On the Lagrangian structure of transport equations: the Vlasov-Poisson system

Luigi Ambrosio [*] Maria Colombo [†] Alessio Figalli [‡]

Abstract

The Vlasov-Poisson system is an important non-linear transport equation, used to describe the evolution of particles under their self-consistent electric or gravitational field. The existence of classical solutions is limited to dimensions $d \leq 3$ under strong assumptions on the initial data, while weak solutions are known to exist under milder conditions. However, in the setting of weak solutions it is unclear whether the Eulerian description provided by the equation physically corresponds to a Lagrangian evolution of the particles. In this paper we develop several general tools concerning the Lagrangian structure of transport equations with non-smooth vector fields and we apply these results to show that weak/renormalized solutions of Vlasov-Poisson are Lagrangian, and actually that the concepts of renormalized and Lagrangian solutions are equivalent. As a corollary, we prove that finite energy solutions in dimension $d \leq 4$ are transported by a global flow (in particular, they preserve all the natural Casimir invariants), and we obtain global existence of weak solutions in any dimension under minimal assumptions on the initial data.

Contents

[*]Scuola Normale Superiore, Pisa. E-mail: luigi.ambrosio@sns.it

[†]ETH Zürich-ITS. E-mail: maria.colombo@eth-its.ethz.ch

[‡]ETH Zürich. E-mail: alessio.figalli@math.ethz.ch

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1 Introduction

The d-dimensional Vlasov-Poisson system describes the evolution of a nonnegative distribution function $f : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ according to Vlasov's equation, under the action of a self-consistent force determined by the Poisson's equation:

$$\begin{cases} \partial_t f_t + v \cdot \nabla_x f_t + E_t \cdot \nabla_v f_t = 0 & \text{in } (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \\ \rho_t(x) = \displaystyle\int_{\mathbb{R}^d} f_t(x,v)\,dv & \text{in } (0,\infty) \times \mathbb{R}^d \\ E_t(x) = \sigma\, c_d \displaystyle\int_{\mathbb{R}^d} \rho_t(y) \frac{x-y}{|x-y|^d}\,dy & \text{in } (0,\infty) \times \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here $f_t(x,v)$ stands for the density of particles having position x and velocity v at time t, $\rho_t(x)$ is the distribution of particles in the physical space, $E_t = \sigma\, \nabla(\Delta^{-1}\rho_t)$ is the force field, $c_d > 0$ is a dimensional constant chosen in such a way that $c_d \operatorname{div}\left(\frac{x}{|x|^d}\right) = \delta_0$, and $\sigma \in \{\pm 1\}$. The case $\sigma = 1$ corresponds to the case of electrostatic forces between charged particles with the same sign (repulsion) while $\sigma = -1$ corresponds to the gravitational case (attraction).

When $d = 3$, this system appears in several physical models. For instance, when $\sigma = 1$ it describes in plasma physics the evolution of charged particles under their self-consistent electric field, while when $\sigma = -1$ the same system is used in astrophysics to describe the motion of galaxy clusters under the gravitational field. Many different models have been developed in connection with the Vlasov-Poisson equation: amongst others, we mention the relativistic version of (1.1) (where the velocity of particles is given by $v/\sqrt{1+|v|^2}$) and the Vlasov-Maxwell system (which takes into account both the electric and magnetic fields of the Maxwell equations).

Regarding the existence of classical solutions, namely, solutions where all the relevant derivatives exist, the first contributions were given by Iordanskii [22] in dimension 1, by Ukai and Okabe [32] in dimension 2, and by Bardos and Degond [6] in dimension 3 for small data. For symmetric initial data, more existence results have been proven in [7, 33, 19, 31] (see also the presentation in [30] for an overview of the topic and the references quoted therein). Finally, in 1989 Pfaffelmoser [29] and Lions and Perthame [25] were able to prove global existence of classical solutions starting from a pretty general class of initial data. Moreover, in [25] the problem of uniqueness is also addressed: there the authors show uniqueness in the class of solutions with bounded space densities in $[0,\infty)\times\mathbb{R}^3$ by considering the Lagrangian flow associated to the vector field $\boldsymbol{b}_t(x,v) := (v, E_t(x))$ (see also [26] for a different proof based on stability in the Wasserstein metric).

The above mentioned results require strong integrability and moment conditions on the initial data (see also [18, 27], where further analysis of the propagation of moments for the Vlasov-Poisson system is carried out), and it would be very desirable to get global existence of solutions under more physical assumptions (for instance, assuming only finite total energy). In the classical paper [5], Arsen'ev proved global existence of weak solutions under the hypothesis that the initial datum is bounded and has finite kinetic energy (see also [21]). This result has then been improved in [20], where the authors relaxed the boundedness assumption to an L^p bound for some suitable $p > 1$.

Notice that these higher integrability hypotheses are needed even to give a meaning to the equation in the distributional sense: indeed, when f_t is merely L^1 the product $E_t f_t$ does not belong to L^1_{loc}. To overcome this difficulty, in [14, 15] the authors considered the concept of renormalized solutions and obtained global existence in the case $\sigma = 1$ under the assumption that the total energy is finite and $f_0 \log(1 + f_0) \in L^1$ (in the case $\sigma = -1$ they still need some L^p assumption on f). Also, under some suitable integrability assumptions on f_t, they can show that the concepts of weak and renormalized solutions are equivalent.

It is important to observe that the Vlasov-Poisson system has a transport structure which allows one to prove that, when the solutions is sufficiently smooth, f_t is transported along the characteristics of the vector field $\boldsymbol{b}_t(x,v) := (v, E_t(x))$. However, when dealing with weak or renormalized solutions, it is not clear whether such a vector field defines a flow on the phase-space, and one loses the relation between the Eulerian and Lagrangian picture.

The main goal of this paper is to show that the Lagrangian picture is still valid even for weak/renormalized solutions (Theorem 2.2), and that the concepts of renormalized and Lagrangian solutions are equivalent. Under such generality, we need to employ a suitable notion of Lagrangian solution (see the precise Definition 4.6 below) that allows for blow up in finite time of the trajectories. More precisely, the idea is that particles evolve along integral curves of the vector field $\boldsymbol{b}_t$, and they can escape to infinity and/or appear from infinity in finite time. However, under some suitable assumptions on the initial data, we can prevent such a finite-time blow-up phenomenon. These results have several interesting consequences.

First of all, in Theorem 2.3 we show that, in the repulsive case with $d = 3, 4$, renormalized solutions with finite energy are transported by a global flow. In particular, they preserve all the natural Casimir invariants.

A second important consequence of our Theorem 2.2 is the proof of the global existence of renormalized/Lagrangian solutions under minimal assumptions on the initial data (Theorems 2.7 and 2.8, and Remark 2.9). Indeed, the proof of Theorem 2.7 is based on a standard approximation argument where one regularizes the Poisson kernel to build approximate solutions, and then one tries to obtain a solution by some limiting procedure. Usually, in all proofs of this kind, this passage to the limit is an extremely delicate step, but our Theorem 2.2 allows us to perform this limiting step in a rather simple way (see Step 5 in the proof of Theorem 2.7).

To prove Theorem 2.2 we rely on the following tools, which we believe have their own interest:

(i) the local version of the DiPerna-Lions theory developed in [2];

(ii) the uniqueness of bounded compactly supported solutions to the continuity equation for a special class of vector fields obtained by convolving a singular kernel with a measure (this is based on the techniques developed in [10, 11, 8], see Section 4.2);

(iii) the fact that the concept of Lagrangian solution is equivalent to the one of renormalized solution (see Sections 4.4 and 5);

(iv) a general superposition principle stating that every nonnegative solution of the continuity equation has a Lagrangian structure without any regularity or growth assumption on the vector field (see Section 5).

While points *(i)* and *(ii)* above are essentially contained in [2, 10, 11, 8] (although some modifications/improvements are needed), points *(iii)* and *(iv)* are completely new, both in terms of ideas and techniques. In particular, the validity of a superposition principle without any growth assumption on the vector field was a long standing open problem in the theory of transport equations with rough vector fields, and the lack of this tool prevented the application of this theory to many equations from physics. Here we completely solve this issue and, although in this paper we focus only on the Vlasov-Poisson system, it is important to point out that our results are completely general.

All the machinery from *(i)*-*(iv)* is needed to prove a general result on the renormalization property for solutions of transport equations which is crucial in our proof. However, from a PDE viewpoint this renormalization property is all we shall need, so in order to keep the presentation as much as possible independent of this heavy machinery we shall organize the paper as follows: in the next section we state our results keeping the presentation on the Lagrangian structure of solutions at an informal level. Then in Sections 3.1 and 3.2 we prove our PDE results without introducing the tools mentioned above but simply using the consequences of them, and we postpone points *(i)*-*(iv)* above to Sections 4 and 5.

We mention that the authors of [9] obtained global existence of Lagrangian solutions in dimension $d \leq 3$ assuming only finite energy of the initial datum (cf. Theorem 2.8 below), with a different argument from ours.

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2 Statement of the results

As already observed in the introduction, the Vlasov-Poisson system has a transport structure: indeed we can rewrite it as

$$\partial_t f_t + \boldsymbol{b}_t \cdot \nabla_{x,v} f_t = 0, \tag{2.1}$$

where the vector field $\boldsymbol{b}_t(x,v) = (v, E_t(x)) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is divergence-free, and is coupled to f_t via the relation $E_t = \sigma\, c_d\, \rho_t * \frac{x}{|x|^d}$. Recalling that $c_d \operatorname{div}\left(\frac{x}{|x|^d}\right) = \delta_0$, the vector field E_t can also be found as $E_t = -\nabla_x V_t$ where the potential $V_t : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ solves[1]

$$-\Delta V_t = \sigma\, \rho_t \quad \text{in } \mathbb{R}^d, \qquad \lim_{|x|\to\infty} V_t(x) = 0. \tag{2.2}$$

Notice that, because the kernel $\frac{x}{|x|^d}$ is locally integrable, the convolution $\frac{x}{|x|^d} * \mu$ makes sense (in the sense of distributions) for any finite measure μ. Actually, since $\rho_t \in L^1(\mathbb{R}^d)$ and $\frac{x}{|x|^d} \in L^p_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ for any $p < d/(d-1)$, the electric field E_t belongs to $L^p_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, therefore $\boldsymbol{b}_t \in L^p_{\rm loc}(\mathbb{R}^{2d};\mathbb{R}^{2d})$ (see also Step 4 of the proof of Theorem 2.7 for more details).

Now, since $\boldsymbol{b}_t$ is divergence-free, the above equation can be rewritten as

$$\partial_t f_t + \operatorname{div}_{x,v}(\boldsymbol{b}_t f_t) = 0,$$

and the equation can be reinterpreted in the distributional sense provided the product $\boldsymbol{b}_t f_t$ belongs to $L^1_{\rm loc}$. However, as mentioned in the introduction, this is not true if f_t is merely L^1. To overcome this difficulty one notices that if f_t is a smooth solution of (2.1) then also $\beta(f_t)$ is a solution for all C^1 functions $\beta : \mathbb{R} \to \mathbb{R}$; indeed

$$\partial_t \beta(f_t) + \boldsymbol{b}_t \cdot \nabla_{x,v} \beta(f_t) = \left[\partial_t f_t + \boldsymbol{b}_t \cdot \nabla_{x,v} f_t\right] \beta'(f_t) = 0,$$

or equivalently (since $\operatorname{div}_{x,v}(\boldsymbol{b}_t) = 0$)

$$\partial_t \beta(f_t) + \operatorname{div}_{x,v}(\boldsymbol{b}_t \beta(f_t)) = 0. \tag{2.3}$$

Notice that, since β is bounded by assumption, $\beta(f_t) \in L^\infty$ so $\boldsymbol{b}_t \beta(f_t) \in L^1_{\rm loc}$ whenever $\boldsymbol{b}_t \in L^1_{\rm loc}$, and (2.3) is well defined in the sense of distribution. This motivates the introduction of the concept of renormalized solution [14]:

Definition 2.1 (Renormalized solutions)**.** *Let $\boldsymbol{b} \in L^1_{\rm loc}([0,T] \times \mathbb{R}^{2d};\mathbb{R}^{2d})$ be a Borel vector field. A Borel function $f \in L^1_{\rm loc}([0,T] \times \mathbb{R}^{2d})$ is a* renormalized solution *of* (2.1) *(starting from f_0) if* (2.3) *holds in the sense of distributions for every $\beta \in C^1 \cap L^\infty(\mathbb{R})$, namely, for every $\phi \in C^1_c([0,T) \times \mathbb{R}^{2d})$,*

$$\int_{\mathbb{R}^{2d}} \phi_0(x,v)\, \beta(f_0(x,v))\, dx\, dv + \int_0^T \int_{\mathbb{R}^{2d}} \left[\partial_t \phi_t(x,v) + \nabla_{x,v}\phi_t(x,v) \cdot \boldsymbol{b}_t(x,v)\right] \beta(f_t(x,v))\, dx\, dv\, dt = 0. \tag{2.4}$$

[1]This description is correct in dimension $d \geq 3$ since the fundamental solution of the Laplacian decays at infinity, while in dimension 2 the function V_t is given by the convolution of ρ_t with $-\frac{1}{2\pi}\log|x|$, see also Remark 2.10.

In the case of the Vlasov-Poisson system, a function $f \in L^\infty((0,T); L^1(\mathbb{R}^{2d}))$ is a renormalized solution of (1.1) *(starting from f_0) if, setting*

$$\rho_t(x) := \int_{\mathbb{R}^d} f_t(x,v)\,dv, \quad E_t(x) := \sigma\, c_d \int_{\mathbb{R}^d} \rho_t(y)\,\frac{x-y}{|x-y|^d}\,dy, \quad \boldsymbol{b}_t(x,v) := (v, E_t(x)), \tag{2.5}$$

the function f_t solves (2.4) *with the vector field $\boldsymbol{b}_t$ given by* (2.5)*.*

Notice that, in the case of the Vlasov-Poisson system, the global integrability of f_t is needed to make sense of ρ_t and E_t.

This definition takes care of the integrability of the term $E_t f_t$ appearing in the equation. However a second problem comes when dealing with weak solutions: the vector field $\boldsymbol{b}_t$ is not Lipschitz in general, so one cannot use the standard Cauchy-Lipschitz theory to construct a flow for such a vector field. In the seminal paper [16], DiPerna and Lions showed that, even for Sobolev vector fields, one can introduce a suitable notion of flow (this result has then been extended in several directions, see for instance [1, 13, 11]). However this theory requires the a priori assumption that the trajectories of the flow do not blow up in finite time, which is expressed in terms of the vector field by the following global hypothesis:

$$\frac{|\boldsymbol{b}_t|(x,v)}{1+|x|+|v|} \in L^1\big((0,T); L^1(\mathbb{R}^{2d})\big) + L^1\big((0,T); L^\infty(\mathbb{R}^{2d})\big).$$

We notice that for Vlasov-Poisson (or more in general for any Hamiltonian system where $\boldsymbol{b}_t(x,v)$ is of the form $(v, E_t(x))$) the above assumption is equivalent to

$$E_t = E_t^1 + E_t^2, \quad \text{with } |E_t^1| \in L^1\big((0,T); L^d(\mathbb{R}^d)\big) \text{ and } \frac{|E_t^2|(x)}{1+|x|} \in L^1\big((0,T); L^\infty(\mathbb{R}^d)\big)$$

(see Lemma A.1 in the appendix). This splitting result is reminiscent of the one used in [9] to prove global existence of a flow in dimension three. Unfortunately, in general the above assumption is rather restrictive, as it requires both some integrability and moment (in v) conditions on f_t, so one cannot apply the classical DiPerna-Lions' theory in our context (notice also that the weaker growth conditions considered in [24] in connection with the linearization of the flow do not seem to apply to the Hamiltonian setting).

In our recent paper [2] we developed a local version of the DiPerna-Lions' theory under no global assumptions on the vector field, and this will be a crucial tool for us to give a Lagrangian description of solutions. More precisely, in Theorem 5.1 we shall first prove that every bounded nonnegative solution of a continuity equation can be always represented as a superposition of mass transported along integral curves of the vector field (notice that a priori these curves may split/intersect). Then, by a modification of the argument in [8] we shall prove that for any vector field of the form $(v, \mu_t * x/|x|^d)$, with μ_t a time-dependent measure, there is uniqueness of bounded compactly supported solutions of the continuity equation (see Theorem 4.4). Finally, combining these facts with the theory from [2], we can show that all bounded/renormalized solutions of Vlasov-Poisson are Lagrangian.

As mentioned before, to express the fact that solutions are Lagrangian we shall need to introduce the concept of Maximal Regular Flow. Roughly speaking this is a (uniquely defined) incompressible flow on the phase-space composed of integral curves of $\boldsymbol{b}_t$ that "transport" the density f_t (notice that, since trajectories may blow-up in finite time, mass of f_t can disappear at infinity and/or come from infinity, but it has to follow the integral curves of $\boldsymbol{b}_t$). However, since the definition is rather technical, in order to keep the presentation simpler we shall not introduce now the concepts of Maximal Regular Flow and of Lagrangian solutions, but postpone them to Section 4. This will leave the general reader with the intuitive concept of what is going on, and only the interested readers may decide to enter into the details of the definition and the proofs.

Our first main result shows that bounded or renormalized solutions of Vlasov-Poisson are Lagrangian. As shown in Theorem 4.10, the concept of Lagrangian solutions is a priori stronger than the one of renormalized solutions as all Lagrangian solutions of Vlasov-Poisson are renormalized, but thanks to our general superposition principle (Theorem 5.1) we can prove that the two concepts are actually equivalent.

Here and in the sequel we shall use the notation L^1_+ to denote the space of nonnegative integrable functions. Also, by weakly continuous solutions we shall always mean that the map $t \mapsto \int_{\mathbb{R}^{2d}} f_t \, \varphi \, dx \, dv$ is continuous for any $\varphi \in C_c(\mathbb{R}^{2d})$.

Theorem 2.2. *Let $T > 0$, $f_0 \in L^1_+(\mathbb{R}^{2d})$ and $f_t \in L^\infty((0,T); L^1_+(\mathbb{R}^{2d}))$ be a weakly continuous function on $[0,T)$. Assume that:*
(i) either $f_t \in L^\infty((0,T); L^\infty(\mathbb{R}^{2d}))$ and f_t is a distributional solution of the Vlasov-Poisson equation (1.1) *starting from f_0;*
(ii) or f_t is a renormalized solution of the Vlasov-Poisson equation (1.1) *starting from f_0' (according to Definition 2.1).*
Then f_t is a Lagrangian solution transported by the Maximal Regular Flow $\boldsymbol{X}(t,x)$ associated to $\boldsymbol{b}_t(x,v) = (v, E_t(x))$ starting from 0, according to Definition 4.6. In particular f_t is renormalized.

The next result provides conditions in dimension $d = 3, 4$ in order to avoid the blow up in finite time of the flow that transports f_t. As we shall explain in Remark 2.10, the case $d = 2$ is slightly different from $d \geq 3$ because of the slower decay at infinity of the kernel $x/|x|^d$. For this reason we restrict the next statements to the case $d \geq 3$, while in Remark 2.10 we mention a possible way to deal with the case $d = 2$.

Theorem 2.3. *Let $d = 3, 4$, fix $T > 0$, and let $f_t \in L^\infty((0,T); L^1_+(\mathbb{R}^{2d}))$ be a renormalized solution of the Vlasov-Poisson equation* (1.1) *(according to Definition 2.1). Assume that both the kinetic energy and the potential energy are integrable in time, that is*

$$\int_0^T \int_{\mathbb{R}^{2d}} |v|^2 f_t(x,v) \, dx \, dv \, dt + \int_0^T \int_{\mathbb{R}^d} |E_t(x)|^2 \, dx \, dt < \infty, \tag{2.6}$$

Then:

(i) the Maximal Regular Flow $\boldsymbol{X}(t,\cdot)$ associated to $\boldsymbol{b}_t = (v, E_t)$ and starting from 0 is globally defined on $[0,T]$ for f_0-a.e. (x,v), that is,

for f_0-a.e. (x,v), the trajectory $t \mapsto \boldsymbol{X}(t,x,v) \in \mathbb{R}^{2d}$ does not blow-up on $[0,T]$;

(ii) *f_t is the image of f_0 through this flow, that is $f_t = \boldsymbol{X}(t,\cdot)_\# f_0 = f_0 \circ \boldsymbol{X}(t,\cdot)^{-1}$, for all $t \in [0,T]$:*

$$\int_{\mathbb{R}^{2d}} \phi(x,v) f_t(x,v)\, dx\, dv = \int_{\mathbb{R}^{2d}} \phi(\boldsymbol{X}(t,x,v)) f_0(x,v)\, dx\, dv \qquad \forall\, \phi \geq 0,\ \forall\, t \in [0,T].$$

(iii) *the map*

$$[0,T] \ni t \mapsto \int_{\mathbb{R}^{2d}} \psi\big(f_t(x,v)\big)\, dx\, dv$$

is constant in time for all $\psi : [0,\infty) \to [0,\infty)$ Borel.

Remark 2.4. *By uniqueness of the flow, one can prove that the semigroup property*

$$\boldsymbol{X}(s,\cdot) \circ \boldsymbol{X}(t,\cdot)^{-1} = [\boldsymbol{X}(t,s,\cdot)]^{-1} \qquad \forall\, 0 \leq s \leq t \leq T$$

holds, where $t \mapsto \boldsymbol{X}(t,s,\cdot)$ is the flow of $\boldsymbol{b}$ starting at time s, namely $\boldsymbol{X}(s,s,\cdot) = \mathrm{id}$ (see Section 4 for more details). Using the fact that f_t is the image of f_0 through the flow, and by the semigroup property just mentioned, it follows that

$$f_t = \boldsymbol{X}(t,\cdot)_\#(f_s \circ \boldsymbol{X}(s)) = \boldsymbol{X}(t,s,\cdot)_\#(f_s) \qquad \forall\, 0 \leq s \leq t \leq T.$$

In other words, f_t can be reconstructed from any f_s using the flow.

Remark 2.5. In dimension $d \geq 3$, as can be formally seen performing an integration by parts, the quantity

$$\int_{\mathbb{R}^{2d}} |v|^2 f_t(x,v)\, dx\, dv + \sigma \int_{\mathbb{R}^d} |E_t(x)|^2\, dx$$

coincides with the total energy of the system (i.e., the sum of the kinetic and potential energy), namely

$$\int_{\mathbb{R}^{2d}} |v|^2 f_t\, dx\, dv + \sigma \int_{\mathbb{R}^d} H * \rho_t\, \rho_t\, dx, \qquad H(x) := \frac{c_d}{d-2} |x|^{2-d},$$

see (3.23) and Lemma 3.4. This quantity is formally conserved in time along solutions of the Vlasov-Poisson system; whether this property holds also for distributional/renormalized solutions is an important open problem in the theory. However, since weak solutions are usually built by approximation, a lower semicontinuity argument shows that the energy at time t is controlled from above by the initial energy. Hence, when $\sigma = 1$ the validity of (2.6) is often guaranteed by the assumption on the initial datum

$$\int_{\mathbb{R}^{2d}} |v|^2 f_0\, dx\, dv + \int_{\mathbb{R}^d} H * \rho_0\, \rho_0\, dx < \infty,$$

see Theorem 2.8 and Remark 2.10 below.

Notice that, in the case $\sigma = -1$, a bound on the total energy does not provide in general a control on both the kinetic energy and the potential energy. Still, one can prove the validity of (2.6) under some additional integrability assumptions on f_0 (see Remark 2.9).

Our second result deals with existence of global Lagrangian solutions under minimal assumptions on the initial data. In this case the sign of σ (i.e., whether the potential is attractive or repulsive) plays a crucial role, since in the repulsive case the total energy controls the kinetic part, while in the attractive case the loss of an a priori bound of the kinetic energy prevents us for showing such a result unless we impose some higher integrability on the initial data (see Remark 2.9). Still, we can state a general existence theorem that holds both in the attractive and repulsive case, and then show that in the repulsive case it gives us what we want.

The basic idea behind our general existence result is the following: when proving existence of solutions by approximation it may happen that, in the approximating sequence, there are some particles that move at higher and higher speed while still remaining localized in a compact set in space (think of a family of particle rotating faster and faster along circles around the origin). Then, while in the limit these particles will disappear from the phase-space (having infinite velocity), the electric field generated by them will survive, since they are still in the physical space. Hence the electric field is not anymore generated by the marginal of f_t in the v-variable, instead it is generated by an "effective density" $\rho_t^{\mathrm{eff}}(x)$ that may be larger than $\rho_t(x)$.

So, our strategy will be first to prove global existence of Lagrangian (hence renormalized) solutions for a generalized Vlasov-Poisson system where the electric field is generated by ρ_t^{eff}. Then, in the particular case $\sigma = 1$, we show that if the initial datum has finite total energy then $\rho_t^{\mathrm{eff}} = \rho_t$ and our solution solves the classical Vlasov-Poisson system.

We begin by introducing the concept of generalized solutions to Vlasov-Poisson. We use the notation $\mathscr{M}_+$ to denote the space of nonnegative measures with finite total mass.

