On flows associated to Sobolev vector fields in Wiener spaces: an approach à la DiPerna-Lions

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1 Introduction

The aim of this paper is the extension to an infinite-dimensional framework of the theory of flows associated to weakly differentiable (with respect to the spatial variable x) vector fields $\boldsymbol{b}(t,x)$. Starting from the seminal paper [30], the finite-dimensional theory had in recent times many developments, with applications to fluid dynamics [40], [41], [26], to the theory of conservation laws [5], [3], and it covers by now Sobolev and even bounded variation [1] vectorfields, under suitable bounds on the distributional divergence of $\boldsymbol{b}_t(x) := \boldsymbol{b}(t,x)$. Furthermore, in the case of $W^{1,p}_{\rm loc}$ vector fields with $p > 1$, even quantitative error estimates have been found in [22]; we refer to the Lecture Notes [2] and [6], and to the bibliographies therein, for the most recent developments on this subject. Our paper fills the gap, pointed out in [2], between this family of results and those available in infinite-dimensional spaces, where only exponential integrability assumptions on $\nabla \boldsymbol{b}_t$ have been considered so far.

Before passing to the description of our results in Wiener spaces, we briefly illustrate the heuristic ideas underlying the above-mentioned finite-dimensional results. The first basic idea is not to look for pointwise uniqueness statements, but rather to the family of solutions to the ODE as a whole. This leads to the concept of flow map $\boldsymbol{X}(t,x)$ associated to $\boldsymbol{b}$ i.e. a map satisfying $\boldsymbol{X}(0,x) = x$ and $\dot{\boldsymbol{X}}(t,x) = \boldsymbol{b}_t(\boldsymbol{X}(t,x))$. It is easily seen that this is not an invariant concept, under modification of $\boldsymbol{b}$ in negligible sets. This leads to the concept of L^r-regular flow: we give here the definition adopted in this paper when $(E, \|\cdot\|)$ is a separable Banach space endowed with a Gaussian measure γ; in the finite-dimensional theory ($E = \mathbb{R}^N$) other reference measures γ could be considered as well (for instance the Lebesgue measure [30], [1]).

Definition 1.1 (L^r-regular $\boldsymbol{b}$-flow). *Let $\boldsymbol{b} : (0,T) \times E \to E$ be a Borel vector field. If $\boldsymbol{X} : [0,T] \times E \to E$ is Borel and $1 \le r \le \infty$, we say that $\boldsymbol{X}$ is a L^r-regular* flow *associated to $\boldsymbol{b}$ if the following two conditions hold:*

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(i) for γ-a.e. $x \in \boldsymbol{X}$ the map $t \mapsto \|\boldsymbol{b}_t(\boldsymbol{X}(t,x))\|$ belongs to $L^1(0,T)$ and

$$\boldsymbol{X}(t,x) = x + \int_0^t \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,x))\,d\tau \qquad \forall\, t \in [0,T]. \tag{1}$$

(ii) for all $t \in [0,T]$ the law of $\boldsymbol{X}(t,\cdot)$ under γ is absolutely continuous with respect to γ, with a density ρ_t in $L^r(\gamma)$, and $\sup_{t\in[0,T]} \|\rho_t\|_r < \infty$.

In (1), the integral is understood in Bochner's sense, namely

$$\langle e^*, \boldsymbol{X}(t,x) - x\rangle = \int_0^t \langle e^*, \boldsymbol{b}_\tau(\boldsymbol{X}(\tau,x))\rangle\,d\tau \qquad \forall\, e^* \in E^*.$$

It is not hard to show that (see Remark 4.2), because of condition (ii), this concept is indeed invariant under modifications of $\boldsymbol{b}$, and so it is appropriate to deal with vector fields belonging to L^p spaces. On the other hand, condition (ii) involves all trajectories $\boldsymbol{X}(\cdot,x)$ up to γ-neglibigle sets, so the best we can hope for, using this concept, is existence and uniqueness of $\boldsymbol{X}(\cdot,x)$ up to γ-negligible sets.

The second basic idea is the the concept of flow is directly linked, via the theory of characteristics, to the transport equation

$$\frac{d}{dt} f(s,x) + \langle \boldsymbol{b}_s(x), \nabla_x f(s,x)\rangle = 0 \tag{2}$$

and to the continuity equation

$$\frac{d}{dt}\mu_t + \mathrm{div}(\boldsymbol{b}_t\mu_t) = 0. \tag{3}$$

The first link has been exploited in [30] to transfer well-posedness results from the transport equation to the ODE, getting uniqueness of L^∞-regular (with respect to Lebesgue measure) $\boldsymbol{b}$-flows in $\mathbb{R}^N$ (see [19] for the generalization of this approch to the case of a Gaussian measure). This is possible because the flow maps $(s,x) \mapsto \boldsymbol{X}(t,s,x)$ (here we made also explicit the dependence on the initial time s, that we kept equal to 0 in Definition 1.1) solve (2) for all $t \in [0,T]$.

Here, in analogy with the approach initiated in [1] (see also [33] for a stochastic counterpart of it, where (3) becomes the forward Kolmogorov equation), we prefer to deal with the continuity equation, which seems to be more natural in a probabilistic framework. The link between the ODE and (3) is based on the fact that any positive finite measure $\boldsymbol{\eta}$ in $C\big([0,T];E\big)$ concentrated on solutions to the ODE is expected to give rise to a weak solution to (3) (if the divergence operator is properly understood), with μ_t given by the marginals of $\boldsymbol{\eta}$ at time t: indeed, (3) describes the evolution of a probability density under the action of the "velocity field" $\boldsymbol{b}$. We shall call these measures $\boldsymbol{\eta}$ *generalized* $\boldsymbol{b}$-flows. Our goal will be, as in [1], [33], to transfer well-posedness informations from the continuity equation to the ODE, getting existence and uniqueness results of the L^r-regular $\boldsymbol{b}$-flows, under suitable assumptions on $\boldsymbol{b}$.

We have to take into account an intrinsic limitation of the theory of L^r-regular $\boldsymbol{b}$-flows that is typical of infinite-dimensional spaces (see for instance [47]): even if $\boldsymbol{b}(t,x) \equiv v$ were constant, the flow map $\boldsymbol{X}(t,x)=x+tv$ would not leave γ quasi-invariant, unless v belongs to a particular subspace of E, the so-called Cameron-Martin space $\mathcal{H}$ of (E,γ), see (7) for its precise definition. So, from now on we shall assume that $\boldsymbol{b}$ takes its values in $\mathcal{H}$. However, thanks to a suitable change of variable, we will treat also some non $\mathcal{H}$-valued vector fields, in the same spirit as in [43], [13] (see also [35], [47]).

We recall that $\mathcal{H}$ can be endowed with a canonical Hilbertian structure $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ that makes the inclusion of $\mathcal{H}$ in E compact; we fix an orthonormal basis (e_i) of $\mathcal{H}$ and we shall denote by $\boldsymbol{b}^i$ the components of $\boldsymbol{b}$ relative to this basis (however, all our results are independent of the choice of (e_i)).

With this choice of the range of $\boldsymbol{b}$, whenever $\mu_t=u_t\gamma$ the equation (3) can be written in the weak sense as

$$\frac{d}{dt}\int_E u_t\,d\gamma=\int_E\langle\boldsymbol{b}_t,\nabla\phi\rangle_{\mathcal{H}}u_t\,d\gamma\qquad\forall\,\phi\in\mathrm{Cyl}(E,\gamma),\tag{4}$$

where $\mathrm{Cyl}(E,\gamma)$ is a suitable space of cylindrical functions induced by (e_i) (see Definition 2.3). Furthermore, a Gaussian divergence operator $\mathrm{div}_\gamma\boldsymbol{c}$ can be defined as the adjoint in $L^2(\gamma)$ of the gradient along $\mathcal{H}$:

$$\int_E\langle\boldsymbol{c},\nabla\phi\rangle_{\mathcal{H}}\,d\gamma=-\int_E\phi\,\mathrm{div}_\gamma\boldsymbol{c}\,d\gamma\qquad\forall\,\phi\in\mathrm{Cyl}(E,\gamma).$$

Another typical feature of our Gaussian framework is that L^∞-bounds on div_γ do not seem natural, unlike those on the Euclidean divergence in $\mathbb{R}^N$ when the reference measure is the Lebesgue measure: indeed, even if $\boldsymbol{b}(t,x)=\boldsymbol{c}(x)$, with $\boldsymbol{c}:\mathbb{R}^N\to\mathbb{R}^N$ smooth and with bounded derivatives, we have $\mathrm{div}_\gamma\boldsymbol{c}=\mathrm{div}\boldsymbol{c}-\langle\boldsymbol{c},x\rangle$ which is unbounded, but exponentially integrable with respect to γ.

We can now state the main result of this paper:

Theorem 1.2 (Existence and uniqueness of L^r-regular $\boldsymbol{b}$-flows)**.** *Let p, $q>1$ and let $\boldsymbol{b}:(0,T)\times E\to\mathcal{H}$ be satisfying:*

(i) $\|\boldsymbol{b}_t\|_{\mathcal{H}}\in L^1\big((0,T);L^p(\gamma)\big)$;

(ii) for a.e. $t\in(0,T)$ we have $\boldsymbol{b}_t\in LD^q_{\mathcal{H}}(\gamma;\mathcal{H})$ with

$$\int_0^T\left(\int_E\|(\nabla\boldsymbol{b}_t)^{\mathrm{sym}}(x)\|^q_{HS}\,d\gamma(x)\right)^{1/q}dt<\infty,\tag{5}$$

and $\mathrm{div}_\gamma\boldsymbol{b}_t\in L^1\big((0,T);L^q(\gamma)\big)$;

(iii) $\exp(c[\mathrm{div}_\gamma\boldsymbol{b}_t]^-)\in L^\infty\big((0,T);L^1(\gamma)\big)$ *for some $c>0$.*

If $r:=\max\{p',q'\}$ and $c\geq rT$, then the L^r-regular flow exists and is unique in the following sense: any two L^r-regular flows $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$ satisfy

$$\boldsymbol{X}(\cdot,x)=\tilde{\boldsymbol{X}}(\cdot,x)\quad\textit{in }[0,T],\textit{ for }\gamma\textit{-a.e. }x\in E.$$

Furthermore, $\boldsymbol{X}$ is L^s-regular for all $s \in [1, \frac{c}{T}]$ and the density u_t of the law of $\boldsymbol{X}(t, \cdot)$ under γ satisfies

$$\int (u_t)^s \, d\gamma \le \left\| \int_E \exp\bigl(Ts[\mathrm{div}_\gamma \boldsymbol{b}_t]^-\bigr) \, d\gamma \right\|_{L^\infty(0,T)} \qquad \forall \, s \in [1, \frac{c}{T}].$$

In particular, if $\exp(c[\mathrm{div}_\gamma \boldsymbol{b}_t]^-) \in L^\infty\bigl((0,T); L^1(\gamma)\bigr)$ for all $c > 0$, then the L^r-regular flow exists globally in time, and is L^s-regular for all $s \in [1, \infty)$.

The symmetric matrix $(\nabla \boldsymbol{b}_t)^{\mathrm{sym}}$, whose Hilbert-Schmidt norm appears in (5), corresponds to the symmetric part of the derivative of $\boldsymbol{b}_t$, defined in a weak sense by (22): notice that, in analogy with the finite dimensional result [18], *no* condition is imposed on the antisymmetric part of the derivative, which need not be given by a function; this leads to a particular function space $LD^q(\gamma; \mathcal{H})$ (well studied in linear elasticity in finite dimensions, see [46]) which is for instance larger than the Sobolev space $W^{1,q}_{\mathcal{H}}(\gamma; \mathcal{H})$, see Definitions 2.4 and 2.6. Also, we will prove that uniqueness of $\boldsymbol{X}$ holds even within the larger class of generalized $\boldsymbol{b}$-flows.

Let us explain first the main differences between our strategy and the techniques used in [23], [24], [25], [43], [13] for autonomous (i.e. time independent) vector fields in infinite-dimensional spaces. The standard approach for the existence of a flow consists in approximating the vector field $\boldsymbol{b}$ with finite-dimensional vector fields $\boldsymbol{b}_N$, constructing a finite-dimensional flow $\boldsymbol{X}_N$, and then passing to the limit as $N \to \infty$. This part of the proof requires quite strong a-priori estimates on the flows to have enough compactness to pass to the limit. To get these a-priori estimates, the assumptions on the vector field, instead of the hypotheses (i)-(iii) in Theorem 1.2, are:

$$\|\boldsymbol{b}\|_{\mathcal{H}} \in \bigcap_{p \in [1,\infty)} L^p(\gamma),$$

$$\exp(c\|\nabla \boldsymbol{b}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}) \in L^1(\gamma) \qquad \text{for all } c > 0,$$

$$\exp(c|\mathrm{div}_\gamma \boldsymbol{b}|) \in L^1(\gamma) \qquad \text{for some } c > 0,$$

where $\|\nabla \boldsymbol{b}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}$ denotes the operator norm of $\nabla \boldsymbol{b}$ from $\mathcal{H}$ to $\mathcal{H}$. So, apart from the minor fact that we allow a measurable time dependence of $\boldsymbol{b}$, the main difference between these results and ours is that we replace exponential integrability of the *operator* norm of $\nabla \boldsymbol{b}$ by q-integrability of the (stronger) *Hilbert-Schmidt* norm of $\nabla \boldsymbol{b}_t$ (or, as we said, of its symmetric part).

Let us remark for instance that, just for the existence part of a generalized $\boldsymbol{b}$-flow, the hypothesis on $\mathrm{div}_\gamma \boldsymbol{b}$ could be relaxed to a one sided bound, as we did. Indeed, this assumption allows to prove uniform estimates on the density of the approximating flows, see for instance Theorem 6.1. On the other hand, the proof of the uniqueness of the flow strongly relies on the fact that one can use the approximating flows $\boldsymbol{X}_N$ also for negative times.

Our strategy is quite different from the above one: the existence and uniqueness of a regular flow will be proved at once in the following way. First of all, the existence of a generalized $\boldsymbol{b}$-flow $\boldsymbol{\eta}$, even without the regularity assumption (5), can be obtained thanks to a tightness argument for measures in $C\bigl([0,T]; E\bigr)$ and proving uniform estimates on the density of the finite-dimensional approximating flows. Then we prove uniqueness in the class of generalized $\boldsymbol{b}$-flows. This implies as a byproduct that $\boldsymbol{\eta}$ is induced by a "deterministic" $\boldsymbol{X}$, thus providing the desired existence and uniqueness result. Moreover the flexibility of this approach allows us to prove the stability of the L^r-regular flow under smooth approximations of the vector field, and thanks to the uniqueness we can also easily deduce the semigroup property.

The main part of the paper is therefore devoted to the proof of uniqueness. As we already said, this depends on the well-posedness of the continuity equation (4). Specifically, we will show uniqueness of solutions u_t in the class $L^\infty\big((0,T);L^r(\gamma)\big)$. The key point, as in the finite-dimensional theory, is to pass from (4) to

$$\frac{d}{dt}\int_E \beta(u_t)\,d\gamma = \int_E \langle \boldsymbol{b}_t,\nabla\phi\rangle_{\mathcal{H}}\beta(u_t)\,d\gamma + \int_E [\beta(u_t)-u_t\beta'(u_t)]\mathrm{div}_\gamma \boldsymbol{b}_t\,d\gamma \qquad \forall\,\phi\in \mathrm{Cyl}(E,\gamma), \tag{6}$$

for all $\beta\in C^1(\mathbb{R})$ with $\beta'(z)$ and $z\beta'(z)-\beta(z)$ bounded, and then to choose as function β suitable C^1 approximations of the positive or of the negative part, to show that the equation preserves the sign of the initial condition. The passage from (4) to (6) can be *formally* justified using the rule

$$\mathrm{div}_\gamma(v\boldsymbol{c}) = v\,\mathrm{div}_\gamma\boldsymbol{c} + \langle\nabla v,\boldsymbol{c}\rangle_{\mathcal{H}}$$

and the chain rule $\nabla\beta(u)=\beta'(u)\nabla u$, but it is not always possible. It is precisely at this place that the regularity assumptions on $\boldsymbol{b}_t$ enter. The finite-dimensional strategy involves a regularization argument (in the space variable only) and a careful analysis of the "commutators" (with $v=u_t$, $\boldsymbol{c}=\boldsymbol{b}_t$)

$$r^\varepsilon(\boldsymbol{c},v) := e^\varepsilon\langle\boldsymbol{c},\nabla T_\varepsilon v\rangle_{\mathcal{H}} - T_\varepsilon(\mathrm{div}_\gamma(v\boldsymbol{c})),$$

where ε is the regularization parameter and T_ε is the regularizing operator. Already in the finite-dimensional theory (see [30], [1]) a careful estimate of r^ε is needed, taking into account some cancellation effects. These effects become even more important in this framework, where we use as a regularizing operator the Ornstein-Uhlenbeck operator (32) (in particular the semigroup property and the fact that T_t is self-adjoint from $L^p(\gamma)$ to $L^{p'}(\gamma)$ will play an important role). The core of our proof is indeed Section 6.2, where we obtain commutator estimates in $\mathbb{R}^N$ independent of N, and therefore suitable for an extension, via the canonical cylindrical approximation, to E.

The paper is structured as follows: first we recall the main notation needed in the paper. In Section 3 we prove the well-posedness of the continuity equation, while in Section 4 we prove existence, uniqueness and stability of regular flows. The results of both sections rely on some finite dimensional a-priori estimates that we postpone to Section 6. Finally, to apply our results also in more general situations, in Section 5 we see how our results can be extended to the case non $\mathcal{H}$-valued vector fields.

