

# PARTIAL REGULARITY FOR OPTIMAL TRANSPORT MAPS

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ABSTRACT. We prove that, for general cost functions on  $\mathbb{R}^n$ , or for the cost  $d^2/2$  on a Riemannian manifold, optimal transport maps between smooth densities are always smooth outside a closed singular set of measure zero.

## 1. INTRODUCTION

A natural and important issue in optimal transport theory is the regularity of optimal transport maps. Indeed, apart from being a typical PDE/analysis question, knowing whether optimal maps are smooth or not is an important step towards a qualitative understanding of them.

It is by now well known that, for the smoothness of optimal maps, conditions on both the cost function and on the geometry of the supports of the measures are needed.

In the special case  $c(x, y) = |x - y|^2/2$  on  $\mathbb{R}^n$ , Caffarelli [3, 4, 5, 6] proved regularity of optimal maps under suitable assumptions on the densities and on the geometry of their support. More precisely, in its simplest form, Caffarelli's result states as follows:

**Theorem 1.1.** *Let  $f$  and  $g$  be smooth probability densities, respectively bounded away from zero and infinity on two bounded open sets  $X$  and  $Y$ , and let  $T : X \rightarrow Y$  denote the unique optimal transport map from  $f$  to  $g$  for the quadratic cost  $|x - y|^2/2$ . If  $Y$  is convex, then  $T$  is smooth inside  $X$ . On the other hand, if  $Y$  is not convex, then there exist smooth densities  $f$  and  $g$  (both bounded away from zero and infinity on  $X$  and  $Y$ , respectively) for which the map  $T$  is not continuous.*

A natural question which arises from the previous result is whether one may prove some partial regularity on  $T$  when the convexity assumption on  $Y$  is removed. In [16, 18] the authors proved the following result:

**Theorem 1.2.** *Let  $f$  and  $g$  be smooth probability densities, respectively bounded away from zero and infinity on two bounded open sets  $X$  and  $Y$ , and let  $T : X \rightarrow Y$  denote the unique optimal transport map from  $f$  to  $g$  for the quadratic cost  $|x - y|^2/2$ . Then there exist two open sets  $X' \subset X$  and  $Y' \subset Y$ , with  $|X \setminus X'| = |Y \setminus Y'| = 0$ , such that  $T : X' \rightarrow Y'$  is a smooth diffeomorphism.*

In the case of general cost functions on  $\mathbb{R}^n$ , or when  $c(x, y) = d(x, y)^2/2$  on a Riemannian manifold  $M$  ( $d(x, y)$  being the Riemannian distance), the situation is much more complicated. Indeed, as shown by Ma, Trudinger, and Wang [33], and Loeper [31], in addition to suitable convexity assumptions on the support of the target density (or on the cut locus of the manifold when  $\text{supp}(g) = M$  [24]), a very strong structural condition on the cost function, the so-called *MTW condition*, is needed to ensure the smoothness of the map.

More precisely, if the MTW condition holds (together with some suitable convexity assumptions on the target domain), then the optimal map is smooth [35, 36, 21, 30, 19]. On the other hand, if the MTW condition fails at one point, then one can construct smooth densities (both supported

on domains which satisfy the needed convexity assumptions) for which the optimal transport map is not continuous [31] (see also [15]).

In the case of Riemannian manifolds, the MTW condition for  $c = d^2/2$  is very restrictive: indeed, as shown by Loeper [31], it implies that  $M$  has non-negative sectional curvature, and actually it is much stronger than the latter [28, 23]. In particular, if  $M$  has negative sectional curvature, then the MTW condition fails at *every* point. Let us also mention that, up to now, the MTW condition is known to be satisfied only for very special classes of Riemannian manifolds, such as spheres, their products, their quotients and submersions, and their perturbations [32, 22, 10, 29, 25, 20, 11], and for instance it is known to fail on sufficiently flat ellipsoids [23].

The goal of the present paper is to show that, even without any condition on the cost function or on the supports of the densities, optimal transport maps are always smooth outside a closed singular set of measure zero. In order to state our results, we first have to introduce some basic assumptions on the cost functions which are needed to ensure existence and uniqueness of optimal maps. As before,  $X$  and  $Y$  denote two open subsets of  $\mathbb{R}^n$ .

- (C0) The cost function  $c : X \times Y \rightarrow \mathbb{R}$  is of class  $C^{2,\alpha}$  for some  $\alpha \in (0, 1)$ , with  $\|c\|_{C^{2,\alpha}(X \times Y)} < \infty$ .
- (C1) For any  $x \in X$ , the map  $Y \ni y \mapsto -D_x c(x, y) \in \mathbb{R}^n$  is injective.
- (C2) For any  $y \in Y$ , the map  $X \ni x \mapsto -D_y c(x, y) \in \mathbb{R}^n$  is injective.
- (C3)  $\det(D_{xy}c)(x, y) \neq 0$  for all  $(x, y) \in X \times Y$ .

Here are our main results:

**Theorem 1.3.** *Let  $X, Y \subset \mathbb{R}^n$  be two bounded open sets, and let  $f : X \rightarrow \mathbb{R}^+$  and  $g : Y \rightarrow \mathbb{R}^+$  be two continuous probability densities, respectively bounded away from zero and infinity on  $X$  and  $Y$ . Assume that the cost  $c : X \times Y \rightarrow \mathbb{R}$  satisfies (C0)-(C3), and denote by  $T : X \rightarrow Y$  the unique optimal transport map sending  $f$  onto  $g$ . Then there exist two relatively closed sets  $\Sigma_X \subset X, \Sigma_Y \subset Y$  of measure zero such that  $T : X \setminus \Sigma_X \rightarrow Y \setminus \Sigma_Y$  is a homeomorphism of class  $C_{\text{loc}}^{0,\beta}$  for any  $\beta < 1$ . In addition, if  $c \in C_{\text{loc}}^{k+2,\alpha}(X \times Y)$ ,  $f \in C_{\text{loc}}^{k,\alpha}(X)$ , and  $g \in C_{\text{loc}}^{k,\alpha}(Y)$  for some  $k \geq 0$  and  $\alpha \in (0, 1)$ , then  $T : X \setminus \Sigma_X \rightarrow Y \setminus \Sigma_Y$  is a diffeomorphism of class  $C_{\text{loc}}^{k+1,\alpha}$ .*

**Theorem 1.4.** *Let  $M$  be a smooth Riemannian manifold, and let  $f, g : M \rightarrow \mathbb{R}^+$  be two continuous probability densities, locally bounded away from zero and infinity on  $M$ . Let  $T : M \rightarrow M$  denote the optimal transport map for the cost  $c = d^2/2$  sending  $f$  onto  $g$ . Then there exist two closed sets  $\Sigma_X, \Sigma_Y \subset M$  of measure zero such that  $T : M \setminus \Sigma_X \rightarrow M \setminus \Sigma_Y$  is a homeomorphism of class  $C_{\text{loc}}^{0,\beta}$  for any  $\beta < 1$ . In addition, if both  $f$  and  $g$  are of class  $C^{k,\alpha}$ , then  $T : M \setminus \Sigma_X \rightarrow M \setminus \Sigma_Y$  is a diffeomorphism of class  $C_{\text{loc}}^{k+1,\alpha}$ .*

The paper is structured as follows: in the next section we introduce some notation and preliminary results. Then, in Section 3, we show how both Theorem 1.3 and Theorem 1.4 are a direct consequence of some local regularity results around differentiability points of  $T$ , see Theorems 4.3 and 5.3. Finally, Sections 4 and 5 are devoted to the proof of these local results.

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## 2. NOTATION AND PRELIMINARY RESULTS

Through a well established procedure, maps that solve optimal transport problems derive from a  $c$ -convex potential, itself solution to a Monge-Ampère type equation.

More precisely, given a cost function  $c : X \times Y \rightarrow \mathbb{R}$ , a function  $u : X \rightarrow \mathbb{R}$  is said  $c$ -convex if it can be written as

$$(2.1) \quad u(x) = \sup_{y \in Y} \{-c(x, y) + \lambda_y\},$$

for some constants  $\lambda_y \in \mathbb{R} \cup \{-\infty\}$ .

Similarly to the subdifferential for convex function, for  $c$ -convex functions one can talk about their  $c$ -subdifferential: if  $u : X \rightarrow \mathbb{R}$  is a  $c$ -convex function as above, the  $c$ -subdifferential of  $u$  at  $x$  is the (nonempty) set

$$(2.2) \quad \partial_c u(x) := \{y \in \bar{Y} : u(z) \geq -c(z, y) + c(x, y) + u(x) \quad \forall z \in X\}.$$

If  $x_0 \in X$  and  $y_0 \in \partial_c u(x_0)$ , we will say that the function

$$(2.3) \quad C_{x_0, y_0}(\cdot) := -c(\cdot, y_0) + c(x_0, y_0) + u(x_0)$$

is a  $c$ -support for  $u$  at  $x_0$ . We also define the *Frechet subdifferential* of  $u$  at  $x$  as

$$\partial^- u(x) := \{p \in \mathbb{R}^n : u(z) \geq u(x) + p \cdot (z - x) + o(|z - x|)\}.$$

We will use the following notation: if  $E \subset X$  then

$$\partial_c u(E) := \bigcup_{x \in E} \partial_c u(x), \quad \partial^- u(E) := \bigcup_{x \in E} \partial^- u(x).$$

It is easy to check that, if  $c$  is of class  $C^1$ , then the following inclusion holds:

$$(2.4) \quad y \in \partial_c u(x) \implies -D_x c(x, y) \in \partial^- u(x).$$

In addition, if  $c$  satisfies **(C0)**-**(C2)**, then we can define the  $c$ -exponential map:

$$(2.5) \quad \text{for any } x \in X, y \in Y, p \in \mathbb{R}^n, \quad \begin{cases} c\text{-exp}_x(p) = y & \Leftrightarrow p = -D_x c(x, y) \\ c^*\text{-exp}_y(p) = x & \Leftrightarrow p = -D_y c(x, y) \end{cases}$$

Using (2.5), we can rewrite (2.4) as

$$(2.6) \quad \partial_c u(x) \subset c\text{-exp}_x(\partial^- u(x)).$$

Notice that, if  $c \in C^1$  and  $Y$  is bounded, it follows immediately from (2.1) that  $c$ -convex functions are Lipschitz, so in particular they are differentiable a.e.

The following notation will be convenient: given a  $c$ -convex function  $u : X \rightarrow \mathbb{R}$ , we define (at almost every point) the map  $T_u : X \rightarrow Y$  as

$$(2.7) \quad T_u(x) := c\text{-exp}_x(\nabla u(x)).$$

(Of course  $T_u$  depends also on  $c$ , but to keep the notation lighter we prefer not to make this dependence explicit. The reader should keep in mind that, whenever we write  $T_u$ , the cost  $c$  is always the one for which  $u$  is  $c$ -convex.)

Finally, let us observe that if  $c$  satisfies **(C0)** and  $Y$  is bounded, then it follows from (2.1) that  $u$  is semiconvex (i.e., there exists a constant  $C > 0$  such that  $u + C|x|^2/2$  is convex, see for instance [13]). In particular, by Alexandrov's Theorem,  $c$ -convex functions are twice differentiable a.e. (see [37, Theorem 14.25] for a list of different equivalent definitions of this notion).

The following is a basic result in optimal transport theory (see for instance [37, Chapter 10]):

**Theorem 2.1.** *Let  $c : X \times Y \rightarrow \mathbb{R}$  satisfy **(C0)**-**(C1)**. Given two probability densities  $f$  and  $g$  supported on  $X$  and  $Y$  respectively, there exists a  $c$ -convex function  $u : X \rightarrow \mathbb{R}$  such that  $T_u : X \rightarrow Y$  is the unique optimal transport map sending  $f$  onto  $g$ .*

In the particular case  $c(x, y) = -x \cdot y$  (which is equivalent to the quadratic cost  $|x - y|^2/2$ ),  $c$ -convex functions are convex and the above result takes the following simple form [2]:

**Theorem 2.2.** *Let  $c(x, y) = -x \cdot y$ . Given two probability densities  $f$  and  $g$  supported on  $X$  and  $Y$  respectively, there exists a convex function  $v : X \rightarrow \mathbb{R}$  such that  $T_v = \nabla v : X \rightarrow Y$  is the unique optimal transport map sending  $f$  onto  $g$ .*

Although on Riemannian manifolds the cost function  $c = d^2/2$  is not smooth everywhere, one can still prove existence of optimal maps [34, 13, 17] (let us remark that, in this case, the  $c$ -exponential map coincides with the classical exponential map in Riemannian geometry):

**Theorem 2.3.** *Let  $M$  be a smooth Riemannian manifold, and  $c = d^2/2$ . Given two probability densities  $f$  and  $g$  supported on  $M$ , there exists a  $c$ -convex function  $u : M \rightarrow \mathbb{R} \cup \{+\infty\}$  such that  $u$  is differentiable  $f$ -a.e., and  $T_u(x) = \exp_x(\nabla u(x))$  is the unique optimal transport map sending  $f$  onto  $g$ .*

We conclude this section by recalling that  $c$ -convex functions arising in optimal transport problems solve a Monge-Ampère type equation almost everywhere, referring to [1, Section 6.2], [37, Chapters 11 and 12], and [15] for more details.