Definition 2.6 (Generalized solution of the Vlasov-Poisson equation)**.** *Given $\overline{f} \in L^1(\mathbb{R}^{2d})$, let $f_t \in L^\infty((0,\infty); L^1_+(\mathbb{R}^{2d}))$ and $\rho_t^{\mathrm{eff}} \in L^\infty((0,\infty); \mathscr{M}_+(\mathbb{R}^d))$. We say that the couple $(f_t, \rho_t^{\mathrm{eff}})$ is a (global in time) generalized solution of the Vlasov-Poisson system starting from $\overline{f}$ if, setting*

$$\rho_t(x) := \int_{\mathbb{R}^d} f_t(x,v)\,dv, \quad E_t^{\mathrm{eff}}(x) := \sigma\, c_d \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d}\,d\rho_t^{\mathrm{eff}}(y), \quad \boldsymbol{b}_t(x,v) := (v, E_t^{\mathrm{eff}}(x)), \tag{2.7}$$

then the following holds: f_t is a renormalized solution of the continuity equation with vector field $\boldsymbol{b}_t$ starting from $\overline{f}$,

$$\rho_t \le \rho_t^{\mathrm{eff}} \qquad \text{as measures for a.e. } t \in (0,\infty), \tag{2.8}$$

and

$$|\rho_t^{\mathrm{eff}}|(\mathbb{R}^d) \le \|f_0\|_{L^1(\mathbb{R}^{2d})} \qquad \text{for a.e. } t \in (0,\infty). \tag{2.9}$$

As in the case of classical solutions, also for generalized solutions the formula (2.7) defining E_t^{eff} should be understood in the sense of distributions and provides a vector field in L^p_{loc} for any $p < d/(d-1)$. Notice that, since $\|\rho_t\|_{L^1(\mathbb{R}^d)} = \|f_t\|_{L^1(\mathbb{R}^{2d})}$, it follows by (2.8) and (2.9) that whenever the mass of f_t is conserved in time, that is $\|f_t\|_{L^1(\mathbb{R}^{2d})} = \|f_0\|_{L^1(\mathbb{R}^{2d})}$ for a.e. $t \in (0,\infty)$, then $\rho_t^{\mathrm{eff}} = \rho_t$ and generalized solutions of the Vlasov-Poisson system are just standard renormalized solutions. The notion of generalized solution suggests that the only way of possible failure for the existence of renormalized solutions of the Vlasov-Poisson system is by losing mass at large velocities; this phenomenon is well-known in the analysis of kinetic equations.

We prove here that generalized solutions of the Vlasov-Poisson equation exist globally for any L^1 initial datum, both in the attractive and in the repulsive case.

Theorem 2.7. *Let $d \geq 3$, and consider $f_0 \in L^1_+(\mathbb{R}^{2d})$. Then there exists a generalized solution $(f_t, \rho_t^{\text{eff}})$ of the Vlasov-Poisson system starting from f_0. In addition, the map*

$$[0, \infty) \ni t \mapsto f_t \in L^1_{\text{loc}}(\mathbb{R}^{2d})$$

is continuous, and the solution f_t is transported by the Maximal Regular Flow associated to $\boldsymbol{b}_t(x, v) = (v, E_t^{\text{eff}}(x))$.

As observed before, if $\rho_t^{\text{eff}} = \rho_t$ then f_t is a renormalized solution of the Vlasov-Poisson system. When $\sigma = 1$ (i.e., in the repulsive case) the equality $\rho^{\text{eff}} = \rho_t$ is satisfied in many cases of interest, for instance whenever the total initial energy is finite (see Theorem 2.8 below), or in the case of infinite energy if other weaker conditions are satisfied as it happens in the context of [34] and [25] (see Remark 3.6).

The following result improves the one announced in [14], generalizing that statement to any dimension and under weaker conditions on the initial data. In the first part of the statement, we show the global-in-time existence and some natural properties of renormalized solutions of the Vlasov-Poisson system in the repulsive case under only the finite-energy assumption on the initial datum. In the last part of the statement, we show that the trajectories of the Regular Lagrangian Flow associated to the renormalized solution starting from $t = 0$ cannot blow up if $d = 3, 4$.

Theorem 2.8. *Let $d \geq 3$, and let $f_0 \in L^1_+(\mathbb{R}^{2d})$ satisfy*

$$\int_{\mathbb{R}^{2d}} |v|^2 f_0 \, dx \, dv + \int_{\mathbb{R}^d} H * \rho_0 \, \rho_0 \, dx < \infty, \qquad H(x) := \frac{c_d}{d-2} |x|^{2-d}.$$

Assume that $\sigma = 1$. Then there exists a global Lagrangian (hence renormalized) solution $f_t \in C([0, \infty); L^1_{\text{loc}}(\mathbb{R}^{2d}))$ of the Vlasov-Poisson system (1.1) with initial datum f_0. Moreover, the following properties hold:

(i) the density ρ_t and the electric field E_t are strongly continuous in $L^1_{\text{loc}}(\mathbb{R}^d)$;

(ii) for every $t \geq 0$, we have the energy bound

$$\int_{\mathbb{R}^{2d}} |v|^2 f_t \, dx \, dv + \int_{\mathbb{R}^d} H * \rho_t \, \rho_t \, dx \leq \int_{\mathbb{R}^{2d}} |v|^2 f_0 \, dx \, dv + \int_{\mathbb{R}^d} H * \rho_0 \, \rho_0 \, dx; \qquad (2.10)$$

(iii) if $d = 3, 4$ then the flow is globally defined on $[0, \infty)$ for f_0-a.e. $(x, v) \in \mathbb{R}^{2d}$ (i.e., trajectories do not blow-up in finite time) and f_t is the image of f_0 through an incompressible flow.

Remark 2.9. When $d = 3$ (resp. $d = 4$), the above result can be generalized to the attractive case $\sigma = -1$ under the additional assumption $f_0 \in L^{9/7}(\mathbb{R}^6)$ (resp. $f_0 \in L^2(\mathbb{R}^8)$ with small L^2-norm). Indeed, in dimension 4 the solution $V_t = H * \rho_t$ of the equation $\Delta V_t = -\rho_t$ satisfies, by Calderón-Zygmund estimates and the Sobolev embedding,

$$\|V_t\|_{L^4(\mathbb{R}^3)} \leq C\|D^2 V_t\|_{L^{4/3}(\mathbb{R}^3)} \leq C\|\rho_t\|_{L^{4/3}(\mathbb{R}^3)}.$$

Thanks to this fact, Hölder inequality, and a classical interpolation lemma (see [14] or [12, Lemma 8.15]) applied with $\alpha = 0$, $q = 9/7$, $p_0 = 6/5$, we estimate

$$\begin{aligned}\frac{1}{4\pi}\int_{\mathbb{R}^4} H * \rho_t\, \rho_t\, dx &\leq C\|H * \rho_t\|_{L^6(\mathbb{R}^4)}\|\rho_t\|_{L^{4/3}(\mathbb{R}^4)} \leq C\|\rho_t\|^2_{L^{4/3}(\mathbb{R}^4)} \\ &\leq C\|f_t\|_{L^2(\mathbb{R}^8)}\Big(\int_{\mathbb{R}^8} |v|^2 f_t\, dx\, dv\Big).\end{aligned} \tag{2.11}$$

where C is a universal constant. If the total energy is bounded by a constant C_0, we deduce that

$$\begin{aligned}C_0 &\geq \int_{\mathbb{R}^8} |v|^2 f_t\, dx\, dv - \frac{1}{4\pi}\int_{\mathbb{R}^4} H * \rho_t\, \rho_t\, dx \\ &\geq \int_{\mathbb{R}^8} |v|^2 f_t(x, v)\, dx\, dv - C \sup_{t\in[0,\infty)} \|f_t\|_{L^2(\mathbb{R}^8)}\Big(\int_{\mathbb{R}^8} |v|^2 f_t\, dx\, dv\Big).\end{aligned}$$

Since $\sup_{t\in[0,\infty)} \|f_t\|_{L^2(\mathbb{R}^8)} \leq \|f_0\|_{L^2(\mathbb{R}^8)}$ (at least when the solution is built by approximation), we deduce that, provided $\|f_0\|_{L^2(\mathbb{R}^8)} < 1/C$, we have a control of the kinetic energy, and therefore of the full energy, thanks to (2.11), uniformly in time. Then Theorem 2.3 can be applied. *Note that the smallness assumption on the L^2 norm of f_0 is necessary (see [20, Part II, Corollary 5.3]). Also, a similar argument to the one described above works in dimension* $d = 3$ *(see also [14] or [12, Remark 8.5]).*

Remark 2.10. In dimension $d = 2$, even with an initial datum $f_0 \in C_c^\infty(\mathbb{R}^d)$, the electric field E_0 cannot belong to L^2 (this is due to the fact that the kernel $x/|x|^d$ does not belong to L^2 at infinity). Moreover, the potential energy does not have a definite sign. For these reasons one needs to slightly modify the equation adding a fixed background density ρ_b satisfying

$$\int_{\mathbb{R}^d} \rho_b(x)\, dx = \int_{\mathbb{R}^d} \rho_0(x)\, dx,$$

giving rise to the following system:

$$\begin{cases} \partial_t f_t + v \cdot \nabla_x f_t + E_t \cdot \nabla_v f_t = 0 & \text{in } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \\ \rho_t(x) = \displaystyle\int_{\mathbb{R}^d} f_t(x, v)\, dv & \text{in } (0, \infty) \times \mathbb{R}^d \\ E_t(x) = \sigma\, c_d \displaystyle\int_{\mathbb{R}^d} \big(\rho_t(y) - \rho_b(y)\big)\, \frac{x - y}{|x - y|^d}\, dy & \text{in } (0, \infty) \times \mathbb{R}^d, \end{cases} \tag{2.12}$$

The presence of ρ_b allows for cancellations in the expression for the L^2 norm of E_0, which turns out to be finite if ρ_b and ρ_0 are sufficiently nice. In this setting, when $\sigma = 1$ one can show that an analogous statement to Theorem 2.8 holds also for $d = 2$. On the other hand, when $\sigma = -1$ one needs to assume that $f_0 \in L \log L(\mathbb{R}^4)$ (compare with Remark 2.9 above).

Remark 2.11. In this paper we restricted ourselves to the Vlasov-Poisson system, but the argument and techniques introduced here generalize to other equations. For instance, a minor modification of our proofs allows one to obtain the same results in the context of the relativistic Vlasov-Poisson system.

The proofs of Theorems 2.2, 2.3, 2.7, and 2.8 are given in the next section. The reader familiar with the Vlasov-Poisson literature may note that some techniques, such as the smoothing of the initial data and of the kernel, are classical. However, many new ideas are introduced in the following; indeed, besides the results of Section 4 and 5 (which are a novelty of this paper) also many technical bounds in Section 3 are nontrivial because of the low-regularity setting. In addition, new techniques are introduced, such as the decomposition on level sets both of the approximating sequence and of the limit.

3 Vlasov-Poisson: Lagrangian solutions and global existence

3.1 The flow associated to Vlasov-Poisson: proof of Theorems 2.2 and 2.3

In this section we collect the proofs of Theorems 2.2, 2.3, 2.7, and 2.8. Note that, as also explained in the introduction, to keep the exposition as much as possible independent of the machinery developed in Sections 4 and 5, we do not introduce here the exact definitions of Maximal Regular Flow and of Lagrangian solution. Instead, to help the reader better understanding the underlying techniques, we just give a very rough and informal presentation of the results from Sections 4 and 5 that we use here. Note that the notation in Sections 4 and 5 is different from the one here, as there the results are stated for general transport equations in $\mathbb{R}^d$. For this reason, we summarize here the results from the next sections only in the particular case that is of interest to us, namely $\boldsymbol{b}_t = (v, E_t(x))$.

First of all, the basic conditions that our vector field need to satisfy are assumptions **(A1)** and **(A2)** of Section 4.1, namely $\boldsymbol{b}_t$ is locally integrable and the continuity equation associated to $\boldsymbol{b}_t$, namely the Cauchy problem for

$$\frac{d}{dt} g_t + \mathrm{div}_{x,v}(\boldsymbol{b}_t g_t) = 0, \qquad \boldsymbol{b}_t = (v, E_t(x)), \tag{3.1}$$

has forward uniqueness in any domain $[a,b] \times \mathbb{R}^d$ in the class of bounded and compactly supported solutions (and boundary data either at $t = a$, or at $t = b$). As proved in Theorem 4.4, because of its special structure, $\boldsymbol{b}_t$ satisfies these properties independently of the density $\rho_t = \int f_t \, dv$ that generates the electric field.

Secondly, Theorem 5.1 shows that, in our situation, every bounded (or renormalized) solution of (3.1) is Lagrangian: informally speaking, this means that the mass of g_t moves along integral curves of the vector-field $\boldsymbol{b}_t$. In particular, because all Lagrangian solutions are renormalized (see Theorem 4.10), this proves that all bounded solutions of the Vlasov-Poisson equation (or, more in general, all bounded densities g_t solving (3.1)) are renormalized.

Finally, one of the consequences of Proposition 4.11 is that, for any g_t lagrangian solution of (3.1), the further global condition

$$\int_0^T \int_{\mathbb{R}^{2d}} \frac{|\boldsymbol{b}_t|(x,v)}{(1+|(x,v)|)\log(2+|(x,v)|)} g_t(x,v)\,dx\,dv\,dt < \infty.$$

allows to deduce that the trajectories along which g_t is transported do not blow up in finite time. As a consequence, we can reconstruct the value of g_t at any time $t>0$ from g_0 and through the formula

$$g_t = g_0 \circ \boldsymbol{X}(t,\cdot)^{-1} \qquad g_t\text{-a.e. in } \mathbb{R}^{2d}.$$

After this informal presentation, we now proceed with the proofs of Theorems 2.2, 2.3, 2.7, and 2.8.

Proof of Theorem 2.2. Notice that the vector field $\boldsymbol{b}$ satisfies assumption **(A1)** of Section 4.1 and is divergence-free. Also, by Theorem 4.4 it satisfies assumption **(A2)**. Therefore by Theorem 5.1 we deduce that f_t (resp. $\beta(f_t)$ with $\beta(s)=\arctan(s)$ if f_t is not bounded but is renormalized) is a Lagrangian solution, and Theorem 4.10 ensures in particular that f_t is a renormalized solution. □

Proof of Theorem 2.3. Thanks to Theorem 2.2 we know that the solution is transported by the maximal regular flow associated to $\boldsymbol{b}_t=(v,E_t)$. Also, since f_t is renormalized, the function $g_t := \frac{2}{\pi}\arctan f_t : (0,T)\times\mathbb{R}^d \to [0,1]$ is a solution of the continuity equation with vector field $\boldsymbol{b}$. Hence, in order to prove that trajectories do not blow up, it is enough to apply the criterion stated in Proposition 4.11 with $\mu_t = g_t\,dx$, that is

$$\int_0^T\int_{\mathbb{R}^{2d}} \frac{|\boldsymbol{b}_t(x,v)|\,g_t(x,v)}{\big(1+(|x|^2+|v|^2)^{1/2}\big)\log\big(2+(|x|^2+|v|^2)^{1/2}\big)}\,dx\,dv\,dt < \infty. \tag{3.2}$$

To this end, we observe that $g_t^2 \le g_t \le f_t$, hence

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^{2d}} \frac{|\boldsymbol{b}_t|\,g_t}{\big(1+(|x|^2+|v|^2)^{1/2}\big)\log\big(2+(|x|^2+|v|^2)^{1/2}\big)}\,dx\,dv\,dt \\
&\le \int_0^T\int_{\mathbb{R}^{2d}} f_t\,dx\,dv\,dt + \int_0^T\int_{\mathbb{R}^{2d}} |E_t|\,\frac{g_t}{(1+|v|)\log(2+|v|)}\,dx\,dv\,dt \\
&\le \int_0^T\int_{\mathbb{R}^{2d}} f_t\,dx\,dv\,dt + \int_0^T\int_{\mathbb{R}^{2d}} \left(\frac{|E_t|^2}{(1+|v|)^4\log^2(2+|v|)} + (1+|v|)^2 g_t^2\right)dx\,dv\,dt \\
&\le \left(\int_{\mathbb{R}^d} \frac{1}{(1+|v|)^4\log^2(2+|v|)}\,dv\right)\left(\int_0^T\int_{\mathbb{R}^d} |E_t|^2\,dx\,dt\right) + 2\int_0^T\int_{\mathbb{R}^{2d}} (1+|v|)^2 f_t\,dx\,dv\,dt.
\end{aligned}$$

Also, since $d\le 4$,

$$\int_{\mathbb{R}^d} \frac{1}{(1+|v|)^4\log^2(2+|v|)}\,dv < \infty,$$

thus (3.2) follows from (2.6).

Now, by the no blow-up criterion in Proposition 4.11 we obtain that the Maximal Regular Flow $\boldsymbol{X}$ of $\boldsymbol{b}$ is globally defined on $[0,T]$, namely its trajectories $\boldsymbol{X}(\cdot,x,v)$ belong to $AC([0,T];\mathbb{R}^{2d})$ for g_0-a.e. $(x,v)\in\mathbb{R}^{2d}$, and $g_t=\boldsymbol{X}(t,\cdot)_\# g_0=g_0\circ\boldsymbol{X}(t,\cdot)^{-1}$. Also, no integral curves of $\boldsymbol{b}_t$ that transport f_t can appear from and/or disappear at infinity during the time interval $[0,T]$. Since $f_t=\tan\left(\frac{\pi}{2}g_t\right)$ and the map $[0,1)\ni s\to\tan\left(\frac{\pi}{2}s\right)\in[0,\infty)$ is a diffeomorphism, we obtain that $f_t=\boldsymbol{X}(t,\cdot)_\# f_0=f_0\circ\boldsymbol{X}(t,\cdot)^{-1}$ as well. In particular, for all Borel functions $\psi:[0,\infty)\to[0,\infty)$ we have

$$\int_{\mathbb{R}^{2d}}\psi(f_t)\,dx\,dv=\int_{\mathbb{R}^{2d}}\psi(f_0)\circ\boldsymbol{X}(t,\cdot)^{-1}\,dx\,dv=\int_{\mathbb{R}^{2d}}\psi(f_0)\,dx\,dv,$$

where the second equality follows by the incompressibility of the flow. □

3.2 Global existence results: proof of Theorems 2.7 and 2.8

Proof of Theorem 2.7. To prove global existence of generalized Lagrangian solutions of Vlasov-Poisson we shall use an approximation procedure. Since the argument is rather long and involved, we divide the proof in five steps that we now describe briefly: In Step 1 we start from approximate solutions f^n, obtained by smoothing the initial datum and the kernel, and we decompose them along their level sets. Exploiting the incompressibility of the flow, these functions are still solutions of the continuity equation with the same vector field and, when n varies, they are uniformly bounded. This allows us to take their limit as $n\to\infty$ in Step 2, and show that the limit belongs to L^1. In Step 3 we introduce ρ^{eff} as the limit as $n\to\infty$ of the approximate densities ρ^n, and we motivate its properties. In Step 4 we show that the vector fields E^n converge to the vector field obtained by convolving ρ^{eff} with the Poisson kernel. Finally, in Step 5 we combine stability results for continuity equations with the results of Section 5 to take the limit in the approximate Vlasov-Poisson equation and show that the limiting solution is transported by the limiting incompressible flow. We now enter into the details of the proof.

Step 1: approximating solutions. Let $K(x):=\sigma\,c_d\,x/|x|^d$ and let us consider approximating kernels $K_n:=K*\psi_n$, where $\psi_n(x):=n^d\psi(nx)$ and $\psi\in C_c^\infty(\mathbb{R}^d)$ is a standard even convolution kernel in $\mathbb{R}^d$. Let $f_0^n\in C_c^\infty(\mathbb{R}^{2d})$ be a sequence of functions such that

$$f_0^n\to f_0\qquad\text{in } L^1(\mathbb{R}^{2d}),\tag{3.3}$$

and denote by f_t^n distributional solutions of the Vlasov system with initial datum f_0^n and kernel K_n (see for instance [17] or [30] for this classical construction based on a fixed point argument in the Wasserstein metric). Also, define $\rho_t^n:=\int f_t^n\,dv$ and $E_t^n:=K_n*\rho_t^n$. Notice that since K_n is smooth and decays at infinity, both E_t^n and ∇E_t^n are bounded on $[0,\infty)\times\mathbb{R}^d$ (with a bound that depends on n). Hence, since $\boldsymbol{b}_t^n:=(v,E_t^n)$ is a Lipschitz divergence-free vector field, its flow $\boldsymbol{X}^n(t):\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ is well defined and incompressible, and by standard theory for the transport equation we obtain that

$$f_t^n=f_0^n\circ\boldsymbol{X}^n(t)^{-1}\qquad\forall\,t\in[0,\infty),\tag{3.4}$$

and

$$\|\rho^n_t\|_{L^1(\mathbb{R}^d)} = \|f^n_t\|_{L^1(\mathbb{R}^{2d})} = \|f^n_0\|_{L^1(\mathbb{R}^{2d})} \qquad \forall\, t \in [0,\infty). \tag{3.5}$$

Assuming without loss of generality that $\mathscr{L}^{2d}(\{f_0 = k\}) = 0$ for every $k \in \mathbb{N}$ (otherwise we consider as level sets the values $\tau + k$ in place of k, for some $\tau \in (0,1)$), from (3.3) we deduce that

$$f^{n,k}_0 := 1_{\{k \le f^n_0 < k+1\}} f^n_0 \to f^k_0 := 1_{\{k \le f_0 < k+1\}} f_0 \quad \text{in } L^1(\mathbb{R}^{2d}) \qquad \forall\, k \in \mathbb{N}. \tag{3.6}$$

Now, for any $k, n \in \mathbb{N}$ we consider $f^{n,k}_t := 1_{\{k \le f^n_t < k+1\}} f^n_t$. Then it follows by (3.4) that

$$f^{n,k}_t = 1_{\{k \le f^n_0 \circ \boldsymbol{X}^n(t)^{-1} < k+1\}} f^n_0 \circ \boldsymbol{X}^n(t)^{-1} \qquad \forall\, t \in [0,\infty), \tag{3.7}$$

$f^{n,k}_t$ is a distributional solution of the continuity equation with vector field $\boldsymbol{b}^n_t$, and

$$\|f^{n,k}_t\|_{L^1(\mathbb{R}^{2d})} = \|f^{n,k}_0\|_{L^1(\mathbb{R}^{2d})} \qquad \forall\, t \in [0,\infty). \tag{3.8}$$

Step 2: limit in the phase-space. By construction the functions $\{f^{n,k}\}_{n\in\mathbb{N}}$ are nonnegative and bounded by $k+1$, hence there exists $f^k \in L^\infty((0,\infty) \times \mathbb{R}^{2d})$ nonnegative such that, up to subsequences,

$$f^{n,k} \rightharpoonup f^k \qquad \text{weakly* in } L^\infty((0,\infty) \times \mathbb{R}^{2d}) \text{ as } n \to \infty \qquad \forall\, k \in \mathbb{N}. \tag{3.9}$$

Moreover, for any K compact subset of $\mathbb{R}^{2d}$ and any bounded function $\phi : (0,\infty) \to [0,\infty)$ with compact support we can use the test function $\phi(t)\, 1_K(x,v)\, \mathrm{sign}(f^k_t)(x,v)$ in the previous weak convergence, and thanks to Fatou's Lemma, (3.8), and (3.6), we get

$$\begin{aligned}
\int_0^\infty \phi(t)\, \|f^k_t\|_{L^1(K)}\, dt &\le \liminf_{n\to\infty} \int_0^\infty \phi(t)\, \|f^{n,k}_t\|_{L^1(K)}\, dt \\
&\le \liminf_{n\to\infty} \int_0^\infty \phi(t)\, \|f^{n,k}_t\|_{L^1(\mathbb{R}^{2d})}\, dt \\
&= \liminf_{n\to\infty} \int_0^\infty \phi(t)\, \|f^{n,k}_0\|_{L^1(\mathbb{R}^{2d})}\, dt \\
&= \Big(\int_0^\infty \phi(t)\, dt\Big)\, \|f^k_0\|_{L^1(\mathbb{R}^{2d})}.
\end{aligned} \tag{3.10}$$

Since ϕ was arbitrary, taking the supremum among all compact subsets $K \subset \mathbb{R}^{2d}$ we obtain

$$\|f^k_t\|_{L^1(\mathbb{R}^{2d})} \le \|f^k_0\|_{L^1(\mathbb{R}^{2d})} \qquad \text{for a.e. } t \in (0,\infty), \tag{3.11}$$

so, in particular, $f^k \in L^\infty((0,\infty); L^1(\mathbb{R}^{2d}))$.

Thanks to (3.11) we see that, if we define

$$f := \sum_{k=0}^{\infty} f^k \qquad \text{in } (0,\infty) \times \mathbb{R}^{2d}, \tag{3.12}$$

then

$$\|f_t\|_{L^1(\mathbb{R}^{2d})} \le \sum_{k=0}^{\infty} \|f_t^k\|_{L^1(\mathbb{R}^{2d})} \le \sum_{k=0}^{\infty} \|f_0^k\|_{L^1(\mathbb{R}^{2d})} = \|f_0\|_{L^1(\mathbb{R}^{2d})} \qquad \text{for a.e. } t \in [0,\infty), \tag{3.13}$$

which implies that $f \in L^\infty((0,\infty); L^1(\mathbb{R}^{2d}))$.