2 Main notation and preliminary results

Measure-theoretic notation. All measures considered in this paper are positive, finite and defined on the Borel σ-algebra. Given $f:E\to F$ Borel and a measure μ in E, we denote by $f_\#\mu$ the push-forward measure in F, i.e. the law of f under μ. We denote by χ_A the characteristic function of a set A, equal to 1 on A, and equal to 0 on its complement.

We consider a separable Banach space $(E,\|\cdot\|)$ endowed with a Gaussian measure γ, i.e. $(e^*)_\#\gamma$ is a Gaussian measure in $\mathbb{R}$ for all $e^*\in E^*$. We shall assume that γ is centered and non-degenerate, i.e. that $\int_E x\,d\gamma(x)=0$ and γ is not supported in a proper subspace of E. We recall (see [38]) that, by Fernique's theorem, $\int_E \exp(c\|x\|^2)\,d\gamma(x)<\infty$, whenever $2c<\sup_{\|e^*\|\leq 1}\|\langle e^*,x\rangle\|_{L^2(\gamma)}$.

Cameron-Martin space. We shall denote by $\mathcal{H}\subset E$ the Cameron Martin space associated to (E,γ). It can be defined [12, 38] as

$$\mathcal{H}:=\left\{\int_E \phi(x)x\,d\gamma(x)\,:\,\phi\in L^2(\gamma)\right\}. \tag{7}$$

The non-degeneracy assumption assumption on γ easily implies that $\mathcal{H}$ is a dense subset of E. If we denote by $i:L^2(\gamma)\to\mathcal{H}\subset E$ the map $\phi\mapsto\int_E\phi(x)x\,d\gamma(x)$, and by K the kernel of i, we can define the Cameron-Martin norm

$$\|i(\phi)\|_{\mathcal{H}}:=\min_{\psi\in K}\|\phi-\psi\|_{L^2(\gamma)},$$

whose induced scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ satisfies

$$\langle i(\phi),i(\psi)\rangle_{\mathcal{H}}=\int_E\phi\psi\,d\gamma\qquad\forall\,\phi\in L^2(\gamma),\ \forall\,\psi\in K^\perp. \tag{8}$$

Notice also that $i(\langle e^*,x\rangle)\in K^\perp$ for all $e^*\in E^*$, because

$$\int_E\langle e^*,x\rangle\psi(x)\,d\gamma(x)=\langle e^*,\int_E x\psi(x)\,d\gamma(x)\rangle=0\qquad\forall\,\psi\in K.$$

Since i is not injective in general, it is often more convenient to work with the map $j:E^*\to\mathcal{H}$, dual of the inclusion map of $\mathcal{H}$ in E (i.e. $j(e^*)$ is defined by $\langle j(e^*),h\rangle_{\mathcal{H}}=\langle e^*,h\rangle$ for all $h\in\mathcal{H}$). The set $j(E^*)$ is obviously dense in $\mathcal{H}$ (for the norm $\|\cdot\|_{\mathcal{H}}$), and j is injective thanks to the density of $\mathcal{H}$ in E; furthermore, choosing $\phi(x)=\langle e^*,x\rangle$ in (8), we see that $i(\langle e^*,x\rangle)=j(e^*)$. As a consequence the vector space $\{\langle e^*,x\rangle\,:\,e^*\in E^*\}$ is dense in $K^\perp$. Since $\|i(\langle e^*,x\rangle)\|\leq\left(\int_E\|x\|^2\,d\gamma\right)^{1/2}\|\langle e^*,x\rangle\|_{L^2(\gamma)}=\|i(\langle e^*,x\rangle)\|_{\mathcal{H}}$, the inclusion of $\mathcal{H}$ in E is continuous, and it is not hard to show that it is also compact (see [12, Corollary 3.2.4]).

This setup becomes much simpler when $(E,\|\cdot\|)$ is an Hilbert space:

Remark 2.1 (The Hilbert case)**.** Assume that $(E,\|\cdot\|)$ is an Hilbert space. Then, after choosing an orthonormal basis in which the covariance operator $(x,y)\mapsto\int_E\langle x,z\rangle\langle y,z\rangle\,d\gamma(z)$ is diagonal, we can identify E with ℓ^2, endowed with the canonical basis ϵ_i, and the coordinates x_i of $x\in\ell^2$ relative to ϵ_i are independent, Gaussian and with variance λ_i^2 (with $\lambda_i>0$ by the non-degeneracy assumption). Then, the integrability of $\|x\|^2$ implies that $\sum_i\lambda_i^2$ is convergent, $e_i^*=\epsilon_i$ (here we are using the Riesz isomorphism to identify ℓ^2 with its dual), $e_i=\lambda_i\epsilon_i$ and the Cameron-Martin space is

$$\mathcal{H}:=\left\{x\in\ell^2\,:\,\sum_{i=1}^\infty\frac{(x^i)^2}{\lambda_i^2}<\infty\right\}.$$

The map $j:\ell^2\to\mathcal{H}$ is given by $(x_i)\mapsto(\lambda_i x_i)$.

Let us remark that, although we constructed $\mathcal{H}$ starting from E, it is indeed $\mathcal{H}$ which plays a central role in our results; according to the Gross viewpoint, this space might have been taken as the starting point, see [12, §3.9] and Section 4.4 for a discussion of this fact.

Finite-dimensional projections. The above-mentioned properties of j allow the choice of $(e_n^*) \subset E^*$ such that $(j(e_n^*))$ is a complete orthonormal system in $\mathcal{H}$. Then, setting $e_n := j(e_n^*)$, we can define the continuous linear projections $\pi_N : E \to \mathcal{H}$ by

$$\pi_N(x) := \sum_{k=1}^{N} \langle e_k^*, x\rangle e_k \Big(= \sum_{k=1}^{N} \langle e_k, x\rangle_{\mathcal{H}} e_k \quad \text{for } x \in \mathcal{H}\Big). \tag{9}$$

The term "projection" is justified by the fact that, by the second equality in (9), $\pi_N|_{\mathcal{H}}$ is indeed the orthogonal projection on

$$\mathcal{H}_N := \operatorname{span}\big(e_1, \ldots, e_N\big). \tag{10}$$

From now such a basis (e_i) of $\mathcal{H}$ will be fixed, and we shall denote by v^i the components of $v \in \mathcal{H}$ relative to this basis. Also, for a given Borel function $u : E \to \mathbb{R}$, we shall denote by $\mathbb{E}_N u$ the conditional expectation of u relative to the σ-algebra generated by $\langle e_1^*, x\rangle, \ldots, \langle e_N^*, x\rangle$. The following result follows by martingale convergence theorems, because the σ–algebra generated by $\langle e_i^*, x\rangle$ is the Borel σ-algebra (see also [12, Corollary 3.5.2]):

Lemma 2.2. *For all $p \in [1, \infty)$ and $u \in L^p(\gamma)$ we have $\mathbb{E}_N u \to u$ γ-a.e. and in $L^p(\gamma)$.*

According to these projections, we can define the space $\mathrm{Cyl}(E, \gamma)$ of smooth cylindrical functions (notice that this definition depends on the choice of the basis (e_n)).

Definition 2.3 (Smooth cylindrical functions)**.** *Let $C_b^\infty(\mathbb{R}^N)$ be the space of smooth functions in $\mathbb{R}^N$, bounded together with all their derivatives. We say that $\phi : E \to \mathbb{R}$ is* cylindrical *if*

$$\phi(x) = \psi\big(\langle e_1^*, x\rangle, \ldots, \langle e_N^*, x\rangle\big) \tag{11}$$

for some integer N and some $\psi \in C_b^\infty(\mathbb{R}^N)$.

If $v \in E$ and $\phi : E \to \mathbb{R}$ we shall denote by $\partial_v \phi$ the partial derivative of ϕ along v, wherever this exists. Obviously, cylindrical functions are differentiable infinitely many times in all directions: if ϕ is as in (11), the first order derivative is given by

$$\partial_v \phi(x) = \sum_{i=1}^{N} \frac{\partial \psi}{\partial z_i}\big(\langle e_1^*, x\rangle, \ldots, \langle e_N^*, x\rangle\big)\langle e_i^*, v\rangle. \tag{12}$$

If $v \in \mathcal{H}$ the above formula becomes

$$\partial_v \phi(x) = \sum_{i=1}^{N} \frac{\partial \psi}{\partial z_i}\big(\langle e_1^*, x\rangle, \ldots, \langle e_N^*, x\rangle\big)\langle e_i, v\rangle_{\mathcal{H}},$$

and this allows to define the gradient of ϕ as an element of $\mathcal{H}$:

$$\nabla \phi(x) := \sum_{i=1}^{N} \frac{\partial \psi}{\partial z_i}\big(\langle e_1^*, x\rangle, \ldots, \langle e_N^*, x\rangle\big) e_i \in \mathcal{H}.$$

Gaussian divergence and differentiability along $\mathcal{H}$. Let $\boldsymbol{b}: E \to \mathcal{H}$ be a vector field with $\|\boldsymbol{b}\|_{\mathcal{H}} \in L^1(\gamma)$; we say that a function $\mathrm{div}_\gamma \boldsymbol{b} \in L^1(\gamma)$ is the *Gaussian divergence* of b (see for instance [12, §5.8]) if

$$\int_E \langle \nabla\phi, \boldsymbol{b}\rangle_{\mathcal{H}}\, d\gamma = -\int_E \phi\, \mathrm{div}_\gamma \boldsymbol{b}\, d\gamma \qquad \forall\, \phi \in \mathrm{Cyl}(E,\gamma). \tag{13}$$

In the finite-dimensional space $E = \mathbb{R}^N$ endowed with the standard Gaussian we have, by an integration by parts,

$$\mathrm{div}_\gamma \boldsymbol{b} = \mathrm{div}\, \boldsymbol{b} - \langle \boldsymbol{b}, x\rangle. \tag{14}$$

We recall the integration by parts formula

$$\int_E \partial_{j(e^*)}\phi\, d\gamma = \int_E \phi\langle e^*, x\rangle\, d\gamma \qquad \forall\, \phi \in \mathrm{Cyl}(E,\gamma),\ \forall\, e^* \in E^*. \tag{15}$$

This motivates the following definitions: if both $u(x)$ and $u(x)\langle e^*, x\rangle$ belong to $L^1(\gamma)$, we call weak derivative of u along $j(e^*)$ the linear functional on $\mathrm{Cyl}(E,\gamma)$

$$\phi \mapsto -\int_E u\partial_{j(e^*)}\phi\, d\gamma + \int_E u\phi\langle e^*, x\rangle\, d\gamma. \tag{16}$$

As in the classical finite-dimensional theory, we can define Sobolev spaces by requiring that these functionals are representable by $L^q(\gamma)$ functions, see Chapter 5 of [12] for a more complete discussion of this topic.

Definition 2.4 (Sobolev space $W^{1,q}_{\mathcal{H}}(\gamma)$). *If $1 \le q \le \infty$, we say that $u \in L^1(\gamma)$ belongs to $W^{1,q}_{\mathcal{H}}(E,\gamma)$ if $u(x)\langle e^*, x\rangle \in L^1(\gamma)$ for all $e^* \in E^*$ and there exists $g \in L^q(\gamma;\mathcal{H})$ satisfying*

$$\int_E u\partial_{j(e^*)}\phi\, d\gamma + \int_E \phi\langle g, j(e^*)\rangle_{\mathcal{H}}\, d\gamma = \int_E u\phi\langle e^*, x\rangle\, d\gamma \qquad \forall\, e^* \in E^*,\ \forall\, \phi \in \mathrm{Cyl}(E,\gamma). \tag{17}$$

The condition $u(x)\langle e^*, x\rangle \in L^1(\gamma)$ is automatically satisfied whenever $u \in L^p(\gamma)$ for some $p > 1$, thanks to the fact that the law of $\langle e^*, x\rangle$ under γ is Gaussian, so that $\langle e^*, x\rangle \in L^r(\gamma)$ for all $r < \infty$.

We shall denote, as usual, the (unique) weak derivative g by ∇u and its components $\langle g, e_i\rangle_{\mathcal{H}}$ by $\partial_i u$, so that (17) becomes

$$\int_E u\partial_i\phi\, d\gamma + \int_E \phi\partial_i u\, d\gamma = \int_E u\phi\langle e_i^*, x\rangle\, d\gamma \qquad \forall\, i \ge 1,\ \forall\, \phi \in \mathrm{Cyl}(E,\gamma). \tag{18}$$

We recall that a continuous linear operator $L: \mathcal{H} \to \mathcal{H}$ is said to be Hilbert-Schmidt if $\|L\|_{HS}$, defined as the square root of the trace of L^tL, is finite. Accordingly, if $L_{ij} = \langle L(e_i), e_j\rangle_{\mathcal{H}}$ is the symmetric matrix representing $L: \mathcal{H} \to \mathcal{H}$ in the basis (e_i), we have that L is of Hilbert-Schmidt class if and only if $\sum_{ij} L_{ij}^2$ is convergent, and

$$\|L\|_{HS} = \sqrt{\sum_{ij} L_{ij}^2}. \tag{19}$$

The following proposition shows that bounded continuous operators from E to $\mathcal{H}$ are of Hilbert-Schmidt class, when restricted to $\mathcal{H}$. In particular our results apply under p-integrability assumptions on $\nabla \boldsymbol{b}_t$ when the operator norm between E and $\mathcal{H}$ is used.

Proposition 2.5. *Let $L : E \to \mathcal{H}$ be a linear continuous operator. Then the restriction of L to $\mathcal{H}$ is of Hilbert-Schmidt class and $\|L\|_{HS} \le C\|L\|_{\mathcal{L}(E,\mathcal{H})}$, with C depending only on E and γ.*

Proof. By [12, Theorem 3.5.10] we can find a complete orthonormal system (f_n) of $\mathcal{H}$ such that $\sum_n \|f_n\|^2 =: C < +\infty$. Denoting by $\|L\|$ the operator norm of L from E to $\mathcal{H}$, we have then

$$\|L\|_{HS}^2 = \sum_{i,j} (\langle L(f_i), f_j\rangle_{\mathcal{H}})^2 = \sum_i \|L(f_i)\|_{\mathcal{H}}^2 \le \|L\|^2 \sum_i \|f_i\|^2 = C\|L\|^2.$$

$\square$

From now on, we shall denote by $L^p(\gamma;\mathcal{H})$ the space of Borel maps $\boldsymbol{c} : E \to \mathcal{H}$ such that $\|\boldsymbol{c}\|_{\mathcal{H}} \in L^p(\gamma)$. Given the basis (e_i) of $\mathcal{H}$, we shall denote by $\boldsymbol{c}^i$ the components of $\boldsymbol{c}$ relative to this basis.

Definition 2.6 (The space $LD(\gamma;\mathcal{H})$)**.** *If $1 \le q \le \infty$, we say that $\boldsymbol{c} \in L^1(\gamma;\mathcal{H})$ belongs to $LD^q(\gamma;\mathcal{H})$ if:*

(a) for all $h = j(e^*) \in \mathcal{H}$, the function $\langle \boldsymbol{c}, h\rangle_{\mathcal{H}}$ has a weak derivative in $L^q(\gamma)$ along h, that we shall denote by $\partial_h \langle \boldsymbol{c}, h\rangle_{\mathcal{H}}$, namely

$$\int_E \langle \boldsymbol{c}, h\rangle_{\mathcal{H}} \partial_h \phi \, d\gamma + \int_E \phi \partial_h \langle \boldsymbol{c}, h\rangle_{\mathcal{H}} \, d\gamma = \int_E \langle \boldsymbol{c}, h\rangle_{\mathcal{H}} \phi \langle e^*, x\rangle \, d\gamma \qquad \forall\, \phi \in \mathrm{Cyl}(E,\gamma); \quad (20)$$

(b) the symmetric matrices

$$(\nabla \boldsymbol{c})_{ij}^{\mathrm{sym}}(x) := \frac{1}{4}\left[\partial_{(e_i+e_j)}(\boldsymbol{c}^i + \boldsymbol{c}^j)(x) - \partial_{(e_i-e_j)}(\boldsymbol{c}^i - \boldsymbol{c}^j)(x)\right] \qquad (21)$$

satisfy

$$\int_E \|(\nabla \boldsymbol{c})^{\mathrm{sym}}\|_{HS}^q \, d\gamma < \infty.$$

If all components $\boldsymbol{c}^i$ of $\boldsymbol{c}$ belongs to $W^{1,q}_{\mathcal{H}}(\gamma)$ then the function $(\nabla \boldsymbol{c})_{ij}^{\mathrm{sym}}$ in (21) really corresponds to the symmetric part of $(\nabla c)_{ij} = \partial_j \boldsymbol{c}^i$, and this explains our choice of notation. However, according to our definition of $LD^q(\gamma;\mathcal{H})$, the vector fields $\boldsymbol{c}$ in this space need not have components $\boldsymbol{c}^i$ in $W^{1,q}_{\mathcal{H}}(\gamma)$. Moreover, from (21) we obtain that $(\partial_i \boldsymbol{c}^j + \partial_j \boldsymbol{c}^i)/2$ are representable by the $L^q(\gamma)$ functions $(\nabla \boldsymbol{c})_{ij}^{\mathrm{sym}}$, namely

$$\int_E \frac{1}{2}(\boldsymbol{c}^i \partial_j \phi + \boldsymbol{c}^j \partial_i \phi) \, d\gamma + \int_E \phi (\nabla \boldsymbol{c})_{ij}^{\mathrm{sym}} \, d\gamma = \int_E \frac{1}{2}(\boldsymbol{c}^i \langle e_j^*, x\rangle + \boldsymbol{c}^j \langle e_i^*, x\rangle)\phi \, d\gamma \qquad \forall\, \phi \in \mathrm{Cyl}(E,\gamma). \quad (22)$$

Remark 2.7 (Density of cylindrical functions)**.** We recall that $\mathrm{Cyl}(E,\gamma)$ is dense in all spaces $W^{1,p}_{\mathcal{H}}(\gamma)$, $1 \le p < \infty$. More precisely, if $1 \le p,q < \infty$, any function $u \in W^{1,p}_{\mathcal{H}}(\gamma) \cap L^q(\gamma)$ can be approximated in $L^q(\gamma)$ by cylindrical functions u_n with $\nabla u_n \to \nabla u$ strongly in $L^p(\gamma;\mathcal{H})$. In the case $p=\infty$, convergence of the gradients occurs in the weak* topology of $L^\infty(\gamma;\mathcal{H})$. These density results can be proved first in the finite-dimensional case and then, thanks to Lemma 2.2, in the general case.