Whenever  $c$  satisfies **(C0)**-**(C3)**, then the transport condition  $(T_u)_\#f = g$  gives

$$(2.8) \quad |\det(DT_u(x))| = \frac{f(x)}{g(T_u(x))} \quad \text{a.e.}$$

In addition, the  $c$ -convexity of  $u$  implies that, at every point  $x$  where  $u$  is twice differentiable,

$$(2.9) \quad D^2u(x) + D_{xx}c(x, c\text{-exp}_x(\nabla u(x))) \geq 0.$$

Hence, writing (2.7) as

$$-D_x c(x, T_u(x)) = \nabla u(x),$$

differentiating the above relation with respect to  $x$ , and using (2.8) and (2.9), we obtain

$$(2.10) \quad \det\left(D^2u(x) + D_{xx}c(x, c\text{-exp}_x(\nabla u(x)))\right) = |\det(D_{xy}c(x, c\text{-exp}_x(\nabla u(x))))| \frac{f(x)}{g(c\text{-exp}_x(\nabla u(x)))}$$

at every point  $x$  where  $u$  is twice differentiable. In particular, when  $c(x, y) = -x \cdot y$ , the convex function  $v$  provided by Theorem 2.2 solves the classical Monge-Ampère equation

$$\det(D^2v(x)) = \frac{f(x)}{g(\nabla v(x))} \quad \text{a.e.}$$

### 3. THE LOCALIZATION ARGUMENT AND PROOF OF THE RESULTS

The goal of this section is to prove Theorems 1.3 and 1.4 by showing that the assumptions of Theorems 4.3 and 5.3 below are satisfied near almost every point.

The rough idea is the following: if  $\bar{x}$  is a point where the semiconvex function  $u$  is twice differentiable, then around that point  $u$  looks like a parabola. In addition, by looking close enough to  $\bar{x}$ , the cost function  $c$  will be very close to the linear one and the densities will be almost constant there. Hence we can apply Theorem 4.3 to deduce that  $u$  is of class  $C^{1,\beta}$  in neighborhood of  $\bar{x}$  (resp.

$u$  is of class  $C^{k+2,\alpha}$  by Theorem 5.3, if  $c \in C_{\text{loc}}^{k+2,\alpha}$  and  $f, g \in C_{\text{loc}}^{k,\alpha}$ , which implies in particular that  $T_u$  is of class  $C^{0,\beta}$  in neighborhood of  $\bar{x}$  (resp.  $T_u$  is of class  $C^{k+1,\alpha}$  by Theorem 5.3, if  $c \in C_{\text{loc}}^{k+2,\alpha}$  and  $f, g \in C_{\text{loc}}^{k,\alpha}$ ). Being our assumptions completely symmetric in  $x$  and  $y$ , we can apply the same argument to the optimal map  $T^*$  sending  $g$  onto  $f$ . Since  $T^* = (T_u)^{-1}$  (see the discussion below), it follows that  $T_u$  is a global homeomorphism of class  $C_{\text{loc}}^{0,\beta}$  (resp.  $T_u$  is a global diffeomorphism of class  $C_{\text{loc}}^{k+1,\alpha}$ ) outside a closed set of measure zero.

We now give a detailed proof.

*Proof of Theorem 1.3.* Let us introduce the “ $c$ -conjugate” of  $u$ , that is, the function  $u^c : Y \rightarrow \mathbb{R}$  defined as

$$u^c(y) := \sup_{x \in X} \{ -c(x, y) - u(x) \}.$$

Then  $u^c$  is  $c^*$ -convex, where

$$(3.1) \quad c^*(y, x) := c(x, y), \quad \text{and} \quad x \in \partial_{c^*} u^c(y) \Leftrightarrow y \in \partial_c u(x)$$

(see for instance [37, Chapter 5]).

Being our assumptions completely symmetric in  $x$  and  $y$ ,  $c^*$  satisfies the same assumptions as  $c$ . In particular, by Theorem 2.1, there exists an optimal map  $T^*$  (with respect to  $c^*$ ) sending  $g$  onto  $f$ . In addition, it is well-known that  $T^*$  is actually equal to

$$T_{u^c}(y) = c^*\text{-exp}_y(\nabla u^c(y)),$$

and that  $T_u$  and  $T_{u^c}$  are inverse to each other, that is

$$(3.2) \quad T_{u^c}(T_u(x)) = x, \quad T_u(T_{u^c}(y)) = y \quad \text{for a.e. } x \in X, y \in Y$$

(see, for instance, [1, Remark 6.2.11]).

Since semiconvex functions are twice differentiable a.e., there exist sets  $X_1 \subset X, Y_1 \subset Y$  of full measure such that (3.2) holds for every  $x \in X_1$  and  $y \in Y_1$ , and in addition  $u$  is twice differentiable for every  $x \in X_1$  and  $u^c$  is twice differentiable for every  $y \in Y_1$ . Let us define

$$X' := X_1 \cap (T_u)^{-1}(Y_1).$$

Using that  $T_u$  transports  $f$  on  $g$  and that the two densities are bounded away from zero and infinity, we see that  $X'$  is of full measure in  $X$ .

We fix a point  $\bar{x} \in X'$ . Since  $u$  is differentiable at  $\bar{x}$  (being twice differentiable), it follows by (2.6) that the set  $\partial_c u(\bar{x})$  is a singleton, namely  $\partial_c u(\bar{x}) = \{c\text{-exp}_{\bar{x}}(\nabla u(\bar{x}))\}$ . Set  $\bar{y} := c\text{-exp}_{\bar{x}}(\nabla u(\bar{x}))$ . Since  $\bar{y} \in Y_1$  (by definition of  $X'$ ),  $u^c$  is twice differentiable at  $\bar{y}$  and  $\bar{x} = T_{u^c}(\bar{y})$ . Up to a translation in the system of coordinates (both in  $x$  and  $y$ ) we can assume that both  $\bar{x}$  and  $\bar{y}$  coincide with the origin  $\mathbf{0}$ .

Let us define

$$\begin{aligned} \bar{u}(z) &:= u(z) - u(\mathbf{0}) + c(z, \mathbf{0}) - c(\mathbf{0}, \mathbf{0}), \\ \bar{c}(z, w) &:= c(z, w) - c(z, \mathbf{0}) - c(\mathbf{0}, w) + c(\mathbf{0}, \mathbf{0}), \\ \bar{u}^{\bar{c}}(w) &:= u^c(w) - u(\mathbf{0}) + c(\mathbf{0}, w) - c(\mathbf{0}, \mathbf{0}). \end{aligned}$$

Then  $\bar{u}$  is a  $\bar{c}$ -convex function,  $\bar{u}^{\bar{c}}$  is its  $\bar{c}$ -conjugate,  $T_{\bar{u}} = T_u$ , and  $T_{\bar{u}^{\bar{c}}} = T_{u^c}$ , so in particular  $(T_{\bar{u}})_{\#} f = g$  and  $(T_{\bar{u}^{\bar{c}}})_{\#} g = f$ . In addition, because by assumption  $\mathbf{0} \in X'$ ,  $\bar{u}$  is twice differentiable

at  $\mathbf{0}$  and  $\bar{u}^c$  is twice differentiable at  $\mathbf{0} = T_{\bar{u}}(\mathbf{0})$ . Let us define  $P := D^2\bar{u}(\mathbf{0})$ , and  $M := D_{xy}\bar{c}(\mathbf{0}, \mathbf{0})$ . Then, since  $\bar{c}(\cdot, \mathbf{0}) = \bar{c}(\mathbf{0}, \cdot) \equiv 0$  and  $\bar{c} \in C^{2,\alpha}$ , a Taylor expansion gives

$$\bar{u}(z) = \frac{1}{2}Pz \cdot z + o(|z|^2), \quad \bar{c}(z, w) = Mz \cdot w + O(|z|^{2+\alpha} + |w|^{2+\alpha}),$$

Let us observe that, since by assumption  $f$  and  $g$  are bounded away from zero and infinity, by **(C3)** and (2.10) applied to  $\bar{u}$  and  $\bar{c}$  we get that  $\det(P), \det(M) \neq 0$ . In addition (2.9) implies that  $P$  is a positive definite symmetric matrix. Hence, we can perform a second change of coordinates:  $z \mapsto \tilde{z} := P^{1/2}z$ ,  $w \mapsto \tilde{w} := -P^{-1/2}M^*w$  ( $M^*$  being the transpose of  $M$ ), so that, in the new variables,

$$(3.3) \quad \tilde{u}(\tilde{z}) := \bar{u}(z) = \frac{1}{2}|\tilde{z}|^2 + o(|\tilde{z}|^2), \quad \tilde{c}(\tilde{z}, \tilde{w}) := \bar{c}(z, w) = -\tilde{z} \cdot \tilde{w} + O(|\tilde{z}|^{2+\alpha} + |\tilde{w}|^{2+\alpha}).$$

By an easy computation it follows that  $(T_{\tilde{u}})_{\#}\tilde{f} = \tilde{g}$ , where <sup>1</sup>

$$(3.4) \quad \tilde{f}(\tilde{z}) := \det(P^{-1/2}) f(P^{-1/2}\tilde{z}), \quad \tilde{g}(\tilde{w}) := |\det((M^*)^{-1}P^{1/2})| g(-(M^*)^{-1}P^{1/2}\tilde{w}).$$

Notice that

$$(3.5) \quad D_{\tilde{z}\tilde{z}}\tilde{c}(\mathbf{0}, \mathbf{0}) = D_{\tilde{w}\tilde{w}}\tilde{c}(\mathbf{0}, \mathbf{0}) = \mathbf{0}_{n \times n}, \quad -D_{\tilde{z}\tilde{w}}\tilde{c}(\mathbf{0}, \mathbf{0}) = \text{Id}, \quad D^2\tilde{u}(\mathbf{0}) = \text{Id},$$

so, using (2.10), we deduce that

$$(3.6) \quad \frac{\tilde{f}(\mathbf{0})}{\tilde{g}(\mathbf{0})} = \frac{\det(D^2\tilde{u}(\mathbf{0}) + D_{\tilde{z}\tilde{z}}\tilde{c}(\mathbf{0}, \mathbf{0}))}{|\det(D_{\tilde{z}\tilde{w}}\tilde{c}(\mathbf{0}, \mathbf{0}))|} = 1.$$

To ensure that we can apply Theorems 4.3 and 5.3, we now perform the following dilation: for  $\rho > 0$  we define

$$u_{\rho}(\tilde{z}) := \frac{1}{\rho^2}\tilde{u}(\rho\tilde{z}), \quad c_{\rho}(\tilde{z}, \tilde{w}) := \frac{1}{\rho^2}\tilde{c}(\rho\tilde{z}, \rho\tilde{w}).$$

We claim that, provided  $\rho$  is sufficiently small,  $u_{\rho}$  and  $c_{\rho}$  satisfy the assumptions of Theorems 4.3 and 5.3.

Indeed, it is immediate to check that  $u_{\rho}$  is a  $c_{\rho}$ -convex function. Also, by the same argument as above, from the relation  $(T_{\tilde{u}})_{\#}\tilde{f} = \tilde{g}$  we deduce that  $T_{u_{\rho}}$  sends  $\tilde{f}(\rho\tilde{z})$  onto  $\tilde{g}(\rho\tilde{w})$ . In addition, since we can freely multiply both densities by a same constant, it actually follows from (3.6) that  $(T_{u_{\rho}})_{\#}f_{\rho} = g_{\rho}$ , where

$$f_{\rho}(\tilde{z}) := \frac{\tilde{f}(\rho\tilde{z})}{\tilde{f}(\mathbf{0})}, \quad g_{\rho}(\tilde{w}) := \frac{\tilde{g}(\rho\tilde{w})}{\tilde{g}(\mathbf{0})}.$$

In particular, since  $f$  and  $g$  are continuous, we get

$$(3.7) \quad |f_{\rho} - 1| + |g_{\rho} - 1| \rightarrow 0 \quad \text{inside } B_3$$

as  $\rho \rightarrow 0$ . Also, by (3.3) we get that, for any  $\tilde{z}, \tilde{w} \in B_3$ ,

$$(3.8) \quad u_{\rho}(\tilde{z}) = \frac{1}{2}|\tilde{z}|^2 + o(1), \quad c_{\rho}(\tilde{z}, \tilde{w}) = -\tilde{z} \cdot \tilde{w} + O(\rho^{\alpha}|z|^{2+\alpha} + \rho^{\alpha}|w|^{2+\alpha}),$$

<sup>1</sup>An easy way to check this is to observe that the measures  $\mu := f(x)dx$  and  $\nu := g(y)dy$  are independent of the choice of coordinates, hence (3.4) follows from the identities

$$f(x)dx = \tilde{f}(\tilde{x})d\tilde{x}, \quad g(y)dy = \tilde{g}(\tilde{y})d\tilde{y}.$$

where  $o(1) \rightarrow 0$  as  $\rho \rightarrow 0$ . In particular, it follows easily that (4.9) and (4.10) hold with any positive constants  $\delta_0, \eta_0$  provided  $\rho$  is small enough.