We now claim that

$$f^n \rightharpoonup f \qquad \text{weakly in } L^1((0,T)\times\mathbb{R}^{2d}) \tag{3.14}$$

for every $T > 0$. Indeed, fix $\varphi \in L^\infty((0,T)\times\mathbb{R}^{2d})$. Noticing that $f^n = \sum_{k=0}^{\infty} f^{n,k}$ and $f = \sum_{k=0}^{\infty} f^k$, by the triangle inequality we have that, for every $k_0 \ge 1$,

$$\begin{aligned} \Big|\int_0^T\!\!\int_{\mathbb{R}^{2d}} \varphi\,(f^n - f)\,dx\,dv\,dt\Big| &= \Big|\sum_{k=0}^{\infty}\int_0^T\!\!\int_{\mathbb{R}^{2d}} \varphi\,(f^{n,k} - f^k)\,dx\,dv\,dt\Big| \\ &\le \Big|\sum_{k=0}^{k_0-1}\int_0^T\!\!\int_{\mathbb{R}^{2d}} \varphi\,(f^{n,k} - f^k)\,dx\,dv\,dt\Big| \\ &\quad + \sum_{k=k_0}^{\infty}\int_0^T\!\!\int_{\mathbb{R}^{2d}} |\varphi|\,|f^{n,k}|\,dx\,dv\,dt + \sum_{k=k_0}^{\infty}\int_0^T\!\!\int_{\mathbb{R}^{2d}} |\varphi|\,|f^k|\,dx\,dv\,dt. \end{aligned}$$

Using (3.8) and (3.11), the last two terms can be estimated by

$$\begin{aligned} &\sum_{k=k_0}^{\infty}\int_0^T\!\!\int_{\mathbb{R}^{2d}} |\varphi|\,|f^{n,k}|\,dx\,dv\,dt + \sum_{k=k_0}^{\infty}\int_0^T\!\!\int_{\mathbb{R}^{2d}} |\varphi|\,|f^k|\,dx\,dv\,dt \\ &\qquad \le T\|\varphi\|_\infty \sum_{k=k_0}^{\infty}\int_{\mathbb{R}^{2d}} |f_0^{n,k}|\,dx\,dv + T\|\varphi\|_\infty \sum_{k=k_0}^{\infty}\int_{\mathbb{R}^{2d}} |f_0^k|\,dx\,dv \\ &\qquad \le T\|\varphi\|_\infty \int_{\{f_0^n \ge k_0\}} |f_0^n|\,dx\,dv + T\|\varphi\|_\infty \int_{\{f_0 \ge k_0\}} |f_0|\,dx\,dv \\ &\qquad = T\|\varphi\|_\infty \Big(\|f_0^n 1_{\{f_0^n \ge k_0\}}\|_{L^1(\mathbb{R}^{2d})} + \|f_0 1_{\{f_0 \ge k_0\}}\|_{L^1(\mathbb{R}^{2d})}\Big). \end{aligned}$$

Notice that, thanks to (3.6) and (3.3), it follows that

$$f_0^n 1_{\{f_0^n \ge k_0\}} \to f_0 1_{\{f_0 \ge k_0\}} \qquad \text{in } L^1(\mathbb{R}^{2d}),$$

so by letting $n \to \infty$ and using (3.9) we deduce that

$$\begin{aligned} \limsup_{n\to\infty}\Big|\int_0^T\!\!\int_{\mathbb{R}^{2d}} \varphi\,(f^n - f)\,dx\,dt\Big| &\le \limsup_{n\to\infty}\Big|\sum_{k=0}^{k_0-1}\int_0^T\!\!\int_{\mathbb{R}^{2d}} \varphi(f^{n,k} - f^k)\,dx\,dv\,dt\Big| \\ &\quad + 2T\|\varphi\|_\infty \|f_0 1_{\{f_0 \ge k_0\}}\|_{L^1(\mathbb{R}^{2d})} \\ &= 2T\|\varphi\|_\infty \|f_0 1_{\{f_0 \ge k_0\}}\|_{L^1(\mathbb{R}^{2d})}. \end{aligned}$$

Hence, letting $k_0 \to \infty$, since $\varphi \in L^\infty$ was arbitrary we obtain (3.14).

Step 3: limit of the physical densities. Since the sequence $\{\rho^n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty((0,\infty);\mathscr{M}_+(\mathbb{R}^d)) \subset \left[L^1((0,\infty),C_0(\mathbb{R}^d))\right]^*$ (see (3.5)), there exists $\rho^{\text{eff}} \in L^\infty((0,\infty);\mathscr{M}_+(\mathbb{R}^d))$ such that

$$\rho^n \rightharpoonup \rho^{\text{eff}} \qquad \text{weakly* in } L^\infty((0,\infty);\mathscr{M}_+(\mathbb{R}^d)). \tag{3.15}$$

Moreover, by the lower semicontinuity of the norm under weak* convergence, using (3.5) again we deduce that

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} |\rho_t^{\text{eff}}|(\mathbb{R}^d) \le \lim_{n\to\infty}\left(\sup_{t\in(0,\infty)} \|\rho_t^n\|_{L^1(\mathbb{R}^d)}\right) = \lim_{n\to\infty} \|f_0^n\|_{L^1(\mathbb{R}^{2d})} = \|f_0\|_{L^1(\mathbb{R}^{2d})}. \tag{3.16}$$

Now, let us consider any nonnegative function $\varphi \in C_c((0,\infty)\times\mathbb{R}^d)$. By (3.15) and (3.14) we obtain that, for any $R>0$,

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^d} \varphi_t(x)\, d\rho_t^{\text{eff}}(x)\, dt &= \lim_{n\to\infty} \int_0^\infty \int_{\mathbb{R}^d} \rho_t^n(x)\, \varphi_t(x)\, dx\, dt \\
&= \lim_{n\to\infty} \int_0^\infty \int_{\mathbb{R}^{2d}} f_t^n(x,v)\, \varphi_t(x)\, dv\, dx\, dt \\
&\ge \liminf_{n\to\infty} \int_0^\infty \int_{\mathbb{R}^d\times B_R} f_t^n(x,v)\, \varphi_t(x)\, dv\, dx\, dt \\
&= \int_0^\infty \int_{\mathbb{R}^d\times B_R} f_t(x,v)\, \varphi_t(x)\, dv\, dx\, dt,
\end{aligned}$$

so by letting $R\to\infty$ we get

$$\int_0^\infty \int_{\mathbb{R}^d} \varphi_t(x)\, d\rho_t^{\text{eff}}(x)\, dt \ge \int_0^\infty \int_{\mathbb{R}^{2d}} f_t(x,v)\, \varphi_t(x)\, dv\, dx\, dt = \int_0^\infty \int_{\mathbb{R}^d} \varphi_t(x)\, d\rho_t(x)\, dt.$$

By the arbitrariness of φ we deduce that

$$\rho_t \le \rho_t^{\text{eff}} \qquad \text{as measures for a.e. } t\in(0,\infty), \tag{3.17}$$

as desired.

Step 4: limit of the vector fields. Set $E_t^{\text{eff}} := K * \rho_t^{\text{eff}}$ and $\boldsymbol{b}_t(x,v) := (v, E_t^{\text{eff}}(x))$. We claim that

$$\boldsymbol{b}^n \rightharpoonup \boldsymbol{b} \qquad \text{weakly in } L^1_{\text{loc}}((0,\infty)\times\mathbb{R}^{2d};\mathbb{R}^{2d}) \tag{3.18}$$

and that, for every ball $B_R \subset \mathbb{R}^d$,

$$[\rho_t^n * K_n](x+h) \to [\rho_t^n * K_n](x) \quad \text{as } |h|\to 0 \text{ in } L^1_{\text{loc}}((0,\infty);L^1(B_R)), \text{ uniformly in } n. \tag{3.19}$$

To show this we first prove that the sequence $\{\boldsymbol{b}^n\}_{n\in\mathbb{N}}$ is bounded in $L^p_{\text{loc}}((0,\infty)\times\mathbb{R}^{2d};\mathbb{R}^{2d})$ for every $p\in[1,d/(d-1))$. Indeed, using Young's inequality, for every $t\ge 0$, $n\in\mathbb{N}$, and $r>0$,

$$\begin{aligned}
&\|\rho_t^n * K_n\|_{L^p(B_r)} = \|(\rho_t^n * \psi_n) * K\|_{L^p(B_r)} \\
&\le \|(\rho_t^n * \psi_n) * (K1_{B_1})\|_{L^p(B_r)} + \|(\rho_t^n * \psi_n) * (K1_{\mathbb{R}^d\setminus B_1})\|_{L^p(B_r)} \\
&\le \|(\rho_t^n * \psi_n) * (K1_{B_1})\|_{L^p(\mathbb{R}^d)} + \mathscr{L}^d(B_r)^{1/p} \|(\rho_t^n * \psi_n) * (K1_{\mathbb{R}^d\setminus B_1})\|_{L^\infty(\mathbb{R}^d)} \\
&\le \|\rho_t^n\|_{L^1(\mathbb{R}^d)} \|\psi_n\|_{L^1(\mathbb{R}^d)} \|K\|_{L^p(B_1)} + \mathscr{L}^d(B_r)^{1/p} \|\rho_t^n\|_{L^1(\mathbb{R}^d)} \|\psi_n\|_{L^1(\mathbb{R}^d)} \|K\|_{L^\infty(\mathbb{R}^d\setminus B_1)}
\end{aligned}$$

hence, up to subsequences, the sequence $\{\boldsymbol{b}^n\}_{n\in\mathbb{N}}$ converges weakly in L^p_{loc}. In order to identify the limit we now show that for every $\varphi \in C_c((0,\infty)\times\mathbb{R}^d)$

$$\lim_{n\to\infty}\int_0^\infty\int_{\mathbb{R}^d}\rho^n_t * K_n\,\varphi_t\,dx\,dt = \int_0^\infty\int_{\mathbb{R}^d}\rho^{\mathrm{eff}}_t * K\,\varphi_t\,dx\,dt.$$

Indeed, by standard properties of convolution,

$$\begin{aligned}
&\Big|\int_0^\infty\int_{\mathbb{R}^d}\rho^n_t * K_n\,\varphi_t\,dx\,dt - \int_0^\infty\int_{\mathbb{R}^d}\rho^{\mathrm{eff}}_t * K\,\varphi_t\,dx\,dt\Big| \\
&= \Big|\int_0^\infty\int_{\mathbb{R}^d}\rho^n_t\,\varphi_t * K_n\,dx\,dt - \int_0^\infty\int_{\mathbb{R}^d}\rho^{\mathrm{eff}}_t\,\varphi_t * K\,dx\,dt\Big| \\
&\le \Big|\int_0^\infty\int_{\mathbb{R}^d}(\rho^n_t - \rho^{\mathrm{eff}}_t)\,\varphi_t * K\,dx\,dt\Big| + \Big|\int_0^\infty\int_{\mathbb{R}^d}\rho^n_t\,(\varphi_t * K - \varphi_t * K * \psi_n)\,dx\,dt\Big| \\
&\le \Big|\int_0^\infty\int_{\mathbb{R}^d}(\rho^n_t - \rho^{\mathrm{eff}}_t)\,\varphi_t * K\,dx\,dt\Big| + \Big(\sup_{t\in(0,\infty)}\|\rho^n_t\|_{L^1(\mathbb{R}^d)}\Big)\,\|\varphi_t * K - \varphi_t * K * \psi_n\|_{L^\infty((0,\infty)\times\mathbb{R}^d)}.
\end{aligned}$$

Letting $n\to\infty$, the first term converges to 0 thanks to the weak convergence (3.15) of ρ^n_t to ρ^{eff}_t and the fact that $\varphi * K = \varphi * (1_{B_1}K) + \varphi * (1_{\mathbb{R}^d\setminus B_1}K)$ is a bounded continuous function, compactly supported in time and decaying at infinity in space. The second term, in turn, converges to 0 since the first factor is bounded (see (3.16)) and $\varphi_t * K * \psi_n$ converges to $\varphi_t * K$ uniformly in $(0,\infty)\times\mathbb{R}^d$.

This computation identifies the weak limit of $\rho^n_t * K_n$ in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^{2d})$, showing that it coincides with $\rho^{\mathrm{eff}}_t * K$ and proving (3.18).

We now prove (3.19). First of all, since $K \in W^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ for every $\alpha < 1$ and $p < d/(d-1+\alpha)$,[2] using Young's inequality we deduce that, for any $t\in(0,\infty)$,

$$\|\rho^n_t * K_n\|_{W^{\alpha,p}(B_R;\mathbb{R}^d)} = \|(\rho^n_t * \psi_n) * K\|_{W^{\alpha,p}(B_R;\mathbb{R}^d)} \le C(R)\|\rho^n_t * \psi_n\|_{L^1(\mathbb{R}^d)}.$$

Since $\|\psi_n\|_{L^1(\mathbb{R}^d)} = 1$, thanks to (3.5) we deduce that the last term is bounded independently of t and n, that is, for every $R>0$,

$$\sup_{t\in(0,\infty)}\sup_{n\in\mathbb{N}}\|\rho^n_t * K_n\|_{W^{\alpha,p}(B_R;\mathbb{R}^d)} < \infty. \tag{3.20}$$

Hence, by a classical embedding between fractional Sobolev spaces and Nikolsky spaces (see for instance [23, Lemma 2.3]) we find that, for $|h|\le R$,

$$\int_{B_R}|\rho^n_t * K_n(x+h) - \rho^n_t * K_n(x)|^p\,dx \le C\big(p,\alpha,R,\|\rho^n_t * K_n\|_{W^{\alpha,p}(B_{2R};\mathbb{R}^d)}\big)\,|h|^{\alpha p},$$

from which (3.19) follows.

Step 5: conclusion. Thanks to (3.18) and (3.19), we can apply the stability result from [16, Theorem II.7] (which does not require any growth condition on the vector fields, see

[2] This can be seen by a direct computation, using the definition of fractional Sobolev spaces.

also [2, Proposition 6.5] for the stability of the associated flows) to deduce that, for every $k \in \mathbb{N}$, f^k is a weakly continuous distributional solution of the continuity equation starting from f_0^k, so by linearity also $F^m := \sum_{k=1}^m f^k$ is a distributional solution for every $m \in \mathbb{N}$.

Since F^m is bounded, the proof of Theorem 2.2 shows that F^m is a renormalized solution for every $m \in \mathbb{N}$. Letting $m \to \infty$, because $F^m \to f$ strongly in $L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^{2d})$ we obtain that f is a renormalized solution of the continuity equation starting from f_0 with vector field $\boldsymbol{b}$. Together with (3.17), (3.13), and (3.16), this proves that $(f_t, \rho_t^{\mathrm{eff}})$ is a generalized solution of the Vlasov-Poisson equation starting from f_0 according to Definition 2.6.

Finally, the fact that f is transported by the Maximal Regular Flow associated to $\boldsymbol{b}_t$ simply follows by the fact that each density f^k is transported by Maximal Regular Flow associated to $\boldsymbol{b}_t$ (using again the argument in the proof of Theorem 2.2) and that $f = \sum_{k=0}^\infty f^k$ is an absolutely convergent series (see (3.13)). Also, thanks to Theorem 4.10 we deduce that f_t belongs to $C([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}^{2d}))$. □

The proof of Theorem 2.8 follows the lines of the proof of Theorem 2.7, obtained by approximating both the initial datum and the kernel with a sequence of smooth data with uniformly bounded energy. In turn, this bound ensures that the approximating sequence of phase-space distributions is tight in the v variable uniformly in time, allowing us to show that $\rho_t^{\mathrm{eff}} = \rho_t$ for a.e. $t \in (0,\infty)$. The approximation of the initial datum with a smooth sequence having uniformly bounded energy is a technical task that we describe in the next lemma.

Lemma 3.1. *Let $d \geq 3$, let ψ be a standard convolution kernel, and set $\psi_k(x) := k^d\psi(kx)$ for every $k \geq 1$. Let $f_0 \in L^1(\mathbb{R}^{2d})$ be an initial datum of finite energy, namely*

$$\int_{\mathbb{R}^{2d}} |v|^2 f_0(x,v)\,dx\,dv + \int_{\mathbb{R}^d} [H * \rho_0](x)\,\rho_0(x)\,dx < \infty,$$

where $\rho_0(x) := \int_{\mathbb{R}^d} f_0(x,v)\,dv$ and $H(x) := \frac{c_d}{d-2}|x|^{2-d}$ for every $x \in \mathbb{R}^d$. Then there exist a sequence of functions $\{f_0^n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^{2d})$ and a sequence $\{k_n\}_{n\in\mathbb{N}}$ such that $k_n \to \infty$ and, setting $\rho_0^n(x) = \int_{\mathbb{R}^d} f_0^n(x,v)\,dv$,

$$\lim_{n\to\infty}\Big(\int_{\mathbb{R}^{2d}} |v|^2 f_0^n\,dx\,dv + \int_{\mathbb{R}^d} H * \psi_{k_n} * \rho_0^n\,\rho_0^n\,dx\Big) = \int_{\mathbb{R}^{2d}} |v|^2 f_0\,dx\,dv + \int_{\mathbb{R}^d} H * \rho_0\,\rho_0\,dx. \quad (3.21)$$

Proof. We split the approximation procedure in three steps. Here and in the sequel we use the notation L_c^∞ to denote the space of bounded functions with compact support.

Step 1: approximation of the initial datum when $f_0 \in L_c^\infty(\mathbb{R}^{2d})$. Assuming that $f_0 \in L_c^\infty(\mathbb{R}^{2d})$, we claim that there exists $\{f_0^n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^{2d})$ such that

$$\lim_{n\to\infty}\Big(\int_{\mathbb{R}^{2d}} |v|^2 f_0^n\,dx\,dv + \int_{\mathbb{R}^d} H * \rho_0^n\,\rho_0^n\,dx\Big) = \int_{\mathbb{R}^{2d}} |v|^2 f_0\,dx\,dv + \int_{\mathbb{R}^d} H * \rho_0\,\rho_0\,dx. \quad (3.22)$$

To this end, consider smooth functions f_0^n which converge to f_0 pointwise, whose L^∞ norms are bounded by $\|f_0\|_{L^\infty(\mathbb{R}^{2d})}$, and whose supports are all contained in the same ball. By construction the densities ρ_0^n are bounded as well and their supports are also contained in a fixed ball; moreover, the functions $H * \rho_0^n$ are bounded and converge to $H * \rho_0$ locally in every $L^p_{\rm loc}$. By dominated convergence, these observations show the validity of (3.22).

Step 2: approximation of the initial datum when $f_0 \in L^1(\mathbb{R}^{2d})$. Assuming that $f_0 \in L^1(\mathbb{R}^{2d})$, we claim that there exists a sequence of functions $\{f_0^n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^{2d})$ such that (3.22) holds.

Indeed, by Step 1 it is enough to approximate f_0 with a sequence in $L_c^\infty(\mathbb{R}^{2d})$ with converging energies. To this aim, for every $n \in \mathbb{N}$ we define the truncations of f_0 given by

$$f_0^n(x,v) := \min\{n, 1_{B_n}(x,v) f_0(x,v)\} \qquad (x,v) \in \mathbb{R}^{2d}.$$

Since $H \geq 0$ the integrands in the left-hand side of (3.22) converge monotonically, hence the integrals converge by monotone convergence.

Step 3: approximation of the kernel. We conclude the proof of the lemma. In order to approximate the kernel, we notice that, given the sequence of functions $f_0^n \in C_c^\infty(\mathbb{R}^d)$ provided by Steps 1-2, for $n \in \mathbb{N}$ fixed we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} H * \psi_k * \rho_0^n \, \rho_0^n \, dx = \int_{\mathbb{R}^d} H * \rho_0^n \, \rho_0^n \, dx.$$

Hence, choosing k_n sufficiently large so that

$$\Big| \int_{\mathbb{R}^d} H * \psi_{k_n} * \rho_0^n \, \rho_0^n \, dx - \int_{\mathbb{R}^d} H * \rho_0^n \, \rho_0^n \, dx \Big| \leq \frac{1}{n},$$

we conclude the proof of the approximation lemma. □

In order to prove Theorem 2.8, in particular properties (i)-(ii)-(iii), we need some further preliminary estimates. As we shall see, the proof of the energy inequality (2.10) is based on the conservation of energy along approximate solutions and on a lower semicontinuity argument. Notice that, since $-\Delta H = \delta_0$, a formal integration by parts (rigorously justified in the case that μ has a smooth compactly supported density with respect to the Lebesgue measure) shows that, for $d \geq 3$ and every $\mu \in \mathscr{M}_+(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} H * \mu(x) \, d\mu(x) = \int_{\mathbb{R}^d} |\nabla H * \mu(x)|^2 \, dx, \tag{3.23}$$

meaning that, if one of the two sides is finite, then so is the other and they coincide. The above identity would immediately imply the convexity of the potential energy and its lower semicontinuity with respect to the weak* convergence of measures. However, since the justification of (3.23) requires some work, we shall prove directly the lower semicontinuity.

Lemma 3.2. *Let $d \geq 3$ and $H(x) := \frac{c_d}{d-2}|x|^{2-d}$, with the convention $H(0) = +\infty$. Then the functional*

$$\mathcal{F}(\mu) := \int_{\mathbb{R}^d} H * \mu(x) \, d\mu(x), \qquad \mu \in \mathscr{M}_+(\mathbb{R}^d),$$

is lower semicontinuous with respect to the weak topology of $\mathscr{M}(\mathbb{R}^d)$.*

Proof. Given a sequence of nonnegative measures μ^n weakly* converging to μ in $\mathscr{M}(\mathbb{R}^d)$, the measures $d\mu^n(x)\,d\mu^n(y) \in \mathscr{M}(\mathbb{R}^{2d})$ weakly* converge to $d\mu(x)\,d\mu(y)$. Hence, since the function $\hat{H}(x,y) := H(x-y)$ is continuous as a map from $\mathbb{R}^{2d}$ to $[0,+\infty]$, we deduce that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H(x-y)\,d\mu(x)\,d\mu(y) \le \liminf_{n\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H(x-y)\,d\mu^n(x)\,d\mu^n(y).$$

□

The following lemma adapts the previous one to the time-dependent framework. In particular it takes care of a further approximation of the kernel in the right-hand side of (3.25) below and involves the time dependence of the functional. We need this kind of lemma since, at the level of generality of Theorem 2.7, the weak convergence of the approximating solutions is not pointwise in time, but it happens only as functions in space-time.

Lemma 3.3. *Let $d \ge 3$, $T > 0$, $\phi \in C_c((0,T))$ nonnegative, $\psi \ge 0$ an even convolution kernel, and $\psi_n(x) := n^d\psi(nx)$ for every $n \ge 1$. Then, for every sequence $\{\rho^n\}_{n\in\mathbb{N}} \subset C^0([0,T];\mathscr{M}_+(\mathbb{R}^d))$ such that $\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\rho^n_t(\mathbb{R}^d) < \infty$ and*

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\Big|\int_{\mathbb{R}^d}\varphi\,d(\rho^n_t-\rho_t)\Big| = 0 \qquad \text{for every } \varphi \in C^\infty_c(\mathbb{R}^d), \tag{3.24}$$

we have

$$\int_0^T \phi(t)\int_{\mathbb{R}^d} H*\rho_t(x)\,d\rho_t(x)\,dt \le \liminf_{n\to\infty}\int_0^T\phi(t)\int_{\mathbb{R}^d} H*\psi_n*\rho^n_t(x)\,d\rho^n_t(x)\,dt. \tag{3.25}$$

Proof. We observe that, given any function $\phi_1 \in C^\infty_c(\mathbb{R}^d)$, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}\phi_1\,d(\rho^n_t*\psi^n-\rho_t)\Big| &\le \Big|\int_{\mathbb{R}^d}(\phi_1*\psi^n-\phi_1)\,d\rho^n_t\Big| + \Big|\int_{\mathbb{R}^d}\phi_1\,d(\rho^n_t-\rho_t)\Big| \\ &\le \|\phi_1*\psi^n-\phi_1\|_{C^0(\mathbb{R}^d)}\rho^n_t(\mathbb{R}^d) + \Big|\int_{\mathbb{R}^d}\phi_1\,d(\rho^n_t-\rho_t)\Big|.\end{aligned}$$

Hence, since $\phi_1*\psi^n$ converge uniformly to ϕ_1 as $n\to\infty$, $\rho^n_t(\mathbb{R}^d)$ is uniformly bounded in n and t, and (3.24) holds, we get

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\Big|\int_{\mathbb{R}^d}\phi_1\,d(\rho^n_t*\psi^n-\rho_t)\Big| = 0. \tag{3.26}$$

Now, let us consider the sequence of measures $\psi_n*\rho^n_t(x)\,\rho^n_t(y)\,dt \in \mathscr{M}((0,T)\times\mathbb{R}^{2d})$. We know that they are weakly* precompact because they have bounded masses; we claim that the weak* limit is $\rho_t(x)\,\rho_t(y)\,dt$. Indeed, testing with functions of the form $\phi_1(x)\phi_2(y)\psi(t)$, with $\phi_1,\phi_2 \in C^\infty_c(\mathbb{R}^d)$ and $\psi \in C^0([0,T])$ and using (3.24) and (3.26), we find that

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(t)\phi_1(x)\phi_2(y)\psi_n*\rho^n_t(x)\,dx\,d\rho^n_t(y)\,dt \\ &\quad= \lim_{n\to\infty}\int_0^T\psi(t)\Big(\int_{\mathbb{R}^d}\phi_1(x)\psi_n*\rho^n_t(x)\,dx\Big)\Big(\int_{\mathbb{R}^d}\phi_2(y)\,d\rho^n_t(y)\Big)\,dt \\ &\quad= \int_0^T\psi(t)\Big(\int_{\mathbb{R}^d}\phi_1(x)\,d\rho_t(x)\Big)\Big(\int_{\mathbb{R}^d}\phi_2(y)\,d\rho_t(y)\Big)\,dt.\end{aligned}$$

Since the function $\phi(t)H(x-y)$ is continuous as a map from $(0,T)\times\mathbb{R}^{2d}$ to $[0,+\infty]$, this implies that (3.25) holds. $\square$

In the following lemma we establish a general inequality between the potential energy and the L^2-norm of the force field, that will be used to show the validity of property (iii) in Theorem 2.8.

Lemma 3.4. *Let $d \geq 3$ and $H(x) := \frac{c_d}{d-2}|x|^{2-d}$. Then, for every $\rho \in L^1(\mathbb{R}^d)$ nonnegative,*

$$\int_{\mathbb{R}^d} H * \rho\, \rho\, dx \geq \int_{\mathbb{R}^d} |\nabla H * \rho|^2\, dx. \tag{3.27}$$

Proof. We split the approximation procedure in three steps.