Remark 2.8. In the sequel we shall use the simple rule

$$\mathrm{div}_\gamma(\boldsymbol{b}u) = u\,\mathrm{div}_\gamma \boldsymbol{b} + \langle \boldsymbol{b}, \nabla u\rangle_{\mathcal{H}},$$

valid whenever $\mathrm{div}_\gamma \boldsymbol{b} \in L^p(\gamma)$, $u \in L^{p'}(\gamma)$, $\boldsymbol{b} \in L^q(\gamma;\mathcal{H})$ and $u \in W^{1,q'}_{\mathcal{H}}(\gamma)$. The proof is a direct consequence of Remark 2.7.

Remark 2.9 (Invariance of div_γ, $W^{1,q}_{\mathcal{H}}(\gamma)$, $LD^q(\gamma)$)**.** The definitions of Gaussian divergence, Sobolev space and LD space, as given, involve the space $\mathrm{Cyl}(E,\gamma)$, which depends on the choice of the complete orthonormal basis (e_i). However, an equivalent formulation could be given using the space $C^1_b(E,\gamma)$ of functions that are Frechet differentiable along all directions in $\mathcal{H}$, with a bounded continuous gradient: indeed, cylindrical functions belong to $C^1_b(E,\gamma)$, and since $C^1_b(E,\gamma)$ is contained in $W^{1,\infty}_{\mathcal{H}}(\gamma)$, thanks to Remark 2.7 the functions in this space can be well approximated (in all spaces $L^p(\gamma)$ with $p<\infty$, and with weak* convergence in $L^\infty(\gamma)$ of gradients) by cylindrical functions. A similar remark applies to the continuity equation, discussed in the next section.

3 Well posedness of the continuity equation

Let $I \subset \mathbb{R}$ be an open interval. In this section we shall consider the *continuity equation* in $I \times E$, possibly with a source term f, i.e.

$$\frac{d}{dt}(u_t\gamma) + \mathrm{div}_\gamma(\boldsymbol{b}_t u_t \gamma) = f\gamma. \tag{23}$$

This equation has to be understood in the weak sense, namely we require that $t \mapsto \int_E u_t\phi\,d\gamma$ is absolutely continuous in I and

$$\frac{d}{dt}\int_E u_t\phi\,d\gamma = \int_E \langle \boldsymbol{b}_t, \nabla\phi\rangle_{\mathcal{H}} u_t\,d\gamma + \int_E f\phi\,d\gamma \qquad \text{a.e. in } I,\ \forall\,\phi \in \mathrm{Cyl}(E,\gamma). \tag{24}$$

The minimal requirement necessary to give a meaning to (24) is that u, f and $|u|\|\boldsymbol{b}\|_{\mathcal{H}}$ belong to $L^1\big(I;L^1(\gamma)\big)$, and we shall always make assumptions on u, f and $\boldsymbol{b}$ to ensure that these properties are satisfied.

Sometimes, to simplify our notation, with a slight abuse we drop γ and write (23) just as

$$\frac{d}{dt}u_t + \mathrm{div}_\gamma(\boldsymbol{b}_t u_t) = f.$$

However, we always have in mind the weak formulation (24), and we shall always assume that $f \in L^1\big(I; L^1(\gamma)\big)$.

Since we are, in particular, requiring all maps $t \mapsto \int_E u_t\phi \, d\gamma$ to be uniformly continuous in I, the map $t \mapsto u_t$ is weakly continuous in I, with respect to the duality of $L^1(\gamma)$ with $\mathrm{Cyl}(E, \gamma)$. Therefore, if $I = (0, T)$, it makes sense to say that a solution u_t of the continuity equation starts from $\bar{u} \in L^1(\gamma)$ at $t = 0$:

$$\lim_{t \downarrow 0} \int_E u_t \phi \, d\gamma = \int_E \bar{u}\phi \, d\gamma \qquad \forall\, u \in \mathrm{Cyl}(E, \gamma). \tag{25}$$

Theorem 3.1 (Well-posedness of the continuity equation)**.** *(Existence) Let $\boldsymbol{b} : (0,T) \times E \to \mathcal{H}$ be satisfying*

$$\|\boldsymbol{b}_t\|_{\mathcal{H}} \in L^1\big((0,T); L^p(\gamma)\big) \qquad \textit{for some } p > 1 \tag{26}$$

and

$$\exp(c[\mathrm{div}_\gamma \boldsymbol{b}_t]^-) \in L^\infty\big((0,T); L^1(\gamma)\big) \qquad \textit{for some } c > Tp'. \tag{27}$$

Then, for any nonnegative $\bar{u} \in L^\infty(\gamma)$, the continuity equation has a nonnegative solution u_t with $u_0 = \bar{u}$ satisfying (as a byproduct of its construction)

$$\int (u_t)^r \, d\gamma \leq \|\bar{u}\|^r_{L^\infty(\gamma)} \left\| \int_E \exp\big(Tr[\mathrm{div}_\gamma \boldsymbol{b}_t]^-\big) \, d\gamma \right\|_{L^\infty(0,T)} \qquad \forall\, r \in [1, \frac{c}{T}],\ t \in [0,T]. \tag{28}$$

(Uniqueness) Let $\boldsymbol{b} : (0,T) \times E \to \mathcal{H}$ be satisfying (26)*, $\boldsymbol{b}_t \in LD^q(\gamma; \mathcal{H})$ for a.e. $t \in (0,T)$ with*

$$\int_0^T \left(\int_E \|(\nabla \boldsymbol{b}_t)^{\mathrm{sym}}\|^q_{HS} \, d\gamma \right)^{1/q} dt < \infty \tag{29}$$

for some $q > 1$, and

$$\mathrm{div}_\gamma \boldsymbol{b}_t \in L^1\big((0,T); L^q(\gamma)\big). \tag{30}$$

Then, setting $r = \max\{p', q'\}$, if $c \geq Tr$ the continuity equation (23) *in $(0,T) \times E$ has at most one solution in the function space $L^\infty\big((0,T); L^r(\gamma)\big)$.*

Definition 3.2 (Renormalized solutions)**.** *We say that a solution u_t of* (23) *in $I \times E$ is* renormalized *if*

$$\frac{d}{dt}\beta(u_t) + \mathrm{div}_\gamma(\boldsymbol{b}_t \beta(u_t)) = [\beta(u_t) - u_t\beta'(u_t)]\mathrm{div}_\gamma \boldsymbol{b}_t + f\beta'(u_t) \tag{31}$$

in the sense of distributions in $I \times E$, for all $\beta \in C^1(\mathbb{R})$ with $\beta'(z)$ and $z\beta'(z) - \beta(z)$ bounded.

In the sequel we shall often use the Ornstein-Uhlenbeck operator T_t, defined for $u \in L^1(\gamma)$ by Mehler's formula

$$T_t u(x) := \int_E u(e^{-t}x + \sqrt{1 - e^{-2t}}y) \, d\gamma(y). \tag{32}$$

In the next proposition we summarize the main properties of the OU operator used in this paper, see Theorem 1.4.1, Theorem 2.9.1 and Proposition 5.4.8 of [12].

Proposition 3.3 (Properties of the OU semigroup). *Let T_t be as in* (32)*.*

(i) $\|T_t u\|_{L^p(\gamma)} \le \|u\|_{L^p(\gamma)}$ *for all* $u \in L^p(\gamma)$*,* $p \in [1,\infty]$*,* $t \ge 0$*, and equality holds if u is nonnegative and* $p = 1$*.*

(ii) T_t *is self-adjoint in* $L^2(\gamma)$ *for all* $t \ge 0$*. More generally, if* $1 \le p \le \infty$*, we have*

$$\int_E vT_t u\, d\gamma = \int_E uT_t v\, d\gamma \qquad \forall\, u \in L^p(\gamma),\ \forall\, v \in L^{p'}(\gamma). \tag{33}$$

(iii) For all $p \in (1,\infty)$*,* $t > 0$ *and* $u \in L^p(\gamma)$ *we have* $T_t u \in W^{1,p}_{\mathcal{H}}(\gamma)$ *and*

$$\|\nabla T_t u\|_{L^p(\gamma;\mathcal{H})} \le C(p,t)\|u\|_{L^p(\gamma)}. \tag{34}$$

(iv) For all $p \in [1,\infty]$ *and* $u \in W^{1,p}_{\mathcal{H}}(\gamma)$ *we have* $\nabla T_t u = e^{-t} T_t \nabla u$*.*

(v) T_t *maps* $\mathrm{Cyl}(E,\gamma)$ *in* $\mathrm{Cyl}(E,\gamma)$ *and* $T_t u \to u$ *in* $L^p(\gamma)$ *as* $t \downarrow 0$ *for all* $u \in L^p(\gamma)$*,* $1 \le p < \infty$*.*

In the same spirit of (16), we can now extend the action of the semigroup from $L^1(\gamma)$ to elements ℓ in the algebraic dual of $\mathrm{Cyl}(E,\gamma)$ as follows:

$$\langle T_t \ell, \phi\rangle := \langle \ell, T_t \phi\rangle \qquad \phi \in \mathrm{Cyl}(E,\gamma).$$

This is an extension, because if ℓ is induced by some function $u \in L^1(\gamma)$, i.e. $\langle \ell, \phi\rangle = \int_E \phi u\, d\gamma$ for all $\phi \in \mathrm{Cyl}(E,\gamma)$, then because of (33) $T_t\ell$ is induced by $T_t u$, i.e. $\langle T_t\ell, \phi\rangle = \int_E \phi T_t u\, d\gamma$ for all $\phi \in \mathrm{Cyl}(E,\gamma)$. In general we shall say that $T_t\ell$ is a function whenever there exists (a unique) $v \in L^1(\gamma)$ such that $\langle T_t\ell, \phi\rangle = \int_E v\phi\, d\gamma$ for all $\phi \in \mathrm{Cyl}(E,\gamma)$.

In the next lemma we will use this concept when ℓ is the Gaussian divergence of a vector field $\boldsymbol{c}$: indeed, ℓ can be thought via the formula $-\int_E \langle \boldsymbol{c}, \nabla\phi\rangle_{\mathcal{H}}\, d\gamma$ as an element of the dual of $\mathrm{Cyl}(E,\gamma)$. Our first proposition provides a sufficient condition ensuring that $T_t(\mathrm{div}_\gamma \boldsymbol{c})$ is a function.

Lemma 3.4. *Assume that* $r \in (1,\infty)$ *and* $\boldsymbol{c} \in L^r(\gamma;\mathcal{H})$*. Then* $T_t(\mathrm{div}_\gamma \boldsymbol{c})$ *is a function in* $L^r(\gamma)$ *for all* $t > 0$*.*

Proof. We use Proposition 3.3(iii) to obtain

$$|\langle T_t(\mathrm{div}_\gamma \boldsymbol{c}), \phi\rangle| = |\langle \mathrm{div}_\gamma \boldsymbol{c}, T_t\phi\rangle| \le \int_E \|\boldsymbol{c}\|_{\mathcal{H}} \|\nabla T_t \phi\|_{\mathcal{H}}\, d\gamma \le C(q,t)\|\boldsymbol{c}\|_{L^r(\gamma;\mathcal{H})}\|\phi\|_{L^{r'}(\gamma)}$$

for all $\phi \in \mathrm{Cyl}(E,\gamma)$, and we conclude. □

In the sequel we shall denote by $(\Lambda(p))^p$ the p-th moment of the standard Gaussian in $\mathbb{R}$, i.e.

$$\Lambda(p) := \left((2\pi)^{-1/2}\int_{\mathbb{R}} |x|^p e^{-|x|^2/2}\, dx\right)^{1/p}. \tag{35}$$

Proposition 3.5 (Commutator estimate). *Let $\boldsymbol{c} \in L^p(\gamma;\mathcal{H}) \cap LD^q(\gamma;\mathcal{H})$ for some $p > 1$, $1 \le q \le 2$, with $\mathrm{div}_\gamma \boldsymbol{c} \in L^q(\gamma)$. Let $r = \max\{p', q'\}$ and set*

$$r^\varepsilon = r^\varepsilon(v, \boldsymbol{c}) := e^\varepsilon \langle \boldsymbol{c}, \nabla T_\varepsilon(v)\rangle - T_\varepsilon(\mathrm{div}_\gamma(v\boldsymbol{c})). \tag{36}$$

Then, for $\varepsilon > 0$ and $v \in L^r(\gamma)$ we have

$$\|r^\varepsilon\|_{L^1(\gamma)} \le \|v\|_{L^r(\gamma)} \left[\frac{\Lambda(p)\varepsilon}{\sqrt{1-e^{-2\varepsilon}}} \|\boldsymbol{c}\|_{L^p(\gamma;\mathcal{H})} + \sqrt{2}\|\mathrm{div}_\gamma \boldsymbol{c}\|_{L^q(\gamma)} + 2\| \|(\nabla \boldsymbol{c})^{\mathrm{sym}}\|_{HS}\|_{L^q(\gamma)} \right]. \tag{37}$$

Finally, $-r^\varepsilon \to v\mathrm{div}_\gamma \boldsymbol{c}$ in $L^1(\gamma)$ as $\varepsilon \downarrow 0$.

Proof. The a-priori estimate (37), which is indeed the main technical point of this paper, will be proved in the Section 6 in finite-dimensional spaces. Here we will just mention how the finite-dimensional approximation can be performed.

Let us first assume that $v \in L^\infty$. Since $v\boldsymbol{c} \in L^p(\gamma;\mathcal{H})$, the previous lemma ensures that r^ε is a function. Keeping $\boldsymbol{c}$ fixed, we see that if $v_n \to v$ strongly in $L^r(\gamma)$ then $r^\varepsilon(v_n, \boldsymbol{c}) \to r^\varepsilon(v, \boldsymbol{c})$ in the duality with $\mathrm{Cyl}(E, \gamma)$, and since the $L^1(\gamma)$ norm is lower semicontinuous with respect to convergence in this duality, thanks to the density of cylindrical functions we see that it suffices to prove (37) when v is cylindrical. Keeping now $v \in \mathrm{Cyl}(E, \gamma)$ fixed, we consider the vector fields

$$\boldsymbol{c}_N := \sum_{i=1}^N \mathbb{E}_N \boldsymbol{c}^i e_i.$$

We observe that (13) gives $\mathrm{div}_\gamma \boldsymbol{c}_N = \mathbb{E}_N(\mathrm{div}_\gamma \boldsymbol{c})$, while (22) gives $(\nabla \boldsymbol{c}_N)^{\mathrm{sym}} = \mathbb{E}_N(\nabla \boldsymbol{c})^{\mathrm{sym}}$. Thus, by Jensen's inequality for conditional expectations we obtain $\|\boldsymbol{c}_N\|_{L^p(\gamma;\mathcal{H})} \le \|\boldsymbol{c}\|_{L^p(\gamma;\mathcal{H})}$ and

$$\int_E |\mathrm{div}_\gamma \boldsymbol{c}_N|^q \, d\gamma \le \int_E |\mathrm{div}_\gamma \boldsymbol{c}|^q \, d\gamma, \qquad \int_E \|(\nabla \boldsymbol{c}_N)^{\mathrm{sym}}\|_{HS}^q \, d\gamma \le \int_E \|(\nabla \boldsymbol{c})^{\mathrm{sym}}\|_{HS}^q \, d\gamma.$$

Now, assuming that v depends only on $\langle e_1^*, x\rangle, \ldots, \langle e_M^*, x\rangle$, if we choose a cylindrical test function ϕ depending only on $\langle e_1^*, x\rangle, \ldots, \langle e_N^*, x\rangle$, with $N \ge M$ (with no loss of generality, because v is fixed), we get

$$\begin{aligned}
&\int_E r^\varepsilon(v, \boldsymbol{c})\phi \, d\gamma = \int_E r^\varepsilon(v, \boldsymbol{c}_N)\phi \, d\gamma \le \sup|\phi| \int_E |r^\varepsilon(v, \boldsymbol{c}_N)| \, d\gamma \\
\le \ & \sup|\phi| \|v\|_{L^r(\gamma)} \left[\frac{\Lambda(p)\varepsilon}{\sqrt{1-e^{-2\varepsilon}}} \|\boldsymbol{c}_N\|_{L^p(\gamma;\mathcal{H})} + \sqrt{2}\|\mathrm{div}_\gamma \boldsymbol{c}_N\|_{L^q(\gamma)} + 2\| \|(\nabla \boldsymbol{c}_N)^{\mathrm{sym}}\|_{HS}\|_{L^q(\gamma)} \right] \\
\le \ & \sup|\phi| \|v\|_{L^r(\gamma)} \left[\frac{\Lambda(p)\varepsilon}{\sqrt{1-e^{-2\varepsilon}}} \|\boldsymbol{c}\|_{L^p(\gamma;\mathcal{H})} + \sqrt{2}\|\mathrm{div}_\gamma \boldsymbol{c}\|_{L^q(\gamma)} + 2\| \|(\nabla \boldsymbol{c})^{\mathrm{sym}}\|_{HS}\|_{L^q(\gamma)} \right].
\end{aligned}$$

This means that, once we know (37) in finite-dimensional spaces, we obtain that the same inequality holds in all Wiener spaces for all $v \in L^\infty(\gamma)$. Finally, to remove also this restriction on v, we consider a sequence $(v_n) \subset L^\infty(\gamma)$ converging in $L^r(\gamma)$ to v and we notice that, because of (37), $r^\varepsilon(v_n, \boldsymbol{c})$ is a Cauchy sequence in L^1 converging in the duality with $\mathrm{Cyl}(E, \gamma)$ to $r^\varepsilon(v, \boldsymbol{c})$.