Furthermore, by the second order differentiability of  $\tilde{u}$  at  $\mathbf{0}$  it follows that the multivalued map  $\tilde{z} \mapsto \partial^- \tilde{u}(\tilde{z})$  is differentiable at  $\mathbf{0}$  (see [37, Theorem 14.25]) with gradient equal to the identity matrix (see (3.3)), hence

$$\partial^- u_\rho(\tilde{z}) \subset B_{\gamma_\rho}(\tilde{z}) \quad \forall \tilde{z} \in B_2,$$

where  $\gamma_\rho \rightarrow 0$  as  $\rho \rightarrow 0$ . Since  $\partial_{c_\rho} u_\rho \subset c_\rho\text{-exp}(\partial^- u_\rho)$  (by (2.6)) and  $\|c_\rho\text{-exp} - \text{Id}\|_\infty = o(1)$  (by (3.8)), we get

$$(3.9) \quad \partial_{c_\rho} u_\rho(\tilde{z}) \subset B_{\delta_\rho}(\tilde{z}) \quad \forall \tilde{z} \in B_3,$$

with  $\delta_\rho = o(1)$  as  $\rho \rightarrow 0$ . Moreover, the  $c_\rho$ -conjugate of  $u_\rho$  is easily seen to be

$$u_\rho^{c_\rho}(\tilde{w}) = \tilde{u}^c(\rho(M^*)^{-1}P^{1/2}\tilde{w}).$$

Since  $u^c$  is twice differentiable at  $\mathbf{0}$ , so is  $u_\rho^{c_\rho}$ . In addition, an easy computation <sup>2</sup> shows that  $D^2 u_\rho^{c_\rho}(\mathbf{0}) = \text{Id}$ . Hence, arguing as above we obtain that

$$(3.10) \quad \partial_{c_\rho^*} u_\rho^{c_\rho}(\tilde{w}) \subset B_{\delta'_\rho}(\tilde{w}) \quad \forall \tilde{w} \in B_3,$$

with  $\delta'_\rho = o(1)$  as  $\rho \rightarrow 0$ .

We now define

$$\mathcal{C}_1 := \overline{B}_1, \quad \mathcal{C}_2 := \partial_{c_\rho} u_\rho(\mathcal{C}_1).$$

Observe that both  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are closed (since the  $c$ -subdifferential of a compact set is closed). Also, thanks to (3.9), by choosing  $\rho$  small enough we can ensure that  $B_{1/3} \subset \mathcal{C}_2 \subset B_3$ . Finally, it follows from (2.6) that

$$(T_{u_\rho})^{-1}(\mathcal{C}_2) \setminus \mathcal{C}_1 \subset (T_{u_\rho})^{-1}(\{\text{points of non-differentiability of } u_\rho^{c_\rho}\}),$$

and since this latter set has measure zero, a simple computation shows that

$$(T_{u_\rho})_\#(f_\rho \mathbf{1}_{\mathcal{C}_1}) = g_\rho \mathbf{1}_{\mathcal{C}_2}.$$

Thus, thanks to (4.8), we get that for any  $\beta < 1$  the assumptions of Theorem 4.3 are satisfied, provided we choose  $\rho$  sufficiently small. Moreover, if in addition  $c \in C_{\text{loc}}^{k+2, \alpha}(X \times Y)$ ,  $f \in C_{\text{loc}}^{k, \alpha}(X)$ , and  $g \in C_{\text{loc}}^{k, \alpha}(Y)$ , then also the assumptions of Theorem 5.3 are satisfied.

Hence, by applying Theorem 4.3 (resp. Theorem 5.3) we deduce that  $u_\rho \in C^{1, \beta}(B_{1/7})$  (resp.  $u_\rho \in C^{k+2, \alpha}(B_{1/9})$ ), so going back to the original variables we get the existence of a neighborhood  $\mathcal{U}_{\tilde{x}}$  of  $\tilde{x}$  such that  $u \in C^{1, \beta}(\mathcal{U}_{\tilde{x}})$  (resp.  $u \in C^{k+2, \alpha}(\mathcal{U}_{\tilde{x}})$ ). This implies in particular that  $T_u \in C^{0, \beta}(\mathcal{U}_{\tilde{x}})$  (resp.  $T_u \in C^{k+1, \alpha}(\mathcal{U}_{\tilde{x}})$ ). Moreover, it follows by Corollary 4.6 that  $T_u(\mathcal{U}_{\tilde{x}})$  contains a neighborhood of  $\tilde{y}$ .

We now observe that, by symmetry, we can also apply Theorem 4.3 (resp. Theorem 5.3) to  $u_\rho^{c_\rho}$ . Hence, there exists a neighborhood  $\mathcal{V}_{\tilde{y}}$  of  $\tilde{y}$  such that  $T_{u^c} \in C^{0, \beta}(\mathcal{V}_{\tilde{y}})$ . Since  $T_u$  and  $T_{u^c}$  are inverse

<sup>2</sup>For instance, this follows by differentiating both relations

$$D_{\tilde{z}} c_\rho(\tilde{z}, T_{u_\rho}(\tilde{z})) = -\nabla u_\rho(\tilde{z}) \quad \text{and} \quad D_{\tilde{w}} c_\rho(T_{u_\rho^{c_\rho}}(\tilde{w}), \tilde{w}) = -\nabla u_\rho^{c_\rho}(\tilde{w})$$

at  $\mathbf{0}$ , and using then (3.5) and the fact that  $\nabla T_{u_\rho^{c_\rho}}(\mathbf{0}) = [\nabla T_{u_\rho}(\mathbf{0})]^{-1}$  and  $D^2 u_\rho(\mathbf{0}) = \text{Id}$ .

to each other (see (3.2)) we deduce that, possibly reducing the size of  $\mathcal{U}_{\bar{x}}$ ,  $T_u$  is a homeomorphism (resp. diffeomorphism) between  $\mathcal{U}_{\bar{x}}$  and  $T_u(\mathcal{U}_{\bar{x}})$ . Let us consider the open sets

$$X'' := \bigcup_{\bar{x} \in X'} \mathcal{U}_{\bar{x}}, \quad Y'' := \bigcup_{\bar{x} \in X'} T_u(\mathcal{U}_{\bar{x}}),$$

and define the (relatively) closed  $\Sigma_X := X \setminus X''$ ,  $\Sigma_Y := Y \setminus Y''$ . Since  $X'' \supset X'$ ,  $X''$  is a set of full measure, so  $|\Sigma_X| = 0$ . In addition, since  $\Sigma_Y = Y \setminus Y'' \subset Y \setminus T_u(X')$  and  $T_u(X')$  has full measure in  $Y$ , we also get that  $|\Sigma_Y| = 0$ .

Finally, since  $T_u : X \setminus \Sigma_X \rightarrow Y \setminus \Sigma_Y$  is a local homeomorphism (resp. diffeomorphism), by (3.2) it follows that  $T_u : X \setminus \Sigma_X \rightarrow Y \setminus \Sigma_Y$  is a global homeomorphism (resp. diffeomorphism), which concludes the proof.  $\square$

*Proof of Theorem 1.4.* The only difference with respect to the situation in Theorem 1.3 is that now the cost function  $c = d^2/2$  is not smooth on the whole  $M \times M$ . However, even if  $d^2/2$  is not everywhere smooth and  $M$  is not necessarily compact, it is still true that the  $c$ -convex function  $u$  provided by Theorem 2.3 is locally semiconvex (i.e., it is locally semiconvex when seen in any chart) [13, 17]. In addition, as shown in [9, Proposition 4.1] (see also [14, Section 3]), if  $u$  is twice differentiable at  $x$ , then the point  $T_u(x)$  is not in the cut-locus of  $x$ . Since the cut-locus is closed and  $d^2/2$  is smooth outside the cut-locus, we deduce the existence of a set  $X$  of full measure such that, if  $x_0 \in X$ , then: (1)  $u$  is twice differentiable at  $x_0$ ; (2) there exists a neighborhood  $\mathcal{U}_{x_0} \times \mathcal{V}_{T_u(x_0)} \subset M \times M$  of  $(x_0, T_u(x_0))$  such that  $c \in C^\infty(\mathcal{U}_{x_0} \times \mathcal{V}_{T_u(x_0)})$ . Hence, by taking a local chart around  $(x_0, T_u(x_0))$ , the same proof as the one of Theorem 1.3 shows that  $T_u$  is a local homeomorphism (resp. diffeomorphism) around almost every point. Using as before that  $T_u : M \rightarrow M$  is invertible a.e., it follows that  $T_u$  is a global homeomorphism (resp. diffeomorphism) outside a closed singular set of measure zero. We leave the details to the interested reader.  $\square$

#### 4. $C^{1,\beta}$ REGULARITY AND STRICT $c$ -CONVEXITY

In this and the next section we prove that, if in some open set a  $c$ -convex function  $u$  is sufficiently close to a parabola and the cost function is close to the linear one, then  $u$  is smooth in some smaller set.

The idea of the proof (which is reminiscent of the argument introduced by Caffarelli in [6] to show  $W^{2,p}$  and  $C^{2,\alpha}$  estimates for the classical Monge-Ampère equation, though several additional complications arise in our case) is the following: since the cost function is close to the linear one and both densities are almost constant,  $u$  is close to a convex function  $v$  solving an optimal transport problem with linear cost and constant densities (Lemma 4.1). In addition, since  $u$  is close to a parabola, so is  $v$ . Hence, by [18] and Caffarelli's regularity theory,  $v$  is smooth, and we can use this information to deduce that  $u$  is even closer to a second parabola (given by the second order Taylor expansion of  $v$  at the origin) inside a small neighborhood around of origin. By rescaling back this neighborhood at scale 1 and iterating this construction, we obtain that  $u$  is  $C^{1,\beta}$  at the origin for some  $\beta \in (0, 1)$ . Since this argument can be applied at every point in a neighborhood of the origin, we deduce that  $u$  is  $C^{1,\beta}$  there, see Theorem 4.3. (A similar strategy has also been used in [7] to show regularity optimal transport maps for the cost  $|x - y|^p$ , either when  $p$  is close to 2 or when  $X$  and  $Y$  are sufficiently far from each other.)

Once this result is proved, we know that  $\partial^- u$  is a singleton at every point, so it follows from (2.6) that

$$\partial_c u(x) = c\text{-exp}_x(\partial^- u(x)),$$



see Remark 4.4 below. (The above identity is exactly what in general may fail for general  $c$ -convex functions, unless the MTW condition holds [31].) Thanks to this fact, we obtain that  $u$  enjoys a comparison principle (Proposition 5.2), and this allows us to use a second approximation argument with solutions of the classical Monge-Ampère equation (in the spirit of [6, 27]) to conclude that  $u$  is  $C^{2,\sigma'}$  in a smaller neighborhood, for some  $\sigma' > 0$ . Then higher regularity follows from standard elliptic estimates, see Theorem 5.3.

**Lemma 4.1.** *Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two closed sets such that*

$$(4.1) \quad B_{1/K} \subset \mathcal{C}_1, \mathcal{C}_2 \subset B_K$$

for some  $K \geq 1$ ,  $f$  and  $g$  two densities supported respectively in  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , and  $u : \mathcal{C}_1 \rightarrow \mathbb{R}$  a  $c$ -convex function such that  $\partial_c u(\mathcal{C}_1) \subset B_K$  and  $(T_u)_\# f = g$ . Let  $\rho > 0$  be such that  $|\mathcal{C}_1| = |\rho \mathcal{C}_2|$  (where  $\rho \mathcal{C}_2$  denotes the dilation of  $\mathcal{C}_2$  with respect to the origin), and let  $v$  be a convex function such that  $\nabla v \# \mathbf{1}_{\mathcal{C}_1} = \mathbf{1}_{\rho \mathcal{C}_2}$  and  $v(\mathbf{0}) = u(\mathbf{0})$ . Then there exists an increasing function  $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , depending only  $K$ , and satisfying  $\omega(\delta) \geq \delta$  and  $\omega(0^+) = 0$ , such that, if

$$(4.2) \quad \|f - \mathbf{1}_{\mathcal{C}_1}\|_\infty + \|g - \mathbf{1}_{\mathcal{C}_2}\|_\infty \leq \delta$$

and

$$(4.3) \quad \|c(x, y) + x \cdot y\|_{C^2(B_K \times B_K)} \leq \delta,$$

then

$$\|u - v\|_{C^0(B_{1/K})} \leq \omega(\delta).$$

*Proof.* Assume the lemma is false. Then there exists  $\varepsilon_0 > 0$ , a sequence of closed sets  $\mathcal{C}_1^h, \mathcal{C}_2^h$  satisfying (4.1), functions  $f_h, g_h$  satisfying (4.2) with  $\delta = 1/h$ , and costs  $c_h$  converging in  $C^2$  to  $-x \cdot y$ , such that

$$u_h(\mathbf{0}) = v_h(\mathbf{0}) = 0 \quad \text{and} \quad \|u_h - v_h\|_{C^0(B_{1/K})} \geq \varepsilon_0,$$

where  $u_h$  and  $v_h$  are as in the statement. First, we extend  $u_h$  and  $v_h$  to  $B_K$  as

$$u_h(x) := \sup_{z \in \mathcal{C}_1^h, y \in \partial_{c_h} u_h(z)} \{u_h(z) - c_h(x, y) + c_h(z, y)\}, \quad v_h(x) := \sup_{z \in \mathcal{C}_1^h, p \in \partial^- v_h(z)} \{v_h(z) + p \cdot (x - z)\}.$$