Step 1: Proof of equality in (3.27) **for** $\rho \in L^\infty_c(\mathbb{R}^d)$**.** Consider first ρ a smooth, compactly supported function. For every $R > 0$, the integration by parts formula gives

$$\int_{B_R} H * \rho\, \rho\, dx = \int_{B_R} |\nabla H * \rho|^2\, dx - \int_{\partial B_R} H * \rho\, \nabla (H * \rho) \cdot \nu_{B_R}\, d\mathcal{H}^{d-1}.$$

By approximation, the same identity holds when ρ is bounded and compactly supported. Now, since $H * \rho$ and $\nabla H * \rho$ respectively decay as R^{2-d} and R^{1-d} when evaluated on ∂B_R, we see that the boundary term in the previous equality disappears as $R \to \infty$ (recall that $d \geq 3$). This proves that equality holds in (3.27) for $\rho \in L^\infty_c(\mathbb{R}^d)$

Step 2: Proof of (3.27) **for** $\rho \in L^1(\mathbb{R}^d)$**.** Given $\rho \in L^1(\mathbb{R}^d)$, for every $n \in \mathbb{N}$ consider the truncations of ρ given by $\rho^n := \min\{n, 1_{B_n}\rho\}$. Since $H \geq 0$, it follows by monotone convergence and Step 1 that

$$\int_{\mathbb{R}^d} H * \rho\, \rho\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} H * \rho^n\, \rho^n\, dx \geq \lim_{n\to\infty} \int_{\mathbb{R}^d} |\nabla H * \rho^n|^2\, dx.$$

Assuming without loss of generality that the left hand side is finite, we see that the sequence $\{\nabla H * \rho^n\}_{n\in\mathbb{N}}$ is bounded in L^2. Hence, since its limit in the sense of distribution is $\nabla H * \rho$, the lower semicontinuity of the L^2-norm with respect to weak convergence implies that $\nabla H * \rho \in L^2(\mathbb{R}^d)$ and that (3.27) holds. $\square$

Proof of Theorem 2.8. We start by showing the existence of renormalized solutions. Given f_0 with finite energy, let $\{f_0^n\}_{n\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^{2d})$ and $\{k_n\}_{n\in\mathbb{N}}$ be as in Lemma 3.1. Also let $K := c_d\, x/|x|^d$ and $K_n := K * \psi_{k_n}$. Applying verbatim the arguments in Steps 1-3 in the proof of Theorem 2.7 we get a sequence f_n of smooth solutions with kernels K_n such that

$$f^n \rightharpoonup f \qquad \text{weakly in } L^1([0,T]\times\mathbb{R}^{2d}) \quad \text{for any } T > 0, \tag{3.28}$$

and

$$\rho^n \rightharpoonup \rho^{\text{eff}} \qquad \text{weakly* in } L^\infty((0,T);\mathscr{M}_+(\mathbb{R}^d)),$$

where $\rho_t^n(x) := \int_{\mathbb{R}^d} f_t^n(x,v)\,dv$. In addition, the conservation of the energy along classical solutions gives that, for every $n \in \mathbb{N}$ and $t \in [0,\infty)$

$$\int_{\mathbb{R}^{2d}} |v|^2 f_t^n \,dx\,dv + \int_{\mathbb{R}^d} H * \psi_{k_n} * \rho_t^n\, \rho_t^n \,dx = \int_{\mathbb{R}^{2d}} |v|^2 f_0^n \,dx\,dv + \int_{\mathbb{R}^d} H * \psi_{k_n} * \rho_0^n\, \rho_0^n \,dx \le C, \tag{3.29}$$

Hence, since $H \ge 0$ we deduce that

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,\infty)} \int_{\mathbb{R}^{2d}} |v|^2 f_t^n \,dx\,dv \le C, \tag{3.30}$$

and by lower semicontinuity of the kinetic energy we deduce that, for every $T > 0$,

$$\int_0^T \int_{\mathbb{R}^{2d}} |v|^2 f_t \,dx\,dv\,dt \le \liminf_{n\to\infty} \int_0^T \int_{\mathbb{R}^{2d}} |v|^2 f_t^n \,dx\,dv\,dt \le C\,T. \tag{3.31}$$

We now want to exploit (3.30) and (3.31) to show that $\rho^{\text{eff}} = \rho$, where $\rho_t(x) := \int_{\mathbb{R}^d} f_t(x,v)\,dv \in L^\infty((0,T); L^1(\mathbb{R}^d))$. For this, we want to show that for any $\varphi \in C_c((0,\infty)\times\mathbb{R}^d)$

$$\lim_{n\to\infty} \int_0^\infty \int_{\mathbb{R}^d} \varphi\, \rho_t^n \,dx\,dt = \int_0^\infty \int_{\mathbb{R}^d} \varphi\, \rho_t \,dx\,dt. \tag{3.32}$$

To this aim, for every $k \in \mathbb{N}$ we consider a continuous nonnegative function $\zeta_k : \mathbb{R}^d \to [0,1]$ which equals 1 inside B_k and 0 outside B_{k+1}, and observe that

$$\begin{aligned}\int_0^\infty\!\!\int_{\mathbb{R}^d} \varphi\,(\rho_t^n - \rho_t)\,dx\,dt = &\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} \varphi_t(x)\, f_t^n(x,v)\,(1-\zeta_k(v))\,dx\,dv\,dt \\ &+ \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} \varphi_t(x)\,(f_t^n(x,v) - f(x,v))\,\zeta_k(v)\,dx\,dv\,dt \\ &+ \int_0^\infty\!\!\int_{\mathbb{R}^{2d}} \varphi_t(x)\, f_t(x,v)\,(\zeta_k(v)-1)\,dx\,dv\,dt.\end{aligned}$$

The second term in the right-hand side converges to 0 by the weak convergence of f^n to f in L^1, while, thanks to (3.30) and (3.31), the other two terms are estimated as

$$\Big|\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} \varphi\, f_t^n(x,v)\,(1-\zeta_k(v))\,dx\,dv\,dt\Big| \le \frac{\|\varphi\|_\infty}{k^2} \int_0^T\!\!\int_{\mathbb{R}^{2d}} f_t^n(x,v)|v|^2\,dx\,dv\,dt \le \frac{C\,T\|\varphi\|_\infty}{k^2},$$

and

$$\Big|\int_0^\infty\!\!\int_{\mathbb{R}^{2d}} \varphi\, f_t(x,v)\,(1-\zeta_k(v))\,dx\,dv\,dt\Big| \le \frac{C\,T\|\varphi\|_\infty}{k^2}.$$

Letting $k \to \infty$, this proves (3.32). Thanks to this fact, the conclusion of the proof proceeds exactly as in Steps 4 and 5 in the proof of Theorem 2.7 with $\rho_t^{\text{eff}} = \rho_t$. In this way, we obtain a global Lagrangian (hence renormalized) solution $f_t \in C([0,\infty); L^1_{\text{loc}}(\mathbb{R}^{2d}))$ of the Vlasov-Poisson system (1.1) with initial datum f_0.

In order to prove properties (i)-(ii)-(iii) (and in particular (2.10)) we perform a lower semicontinuity argument on the energy of the approximate solutions f^n constructed in the first part of the proof of Theorem 2.8. **Step 1: bound on the total energy for $\mathscr{L}^1$-almost every time.** Consider a nonnegative function $\phi \in C_c((0,\infty))$. Testing the weak convergence (3.28) with $\phi(t)\,|v|^2\chi_r(x,v)$ where $\chi_r \in C_c^\infty(\mathbb{R}^{2d})$ is a nonnegative cutoff function between B_r and B_{r+1}, we find that, for every $r>0$,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^{2d}} \phi(t)\,|v|^2\chi_r(x,v)\,f_t\,dx\,dv\,dt &= \lim_{n\to\infty}\int_0^\infty \int_{\mathbb{R}^{2d}} \phi(t)\,|v|^2\chi_r(x,v)\,f_t^n\,dx\,dv\,dt\\ &\le \liminf_{n\to\infty}\int_0^\infty \phi(t)\int_{\mathbb{R}^{2d}} |v|^2 f_t^n\,dx\,dv\,dt.\end{aligned}$$

Taking the supremum with respect to r, we deduce that

$$\int_0^\infty \phi(t)\int_{\mathbb{R}^{2d}} |v|^2 f_t\,dx\,dv\,dt \le \liminf_{n\to\infty}\int_0^\infty \phi(t)\int_{\mathbb{R}^{2d}} |v|^2 f_t^n\,dx\,dv\,dt. \tag{3.33}$$

By the arbitrariness of ϕ it follows that $|v|^2 f_t \in L^1_{\mathrm{loc}}(\mathbb{R}^{2d})$ for a.e. t, with a uniform bound in t (notice also that the same bound applies uniformly in n to $|v|^2 f_t^n$). In particular this allows us to integrate the transport equation $\partial_t f_t + \mathrm{div}_{x,v}(\boldsymbol{b}_t f_t)=0$ with respect to v on the whole $\mathbb{R}^d$ and obtain

$$\partial_t\rho_t + \mathrm{div}_x(V_t\rho_t)=0, \qquad V_t(x) := \frac{\int_{\mathbb{R}^d} v\,f_t(x,v)\,dv}{\rho_t(x)}.$$

Note that, since $\rho_t = \int f_t\,dv$, it follows by Jensen's inequality that

$$\begin{aligned}\int_{\mathbb{R}^d} |V_t|^2\rho_t\,dx &= \int_{\mathbb{R}^d}\left|\frac{\int_{\mathbb{R}^d} v\,f_t\,dv}{\int_{\mathbb{R}^d} f_t\,dv}\right|^2\rho_t\,dx\\ &\le \int_{\mathbb{R}^d}\frac{\int_{\mathbb{R}^d}|v|^2\,f_t\,dv}{\int_{\mathbb{R}^d} f_t\,dv}\rho_t\,dx = \int_{\mathbb{R}^{2d}}|v|^2\,f_t\,dx\,dv \le \int_{\mathbb{R}^{2d}}|v|^2\,f_0\,dx\,dv,\end{aligned}$$

thus $\|V_t\|_{L^2(\rho_t)} \in L^\infty((0,\infty))$. By classical results on continuity equations, this implies that ρ_t has a weakly* continuous representative (see for instance [4, Lemma 8.1.2]). In addition, since ρ_t^n satisfy a similar continuity equation with $V_t^n(x) := \frac{\int_{\mathbb{R}^d} v\,f_t^n(x,v)\,dv}{\rho_t^n}$ with

$$\sup_{t,n}\int_{\mathbb{R}^d}|V_t^n|^2\rho_t^n\,dx \le C_0,$$

(as above, this follows by the uniform bound on the kinetic energy of f_t^n), we deduce that

$$\left|\int \rho_t^n\varphi\,dx - \int \rho_s^n\varphi\,dx\right| \le \|\varphi\|_{C^1}\int_s^t\int_{\mathbb{R}^d}|V_t^n|\rho_t^n\,dx \le C_0\|\varphi\|_{C^1}|t-s|$$

(compare with the proof of [4, Lemma 8.1.2]). This proves that, for all test functions $\varphi \in C_c^\infty(\mathbb{R}^d)$, the maps $t \mapsto \int_{\mathbb{R}^d}\rho_t^n\varphi\,dx$ are equicontinuous, so the weak* convergence of ρ_t^n to ρ_t in $L^\infty([0,T];\mathscr{M}_+(\mathbb{R}^d))$ gives

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left|\int_{\mathbb{R}^d}\varphi\,d(\rho_t^n-\rho_t)\right| = 0 \qquad \text{for every } \varphi \in C_c^\infty(\mathbb{R}^d).$$

Hence, since the mass of ρ_t^n is uniformly bounded with respect to n and t, it follows from Lemma 3.3 that

$$\int_0^\infty \phi(t) \int_{\mathbb{R}^d} H * \rho_t \, \rho_t \, dx \, dt \le \liminf_{n\to\infty} \int_0^\infty \phi(t) \int_{\mathbb{R}^d} H * \psi_{k_n} * \rho_t^n \, \rho_t^n \, dx \, dt. \tag{3.34}$$

Adding (3.33) and (3.34), by the subadditivity of the lim inf and by the energy identity (3.29) on the approximating solutions, we find that

$$\begin{aligned}
&\int_0^\infty \phi(t) \Big(\int_{\mathbb{R}^{2d}} |v|^2 f_t \, dx \, dv + \int_{\mathbb{R}^d} H * \rho_t \, \rho_t \, dx \Big) \, dt \\
&\le \liminf_{n\to\infty} \int_0^\infty \phi(t) \Big(\int_{\mathbb{R}^{2d}} |v|^2 f_0^n \, dx \, dv + \int_{\mathbb{R}^d} H * \psi_{k_n} * \rho_0^n \, \rho_0^n \, dx \Big) \, dt \\
&= \Big(\int_0^\infty \phi(t) \, dt \Big) \Big(\int_{\mathbb{R}^{2d}} |v|^2 f_0 \, dx \, dv + \int_{\mathbb{R}^d} H * \rho_0 \, \rho_0 \, dx \Big).
\end{aligned}$$

By the arbitrariness of ϕ it follows that (2.10) holds for $\mathscr{L}^1$-a.e. $t \in (0, \infty)$.

Step 2: boundedness of the total energy for every time. Observe that the kinetic energy (resp. the potential energy) is lower semicontinuous with respect to strong $L^1_{\mathrm{loc}}(\mathbb{R}^{2d})$-convergence of f (resp. weak* convergence in $\mathscr{M}(\mathbb{R}^d)$ of ρ). Since (2.10) holds true for a.e. $t \in (0, \infty)$ by Step 1, and the maps $t \mapsto f_t \in L^1(\mathbb{R}^{2d})$ and $t \mapsto \rho_t \in \mathscr{M}(\mathbb{R}^d)$ are continuous for the L^1_{loc} and the weak* convergence respectively, given any time $\bar t \in [0, \infty)$ we approximate it with a sequence $t_n \to \bar t$ such that the energy bound (2.10) holds for every t_n and let $n \to \infty$ to obtain that (2.10) holds with $t = \bar t$.

Step 3: strong L^1_{loc}-continuity of the physical density and the electric fields. Given $t \in [0, \infty)$, consider a sequence of times $t_n \to t$. Fix $r > 0$, and notice that for any $R > 0$

$$\int_{B_r} \int_{\mathbb{R}^d} |f_{t_n} - f_t| \, dv \, dx \le \int_{B_r} \int_{B_R} |f_{t_n} - f_t| \, dv \, dx + \int_{B_r} \int_{\mathbb{R}^d \setminus B_R} \frac{|v|^2}{R^2} (f_{t_n} + f_t) \, dv \, dx.$$

Thanks to (2.10) and the strong L^1_{loc} continuity of f_t, we can first let $n \to \infty$ and then $R \to \infty$ to deduce that

$$\lim_{n\to\infty} \int_{B_r} |\rho_{t_n} - \rho_t| \, dx \le \lim_{n\to\infty} \int_{B_r} \int_{\mathbb{R}^d} |f_{t_n} - f_t| \, dv \, dx = 0.$$

This proves the strong L^1_{loc}-continuity of ρ_t. Since $E_t = K * \rho_t$ and $\|\rho_t\|_{L^1(\mathbb{R}^d)} \le C$, it is simple to see that also E_t is strongly continuous in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Step 4: global characteristics in dimension 3 and 4. The bound (2.10) and Lemma 3.4 imply that $E_t = \nabla H * \rho_t \in L^\infty((0, \infty), L^2(\mathbb{R}^d))$, so we can apply Theorem 2.3 to deduce that the trajectories of the Maximal Regular Flow starting at any time t do not blow up for f_t-a.e. $(x, v) \in \mathbb{R}^{2d}$. □

Remark 3.5. As a consequence of Theorems 2.2 and 2.8, and Remarks 2.9 and 2.10, we deduce that, for $d = 2, 3, 4$ and $\sigma = 1$, finite energy solutions conserve the mass, namely

$\|f_t\|_{L^1(\mathbb{R}^{2d})} = \rho_t(\mathbb{R}^d) = \rho_0(\mathbb{R}^d) = \|f_0\|_{L^1(\mathbb{R}^{2d})}$ for every $t \in [0, \infty)$. In particular, in this case solutions are strongly continuous in $L^1(\mathbb{R}^{2d})$ and not only in $L^1_{\text{loc}}(\mathbb{R}^{2d})$ (see for instance the argument in Step 2 of the proof of Theorem 4.10).

Remark 3.6. As shown in the first part of Theorem 2.8, the construction in the proof of Theorem 2.7 provides renormalized solutions of the Vlasov-Poisson system if further assumptions are made on the initial datum, such as finiteness of the total energy. Still, there are examples of infinite energy data such that the generalized solutions built in Theorem 2.7 solve the Vlasov-Poisson system. For instance, in [28] Perthame considers an initial datum $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$ with $(1+|x|^2)f_0 \in L^1(\mathbb{R}^6)$ and infinite energy, and he shows the existence of a solution $f \in L^\infty([0,\infty); L^1 \cap L^\infty(\mathbb{R}^6))$ of the Vlasov-Poisson system such that the quantities

$$t^{1/2}\|E_t\|_{L^2}, \qquad t^{3/5}\|\rho_t\|_{L^{5/3}}, \qquad \int_{\mathbb{R}^6} \frac{|x-vt|^2}{t} f_t(x,v)\,dx\,dv \tag{3.35}$$

are bounded for all $t \in (0, \infty)$. It can be easily seen that, under Perthame's assumptions, the construction in the proof of Theorem 2.7 provides a solution of the Vlasov-Poisson equation as the one built in [28]. In particular, thanks to the a priori estimate (3.35) on the approximating sequence, it is easy to see that $\rho^{\text{eff}} = \rho$, therefore providing a Lagrangian (and therefore renormalized and distributional) solution of Vlasov-Poisson.

Analogously, under the assumptions of [34], a similar argument shows that the generalized solutions built in Theorem 2.7 solve the classical Vlasov-Poisson system.

4 Maximal Regular Flows of the state space and renormalized solutions

The aim of this and next section is to develop the abstract theory of Maximal Regular Flows and Lagrangian/renormalized solutions that are behind the results presented in the previous sections. We warn the reader that from now on, since the theory is completely general, we shall often consider flows of vector fields in $\mathbb{R}^d$ and denote by x a point in $\mathbb{R}^d$. Then, for the applications to kinetic equations in the phase-space $\mathbb{R}^{2d}$, one should apply these results replacing d with $2d$ and x with (x, v).

4.1 Preliminaries on Maximal Regular Flows

In this section we recall the basic results in [2], where a local version of the theory of DiPerna-Lions [16] and Ambrosio [1] was developed. First we recall the definition of a local (in space and time) version of the Regular Lagrangian Flow introduced by Ambrosio [1]. Here and in the sequel, $\mathcal{B}(\mathbb{R}^d)$ denotes the collection of Borel sets in $\mathbb{R}^d$, and $AC([\tau_1, \tau_2]; \mathbb{R}^d)$ is the space of absolutely continuous curves on $[\tau_1, \tau_2]$ with values in $\mathbb{R}^d$, $\mathscr{L}^d$ is the Lebesgue measure in $\mathbb{R}^d$.

We denote by $\mathscr{P}(X)$ the set of Borel probability measures on a metric space X, and we use $e_t : C([0,T]; \mathbb{R}^k) \to \mathbb{R}^k$ to denote the evaluation map at time t, that is $e_t(\gamma) := \gamma(t)$ for any continuous curve γ (depending on the context, k may be equal to d or $2d$). Finally, given two metric spaces X and Y, a finite Borel measure μ on X and a μ-measurable map $f : X \to Y$, we by $f_{\#}\mu$ the pushforward measure on Y, so that $\int_Y \phi \, df_{\#}\mu = \int_X \phi \circ f \, d\mu$ whenever the integrals make sense.

Definition 4.1 (Regular Flow). *Let $B \in \mathcal{B}(\mathbb{R}^d)$, $\tau_1 < \tau_2$, and $\boldsymbol{b} : (\tau_1, \tau_2) \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field. We say that a Borel map $\boldsymbol{X} : [\tau_1, \tau_2] \times B \to \mathbb{R}^d$ is a* Regular Flow *(relative to $\boldsymbol{b}$) in $[\tau_1, \tau_2] \times B$ if the following two properties hold:*

(i) for a.e. $x \in B$, $\boldsymbol{X}(\cdot, x) \in AC([\tau_1, \tau_2]; \mathbb{R}^d)$ and solves the ODE $\dot{x}(t) = \boldsymbol{b}_t(x(t))$ a.e. in (τ_1, τ_2), with the initial condition $\boldsymbol{X}(\tau_1, x) = x$;

(ii) there exists a constant $C = C(\boldsymbol{X})$ satisfying $\boldsymbol{X}(t, \cdot)_{\#}(\mathscr{L}^d \llcorner B) \le C\mathscr{L}^d$ for all $t \in [\tau_1, \tau_2]$.

Let $T \in (0, \infty)$ and let $\boldsymbol{b} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field. The main object of our analysis is the Maximal Regular Flow, which takes into account the possibility of blow-up before time T (or after time 0, when an initial condition $s \in (0, T)$ is under consideration).

Definition 4.2 (Maximal Regular Flow). *For every $s \in (0, T)$ we say that a Borel map $\boldsymbol{X}(\cdot, s, \cdot)$ is a* Maximal Regular Flow *starting at time s if there exist two Borel maps $T^+_{s,\boldsymbol{X}} : \mathbb{R}^d \to (s, T]$, $T^-_{s,\boldsymbol{X}} : \mathbb{R}^d \to [0, s)$ such that $\boldsymbol{X}(\cdot, s, x)$ is defined in $(T^-_{s,\boldsymbol{X}}(x), T^+_{s,\boldsymbol{X}}(x))$ and the following two properties hold:*

(i) for a.e. $x \in \mathbb{R}^d$, $\boldsymbol{X}(\cdot, s, x) \in AC_{\mathrm{loc}}((T^-_{s,\boldsymbol{X}}(x), T^+_{s,\boldsymbol{X}}(x)); \mathbb{R}^d)$ and solves the ODE $\dot{x}(t) = \boldsymbol{b}_t(x(t))$ a.e. in $(T^-_{s,\boldsymbol{X}}(x), T^+_{s,\boldsymbol{X}}(x))$, with the initial condition $\boldsymbol{X}(s, s, x) = x$;

(ii) there exists a constant $C = C(s, \boldsymbol{X})$ such that

$$\boldsymbol{X}(t, s, \cdot)_{\#}\big(\mathscr{L}^d \llcorner \{T^-_{s,\boldsymbol{X}} < t < T^+_{s,\boldsymbol{X}}\}\big) \le C\mathscr{L}^d \qquad \forall\, t \in [0, T]; \tag{4.1}$$

(iii) for a.e. $x \in \mathbb{R}^d$, either $T^+_{s,\boldsymbol{X}}(x) = T$ (resp. $T^-_{s,\boldsymbol{X}}(x) = 0$) and $\boldsymbol{X}(\cdot, s, x)$ can be continuously extended up to $t = T$ (resp. $t = 0$) so that $\boldsymbol{X}(\cdot, s, x) \in C([s, T]; \mathbb{R}^d)$ (resp. $\boldsymbol{X}(\cdot, s, x) \in C([0, s]; \mathbb{R}^d)$), or

$$\lim_{t \uparrow T^+_{s,\boldsymbol{X}}(x)} |\boldsymbol{X}(t, s, x)| = \infty \qquad \textit{(resp.} \ \lim_{t \downarrow T^-_{s,\boldsymbol{X}}(x)} |\boldsymbol{X}(t, s, x)| = \infty). \tag{4.2}$$

In particular, $T^+_{s,\boldsymbol{X}}(x) < T$ (resp. $T^-_{s,\boldsymbol{X}}(x) > 0$) implies (4.2).

The definition of Maximal Regular Flow can be extended up to the times $s = 0$ and $s = T$ setting $T^-_{0,\boldsymbol{X}} \equiv 0$ and $T^+_{T,\boldsymbol{X}} \equiv T$.

A Maximal Regular Flow has been built in [2] under general local assumptions on $\boldsymbol{b}$. Before stating the result, we recall these assumptions. For $T \in (0, \infty)$ we are given a Borel vector field $\boldsymbol{b} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying:

(A1) $\int_0^T \int_{B_R} |\boldsymbol{b}_t(x)| \, dx \, dt < \infty$ for any $R > 0$;

(A2) for any nonnegative $\bar\rho \in L^\infty_+(\mathbb{R}^d)$ with compact support and any closed interval $[a,b] \subset [0,T]$, both continuity equations

$$\frac{d}{dt}\rho_t \pm \operatorname{div}(\boldsymbol{b}_t\rho_t) = 0 \qquad \text{in } (a,b)\times\mathbb{R}^d \tag{4.3}$$

have at most one solution in the class of all weakly* nonnegative continuous functions $[a,b] \ni t \mapsto \rho_t$ with $\rho_a = \bar\rho$ and $\cup_{t\in[a,b]} \operatorname{supp}\rho_t \Subset \mathbb{R}^d$.

Since the vector fields that arise in the applications we have in mind are divergence-free, we assume throughout the paper that our velocity field $\boldsymbol{b}$ satisfies

$$\operatorname{div}\boldsymbol{b}_t = 0 \qquad \text{in } \mathbb{R}^d \text{ in the sense of distributions, for a.e. } t \in (0,T). \tag{4.4}$$

The existence and uniqueness of the Maximal Regular Flow after time s, as well as the semigroup property, were proved in [2, Theorems 5.7, 6.1, 7.1] assuming a one sided bound (specifically a lower bound) on the divergence. In this context, uniqueness should be understood as follows: if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are Maximal Regular Flows, for all $s \in [0,T]$ one has

$$\begin{cases} T^\pm_{s,\boldsymbol{X}}(x) = T^\pm_{s,\boldsymbol{Y}}(x) \text{ for a.e. } x \in \mathbb{R}^d \\ \boldsymbol{X}(\cdot,s,x) = \boldsymbol{Y}(\cdot,s,x) \text{ in } (T^-_{s,\boldsymbol{X}}(x), T^+_{s,\boldsymbol{X}}(x)) \text{ for a.e. } x \in \mathbb{R}^d. \end{cases} \tag{4.5}$$

Under our assumptions on the divergence, by simply reversing the time variable, the Maximal Regular Flow can be built both forward and backward in time, so we state the result directly in the time-reversible case.

