the local $H^{1/2}$ norm of $\varphi$ to another norm which depends only on the marginals of $\varphi$ (up to the lower order term on the second line of the formula, which anyhow will vanishes when we take $\varphi=\varphi_\varepsilon$ and let $\varepsilon\to 0$). The key fact is that the choice of the origin is completely arbitrary. So, we argue as follows.

Replace $E$ by $E^{(1)}$, the set of its density one points, i.e.,

$$E^{(1)} := \left\{x\in\mathbb{R}^n : \lim_{r\to 0}\frac{|E\cap B_r(x)|}{|B_r|}=1\right\},$$

and define $\mathcal{C}$ as the convex hull of $E^{(1)}$. If $E^{(1)}$ is not convex, then we can find a point $x_0\in\partial^*E\setminus\partial\mathcal{C}$ which belongs to the interior of $\mathcal{C}$. Let us fix a system of coordinates so that $x_0$ is the origin. With this choice, for any $\theta\in[0,\pi]$ the set $(a_\theta,b_\theta)=\{w_\theta>0\}$ coincides with the projection of $\mathcal{C}$ onto the line in the direction of $\mathbf{e}_\theta$. This shows that the set $S$ defined above is closed, and because $x_0$ belongs to the interior of $\mathcal{C}$ there exists a small constant $\rho>0$ such that

$$[-2\rho,2\rho]\subset(a_\theta,b_\theta) \qquad \forall\,\theta\in[0,\pi].$$

Let us take a cut-off function $\psi(t)$ supported inside $[-\rho,\rho]$, and then choose a cut-off function $0\le\chi\le 1$ such that the support of $\chi(x)\chi(y)$ is contained inside the one of $\psi(\mathbf{e}_\theta\cdot x)\psi(\mathbf{e}_\theta\cdot y)$ for any $\theta\in[0,\pi]$, and $\chi=1$ inside $B_{r_0}(x_0)$ for some small $r_0$. Then, since $\nu$ is concentrated on $S$, by (3.6) applied to $w_{\theta,\varepsilon}$ and $\varphi_\varepsilon$, Lemma 2.2, (2.1), and the fact that $\mu$ is concentrated on $\{x = y\}$, we get

$$
\begin{aligned}
0 &= \int_0^\pi \int_{\mathbb{R}} \psi(t)^2 \nu(dt, d\theta) \\
&\geq \liminf_{\varepsilon \to 0} \frac{\hat{c}}{|\log(\varepsilon)|} \int_{B_{r_0}(x_0)} \int_{B_{r_0}(x_0)} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^3}\, dx\, dy \\
&\qquad\qquad - \frac{C}{|\log(\varepsilon)|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^2}\, dx\, dy \\
&\gtrsim r_0^{n-1} - C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x-y|\, \mu(dx, dy) = r_0^{n-1} > 0
\end{aligned}
$$

a contradiction which concludes the proof in the two dimensional case.

For the general case we argue as follows: let $\pi \subset \mathbb{R}^n$ be a two dimensional plane passing through the origin, and for any $\mathbf{e} \in \mathbb{S}^{n-1} \cap \pi$ consider the projection of $E$ onto the hyperplane $\mathbf{e}^\perp$. The hypothesis on $E$ implies that, if we slice $E$ with some translate $\pi_v := \pi + v$ ($v \in \mathbb{R}^n$) of $\pi$, by the slicing formula (see for instance [7, Theorem 18.11 and Remark 18.13]) the set $E \cap \pi_v \subset \pi_v \simeq \mathbb{R}^2$ is a bounded set of finite perimeter for $\mathcal{H}^n$-a.e. $v \in \mathbb{R}^n$. In addition, $E \cap \pi_v$ satisfies the assumptions of our theorem with $n = 2$. Hence, by what we proved above, $E \cap \pi_v$ coincides with a convex set up to a set of $\mathcal{H}^2$-measure zero.

We now show that $E^{(1)}$ is convex. Fix $x, y \in E^{(1)}$ and $t \in (0,1)$, and pick a plane $\pi$ such that $x - y \in \pi$. By the discussion above we deduce that, for $\mathcal{H}^n$-a.e. $v \in B_r(y)$,

$$E \cap \pi_v \quad \text{is equal to a convex set up to a set of } \mathcal{H}^2\text{-measure zero.}$$

Hence, by Fubini's Theorem we obtain the following: for $\mathcal{H}^n$-a.e. $v \in B_r(y)$, for $\mathcal{H}^2$-a.e. $z \in B_r(y) \cap E \cap (E - (x-y)) \cap \pi_v$, the set

$$[z, z + (x-y)] \cap E \quad \text{is equal to a segment up to a set of } \mathcal{H}^1\text{-measure zero}$$

(here we use $[z, z + (x-y)]$ to denote the segment from $z$ to $z + (x-y)$). From this fact and Fubini's theorem (again), it follows that

$$\mathcal{H}^1\Big(\big([z, z+(x-y)] \cap B_r(tx+(1-t)y)\big) \setminus E\Big) = 0 \qquad \text{for } \mathcal{H}^n\text{-a.e. } z \in B_r(y) \cap E \cap (E - (x-y)). \tag{4.1}$$

Since both $x, y \in E^{(1)}$ we see that

$$\liminf_{r \to 0} \frac{|B_r(y) \cap E \cap (E - (x-y))|}{|B_r|} \geq 1 - \lim_{r \to 0} \frac{|B_r(y) \setminus E|}{|B_r|} - \lim_{r \to 0} \frac{|B_r(x) \setminus E|}{|B_r|} = 1,$$

so it follows from (4.1) that

$$\lim_{r \to 0} \frac{|B_r(tx + (1-t)y) \setminus E|}{|B_r|} = 0,$$

proving that $tx + (1-t)y \in E^{(1)}$. By the arbitrariness of $x, y, t$ we deduce that $E^{(1)}$ is convex, concluding the proof.

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