Notice that, since by assumption  $\partial_{c_h} u_h(\mathcal{C}_1^h) \subset B_K$ , we have  $\partial_{c_h} u_h(B_K) \subset B_K$ . Also,  $(T_{u_h})_\# f_h = g_h$  gives that  $\int f_h = \int g_h$ , so it follows from (4.2) that

$$\rho_h = \left( |\mathcal{C}_1^h| / |\mathcal{C}_2^h| \right)^{1/n} \rightarrow 1 \quad \text{as } h \rightarrow \infty,$$

which implies that  $\partial^- v_h(B_K) \subset B_{\rho_h K} \subset B_{2K}$  for  $h$  large. Thus, since the  $C^1$ -norm of  $c_h$  is uniformly bounded, we deduce that both  $u_h$  and  $v_h$  are uniformly Lipschitz. Recalling that  $u_h(\mathbf{0}) = v_h(\mathbf{0}) = 0$ , we get that, up to a subsequence,  $u_h$  and  $v_h$  uniformly converge inside  $B_K$  to  $u_\infty$  and  $v_\infty$  respectively, where

$$(4.4) \quad u_\infty(\mathbf{0}) = v_\infty(\mathbf{0}) = 0 \quad \text{and} \quad \|u_\infty - v_\infty\|_{C^0(B_{1/K})} \geq \varepsilon_0.$$

In addition  $f_h$  (resp.  $g_h$ ) weak-\* converge in  $L^\infty$  to some density  $f_\infty$  (resp.  $g_\infty$ ) supported in  $\overline{B_K}$ . Also, since  $\rho_h \rightarrow 1$ , using (4.2) we get that  $\mathbf{1}_{\mathcal{C}_1^h}$  (resp.  $\mathbf{1}_{\rho_h \mathcal{C}_2^h}$ ) weak-\* converges in  $L^\infty$  to  $f_\infty$  (resp.  $g_\infty$ ). Finally we remark that, because of (4.2) and the fact that  $\mathcal{C}_1^h \supset B_{1/K}$ , we also have

$$f_\infty \geq \mathbf{1}_{B_{1/K}}.$$

In order to get a contradiction we have to show that  $u_\infty = v_\infty$  in  $B_{1/K}$ . To see this, we apply [37, Theorem 5.20] to deduce that both  $\nabla u_\infty$  and  $\nabla v_\infty$  are optimal transport maps for the linear cost  $-x \cdot y$  sending  $f_\infty$  onto  $g_\infty$ . By uniqueness of the optimal map (see Theorem 2.2) we deduce that  $\nabla v_\infty = \nabla u_\infty$  almost everywhere inside  $B_{1/K} \subset \text{spt } f_\infty$ , hence  $u_\infty = v_\infty$  in  $B_{1/K}$  (since  $u_\infty(\mathbf{0}) = v_\infty(\mathbf{0}) = 0$ ), contradicting (4.4).  $\square$

Here and in the sequel, we use  $\mathcal{N}_r(E)$  to denote the  $r$ -neighborhood of a set  $E$ .

**Lemma 4.2.** *Let  $u$  and  $v$  be, respectively,  $c$ -convex and convex, let  $D \in \mathbb{R}^{n \times n}$  be a symmetric matrix satisfying*

$$(4.5) \quad \text{Id}/K \leq D \leq K \text{Id}$$

for some  $K \geq 1$ , and define the ellipsoid

$$E(x_0, h) := \{x : D(x - x_0) \cdot (x - x_0) \leq h\}, \quad h > 0.$$

Assume that there exist small positive constants  $\varepsilon, \delta$  such that

$$(4.6) \quad \|v - u\|_{C^0(E(x_0, h))} \leq \varepsilon, \quad \|c + x \cdot y\|_{C^2(E(x_0, h) \times \partial_c u(E(x_0, h)))} \leq \delta.$$

Then

$$(4.7) \quad \partial_c u(E(x_0, h - \sqrt{\varepsilon})) \subset \mathcal{N}_{K'(\delta + \sqrt{h\varepsilon})}(\partial v(E(x_0, h))) \quad \forall 0 < \varepsilon < h^2 \leq 1,$$

where  $K'$  depends only on  $K$ .

*Proof.* Up to a change of coordinates we can assume that  $x_0 = \mathbf{0}$ , and to simplify notation we set  $E_h := E(x_0, h)$ . Let us define

$$\bar{v}(x) := v(x) + \varepsilon + 2\sqrt{\varepsilon}(Dx \cdot x - h),$$

so that  $\bar{v} \geq u$  outside  $E_h$ , and  $\bar{v} \leq u$  inside  $E_{h-\sqrt{\varepsilon}}$ . Then, taking a  $c$ -support to  $u$  in  $E_{h-\sqrt{\varepsilon}}$  (i.e., a function  $C_{x,y}$  as in (2.3), with  $x \in E_{h-\sqrt{\varepsilon}}$  and  $y \in \partial_c u(x)$ ), moving it down and then lifting it up until it touches  $\bar{v}$  from below, we see that it has to touch the graph of  $\bar{v}$  at some point  $\bar{x} \in E_h$ : in other words <sup>3</sup>

$$\partial_c u(E_{h-\sqrt{\varepsilon}}) \subset \partial_c \bar{v}(E_h).$$

By (4.5) we see that  $\text{diam } E_h \leq 2\sqrt{Kh}$ , so by a simple computation (using again (4.5)) we get

$$\partial^- \bar{v}(E_h) \subset \mathcal{N}_{4K\sqrt{Kh\varepsilon}}(\partial^- v(E_h)).$$

Thus, since  $\partial_c \bar{v}(E_h) \subset c\text{-exp}(\partial^- \bar{v}(E_h))$  (by (2.6)) and  $\|c\text{-exp} - \text{Id}\|_{C^0} \leq \delta$  (by (4.6)), we easily deduce that

$$\partial_c u(E_{h-\sqrt{\varepsilon}}) \subset \mathcal{N}_{K'(\delta + \sqrt{h\varepsilon})}(\partial^- v(E_h)),$$

proving (4.7).  $\square$

**Theorem 4.3.** *Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two closed sets satisfying*

$$B_{1/3} \subset \mathcal{C}_1, \mathcal{C}_2 \subset B_3,$$

let  $f, g$  be two densities supported in  $\mathcal{C}_1$  and  $\mathcal{C}_2$  respectively, and let  $u : \mathcal{C}_1 \rightarrow \mathbb{R}$  be a  $c$ -convex function such that  $\partial_c u(\mathcal{C}_1) \subset B_3$  and  $(T_u)_\# f = g$ . Assume also that  $c \in C^{2,\alpha}(B_3 \times B_3)$  for some

<sup>3</sup>Even if  $\bar{v}$  is not  $c$ -convex, it still makes sense to consider his  $c$ -subdifferential (notice that the  $c$ -subdifferential of  $\bar{v}$  may be empty at some points). In particular, the inclusion  $\partial_c \bar{v}(x) \subset c\text{-exp}_x(\partial^- \bar{v}(x))$  still holds.

$\alpha \in (0, 1)$ . Then, for every  $\beta \in (0, 1)$  there exist constants  $\delta_0, \eta_0 > 0$  such that the following holds: if

$$(4.8) \quad \|f - \mathbf{1}_{\mathcal{C}_1}\|_\infty + \|g - \mathbf{1}_{\mathcal{C}_2}\|_\infty \leq \delta_0,$$

$$(4.9) \quad \|c(x, y) + x \cdot y\|_{C^{2,\alpha}(B_3 \times B_3)} \leq \delta_0,$$

and

$$(4.10) \quad \left\| u - \frac{1}{2}|x|^2 \right\|_{C^0(B_3)} \leq \eta_0,$$

then  $u \in C^{1,\beta}(B_{1/7})$ .

*Proof.* We divide the proof into several steps.

• *Step 1:  $u$  is close to a strictly convex solution of the Monge-Ampère equation.* Let  $v : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function such that  $\nabla v \# \mathbf{1}_{\mathcal{C}_1} = \mathbf{1}_{\rho\mathcal{C}_2}$  with  $\rho = (|\mathcal{C}_1|/|\mathcal{C}_2|)^{1/n}$  (see Theorem 2.2). Up to adding a constant to  $v$ , without loss of generality we can assume that  $v(\mathbf{0}) = u(\mathbf{0})$ . Hence, we can apply Lemma 4.1 to obtain

$$(4.11) \quad \|v - u\|_{C^0(B_{1/3})} \leq \omega(\delta_0)$$

for some (universal) modulus of continuity  $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , which combined with (4.10) gives

$$\left\| v - \frac{1}{2}|x|^2 \right\|_{C^0(B_{1/3})} \leq \eta_0 + \omega(\delta_0).$$

Also, since  $\int_{\mathcal{C}_1} f = \int_{\mathcal{C}_2} g$ , it follows easily from (4.8) that  $|\rho - 1| \leq 3\delta_0$ . By these two facts we get that  $\partial^- v(B_{1/4}) \subset B_{7/24} \subset \rho\mathcal{C}_2$  provided  $\delta_0$  and  $\eta_0$  are small enough (recall that  $v$  is convex and that  $B_{1/3} \subset \mathcal{C}_2$ ), so we can apply [18, Proposition 3.4] to deduce that  $v$  is a strictly convex Alexandrov solution to the Monge-Ampère equation

$$(4.12) \quad \det D^2 v = 1 \quad \text{in } B_{1/4}.$$

In addition, by a simple compactness argument, we see that the modulus of strict convexity of  $v$  inside  $B_{1/4}$  is universal. So, by classical Pogorelov and Schauder estimates, we obtain the existence of a universal constant  $K_0 \geq 1$  such that

$$(4.13) \quad \|v\|_{C^3(B_{1/5})} \leq K_0, \quad \text{Id}/K_0 \leq D^2 v \leq K_0 \text{Id} \quad \text{in } B_{1/5}.$$

In particular, there exists a universal value  $\bar{h} > 0$  such that, for all  $x \in B_{1/7}$ ,

$$Q(x, v, h) := \{z : v(z) \leq v(x) + \nabla v(x) \cdot (z - x) + h\} \subset \subset B_{1/6} \quad \forall h \leq \bar{h}.$$

• *Step 2: Sections of  $u$  are close to sections of  $v$ .* Given  $x \in B_{1/7}$  and  $y \in \partial_c u(x)$ , we define

$$S(x, y, u, h) := \{z : u(z) \leq u(x) - c(z, y) + c(x, y) + h\}.$$

We claim that, if  $\delta_0$  is small enough, then for all  $x \in B_{1/7}$ ,  $y \in \partial_c u(x)$ , and  $h \leq \bar{h}/2$ , it holds

$$(4.14) \quad Q(x, v, h - K_1\sqrt{\omega(\delta_0)}) \subset S(x, y, u, h) \subset Q(x, v, h + K_1\sqrt{\omega(\delta_0)}),$$

where  $K_1 > 0$  is a universal constant.

Let us show the first inclusion. For this, take  $x \in B_{1/7}$ ,  $y \in \partial_c u(x)$ , and define

$$p_x := -D_x c(x, y) \in \partial^- u(x).$$

Since  $v$  has universal  $C^2$  bounds (see (4.13)) and  $u$  is semi-convex (with a universal bound), a simple interpolation argument gives

$$(4.15) \quad |p_x - \nabla v(x)| \leq K' \sqrt{\|u - v\|_{C^0(B_{1/5})}} \leq K' \sqrt{\omega(\delta_0)} \quad \forall x \in B_{1/7}.$$

In addition, by (4.9),

$$(4.16) \quad |y - p_x| \leq \|D_x c + \text{Id}\|_{C^0(B_3 \times B_3)} \leq \delta_0,$$

hence

$$(4.17) \quad |z \cdot p_x + c(z, y)| \leq |z \cdot p_x - z \cdot y| + |z \cdot y + c(z, y)| \leq 2\delta_0 \quad \forall x, z \in B_{1/7}.$$

Thus, if  $z \in Q(x, v, h - K_1 \sqrt{\omega(\delta_0)})$ , by (4.11), (4.15), and (4.17) we get

$$\begin{aligned} u(z) &\leq v(z) + \omega(\delta_0) \leq v(x) + \nabla v(x) \cdot (z - x) + h - K_1 \sqrt{\omega(\delta_0)} + \omega(\delta_0) \\ &\leq u(x) + p_x \cdot z - p_x \cdot x + h - K_1 \sqrt{\omega(\delta_0)} + 2\omega(\delta_0) + 2K' \sqrt{\omega(\delta_0)} \\ &\leq u(x) - c(z, y) + c(x, y) + h - K_1 \sqrt{\omega(\delta_0)} + 2\omega(\delta_0) + 2K' \sqrt{\omega(\delta_0)} + 4\delta_0 \\ &\leq u(x) - c(z, y) + c(x, y) + h, \end{aligned}$$

provided  $K_1 > 0$  is sufficiently large. This proves the first inclusion, and the second is analogous.