Theorem 4.3 (Existence, uniqueness, and semigroup property)**.** *Let $\boldsymbol{b} : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field satisfying* **(A1)** *and* **(A2)***. Then the Maximal Regular Flow starting from any $s \in [0,T]$ is unique according to* (4.5)*, and existence is ensured under the additional assumption* (4.4)*. In addition, still assuming* (4.4)*, for all $s \in [0,T]$ the following properties hold:*

(i) the compressibility constant $C(s,\boldsymbol{X})$ in Definition 4.2 equals 1 and for every $t \in [0,T]$

$$\boldsymbol{X}(t,s,\cdot)_\#\big(\mathscr{L}^d \llcorner \{T^-_{s,\boldsymbol{X}} < t < T^+_{s,\boldsymbol{X}}\}\big) = \mathscr{L}^d \llcorner \big(\boldsymbol{X}(t,s,\cdot)(\{T^-_{s,\boldsymbol{X}} < t < T^+_{s,\boldsymbol{X}}\})\big); \tag{4.6}$$

(ii) if $\tau_1 \in [0,s]$, $\tau_2 \in [s,T]$, and $\boldsymbol{Y}$ is a Regular Flow in $[\tau_1,\tau_2]\times B$, then $T^+_{s,\boldsymbol{X}} > \tau_2$, $T^-_{s,\boldsymbol{X}} < \tau_1$ a.e. in B; moreover

$$\boldsymbol{X}(\cdot,s,x) = \boldsymbol{Y}(\cdot,\boldsymbol{X}(\tau_1,s,x)) \quad \text{in } [\tau_1,\tau_2], \text{ for a.e. } x \in B; \tag{4.7}$$

(iii) the Maximal Regular Flow satisfies the semigroup property, namely for all s, $s' \in [0,T]$

$$T^\pm_{s',\boldsymbol{X}}(\boldsymbol{X}(s',s,x)) = T^\pm_{s,\boldsymbol{X}}(x), \qquad \text{for } \mathscr{L}^d\text{-a.e. } x \in \{T^+_{s,\boldsymbol{X}} > s' > T^-_{s,\boldsymbol{X}}\}, \tag{4.8}$$

and, for a.e. $x \in \{T^+_{s,\boldsymbol{X}} > s' > T^-_{s,\boldsymbol{X}}\}$,

$$\boldsymbol{X}\big(t,s',\boldsymbol{X}(s',s,x)\big) = \boldsymbol{X}(t,s,x) \qquad \forall\, t \in (T^-_{s,\boldsymbol{X}}(x), T^+_{s,\boldsymbol{X}}(x)). \tag{4.9}$$

4.2 Uniqueness for the continuity equation and singular integrals

In this section we deal with uniqueness of solutions to the continuity equation when the gradient of the vector field is given by the singular integral of a time dependent family of measures. The theorem is a minor variant of a result by Bohun, Bouchut, and Crippa [8] (see also [10, 11], where the uniqueness is proved for vector fields whose gradient is the singular integral of an L^1 function). We give the proof of the theorem under the precise assumptions that we need later on, since [8] deals with globally defined regular flows (hence the authors need to assume global growth conditions on the vector field), whereas here we present a local version of such result.

Theorem 4.4. *Let* $\boldsymbol{b} : (0,T) \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ *be given by* $\boldsymbol{b}_t(x,v) = (\boldsymbol{b}_{1t}(v), \boldsymbol{b}_{2t}(x))$*, where*

$$\boldsymbol{b}_1 \in L^\infty((0,T); W^{1,\infty}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)), \qquad \boldsymbol{b}_{2t} = K * \rho_t,$$

with $\rho \in L^\infty((0,T); \mathscr{M}_+(\mathbb{R}^d))$ *and* $K(x) = x/|x|^d$*.*
Then $\boldsymbol{b}$ *satisfies* (**A2**) *of Section 4.1, namely the uniqueness of bounded compactly supported nonnegative distributional solutions of the continuity equation.*

Proof. To simplify the notation we give the proof in the case of autonomous vector fields (in particular $\rho_t = \rho$ is independent of time), but the exact same computations work for the general statement.

It is enough to show that, given $B_R \subset \mathbb{R}^d$ and $\boldsymbol{\eta} \in \mathscr{P}\big(C([0,T]; B_R \times B_R)\big)$ concentrated on integral curves of $\boldsymbol{b}$ and such that $(e_t)_\# \boldsymbol{\eta} \le C_0 \mathscr{L}^{2d}$ for all $t \in [0,T]$, the disintegration $\boldsymbol{\eta}_x$ of $\boldsymbol{\eta}$ with respect to the map e_0 is a Dirac delta for $e_{0\#}\boldsymbol{\eta}$-a.e. x. Indeed, thanks to Theorem 5.1 below, any two bounded nonnegative distributional solutions supported inside $B_R \times B_R$ and starting from the same initial datum $\bar\rho$ can be represented by $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in \mathscr{P}\big(C([0,T]; B_R \times B_R)\big)$. Hence, setting $\boldsymbol{\eta} = (\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2)/2$, if we can prove that $\boldsymbol{\eta}_x$ is a Dirac delta for $\bar\rho$-a.e. x we deduce that $(\boldsymbol{\eta}_1)_x = (\boldsymbol{\eta}_2)_x = \boldsymbol{\eta}_x$ for $\bar\rho$-a.e. x, thus $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2$.

To show that $\boldsymbol{\eta}_x$ is a Dirac delta for $e_{0\#}\boldsymbol{\eta}$-a.e. x, let us consider the function

$$\Phi_{\delta,\zeta}(t) := \iiint \log\Big(1 + \frac{|\gamma^1(t) - \eta^1(t)|}{\zeta\,\delta} + \frac{|\gamma^2(t) - \eta^2(t)|}{\delta}\Big)\, d\boldsymbol{\eta}_x(\gamma)\, d\boldsymbol{\eta}_x(\eta)\, d\bar\rho(x),$$

where δ, $\zeta \in (0,1)$ are small parameters to be chosen later, $t \in [0,T]$, $\bar\rho := (e_0)_\#\boldsymbol{\eta}$, and we use the notation $\gamma(t) = (\gamma^1(t), \gamma^2(t)) \in \mathbb{R}^d \times \mathbb{R}^d$. It is clear that $\Phi_{\delta,\zeta}(0) = 0$.

Let us define the probability measure $\mu \in \mathscr{P}\big(\mathbb{R}^d \times C([0,T];\mathbb{R}^d)^2\big)$ by $d\mu(x,\eta,\gamma) := d\boldsymbol{\eta}_x(\eta)\, d\boldsymbol{\eta}_x(\gamma)\, d\bar\rho(x)$, and assume by contradiction that $\boldsymbol{\eta}_x$ is not a Dirac delta for $\bar\rho$-a.e. x. This means that there exists a constant $a > 0$ such that

$$\iiint \Big(\int_0^T \min\big\{|\gamma(t) - \eta(t)|, 1\big\}\, dt \Big)\, d\mu(x,\eta,\gamma) \ge a.$$

By Fubini's Theorem this implies that there exists a time $t_0 \in (0,T]$ such that

$$\iiint \min\big\{|\gamma(t_0) - \eta(t_0)|, 1\big\}\, d\mu(x,\eta,\gamma) \ge \frac{a}{T}.$$

Since the integrand is bounded by 1 and the measure μ has mass 1, this means that the set

$$A := \left\{(x,\eta,\gamma) \,:\, \min\bigl\{|\gamma(t_0)-\eta(t_0)|,1\bigr\} \geq \frac{a}{2T}\right\}$$

has μ-measure at least $a/(2T)$. Then, assuming without loss of generality that $a \leq 2T$, this implies that $|\gamma(t_0)-\eta(t_0)| \geq a/(2T)$ for all $(x,\eta,\gamma)\in A$, hence

$$\begin{aligned}\Phi_{\delta,\zeta}(t_0) &\geq \iiint_A \log\Bigl(1+\frac{|\gamma^1(t_0)-\eta^1(t_0)|}{\zeta\,\delta}+\frac{|\gamma^2(t_0)-\eta^2(t_0)|}{\delta}\Bigr)\,d\mu(x,\eta,\gamma)\\ &\geq \frac{a}{2T}\log\Bigl(1+\frac{a}{2\delta T}\Bigr).\end{aligned} \tag{4.10}$$

We now want to show that this is impossible.

Computing the time derivative of $\Phi_{\delta,\zeta}$ we see that

$$\frac{d\Phi_{\delta,\zeta}}{dt}(t) \leq \int_{\mathbb{R}^d}\int\int\left(\frac{|\boldsymbol{b}_1(\gamma^2(t))-\boldsymbol{b}_1(\eta^2(t))|}{\zeta\bigl(\delta+|\gamma^2(t)-\eta^2(t)|\bigr)}+\frac{\zeta|\boldsymbol{b}_2(\gamma^1(t))-\boldsymbol{b}_2(\eta^1(t))|}{\zeta\,\delta+|\gamma^1(t)-\eta^1(t)|}\right)d\mu(x,\eta,\gamma). \tag{4.11}$$

By our assumption on $\boldsymbol{b}_1$, the first summand is easily estimated using the Lipschitz regularity of $\boldsymbol{b}_1$ in B_R:

$$\int_{\mathbb{R}^d}\int\int\frac{|\boldsymbol{b}_1(\gamma^2(t))-\boldsymbol{b}_1(\eta^2(t))|}{\zeta(\delta+|\gamma^2(s)-\eta^2(s)|)}\,d\mu(x,\eta,\gamma) \leq \frac{\|\nabla\boldsymbol{b}_1\|_{L^\infty(B_R)}}{\zeta}. \tag{4.12}$$

To estimate the second integral we show that for some constant C, which depends only on d, $\rho(\mathbb{R}^d)$, and R, one has

$$\iiint\frac{\zeta\,|K*\rho(\gamma^1(t))-K*\rho(\eta^1(t))|}{\zeta\,\delta+|\gamma^1(t)-\eta^1(t)|}\,d\mu(x,\eta,\gamma)\leq C\,\zeta\left(1+\log\Bigl(\frac{C}{\zeta\,\delta}\Bigr)\right). \tag{4.13}$$

To this end, we first recall the definition of weak L^p norm of a μ-measurable function $f: X\to\mathbb{R}$ in a measure space (X,μ):

$$|||f|||_{M^p(X,\mu)} := \sup\bigl\{\lambda\,\mu(\{|f|>\lambda\})^{1/p} : \lambda>0\bigr\}.$$

By [11, Proposition 4.2 and Theorem 3.3(ii)] there exists a modified maximal operator $\tilde{M}$, which associates to every $\sigma\in\mathscr{M}_+(\mathbb{R}^d)$ a function $\tilde{M}(DK*\sigma)\in L^1(\mathbb{R}^d)$ satisfying the following properties: there exists a set L with $\mathscr{L}^d(L)=0$ such that

$$|K*\sigma(x)-K*\sigma(y)|\leq C\bigl[\tilde{M}(DK*\sigma)(x)+\tilde{M}(DK*\sigma)(y)\bigr]\,|x-y| \qquad \forall\, x,\, y\in\mathbb{R}^d\setminus L, \tag{4.14}$$

and the weak-L^1 estimate

$$|||\tilde{M}(DK*\sigma)|||_{M^1(B_R)}\leq C\,\sigma(\mathbb{R}^d) \tag{4.15}$$

holds with a constant C which depends only on d and R. Applying (4.14), we see that

$$\iiint\frac{|K*\rho(\gamma^1(t))-K*\rho(\eta^1(t))|}{\zeta\,\delta+|\gamma^1(t)-\eta^1(t)|}\,d\mu\leq\int g_t(x,\eta,\gamma)\,d\mu, \tag{4.16}$$

? where

$$g_t(x,\eta,\gamma) := \min\left\{C\,\tilde{M}(DK*\rho)(\gamma^1(t))+C\,\tilde{M}(DK*\rho)(\eta^1(t)),\,\frac{|K*\rho|(\gamma^1(t))+|K*\rho|(\eta^1(t))}{\zeta\,\delta}\right\}.$$

Let us fix $p := \frac{d}{d-1/2} \in \left(1, \frac{d}{d-1}\right)$, so that $|K| \in L^p_{\text{loc}}(\mathbb{R}^d)$. The last term in (4.16) can be estimated thanks to the following interpolation inequality (see [11, Lemma 2.2])

$$\|g_t\|_{L^1(\mu)} \le \frac{p}{p-1}|||g_t|||_{M^1(\mu)}\left(1+\log\left(\frac{|||g_t|||_{M^p(\mu)}}{|||g_t|||_{M^1(\mu)}}\right)\right).$$

Also, the first term in the right-hand side above can be estimated using our assumption $(e_t)_\#\boldsymbol{\eta} \le C_0\mathscr{L}^d$ and (4.15):

$$\begin{aligned}
|||g_t|||_{M^1(\mu)} &\le 2\,|||\tilde{M}(DK*\rho)(\eta^1(t))|||_{M^1(\mu)}\\
&= 2\,|||\tilde{M}(DK*\rho)(\eta^1(t))|||_{M^1(\boldsymbol{\eta})}\\
&= 2\,|||\tilde{M}(DK*\rho)(x)|||_{M^1(B_R\times B_R,e_{t\#}\boldsymbol{\eta})}\\
&\le 2\,C_0\,|||\tilde{M}(DK*\rho)(x)|||_{M^1(B_R\times B_R,\mathscr{L}^{2d})}\\
&\le 2\,C_0\,\mathscr{L}^d(B_R)\,|||\tilde{M}(DK*\rho)(x)|||_{M^1(B_R,\mathscr{L}^d)}\\
&\le 2\,C_0\,C\,\mathscr{L}^d(B_R)\,\rho(\mathbb{R}^d).
\end{aligned}$$

Similarly, the second term in the right hand side can be estimated using $(e_t)_\#\boldsymbol{\eta} \le C_0\mathscr{L}^d$ and Young's inequality:

$$\begin{aligned}
|||g_t|||_{M^p(\mu)} &\le 2\,(\zeta\,\delta)^{-1}\|K*\rho(\eta^1(t))\|_{L^p(\mu)}\\
&= 2\,(\zeta\,\delta)^{-1}\|K*\rho(\eta^1(t))\|_{L^p(\boldsymbol{\eta})}\\
&\le 2\,C_0\,(\zeta\,\delta)^{-1}\|K*\rho(x)\|_{L^p(B_R\times B_R)}\\
&\le 2\,C_0\,(\zeta\,\delta)^{-1}\mathscr{L}^d(B_R)\,\|K*\rho\|_{L^p(B_R)}\\
&\le 2\,C_0\,(\zeta\,\delta)^{-1}\mathscr{L}^d(B_R)\,\|K\|_{L^p(B_R)}\,\rho(\mathbb{R}^d)\\
&\le C\,(\zeta\,\delta)^{-1},
\end{aligned}$$

where C depends on d, R, and $\rho(\mathbb{R}^d)$. Combining these last estimates with (4.16), we obtain (4.13).

Then, using (4.11), (4.12), and (4.13), we deduce that

$$\frac{d\Phi_{\delta,\zeta}}{dt}(t) \le \frac{C}{\zeta} + C\,\zeta + C\,\zeta\,\log\left(\frac{C}{\zeta\,\delta}\right)$$

for some constant C depending only on d, R, $\rho(\mathbb{R}^d)$, and $\|\nabla b_1\|_{L^\infty(\mathbb{R}^d)}$. Integrating with respect to time in $[0,t_0]$, we find that

$$\Phi_{\delta,\zeta}(t_0) \le C\,t_0\left(\frac{1}{\zeta} + \zeta + \zeta\,\log\left(\frac{C}{\zeta}\right) + \zeta\,\log\left(\frac{1}{\delta}\right)\right).$$

Choosing first $\zeta > 0$ small enough in order to have $C\,t_0\,\zeta < a/(2T)$ and then letting $\delta \to 0$, we find a contradiction with (4.10), which concludes the proof. □

4.3 Generalized flows and Maximal Regular Flows

We denote by $\mathring{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ the one-point compactification of $\mathbb{R}^d$ and we recall the definition of generalized flow and of regular generalized flow in our context, as introduced in [2, Definition 5.3]. Given an open subset $A \subset [0,\infty)$, we denote by $AC_{\mathrm{loc}}(A;\mathbb{R}^d)$ the set of continuous curves $\gamma : A \to \mathbb{R}$ that are absolutely continuous when restricted to any closed interval contained in A.

Definition 4.5 (Generalized flow)**.** *Let $\boldsymbol{b} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field. The measure $\boldsymbol{\eta} \in \mathscr{M}_+\big(C([0,T];\mathring{\mathbb{R}}^d)\big)$ is said to be a* generalized flow *of $\boldsymbol{b}$ if $\boldsymbol{\eta}$ is concentrated on the set*[3]

$$\Gamma := \big\{\eta \in C([0,T];\mathring{\mathbb{R}}^d) : \ \eta \in AC_{\mathrm{loc}}(\{\eta \neq \infty\};\mathbb{R}^d) \ \textit{and} \\ \dot\eta(t) = \boldsymbol{b}_t(\eta(t)) \ \textit{for a.e. } t \ \textit{such that } \eta(t) \neq \infty\big\}. \quad (4.17)$$

We say that a generalized flow $\boldsymbol{\eta}$ is regular *if there exists $L_0 \geq 0$ satisfying*

$$(e_t)_\#\boldsymbol{\eta} \llcorner \mathbb{R}^d \leq L_0 \mathscr{L}^d \qquad \forall\, t \in [0,T]. \quad (4.18)$$

In the case of a smooth bounded vector field, whose associated flow is denoted by $\boldsymbol{X}$ a particular class of generalized flows is the one generated by transporting a given measure $\mu_0 \in \mathscr{P}\big(\mathbb{R}^d\big)$ along the integral lines of the flow:

$$\boldsymbol{\eta} = \int_{\mathbb{R}^d} \delta_{\boldsymbol{X}(\cdot,x)}\, d\mu_0(x),$$

where $\delta_{\boldsymbol{X}(\cdot,x)} \in \mathscr{P}\big(AC([0,T];\mathbb{R}^d)\big)$ denotes the Dirac delta on the path $\boldsymbol{X}(\cdot,x)$.

In the next definition we propose a generalization of this construction involving Maximal Regular Flows.

[3]In connection with the definition of generalized flow, let us provide a sketch of proof of the fact that the set Γ in (4.17) is Borel in $C([0,T];\mathring{\mathbb{R}}^d)$.

First of all one notices that, for all intervals $[a,b] \subset [0,T]$, the set $\{\eta : \ \eta([a,b]) \subset \mathbb{R}^d\}$ is Borel. Then, considering the absolute continuity of a curve η in the integral form

$$|\eta(t) - \eta(s)| \leq \int_s^t |\boldsymbol{b}_r(\eta(r))|\, dr \qquad \forall\, s,t \in [a,b], \ s \leq t,$$

it is sufficient to verify (arguing componentwise and splitting in positive and negative part) that for any nonnegative Borel function $\boldsymbol{c}$ and for any $s,t \in [0,T]$ with $s \leq t$ fixed, the function

$$\eta \mapsto \int_s^t \boldsymbol{c}_r(\eta(r))\, dr$$

is Borel in $\{\eta : \ \eta([a,b]) \subset \mathbb{R}^d\}$. This follows by a monotone class argument, since the property is obviously true for continuous functions and it is stable under equibounded and monotone convergence.

As soon as the absolute continuity property is secured, also the verification of the Borel regularity of

$$\Gamma \cap \{\eta : \ \eta([a,b]) \subset \mathbb{R}^d\} = \big\{\eta \in C([0,T]);\mathring{\mathbb{R}}^d) : \eta \in AC([a,b];\mathbb{R}^d), \ \dot\eta(t) = \boldsymbol{b}_t(\eta(t)) \text{ a.e. in } (a,b)\big\}$$

can be achieved following similar lines. Finally, by letting the endpoints a, b vary in a countable dense set we obtain that Γ is Borel.

Definition 4.6 (Measures transported by the Maximal Regular Flow and Lagrangian solutions)**.** *Let* $\boldsymbol{b} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ *be a Borel vector field having a Maximal Regular Flow* $\boldsymbol{X}$*, and let* $\boldsymbol{\eta} \in \mathscr{M}_+\big(C([0,T];\mathring{\mathbb{R}}^d)\big)$ *with* $(e_t)_\#\boldsymbol{\eta} \ll \mathscr{L}^d$ *for all* $t \in [0,T]$*. We say that* $\boldsymbol{\eta}$ *is transported by* $\boldsymbol{X}$ *if, for all* $s \in [0,T]$*,* $\boldsymbol{\eta}$ *is concentrated on*

$$\big\{\eta \in C([0,T];\mathring{\mathbb{R}}^d) : \eta(s) = \infty \ \text{or}\ \eta(\cdot) = \boldsymbol{X}(\cdot, s, \eta(s)) \ \text{in}\ (T^-_{s,\boldsymbol{X}}(\eta(s)), T^+_{s,\boldsymbol{X}}(\eta(s)))\big\}. \quad (4.19)$$

Correspondingly, let $\rho_t \in L^\infty((0,T); L^1_+(\mathbb{R}^d))$ *be a distributional solution of the continuity equation, weakly continuous on* $[0,T]$ *in duality with* $C_c(\mathbb{R}^d)$*. We say that* ρ_t *is a Lagrangian solution if there exists* $\boldsymbol{\eta} \in \mathscr{M}_+\big(C([0,T];\mathring{\mathbb{R}}^d)\big)$ *transported by* $\boldsymbol{X}$ *with* $(e_t)_\#\boldsymbol{\eta} = \rho_t \mathscr{L}^d$ *for every* $t \in [0,T]$*.*

The absolute continuity assumption $(e_t)_\#\boldsymbol{\eta} \ll \mathscr{L}^d$ on the marginals of $\boldsymbol{\eta}$ is needed to ensure that this notion is invariant with respect to the uniqueness property in (4.5). In other words, if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are related as in (4.5) then $\boldsymbol{\eta}$ is transported by $\boldsymbol{X}$ if and only if $\boldsymbol{\eta}$ is transported by $\boldsymbol{Y}$.

It is easily seen that if $\boldsymbol{\eta}$ is transported by a Maximal Regular Flow, then $\boldsymbol{\eta}$ is a generalized flow according to Definition 4.5, but in connection with the proof of the renormalization property we are more interested in the converse statement. As shown in the next theorem, this holds for regular generalized flows and for divergence-free vector fields satisfying **(A1)**-**(A2)** of Section 4.1.

Theorem 4.7 (Regular generalized flows are transported by $\boldsymbol{X}$)**.** *Let* $\boldsymbol{b} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ *be a divergence-free vector field satisfying* **(A1)**-**(A2)** *of Section 4.1 and let* $\boldsymbol{X}$ *be its Maximal Regular Flow. Let* $\boldsymbol{\eta} \in \mathscr{M}_+\big(C([0,T];\mathring{\mathbb{R}}^d)\big)$ *be a regular generalized flow according to Definition 4.5.*
Given $s \in [0,T]$*, consider a Borel family* $\{\boldsymbol{\eta}^s_x\} \subset \mathscr{P}\big(C([0,T];\mathring{\mathbb{R}}^d)\big)$*,* $x \in \mathring{\mathbb{R}}^d$*, of conditional probability measures representing* $\boldsymbol{\eta}$ *with respect to the marginal* $(e_s)_\#\boldsymbol{\eta}$*, that is,* $\int \boldsymbol{\eta}^s_x \, d[(e_s)_\#\boldsymbol{\eta}](x) = \boldsymbol{\eta}$*. Then for* $(e_s)_\#\boldsymbol{\eta}$*-almost every* $x \in \mathbb{R}^d$ *we have that* $\boldsymbol{\eta}^s_x$ *is concentrated on the set*

$$\hat{\Gamma}_s := \big\{\eta \in C([0,T];\mathring{\mathbb{R}}^d) \,:\, \eta(s) = x,\ \eta(\cdot) = \boldsymbol{X}(\cdot, s, \eta(s)) \ \text{in}\ (T^-_{s,\boldsymbol{X}}(\eta(s)), T^+_{s,\boldsymbol{X}}(\eta(s)))\big\}. \quad (4.20)$$

In particular $\boldsymbol{\eta}$ *is transported by* $\boldsymbol{X}$*.*

Proof. First of all we notice that the set $\hat{\Gamma}_s$ in (4.20) is Borel. Indeed, the maps $\eta \mapsto T^\pm_{s,\boldsymbol{X}}(\eta(s))$ are Borel because $T^\pm_{\boldsymbol{X}}$ are Borel in $\mathbb{R}^d$, and the map $\eta \mapsto \boldsymbol{X}(t, s, \eta(s))$ is Borel as well for any $t \in [0,T]$. Therefore, choosing a countable dense set of times $t \in [0,T]$ the Borel regularity of $\hat{\Gamma}_s$ is achieved.

The fact that $\boldsymbol{\eta}^s_x$ is concentrated on the set $\{\eta \,:\, \eta(s) = x\}$ is immediate from the definition of $\boldsymbol{\eta}^s_x$. We now show that for $(e_s)_\#\boldsymbol{\eta}$-almost every $x \in \mathbb{R}^d$ the measure $\boldsymbol{\eta}^s_x$ is concentrated on the set

$$\big\{\eta \in C([0,T];\mathring{\mathbb{R}}^d) : \eta(\cdot) = \boldsymbol{X}(\cdot, s, x) \ \text{in}\ [s, T^+_{s,\boldsymbol{X}}(x))\big\}. \quad (4.21)$$

Applying the same result backward in time, this will prove that $\boldsymbol{\eta}^s_x$ is concentrated on the set $\hat{\Gamma}_s$ in (4.20).

For $r \in (s,T]$ we denote by $\Sigma^{s,r} : C([0,T];\mathring{\mathbb{R}}^d) \to C([s,r];\mathring{\mathbb{R}}^d)$ the map induced by restriction to $[s,r]$, namely $\Sigma^{s,r}(\eta) := \eta|_{[s,r]}$.