The strong convergence of r^ε can be achieved by a density argument. More precisely, if $q > 1$ (so that $r < \infty$), since $r^\varepsilon(v, \boldsymbol{c}) = r^\varepsilon(v - \phi, \boldsymbol{c}) + r^\varepsilon(\phi, \boldsymbol{c})$, by (37) and the density of cylindrical functions in $L^r(\gamma)$, we need only to consider the case when $v = \phi$ is cylindrical. In this case

$$r^\varepsilon = \langle \boldsymbol{c}, T_\varepsilon \nabla\phi\rangle - T_\varepsilon(\phi \mathrm{div}_\gamma \boldsymbol{c} + \langle \boldsymbol{c}, \nabla\phi\rangle)$$

and its convergence to $-\phi \mathrm{div}_\gamma \boldsymbol{c}$ is an obvious consequence of the continuity properties of T_ε.

In the case $q = 1$ (that is $r = \infty$), the approximation argument is a bit more involved. Since we will never consider L^∞-regular flows, we give here just a sketch of the proof. We argue as in [41]: we write $r^\varepsilon(v, \boldsymbol{c}) = r^\varepsilon(v, \boldsymbol{c} - \tilde{\boldsymbol{c}}) + r^\varepsilon(v - \tilde{v}, \tilde{\boldsymbol{c}}) + r^\varepsilon(\tilde{v}, \tilde{\boldsymbol{c}})$, with $\tilde{v}$ and $\tilde{\boldsymbol{c}}$ smooth and bounded with all their derivatives. Using (37) twice, we first choose $\tilde{\boldsymbol{c}}$ so that $r^\varepsilon(v, \boldsymbol{c} - \tilde{\boldsymbol{c}})$ is small uniformly in ε, and then, since now $\tilde{\boldsymbol{c}}$ is smooth with bounded derivatives, it suffices to choose $\tilde{v}$ close to v in L^s for some $s > 1$ to make $r^\varepsilon(v - \tilde{v}, \tilde{\boldsymbol{c}})$ small. We can now conclude as above. □

The following lemma is standard (both properties can be proved by a smoothing argument; for the second one, see [12, Corollary 5.4.3]):

Lemma 3.6 (Chain rules). *Let $\beta \in C^1(\mathbb{R})$ with β' bounded.*

(i) If $u, f \in L^1\big(I; L^1(\gamma)\big)$ satisfy $\frac{d}{dt}u = f$ in the weak sense, then $\frac{d}{dt}\beta(u) = \beta'(u)f$, still in the weak sense.

(ii) If $u \in W^{1,p}_{\mathcal{H}}(\gamma)$ then $\beta(u) \in W^{1,p}_{\mathcal{H}}(\gamma)$ and $\nabla\beta(u) = \beta'(u)\nabla u$.

Theorem 3.7 (Renormalization property). *Let $\boldsymbol{b} : I \times E \to \mathcal{H}$ be satisfying the assumptions of the uniqueness part of Theorem 3.1, with I in place of $(0, T)$. Then any solution u_t of the continuity equation (23) in $L^\infty\big(I; L^r(\gamma)\big)$, with $r = \max\{p', q'\}$, is renormalized.*

Proof. In the first step we prove the renormalized property assuming that $u_t \in W^{1,r}_{\mathcal{H}}(\gamma)$ for a.e. t, and that both u_t and $\|\nabla u_t\|_{\mathcal{H}}$ belong to $L^\infty\big(I; L^r(\gamma)\big)$. Under this assumption, Remark 2.8 gives that $\mathrm{div}_\gamma(\boldsymbol{b}_t u_t) = u_t \mathrm{div}_\gamma \boldsymbol{b}_t + \langle \boldsymbol{b}_t, \nabla u_t\rangle_{\mathcal{H}}$, therefore

$$\frac{d}{dt}u_t = -u_t \mathrm{div}_\gamma \boldsymbol{b}_t + \langle \boldsymbol{b}_t, \nabla u_t\rangle_{\mathcal{H}} \in L^1\big(I; L^1(\gamma)\big).$$

Now, using Lemma 3.6 and Remark 2.8 again, we get

$$\begin{aligned}
\frac{d}{dt}\beta(u_t) &= -\beta'(u_t)u_t \mathrm{div}_\gamma \boldsymbol{b}_t - \beta'(u_t)\langle \boldsymbol{b}_t, \nabla u_t\rangle_{\mathcal{H}} \\
&= [\beta(u_t) - \beta'(u_t)u_t]\mathrm{div}_\gamma \boldsymbol{b}_t - \beta(u_t)\mathrm{div}_\gamma \boldsymbol{b}_t - \langle \boldsymbol{b}_t, \nabla\beta(u_t)\rangle_{\mathcal{H}} \\
&= [\beta(u_t) - \beta'(u_t)u_t]\mathrm{div}_\gamma \boldsymbol{b}_t - \mathrm{div}_\gamma(\boldsymbol{b}_t \beta(u_t)).
\end{aligned}$$

Now we prove the renormalization property in the general case. Let us define $u^\varepsilon_t := e^{-\varepsilon}T_\varepsilon(u_t)$; since T_ε is self-adjoint in the sense of Proposition 3.3(ii) and T_ε maps cylindrical functions into cylindrical functions, the continuity equation $\frac{d}{dt}u_t + \mathrm{div}_\gamma(\boldsymbol{b}_t u_t) = 0$ gives, still in the weak sense of duality with cylindrical functions,

$$\frac{d}{dt}u^\varepsilon_t + e^{-\varepsilon}T_\varepsilon[\mathrm{div}_\gamma(\boldsymbol{b}_t u_t)] = 0.$$

Recalling the definition (36), we may write

$$\frac{d}{dt}u_t^\varepsilon + \mathrm{div}_\gamma(\boldsymbol{b}_t u_t^\varepsilon) = e^{-\varepsilon} r^\varepsilon + u_t^\varepsilon \mathrm{div}_\gamma \boldsymbol{b}_t.$$

Denoting by f^ε the right hand side, we know from Proposition 3.5 that $f^\varepsilon \to 0$ in $L^1\big((0,T);L^1(\gamma)\big)$. Taking into account that u_t^ε and $\|\nabla u_t^\varepsilon\|_{\mathcal{H}}$ belong to $L^\infty\big(I;L^r\gamma)\big)$ (by Proposition 3.3(iii)), from the first step we obtain

$$\frac{d}{dt}\beta(u_t^\varepsilon) + \mathrm{div}_\gamma(\boldsymbol{b}_t\beta(u_t^\varepsilon)) = [\beta(u_t^\varepsilon) - u_t^\varepsilon\beta'(u_t^\varepsilon)]\mathrm{div}_\gamma\boldsymbol{b}_t + \beta'(u_t^\varepsilon)f^\varepsilon$$

for all $\beta \in C^1(\mathbb{R})$ with $\beta'(z)$ and $z\beta'(z) - \beta(z)$ bounded. So, passing to the limit as $\varepsilon \downarrow 0$ we obtain that u_t is a renormalized solution. $\square$

Proof of Theorem 3.1. (Existence) It can be obtained as a byproduct of the results in Section 4: Theorem 4.5 provides a generalized flow, i.e. a positive finite measure $\boldsymbol{\eta}$ in the space of paths $\Omega(E)$, whose marginals $(e_t)_\#\boldsymbol{\eta}$ at all times have a density uniformly bounded in $L^r(\gamma)$, and $(e_0)_\#\boldsymbol{\eta} = \bar{u}\gamma$. Then, denoting by u_t the density of $(e_t)_\#\boldsymbol{\eta}$ with respect to γ, Proposition 4.8 shows that u_t solve the continuity equation.

(Uniqueness) By the linearity of the equation, it suffices to show that $\bar{u} = 0$ implies $u_t \le 0$ for all $t \in [0,T]$ for all solutions $u \in L^\infty\big((0,T);L^r(\gamma)\big)$. We extend u_t and $\boldsymbol{b}_t$ to the interval $I := (-1,T)$ by setting $u_t = \bar{u}$ and $\boldsymbol{b}_t = 0$ for all $t \in (-1,0]$, and it is easy to check that this extension preserves the validity of the continuity equation (still in the weak form).

We choose, as a C^1 approximation of the positive part, the functions $\beta_\varepsilon(z)$ equal to $\sqrt{z^2+\varepsilon^2}-\varepsilon$ for $z \ge 0$, and null for $z \le 0$. Thanks to Theorem 3.7, we can apply (31) with $\beta = \beta_\varepsilon$, with the test function $\phi \equiv 1$, to obtain

$$\frac{d}{dt}\int_E \beta_\varepsilon(u_t)\,d\gamma = \int_E [\beta_\varepsilon(u_t) - u_t\beta_\varepsilon'(u_t)]\mathrm{div}_\gamma\boldsymbol{b}_t\,d\gamma \le \varepsilon\int_E [\mathrm{div}_\gamma\boldsymbol{b}_t]^-\,d\gamma,$$

where we used the fact that $-\varepsilon \le \beta_\varepsilon(z) - z\beta_\varepsilon'(z) \le 0$. Letting $\varepsilon \downarrow 0$ we obtain that $\frac{d}{dt}\int_E u_t^+\,d\gamma \le 0$ in $(-1,T)$ in the sense of distributions. But since $u_t = 0$ for all $t \in (-1,0)$, we obtain $u_t^+ = 0$ for all $t \in [0,T)$.

4 Existence, uniqueness and stability of the flow

In this section we discuss the problems of existence and uniqueness of a flow associated to $\boldsymbol{b} : [0,T] \times E \to \mathcal{H}$, and we discuss its main properties.

4.1 Existence of a generalized $\boldsymbol{b}$-flow

It will be useful, in order to establish our first existence result, a definition of flow more general than Definition 1.1. In the sequel we shall denote by $\Omega(E)$ the space of continuous maps from

$[0,T]$ to E, endowed with the sup norm. Since E is separable, $\Omega(E)$ is complete and separable. We shall denote by

$$e_t : \Omega(E) \to E, \qquad e_t(\omega) := \omega(t)$$

the evaluation maps at time $t \in [0,T]$.

If $1 \le \alpha \le \infty$, we shall also denote by $AC^\alpha(E) \subset \Omega(E)$ the subspace of functions ω satisfying

$$\omega(t) = \omega(0) + \int_0^t g(s)\, ds \qquad \forall\, t \in [0,T] \tag{38}$$

for some $g \in L^\alpha\big((0,T);E\big)$. The function g, that we shall denote by $\dot\omega$, is uniquely determined up to negligible sets by (38): indeed, if $\bar t$ is a Lebesgue point of g then $\langle e^*, g(\bar t)\rangle$ coincides with the derivative at $t = \bar t$ of the real-valued absolutely continuous function $t \mapsto \langle e^*, \omega(t)\rangle$, for all $e^* \in E^*$.

Definition 4.1 (Generalized $\boldsymbol{b}$-flows and L^r-regularity)**.** *If $\boldsymbol{b} : [0,T] \times E \to E$, we say that a probability measure $\boldsymbol{\eta}$ in $\Omega(E)$ is a flow associated to $\boldsymbol{b}$ if:*

(i) $\boldsymbol{\eta}$ is concentrated on maps $\omega \in AC^1(E)$ satisfying the ODE $\dot\omega = \boldsymbol{b}(t,\omega)$ in the integral sense, namely

$$\omega(t) = \omega(0) + \int_0^t \boldsymbol{b}_\tau(\omega(\tau))\, d\tau \qquad \forall\, t \in [0,T]; \tag{39}$$

(ii) $(e_0)_\# \boldsymbol{\eta} = \gamma$.

If in addition there exists $1 \le r \le \infty$ such that, for all $t \in [0,T]$, the image measures $(e_t)_\# \boldsymbol{\eta}$ are absolutely continuous with respect to γ with a density in $L^r(\gamma)$, then we say that the flow is L^r-regular.

Remark 4.2 (Invariance of $\boldsymbol{b}$-flows)**.** Assume that $\boldsymbol{\eta}$ is a generalized L^1-regular $\boldsymbol{b}$-flow and $\tilde{\boldsymbol{b}}$ is a modification of $\boldsymbol{b}$, i.e., for a.e. $t \in (0,T)$ the set $N_t := \{\boldsymbol{b}_t \ne \tilde{\boldsymbol{b}}_t\}$ is γ-negligible. Then, because of L^1-regularity we know that, for a.e. $t \in (0,T)$, $\omega(t) \notin N_t$ $\boldsymbol{\eta}$-almost surely. By Fubini's theorem, we obtain that, for $\boldsymbol{\eta}$-a.e. ω, the set of times t such that $\omega(t) \in N_t$ is negligible in $(0,T)$. As a consequence $\boldsymbol{\eta}$ is a $\tilde{\boldsymbol{b}}$-flow as well.

Remark 4.3 (Martingale solutions of ODEs)**.** We remark that the notion of generalized flow coincides with the Stroock-Varadhan's notion of martingale solutions for stochastic differential equations in the particular case when there is no noise (so that the stochastic differential equation reduces to an ordinary differential equations), see for instance [45] and [33, Lemma 3.8].

From now on, we shall adopt the convention $\|v\|_{\mathcal H} = +\infty$ for $v \in E \setminus \mathcal{H}$.

Proposition 4.4 (Compactness)**.** *Let $K \subset E$ be a compact set, $C \ge 0$, $\alpha \in (1,\infty)$ and let $\mathcal{F} \subset AC^\alpha(E)$ be the family defined by:*

$$\mathcal{F} := \left\{ \omega \in AC^\alpha(E) \,:\, \omega(0) \in K,\ \int_0^T \|\dot\omega\|_{\mathcal H}^\alpha\, dt \le C \right\}.$$

Then $\mathcal{F}$ is compact in $\Omega(E)$.

Proof. Let us fix an integer h, and split $[0,T]$ in the h equal intervals $I_i := [iT/h,(i+1)T/h]$, $i = 0,\ldots,h-1$. We consider the family $\mathcal{F}_h$ obtained by replacing each curve $\omega(t)$ in $\mathcal{F}$ with the continuous "piecewise affine" curve ω_h coinciding with ω at the endpoints of the intervals I_i and with constant derivative, equal to $\frac{T}{h}\int_{I_i}\dot\omega(t)\,dt$, in all intervals $(iT/h,(i+1)T/h)$. We will check that each family $\mathcal{F}_h$ is relatively compact, and that $\sup|\omega-\omega_h|\to 0$ as $h\to\infty$, uniformly with respect to $\omega\in\mathcal{F}$. These two facts obviously imply, by a diagonal argument, the relative compactness of $\mathcal{F}$.

The family $\mathcal{F}_h$ is easily seen to be relatively compact: indeed, the initial points of the curve lie in the compact set K, and since $\{\int_{I_0}\dot\omega(t)\,dt\}_{\omega\in\mathcal{F}}$ is uniformly bounded in $\mathcal{H}$, the compactness of the embedding of $\mathcal{H}$ in E shows that also the family of points $\{\omega_h(T/h)\}_{\omega\in\mathcal{F}}$ is relatively compact; continuing in this way, we prove that all families of points $\{\omega_h(iT/h)\}_{\omega\in\mathcal{F}}$, $i=0,\ldots,h-1$, and therefore the family $\mathcal{F}_h$, are relatively compact.

Fix $\omega\in\mathcal{F}$; denoting by L the norm of the embedding of $\mathcal{H}$ in E, we have

$$\|\omega(t)-\omega_h(t)\| \le \int_{iT/h}^{t}\|\dot\omega(\tau)-\dot\omega_h(\tau)\|\,d\tau \le 2L\int_{iT/h}^{t}\|\dot\omega(\tau)\|_{\mathcal{H}}\,d\tau \le 2LC^{1/\alpha}\left(\frac{T}{h}\right)^{1-1/\alpha}$$

for all $t\in[iT/h,(i+1)T/h]$. This proves the uniform convergence of ω_h to ω as $h\to\infty$, as ω varies in $\mathcal{F}$.