• *Step 3: Both the sections of  $u$  and their images are close to ellipsoids with controlled eccentricity, and  $u$  is close to a smooth function near  $x_0$ .* We claim that there exists a universal constant  $K_2 \geq 1$  such that the following holds: For every  $\eta_0 > 0$  small, there exist small positive constants  $h_0 = h_0(\eta_0)$  and  $\delta_0 = \delta_0(h_0, \eta_0)$  such that, for all  $x_0 \in B_{1/7}$ , there is a symmetric matrix  $A$  satisfying

$$(4.18) \quad \text{Id} / K_2 \leq A \leq K_2 \text{Id}, \quad \det(A) = 1,$$

and such that, for all  $y_0 \in \partial_c u(x_0)$ ,

$$(4.19) \quad \begin{aligned} A \left( B_{\sqrt{h_0/8}}(x_0) \right) &\subset S(x_0, y_0, u, h_0) \subset A \left( B_{\sqrt{8h_0}}(x_0) \right), \\ A^{-1} \left( B_{\sqrt{h_0/8}}(y_0) \right) &\subset \partial_c u(S(x_0, y_0, u, h_0)) \subset A^{-1} \left( B_{\sqrt{8h_0}}(y_0) \right). \end{aligned}$$

Moreover

$$(4.20) \quad \left\| u - C_{x_0, y_0} - \frac{1}{2} |A^{-1}(x - x_0)|^2 \right\|_{C^0(A(B_{\sqrt{8h_0}}(x_0)))} \leq \eta_0 h_0,$$

where  $C_{x_0, y_0}$  is a  $c$ -support function for  $u$  at  $x_0$ , see (2.3).

In order to prove the claim, take  $h_0 \ll \bar{h}$  small (to be fixed) and  $\delta_0 \ll h_0$  such that  $K_1 \sqrt{\omega(\delta_0)} \leq h_0/2$ , where  $K_1$  is as in Step 2, so that

$$(4.21) \quad Q(x_0, v, h_0/2) \subset S(x_0, y_0, u, h_0) \subset Q(x_0, v, 3h_0/2) \subset\subset B_{1/6}.$$

By (4.13) and Taylor formula we get

$$(4.22) \quad v(x) = v(x_0) + \nabla v(x_0) \cdot (x - x_0) + \frac{1}{2} D^2 v(x_0) (x - x_0) \cdot (x - x_0) + O(|x - x_0|^3),$$

so that defining

$$(4.23) \quad E(x_0, h_0) := \left\{ x : \frac{1}{2} D^2 v(x_0) (x - x_0) \cdot (x - x_0) \leq h_0 \right\}$$

and using (4.13), we deduce that, for every  $h_0$  universally small,

$$(4.24) \quad E(x_0, h_0/2) \subset Q(x_0, v, h_0) \subset E(x_0, 2h_0).$$

Moreover, always for  $h_0$  small, thanks to (4.22) and the uniform convexity of  $v$

$$(4.25) \quad \nabla v(E(x_0, h_0)) \subset E^*(\nabla v(x_0), 2h_0) \subset \nabla v(E(x_0, 3h_0))$$

where we have set

$$E^*(\bar{y}, h_0) := \left\{ y : \frac{1}{2} [D^2 v(\bar{y})]^{-1} (y - \bar{y}) \cdot (y - \bar{y}) \leq h_0 \right\}.$$

By Lemma 4.2, (4.24), and (4.25) applied with  $3h_0$  in place of  $h_0$ , we deduce that for  $\delta_0 \ll h_0 \ll \bar{h}$

$$(4.26) \quad \partial_c u(S(x_0, y_0, u, h_0)) \subset \mathcal{N}_{K'' \sqrt{\omega(\delta_0)}}(\nabla v(E(x_0, 3h_0))) \subset E^*(\nabla v(x_0), 7h_0).$$

Moreover, by (4.15), if  $y_0 \in \partial_c u(x_0)$  and we set  $p_{x_0} := -D_x c(x_0, y_0)$ , then

$$|y_0 - \nabla v(x_0)| \leq |p_{x_0} - \nabla v(x_0)| + \|D_x c + \text{Id}\|_{C^0(B_3 \times B_3)} \leq K' \sqrt{\omega(\delta_0)} + \delta_0.$$

Thus, choosing  $\delta_0$  sufficiently small, it holds

$$(4.27) \quad E^*(\nabla v(x_0), 7h_0) \subset E^*(y_0, 8h_0) \quad \forall y_0 \in \partial_c u(x_0).$$

We now want to show that

$$E^*(y_0, h_0/8) \subset \partial_c u(S(x_0, y_0, u, h_0)) \quad \forall y_0 \in \partial_c u(x_0).$$

Observe that, arguing as above, we get

$$(4.28) \quad E^*(y_0, h_0/8) \subset E^*(\nabla v(x_0), h_0/7) \quad \forall y_0 \in \partial_c u(x_0)$$

provided  $\delta_0$  is small enough, so it is enough to prove that

$$E^*(\nabla v(x_0), h_0/7) \subset \partial_c u(S(x_0, y_0, u, h_0)).$$

For this, let us define the  $c^*$ -convex function  $u^c : B_3 \rightarrow \mathbb{R}$  and the convex function  $v^* : B_3 \rightarrow \mathbb{R}$  as

$$u^c(y) := \sup_{x \in B_{1/5}} \{ -c(x, y) - u(x) \}, \quad v^*(y) := \sup_{x \in B_{1/5}} \{ x \cdot y - v(x) \}$$

(see (3.1)). Then it is immediate to check that

$$(4.29) \quad |u^c - v^*| \leq \omega(\delta_0) + \delta_0 \leq 2\omega(\delta_0) \quad \text{on } B_3.$$

Also, in view of (4.13),  $v^*$  is a uniformly convex function of class  $C^3$  on the open set  $\nabla v(B_{1/5})$ . In addition, since

$$(4.30) \quad F \subset \partial_c u(\partial_{c^*} u^c(F)) \quad \text{for any set } F,$$

thanks to (4.21) and (4.24) it is enough to show

$$(4.31) \quad \partial_{c^*} u^c(E^*(\nabla v(x_0), h_0/7)) \subset E(x_0, h_0/4).$$

For this, we apply Lemma 4.2 to  $u^c$  and  $v^*$  to infer

$$\partial_{c^*} u^c(E^*(\nabla v(x_0), h_0/7)) \subset \mathcal{N}_{K''' \sqrt{\omega(\delta)}}(\nabla v^*(E^*(\nabla v(x_0), h_0/7))) \subset E(x_0, h_0/4),$$

where we used that

$$\nabla v^* = [\nabla v]^{-1} \quad \text{and} \quad D^2 v^*(\nabla v(x_0)) = [D^2 v(x_0)]^{-1}.$$

Thus, recalling (4.26), we have proved that there exist  $h_0$  universally small, and  $\delta_0$  small depending on  $h_0$ , such that

$$(4.32) \quad E^*(\nabla v(x_0), h_0/7) \subset \partial_c u(S(x_0, y_0, u, h_0)) \subset E^*(\nabla v(x_0), 7h_0) \quad \forall x_0 \in B_{1/7}.$$

Using (4.21), (4.24), (4.27), and (4.28), this proves (4.19) with  $A := [D^2 v(x_0)]^{-1/2}$ . Also, thanks to (4.12) and (4.13), (4.18) holds.

In order to prove the second part of the claim, we exploit (4.11), (4.9), (4.16), (4.15), (4.22), and (4.18) (recall that  $C_{x_0, y_0}$  is defined in (2.3) and that  $A = [D^2 v(x_0)]^{-1/2}$ ):

$$\begin{aligned} & \left\| u - C_{x_0, y_0} - \frac{1}{2} |A^{-1}(x - x_0)|^2 \right\|_{C^0(E(x_0, 8h_0))} \\ &= \left\| u - C_{x_0, y_0} - \frac{1}{2} D^2 v(x_0)(x - x_0) \cdot (x - x_0) \right\|_{C^0(E(x_0, 8h_0))} \\ &\leq 2 \|u - v\|_{C^0(E(x_0, 8h_0))} + \|c(x, y_0) + x \cdot y_0\|_{C^0(E(x_0, 8h_0))} + \|c(x_0, y_0) + x_0 \cdot y_0\|_{C^0(E(x_0, 8h_0))} \\ &\quad + \|(y_0 - p_{x_0}) \cdot (x - x_0)\|_{C^0(E(x_0, 8h_0))} + \|(p_{x_0} - \nabla v(x_0)) \cdot (x - x_0)\|_{C^0(E(x_0, 8h_0))} \\ &\quad + \left\| v - v(x_0) - \nabla v(x_0) \cdot (x - x_0) - \frac{1}{2} D^2 v(x_0)(x - x_0) \cdot (x - x_0) \right\|_{C^0(E(x_0, 8h_0))} \\ &\leq 2\omega(\delta_0) + 3\delta_0 + K' \sqrt{\omega(\delta_0)} + K \left( K_2 \sqrt{8h_0} \right)^3 \leq \eta_0 h_0, \end{aligned}$$

where the last inequality follows by choosing first  $h_0$  sufficiently small, and then  $\delta_0$  much smaller than  $h_0$ .

• *Step 4: A first change of variables.* Fix  $x_0 \in B_{1/7}$ ,  $y_0 \in \partial_c u(x_0)$ , define  $M := -D_{xy} c(x_0, y_0)$ , and consider the change of variables

$$\begin{cases} \bar{x} := x - x_0 \\ \bar{y} := M^{-1}(y - y_0). \end{cases}$$

Notice that, by (4.9), it follows that

$$(4.33) \quad |M - \text{Id}| + |M^{-1} - \text{Id}| \leq 3\delta_0$$

for  $\delta_0$  sufficiently small. We also define

$$\begin{aligned} \bar{c}(\bar{x}, \bar{y}) &:= c(x, y) - c(x, y_0) - c(x_0, y) + c(x_0, y_0), \\ \bar{u}(\bar{x}) &:= u(x) - u(x_0) + c(x, y_0) - c(x_0, y_0), \\ \bar{u}^{\bar{c}}(\bar{y}) &:= u^c(y) - u^c(y_0) + c(x_0, y) - c(x_0, y_0). \end{aligned}$$

Then  $\bar{u}$  is  $\bar{c}$ -convex,  $\bar{u}^{\bar{c}}$  is  $\bar{c}^*$ -convex (where  $\bar{c}^*(\bar{y}, \bar{x}) = \bar{c}(\bar{x}, \bar{y})$ ), and

$$(4.34) \quad \bar{c}(\cdot, \mathbf{0}) = \bar{c}(\mathbf{0}, \cdot) \equiv 0, \quad D_{\bar{x}\bar{y}} \bar{c}(\mathbf{0}, \mathbf{0}) = -\text{Id}.$$

We also notice that

$$(4.35) \quad \partial_{\bar{c}} \bar{u}(\bar{x}) = M^{-1}(\partial_c u(\bar{x} + x_0) - y_0).$$

Thus, recalling (4.19), and using (4.33) and (4.35), for  $\delta_0$  sufficiently small we obtain

(4.36)

$$\begin{aligned} A\left(B_{\sqrt{h_0/9}}\right) &\subset S(\mathbf{0}, \mathbf{0}, \bar{u}, h_0) \subset A\left(B_{\sqrt{9h_0}}\right), \\ A^{-1}\left(B_{\sqrt{h_0/9}}\right) &\subset M^{-1}A^{-1}\left(B_{\sqrt{h_0/8}}\right) \subset \partial_{\bar{c}}\bar{u}(S(\mathbf{0}, \mathbf{0}, \bar{u}, h_0)) \subset M^{-1}A^{-1}\left(B_{\sqrt{8h_0}}\right) \subset A^{-1}\left(B_{\sqrt{9h_0}}\right). \end{aligned}$$

Since  $(T_{\bar{u}})_{\#}f = g$ , it follows that  $T_{\bar{u}} = \bar{c}\text{-exp}(\nabla\bar{u})$  satisfies

$$(T_{\bar{u}})_{\#}\bar{f} = \bar{g}, \quad \text{with } \bar{f}(\bar{x}) := f(\bar{x} + x_0), \quad \bar{g}(\bar{y}) := \det(M)g(M\bar{y} + y_0)$$

(see for instance the footnote in the proof of Theorem 1.3). Notice that, since  $|M - \text{Id}| \leq \delta_0$  (by (4.9)), we have  $|\det(M) - 1| \leq (1 + 2n)\delta_0$  (for  $\delta_0$  small), so by (4.8) we get

$$(4.37) \quad \|\bar{f} - \mathbf{1}_{\mathcal{C}_1 - x_0}\|_{\infty} + \|\bar{g} - \mathbf{1}_{M^{-1}(\mathcal{C}_2 - y_0)}\|_{\infty} \leq 2(1 + n)\delta_0.$$

• *Step 5: A second change of variables and the iteration argument.* We now perform a second change of variable: we set

$$(4.38) \quad \begin{cases} \tilde{x} := \frac{1}{\sqrt{h_0}}A^{-1}\bar{x} \\ \tilde{y} := \frac{1}{\sqrt{h_0}}A\bar{y}, \end{cases}$$

and define

$$\begin{aligned} c_1(\tilde{x}, \tilde{y}) &:= \frac{1}{h_0}\bar{c}(\sqrt{h_0}A\tilde{x}, \sqrt{h_0}A^{-1}\tilde{y}), \\ u_1(\tilde{x}) &:= \frac{1}{h_0}\bar{u}(\sqrt{h_0}A\tilde{x}), \\ u_1^{c_1}(\tilde{y}) &:= \frac{1}{h_0}\bar{u}^{\bar{c}}(\sqrt{h_0}A^{-1}\tilde{y}). \end{aligned}$$

We also define

$$f_1(\tilde{x}) := \bar{f}(\sqrt{h_0}A\tilde{x}), \quad g_1(\tilde{y}) := \bar{g}(\sqrt{h_0}A^{-1}\tilde{y}).$$