For every $R > 0$, $r \in (s,T]$, let us consider

$$\boldsymbol{\eta}^{R,r} := \Sigma^{s,r}_{\#}\Big(\boldsymbol{\eta} \llcorner \big\{\eta : \eta(t) \in B_R \text{ for every } t \in [s,r]\big\}\Big).$$

By construction $\boldsymbol{\eta}^{R,r}$ is a regular generalized flow relative to $\boldsymbol{b}$ with compact support, hence our regularity assumption on $\boldsymbol{b}$ allows us to apply [2, Theorem 3.4] to deduce that

$$\boldsymbol{\eta}^{R,r} = \int \delta_{\boldsymbol{Y}(\cdot,x)}\, d[(e_s)_{\#}\boldsymbol{\eta}^{R,r}](x), \tag{4.22}$$

where $\boldsymbol{Y}(\cdot,x)$ is an integral curve of $\boldsymbol{b}$ in $[s,r]$ for $(e_s)_{\#}\boldsymbol{\eta}$-a.e. $x \in \mathbb{R}^d$. Let us denote by $\rho_{R,r}$ the density of $(e_s)_{\#}\boldsymbol{\eta}^{R,r}$ with respect to $\mathscr{L}^d$, which is bounded by L_0 thanks to (4.18). For every $\delta > 0$ we have that

$$\begin{aligned} \boldsymbol{Y}(t,\cdot)_{\#}\big(\mathscr{L}^d \llcorner \{\rho_{R,r} > \delta\}\big) &= (e_t)_{\#} \int_{\{\rho_{R,r} > \delta\}} \delta_{\boldsymbol{Y}(\cdot,x)}\, d\mathscr{L}^d(x) \\ &\le \frac{1}{\delta}\,(e_t)_{\#} \int_{\{\rho_{R,r} > \delta\}} \delta_{\boldsymbol{Y}(\cdot,x)}\, d[(e_s)_{\#}\boldsymbol{\eta}^{R,r}](x) \\ &\le \frac{1}{\delta}\,(e_t)_{\#}\boldsymbol{\eta}^{R,r} \le \frac{1}{\delta}\,(e_t)_{\#}\boldsymbol{\eta} \llcorner \mathbb{R}^d \le \frac{L_0}{\delta}\,\mathscr{L}^d, \end{aligned} \tag{4.23}$$

hence $\boldsymbol{Y}(\cdot,x)$ is a Regular Flow of $\boldsymbol{b}$ in $[s,r] \times \{\rho_{R,r} > \delta\}$ according to Definition 4.1. By Theorem 4.3(ii) we deduce that $\boldsymbol{Y}(\cdot,x) = \boldsymbol{X}(\cdot,s,x)$ for a.e. $x \in \{\rho_{R,s} > \delta\}$ and therefore, letting $\delta \to 0$,

$$\boldsymbol{Y}(\cdot,x) = \boldsymbol{X}(\cdot,s,x) \qquad \text{in } [s,r] \text{ for } (e_s)_{\#}\boldsymbol{\eta}^{R,s}\text{-a.e. } x \in \mathbb{R}^d. \tag{4.24}$$

Letting $R \to \infty$ we have that $\boldsymbol{\eta}^{R,r} \to \boldsymbol{\sigma}^r$ increasingly, where

$$\boldsymbol{\sigma}^r := \Sigma^{s,r}_{\#}\Big(\boldsymbol{\eta} \llcorner \{\eta : \eta(t) \ne \infty \text{ for every } t \in [s,r]\}\Big),$$

and by (4.22) and (4.24) we get that

$$\boldsymbol{\sigma}^r = \int \delta_{\boldsymbol{X}(\cdot,s,x)}\, d[(e_s)_{\#}\boldsymbol{\sigma}^r](x) \qquad \forall\, r \in (s,T]. \tag{4.25}$$

Now, arguing by contradiction, let us assume that there exists a Borel set $E \subset \mathbb{R}^d$ such that $(e_s)_{\#}\boldsymbol{\eta}(E) > 0$ and $\boldsymbol{\eta}^s_x$ is not concentrated on the set (4.21) for every $x \in E$, namely

$$\boldsymbol{\eta}^s_x\Big(\big\{\eta \in C([0,T];\mathring{\mathbb{R}}^d) : \eta \ne \boldsymbol{X}(\cdot,s,x) \text{ as a curve in } [s, T^+_{s,\boldsymbol{X}}(x))\big\}\Big) > 0.$$

Since this is equivalent to

$$\boldsymbol{\eta}^s_x\Bigg(\bigcup_{r \in \mathbb{Q} \cap (s, T^+_{s,\boldsymbol{X}}(x))} \big\{\eta \in C([0,T];\mathring{\mathbb{R}}^d) : \eta \ne \boldsymbol{X}(\cdot,s,x) \text{ in } [s,r],\ \eta([s,r]) \subset \mathbb{R}^d\big\}\Bigg) > 0,$$

we deduce that for every $x \in E$ there exists $r_x \in \mathbb{Q} \cap (s, T^+_{s,\boldsymbol{X}}(x))$ such that

$$\boldsymbol{\eta}^s_x\Big(\big\{\eta \in C([0,T];\mathring{\mathbb{R}}^d) : \eta \neq \boldsymbol{X}(\cdot, s, x) \text{ as a curve in } [s, r_x],\ \eta([s, r_x]) \subset \mathbb{R}^d\big\}\Big) > 0.$$

In other words, for every $x \in E$ there exists a rational number r_x such that

$$\Sigma^{s,r_x}_{\#}\Big(\boldsymbol{\eta}^s_x \llcorner \{\eta : \eta(t) \neq \infty \text{ for every } t \in [s, r_x]\}\Big) \text{ is nonzero and not a multiple of } \delta_{\boldsymbol{X}(\cdot,s,x)}.$$

Therefore, there exist a Borel set $E' \subset E$ of positive $(e_s)_\#\boldsymbol{\eta}$-measure and $r \in (s,T] \cap \mathbb{Q}$ such that for every $x \in E'$

$$\Sigma^{s,r}_{\#}\Big(\boldsymbol{\eta}^s_x \llcorner \{\eta : \eta(t) \neq \infty \text{ for every } t \in [s, r]\}\Big) \text{ is nonzero and not a multiple of } \delta_{\boldsymbol{X}(\cdot,s,x)}.$$

Notice now that, by (4.25) and $(e_s)_\#\boldsymbol{\sigma}^r \leq (e_s)_\#\boldsymbol{\eta}$, it follows that

$$\int \delta_{\boldsymbol{X}(\cdot,s,x)}\, d[(e_s)_\#\boldsymbol{\eta}](x) \geq \boldsymbol{\sigma}^r = \int \Sigma^{s,r}_{\#}\Big(\boldsymbol{\eta}^s_x \llcorner \{\eta : \eta(t) \neq \infty \text{ for every } t \in [s,r]\}\Big)\, d[(e_s)_\#\boldsymbol{\eta}](x),$$

hence $\delta_{\boldsymbol{X}(\cdot,s,x)} \geq \Sigma^{s,r}_{\#}\Big(\boldsymbol{\eta}^s_x \llcorner \{\eta : \eta(t) \neq \infty \text{ for every } t \in [s,r]\}\Big)$ for $(e_s)_\#\boldsymbol{\eta}$-a.e. x, and therefore a contradiction with the existence of E'. This proves that $\boldsymbol{\eta}^s_x$ is concentrated on the set defined in (4.21), as desired.

Finally, in order to prove that $\boldsymbol{\eta}$ is transported by $\boldsymbol{X}$ we apply the definition of disintegration and the fact that for $(e_s)_\#\boldsymbol{\eta}$-a.e. $x \in \mathbb{R}^d$ the measure $\boldsymbol{\eta}^s_x$ is concentrated on the set $\hat{\Gamma}_s$ in (4.20) to obtain that $\boldsymbol{\eta}(\hat{\Gamma}) = \int \boldsymbol{\eta}^s_x(\hat{\Gamma})\, d[(e_s)_\#\boldsymbol{\eta}](x) = 1$, where $\hat{\Gamma}$ is the set in (4.19). □

4.4 Regular generalized flows and renormalized solutions

We now recall the well-known concept of renormalized solution to a continuity equation. This was already introduced in Section 2 in the context of the Vlasov-Poisson system, but we prefer to reintroduce it here in its general formulation for the convenience of the reader. To fix the ideas we consider the interval $(0,T)$ and 0 as initial time, but the definition can be immediately adapted to general intervals, forward and backward in time.

Definition 4.8 (Renormalized solutions)**.** *Let $\boldsymbol{b} \in L^1_{\mathrm{loc}}((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ be a Borel and divergence-free vector field. A Borel function $\rho : (0,T) \times \mathbb{R}^d \to \mathbb{R}$ is a renormalized solution of the continuity equation relative to $\boldsymbol{b}$ if*

$$\partial_t \beta(\rho) + \nabla \cdot (\boldsymbol{b}\beta(\rho)) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d \qquad \forall\, \beta \in C^1 \cap L^\infty(\mathbb{R}) \tag{4.26}$$

in the sense of distributions. Analogously, we say that ρ is a renormalized solutions starting from a Borel function $\rho_0 : \mathbb{R}^d \to \mathbb{R}$ if

$$\int_{\mathbb{R}^d} \phi_0(x)\, \beta(\rho_0(x))\, dx + \int_0^T \int_{\mathbb{R}^d} [\partial_t \phi_t(x) + \nabla\phi_t(x) \cdot \boldsymbol{b}_t(x)]\, \beta(\rho_t(x))\, dx\, dt = 0 \tag{4.27}$$

for all $\phi \in C^\infty_c([0,T) \times \mathbb{R}^d)$ and all $\beta \in C^1 \cap L^\infty(\mathbb{R})$.

Remark 4.9 (Equivalent formulations)**.** As shown for instance in [4, Section 8.1]), an equivalent formulation of (4.27) is the following: for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ the function $t \mapsto \int_{\mathbb{R}^d} \varphi(x)\, \beta(\rho_t(x))\, dx$ coincides a.e. with an absolutely continuous function $t \mapsto A(t)$ such that $A(0) = \int_{\mathbb{R}^d} \varphi(x)\, \beta(\rho_0(x))\, dx$ and

$$\frac{d}{dt} A(t) = \int_{\mathbb{R}^d} \nabla \varphi(x) \cdot \boldsymbol{b}_t(x)\, \beta(\rho_t(x))\, dx \qquad \text{for a.e. } t \in (0, T). \tag{4.28}$$

Moreover, by an easy approximation argument, the same holds for every Lipschitz compactly supported test function $\varphi : \mathbb{R}^d \to \mathbb{R}$. In this way, possibly splitting φ into its positive and negative part, only nonnegative test functions need to be considered. Analogously, by writing every $\beta \in C^1(\mathbb{R}^d)$ as the sum of a C^1 monotone nondecreasing function and of a C^1 monotone nonincreasing function, we can use the linearity of the equation with respect to $\beta(\rho_t)$ to reduce to the case of $\beta \in C^1 \cap L^\infty(\mathbb{R})$ monotone nondecreasing.

In the next theorem we show first that, flowing an initial datum $\rho_0 \in L^1(\mathbb{R}^d)$ through the maximal flow, we obtain a renormalized solution of the continuity equation. In turn, this is a key tool to prove the second part of the lemma, namely that any measure $\boldsymbol{\eta}$ transported by the maximal regular flow induces, through its marginals, a renormalized solution. The proof of these facts heavily relies on the incompressibility of the flow and therefore on the assumption that the vector field is divergence-free. A generalization of this lemma to the case of vector fields with bounded divergence is possible, but rather technical and long. We notice that the assumptions **(A1)** and **(A2)**, as well as the one on the divergence of the vector field $\boldsymbol{b}$, are used only for the existence and uniqueness of a maximal regular flow which preserves the Lebesgue measure on its domain of definition (see Theorem 4.3).

To fix the ideas, in part (i) of the theorem below we consider only 0 as initial time. An analogous statement can be given for any other initial time $s \in [0, T]$, considering intervals $[0, s]$ or $[s, T]$, with no additional assumption on $\boldsymbol{b}$.

Theorem 4.10 (Equivalence of renormalized and Lagrangian solutions)**.** *Let $\boldsymbol{b} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a divergence-free vector field satisfying* **(A1)**-**(A2)** *of Section 4.1. Let $\boldsymbol{X}(t, s, x)$ be the maximal regular flow of $\boldsymbol{b}$ according to Definition 4.2.*

(i) If $\rho_0 \in L^1(\mathbb{R}^d)$, we define $\rho_t \in L^1(\mathbb{R}^d)$ by

$$\rho_t := \boldsymbol{X}(t, 0, \cdot)_\# (\rho_0 \llcorner \{T^+_{0,\boldsymbol{X}} > t\}), \qquad t \in [0, T).$$

Then ρ_t is a renormalized solution of the continuity equation starting from ρ_0. In addition the map $t \mapsto \rho_t$ is strongly continuous on $[0, T)$ with respect to the L^1_{loc} convergence, and it is also strongly L^1 continuous from the right.

(ii) If $\boldsymbol{\eta} \in \mathscr{M}_+\big(C([0, T]; \mathring{\mathbb{R}}^d)\big)$ is transported by $\boldsymbol{X}$, and $(e_t)_\# \boldsymbol{\eta} \llcorner \mathbb{R}^d \ll \mathscr{L}^d$ for every $t \in [0, T]$, then the density ρ_t of $(e_t)_\# \boldsymbol{\eta} \llcorner \mathbb{R}^d$ with respect to $\mathscr{L}^d$ is strongly continuous on $[0, T)$ with respect to the L^1_{loc} convergence and it is a renormalized solution of the continuity equation.

Proof. We split the proof in four steps.

Step 1: proof of (i), renormalization property of ρ_t. In the proof of (i) we set for simplicity $\boldsymbol{X}(t,x) = \boldsymbol{X}(t,0,x)$ and $T^+_{0,\boldsymbol{X}} = T_{\boldsymbol{X}}$. We first notice that, by the incompressibility of the flow (4.6) and by the definition of ρ_t, for every $t \in [0,T)$ and $\varphi \in C_c(\mathbb{R}^d)$ one has

$$\int_{\{T_{\boldsymbol{X}}>t\}} \varphi(\boldsymbol{X}(t,x))\, \rho_t(\boldsymbol{X}(t,x))\, dx = \int_{\boldsymbol{X}(t,\cdot)(\{T_{\boldsymbol{X}}>t\})} \varphi\, \rho_t\, dx = \int_{\{T_{\boldsymbol{X}}>t\}} \varphi(\boldsymbol{X}(t,x))\, \rho_0\, dx,$$

hence, for any $t \in [0,T)$,

$$\rho_t(\boldsymbol{X}(t,x)) = \rho_0(x) \qquad \text{for } \mathscr{L}^d\text{-a.e. } x \in \{T_{\boldsymbol{X}} > t\}. \tag{4.29}$$

Let $\beta \in C^1 \cap L^\infty(\mathbb{R})$. Using again (4.6) and by (4.29) we have that

$$\int_{\mathbb{R}^d} \varphi\, \beta(\rho_t)\, dx = \int_{\boldsymbol{X}(t,\cdot)(\{T_{\boldsymbol{X}}>t\})} \varphi\, \beta(\rho_t)\, dx = \int_{\{T_{\boldsymbol{X}}>t\}} \varphi(\boldsymbol{X}(t,\cdot))\, \beta(\rho_0)\, dx \tag{4.30}$$

for any $\varphi \in C_c(\mathbb{R}^d)$. In addition, the blow-up property (4.2) ensures that $t \mapsto \varphi(\boldsymbol{X}(t,x))$ can be continuously extended to be identically 0 on the time interval $[T_{\boldsymbol{X}}(x), T)$ (in case of blow-up before time T); furthermore, for the same reason, if $\varphi \in C^1_c(\mathbb{R}^d)$ then the extended map is absolutely continuous in $[0,T]$ and

$$\frac{d}{dt}\varphi(\boldsymbol{X}(t,x)) = \chi_{[0,T_{\boldsymbol{X}}(x))}(t)\, \nabla\varphi(\boldsymbol{X}(t,x)) \cdot \boldsymbol{b}_t(\boldsymbol{X}(t,x)) \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,T). \tag{4.31}$$

Therefore, using (4.30) and integrating (4.31), for all $\varphi \in C^1_c(\mathbb{R}^d)$ we find that

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi\, \beta(\rho_t)\, dx = \int_{\{T_{\boldsymbol{X}}>t\}} \nabla\varphi(\boldsymbol{X}(t,\cdot)) \cdot \boldsymbol{b}_t(\boldsymbol{X}(t,\cdot))\, \beta(\rho_0)\, dx = \int_{\mathbb{R}^d} \nabla\varphi \cdot \boldsymbol{b}_t\, \beta(\rho_t)\, dx$$

for $\mathscr{L}^1$-a.e. $t \in (0,T)$, which proves the renormalization property.

Step 2: proof of (i), strong continuity of ρ_t. We notice that, as a consequence of the possibility of continuously extending the map $t \mapsto \varphi(\boldsymbol{X}(\cdot,x))$ after the time $T_{\boldsymbol{X}}(x)$ for $\varphi \in C_c(\mathbb{R}^d)$, the map $[0,T) \ni t \mapsto \rho_t$ is weakly continuous in the duality with $C_c(\mathbb{R}^d)$. Let us prove now the strong continuity of $t \mapsto \rho_t$.

We start with the proof for $t = 0$. Fix $\epsilon > 0$, let $\psi \in C_c(\mathbb{R}^d)$ with $\|\psi - \rho_0\|_{L^1(\mathbb{R}^d)} < \epsilon$, and notice that the positivity $\mathscr{L}^d$-a.e. in $\mathbb{R}^d$ of $T_{\boldsymbol{X}}$ gives

$$\int_{\mathbb{R}^d} |\rho_t(x) - \psi(x)|\, dx \le \int_{\boldsymbol{X}(t,\cdot)(\{T_{\boldsymbol{X}}>t\})} |\rho_t(x) - \psi(x)|\, dx + \int_{\boldsymbol{X}(t,\cdot)(\{0<T_{\boldsymbol{X}}\le t\})} |\psi(x)|\, dx$$

and that the second summand in the right hand side is infinitesimal as $t \downarrow 0$. Changing variables and using (4.29) together with the incompressibility of the flow, it follows that

$$\int_{\boldsymbol{X}(t,\cdot)(\{T_{\boldsymbol{X}}>t\})} |\rho_t(x) - \psi(x)|\, dx = \int_{\{T_{\boldsymbol{X}}>t\}} |\rho_0(x) - \psi(\boldsymbol{X}(t,x))|\, dx,$$

therefore

$$\limsup_{t\downarrow 0}\int_{\mathbb{R}^d}|\rho_t-\psi|\,dx\le\limsup_{t\downarrow 0}\int_{\{T_{\boldsymbol{X}}>t\}}|\rho_0(x)-\psi(\boldsymbol{X}(t,x))|\,dx\le\int_{\mathbb{R}^d}|\rho_0-\psi|\,dx.$$

This proves that $\limsup_t\|\rho_t-\rho_0\|_{L^1(\mathbb{R}^d)}\le 2\epsilon$ and, by the arbitrariness of ϵ, the desired strong continuity at $t=0$ follows.

We now notice that the same argument together with the semigroup property of Theorem 4.3(iii) shows that the map $t\mapsto\rho_t$ is strongly continuous from the right in L^1. In addition, reversing the time variable and using again the semigroup property, we deduce that the identity $\rho_t(x)=\rho_s(\boldsymbol{X}(t,s,x))\,1_{\{T_{\boldsymbol{X}}>t\}}(\boldsymbol{X}(0,s,x))$ holds, therefore

$$\lim_{s\uparrow t}\int_{\mathbb{R}^d}|\rho_t(x)-\rho_s(x)\,1_{\{T_{\boldsymbol{X}}>t\}}(\boldsymbol{X}(0,s,x))|\,dx=0\qquad\forall\,t\in(0,T).$$

Hence, in order to prove that the map $t\mapsto\rho_t$ is strongly continuous in L^1_{loc}, we are left to show that for every $R>0$ and $t\in(0,T)$ one has

$$\lim_{s\uparrow t}\int_{B_R}|\rho_s(x)-\rho_s(x)\,1_{\{T_{\boldsymbol{X}}>t\}}(\boldsymbol{X}(0,s,x))|\,dx=0.\tag{4.32}$$

For this, we observe that by (4.29) and the incompressibility of the flow, we have that

$$\begin{aligned}\int_{B_R}|\rho_s(x)-\rho_s(x)\,1_{\{T_{\boldsymbol{X}}>t\}}(\boldsymbol{X}(0,s,x))|\,dx&=\int_{B_R}|\rho_s|(x)\,1_{\{T_{\boldsymbol{X}}\le t\}}(\boldsymbol{X}(0,s,x))\,dx\\&=\int_{\mathbb{R}^d}|\rho_0|(y)\,1_{\{T_{\boldsymbol{X}}\le t\}}(y)\,1_{B_R}(\boldsymbol{X}(s,0,y))\,dy.\end{aligned}\tag{4.33}$$

Since trajectories go to infinity when the time approaches $T_{\boldsymbol{X}}$ (see (4.2)), it follows that

$$1_{\{T_{\boldsymbol{X}}\le t\}}(y)\,1_{B_R}(\boldsymbol{X}(s,0,y))\to 0\qquad\text{for }\mathscr{L}^d\text{-a.e. }y\text{ as }s\uparrow t,$$

so (4.32) follows by dominated convergence. This concludes the proof of (i).

Step 3: proof of (ii), renormalization property of ρ_t. We begin by showing that ρ_t is a renormalized solution of the continuity equation.

By Remark 4.9 it is enough to prove that, given a bounded monotone nondecreasing function $\beta\in C^1(\mathbb{R})$ and $\varphi\in C^\infty_c(\mathbb{R}^d)$ nonnegative, the function $t\mapsto\int_{\mathbb{R}^d}\varphi\,\beta(\rho_t)\,dx$ is absolutely continuous in $[0,T]$ and

$$\frac{d}{dt}\int_{\mathbb{R}^d}\varphi\,\beta(\rho_t)\,dx=\int_{\mathbb{R}^d}\nabla\varphi\cdot\boldsymbol{b}_t\,\beta(\rho_t)\,dx\qquad\text{for }\mathscr{L}^1\text{-a.e. }t\in(0,T).\tag{4.34}$$

To show that the map is absolutely continuous, let us consider $s,\,t\in[0,T]$ and let $\tilde\rho^t_r$ be the evolution of ρ_t through the flow $\boldsymbol{X}(\cdot,t,x)$, namely

$$\tilde\rho^t_r:=\boldsymbol{X}(r,t,\cdot)_\#(\rho_t\llcorner\{T^+_{t,\boldsymbol{X}}>r>T^-_{t,\boldsymbol{X}}\})\qquad\text{for every }r\in[0,T].\tag{4.35}$$

Since $\boldsymbol{\eta}$ is transported by $\boldsymbol{X}$ (by assumption), we claim that

$$\tilde{\rho}^t_r \leq \rho_r \qquad \text{for every } r \in [0,T]. \tag{4.36}$$

Indeed, with the notation of the statement of Theorem 4.7, since $\delta_{\boldsymbol{X}(r,t,x)} = (e_r)_\# \boldsymbol{\eta}^t_x$ for ρ_t-a.e. $x \in \{T^+_{t,\boldsymbol{X}} > r > T^-_{t,\boldsymbol{X}}\}$, for every $r \in [0,T]$ one has

$$\begin{aligned} \tilde{\rho}^t_r \mathscr{L}^d &= \int_{\{T^-_{t,\boldsymbol{X}} < s\}} \delta_{\boldsymbol{X}(s,t,x)}\, \rho_t(x)\, dx \leq \int_{\mathbb{R}^d} (e_r)_\# \boldsymbol{\eta}^t_x\, \rho_t(x)\, dx \\ &= (e_r)_\# \int_{\mathbb{R}^d} \boldsymbol{\eta}^t_x\, \rho_t(x)\, dx = (e_r)_\# \boldsymbol{\eta} = \rho_r \mathscr{L}^d. \end{aligned}$$

Combining (4.36), the equality $\tilde{\rho}^t_t = \rho_t$, the monotonicity of β, and statement (i), we deduce that

$$\int_{\mathbb{R}^d} [\beta(\rho_t) - \beta(\rho_s)]\, \varphi\, dx \leq \int_{\mathbb{R}^d} [\beta(\tilde{\rho}^t_t) - \beta(\tilde{\rho}^t_s)]\, \varphi\, dx = \int_s^t \int_{\mathbb{R}^d} \beta(\tilde{\rho}^t_r)\, \nabla\varphi \cdot \boldsymbol{b}_r\, dx\, dr \tag{4.37}$$

and similarly

$$\int_{\mathbb{R}^d} [\beta(\rho_t) - \beta(\rho_s)]\, \varphi\, dx \geq \int_{\mathbb{R}^d} [\beta(\tilde{\rho}^s_t) - \beta(\tilde{\rho}^s_s)]\, \varphi\, dx = \int_s^t \int_{\mathbb{R}^d} \beta(\tilde{\rho}^s_r)\, \nabla\varphi \cdot \boldsymbol{b}_r\, dx\, dr.$$

In particular

$$\Big| \int_{\mathbb{R}^d} [\beta(\rho_t) - \beta(\rho_s)]\, \varphi\, dx \Big| \leq \|\beta\|_\infty \int_{\mathbb{R}^d} \int_s^t |\nabla\varphi|\, |\boldsymbol{b}_r|\, dr\, dx,$$

which shows that the function $t \mapsto \int_{\mathbb{R}^d} \varphi\, \beta(\rho_t)\, dx$ is absolutely continuous in $[0,T]$.

Hence, in order to prove (4.34) it suffices to notice that (4.37) and the strong continuity of $r \mapsto \tilde{\rho}^t_r$ at $r = t$ (ensured by statement (i)) give

$$\int_{\mathbb{R}^d} [\beta(\rho_t) - \beta(\rho_s)]\, \varphi\, dx \leq (t-s) \int_{\mathbb{R}^d} \beta(\rho_t)\, \nabla\varphi \cdot \boldsymbol{b}_t\, dx + o(t-s),$$

hence (4.34) holds at any differentiability point of $t \mapsto \int_{\mathbb{R}^d} \varphi\, \beta(\rho_t)\, dx$, thus for a.e. t.