Finally, we have to check that $\mathcal{F}$ is closed. The stability of the condition $\omega(0)\in K$ under uniform convergence is obvious. The stability of the second condition can be easily obtained thanks to the reflexivity of the space $L^\alpha\big((0,T);\mathcal{H}\big)$. □

Theorem 4.5 (Existence of L^r-regular generalized $\boldsymbol{b}$-flows)**.** *Let $\boldsymbol{b}:[0,T]\times E\to\mathcal{H}$ be satisfying the assumptions of the existence part of Theorem 3.1. Then there exists a generalized $\boldsymbol{b}$-flow $\boldsymbol{\eta}$, L^r-regular for all $r\in[1,c/T]$. In addition, the density u_t of $(e_t)_\#\boldsymbol{\eta}$ with respect to γ satisfies*

$$\int (u_t)^r\,d\gamma \le \left\|\int \exp\big(Tr[\mathrm{div}_\gamma \boldsymbol{b}_t]^-\big)\,d\gamma\right\|_{L^\infty(0,T)} \qquad \forall\, t\in[0,T]. \tag{40}$$

Proof. **Step 1.** (finite-dimensional approximation) Let $\boldsymbol{b}_N:[0,T]\times E\to\mathcal{H}_N$ be defined by $\sum_{i=1}^N \boldsymbol{b}_N^i e_i$, where

$$\boldsymbol{b}_N^i(t,\cdot) := \mathbb{E}_N \boldsymbol{b}_t^i, \qquad 1\le i\le N,\ t\in[0,T].$$

Arguing as in the proof of Proposition 3.5, we have the estimates

$$\int_0^T\left(\int_E \|(\boldsymbol{b}_N)_t\|_{\mathcal{H}}^p\,d\gamma(x)\right)^{1/p} dt \le \int_0^T\left(\int_E \|\boldsymbol{b}_t\|_{\mathcal{H}}^p\,d\gamma(x)\right)^{1/p} dt, \tag{41}$$

$$\left\|\int_E \exp\big(c[\mathrm{div}_\gamma(\boldsymbol{b}_N)_t]^-\big)\,d\gamma(x)\right\|_{L^\infty(0,T)} \le \left\|\int_E \exp\big(c[\mathrm{div}_\gamma \boldsymbol{b}_t]^-\big)\,d\gamma(x)\right\|_{L^\infty(0,T)}. \tag{42}$$

By applying Theorem 6.1 to the finite-dimensional fields $\tilde{\boldsymbol{b}}_N$ given by the restriction of $\boldsymbol{b}_N$ to $[0,T]\times\mathcal{H}_N$, we obtain a generalized flow $\boldsymbol{\sigma}_N$ in $\mathcal{H}_N$ (i.e. a positive finite measure in $\Omega(H_N)$)

associated to $\tilde{\boldsymbol{b}}_N$. Using the inclusion map i_N of $\mathcal{H}_N$ in $\mathcal{H}$ we obtain a generalized flow $\boldsymbol{\eta}_N := (i_N)_\#\boldsymbol{\sigma}_N$ associated to $\boldsymbol{b}_N$. In addition, (42) and the finite-dimensional estimate (57) give

$$\sup_{t\in[0,T]}\sup_{N\geq 1}\int_E (u_t^N)^r\,d\gamma \leq \|\bar{u}\|^r_{L^\infty(\gamma)}\left\|\int_E \exp\bigl(Tr[\mathrm{div}_\gamma \boldsymbol{b}_t]^-\bigr)\,d\gamma\right\|_{L^\infty(0,T)}, \tag{43}$$

with u_t^N equal to the density of $(e_t)_\#\boldsymbol{\eta}_N$ with respect to γ.

Step 2. (Tightness and limit flow $\boldsymbol{\eta}$). We call coercive a functional Ψ if its sublevel sets $\{\Psi\leq C\}$ are compact. Since $(\mathbb{E}_N\bar{u}\gamma)$ is a tight family of measures, by Prokhorov theorem we can find (see for instance [45]) a coercive functional $\Phi_1 : E\to[0,+\infty)$ such that $\sup_N\int_E\Phi_1\mathbb{E}_N\bar{u}\,d\gamma<\infty$. We choose $\alpha\in(1,p)$ such that $(p/\alpha)'\leq c/T$ (this is possible because we are assuming that $p'T<c$) and consider the functional

$$\Phi(\omega) := \begin{cases}\Phi_1(\omega(0)) + \int_0^T \|\dot{\omega}(t)\|^\alpha_{\mathcal{H}}\,dt & \text{if } \omega\in AC^p(E);\\ +\infty & \text{if } \omega\in\Omega(E)\setminus AC^\alpha(E).\end{cases} \tag{44}$$

Thanks to Proposition 4.4 and the coercivity of Φ_1, Φ is a coercive functional in $\Omega(E)$. Since

$$\begin{aligned}\int_{\Omega(E)}\Phi(\omega)\,d\boldsymbol{\eta}_N(\omega) &= \int_E\Phi_1(x)\mathbb{E}_N\bar{u}(x)\,d\gamma(x) + \int_0^T\int_{\Omega(E)}\|(\boldsymbol{b}_N)_t(\omega(t))\|^\alpha_{\mathcal{H}}\,d\boldsymbol{\eta}_N(\omega)\,dt\\ &= \int_E\Phi_1(x)\mathbb{E}_N\bar{u}(x)\,d\gamma(x) + \int_0^T\int_E\|(\boldsymbol{b}_N)_t(x)\|^\alpha_{\mathcal{H}}u_t^N(x)\,d\gamma(x)\,dt\end{aligned}$$

we can apply Hölder inequality with the exponents p/α and $(p/\alpha)'$, (41), (42) and (43) to obtain that $\int\Phi\,d\boldsymbol{\eta}_N$ is uniformly bounded. So, we can apply again Prokhorov theorem to obtain that $(\boldsymbol{\eta}_N)$ is tight in $\Omega(E)$. Therefore we can find a positive finite measure $\boldsymbol{\eta}$ in $\Omega(E)$ and a family of integers $N_i\to\infty$ such that $\boldsymbol{\eta}_{N_i}\to\boldsymbol{\eta}$ weakly, in the duality with $C_b\bigl(\Omega(E)\bigr)$. In the sequel, to simplify our notation, we shall assume that convergence occurs as $N\to\infty$. Obviously, because of (43), $\boldsymbol{\eta}$ is L^r-regular and, more precisely, (40) holds.

Step 3. ($\boldsymbol{\eta}$ is a $\boldsymbol{b}$-flow). It suffices to show that

$$\int_{\Omega(E)} 1\wedge\|\omega(t)-\omega(0)-\int_0^t\boldsymbol{b}_s(\omega(s))\,ds\|\,d\boldsymbol{\eta} = 0 \tag{45}$$

for any $t\in[0,T]$. The technical difficulty is the integrand in (45), due to the lack of regularity of $\boldsymbol{b}_t$, is not continuous in $\Omega(E)$; the truncation with the constant 1 is used to have a bounded integrand. To this aim, we prove first that

$$\int_{\Omega(E)} 1\wedge\|\omega(t)-\omega(0)-\int_0^t\boldsymbol{c}_s(\omega(s))\,ds\|\,d\boldsymbol{\eta} \leq \int_0^T\int_E\|\boldsymbol{b}_s(x)-\boldsymbol{c}_s(x)\|u_s(x)\,d\gamma(x)\,ds \tag{46}$$

for any bounded continuous function $\boldsymbol{c}$. Then, choosing a sequence $(\boldsymbol{c}_n)$ converging to $\boldsymbol{b}$ in $L^1\bigl((0,T);L^p(\gamma;E)\bigr)$, and noticing that

$$\int_{\Omega(E)}\int_0^T\|\boldsymbol{b}_s(\omega(s))-(\boldsymbol{c}_n)_s(\omega(s))\|\,ds\,d\boldsymbol{\eta} = \int_0^T\int_E\|\boldsymbol{b}_s(x)-(\boldsymbol{c}_n)_s(x)\|u_s(x)\,d\gamma(x)\,ds\to 0,$$

we can pass to the limit in (46) with $\boldsymbol{c} = \boldsymbol{c}_n$ to obtain (45).

It remains to show (46). This is a limiting argument based on the fact that (45) holds for $\boldsymbol{b}_N$, $\boldsymbol{\eta}_N$:

$$
\begin{aligned}
&\int_{\Omega(E)} 1 \wedge \|\omega(t) - \omega(0) - \int_0^t \boldsymbol{c}_s(\omega(s))\,ds\|\,d\boldsymbol{\eta} \\
= &\lim_{N\to\infty} \int_{\Omega(E)} 1 \wedge \|\omega(t) - \omega(0) - \int_0^t \boldsymbol{c}_s(\omega(s))\,ds\|\,d\boldsymbol{\eta}_N \\
= &\lim_{N\to\infty} \int_{\Omega(E)} 1 \wedge \|\int_0^t (\boldsymbol{b}_N)_s(\omega(s)) - \boldsymbol{c}_s(\omega(s))\,ds\|\,d\boldsymbol{\eta}_N \\
\le &\limsup_{N\to\infty} \int_0^T \int_E \|(\boldsymbol{b}_N)_s(x) - \boldsymbol{c}_s(x)\| u_s^N(x)\,d\gamma(x)\,ds = \int_0^T \int_E \|\boldsymbol{b}_s(x) - \boldsymbol{c}_s(x)\| u_s(x)\,d\gamma(x)\,ds.
\end{aligned}
$$

In order to obtain the last equality we added and subtracted $\|\boldsymbol{b}_s - \boldsymbol{c}_s\| u_s^N$, and we used the strong convergence of $\boldsymbol{b}_N$ to $\boldsymbol{b}$ in $L^1\big((0,T);L^p(\gamma;E)\big)$ and the weak* convergence of u_s^N to u_s in $L^\infty\big((0,T);L^{p'}(\gamma;E)\big)$. □

4.2 Uniqueness of the $\boldsymbol{b}$-flow

The following lemma provides a simple characterization of Dirac masses (i.e. measures concentrated at a single point), for measures in $\Omega(E)$ and for families of measures in E.

Lemma 4.6. *Let $\boldsymbol{\sigma}$ be a positive finite measure in $\Omega(E)$. Then $\boldsymbol{\sigma}$ is a Dirac mass if and only if $(e_t)_\#\boldsymbol{\sigma}$ is a Dirac mass for all $t \in \mathbb{Q} \cap [0,T]$.*
A Borel family $\{\nu_x\}_{x\in E}$ of positive finite measures in E (i.e. $x \mapsto \nu_x(A)$ is Borel in E for all $A \subset E$ Borel) is made, for γ-a.e. x, by Dirac masses if and only if

$$\nu_x(A_1)\nu_x(A_2) = 0 \quad \gamma\text{-a.e. in } E, \text{ for all disjoint Borel sets } A_1,\, A_2 \subset E. \tag{47}$$

Proof. The first statement is a direct consequence of the fact that all elements of $\Omega(E)$ are continuous maps, which are uniquely determined on $\mathbb{Q} \cap [0,T]$. In order to prove the second statement, let us fix an integer $k \ge 1$ and a countable partition (A_i) of Borel sets with $\mathrm{diam}(A_i) \le 1/k$ (its existence is ensured by the separability of E). By (47) we obtain a γ-negligible Borel set N_k satisfying $\nu_x(A_i)\nu_x(A_j) = 0$ for all $x \in E \setminus N_k$. As a consequence, the support of each of the measures ν_x, as x varies in $E \setminus N_k$, is contained in the closure of one of the sets A_i, which has diameter less than $1/k$. It follows that ν_x is a Dirac mass for all $x \in E \setminus \bigcup_k N_k$. □

Theorem 4.7 (Uniqueness of L^r-regular generalized $\boldsymbol{b}$-flows)**.** *Let $\boldsymbol{b} : [0,T]\times E \to \mathcal{H}$ be satisfying the assumptions of the uniqueness part of Theorem 3.1, let $r = \max\{p', q'\}$ and assume that $c \ge rT$. Let $\boldsymbol{\eta}$ be a L^r-regular generalized $\boldsymbol{b}$-flow. Then:*

(i) for γ-a.e. $x \in E$, the measures $\mathbb{E}(\boldsymbol{\eta}|\omega(0) = x)$ are Dirac masses in $\Omega(E)$, and setting

$$\mathbb{E}(\boldsymbol{\eta}|\omega(0) = x) = \delta_{\boldsymbol{X}(\cdot,x)}, \qquad \boldsymbol{X}(\cdot,x) \in \Omega(E), \tag{48}$$

the map $\boldsymbol{X}(t,x)$ is a L^r-regular $\boldsymbol{b}$-flow, according to Definition 1.1.

(ii) Any other L^r-regular generalized $\boldsymbol{b}$-flow coincides with $\boldsymbol{\eta}$. In particular $\boldsymbol{X}$ is the unique L^r-regular $\boldsymbol{b}$-flow.

Proof. (i) We set $\boldsymbol{\eta}_x := \mathbb{E}(\boldsymbol{\eta}|\omega(0) = x)$. Taking into account the first statement in Lemma 4.6, it suffices to show that, for $\bar{t} \in \mathbb{Q} \cap [0,T]$ fixed, the measures $\nu_x := \mathbb{E}((e_{\bar{t}})_\#\boldsymbol{\eta}|\omega(0) = x) = (e_{\bar{t}})_\#\boldsymbol{\eta}_x$ are Dirac masses for γ-a.e. $x \in E$. Still using Lemma 4.6, we will check the validity of (47). Since $\nu_x = \delta_x$ when $\bar{t} = 0$, we shall assume that $\bar{t} > 0$.

Let us argue by contradiction, assuming the existence of a Borel set $L \subset E$ with $\gamma(L) > 0$ and disjoint Borel sets A_1, $A_2 \subset E$ such that both $\nu_x(A_1)$ and $\nu_x(A_2)$ are positive for $x \in L$. We will get a contradiction with Theorem 3.1, building two distinct solutions of the continuity equation with the same initial condition $\bar{u} \in L^\infty(\gamma)$. With no loss of generality, possibly passing to a smaller set L still with positive γ-measure, we can assume that the quotient $\beta(x) := \nu_x(A_1)/\nu_x(A_2)$ is uniformly bounded in L. Let $\Omega_i \subset \Omega(E)$ be the set of trajectories ω which belong to A_i at time $\bar{t}$; obviously $\Omega_1 \cap \Omega_2 = \emptyset$ and we can define positive finite measures $\boldsymbol{\eta}_i$ in $\Omega(E)$ by

$$\boldsymbol{\eta}_1 := \int_L \chi_{\Omega_1}\boldsymbol{\eta}_x\, d\gamma(x), \qquad \boldsymbol{\eta}_2 := \int_L \beta(x)\chi_{\Omega_2}\boldsymbol{\eta}_x\, d\gamma(x).$$

By Proposition 4.8, both $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ induce, via the identity $u^i_t\gamma = (e_t)_\#\boldsymbol{\eta}_i$, a solution to the continuity equation which is uniformly bounded (just by comparison with the one induced by $\boldsymbol{\eta}$) in $L^r(\gamma)$. Moreover, both solutions start from the same initial condition $\bar{u}(x) = \nu_x(A_1)\chi_L(x)$. On the other hand, by the definition of Ω_i, $u^1_{\bar{t}}\gamma$ is concentrated in A_1 while $u^2_{\bar{t}}\gamma$ is concentrated in A_2, therefore $u^1_{\bar{t}} \neq u^2_{\bar{t}}$. So, uniqueness of solutions to the continuity equation is violated.
(ii) If $\boldsymbol{\sigma}$ is any other L^r-regular generalized $\boldsymbol{b}$-flow, we may apply statement (i) to the flows $\boldsymbol{\sigma}$, to obtain that for γ-a.e. x also the measures $\mathbb{E}(\boldsymbol{\sigma}|\omega(0) = x)$ are Dirac masses; but since the property of being a generalized flow is stable under convex combinations, also the measures

$$\frac{1}{2}\mathbb{E}(\boldsymbol{\eta}|\omega(0) = x) + \frac{1}{2}\mathbb{E}(\boldsymbol{\sigma}|\omega(0) = x) = \mathbb{E}\big(\frac{\boldsymbol{\eta}+\boldsymbol{\sigma}}{2}|\omega(0) = x\big)$$

must be Dirac masses for γ-a.e. x. This can happen only if $\mathbb{E}(\boldsymbol{\eta}|\omega(0) = x) = \mathbb{E}(\boldsymbol{\sigma}|\omega(0) = x)$ for γ-a.e. x. ☐

The connection between solutions to the ODE $\dot{\boldsymbol{X}} = \boldsymbol{b}_t(\boldsymbol{X})$ and the continuity equation is classical: in the next proposition we present it under natural regularity assumptions in this setting.

Proposition 4.8. *Let $\boldsymbol{\eta}$ be a positive finite measure in $\Omega(E)$ satisfying:*

(a) $\boldsymbol{\eta}$ is concentrated on paths $\omega \in AC^1(E)$ such that $\omega(t) = \omega(0) + \int_0^t \boldsymbol{b}_s(\omega(s))\, ds$ for all $t \in [0,T]$;

(b) $\int_0^T \int_{\Omega(E)} \|\dot{\omega}(t)\|_{\mathcal{H}}\, d\boldsymbol{\eta}(\omega)\, dt < \infty$.

Then the measures $\mu_t := (e_t)_\#\boldsymbol{\eta}$ satisfy $\frac{d}{dt}\mu_t + \mathrm{div}_\gamma(\boldsymbol{b}_t\mu_t) = 0$ in $(0,T) \times E$ in the weak sense.