Since  $\det(A) = 1$  (see (4.18)), it is easy to check that  $(T_{u_1})_{\#}f_1 = g_1$  (see the footnote in the proof of Theorem 1.3). Also, since  $(\|A\| + \|A^{-1}\|)\sqrt{h_0} \ll 1$ , it follows from (4.37) that

$$(4.39) \quad |f_1 - 1| + |g_1 - 1| \leq 2(1 + n)\delta_0 \quad \text{inside } B_3.$$

Moreover, defining

$$\mathcal{C}_1^{(1)} := S(\mathbf{0}, \mathbf{0}, u_1, 1), \quad \mathcal{C}_2^{(1)} := \partial_{c_1}u_1(S(\mathbf{0}, \mathbf{0}, u_1, 1)),$$

both  $\mathcal{C}_1^{(1)}$  and  $\mathcal{C}_2^{(1)}$  are closed, and thanks to (4.36)

$$(4.40) \quad B_{1/3} \subset \mathcal{C}_1^{(1)}, \mathcal{C}_2^{(1)} \subset B_3.$$

Also, since  $(T_{u_1})_{\#}f_1 = g_1$ , arguing as in the proof of Theorem 1.3 we get

$$(T_{u_1})_{\#}(f_1\mathbf{1}_{\mathcal{C}_1^{(1)}}) = (g_1\mathbf{1}_{\mathcal{C}_2^{(1)}}),$$

and by (4.39)

$$\|f_1\mathbf{1}_{\mathcal{C}_1^{(1)}} - \mathbf{1}_{\mathcal{C}_1^{(1)}}\|_{\infty} + \|g_1\mathbf{1}_{\mathcal{C}_2^{(1)}} - \mathbf{1}_{\mathcal{C}_2^{(1)}}\|_{\infty} \leq 2(1 + n)\delta_0.$$

Finally, using (4.34), (4.9), and (4.20), it is easy to check that

$$\|c_1(\tilde{x}, \tilde{y}) + \tilde{x} \cdot \tilde{y}\|_{C^{2,\alpha}(B_3 \times B_3)} \leq \delta_0, \quad \left\|u_1 - \frac{1}{2}|\tilde{x}|^2\right\|_{C^0(B_3)} \leq \eta_0$$

provided  $h_0$  is small enough. Indeed, the second inequality is just a direct consequence of (4.20), while (4.34) and a Taylor expansion yield

$$c_1(\tilde{x}, \tilde{y}) = \frac{1}{h_0} \bar{c}(\sqrt{h_0} A \tilde{x}, \sqrt{h_0} A^{-1} \tilde{y}) = \tilde{x} \cdot \tilde{y} + O(\delta_0 K_2^2 (K_2 \sqrt{h_0})^\alpha).$$

so we can choose  $h_0$  small enough to ensure that

$$(4.41) \quad O(\delta_0 K_2^2 (K_2 \sqrt{h_0})^\alpha) \leq \delta_0.$$

This shows that  $u_1$  satisfies the same assumptions as  $u$  with  $\delta_0$  replaced by  $2(1+n)\delta_0$ . Hence, up to take  $\delta_0$  slightly smaller, we can apply Step 3 to  $u_1$ , and we find a symmetric matrix  $A_1$  satisfying

$$\begin{aligned} \text{Id} / K_2 &\leq A_1 \leq K_2 \text{Id}, & \det(A_1) &= 1, \\ A_1 \left( B_{\sqrt{h_0/8}} \right) &\subset S(\mathbf{0}, \mathbf{0}, u_1, h_0) \subset A_1 \left( B_{\sqrt{8h_0}} \right), \\ A_1^{-1} \left( B_{\sqrt{h_0/8}} \right) &\subset \partial_{c_1} u_1(S(\mathbf{0}, \mathbf{0}, u_1, h_0)) \subset A_1^{-1} \left( B_{\sqrt{8h_0}} \right), \\ \left\| u_1 - \frac{1}{2} |A_1^{-1} \tilde{x}|^2 \right\|_{C^0(A_1(B(0, \sqrt{8h_0})))} &\leq \eta_0 h_0. \end{aligned}$$

(Here  $K_2$  and  $h_0$  are as in Step 3.)

This allows us to apply to  $u_1$  the very same construction as the one used above to define  $u_1$  from  $\bar{u}$ : we set

$$c_2(\tilde{x}, \tilde{y}) := \frac{1}{h_0} c_1(\sqrt{h_0} A_1 \tilde{x}, \sqrt{h_0} A_1^{-1} \tilde{y}), \quad u_2(\tilde{x}) := \frac{1}{h_0} u_1(\sqrt{h_0} A_1 \tilde{x}),$$

so that  $(T_{u_2})_\# f_2 = g_2$  with

$$f_2(\tilde{x}) := f_1(\sqrt{h_0} A_1 \tilde{x}), \quad g_2(\tilde{y}) := \bar{g}(\sqrt{h_0} A_1^{-1} \tilde{y}).$$

Arguing as before, it is easy to check that  $u_2, c_2, f_2, g_2$  satisfy the same assumptions as  $u_1, c_1, f_1, g_1$  with exactly the same constants, provided  $h_0 \ll 1$  satisfies (4.41).

So we can keep iterating this construction, defining for any  $k \in \mathbb{N}$

$$c_{k+1}(\tilde{x}, \tilde{y}) := \frac{1}{h_0} c_k(\sqrt{h_0} A_k \tilde{x}, \sqrt{h_0} A_k^{-1} \tilde{y}), \quad u_{k+1}(\tilde{x}) := \frac{1}{h_0} u_k(\sqrt{h_0} A_k \tilde{x}),$$

where  $A_k$  is the matrix constructed in the  $k$ -th iteration. In this way, if we set

$$M_k := A_k \cdot \dots \cdot A_1, \quad \forall k \geq 1,$$

we obtain a sequence of symmetric matrices satisfying

$$(4.42) \quad \text{Id} / K_2^k \leq M_k \leq K_2^k \text{Id}, \quad \det(M_k) = 1,$$

and such that

$$(4.43) \quad M_k \left( B_{(h_0/8)^{k/2}} \right) \subset S(\mathbf{0}, \mathbf{0}, u_k, h_0^k) \subset M_k \left( B_{(8h_0)^{k/2}} \right).$$

• *Step 6:  $C^{1,\beta}$  regularity.* We now show that, for any  $\beta \in (0, 1)$ , we can choose  $h_0$  and  $\delta_0 = \delta_0(h_0)$  small enough so that  $u_1$  is  $C^{1,\beta}$  at the origin (here  $u_1$  is the function constructed in the previous step). This will imply that  $u$  is  $C^{1,\beta}$  at  $x_0$  with universal bounds, which by the arbitrariness of  $x_0 \in B_{1/7}$  gives  $u \in C^{1,\beta}(B_{1/7})$ .



Fix  $\beta \in (0, 1)$ . Then by (4.42) and (4.43) we get

$$(4.44) \quad B_{(\sqrt{h_0}/(\sqrt{8}K_2))^k} \subset S(\mathbf{0}, \mathbf{0}, u_1, h_0^k) \subset B_{(K_2\sqrt{8h_0})^k},$$

so defining  $r_0 := \sqrt{h_0}/(\sqrt{8}K_2)$  we obtain

$$\|u_1\|_{C^0(B_{r_0^k})} \leq h_0^k = (\sqrt{8}K_2r_0)^{2k} \leq r_0^{(1+\beta)k},$$

provided  $h_0$  (and so  $r_0$ ) is sufficiently small. This implies the  $C^{1,\beta}$  regularity of  $u_1$  at  $\mathbf{0}$ , concluding the proof.  $\square$

**Remark 4.4** (Local to global principle). If  $u$  is differentiable at  $x$  and  $c$  satisfies **(C0)**-**(C1)**, then every ‘‘local support’’ at  $x$  is also a ‘‘global  $c$ -support’’ at  $x$ , that is,  $\partial_c u(x) = c\text{-exp}_x(\partial^- u(x))$ . To see this, just notice that

$$\emptyset \neq \partial_c u(x) \subset c\text{-exp}_x(\partial^- u(x)) = \{c\text{-exp}_x(\nabla u(x))\}$$

(recall (2.6)), so necessarily the two sets have to coincide.

**Corollary 4.5.** *Let  $u$  be as in Theorem 4.3. Then  $u$  is strictly  $c$ -convex in  $B_{1/7}$ . More precisely, for every  $\gamma > 2$  there exist  $\eta_0, \delta_0 > 0$  depending only on  $\gamma$  such that, if the hypotheses of Theorem 4.3 are satisfied, then, for all  $x_0 \in B_{1/7}$ ,  $y_0 \in \partial_c u(x_0)$ , and  $C_{x_0, y_0}$  as in (2.3), we have*

$$(4.45) \quad \inf_{\partial B_r(x_0)} \{u - C_{x_0, y_0}\} \geq c_0 r^\gamma \quad \forall r \leq \text{dist}(x_0, \partial B_{1/7}),$$

with  $c_0 > 0$  universal.

*Proof.* With the same notation as in the proof of Theorem 4.3, it is enough to show that

$$\inf_{\partial B_r} u_1 \geq r^{1/\beta},$$

where  $u_1$  is the function constructed in Step 5 of the proof of Theorem 4.3. Defining  $\varrho_0 := K_2\sqrt{8h_0}$ , it follows from (4.44) that

$$\inf_{\partial B_{\varrho_0^k}} u_1 \geq h_0^k = (\varrho_0/(\sqrt{8}K_2))^{2k} \geq \varrho_0^{\gamma k},$$

provided  $h_0$  is small enough.  $\square$

A simple consequence of the above results is the following:

**Corollary 4.6.** *Let  $u$  be as in Theorem 4.3, then  $T_u(B_{1/7})$  is open.*

*Proof.* Since  $u \in C^{1,\beta}(B_{1/7})$  we have that  $T_u(B_{1/7}) = \partial_c u(B_{1/7})$  (see Remark 4.4). We claim that it is enough to show that if  $y_0 \in \partial_c u(B_{1/7})$ , then there exists  $\varepsilon = \varepsilon(y_0) > 0$  small such that, for all  $|y - y_0| < \varepsilon$ , the function  $u(\cdot) + c(\cdot, y)$  has a local minimum at some point  $\bar{x} \in B_{1/7}$ . Indeed, if this is the case, then

$$\nabla u(\bar{x}) = -D_x c(\bar{x}, y),$$

and so  $y \in \partial_c u(\bar{x})$  (by Remark 4.4), hence  $B_\varepsilon(y_0) \subset T_u(B_{1/7})$ .

To prove the above fact, fix  $r > 0$  such that  $B_r(x_0) \subset B_{1/7}$ , and pick  $\bar{x}$  a point in  $\overline{B}_r(x_0)$  where the function  $u(\cdot) + c(\cdot, y)$  attains its minimum, i.e.,

$$\bar{x} \in \underset{\overline{B}_r(x_0)}{\text{argmin}} \{u(x) + c(x, y)\}.$$

Since, by (4.45),

$$\begin{aligned} \min_{x \in \partial B_r(x_0)} \{u(x) + c(x, y)\} &\geq \min_{x \in \partial B_r(x_0)} \{u(x) + c(x, y_0)\} - \varepsilon \|c\|_{C^1} \\ &\geq u(x_0) + c(x_0, y_0) + c_0 r^\gamma - \varepsilon \|c\|_{C^1}, \end{aligned}$$

while

$$u(x_0) + c(x_0, y) \leq c(x_0, y_0) + u(x_0) + \varepsilon \|c\|_{C^1},$$

choosing  $\varepsilon < \frac{c_0}{2\|c\|_{C^1}} r^\gamma$  we obtain that  $\bar{x} \in B_r(x_0) \subset B_{1/7}$ . This implies that  $\bar{x}$  is a local minimum for  $u(\cdot) + c(\cdot, y)$ , concluding the proof.  $\square$

## 5. COMPARISON PRINCIPLE AND $C^{2,\alpha}$ REGULARITY

We begin this section with a change of variable formula for the  $c$ -exponential map.

**Lemma 5.1.** *Let  $\Omega$  be an open set,  $v \in C^2(\Omega)$ , and assume that  $\nabla v(\Omega) \subset \text{Dom } c\text{-exp}$  and that*

$$D^2v(x) + D_{xx}c(x, c\text{-exp}_x(\nabla v(x))) \geq 0 \quad \forall x \in \Omega.$$

*Then, for every Borel set  $A \subset \Omega$ ,*

$$|c\text{-exp}(\nabla v(A))| \leq \int_A \frac{\det(D^2v(x) + D_{xx}c(x, c\text{-exp}_x(\nabla v(x))))}{|\det(D_{xy}c(x, c\text{-exp}_x(\nabla v(x))))|} dx.$$

*In addition, if the map  $x \mapsto c\text{-exp}_x(\nabla v(x))$  is injective, then equality holds.*

*Proof.* The result follows from a direct application of the Area Formula [12, Section 3.3.2, Theorem 1] once one notices that, differentiating the identity

$$\nabla v(x) = -D_x c(x, c\text{-exp}_x(\nabla v(x)))$$

(see (2.5)), the Jacobian determinant of the  $C^1$  map  $x \mapsto c\text{-exp}_x(\nabla v(x))$  is given precisely by

$$\frac{\det(D^2v(x) + D_{xx}c(x, c\text{-exp}_x(\nabla v(x))))}{|\det(D_{xy}c(x, c\text{-exp}_x(\nabla v(x))))|}.$$

$\square$

In the next proposition we show a comparison principle between  $C^1$   $c$ -convex functions and smooth solutions to the Monge-Ampère equation.<sup>4</sup> As already mentioned at the beginning of Section 4 (see also Remark 4.4), the  $C^1$  regularity of  $u$  is crucial to ensure that the  $c$ -subdifferential coincides with its local counterpart  $c\text{-exp}(\partial^- u)$ .