Step 4: proof of (ii), strong continuity of ρ_t. We now show that ρ_t is strongly continuous on $[0,T)$ with respect to the L^1_{loc} convergence; more precisely we show that, for every $t \in [0,T)$ and for every $r > 0$,

$$\lim_{s \uparrow t} \int_{B_r} |\rho_s - \rho_t|\, dx = 0 \tag{4.38}$$

(reversing the time variable, the same argument gives the right-continuity). To this end, let us define $\tilde{\rho}^t$ as in (4.35) for every $t \in [0,T]$; we claim that

$$\tilde{\rho}^t_s = \rho_s \llcorner \{T^+_{s,\boldsymbol{X}} > t\} \qquad \text{for every } s \in [0,t]. \tag{4.39}$$

Indeed, let us fix $s, t \in [0,T]$ and $s \le t$. Denoting with $\boldsymbol{\eta}_x^t$ the disintegration of $\boldsymbol{\eta}$ with respect to the map e_t, recalling that $\boldsymbol{\eta}_x^t$ is concentrated on curves $\eta \in C([0,T]; \mathring{\mathbb{R}}^d)$ with $\eta(t) = x$, by Theorem 4.7 we have that, for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$,

$$
\begin{aligned}
1_{\{T^-_{t,\boldsymbol{X}} < s\}}(x)\, \delta_{\boldsymbol{X}(s,t,x)} &= (e_s)_\# \Big(\boldsymbol{\eta}_x^t \llcorner \big\{ \eta \in C([0,T]; \mathring{\mathbb{R}}^d) : \eta(t) = x \text{ and } T^-_{t,\boldsymbol{X}}(x) < s \big\} \Big) \\
&= (e_s)_\# \Big(\boldsymbol{\eta}_x^t \llcorner \big\{ \eta \in C([0,T]; \mathring{\mathbb{R}}^d) : \eta(t) \neq \infty \text{ and } T^-_{t,\boldsymbol{X}}(\eta(t)) < s \big\} \Big),
\end{aligned}
$$

hence we can rewrite $\tilde{\rho}_s^t$ in terms of $\boldsymbol{\eta}$ as

$$
\begin{aligned}
\tilde{\rho}_s^t \, \mathscr{L}^d &= \int_{\{T^-_{t,\boldsymbol{X}} < s\}} \delta_{\boldsymbol{X}(s,t,x)}\, \rho_t(x)\, dx \\
&= \int_{\mathbb{R}^d} (e_s)_\# \Big(\boldsymbol{\eta}_x^t \llcorner \big\{ \eta \in C([0,T]; \mathring{\mathbb{R}}^d) : \eta(t) \neq \infty \text{ and } T^-_{t,\boldsymbol{X}}(\eta(t)) < s \big\} \Big)\, \rho_t(x)\, dx \\
&= (e_s)_\# \Big(\boldsymbol{\eta} \llcorner \big\{ \eta \in C([0,T]; \mathring{\mathbb{R}}^d) : \eta(t) \neq \infty \text{ and } T^-_{t,\boldsymbol{X}}(\eta(t)) < s \big\} \Big).
\end{aligned}
\tag{4.40}
$$

By the semigroup property (Theorem 4.3(iii)) there exists a set $E_{s,t} \subset \mathbb{R}^d$ of $\mathscr{L}^d$-measure 0 such that

$$
\begin{aligned}
T^{\pm}_{s,\boldsymbol{X}}(\boldsymbol{X}(s,t,x)) = T^{\pm}_{t,\boldsymbol{X}}(x) \qquad & \forall\, x \in \{T^+_{t,\boldsymbol{X}} > s > T^-_{t,\boldsymbol{X}}\} \setminus E_{s,t}, \\
T^{\pm}_{t,\boldsymbol{X}}(\boldsymbol{X}(t,s,x)) = T^{\pm}_{s,\boldsymbol{X}}(x) \qquad & \forall\, x \in \{T^+_{s,\boldsymbol{X}} > t > T^-_{s,\boldsymbol{X}}\} \setminus E_{s,t}, \\
\boldsymbol{X}\big(\cdot, s, \boldsymbol{X}(s,t,x)\big) = \boldsymbol{X}(\cdot, t, x) \quad \text{in } (T^-_{t,\boldsymbol{X}}(x), T^+_{t,\boldsymbol{X}}(x)) \qquad & \forall\, x \in \{T^+_{t,\boldsymbol{X}} > s > T^-_{t,\boldsymbol{X}}\} \setminus E_{s,t}, \\
\boldsymbol{X}\big(\cdot, t, \boldsymbol{X}(t,s,x)\big) = \boldsymbol{X}(\cdot, s, x) \quad \text{in } (T^-_{s,\boldsymbol{X}}(x), T^+_{s,\boldsymbol{X}}(x)) \qquad & \forall\, x \in \{T^+_{s,\boldsymbol{X}} > t > T^-_{s,\boldsymbol{X}}\} \setminus E_{s,t}.
\end{aligned}
$$

Since $(e_s)_\# \boldsymbol{\eta} \llcorner \mathbb{R}^d$ is absolutely continuous with respect to $\mathscr{L}^d$ (hence the set of curves η such that $\eta(s) \in E_{s,t}$ is $\boldsymbol{\eta}$-negligible) and $\boldsymbol{\eta}$ is transported by the maximal regular flow, we have the following equalities, which hold up to a set of $\boldsymbol{\eta}$-measure 0:

$$
\begin{aligned}
&\big\{ \eta \in C([0,T]; \mathring{\mathbb{R}}^d) : \eta(s) \neq \infty \text{ and } T^+_{s,\boldsymbol{X}}(\eta(s)) > t \big\} \\
&= \big\{ \eta \in C([0,T]; \mathring{\mathbb{R}}^d) : \eta(s) \neq \infty,\ \eta(s) \notin E_{s,t},\ T^+_{s,\boldsymbol{X}}(\eta(s)) > t \\
&\qquad\qquad \text{and } \eta(\cdot) = \boldsymbol{X}(\cdot, s, \eta(s)) \text{ in } (T^-_{s,\boldsymbol{X}}(\eta(s)), T^+_{s,\boldsymbol{X}}(\eta(s)) \big\} \\
&= \big\{ \eta \in C([0,T]; \mathring{\mathbb{R}}^d) : \eta(t) \neq \infty,\ \eta(t) \notin E_{s,t},\ T^-_{t,\boldsymbol{X}}(\eta(t)) < s \\
&\qquad\qquad \text{and } \eta(\cdot) = \boldsymbol{X}(\cdot, t, \eta(t)) \text{ in } (T^-_{t,\boldsymbol{X}}(\eta(t)), T^+_{t,\boldsymbol{X}}(\eta(t)) \big\} \\
&= \big\{ \eta \in C([0,T]; \mathring{\mathbb{R}}^d) : \eta(t) \neq \infty \text{ and } T^-_{t,\boldsymbol{X}}(\eta(t)) < s \big\}.
\end{aligned}
\tag{4.41}
$$

This implies that

$$
\begin{aligned}
\rho_s \llcorner \{T^+_{s,\boldsymbol{X}} > t\} &= (e_s)_\# \Big(\boldsymbol{\eta} \llcorner \big\{ \eta \in C([0,T]; \mathring{\mathbb{R}}^d) : \eta(s) \neq \infty \text{ and } T^+_{s,\boldsymbol{X}}(\eta(s)) > t \big\} \Big) \\
&= (e_s)_\# \Big(\boldsymbol{\eta} \llcorner \big\{ \eta \in C([0,T]; \mathring{\mathbb{R}}^d) : \eta(t) \neq \infty \text{ and } T^-_{t,\boldsymbol{X}}(\eta(t)) < s \big\} \Big),
\end{aligned}
$$

that combined with (4.40) gives (4.39).

Now, in order to prove (4.38), we apply the triangular inequality to infer that

$$\int_{B_r} |\rho_s - \rho_t|\,dx \le \int_{B_r} |\rho_s - \tilde\rho^t_s|\,dx + \int_{B_r} |\tilde\rho^t_s - \rho_t|\,dx.$$

The second term in the right-hand side converges to 0 when $s \uparrow t$ by the strong L^1_{loc} continuity of ρ^t_s with respect to s proved in statement (i). To see that also the first term converges to 0, we use (4.39), the identity $\rho_t \mathscr{L}^d = (e_t)_\# \boldsymbol{\eta} \llcorner \mathbb{R}^d$, and the fact that $\boldsymbol{\eta}$ is transported by the maximal flow, to obtain

$$\begin{aligned}\int_{B_r} |\rho_s - \tilde\rho^t_s|\,dx &= \int_{B_r} \rho_s\, 1_{\{T^+_{s,\boldsymbol{X}} \le t\}}\,dx = \int 1_{B_r \cap \{T^+_{s,\boldsymbol{X}} \le t\}}(\eta(s))\,d\boldsymbol{\eta}(\eta)\\ &= \boldsymbol{\eta}\Big(\big\{\eta : \eta(s) \in B_r \cap \{T^+_{s,\boldsymbol{X}} \le t\} \text{ and } \eta(\cdot) = \boldsymbol{X}(\cdot, s, \eta(s)) \text{ in } [s, T^+_{s,\boldsymbol{X}}(\eta(s))\big\}\Big).\end{aligned}$$

Notice that, if η is a curve which belongs to the set in the last line, then it belongs to B_r at time s and blows up in $[s,t]$, thus

$$\int_{B_r} |\rho_s - \rho^t_s|\,dx \le \boldsymbol{\eta}\Big(\big\{\eta : \eta(s') \in B_r \text{ and } \eta(s'') = \infty \text{ for some } s', s'' \in [s,t]\big\}\Big).$$

Since set in the right-hand side monotonically decreases to the empty set as $s \uparrow t$, its $\boldsymbol{\eta}$-measure converges to 0, which proves (4.38) and concludes the proof. □

We now discuss a general no blow-up criterion for a generalized flow $\boldsymbol{\eta}$. Given $\boldsymbol{\eta}$, we make an assuption on the integrability of $\boldsymbol{b}$ along $\boldsymbol{\eta}$, without requiring a growth condition on the vector field as in the more classical statements of DiPerna and Lions, and we deduce that $\boldsymbol{\eta}$-a.e. trajectory of the maximal regular flow does not blow up in finite time. We also underline that, with respect to more classical statements preventing blow-up, we make a logarithmic improvement in the denominator in (4.42) below. This is based on the fact that the ordinary differential equation $\dot r = (1+r)\log(2+r)$ in $[0,\infty)$ does not allow for blow up in finite time. This result (in particular also the logarithmic improvement) plays an important role in the proof of Theorem 2.3.

Proposition 4.11 (No blow-up criterion)**.** *Let $\boldsymbol{b} \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ be a Borel vector field, let $\boldsymbol{\eta} \in \mathscr{M}_+\big(C([0,T]; \mathring{\mathbb{R}}^d)\big)$ be a generalized flow of $\boldsymbol{b}$, and for $t \in [0,T]$ let $\mu_t := (e_t)_\# \boldsymbol{\eta} \llcorner \mathbb{R}^d$. Let η_∞ denote the constant curve $\eta \equiv \infty$, and assume that $\boldsymbol{\eta}(\{\eta_\infty\}) = 0$ and*

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t|(x)}{(1+|x|)\log(2+|x|)}\,d\mu_t(x)\,dt < \infty. \tag{4.42}$$

Then $\boldsymbol{\eta}$ is concentrated on curves that do not blow up, namely

$$\boldsymbol{\eta}\big(\{\eta \in C([0,T]; \mathring{\mathbb{R}}^d)) : \eta(t) = \infty \text{ for some } t \in [0,T]\}\big) = 0.$$

In particular, if we assume that $\mu_t \ll \mathscr{L}^d$ for every $t \in [0,T]$ and that $\boldsymbol{\eta}$ is concentrated on the maximal regular flow $\boldsymbol{X}$ associated to $\boldsymbol{b}$, then $\boldsymbol{X}$ is globally defined on $[0,T]$ for μ_0-a.e. x, namely the trajectories $\boldsymbol{X}(\cdot, x)$ belong to $AC([0,T]; \mathbb{R}^d)$ for μ_0-a.e. $x \in \mathbb{R}^d$.

Proof. Since $\boldsymbol{\eta}(\{\eta_\infty\}) = 0$ we know that $\boldsymbol{\eta}$-a.e. curve is finite at some time. In particular, if we fix a countable dense set of times $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$, we see that (by continuity of the curves) $\boldsymbol{\eta}$ is concentrated on $\cup_{n\in\mathbb{N}}\Gamma_n$ with

$$\Gamma_n := \{\eta \in C([0,T]; \mathring{\mathbb{R}}^d)) : \eta(t_n) \in \mathbb{R}^d\},$$

so it is enough to show that $\boldsymbol{\eta}\llcorner\Gamma_n$ is concentrated on curves that do not blow up.

By applying Theorem 4.7 with $s = t_n$ it follows that $\boldsymbol{\eta}\llcorner\Gamma_n$ is concentrated on curves η that are finite on the time interval $(T^-_{t_n,\boldsymbol{X}}(\eta(t_n)), T^+_{t_n,\boldsymbol{X}}(\eta(t_n))) \subset [0,T]$. Hence, since $(e_t)_\#(\boldsymbol{\eta}\llcorner\Gamma_n) \le \mu_t$, by Fubini theorem and assumption (4.42) we get

$$\begin{aligned}
&\int \int_{T^-_{t_n,\boldsymbol{X}}(\eta(t_n))}^{T^+_{t_n,\boldsymbol{X}}(\eta(t_n))} \left|\frac{d}{dt}\big[\log\log(2+|\eta(t)|)\big]\right| dt\, d[\boldsymbol{\eta}\llcorner\Gamma_n](\eta) \\
&\qquad \le \int \int_{T^-_{t_n,\boldsymbol{X}}(\eta(t_n))}^{T^+_{t_n,\boldsymbol{X}}(\eta(t_n))} \frac{|\dot\eta(t)|}{(1+|\eta(t)|)\log(2+|\eta(t)|)}\, dt\, d[\boldsymbol{\eta}\llcorner\Gamma_n](\eta) \\
&\qquad = \int \int_{T^-_{t_n,\boldsymbol{X}}(\eta(t_n))}^{T^+_{t_n,\boldsymbol{X}}(\eta(t_n))} \frac{|\boldsymbol{b}_t|(\eta(t))}{(1+|\eta(t)|)\log(2+|\eta(t)|)}\, dt\, d[\boldsymbol{\eta}\llcorner\Gamma_n](\eta) \\
&\qquad \le \int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t|(x)}{(1+|x|)\log(2+|x|)}\, d\mu_t(x)\, dt < \infty.
\end{aligned}$$

This implies that, for $\boldsymbol{\eta}$-a.e. curve $\eta \in \Gamma_n$,

$$\begin{aligned}
&\sup_{T^-_{t_n,\boldsymbol{X}}(\eta(t_n)) \le s < \tau \le T^+_{t_n,\boldsymbol{X}}(\eta(t_n))} \big|\log\log(2+|\eta(s)|) - \log\log(2+|\eta(\tau)|)\big| \\
&\qquad\qquad \le \int_{T^-_{t_n,\boldsymbol{X}}(\eta(t_n))}^{T^+_{t_n,\boldsymbol{X}}(\eta(t_n))} \left|\frac{d}{dt}\big[\log\log(2+|\eta(t)|)\big]\right| dt < \infty,
\end{aligned}$$

which in turn says that $T^-_{t_n,\boldsymbol{X}}(\eta(t_n)) = 0$, $T^+_{t_n,\boldsymbol{X}}(\eta(t_n)) = T$, and the curve η cannot blow up in $[0,T]$, as desired.

To show the second part of the statement, le us consider the disintegration of $\boldsymbol{\eta}$ with respect to e_0. By the properties of $\boldsymbol{\eta}$ we have that, for μ_0-a.e. x, the probability measure $\boldsymbol{\eta}_x$ is concentrated on the set

$$\big\{\eta : \eta(0) = x,\ \eta \ne \infty \text{ in } [0,T], \eta = \boldsymbol{X}(\cdot, x) \text{ in } [0, T_{\boldsymbol{X}}(x))\big\}.$$

Since $\boldsymbol{\eta}_x$ is a probability measure it follows that this set is nonempty, that $T_{\boldsymbol{X}}(x) = T$, and this set has to coincide with $\{\boldsymbol{X}(\cdot,x)\}$, thus $\boldsymbol{\eta}_x = \delta_{\boldsymbol{X}(\cdot,x)}$, as desired. In particular, we deduce that for every $t > 0$

$$\mu_t = (e_t)_\#\boldsymbol{\eta}\llcorner\mathbb{R}^d = (e_t)_\#\boldsymbol{\eta} = (e_t)_\# \int_{\mathbb{R}^d} \delta_{\boldsymbol{X}(\cdot,x)}\, d\mu_0(x) = \boldsymbol{X}(t,\cdot)_\#\mu_0.$$

□

5 The superposition principle under local integrability bounds

In order to represent the solution to the continuity equation by means of a generalized flow we would like to apply the so-called superposition principle (see [3, Theorem 12] or [2, Theorem 2.1]). However, the lack of global bounds makes this approach very difficult to implement. An analogue of the classical superposition principle is the content of the following theorem.

Theorem 5.1 (Extended superposition principle)**.** *Let $\boldsymbol{b} \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ be a Borel vector field, and let $\rho_t \in L^\infty((0,T); L^1_+(\mathbb{R}^d))$ be a distributional solution of the continuity equation, weakly continuous on $[0,T]$ in duality with $C_c(\mathbb{R}^d)$. Assume that:*
(i) either $|\boldsymbol{b}_t|\rho_t \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) or $\operatorname{div} \boldsymbol{b}_t = 0$ and ρ_t is a renormalized solution.
Then there exists $\boldsymbol{\eta} \in \mathscr{M}_+\big(C([0,T]; \mathring{\mathbb{R}}^d)\big)$, concentrated on the set Γ defined in (4.17), which satisfies

$$|\boldsymbol{\eta}|(C([0,T]; \mathring{\mathbb{R}}^d)) \leq \sup_{t \in [0,T]} \|\rho_t\|_{L^1(\mathbb{R}^d)}$$

and

$$(e_t)_\# \boldsymbol{\eta} \llcorner \mathbb{R}^d = \rho_t \mathscr{L}^d \qquad \text{for every } t \in [0,T].$$

In addition, if ρ_t belongs also to $L^\infty((0,T); L^\infty_+(\mathbb{R}^d))$ (or ρ_t is a renormalized solution) and $\boldsymbol{b}$ is divergence-free and satisfies **(A1)**-**(A2)** *of Section 4.1, then $\boldsymbol{\eta}$ is transported by the Maximal Regular Flow $\boldsymbol{X}$ of $\boldsymbol{b}$. In particular, ρ_t is a Lagrangian solution.*

Remark 5.2. Noticing that the assumption $|\boldsymbol{\eta}|(C([0,T]; \mathring{\mathbb{R}}^d)) \leq \sup_{t \in [0,T]} \mu_t(\mathbb{R}^d)$ implies that the curve $\eta \equiv \infty$ has $\boldsymbol{\eta}$-measure 0, if we assume that

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\boldsymbol{b}_t|(x)}{(1+|x|)\log(2+|x|)}\, \rho_t(x)\, dx\, dt < \infty \tag{5.1}$$

then it follows by Proposition 4.11 that ρ_t is Lagrangian, namely $T^+_{0,\boldsymbol{X}}(x) = T$, $\boldsymbol{X}(\cdot, 0, x) \in AC([0,T]; \mathbb{R}^d)$ for a.e. $x \in \{\rho_0 > 0\}$, and $\rho_t \mathscr{L}^d = \boldsymbol{X}(t, \cdot)_\# \rho_0 \mathscr{L}^d$.

Let us first briefly explain the idea behind the proof of the theorem above. To overcome the lack of global bounds on $\boldsymbol{b}$ we introduce a kind of "damped" stereographic projection, with a damping depending on the growth of $|\boldsymbol{b}|$ at ∞, and we look at the flow of $\boldsymbol{b}$ on the d-dimensional sphere $\mathbb{S}^d$ in such a way that the north pole N of the sphere corresponds to the points at infinity of $\mathbb{R}^d$. Then we apply the superposition principle in these new variables and eventually, going back to the original variables, we obtain a representation of the solution as a generalized flow. Let us observe that it is crucial for us that the map sending $\mathbb{R}^d$ onto $\mathbb{S}^d$ is chosen a function of $\boldsymbol{b}$: indeed, as we shall see, by shrinking enough distances at infinity we can ensure that the vector field read on the sphere becomes *globally* integrable.

We denote by N be the north pole of the d-dimensional sphere $\mathbb{S}^d$, thought of as a subset of $\mathbb{R}^{d+1}$. For our constructions we will use a smooth diffeomorphism which maps $\mathbb{R}^d$ onto $\mathbb{S}^d \setminus \{N\}$ and whose derivative has a prescribed decay at ∞.

Lemma 5.3. *Let $D : [0, \infty) \to (0, 1]$ be a monotone nonincreasing function. Then there exist $r_0 > 0$ and a smooth diffeomorphism $\psi : \mathbb{R}^d \to \mathbb{S}^d \setminus \{N\} \subset \mathbb{R}^{d+1}$ such that*

$$\psi(x) \to N \text{ as } |x| \to \infty, \tag{5.2}$$

$$|\nabla\psi(x)| \le D(0) \qquad \forall\, x \in \mathbb{R}^d, \tag{5.3}$$

$$|\nabla\psi(x)| \le D(|x|) \qquad \forall\, x \in \mathbb{R}^d \setminus B_{r_0}. \tag{5.4}$$

Proof. We split the construction in two parts: first we perform a 1-dimensional construction, and then we use this construction to build the desired diffeomorphism.

Step 1: 1-dimensional construction. Let $D_0 : [0, \infty) \to (0, 1]$ be a monotone nonincreasing function. We claim that there exists a smooth diffeomorphism $\psi_0 : [0, \infty) \to [0, \pi)$ such that

$$\lim_{r\to\infty} \psi_0(r) = \pi, \qquad \lim_{r\to\infty} \psi_0'(r) = 0, \tag{5.5}$$

$$\psi_0(r) = c_0\, D_0(0)\, r \qquad \forall\, r \in [0, \pi/D_0(0)), \quad \text{for some } c_0 \in (0, 1), \tag{5.6}$$

$$|\psi_0'(r)| \le D_0(0) \qquad \forall\, r \in [0, \infty), \tag{5.7}$$

$$|\psi_0'(r)| \le D_0(r) \qquad \forall\, r \in [2\pi/D_0(0), \infty). \tag{5.8}$$

Indeed, define the nonincreasing L^1 function $D_1 : [0, \infty) \to (0, \infty)$ as

$$D_1(r) := \begin{cases} D_0(0) & \text{if } r \in [0, 1 + \pi/D_0(0)] \\ \min\{D_0(r), r^{-2}\} & \text{if } r \in (1 + \pi/D_0(0), \infty). \end{cases}$$

We then consider an asymmetric convolution kernel, namely a nonnegative function $\sigma \in C_c^\infty((0, 1))$ with $\int_{\mathbb{R}} \sigma = 1$, and consider the convolution of $D_1(r)$ with $\sigma(-r)$:

$$\psi_1(r) := \int_0^1 \sigma(r')D_1(r + r')\, dr' \qquad \forall\, r \in [0, \infty).$$

Notice that ψ_1 is smooth on $(0, \infty)$, positive, nonincreasing, and $\psi_1 \le D_1$ in $[0, \infty)$. In particular $\psi_1 \in L^1((0, \infty))$. Moreover we have that $\psi_1 \equiv D_0(0)$ in $[0, \pi/D_0(0)]$, hence $\|\psi_1\|_{L^1((0,\infty))} \ge \pi$ and $c_0 := \pi\|\psi_1\|^{-1}_{L^1((0,\infty))} \in (0, 1)$. Finally, we define ψ_0 as

$$\psi_0(r) := c_0 \int_0^r \psi_1(s)\, ds \qquad \forall\, r \in [0, \infty).$$

Since $|\psi_0'(r)| = c_0|\psi_1(r)| \le D_1(r)$, taking into account that $\pi/D_0(0) > 1$ it is easy to check that all the desired properties are satisfied.

Step 2: "radial" diffeomorphism in any dimension. Let $D_0 : [0, \infty) \to (0, 1]$ to be chosen later and consider ψ_0 and c_0 as in Step 1. We define $\psi : \mathbb{R}^d \to \mathbb{S}^d \setminus \{N\} \subset \mathbb{R}^{d+1}$ which maps every half-line starting at the origin to an arc of sphere between the south pole and the north pole:

$$\psi(x) := \sin(\psi_0(|x|))\Big(\frac{x}{|x|}, 0\Big) - \cos(\psi_0(|x|))\big(0, \ldots, 0, 1\big).$$

Thanks to (5.6) and to the fact that the functions $x \mapsto |x|^2$, $t \mapsto \sin(\sqrt{t})/\sqrt{t}$, and $t \mapsto \cos(\sqrt{t})$ are all of class C^∞, we obtain that $\psi \in C^\infty(\mathbb{R}^d;\mathbb{R}^{d+1})$. We also notice that its inverse $\phi : \mathbb{S}^d \setminus \{N\} \to \mathbb{R}^d$ can be explicitly computed:

$$\begin{aligned}\phi(x_1,\dots,x_{d+1}) &= \psi_0^{-1}(\arccos(-x_{d+1}))\,\frac{(x_1,\dots,x_d)}{|(x_1,\dots,x_d)|}\\ &= \psi_0^{-1}(\arcsin(|(x_1,\dots,x_d)|))\,\frac{(x_1,\dots,x_d)}{|(x_1,\dots,x_d)|}.\end{aligned}$$

Writing $r = |x|$ and denoting by I_d the identity matrix on the first d components, we compute the gradient of ψ:

$$\begin{aligned}\nabla\psi(x) = \frac{\cos(\psi_0(r))\,\psi_0'(r)\,r - \sin(\psi_0(r))}{r^3}\,(x,0)\otimes(x,0) + \frac{\sin(\psi_0(r))}{r}\,I_d\\ - \frac{\sin(\psi_0(r))\,\psi_0'(r)}{r}\,(x,0)\otimes(0,\dots,0,1).\end{aligned}$$

It is immediate to check that $|\nabla\psi(x)| \neq 0$ for all $x \in \mathbb{R}^d$, so it follows by the Inverse Function Theorem that ϕ is smooth as well. Also, we can estimate

$$|\nabla\psi(x)| \le 2\,|\psi_0'(r)| + 2\,\frac{\sin(\psi_0(r))}{r}. \tag{5.9}$$

Using now (5.7) and (5.8), the first term in the right hand side above can be bounded by $2D_0(0)$ for every $x \in \mathbb{R}^d$, and by $2D_0(r)$ for every $x \in \mathbb{R}^d$ such that $r = |x| \ge 2\pi/D_0(0)$. As regards the second term, for $r \in [0,\pi/D_0(0)]$ we have that

$$\frac{\sin(\psi_0(r))}{r} = \frac{\sin(c_0\,D_0(0)\,r)}{r} \le c_0\,D_0(0), \tag{5.10}$$

while for $r \in [\pi/D_0(0),\infty)$ we estimate the numerator with 1 to get

$$\frac{\sin(\psi_0(r))}{r} \le \frac{D_0(0)}{\pi}. \tag{5.11}$$

Therefore, since $c_0 < 1$, by (5.9), (5.10), and (5.11) we get

$$|\nabla\psi(x)| \le 4\,D_0(0) \qquad \forall\, x \in \mathbb{R}^d. \tag{5.12}$$

Now, for $r \in [2\pi/D_0(0),\infty)$, thanks to (5.5) and (5.8) we can estimate

$$\frac{\sin(\psi_0(r))}{r} = \frac{1}{r}\int_r^\infty -\cos(\psi_0(s))\,\psi_0'(s)\,ds \le \frac{1}{r}\int_r^\infty |\psi_0'(s)|\,ds \le \frac{1}{r}\int_r^\infty D_0(s)\,ds, \tag{5.13}$$

thus by (5.8), (5.9), and (5.13), we obtain

$$|\nabla\psi(x)| \le 2\,D_0(r) + \frac{2}{r}\int_r^\infty D_0(s)\,ds \qquad \forall\, x \in \mathbb{R}^d \setminus B_{2\pi/D_0(0)}. \tag{5.14}$$

So, provided we choose $D_0(r) := \min\{4^{-1}, r^{-2}\}\, D(r)$ we obtain that (5.12) implies (5.3). Also, by choosing $r_0 := 2\pi/D_0(0) > 2$, from (5.14) and because D is monotone nonincreasing we deduce that

$$|\nabla\psi(x)| \le \frac{D(r)}{2} + \frac{1}{r}\int_r^\infty \frac{D(r)}{s^2}\, ds \le \frac{D(r)}{2} + \frac{D(r)}{r^2} \le D(r) \qquad \forall\, x \in \mathbb{R}^d \setminus B_{r_0},$$

proving (5.4) and concluding the proof. □

Proof of Theorem 5.1. We first assume that $|\boldsymbol{b}_t|\rho_t \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ and we prove the result in this case. This is done in two steps:
- In Step 1, based on Lemma 5.3, we construct a diffeomorphism between $\mathbb{R}^d$ and $\mathbb{S}^d \setminus \{N\}$ with the property that the vector field $\boldsymbol{b}$, read on the sphere, becomes globally integrable.
- In Step 2 we associate to ρ_t a solution of the continuity equation on the sphere; this is done by adding a time-dependent mass in the north pole. Then the classical superposition principle applies on the sphere, and this implies the desired superposition result for ρ_t.