Proof. Let $\phi(x) = \psi(\langle e_1^*, x\rangle, \ldots, \langle e_N^*, x\rangle)$ be cylindrical. By (a) and Fubini's theorem, for a.e. t the following property holds: the maps $\langle e_i^*, \omega(t)\rangle$, $1 \le i \le N$, are differentiable at t, with derivative equal to $\langle e_i^*, \boldsymbol{b}_t(\omega(t))\rangle$, for $\boldsymbol{\eta}$-a.e. ω. Taking (12) into account, for a.e. t we have

$$\begin{aligned}
\frac{d}{dt}\int_E \phi\, d\mu_t &= \frac{d}{dt}\int_{\Omega(E)} \psi(\langle e_1^*, \omega(t)\rangle, \ldots, \langle e_N^*, \omega(t)\rangle)\, d\boldsymbol{\eta} \\
&= \sum_{i=1}^N \int_{\Omega(E)} \frac{\partial \psi}{\partial z_i}(\langle e_1^*, \omega(t)\rangle, \ldots, \langle e_N^*, \omega(t)\rangle)\langle e_i^*, \dot\omega(t)\rangle\, d\boldsymbol{\eta} \\
&= \sum_{i=1}^N \int_{\Omega(E)} \frac{\partial \psi}{\partial z_i}(\langle e_1^*, \omega(t)\rangle, \ldots, \langle e_N^*, \omega(t)\rangle)\langle e_i, \boldsymbol{b}_t(\omega(t))\rangle_{\mathcal{H}}\, d\boldsymbol{\eta} \\
&= \int_E \langle \nabla\phi, \boldsymbol{b}_t\rangle_{\mathcal{H}}\, d\mu_t.
\end{aligned}$$

In the previous identity we used, to pass to the limit under the integral sign, the property

$$\lim_{h\to 0}\langle e_i^*, \frac{\omega(t+h)-\omega(t)}{h}\rangle = \langle e_i^*, \dot\omega(t)\rangle \quad \text{in } L^1(\boldsymbol{\eta}), \text{ for } 1 \le i \le N,$$

whose validity for a.e. t is justified by assumption (b). The same assumption also guaranteees (see for instance [2, §3] for a detailed proof) that $t \mapsto \int_E \phi\, d\mu_t$ is absolutely continuous, so its pointwise a.e. derivative coincides with the distributional derivative. □

4.3 Stability of the $\boldsymbol{b}$-flow and semigroup property

The methods we used to show existence and uniqueness of the flow also yield stability of the flow with respect to approximations (not necessarily finite-dimensional ones) of the vector field. In the proof we shall use the following simple lemma (see for instance Lemma 22 of [2] for a proof), where we use the notation $\mathrm{id} \times f$ for the map $x \mapsto (x, f(x))$.

Lemma 4.9 (Convergence in law and in probability). *Let F be a metric space and let f_n, $f : E \to F$ be Borel maps. Then $f_n \to f$ in γ-probability if and only if $\mathrm{id} \times f_n \to \mathrm{id} \times f$ in law.*

Theorem 4.10 (Stability of L^r-regular $\boldsymbol{b}$-flows). *Let p, $q > 1$, $r = \max\{p', q'\}$ and let $\boldsymbol{b}_n$, $\boldsymbol{b} : (0,T) \times E \to \mathcal{H}$ be satisfying:*

(i) $\boldsymbol{b}_n \to \boldsymbol{b}$ *in* $L^1\big((0,T); L^p(\gamma;\mathcal{H})\big)$;

(ii) for a.e. $t \in (0,T)$ *we have* $(\boldsymbol{b}_n)_t$, $\boldsymbol{b}_t \in LD^q_{\mathcal{H}}(\gamma;\mathcal{H})$ *with*

$$\sup_{n\in\mathbb{N}} \int_0^T \left(\int_E \|(\nabla(\boldsymbol{b}_n)_t)^{\mathrm{sym}}(x)\|_{HS}^q\, d\gamma(x)\right)^{1/q} dt < \infty \tag{49}$$

and $\mathrm{div}_\gamma(\boldsymbol{b}_n)_t$ *and* $\mathrm{div}_\gamma \boldsymbol{b}_t$ *belong to* $L^1\big((0,T); L^q(\gamma)\big)$;

(iii) $\exp(c[\mathrm{div}_\gamma(\boldsymbol{b}_n)_t]^-)$ *are uniformly bounded in* $L^\infty\big((0,T); L^1(\gamma)\big)$ *for some* $c \ge Tr$.

Then, denoting by $\boldsymbol{X}_n$ (resp. $\boldsymbol{X}$) the unique L^r regular $\boldsymbol{b}_n$-flows (resp. $\boldsymbol{b}$-flow) we have

$$\lim_{n\to\infty}\int_E \sup_{[0,T]} \|\boldsymbol{X}_n(\cdot,x)-\boldsymbol{X}(\cdot,x)\|\,d\gamma(x)=0. \tag{50}$$

Proof. Let us denote the generalized $\boldsymbol{b}_n$-flows $\boldsymbol{\eta}_n$ induced by $\boldsymbol{X}_n$, namely the law under γ of $x\mapsto \boldsymbol{X}_n(\cdot,x)$. The uniform estimates (iii), together with the boundedness of $\|\boldsymbol{b}_n\|_{\mathcal{H}}$ in $L^1\big((0,T);L^p(\gamma)\big)$ imply, in view of (40),

$$\sup_{n\in\mathbb{N}}\int (u^n_t)^r\,d\gamma \le \sup_{n\in\mathbb{N}}\bigg\|\int \exp\big(Tr[\mathrm{div}_\gamma \boldsymbol{b}^n_t]^-\big)\,d\gamma\bigg\|_{L^\infty(0,T)}<\infty \qquad \forall\, t\in[0,T], \tag{51}$$

where u^n_t is the density of $(e_t)_\#\boldsymbol{\eta}_n = X(t,\cdot)_\#\gamma$ with respect to γ. In addition, by the same argument used in Step 2 of the proof of Theorem 4.5 we have

$$\sup_{n\in\mathbb{N}}\int_{\Omega(E)}\Phi(\omega)\,d\boldsymbol{\eta}_n(\omega)<\infty,$$

where Φ is defined as in (44), with $\alpha\in(1,p)$ and $\Phi_1:E\to[0,\infty)$ γ-integrable and coercive.

This estimate implies the tightness of $(\boldsymbol{\eta}_n)$. If $\boldsymbol{\eta}$ is a limit point, in the duality with $C_b(\Omega(E))$, of $\boldsymbol{\eta}_n$, the same argument used in Step 3 of the proof of Theorem 4.5 gives that $\boldsymbol{\eta}$ is a generalized $\boldsymbol{b}$-flow. In addition, the uniform estimates (51) imply that $\boldsymbol{\eta}$ is L^r-regular. As a consequence we can apply Theorem 4.7 to obtain that $\boldsymbol{\eta}$ is the law of the $\Omega(E)$-valued map $x\mapsto \boldsymbol{X}(\cdot,x)$, and more precisely that $\mathbb{E}(\boldsymbol{\eta}|\omega(0)=x)=\delta_{\boldsymbol{X}(\cdot,x)}$ for γ-a.e. x. Therefore, by the uniqueness of $\boldsymbol{X}$, the whole sequence $(\boldsymbol{\eta}_n)$ converges to $\boldsymbol{\eta}$ and $\boldsymbol{X}_n$ converge in law to $\boldsymbol{X}$.

In order to obtain that $x\mapsto \boldsymbol{X}_n(\cdot,x)$ converge in γ-probability to $x\mapsto \boldsymbol{X}(\cdot,x)$ we use Lemma 4.9 with $F=\Omega(E)$, so we have to show that $\mathrm{id}\times \boldsymbol{X}_n(\cdot,x)$ converge in law to $\mathrm{id}\times\boldsymbol{X}(\cdot,x)$. For all $\psi\in C_b\big(E\times\Omega(E)\big)$ we have

$$\int_E \psi(x,\boldsymbol{X}_n(\cdot,x))\,d\gamma(x)=\int_{\Omega(E)}\psi(e_0(\omega),\omega)\,d\boldsymbol{\eta}_n\to\int_{\Omega(E)}\psi(e_0(\omega),\omega)\,d\boldsymbol{\eta}=\int_E\psi(x,\boldsymbol{X}(\cdot,x))\,d\gamma(x),$$

and this proves the convergence in law.

Finally, by adding and subtracting x, we can prove (50) provided we show that $\sup_{[0,T]}|\boldsymbol{X}(\cdot,x)-x|\in L^1(\gamma)$ and $\sup_{[0,T]}|\boldsymbol{X}_n(\cdot,x)-x|$ are equi-integrable in $L^1(\gamma)$. We prove the second property only, because the proof of the first one is analogous. Starting from the integral formulation of the ODE, Jensen's inequality gives $\sup_{[0,T]}|\boldsymbol{X}_n(\cdot,x)-x|^\alpha\le T^{\alpha-1}\int_0^T\|\boldsymbol{b}_\tau(\boldsymbol{X}_n(\tau,x))\|\,d\tau$ and by integrating both sides with respect to γ, Fubini's theorem gives

$$\int_E\sup_{[0,T]}|\boldsymbol{X}_n(\cdot,x)-x|^\alpha\,d\gamma(x)\le T^{\alpha-1}\int_E\int_0^T\int_E\|\boldsymbol{b}_\tau\|^\alpha u^n_\tau\,d\gamma\,d\tau.$$

Choosing $\alpha>1$ such that $(p/\alpha)'\le c/T$ (this is possible because we are assuming that $c>p'T$) and applying the Hölder inequality with the exponents p/α and $(p/\alpha)'$ we obtain that $\sup_{[0,T]}|\boldsymbol{X}_n(\cdot,x)-x|$ are equibounded in $L^\alpha(\gamma)$. $\square$

Under the same assumptions of Theorem 4.7, for all $s \in [0,T]$ also a unique L^r-regular flow $\boldsymbol{X}^s : [s,T] \times E \to E$ exists, characterized by the properties that $\tau \mapsto \boldsymbol{X}^s(\tau,x)$ is an absolutely continuous map in $[s,T]$ satisfying

$$\boldsymbol{X}^s(t,x) = x + \int_s^t \boldsymbol{b}_\tau\big(\boldsymbol{X}^s(\tau,x)\big)\,d\tau \qquad \forall\, t \in [s,T] \tag{52}$$

for γ-a.e. $x \in E$, and the regularity condition $\boldsymbol{X}^s(\tau,\cdot)_\#\gamma = f_\tau\gamma$, with $f_\tau \in L^r(\gamma)$ uniformly for $\tau \in [s,T]$. This family of flow maps satisfies the semigroup property:

Proposition 4.11 (Semigroup property). *Under the same assumptions of Theorem 4.7, the unique L^r-regular flows $\boldsymbol{X}^s$ starting at time s satisfy the semigroup property*

$$\boldsymbol{X}^s\left(t, \boldsymbol{X}^r(s,x)\right) = \boldsymbol{X}^r(t,x) \qquad \textit{for } \gamma\textit{-a.e. } x \in E,\ \forall\, 0 \le r \le s \le t \le T. \tag{53}$$

Proof. Let r, s, t be fixed. By combining the finite-dimensional projection argument of Step 1 of the proof of Theorem 4.5, with the smoothing argument used in Step 2 of the proof of Theorem 6.1 we can find a family of vector fields $\boldsymbol{b}_n$ converging to $\boldsymbol{b}$ in $L^1\big((0,T);L^p(\gamma;\mathcal{H})\big)$ and satisfying the uniform bounds of Theorem 4.10, whose (classical) flows $\boldsymbol{X}_n$ satisfy the semigroup property (see (62))

$$\boldsymbol{X}_n^s\left(t, \boldsymbol{X}_n^r(s,x)\right) = \boldsymbol{X}_n^r(t,x) \qquad \text{for } \gamma\text{-a.e. } x \in E,\ \forall\, 0 \le r \le s \le t \le T. \tag{54}$$

We will pass to the limit in (54), to obtain (53). To this aim, notice that (50) of Theorem 4.10 immediately provides the convergence in $L^1(\gamma)$ of the right hand sides, so that we need just to show convergence in γ-measure of the left hand sides. Notice first that the convergence in γ-measure of $\boldsymbol{X}_n^r(s,\cdot)$ to $\boldsymbol{X}^r(s,\cdot)$ implies the convergence in γ-measure of $\psi(\boldsymbol{X}_n^r(s,\cdot))$ to $\psi(\boldsymbol{X}^r(s,\cdot))$ for any Borel function $\psi : E \to \mathbb{R}$ (this is a simple consequence of the fact that, by Lusin's theorem, we can find a nondecreasing sequence of compact sets $K_n \subset E$ such that $\psi|_{K_n}$ is uniformly continuous and $\gamma(E \setminus K_n) \downarrow 0$, and of the fact that the laws of $\boldsymbol{X}_x^r(s,\cdot)$ are uniformly bounded in $L^r(\gamma)$), so that choosing $\psi(z) := \boldsymbol{X}^s(t,z)$, and adding and subtracting $\boldsymbol{X}^s(t,\boldsymbol{X}_n(s,x))$, the convergence in γ-measure of the right hand sides of (54) to $\boldsymbol{X}^s\left(t,\boldsymbol{X}^r(s,x)\right)$ follows by the convergence in γ-measure to 0 of

$$\boldsymbol{X}_n^s\left(t,\boldsymbol{X}_n^r(s,x)\right) - \boldsymbol{X}^s\left(t,\boldsymbol{X}_n^r(s,x)\right).$$

Denoting by ρ_n the density of the law of $\boldsymbol{X}_n^r(s,\cdot)$, we have

$$\int_E 1 \wedge \|\boldsymbol{X}_n^s\left(t,\boldsymbol{X}_n^r(s,x)\right) - \boldsymbol{X}^s\left(t,\boldsymbol{X}_n^r(s,x)\right)\|\,d\gamma(x) = \int_E 1 \wedge \|\boldsymbol{X}_n^s(t,y) - \boldsymbol{X}^s(t,y)\|\rho_n(y)\,d\gamma(y),$$

and the right hand side tends to 0 thanks to (50) and to the equi-integrability of (ρ_n). □

The semigroup property allows also to construct a unique family of flows $\boldsymbol{X}^s : [s,T] \times E \times E$ even in the case when the assumption (27) is replaced by

$$\exp(c[\mathrm{div}_\gamma \boldsymbol{b}_t]^-) \in L^\infty\big((0,T);L^1(\gamma)\big) \qquad \text{for } \textit{some } c > 0.$$

The idea is to compose the flows defined on sufficiently short intervals, with length T' satisfying $c > rT'$. It is easy to check that this family of flow maps is uniquely determined by the semigroup property (53) and by the *local* regularity property

$$\boldsymbol{X}^s(t,\cdot)_\#\gamma \ll \gamma \text{ with a density in } L^r(\gamma) \text{ for all } t \in [s, \min\{s+T', T\}],\ s \in [0,T].$$

Globally in time, the only property retained is $\boldsymbol{X}^s(t,\cdot)_\#\gamma \ll \gamma$ for all $t \in [s,T]$.

4.4 Convergence of finite-dimensional flows

Assume that we are given vector fields $\boldsymbol{b}_N : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying, for some p, $q > 1$ the assumptions (i), (ii), (iii) of Theorem 1.2 (with $E = \mathcal{H} = \mathbb{R}^N$) relative to the standard Gaussian γ_N in $\mathbb{R}^N$, with norms uniformly bounded by constants independent of N. Let us assume that $\boldsymbol{b}_N$ is a consistent family, namely the conditional expectation of the projection of $(\boldsymbol{b}_{N+1})_t$ on $\mathbb{R}^N$, given $x^1, \ldots, x^N$, is $(\boldsymbol{b}_N)_t$. Let $\boldsymbol{X}_N : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ be the associated $\boldsymbol{b}_N$-flows.

In this section we briefly illustrate how the stability results of this paper can be used to prove the convergence of $\boldsymbol{X}_N$ and to characterize their limit.

To this aim, let us denote by γ_p the product of standard Gaussians in the countable product $\mathbb{R}^\infty$, and notice that the consistency assumption provides us with a unique vector field $\boldsymbol{b} : [0,T] \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ such that, denoting by $\mathbb{E}_N$ the conditional expectation with respect to $x^1, \ldots, x^N$ and by $\pi_N : \mathbb{R}^\infty \to \mathbb{R}^N$ the canonical projections, the identities $\mathbb{E}_N \pi_N \boldsymbol{b}_t = (\boldsymbol{b}_N)_t$ hold. In order to recover a Wiener space we fix a sequence $(\lambda_i) \in \ell^2$ and define

$$E := \left\{ (x^i) : \sum_{i=1}^\infty \lambda_i^2 (x^i)^2 < \infty \right\}.$$

The space E can be endowed with the canonical scalar product, and obviously $\gamma_p(E) = 1$, so that $\boldsymbol{b}$ can be also viewed as a vector field in E and the induced measure γ in E is Gaussian. According to Remark 2.1, its Cameron-Martin space $\mathcal{H}$ can be identified with ℓ^2. Then, we can apply the stability Theorem 4.10 (viewing, with a slight abuse, $\boldsymbol{b}_N$ as vector fields in E and, consequently, their flows $\boldsymbol{X}_N$ as flows in E which leave $x^{N+1}, x^{N+2}, \ldots$ fixed) to obtain that $\boldsymbol{X}_N$ converge to the flow $\boldsymbol{X}$ relative to $\boldsymbol{b}$ in $L^1(\gamma; E)$. It follows that

$$\lim_{N\to\infty} \int_{\mathbb{R}^\infty} \sqrt{\sum_{i=1}^\infty \lambda_i^2 |\boldsymbol{X}_N^i(t,x) - \boldsymbol{X}^i(t,x)|^2}\, d\gamma_p(x) = 0 \qquad \forall\, t \in [0,T],\ \forall\, (\lambda_i) \in \ell^2. \tag{55}$$

Finally, notice that also $\boldsymbol{X}$ could be defined without an explicit mention to E, working in $(\mathbb{R}^\infty, \gamma_p)$ in place of (E, γ). According to this viewpoint, E plays just the role of an auxiliary space, and deliberately we wrote (55) without an explicit mention to it.