Here and in the sequel, we use  $\text{co}[E]$  to denote the convex hull of a set  $E$ . Also, recall that  $\mathcal{N}_r(E)$  denotes the  $r$ -neighborhood of  $E$ .

**Proposition 5.2** (Comparison principle). *Let  $u$  be a  $c$ -convex function of class  $C^1$  inside the set  $S := \{u < 1\}$ , and assume that  $u(\mathbf{0}) = 0$ ,  $B_{1/K} \subset S \subset B_K$ , and that  $\nabla u(S) \subset\subset \text{Dom } c\text{-exp}$ . Let  $f, g$  be two densities such that*

$$(5.1) \quad \|f/\lambda_1 - 1\|_{C^0(S)} + \|g/\lambda_2 - 1\|_{C^0(T_u(S))} \leq \varepsilon$$

<sup>4</sup>A similar result for the case  $c(x, y) = |x - y|^p$  appeared in [7, Theorem 6.2]. Here, however, we have to deal with some additional difficulties due to the fact that the  $c$ -exponential map is not necessarily defined on the whole  $\mathbb{R}^n$ .

for some constants  $\lambda_1, \lambda_2 \in (1/2, 2)$  and  $\varepsilon \in (0, 1/4)$ , and assume that  $(T_u)_\# f = g$ . Furthermore, suppose that

$$(5.2) \quad \|c + x \cdot y\|_{C^2(B_K \times B_K)} \leq \delta.$$

Then there exist a universal constant  $\gamma \in (0, 1)$ , and  $\delta_1 = \delta_1(K) > 0$  small, such that the following holds: Let  $v$  be the solution of

$$\begin{cases} \det(D^2v) = \lambda_1/\lambda_2 & \text{in } \mathcal{N}_{\delta^\gamma}(\text{co}[S]), \\ v = 1 & \text{on } \partial(\mathcal{N}_{\delta^\gamma}(\text{co}[S])). \end{cases}$$

Then

$$(5.3) \quad \|u - v\|_{C^0(S)} \leq C_K \left( \varepsilon + \delta^{\gamma/n} \right) \quad \text{provided } \delta \leq \delta_1,$$

where  $C_K$  is a constant independent of  $\lambda_1, \lambda_2, \varepsilon$ , and  $\delta$  (but which depends on  $K$ ).

*Proof.* First of all we observe that, since  $u(\mathbf{0}) = 0$ ,  $u = 1$  on  $\partial S$ ,  $S \subset B_K$ , and  $\|c + x \cdot y\|_{C^2(B_K)} \leq \delta \ll 1$ , it is easy to check that there exists a universal constant  $a_1 > 0$  such that

$$(5.4) \quad |D_x c(x, y)| \geq a_1 \quad \forall x \in \partial S, y = c\text{-exp}_x(\nabla u(x)).$$

Thanks to (5.4) and (5.2), it follows from the Implicit Function Theorem that, for each  $x \in \partial S$ , the boundary of the set

$$E_x := \{z \in B_K : c(z, y) - c(x, y) + u(x) \leq 1\}$$

is of class  $C^2$  inside  $B_K$ , and its second fundamental form is bounded by  $C_K \delta$ , where  $C_K > 0$  depends only on  $K$ . Hence, since  $S$  can be written as

$$S := \bigcap_{x \in \partial S} E_x,$$

it follows that

$S$  is a  $(C_K \delta)$ -semiconvex set,

that is, for any couple of points  $x_0, x_1 \in S$  the ball centered at  $x_{1/2} := (x_0 + x_1)/2$  of radius  $C_K \delta |x_1 - x_0|^2$  intersects  $S$ . Since  $S \subset B_K$ , this implies that  $\text{co}[S] \subset \mathcal{N}_{C'_K \delta}(S)$  for some positive constant  $C'_K$  depending only on  $K$ . Thus, for any  $\gamma \in (0, 1)$  we obtain

$$\mathcal{N}_{\delta^\gamma}(\text{co}[S]) \subset \mathcal{N}_{(1+C'_K)\delta^\gamma}(S).$$

Since  $v = 1$  on  $\partial(\mathcal{N}_{\delta^\gamma}(\text{co}[S]))$  and  $\lambda_1/\lambda_2 \in (1/4, 4)$ , by standard interior estimates for solution of the Monge-Ampère equation with constant right hand side (see for instance [8, Lemma 1.1]), we obtain

$$(5.5) \quad \text{osc}_S v \leq C''_K$$

$$(5.6) \quad 1 - C''_K \delta^{\gamma/n} \leq v < 1 \quad \text{on } \partial S,$$

$$(5.7) \quad D^2v \geq \delta^{\gamma/\tau} \text{Id} / C''_K \quad \text{in } \text{co}[S],$$

for some  $\tau > 0$  universal, and some constant  $C''_K$  depending only on  $K$ .

Let us define

$$v^+ := (1 + 4\varepsilon + 2\sqrt{\delta})v - 4\varepsilon - 2\sqrt{\delta}, \quad v^- := (1 - 4\varepsilon - \sqrt{\delta}/2)v + 4\varepsilon + \sqrt{\delta}/2 + 2C''_K \delta^{\gamma/n}.$$

Our goal is to show that we can choose  $\gamma$  universally small so that  $v^- \geq u \geq v^+$  on  $S$ . Indeed, if we can do so, then by (5.5) this will imply (5.3), concluding the proof.

First of all notice that, thanks to (5.6),  $v^- > u > v^+$  on  $\partial S$ . Let us show first that  $v^+ \leq v$ . Assume by contradiction this is not the case. Then, since  $u > v^+$  on  $\partial S$ ,

$$\emptyset \neq Z := \{u < v^+\} \subset\subset S.$$

Since  $v^+$  is convex, taking any supporting plane to  $v^+$  at  $x \in Z$ , moving it down and then lifting it up until it touches  $u$  from below, we deduce that

$$(5.8) \quad \nabla v^+(Z) \subset \nabla u(Z)$$

(recall that both  $u$  and  $v^+$  are of class  $C^1$ ), thus by Remark 4.4

$$(5.9) \quad |\mathbf{c}\text{-exp}(\nabla v^+(Z))| \leq |T_u(Z)|.$$

We show that this is impossible. For this, using (5.7) and choosing  $\gamma := \tau/4$ , for any  $x \in Z$  we compute

$$\begin{aligned} D^2v^+(x) + D_{xx}c(x, \mathbf{c}\text{-exp}_x(\nabla v^+(x))) &\geq (1 + \sqrt{\delta} + 4\varepsilon)D^2v + \sqrt{\delta}D^2v - \delta \text{Id} \\ &\geq (1 + \sqrt{\delta} + 4\varepsilon)D^2v + (\delta^{3/4}/C_K'' - \delta) \text{Id} \\ &\geq (1 + \sqrt{\delta} + 4\varepsilon)D^2v, \end{aligned}$$

provided  $\delta$  is sufficiently small, the smallness depending only on  $K$ . Thus, thanks (5.2) we have

$$\begin{aligned} \frac{\det(D^2v^+(x) + D_{xx}c(x, \mathbf{c}\text{-exp}_x(\nabla v^+(x))))}{|\det(D_{xy}c(x, \mathbf{c}\text{-exp}_x(\nabla v^+(x))))|} &\geq \frac{\det((1 + \sqrt{\delta} + 4\varepsilon)D^2v)}{1 + \delta} \\ &\geq (1 + \sqrt{\delta} + 4\varepsilon)^n (1 - 2\delta) \frac{\lambda_1}{\lambda_2} \\ &\geq (1 + 4n\varepsilon) \frac{\lambda_1}{\lambda_2}. \end{aligned}$$

In addition, thanks (5.7) and (5.2), since  $\delta^{\gamma/\tau} = \delta^{1/4} \gg \delta$  we see that

$$D^2v^+ > \|D_{xx}c\|_{C^0(B_K \times B_K)} \text{Id} \quad \text{inside } \text{co}[S].$$

Hence, for any  $x, z \in Z$ ,  $x \neq z$  and  $y = \mathbf{c}\text{-exp}_x(\nabla v^+(x))$  (notice that  $\mathbf{c}\text{-exp}_x(\nabla v^+(x))$  is well-defined because of (5.8) and the assumption  $\nabla u(S) \subset\subset \text{Dom } \mathbf{c}\text{-exp}$ ), it follows

$$\begin{aligned} v^+(z) + c(z, y) &\geq v^+(x) + c(x, y) \\ &\quad + \frac{1}{2} \int_0^1 \left( D^2v^+(tz + (1-t)x) + D_{xx}c(tz + (1-t)x, y) \right) [z - x, z - x] dt \\ &> v^+(x) + c(x, y), \end{aligned}$$

where we used that  $\nabla v^+(x) + D_x c(x, y) = 0$ . This means that the supporting function  $z \mapsto -c(z, y) + c(x, y) + v^+(x)$  can only touch  $v^+$  from below at  $x$ , which implies that the map  $Z \ni x \mapsto \mathbf{c}\text{-exp}_x(\nabla v^+(x))$  is injective. Thus, by Lemma 5.1 we get

$$(5.10) \quad |\mathbf{c}\text{-exp}(\nabla v^+(Z))| \geq (1 + 4n\varepsilon) \frac{\lambda_1}{\lambda_2} |Z|.$$

On the other hand, since  $u$  is  $C^1$ , it follows from  $(T_u)_\# f = g$  and (5.1) that

$$|T_u(Z)| = \int_Z \frac{f(x)}{g(T_u(x))} dx \leq \frac{\lambda_1(1 + \varepsilon)}{\lambda_2(1 - \varepsilon)} |Z| \leq (1 + 3\varepsilon) \frac{\lambda_1}{\lambda_2} |Z|.$$

This estimate combined with (5.10) shows that (5.9) is impossible unless  $Z$  is empty. This proves that  $v^+ \leq u$ .

The proof of the inequality  $v^- \leq u$  follows by the same argument except for a minor modification. More precisely, let us assume by contradiction that  $W := \{u > v^-\}$  is nonempty. In order to apply the previous argument we would need to know that  $\nabla v^-(W) \subset \text{Dom } c\text{-exp}$ . However, since the gradient of  $v$  can be very large near  $\partial S$ , this may be a problem.

To circumvent this issue we argue as follows: since  $W$  is nonempty, there exists a positive constant  $\bar{\mu}$  such that  $u$  touches  $v^- + \bar{\mu}$  from below inside  $S$ . Let  $E$  be the contact set, i.e.,  $E := \{u = v^- + \bar{\mu}\}$ . Since both  $u$  and  $v^-$  are  $C^1$ ,  $\nabla u = \nabla v^-$  on  $E$ . Thus, if  $\eta > 0$  is small enough, then the set  $W_\eta := \{u > v^- + \bar{\mu} - \eta\}$  is nonempty and  $\nabla v^-(W_\eta)$  is contained in a small neighborhood of  $\nabla u(W_\eta)$ , which is compactly contained in  $\text{Dom } c\text{-exp}$ . At this point, one argues exactly as in the first part of the proof, with  $W_\eta$  in place of  $Z$ , to find a contradiction.  $\square$

**Theorem 5.3.** *Let  $u, f, g, \eta_0, \delta_0$  be as in Theorem 4.3, and assume in addition that  $c \in C^{k,\alpha}(B_3 \times B_3)$  and  $f, g \in C^{k,\alpha}(B_{1/3})$  for some  $k \geq 0$  and  $\alpha \in (0, 1)$ . There exist small constants  $\eta_1 \leq \eta_0$  and  $\delta_1 \leq \delta_0$  such that, if*

$$(5.11) \quad \|f - \mathbf{1}_{c_1}\|_\infty + \|g - \mathbf{1}_{c_2}\|_\infty \leq \delta_1,$$

$$(5.12) \quad \|c(x, y) + x \cdot y\|_{C^{2,\alpha}(B_3 \times B_3)} \leq \delta_1,$$

and

$$(5.13) \quad \left\| u - \frac{1}{2}|x|^2 \right\|_{C^0(B_3)} \leq \eta_1,$$

then  $u \in C^{k+2,\alpha}(B_{1/9})$ .

*Proof.* We divide the proof in two steps.

• *Step 1:  $C^{1,1}$  regularity.* Fix a point  $x_0 \in B_{1/8}$ , and set  $y_0 := c\text{-exp}_{x_0}(\nabla u(x_0))$ . Up to replace  $u$  (resp.  $c$ ) with the function  $u_1$  (resp.  $c_1$ ) constructed in Steps 4 and 5 in the proof of Theorem 4.3, we can assume that  $u \geq 0$ ,  $u(\mathbf{0}) = 0$ , that

$$S_h := S(\mathbf{0}, \mathbf{0}, u, h) = \{u \leq h\},$$

and that

$$(5.14) \quad D_{xy}c(\mathbf{0}, \mathbf{0}) = -\text{Id}.$$

Under these assumptions we will show that the sections of  $u$  are of “good shape”, i.e.,

$$(5.15) \quad B_{\sqrt{h}/K} \subset S_h \subset B_{K\sqrt{h}} \quad \forall h \leq h_1,$$

for some universal  $h_1$  and  $K$ . Arguing as in Step 6 of Theorem 4.3, this will give that  $u$  is  $C^{1,1}$  at the origin, and thus at every point in  $B_{1/8}$ .