Once the theorem has been proved for $|\boldsymbol{b}_t|\rho_t \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$, we show in Step 3 how to handle the case when ρ_t is a renormalized solution.

Finally, in Step 4 we exploit the results of Section 4 to show that ρ_t is Lagrangian.

Step 1: construction of a diffeomorphism between $\mathring{\mathbb{R}}^d$ and $\mathbb{S}^d$. We build a diffeomorphism $\psi \in C^\infty(\mathbb{R}^d; \mathbb{S}^d \setminus \{N\})$ such that

$$\lim_{x\to\infty} \psi(x) = N, \tag{5.15}$$

$$\int_0^T \int_{\mathbb{R}^d} |\nabla\psi(x)|\, |\boldsymbol{b}_t(x)|\, \rho_t(x)\, dx\, dt < \infty. \tag{5.16}$$

To this end, we apply Lemma 5.3 with $D(r) = 1$ in $[0,1)$ and $D(r) = (2^n C_n)^{-1}$ for $r \in [2^{n-1}, 2^n)$, where

$$C_n := 1 + \int_0^T \int_{B_{2^n}} |\boldsymbol{b}_t(x)|\, \rho_t(x)\, dx\, dt \qquad \text{for every } n \in \mathbb{N}.$$

In this way we obtain a smooth diffeomorphism $\psi : \mathbb{R}^d \to \mathbb{S}^d \setminus \{N\}$ such that (5.15) holds, $|\nabla\psi(x)| \le 1$ on $\mathbb{R}^d$, and

$$|\nabla\psi(x)| \le \frac{1}{2^n C_n} \qquad \forall\, x \in B_{2^n} \setminus B_{2^{n-1}},\ n \ge n_0, \tag{5.17}$$

for some $n_0 > 0$. Thanks to these facts we deduce that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |\nabla\psi(x)|\, |\boldsymbol{b}_t(x)|\, \rho_t(x)\, dx\, dt \\
&\le \int_0^T \int_{B_{2^{n_0}}} |\boldsymbol{b}_t(x)|\, \rho_t(x)\, dx\, dt + \sum_{i=n_0+1}^\infty \int_0^T \int_{B_{2^i}\setminus B_{2^{i-1}}} |\nabla\psi(x)|\, |\boldsymbol{b}_t(x)|\, \rho_t(x)\, dx\, dt \\
&\le \int_0^T \int_{B_{2^{n_0}}} |\boldsymbol{b}_t(x)|\, \rho_t(x)\, dx\, dt + \sum_{i=n_0+1}^\infty \frac{1}{2^i} < \infty,
\end{aligned} \tag{5.18}$$

which proves (5.16).

Step 2: superposition principle on the sphere. We build $\boldsymbol{\eta} \in \mathscr{M}_+\big(C([0,T];\mathring{\mathbb{R}}^d)\big)$ such that $|\boldsymbol{\eta}|(C([0,T];\mathring{\mathbb{R}}^d)) \leq \sup_{t\in[0,T]} \|\rho_t\|_{L^1(\mathbb{R}^d)}$, $\boldsymbol{\eta}$ is concentrated on curves η which are locally absolutely continuous integral curves of $\boldsymbol{b}$ in $\{\eta \neq \infty\}$, and whose marginal at time t in $\mathbb{R}^d$ is $\rho_t\mathscr{L}^d$.

Without loss of generality, possibly dividing every ρ_t by $\sup_{t\in[0,T]} \|\rho_t\|_{L^1(\mathbb{R}^d)}$, we can assume that $\sup_{t\in[0,T]} \|\rho_t\|_{L^1(\mathbb{R}^d)} = 1$. Let $\phi : \mathbb{S}^d \setminus \{N\}) \to \mathbb{R}^d$ be the inverse of the diffeomorphism ψ constructed in Step 1, set $m_t := \|\rho_t\|_{L^1(\mathbb{R}^d)} \leq 1$, and define

$$\boldsymbol{c}_t(y) := \begin{cases} \nabla\psi(\phi(y))\,\boldsymbol{b}_t(\phi(y)) & \text{if } y \in \mathbb{S}^d \setminus \{N\} \\ 0 & \text{if } y = N \end{cases} \tag{5.19}$$

and

$$\mu_t := \psi_\#(\rho_t\mathscr{L}^d) + (1-m_t)\,\delta_N \in \mathscr{P}\big(\mathbb{S}^d\big), \qquad t \in [0,T].$$

Notice that, since ϕ is the inverse of ψ, we have $\phi_\#\big(\mu_t \llcorner (\mathbb{S}^d \setminus \{N\})\big) = \rho_t$. Hence, since $\boldsymbol{c}_t(N) = 0$ we get

$$\begin{aligned} \int_0^T \int_{\mathbb{S}^d} |\boldsymbol{c}_t|\,d\mu_t\,dt &= \int_0^T \int_{\mathbb{S}^d\setminus\{N\}} |\nabla\psi|(\phi(y))\,|\boldsymbol{b}_t|(\phi(y))\,d\mu_t(y)\,dt \\ &= \int_0^T \int_{\mathbb{R}^d} |\nabla\psi|(x)\,|\boldsymbol{b}_t|(x)\,\rho_t(x)\,dx\,dt < \infty, \end{aligned}$$

where in the last inequality we used (5.16).

We now show that the probability measure μ_t is a solution to the continuity equation on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ with vector field $\boldsymbol{c}_t$. To this end we first notice that, by the weak continuity in duality with $C_c(\mathbb{R}^d)$ of ρ_t and by the fact that all the measures μ_t have unit mass, we deduce that μ_t is weakly continuous in time. Indeed, any limit point of μ_s as $s \to t$ is uniquely determined on $\mathbb{S}^d\setminus\{N\}$, and then the mass normalization gives that it is completely determined. We want to prove that the function $t \mapsto \int_{\mathbb{S}^d} \varphi\,d\mu_t$ is absolutely continuous and satisfies

$$\frac{d}{dt}\int_{\mathbb{S}^d} \varphi\,d\mu_t = \int_{\mathbb{S}^d} \boldsymbol{c}_t \cdot \nabla\varphi\,d\mu_t \qquad \text{a.e. on } (0,T) \tag{5.20}$$

for every $\varphi \in C^\infty(\mathbb{R}^{d+1})$. We remark that, since ρ_t is a solution to the continuity equation in $\mathbb{R}^d$ with vector field $\boldsymbol{b}_t$, changing variables with the diffeomorphism ψ we obtain that (5.20) holds for every $\varphi \in C_c^\infty(\mathbb{R}^{d+1} \setminus \{N\})$, hence we are left to check that (5.20) holds also when φ is not necessarily 0 in a neighborhood of the north pole.

Fix $\varphi \in C^\infty(\mathbb{R}^{d+1})$. Since $\mu_t(N) = 1 - m_t = 1 - \mu_t(\mathbb{S}^d \setminus \{N\})$, for every $t \in [0,T]$ we have that

$$\int_{\mathbb{S}^d} \varphi\,d\mu_t = \int_{\mathbb{S}^d\setminus\{N\}} \varphi\,d\mu_t + \varphi(N)\,\mu_t(N) = \varphi(N) + \int_{\mathbb{S}^d} (\varphi - \varphi(N))\,d\mu_t. \tag{5.21}$$

Now, given $\varepsilon > 0$ let us consider a function $\chi_\varepsilon \in C^\infty(\mathbb{R}^{d+1})$ which is 0 in $B_\varepsilon(N)$, 1 outside $B_{2\varepsilon}(N)$, and whose gradient is bounded by $2/\varepsilon$. Since ρ_t is a solution to the continuity equation in $\mathbb{R}^d$ and since $\chi_\varepsilon(\varphi - \varphi(N))$ is a smooth, compactly supported function in $C_c^\infty(\mathbb{R}^{d+1} \setminus \{N\})$ we deduce that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}^d} \chi_\varepsilon(\varphi - \varphi(N))\,d\mu_t &= \int_{\mathbb{S}^d\setminus\{N\}} \boldsymbol{c}_t \cdot \nabla[\chi_\varepsilon(\varphi - \varphi(N))]\,d\mu_t \\ &= \int_{\mathbb{S}^d\setminus\{N\}} (\varphi - \varphi(N))\,\boldsymbol{c}_t \cdot \nabla\chi_\varepsilon\,d\mu_t + \int_{\mathbb{S}^d\setminus\{N\}} \chi_\varepsilon\,\boldsymbol{c}_t \cdot \nabla\varphi\,d\mu_t.\end{aligned} \tag{5.22}$$

To estimate the first term in the right-hand side of (5.22) we use that $|\varphi - \varphi(N)| \le \varepsilon\|\nabla\varphi\|_\infty$ in $B_\varepsilon(N)$ and that $|\nabla\chi_\varepsilon| \le 2/\varepsilon$ to get that

$$\left|\int_{\mathbb{S}^d\setminus\{N\}} \boldsymbol{c}_t \cdot \nabla\chi_\varepsilon(\varphi - \varphi(N))\,d\mu_t\right| \le 2\,\|\nabla\phi\|_\infty \int_{B_{2\varepsilon}(N)\setminus B_\varepsilon(N)} |\boldsymbol{c}_t|\,d\mu_t,$$

and notice the latter goes to 0 in $L^1(0,T)$ as $\varepsilon \to 0$ since $|\boldsymbol{c}|$ is integrable with respect to $\mu_t\,dt$ in space-time thanks to (5.20). Since the second term in the right-hand side of (5.22) converges in $L^1(0,T)$ to $\int_{\mathbb{S}^d\setminus\{N\}} \boldsymbol{c}_t \cdot \nabla\varphi\,d\mu_t$, taking the limit as $\varepsilon \to 0$ in (5.22) we obtain that $t \mapsto \int_{\mathbb{S}^d}(\varphi - \varphi(N))\,d\mu_t$ is absolutely continuous in $[0,T]$ and that for a.e. $t \in (0,T)$ one has

$$\frac{d}{dt}\int_{\mathbb{S}^d}(\varphi - \varphi(N))\,d\mu_t = \int_{\mathbb{S}^d} \boldsymbol{c}_t \cdot \nabla\varphi\,d\mu_t.$$

Using the identity (5.21), this formula can be rewritten in the form (5.20), as desired.

Since μ_t is a weakly continuous solution of the continuity equation and the integrability condition (5.20) holds, we can apply the superposition principle (see [3, Theorem 12] or [2, Theorem 2.1]) to deduce the existence of a measure $\boldsymbol{\sigma} \in \mathscr{P}\big(C([0,T];\mathbb{S}^d)\big)$ which is concentrated on integral curves of $\boldsymbol{c}$ and such that $(e_t)_\#\boldsymbol{\sigma} = \mu_t$ for all $t \in [0,T]$.

We then consider $\hat\phi : \mathbb{S}^d \to \mathring{\mathbb{R}}^d$ to be the inverse of ψ extended to N as $\hat\phi(N) = \infty$, and define $\Phi : C([0,T];\mathbb{S}^d) \to C([0,T];\mathring{\mathbb{R}}^d)$ as $\Phi(\eta) := \hat\phi \circ \eta$. Then the measure

$$\boldsymbol{\eta} := \Phi_\#\boldsymbol{\sigma} \in \mathscr{P}\big(C([0,T];\mathring{\mathbb{R}}^d)\big)$$

is concentrated on locally absolutely continuous integral curves of $\boldsymbol{b}$ in the sense stated in (4.17), and

$$(e_t)_\#\boldsymbol{\eta}\llcorner\mathbb{R}^d = \big(\hat\phi_\#(e_t)_\#\boldsymbol{\sigma}\big)\llcorner\mathbb{R}^d = \big(\hat\phi_\#\mu_t\big)\llcorner\mathbb{R}^d = \rho_t\mathscr{L}^d.$$

Step 3: the case of renormalized solutions. We now show how to prove the result when $\operatorname{div}\boldsymbol{b}_t = 0$ and ρ_t is a renormalized solution. Notice that in this case we have no local integrability information on $|\boldsymbol{b}_t|\rho_t$, so the argument above does not apply. However, exploiting the fact that ρ_t is renormalized we can easily reduce to that case.

More precisely, we begin by observing that, by a simple approximation argument, the renormalization property (see Definition 4.8) is still true when β is a bounded Lipschitz function. Thanks to this observation we consider, for $k \ge 0$, the functions

$$\beta_k(s) := \begin{cases} 0 & \text{if } s \le k, \\ s - k & \text{if } k \le s \le k+1, \\ 1 & \text{if } s \ge k+1. \end{cases}$$

Since ρ_t is renormalized, $\beta_k(\rho_t)$ is a bounded distributional solution of the continuity equation, hence by Steps 1-2 above there exists a measure $\boldsymbol{\eta}_k \in \mathscr{M}_+\big(C([0,T];\mathring{\mathbb{R}}^d)\big)$ with

$$|\boldsymbol{\eta}_k|(C([0,T];\mathring{\mathbb{R}}^d)) \leq \sup_{t\in[0,T]} \|\beta_k(\rho_t)\|_{L^1(\mathbb{R}^d)},$$

which is concentrated on the set defined in (4.17) and satisfies

$$(e_t)_\#\boldsymbol{\eta}_k \llcorner \mathbb{R}^d = \beta_k(\rho_t)\,\mathscr{L}^d \qquad \text{for every } t\in[0,T].$$

Since $\sum_{k\geq 0}\beta_k(s) = s$, we immediately deduce that the measure $\boldsymbol{\eta} := \sum_{k\geq 0}\boldsymbol{\eta}_k$ satisfies all the desired properties.

Step 4: representation via the Maximal Regular Flow. Under the additional assumption that $\boldsymbol{b}$ is divergence-free and satisfies **(A1)-(A2)** of Section 4.1, if $\rho_t \in L^\infty((0,T)\times\mathbb{R}^d)$ (resp. that ρ_t is renormalized) then $\boldsymbol{\eta}$ (resp. every $\boldsymbol{\eta}_k$) is a regular generalized flow and by Theorem 4.7 it is transported by the Maximal Regular Flow. □

A The assumptions of DiPerna-Lions for Hamiltonian ODEs

In this appendix we characterize Hamiltonian-type systems that fall under the assumptions of the classical DiPerna Lions theory.

We recall that, in the seminal paper [16], DiPerna and Lions showed that for Sobolev vector fields one can introduce a suitable notion of flow provided the trajectories of the flow do not blow up in finite time. This is is expressed in terms of the vector field by the following global hypothesis:

$$\frac{|\boldsymbol{b}_t|(x,v)}{1+|x|+|v|} \in L^1\big((0,T);L^1(\mathbb{R}^{2d})\big) + L^1\big((0,T);L^\infty(\mathbb{R}^{2d})\big). \tag{A.1}$$

In the case when $b_t(x,v)$ takes the form $(v,E_t(x))$ for some time dependent vector-field $E_t:(0,T)\times\mathbb{R}^d\to\mathbb{R}^d$, we see that the first term $\frac{|v|}{1+|x|+|v|}$ is bounded. Hence, one needs to understand under which assumptions on E_t the term $\frac{|E_t|(x)}{1+|x|+|v|}$ satisfies (A.1). The next result gives a complete answer to this question.

Lemma A.1. *Let $T>0$, $E:(0,T)\times\mathbb{R}^d\to\mathbb{R}^d$. Then the following three conditions are equivalent:*

(i) $\frac{|E_t|(x)}{1+|x|+|v|} \in L^1\big((0,T);L^1(\mathbb{R}^{2d})\big) + L^1\big((0,T);L^\infty(\mathbb{R}^{2d})\big)$,

(ii) $E_t = E_t^1 + E_t^2$ *with* $|E_t^1| \in L^1\big((0,T);L^d(\mathbb{R}^d)\big)$ *and* $\frac{|E_t^2|(x)}{1+|x|} \in L^1\big((0,T);L^\infty(\mathbb{R}^d)\big)$,

(iii) there exists $C(t)\in L^1((0,T))$ *such that* $\big(|E_t|(x)-C(t)(1+|x|)\big)_+ \in L^1\big((0,T);L^d(\mathbb{R}^d)\big)$.

Proof. We first prove the equivalence between (ii) and (iii). If (ii) holds, write $E_t = E_t^1 + E_t^2$ as in (ii) and define $C(t) := \big\|\frac{E_t^2(x)}{1+|x|}\big\|_{L^\infty(\mathbb{R}^d)}$. Then

$$\big(|E_t| - C(t)(1+|x|)\big)_+ = \big(|E_t^1| + |E_t^2| - C(t)(1+|x|)\big)_+ \leq |E_t^1| \in L^d\big((0,T);L^d(\mathbb{R}^d)\big).$$

On the other hand, if (iii) holds we choose

$$\begin{aligned} E_t^1(x) &:= \tfrac{E_t(x)}{|E_t(x)|}\big(|E_t|(x) - C(t)(1+|x|)\big)_+, \\ E_t^2(x) &:= \tfrac{E_t(x)}{|E_t(x)|}\Big[C(t)(1+|x|) - \big(|E_t|(x) - C(t)(1+|x|)\big)_-\Big], \end{aligned}$$

and it is easily seen that they satisfy (ii).

We now show that (ii) implies (i). Writing $E_t = E_t^1 + E_t^2$ as in (ii), we notice that, for any function $D \in L^1((0,T))$ with $D(t) \geq 1$,

$$\begin{aligned} \frac{E_t(x)}{1+|x|+|v|} = \frac{E_t^1(x)}{1+|x|+|v|}1_{\{(x,v):D(t)|v|\leq|E_t^1|(x)\}} \\ + \frac{E_t^1(x)}{1+|x|+|v|}1_{\{(x,v):D(t)|v|>|E_t^1|(x)\}} + \frac{E_t^2(x)}{1+|x|+|v|}, \end{aligned}$$

and the last two terms belong to $L^1\big((0,T);L^\infty(\mathbb{R}^{2d})\big)$. To show that the first term belongs to $L^1\big((0,T);L^1(\mathbb{R}^{2d})\big)$, we pass in polar coordinates in v and notice that the function $r^{d-1}/(1+|x|+r)$ is increasing in r for any $x \in \mathbb{R}^d$, to get

$$\begin{aligned} \int_0^T\int_{\mathbb{R}^d}\int_{\big\{v:|v|\leq\frac{|E_t^1|(x)}{D(t)}\big\}} \frac{|E_t^1|(x)}{1+|x|+|v|}\,dv\,dx\,dt &\leq C_d\int_0^T\int_{\mathbb{R}^d}|E_t^1|(x)\int_0^{\frac{|E_t^1|(x)}{D(t)}}\frac{r^{d-1}}{1+|x|+r}\,dr\,dx\,dt \\ &\leq C_d\int_0^T\int_{\mathbb{R}^d}\frac{|E_t^1|(x)^2}{D(t)}\frac{\frac{|E_t^1|(x)^{d-1}}{D(t)^{d-1}}}{1+|x|+\frac{|E_t^1|(x)}{D(t)}}\,dx\,dt \\ &\leq C_d\int_0^T\frac{1}{D(t)^{d-1}}\int_{\mathbb{R}^d}|E_t^1|(x)^d\,dx\,dt. \end{aligned}$$

In particular, choosing $D(t) := 1 + \|E_t^1\|_{L^d(\mathbb{R}^d)}$, we can bound the last term above by

$$\int_0^T \|E_t^1\|_{L^d(\mathbb{R}^d)}\,dt < \infty,$$

which concludes the proof that (iii) implies (i).

Finally, we assume that (i) holds and show (iii). Indeed, write $\frac{|E_t|(x)}{1+|x|+|v|}$ as a sum as in (i), and denote by $C(t) \in L^1((0,T))$ a bound for the $L^\infty(\mathbb{R}^{2d})$-norm of the second addend. With no loss of generality, we can assume that $C(t) \geq 1$. By assumption, with this choice of $C(t)$ we have

$$\int_0^T\int_{\mathbb{R}^{2d}}\Big(\frac{|E_t|(x)}{1+|x|+|v|} - C(t)\Big)_+\,dv\,dx\,dt < \infty.$$

We first rewrite this integral as

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}^{2d}}\Big(\frac{|E_t|(x)}{1+|x|+|v|} - C(t)\Big)_+\,dv\,dx\,dt \\ &= \int_0^T\int_{\big\{x:\frac{|E_t|(x)}{1+|x|}\geq C(t)\big\}}\int_{\big\{v:|v|\leq\frac{|E_t|(x)}{C(t)}-1-|x|\big\}}\Big(\frac{|E_t|(x)}{1+|x|+|v|} - C(t)\Big)_+\,dv\,dx\,dt. \end{aligned} \tag{A.2}$$

Then, we note that

$$A := \left\{x : \frac{|E_t|(x)}{1+|x|} \geq 4C(t)\right\} \cap \left\{\frac{|E_t|(x)}{4C(t)} \leq |v| \leq \frac{|E_t|(x)}{2C(t)}\right\}$$
$$\subset \left\{x : \frac{|E_t|(x)}{1+|x|} \geq C(t)\right\} \cap \left\{0 \leq |v| \leq \frac{|E_t|(x)}{C(t)} - 1 - |x|\right\},$$

and that, inside A,

$$\frac{|E_t|(x)}{1+|x|+|v|} \geq \frac{|E_t|(x)}{\frac{|E_t|(x)}{4C(t)} + \frac{|E_t|(x)}{2C(t)}} = \frac{4}{3}C(t),$$

therefore

$$\frac{|E_t|(x)}{1+|x|+|v|} - C(t) \geq \frac{|E_t|(x)}{4(1+|x|+|v|)} \geq \frac{|E_t|(x)}{8|v|} \qquad \text{inside } A.$$

Thus, we can bound from below the second integral in (A.2) by

$$\begin{aligned}
&\int_0^T \int_{\left\{x: \frac{|E_t|(x)}{1+|x|} \geq 4C(t)\right\}} \int_{\left\{v: \frac{|E_t|(x)}{4C(t)} \leq |v| \leq \frac{|E_t|(x)}{2C(t)}\right\}} \frac{|E_t|(x)}{8|v|} \, dv \, dx \, dt \\
&= c_d \int_0^T \int_{\left\{x: \frac{|E_t|(x)}{1+|x|} \geq 4C(t)\right\}} \int_{\frac{|E_t|(x)}{2C(t)}}^{\frac{|E_t|(x)}{2C(t)}} \frac{|E_t|(x)}{8} r^{d-2} \, dr \, dx \, dt \\
&= \hat{c}_d \int_0^T \int_{\left\{x: \frac{|E_t|(x)}{1+|x|} \geq 4C(t)\right\}} \frac{|E_t|(x)^d}{C(t)^{d-1}} \, dx \, dt \\
&\geq \hat{c}_d \int_0^T \frac{1}{C(t)^{d-1}} \int_{\mathbb{R}^d} \big(|E_t|(x) - 4C(t)(1+|x|)\big)_+^d \, dx \, dt.
\end{aligned} \tag{A.3}$$

Since, by Hölder inequality,

$$\begin{aligned}
&\int_0^T \left\| \big(|E_t|(x) - 4C(t)(1+|x|)\big)_+ \right\|_{L^d(\mathbb{R}^d)} dt \\
&\qquad \leq \left(\int_0^T \frac{1}{C(t)^{d-1}} \int_{\mathbb{R}^d} \big(|E_t|(x) - 4C(t)(1+|x|)\big)_+^d \, dx \, dt\right)^{1/d} \left(\int_0^T C(t) \, dt\right)^{\frac{d-1}{d}},
\end{aligned}$$

it follows by (A.2) and (A.3) that $\big(|E_t|(x) - 4C(t)(1+|x|)\big)_+ \in L^1\big((0,T); L^d(\mathbb{R}^d)\big)$, which proves (iii). □

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