5 An extension to non $\mathcal{H}$-valued vector fields

In [43], [13], the authors consider the following equation:

$$\boldsymbol{X}(t,x) = \tilde{Q}_t x + \int_0^t Q_{t-s} \boldsymbol{b}_s(\boldsymbol{X}(s,x))\, ds. \tag{56}$$

Here $(Q_t)_{t\in\mathbb{R}}$ is a strongly continuous group of orthogonal operator on $\mathcal{H}$, and $\tilde{Q}_t : E \to E$ denotes the measurable linear extension of Q_t to E (which always exists and preserves the measure γ, see for instance [36]). Observe that, thanks to the Duhamel formula, (56) formally corresponds to the equation

$$\dot{\boldsymbol{X}}(t,x) = L\boldsymbol{X}(t,x) + \boldsymbol{b}_t(\boldsymbol{X}(t,x)),$$

where L denotes the generator of the group (i.e. $\dot{Q}_t = LQ_t$).

The definition of L^r-regular flow can be extended in the obvious way to (56). Let us now see how our results allow to prove existence and uniqueness of L^r-regular flows under the assumptions of Theorem 1.2 (observe that this forces in particular $r > 1$).

Let $\boldsymbol{X}(t,x)$ be a solution of (56), and define $\boldsymbol{Y}(t,x) := \tilde{Q}_{-t}\boldsymbol{X}(t,x)$. Then we have

$$\begin{aligned}\boldsymbol{Y}(t,x) &= x + \int_0^t Q_{-s}\boldsymbol{b}_s(\boldsymbol{X}(s,x))\,ds \\ &= x + \int_0^t Q_{-s}\boldsymbol{b}_s(\tilde{Q}_s\boldsymbol{Y}(s,x))\,ds.\end{aligned}$$

Therefore $\boldsymbol{Y}$ is a flow associated to the vector field $\boldsymbol{c}_t(x) := Q_{-t}\boldsymbol{b}_t(\tilde{Q}_t x)$. Moreover $\boldsymbol{Y}$ is still a L^r-regular flow. Indeed, if $u_t \in L^r(\gamma)$ denotes the density of the law of $\boldsymbol{X}(t,\cdot)$, then, for all $\phi \in \mathrm{Cyl}(E,\gamma)$, we have

$$\begin{aligned}\int \phi(\boldsymbol{Y}(t,x))\,d\gamma(x) &= \int \phi(\tilde{Q}_{-t}\boldsymbol{X}(t,x))\,d\gamma(x) = \int \phi(\tilde{Q}_{-t}x)u_t(x)\,d\gamma(x) \\ &\leq \|u_t\|_{L^r(\gamma)}\|\phi\circ\tilde{Q}_t\|_{L^{r'}(\gamma)} = \|u_t\|_{L^r(\gamma)}\|\phi\|_{L^{r'}(\gamma)}.\end{aligned}$$

Since $r > 1$, this implies that $\boldsymbol{Y}$ is L^r-regular. On the other hand we remark that, using the same argument, one obtains that, if $\boldsymbol{Y}$ is a L^r-regular flow associated to $\boldsymbol{c}$, then $\boldsymbol{X}(t,x) := \tilde{Q}_t\boldsymbol{Y}(t,x)$ is a L^r-regular flow for (56).

We have therefore shown that there is a one-to-one correspondence between L^r-regular flows for (56) and L^r-regular flows associated to $\boldsymbol{c}$. To conclude the existence and uniqueness of L^r-regular flows for (56), it suffices to observe that, thanks to the orthogonality of Q_t and the measure-preserving property of $\tilde{Q}_t$, if $\boldsymbol{b}$ satisfies all the assumptions in Theorem 1.2, then so does $\boldsymbol{c}$ thanks to the identities $\|\boldsymbol{c}_t(x)\|_{\mathcal{H}} = \|\boldsymbol{b}_t(\tilde{Q}_t x)\|_{\mathcal{H}}$, $\|(\nabla\boldsymbol{c}_t)^{\mathrm{sym}}(x)\|_{HS} = \|(\nabla\boldsymbol{b}_t)^{\mathrm{sym}}(\tilde{Q}_t x)\|_{HS}$, and $\mathrm{div}_\gamma\,\boldsymbol{c}_t(x) = \mathrm{div}_\gamma\,\boldsymbol{b}_t(\tilde{Q}_t x)$.

Indeed, let us check the formula for the symmetric part of the derivative, the proof of the one concerning the divergence being similar and even simpler. Let $h = j(e^*) \in \mathcal{H}$ and notice that $Q_t h = j(f^*)$, where $\langle f^*, y\rangle = \langle e^*, \tilde{Q}_{-t}(y)\rangle$. Using Remark 2.9 and the fact that $\phi \mapsto \phi\circ\tilde{Q}_t$

maps $\mathrm{Cyl}(E,\gamma)$ into $C^1_b(E,\gamma)$, for $\phi\in\mathrm{Cyl}(E,\gamma)$ we get

$$
\begin{aligned}
&\int_E \langle \boldsymbol{c}_t,h\rangle_{\mathcal{H}}\partial_h\phi\,d\gamma = \int_E \langle \boldsymbol{b}_t(\tilde{Q}_t x),Q_t h\rangle_{\mathcal{H}}\partial_h\phi(x)\,d\gamma(x)\\
= \;& \int_E \langle \boldsymbol{b}_t(y),Q_t h\rangle_{\mathcal{H}}(\partial_h\phi)\circ\tilde{Q}_{-t}(y)\,d\gamma(y) = \int_E \langle \boldsymbol{b}_t(y),Q_t h\rangle_{\mathcal{H}}\partial_{Q_t h}(\phi\circ\tilde{Q}_{-t})(y)\,d\gamma(y)\\
= \;& -\int_E \partial_{Q_t h}\langle \boldsymbol{b}_t,Q_t h\rangle_{\mathcal{H}}\phi\circ\tilde{Q}_{-t}\,d\gamma(y) + \int_E \langle \boldsymbol{b}_t(y),Q_t h\rangle_{\mathcal{H}}\phi\circ\tilde{Q}_{-t}\langle f^*,y\rangle\,d\gamma(y)\\
= \;& -\int_E [\partial_{Q_t h}\langle \boldsymbol{b}_t,Q_t h\rangle_{\mathcal{H}}]\circ\tilde{Q}_t\phi\,d\gamma(x) + \int_E \langle \boldsymbol{c}_t(x),h\rangle_{\mathcal{H}}\phi\langle e^*,x\rangle\,d\gamma(x).
\end{aligned}
$$

This proves that $\partial_h\langle \boldsymbol{c}_t,h\rangle_{\mathcal{H}} = \partial_{Q_t h}\langle \boldsymbol{b}_t,Q_t h\rangle_{\mathcal{H}}\circ\tilde{Q}_t$, and using the fact that Q_t maps orthonormal bases of $\mathcal{H}$ in orthonormal bases of $\mathcal{H}$ we get $\|(\nabla\boldsymbol{c}_t)^{\mathrm{sym}}\|_{HS} = \|(\nabla\boldsymbol{b}_t)^{\mathrm{sym}}\|_{HS}\circ\tilde{Q}_t$.

6 Finite-dimensional estimates

This section is devoted to the proof of the crucial a-priori bounds (28) and (37) in finite-dimensional Wiener spaces. So, we shall assume that $E=\mathcal{H}=\mathbb{R}^N$ and, only in this section, denote by $x\cdot y$ the scalar product in $\mathbb{R}^N$, and by $|x|$ the Euclidean norm (corresponding to the norm of the Cameron-Martin space). Also, only in this section we shall denote by γ the standard Gaussian in $\mathbb{R}^N$, product of N standard Gaussians in $\mathbb{R}$, and by $\int$ integrals on the whole of $\mathbb{R}^N$. The sums $\sum_i$ (resp. $\sum_{i,j}$) will always be understood with i (resp. i and j) running from 1 to N.

6.1 Upper bounds on the flow density

In this subsection we show the existence part of Theorem 3.1 in finite-dimensional Wiener spaces $E=\mathcal{H}=\mathbb{R}^N$.

Theorem 6.1. *Let $\boldsymbol{b}:(0,T)\times\mathbb{R}^N\to\mathbb{R}^N$ be satisfying the assumptions of the existence part of Theorem 3.1. Then, for any $r\in[1,c/T]$ there exists a generalized L^r-regular $\boldsymbol{b}$-flow $\boldsymbol{\eta}$. Its density u_t satisfies also*

$$
\int (u_t)^r\,d\gamma \le \left\|\int \exp\bigl(Tr[\mathrm{div}_\gamma\boldsymbol{b}_t]^-\bigr)\,d\gamma\right\|_{L^\infty(0,T)} \qquad \forall\, t\in[0,T]. \tag{57}
$$

Proof. **Step 1.** Here we consider first the case when $\boldsymbol{b}_t$ are smooth, with $\int_0^T \|\nabla\boldsymbol{b}_t\|_{L^\infty(B)}\,dt$ finite for all bounded open sets $B\subset\mathbb{R}^N$. Under this assumption, for all $x\in\mathbb{R}^N$ the unique solution $\boldsymbol{X}(\cdot,x)$ to the ODE $\dot{\boldsymbol{X}}(t,x)=\boldsymbol{b}_t(\boldsymbol{X}(t,x))$, with the initial condition $\boldsymbol{X}(0,x)=x$, is defined until some maximal time $\tau(x)\in(0,T]$. Obviously, by the maximality of $\tau(x)$, if

$$
\limsup_{t\uparrow\tau(x)} |\boldsymbol{X}(t,x)| < +\infty
$$

then $\tau(x)=T$ and the solution is continuous in $[0,T]$.

Let us fix $s\in[0,T)$. We denote E_s the set $\{\tau>s\}$ and notice that standard stability results for ODE's with a locally Lipschitz vector field ensure that E_s is open and that $x\mapsto \boldsymbol{X}(t,x)$ is smooth in E_s for $t\in[0,s]$. Furthermore, from the identity $\dot\nabla_x\boldsymbol{X}(t,x)=\nabla\boldsymbol{b}_t(\boldsymbol{X}(t,x))\nabla_x\boldsymbol{X}(t,x)$, obtained by spatial differentiation of the ODE (see [2] for details), one obtains

$$\dot J\boldsymbol{X}(t,x)=\mathrm{div}\boldsymbol{b}_t(\boldsymbol{X}(t,x))J\boldsymbol{X}(t,x)\qquad x\in E_s,\ t\in[0,s],\tag{58}$$

where $J\boldsymbol{X}(t,x)$ is the determinant of $\nabla_x\boldsymbol{X}(t,x)$.

We first compute a pointwise expression for the measure $\boldsymbol{X}(t,\cdot)_\#(\chi_{E_s}\gamma)$ for $t\in[0,s]$. By the change of variables formula, the density ρ^s_t of $\boldsymbol{X}(t,\cdot)_\#(\chi_{E_s}\gamma)$ with respect to $\mathscr{L}^N$ is linked to the initial density $\bar\rho^s$ by

$$\rho^s_t(\boldsymbol{X}(t,x))=\frac{\bar\rho^s(x)}{J\boldsymbol{X}(t,x)},$$

where $\bar\rho^s(y):=\chi_{E_s}(y)e^{-|y|^2/2}$. Denoting by u^s_t the density of $\boldsymbol{X}(t,\cdot)_\#(\chi_{E_s}\gamma)$ with respect to γ, we get

$$u^s_t\big(\boldsymbol{X}(t,x)\big)=\frac{\bar\rho^s(x)}{J\boldsymbol{X}(t,x)}e^{|\boldsymbol{X}(t,x)|^2/2}.\tag{59}$$

So, taking the identity (58) into account, we obtain

$$\frac{d}{dt}u^s_t\big(\boldsymbol{X}(t,x)\big)=-\mathrm{div}_\gamma\boldsymbol{b}_t\big(\boldsymbol{X}(t,x)\big)\frac{\bar\rho^s(x)}{J\boldsymbol{X}(t,x)}e^{|\boldsymbol{X}(t,x)|^2/2}=-\mathrm{div}_\gamma\boldsymbol{b}_t\big(\boldsymbol{X}(t,x)\big)u^s_t\big(\boldsymbol{X}(t,x)\big).$$

By integrating the ODE, for $t\in[0,s]$ we get

$$\begin{aligned}u^s_t\big(\boldsymbol{X}(t,x)\big)&=\chi_{E_s}(x)\exp\Big(-\int_0^t\mathrm{div}_\gamma\boldsymbol{b}_\tau\big(\boldsymbol{X}(\tau,x)\big)\,d\tau\Big)\\&\le\chi_{E_s}(x)\exp\Big(\int_0^t[\mathrm{div}_\gamma\boldsymbol{b}_\tau\big(\boldsymbol{X}(\tau,x)\big)]^-\,d\tau\Big).\end{aligned}$$

We can now estimate $\|u^s_t\|_{L^r(\gamma)}$ as follows:

$$\begin{aligned}\int(u^s_t)^r\,d\gamma&=\int(u^s_t)^{r-1}u^s_t\,d\gamma\le\int\exp\Big((r-1)\int_0^t[\mathrm{div}_\gamma\boldsymbol{b}_\tau\big(\boldsymbol{X}(\tau,x)]^-\big)\,d\tau\Big)\chi_{E_s}(x)\,d\gamma(x)\\&\le\int\frac{1}{t}\int_0^t\exp\big(t(r-1)[\mathrm{div}_\gamma\boldsymbol{b}_\tau\big(\boldsymbol{X}(\tau,x)\big)]^-\big)\,d\tau\chi_{E_s}(x)\,d\gamma(x)\\&=\frac{1}{t}\int_0^t\int\exp\big(t(r-1)[\mathrm{div}_\gamma\boldsymbol{b}_\tau\big(\boldsymbol{X}(\tau,x)\big)]^-\big)\chi_{E_s}(x)\,d\gamma(x)\,d\tau\\&\le\frac{1}{t}\int_0^t\int\exp\big(T(r-1)[\mathrm{div}_\gamma\boldsymbol{b}_\tau(y)]^-\big)u^s_\tau(y)\,d\gamma(y)\,d\tau.\end{aligned}$$

Now, set $\Lambda(t):=\int_0^t\|u^s_\tau\|^r_{L^r(\gamma)}\,d\tau$ and apply the Hölder inequality to get

$$\begin{aligned}\Lambda'(t)&\le\frac{1}{t}\Big(\int_0^t\int\exp\big(Tr[\mathrm{div}_\gamma\boldsymbol{b}_\tau(y)]^-\big)\,d\gamma(y)\,d\tau\Big)^{1/r'}\Lambda^{1/r}(t)\\&\le Kt^{1/r'-1}\Lambda^{1/r}(t)=Kt^{-1/r}\Lambda^{1/r}(t),\end{aligned}\tag{60}$$

with $K := \| \int \exp\bigl(Tr[\mathrm{div}_\gamma \boldsymbol{b}_t]^-\bigr)\,d\gamma\|_{L^\infty(0,T)}^{1/r'}$. An integration of this differential inequality yields $\Lambda(t) \le K^{r'} t$, which inserted into (60) gives

$$\int (u_t^s)^r \, d\gamma \le \left\| \int \exp\bigl(Tr[\mathrm{div}_\gamma \boldsymbol{b}_t]^-\bigr)\,d\gamma \right\|_{L^\infty(0,T)} \qquad \forall\, t \in [0,s],\ \forall\, s \in [0,T). \tag{61}$$

Now, let us prove that the flow is globally defined in $[0,T]$ for γ-a.e. x: we have indeed

$$\begin{aligned}\int \sup_{[0,\tau(x))} |\boldsymbol{X}(t,x) - x|\,d\gamma(x) \quad &\le \quad \int\int_0^{\tau(x)} |\boldsymbol{b}_t(\boldsymbol{X}(t,x))|\,dt\,d\gamma(x) = \int_0^T \int_{E_t} |\boldsymbol{b}_t(\boldsymbol{X}(t,x))|\,d\gamma(x)\,dt \\ &= \quad \int_0^T \int |\boldsymbol{b}_t| u_t^t \,d\gamma\,dt.\end{aligned}$$

Using (61) with $s = t$, we obtain that $\int \sup_{[0,\tau(x))} |\boldsymbol{X}(t,x) - x|\,d\gamma(x)$ is finite, so that $\tau(x) = T$ and $\boldsymbol{X}(\cdot,x)$ is continuous up to $t = T$ for γ-a.e. x. Letting $s \uparrow T$ in (61) we obtain (57).

Denoting as in (52) by $\boldsymbol{X}^s$ the flow starting at time s, we also notice (this is useful in the proof, by approximation, of the semigroup property in Proposition 4.11) that the pointwise uniqueness of the flow implies the semigroup property

$$\boldsymbol{X}^s\left(t, \boldsymbol{X}^r(s,x)\right) = \boldsymbol{X}^r(t,x) \qquad \forall\, 0 \le r \le s \le t \le T \tag{62}$$

for all x where $\boldsymbol{X}^r(\cdot,x)$ is globally defined in $[r,T]$.