First of all notice that, thanks to (5.13), for any  $h_1 > 0$  we can choose  $\eta_1 = \eta_1(h_1) > 0$  small enough such that (5.15) holds for  $S_{h_1}$  with  $K = 2$ . Hence, assuming without loss of generality that  $\delta_1 \leq 1$ , we see that

$$B_{\sqrt{h_1}/3} \subset \mathcal{N}_{\delta_1^\gamma \sqrt{h_1}}(\text{co}[S_{h_1}]) \subset B_{3\sqrt{h_1}},$$

where  $\gamma$  is the exponent from Proposition 5.2. Let  $v_1$  solve the Monge-Ampère equation

$$\begin{cases} \det(D^2 v_1) = f(\mathbf{0})/g(\mathbf{0}) & \text{in } \mathcal{N}_{\delta_1^\gamma \sqrt{h_1}}(\text{co}[S_{h_1}]), \\ v_1 = h_1 & \text{on } \partial \mathcal{N}_{\delta_1^\gamma \sqrt{h_1}}(\text{co}[S_{h_1}]). \end{cases}$$

Since  $B_{1/3} \subset N_{\delta_1^\gamma \sqrt{h_1}}(\text{co}[S_{h_1}])/\sqrt{h_1} \subset B_3$ , by standard Pogorelov estimates applied to the function  $v_1(\sqrt{h_1}x)/h_1$  (see for instance [26, Theorem 4.2.1]), it follows that  $|D^2v_1(0)| \leq M$ , with  $M > 0$  some large universal constant.

Let  $h_k := h_1 2^{-k}$  and define  $\bar{K} \geq 3$  to be the largest number such that any solution  $w$  of

$$(5.16) \quad \begin{cases} \det(D^2w) = f(\mathbf{0})/g(\mathbf{0}) & \text{in } Z, \\ w = 1 & \text{on } \partial Z, \end{cases} \quad \text{with } B_{1/\bar{K}} \subset Z \subset B_{\bar{K}},$$

satisfies  $|D^2w(0)| \leq M + 1$ .<sup>5</sup> We prove by induction that (5.15) holds with  $K = \bar{K}$ .

If  $h = h_1$  then we already know that (5.15) holds with  $K = 2$  (and so with  $K = \bar{K}$ ).

Assume now that (5.15) holds with  $h = h_k$  and  $K = \bar{K}$ , and we want to show that it holds with  $h = h_{k+1}$ . For this, for any  $k \in \mathbb{N}$  we consider  $u_k$  the solution of

$$\begin{cases} \det(D^2v_k) = f(\mathbf{0})/g(\mathbf{0}) & \text{in } \mathcal{N}_{\delta_k^\gamma \sqrt{h_k}}(\text{co}[S_{h_k}]), \\ v_k = h_1 2^{-k} & \text{on } \partial \mathcal{N}_{\delta_k^\gamma \sqrt{h_k}}(\text{co}[S_{h_k}]), \end{cases}$$

where

$$\delta_k := \|c(x, y) + x \cdot y\|_{C^2(S_{h_k} \times T_u(S_{h_k}))} \leq \delta_1.$$

Let us consider the rescaled functions

$$\bar{u}_k(x) := u(\sqrt{h_k}x)/h_k, \quad \bar{v}_k(x) := v_k(\sqrt{h_k}x)/h_k.$$

Since by the inductive hypothesis  $B_{1/\bar{K}} \subset \bar{S}_k := \{\bar{u}_k \leq 1\} \subset B_{\bar{K}}$ , we can apply Proposition 5.2 to deduce that

$$(5.17) \quad \|\bar{u}_k - \bar{v}_k\|_{C^0(\bar{S}_k)} \leq C_{\bar{K}} \left( \text{osc}_{S_{h_k}} f + \text{osc}_{T_u(S_{h_k})} g + \delta_k^{\gamma/n} \right) \leq C_{\bar{K}} (\delta_1 + \delta_1^{\gamma/n}).$$

This implies in particular that, if  $\delta_1$  is sufficiently small,  $B_{1/(2\bar{K})} \subset \{\bar{v}_k \leq 1\} \subset B_{2\bar{K}}$ . By standard estimates on the sections of solutions to the Monge-Ampère equation, the shapes of  $\{\bar{v}_k \leq 1\}$  and  $\{\bar{v}_k \leq 1/2\}$  are comparable, and in addition sections are well included into each other [26, Theorem 3.3.8]: there exists a universal constant  $L > 1$  such that

$$B_{1/(L\bar{K})} \subset \{\bar{v}_k \leq 1/2\} \subset B_{L\bar{K}}, \quad \text{dist}(\{\bar{v}_k \leq 1/4\}, \partial\{\bar{v}_k \leq 1/2\}) \geq 1/(LK).$$

Using again (5.17) we deduce that, if  $\delta_1$  is sufficiently small,

$$B_{1/(2L\bar{K})} \subset \{\bar{u}_k \leq 1/2\} \subset B_{2L\bar{K}}, \quad \text{dist}(\{\bar{u}_k \leq 1/4\}, \partial\{\bar{u}_k \leq 1/2\}) \geq 1/(2LK)$$

so, by scaling back,

$$(5.18) \quad B_{\sqrt{h_{k+1}}/(2L\bar{K})} \subset S_{h_{k+1}} \subset B_{2L\bar{K}\sqrt{h_{k+1}}}, \quad \text{dist}(S_{h_{k+2}}, \partial S_{h_{k+1}}) \geq \sqrt{h_k}/(2LK).$$

This allows us to apply Proposition 5.2 also to  $\bar{u}_{k+1}$  to get

$$(5.19) \quad \|\bar{u}_{k+1} - \bar{v}_{k+1}\|_{C^0(\bar{S}_{k+1})} \leq C_{2L\bar{K}} \left( \text{osc}_{S_{h_{k+1}}} f + \text{osc}_{T_u(S_{h_{k+1}})} g + \delta_{k+1}^{\gamma/n} \right).$$

<sup>5</sup>The fact that  $\bar{K}$  is well defined (i.e.,  $3 \leq \bar{K} < \infty$ ) follows by the following facts: first of all, by definition,  $M$  is an a-priori bound for  $|D^2w(0)|$  whenever  $w$  is a solution of (5.16) with  $B_{1/3} \subset Z \subset B_3$ , so  $\bar{K} \geq 3$ . On the other hand  $\bar{K} \leq \sqrt{2(M+1)}$ . Indeed, since  $1/2 \leq f(\mathbf{0})/g(\mathbf{0}) \leq 2$  (by (5.11)) and  $M \geq 1$ , the function

$$\bar{w} := (M+1)x_1^2 + \frac{f(\mathbf{0})}{g(\mathbf{0})} \frac{x_2^2}{M+1} + x_3^2 + \dots + x_n^2$$

is a solution of (5.16) such that  $B_{1/\sqrt{2(M+1)}} \subset B_{1/\sqrt{M+1}} \subset \{\bar{w} \leq 1\} \subset B_{\sqrt{2(M+1)}}$  and  $|D^2\bar{w}(0)| = 2(M+1)$ .

We now observe that, by (5.15) and the  $C^{1,\beta}$  regularity of  $u$  (see Theorem 4.3), it follows that

$$\text{diam}(S_{h_k}) + \text{diam}(T_u(S_{h_k})) \leq Ch_k^{\beta/2},$$

so by the  $C^{0,\alpha}$  regularity of  $f$  and  $g$ , and the  $C^{2,\alpha}$  regularity of  $c$ , we have (recall that  $\gamma < 1$ )

$$(5.20) \quad \text{osc}_{S_{h_k}} f + \text{osc}_{T_u(S_{h_k})} g + \delta_k^{\gamma/n} \leq C' h_k^\sigma, \quad \sigma := \frac{\alpha\beta\gamma}{2n}$$

Hence, by (5.17) and (5.19),

$$\|\bar{u}_k - \bar{v}_k\|_{C^0(\bar{S}_k)} + \|\bar{u}_{k+1} - \bar{v}_{k+1}\|_{C^0(\bar{S}_{k+1})} \leq C(C_{\bar{K}} + C_{2L\bar{K}})h_k^\sigma,$$

from which we deduce (recall that  $h_k = 2h_{k+1}$ )

$$\begin{aligned} \|v_k - v_{k+1}\|_{C^0(S_{h_{k+1}})} &\leq \|v_k - u\|_{C^0(S_{h_k})} + \|u - v_{k+1}\|_{C^0(S_{h_{k+1}})} \\ &= h_k \|\bar{u}_k - \bar{v}_k\|_{C^0(\bar{S}_k)} + h_{k+1} \|\bar{u}_{k+1} - \bar{v}_{k+1}\|_{C^0(\bar{S}_{k+1})} \\ &\leq C(C_{\bar{K}} + C_{2L\bar{K}})h_k^{1+\sigma}. \end{aligned}$$

Since  $v_k$  and  $v_{k+1}$  are two strictly convex solutions of the Monge Ampère equation with constant right hand side inside  $S_{h_{k+1}}$ , and since  $S_{h_{k+2}}$  is “well contained” inside  $S_{h_{k+1}}$ , by classical Pogorelov and Schauder estimates we get

$$(5.21) \quad \|D^2 v_k - D^2 v_{k+1}\|_{C^0(S_{h_{k+2}})} \leq C'_{\bar{K}} h_k^\sigma$$

$$(5.22) \quad \|D^3 v_k - D^3 v_{k+1}\|_{C^0(S_{h_{k+2}})} \leq C'_{\bar{K}} h_k^{\sigma-1/2},$$

where  $C'_{\bar{K}}$  is some constant depending only on  $\bar{K}$ . By (5.21) applied to  $v_j$  for all  $j = 1, \dots, k$  (this can be done since, by the inductive assumption, (5.15) holds for  $h = h_j$  with  $j = 1, \dots, k$ ) we obtain

$$\begin{aligned} |D^2 v_{k+1}(0)| &\leq |D^2 v_1(0)| + \sum_{j=1}^k |D^2 v_j(0) - D^2 v_{j+1}(0)| \\ &\leq M + C'_{\bar{K}} h_1^\sigma \sum_{j=0}^k 2^{-j\sigma} \\ &\leq M + \frac{C'_{\bar{K}}}{1-2^{-\sigma}} h_1^\sigma \leq M + 1, \end{aligned}$$

provided we choose  $h_1$  small enough (recall that  $h_k = h_1 2^{-k}$ ). By the definition of  $\bar{K}$  it follows that also  $S_{h_{k+1}}$  satisfies (5.15), concluding the proof of the inductive step.

• *Step 2: higher regularity.* Now that we know that  $u \in C^{1,1}(B_{1/8})$ , Equation (2.10) becomes uniformly elliptic. So one may use Evans-Krylov Theorem to obtain that  $u \in C_{\text{loc}}^{2,\sigma'}(B_{1/9})$  for some  $\sigma' > 0$ , and then standard Schauder estimates to conclude the proof. However, for the convenience of the reader, we show here how to give a simple direct proof of the  $C^{2,\sigma'}$  regularity of  $u$  with  $\sigma' = 2\sigma$ .

As in the previous step, it suffices to show that  $u$  is  $C^{2,\sigma'}$  at the origin, and for this we have to prove that there exists a sequence of paraboloids  $P_k$  such that

$$(5.23) \quad \sup_{B_{r_0^k/C}} |u - P_k| \leq Cr_0^{k(2+\sigma')}$$

for some  $r_0, C > 0$ .

Let  $v_k$  be as in the previous step, and let  $P_k$  be their second order Taylor expansion at  $\mathbf{0}$ :

$$P_k(x) = v_k(\mathbf{0}) + \nabla v_k(\mathbf{0}) \cdot x + \frac{1}{2} D^2 v_k(\mathbf{0}) x \cdot x.$$

We observe that, thanks to (5.15),

$$(5.24) \quad \|v_k - P_k\|_{C^0(B(0, \sqrt{h_{k+2}/K}))} \leq \|v_k - P_k\|_{C^0(S_{h_{k+2}})} \leq C \|D^3 v_k\|_{C^0(S_{h_{k+2}})} h_k^{3/2}.$$

In addition, by (5.22) applied with  $j = 1, \dots, k$  and recalling that  $h_k = h_1 2^{-k}$  and  $2\sigma < 1$  (see (5.20)), we get

$$(5.25) \quad \begin{aligned} \|D^3 v_k\|_{C^0(S_{h_{k+2}})} &\leq \|D^3 v_1\|_{C^0(S_{h_3})} + \sum_{j=1}^k \|D^3 v_j - D^3 v_{j+1}\|_{C^0(S_{h_{j+2}})} \\ &\leq C \left( 1 + \sum_{j=1}^k h_j^{(\sigma-1/2)} \right) \leq C h_k^{\sigma-1/2}. \end{aligned}$$

Combining (5.15), (5.24), (5.25), and recalling (5.17) and (5.20), we obtain

$$\|u - P_k\|_{C^0(B_{\sqrt{h_{k+2}/K})}} \leq \|v_k - P_k\|_{C^0(S_{h_{k+2}})} + \|v_k - u\|_{C^0(S_{h_{k+2}})} \leq C h_k^{1+\sigma},$$

so (5.23) follows with  $r_0 = 1/\sqrt{2}$  and  $\sigma' = 2\sigma$ . □

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