Step 2. In this step we remove the regularity assumptions made on $\boldsymbol{b}$, considering the vector fields $\boldsymbol{b}_\varepsilon$ defined by $\boldsymbol{b}_\varepsilon^i(t,\cdot) := T_\varepsilon \boldsymbol{b}_t^i$. It is immediate to check that the fields $\boldsymbol{b}_\varepsilon$ satisfy the regularity assumptions made in Step 1, so the existence of a L^r-regular $\boldsymbol{b}_\varepsilon$-flow $\boldsymbol{\eta}_\varepsilon$ satisfying

$$\int (u_t^\varepsilon)^r \, d\gamma \le \left\| \int \exp\bigl(Tr[\mathrm{div}_\gamma (\boldsymbol{b}_\varepsilon)_t]^-\bigr)\,d\gamma \right\|_{L^\infty(0,T)} \tag{63}$$

is ensured by Step 1. In (63) the functions u_t^ε are, as usual, the densities of $(e_t)_\# \boldsymbol{\eta}_\varepsilon$ with respect to γ. Now, since $\mathrm{div}_\gamma((\boldsymbol{b}_\varepsilon)_t) = e^{-\varepsilon} T_\varepsilon(\mathrm{div}_\gamma \boldsymbol{b}_t)$, we may apply Jensen's inequality to get

$$\int (u_t^\varepsilon)^r \, d\gamma \le \left\| \int \exp\bigl(e^{-\varepsilon} Tr[\mathrm{div}_\gamma \boldsymbol{b}_t]^-\bigr)\,d\gamma \right\|_{L^\infty(0,T)}. \tag{64}$$

Since

$$\int_0^T \left(\int \|\boldsymbol{b}_\varepsilon(t,x)\|_{\mathcal{H}}^p \, d\gamma \right)^{1/p} dt \le \int_0^T \left(\int \|\boldsymbol{b}(t,x)\|_{\mathcal{H}}^p \, d\gamma \right)^{1/p} dt,$$

the same tightness argument used in the proof of Theorem 4.5 to pass from finitely many to infinitely many dimensions provides us with a $\boldsymbol{b}$-flow $\boldsymbol{\eta}$ satisfying (57): any weak limit point $\boldsymbol{\eta}$ of $\boldsymbol{\eta}_\varepsilon$ as $\varepsilon \downarrow 0$. □

6.2 Commutator estimate

This subsection is entirely devoted to the proof of the commutator estimate (37) in finite-dimensional Wiener spaces.

We will often use the "Gaussian rotations"

$$(x,y) \mapsto (z,w) := \big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}y, -\sqrt{1-e^{-2\varepsilon}}x + e^{-\varepsilon}y\big), \tag{65}$$

mapping the product measure $\gamma(dx)\times\gamma(dy)$ into $\gamma(dz)\times\gamma(dw)$. Indeed, the transformations above preserve the Lebesgue measure in $\mathbb{R}^N\times\mathbb{R}^N$ (being their Jacobian identically equal to 1) and $|x|^2+|y|^2=|z|^2+|w|^2$.

We now state two elementary Gaussian estimates. The first one

$$\left(\int |l\cdot w|^p\,d\gamma(w)\right)^{1/p} = |l|\left(\int |w_1|^p\,d\gamma(w)\right)^{1/p} = \Lambda(p)|l| \qquad \forall\, l\in\mathbb{R}^N, \tag{66}$$

with Λ depending only on p, is a simple consequence of the rotation invariance of γ.

Lemma 6.2. *Let $A:\mathbb{R}^N\to\mathbb{R}^N$ be a linear map and $c\in\mathbb{R}$. Then, if $q\le 2$, we have*

$$\left(\int \big|\langle Aw,w\rangle - c\big|^q\,d\gamma(w)\right)^{1/q} \le \sqrt{2}\|A^{\rm sym}\|_{HS} + |\mathrm{tr}\,A - c|. \tag{67}$$

Proof. Obviously we can assume that A is symmetric. By rotation invariance, we can also assume that A is diagonal, and denote by $\lambda_1,\ldots,\lambda_N$ its eigenvalues. We have then

$$\begin{aligned}
\int \Big|\sum_i \lambda_i (w^i)^2 - c\Big|^2 d\gamma(w) &= \int \Big[\sum_{ij}\lambda_i\lambda_j (w^i)^2(w^j)^2 - 2c\sum_i \lambda_i (w^i)^2 + c^2\Big]\,d\gamma(w)\\
&= 3\sum_i \lambda_i^2 + \sum_{i\neq j}\lambda_i\lambda_j - 2c\sum_i\lambda_i + c^2\\
&= 2\sum_i \lambda_i^2 + \sum_{ij}\lambda_i\lambda_j - 2c\sum_i\lambda_i + c^2\\
&= 2\sum_i \lambda_i^2 + \big(\sum_i \lambda_i - c\big)^2.
\end{aligned}$$

If $q=2$ we take the square roots of both sides and we conclude; if $q\le 2$ we apply the Hölder inequality. ☐

Henceforth, a vector field $\boldsymbol{c}\in L^p(\gamma;\mathbb{R}^N)\cap LD^q_{\mathcal{H}}(\gamma;\mathbb{R}^N)$ and a function $v\in L^r(\gamma)$ will be fixed, with $r=\max\{p',q'\}$ and $p>1$, $1\le q\le 2$. Our goal is to prove the estimate

$$\|r^\varepsilon\|_{L^1(\gamma)} \le \|v\|_{L^r(\gamma)}\left[\frac{\Lambda(p)\varepsilon}{\sqrt{1-e^{-2\varepsilon}}}\|\boldsymbol{c}\|_{L^p(\gamma;\mathbb{R}^N)} + 2^{1/q'}\|\mathrm{div}_\gamma\boldsymbol{c}\|_{L^q(\gamma)} + 2^{1/q'}\sqrt{2}\|\|(\nabla\boldsymbol{c})^{\rm sym}\|_{HS}\|_{L^q(\gamma)}\right], \tag{68}$$

where

$$r^\varepsilon := e^\varepsilon\boldsymbol{c}\cdot\nabla v_\varepsilon - T_\varepsilon(\mathrm{div}_\gamma(v\boldsymbol{c})). \tag{69}$$

Since $2^{1/q'} \le \sqrt{2}$, this yields the finite-dimensional version of (37).

In this setup the Ornstein-Uhlenbeck operator $v_\varepsilon := T_\varepsilon v$ takes the explicit form

$$v_\varepsilon(x) := \int v(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}y)\,d\gamma(y) = \int v(z)\rho_\varepsilon(x,z)\,d\gamma(z)$$

with

$$\begin{aligned}\rho_\varepsilon(x,z) &:= \frac{1}{(1-e^{-2\varepsilon})^{N/2}}\exp(-\frac{|e^{-\varepsilon}x-z|^2}{2(1-e^{-2\varepsilon})})\exp(\frac{|z|^2}{2}) \\ &= \frac{1}{(1-e^{-2\varepsilon})^{N/2}}\exp(-\frac{|e^{-\varepsilon}x|^2 - 2\varepsilon^{-\varepsilon}x\cdot z + |e^{-\varepsilon}z|^2}{2(1-e^{-2\varepsilon})}).\end{aligned}$$

This implies that

$$\begin{aligned}\nabla v_\varepsilon(x) &= \int v(z)\nabla_x\rho_\varepsilon(x,z)\,d\gamma(z) = -e^{-\varepsilon}\int \frac{e^{-\varepsilon}x-z}{1-e^{-2\varepsilon}}f(z)\rho_\varepsilon(x,z)\,d\gamma(z) \\ &= e^{-\varepsilon}\int v(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)\frac{y}{\sqrt{1-e^{-2\varepsilon}}}\,d\gamma(y). \qquad (70)\end{aligned}$$

Let us look for a more explicit expression of the commutator in (69). To this aim, we show first that $T_\varepsilon(\mathrm{div}_\gamma(v\boldsymbol{c}))$ is a function, and

$$T_\varepsilon(\mathrm{div}_\gamma(v\boldsymbol{c}))(x) = \int (v\boldsymbol{c})(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)\cdot\frac{y}{\sqrt{1-e^{-2\varepsilon}}}\,d\gamma(y) - T_\varepsilon(z\cdot v\boldsymbol{c})(x). \qquad (71)$$

If $\boldsymbol{c}$ and v are smooth, this is immediate to check: indeed, thanks to (14), we need only to show that

$$T_\varepsilon(\mathrm{div}\,(v\boldsymbol{c}))(x) = \int (v\boldsymbol{c})(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)\cdot\frac{y}{\sqrt{1-e^{-2\varepsilon}}}\,d\gamma(y).$$

The latter is a direct consequence of (70) (with v replaced by $v\boldsymbol{c}^i$) and of the relation $\partial_i T_\varepsilon(v\boldsymbol{c}^i) = e^{-\varepsilon}T_\varepsilon(\partial_i(v\boldsymbol{c}^i))$. If v and $\boldsymbol{c}$ are not smooth, we argue by approximation.

Therefore, taking (70) and (71) into account, we have that $r^\varepsilon(x)$ is given by

$$\begin{aligned}&\int v(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)\frac{\boldsymbol{c}(x)-\boldsymbol{c}(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)}{\sqrt{1-e^{-2\varepsilon}}}\cdot y\,d\gamma(y) \\ &+\int v(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)\boldsymbol{c}(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)\cdot(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)\,d\gamma(y) \\ &= \int \frac{v(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)}{\sqrt{1-e^{-2\varepsilon}}}\Big\{\boldsymbol{c}(x)\cdot y - \boldsymbol{c}(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)\cdot(e^{-2\varepsilon}y - e^{-\varepsilon}\sqrt{1-e^{-2\varepsilon}}x)\Big\}\,d\gamma(y).\end{aligned}$$

Now, using the abbreviations $\alpha_\varepsilon(x,y) := v(e^{-\varepsilon}x+\sqrt{1-e^{-2\varepsilon}}y)$, $\beta_\varepsilon := \varepsilon/\sqrt{1-e^{-2\varepsilon}}$, we interpolate and write $-r^\varepsilon(x)$ as

$$
\begin{aligned}
&\frac{1}{\sqrt{1-e^{-2\varepsilon}}}\int \alpha_\varepsilon(x,y)\frac{d}{dt}\int_0^1 \boldsymbol{c}(e^{-t\varepsilon}x+\sqrt{1-e^{-2\varepsilon t}}y)\cdot(e^{-2t\varepsilon}y-e^{-t\varepsilon}\sqrt{1-e^{-2t\varepsilon}}x)\,dt\,d\gamma(y)\\
&=\beta_\varepsilon\int \alpha_\varepsilon(x,y) \qquad (72)\\
&\int_0^1\Big[\sum_{ij}\Big(\partial_j\boldsymbol{c}^i(e^{-t\varepsilon}x+\sqrt{1-e^{-2t\varepsilon}}y)[e^{-t\varepsilon}\sqrt{1-e^{-2t\varepsilon}}x^i-e^{-2t\varepsilon}y^i][e^{-t\varepsilon}x^j-\frac{e^{-2t\varepsilon}}{\sqrt{1-e^{-2t\varepsilon}}}y^j]\Big)\\
&+\sum_i\Big(\boldsymbol{c}^i(e^{-t\varepsilon}x+\sqrt{1-e^{-2t\varepsilon}}y)[(e^{-t\varepsilon}\sqrt{1-e^{-2t\varepsilon}}-\frac{e^{-3t\varepsilon}}{\sqrt{1-e^{-2t\varepsilon}}})x^i-2e^{-2t\varepsilon}y^i]\Big)\Big]\,dt\,d\gamma(y)\\
&=:\beta_\varepsilon\int\alpha_\varepsilon(x,y)(A_\varepsilon(x,y)+B_\varepsilon(x,y))\,d\gamma(y), \qquad (73)
\end{aligned}
$$

where, adding and subtracting

$$
\sum_i\boldsymbol{c}^i(e^{-t\varepsilon}x+\sqrt{1-e^{-2t\varepsilon}}y)\frac{e^{-2t\varepsilon}}{\sqrt{1-e^{-2t\varepsilon}}}(e^{-t\varepsilon}x^i+\sqrt{1-e^{-2t\varepsilon}}y^i),
$$

we have set

$$
\begin{aligned}
A_\varepsilon(x,y):=\int_0^1\Big(\sum_{ij}\quad &\partial_j\boldsymbol{c}^i(e^{-t\varepsilon}x+\sqrt{1-e^{-2t\varepsilon}}y)[e^{-t\varepsilon}\sqrt{1-e^{-2t\varepsilon}}x^i-e^{-2t\varepsilon}y^i][e^{-t\varepsilon}x^j-\frac{e^{-2t\varepsilon}}{\sqrt{1-e^{-2t\varepsilon}}}y^j]\\
&-\sum_i\boldsymbol{c}^i(e^{-t\varepsilon}x+\sqrt{1-e^{-2t\varepsilon}}y)\frac{e^{-2t\varepsilon}}{\sqrt{1-e^{-2t\varepsilon}}}(e^{-t\varepsilon}x^i+\sqrt{1-e^{-2t\varepsilon}}y^i)\Big)\,dt,
\end{aligned}
$$

$$
B_\varepsilon(x,y):=\int_0^1\sum_i\Big(\boldsymbol{c}^i(e^{-t\varepsilon}x+\sqrt{1-e^{-2t\varepsilon}}y)e^{-t\varepsilon}[\sqrt{1-e^{-2t\varepsilon}}x^i-e^{-t\varepsilon}y^i]\Big)\,dt.
$$

Let us estimate $\beta_\varepsilon\int\int|\alpha_\varepsilon B_\varepsilon|\,d\gamma d\gamma$ first: the change of variables (65) and Fubini's theorem give

$$
\beta_\varepsilon\int\int|\alpha_\varepsilon B_\varepsilon|\,d\gamma(x)\,d\gamma(y)\le\beta_\varepsilon\int_0^1 e^{-\varepsilon t}\int\int|v(z)|\big|\sum_i\boldsymbol{c}^i(z)w^i\big|\,d\gamma(z)\,d\gamma(w)\,dt.
$$

Using (66) with $f=\boldsymbol{c}(z)$, we get

$$
\beta_\varepsilon\int\int|\alpha_\varepsilon B_\varepsilon|\,d\gamma(x)\,d\gamma(y)\le\beta_\varepsilon\int\int|v(z)|\Big|\sum_i\boldsymbol{c}^i(z)w^i\Big|\,d\gamma(z)\,d\gamma(w)\le\beta_\varepsilon\Lambda(p)\|\boldsymbol{c}\|_{L^p(\gamma;\mathbb{R}^N)}\|v\|_{L^{p'}(\gamma)}. \qquad (74)
$$

Now, we estimate $\beta_\varepsilon\int\int|\alpha_\varepsilon A_\varepsilon|\,d\gamma\,d\gamma$; again, we use the change of variables (65) to write

$$
e^{-t\varepsilon}\sqrt{1-e^{-2t\varepsilon}}x^i-e^{-2t\varepsilon}y^i=-e^{-t\varepsilon}w^i,\qquad e^{-t\varepsilon}x^j-\frac{e^{-2t\varepsilon}}{\sqrt{1-e^{-2t\varepsilon}}}y^j=-\frac{e^{-t\varepsilon}}{\sqrt{1-e^{-2t\varepsilon}}}w^j.
$$

Therefore we get

$$
\begin{aligned}
&\beta_\varepsilon \int\int |\alpha_\varepsilon A_\varepsilon| \, d\gamma(x)\, d\gamma(y) \\
&\leq \beta_\varepsilon \int_0^1 \int\int |v(z)| \Big| \sum_{ij} \partial_j \boldsymbol{c}^i(z) \frac{e^{-2t\varepsilon}}{\sqrt{1-e^{-2t\varepsilon}}} w^i w^j - \sum_i \boldsymbol{c}^i(z) \frac{e^{-2t\varepsilon}}{\sqrt{1-e^{-2t\varepsilon}}} z^i \Big| \, d\gamma(z)\, d\gamma(w)\, dt \\
&= \int\int |v(z)| \Big| \sum_{ij} \partial_j \boldsymbol{c}^i(z) w^i w^j - \sum_i \boldsymbol{c}^i(z) z^i \Big| \, d\gamma(z)\, d\gamma(w),
\end{aligned}
$$

where we used the identity

$$
\int_0^1 \frac{e^{-2t\varepsilon}}{\sqrt{1-e^{-2t\varepsilon}}} \, dt = \frac{\sqrt{1-e^{-2\varepsilon}}}{\varepsilon} = \beta_\varepsilon^{-1}.
$$

Eventually we use (67) with $A = \nabla \boldsymbol{c}(z)$ and $c = \boldsymbol{c}(z) \cdot z$ to obtain

$$
\begin{aligned}
\beta_\varepsilon \int\int |\alpha_\varepsilon A_\varepsilon| \, d\gamma(x)\, d\gamma(y) &\leq \|v\|_{L^{q'}(\gamma)} \Big(\int\int \big| \sum_{ij} \partial_j \boldsymbol{c}^i(z) w^i w^j - \sum_i \boldsymbol{c}^i(z) z^i \big|^q \, d\gamma(w)\, d\gamma(z) \Big)^{1/q} \\
&\leq 2^{1-1/q} \|v\|_{L^{q'}(\gamma)} \Big(\int \sqrt{2}^q \| \|(\nabla \boldsymbol{c})^{\rm sym}\|_{HS} \|^q + |\mathrm{div}_\gamma \boldsymbol{c}|^q \, d\gamma(z) \Big)^{1/q} \\
&\leq 2^{1-1/q} \|v\|_{L^{q'}(\gamma)} \Big(\sqrt{2} \| \|(\nabla \boldsymbol{c})^{\rm sym}\|_{HS} \|_{L^q(\gamma)} + \|\mathrm{div}_\gamma \boldsymbol{c}\|_{L^q(\gamma)} \Big). \qquad (75)
\end{aligned}
$$

Combining (72), (74) and (75), we have proved (68).

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