Quantitative stability of the Brunn-Minkowski inequality for sets of equal volume

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Abstract

We prove a quantitative stability result for the Brunn-Minkowski inequality on sets of equal volume: if $|A| = |B| > 0$ and $|A + B|^{1/n} = (2 + \delta)|A|^{1/n}$ for some small δ, then, up to a translation, both A and B are close (in terms of δ) to a convex set $\mathcal{K}$. Although this result was already proved in our previous paper [9] even for sets of different volume, we provide here a more elementary proof that we believe has its own interest. Also, in terms of the stability exponent, this result provides a stronger estimate than the result in [9].

1 Introduction

The Brunn-Minkowski inequality is a very classical and powerful inequality in convex geometry that has found important applications in analysis, statistics, and information theory. We refer the reader to [14] for an extended exposition on the Brunn-Minkowski inequality and its relation to several other famous inequalities; see also [6, 7].

To state the inequality, we first need some basic notation. Given two subset $A, B \subset \mathbb{R}^n$, and $c > 0$, we define the set sum and scalar multiple by

$$A + B := \{a + b : a \in A,\ b \in B\}, \quad cA := \{ca : a \in A\} \tag{1.1}$$

We shall use $|E|$ to denote the Lebesgue measure of a set E. (If E is not measurable, $|E|$ denotes the outer Lebesgue measure of E.) The Brunn-Minkowski inequality says that, given $A, B \subset \mathbb{R}^n$ measurable sets,

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}. \tag{1.2}$$

In addition, if $|A|, |B| > 0$, then equality holds if and only if there exist a convex set $\mathcal{K} \subset \mathbb{R}^n$, $\lambda_A, \lambda_B > 0$, and $v_A, v_B \in \mathbb{R}^n$, such that

$$A \subset \lambda_A\mathcal{K} + v_A, \quad B \subset \lambda_B\mathcal{K} + v_B, \qquad \big|(\lambda_A\mathcal{K} + v_A) \setminus A\big| = \big|(\lambda_B\mathcal{K} + v_B) \setminus B\big| = 0.$$

In other words, if equality holds in (1.2), then A and B are subsets of full measure in *homothetic convex* sets.

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Because of the variety of applications of (1.2) as well as the fact the one can characterize the case of equality, a natural stability question that one would like to address is the following:

Let A, B be two sets for which equality in (1.2) *almost holds. Is it true that, up to translations and dilations, A and B are close to the same convex set?*

This question has a long history. First of all, when $n = 1$ and $A = B$, inequality (1.2) reduces to $|A+A| \geq 2|A|$. If one approximates sets in $\mathbb{R}$ with finite unions of intervals, then one can translate the problem to $\mathbb{Z}$, and in the discrete setting the question becomes a well studied problem in additive combinatorics. There are many results on this topic, usually called Freiman-type theorems. The precise statement in one dimension is the following.

Theorem 1.1. *Let $A \subset \mathbb{R}$ be a measurable set, and denote by* $\mathrm{co}(A)$ *its convex hull. Then*

$$|A + A| - 2|A| \geq \min\{|\mathrm{co}(A) \setminus A|, |A|\},$$

or, equivalently, if $|A| > 0$ then

$$\delta(A) \geq \frac{1}{2} \min\left\{\frac{|\mathrm{co}(A) \setminus A|}{|A|}, 1\right\}.$$

This theorem can be obtained as a corollary of a result of G. Freiman [12] about the structure of additive subsets of $\mathbb{Z}$. (See [13] or [17, Theorem 5.11] for a statement and a proof.) However, it turns out that to prove of Theorem 1.1 one only needs weaker results, and one can find an elementary self-contained proof of Theorem 1.1 in [8, Section 2].

In the case $n = 1$ but $A \neq B$, the following sharp stability result holds again as a consequence of classical theorems in additive combinatorics (an elementary proof of this result can be given using Kemperman's theorem [3, 4]):

Theorem 1.2. *Let $A, B \subset \mathbb{R}$ be measurable sets. If $|A + B| < |A| + |B| + \delta$ for some $\delta \leq \min\{|A|, |B|\}$, then $|\mathrm{co}(A) \setminus A| \leq \delta$ and $|\mathrm{co}(B) \setminus B| \leq \delta$.*

Concerning the higher dimensional case, in [1, 2] M. Christ proved a *qualitative* stability result for (1.2), giving a positive answer to the stability question raised above. However, his results do not provide any quantitative control.

On the *quantitative* side, V. I. Diskant [5] and H. Groemer [15] obtained some stability results for *convex sets* in terms of the Hausdorff distance. More recently, in [10, 11], the first author together with F. Maggi and A. Pratelli obtained a sharp stability result in terms of the L^1 distance, still on convex sets. Since this last result will be used later in our proofs, we state it in detail. (Here and from now on, $E\Delta F$ denotes the symmetric difference between sets E and F, that is $E\Delta F = (E \setminus F) \cup (F \setminus E)$.)

Theorem 1.3. *Let $A, B \subset \mathbb{R}^n$ be convex sets, and define*

$$\mathscr{A}(A, B) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|A\Delta(x_0 + \tau B)|}{|A|} \,:\, \tau = \left(\frac{|A|}{|B|}\right)^{1/n} \right\}, \qquad \sigma(A, B) := \max\left\{\frac{|A|}{|B|}, \frac{|B|}{|A|}\right\}.$$

There exists a computable dimensional constant $C_0(n)$ such that

$$|A+B|^{1/n} \geq \left(|A|^{1/n}+|B|^{1/n}\right)\left\{1+\frac{\mathscr{A}(A,B)^2}{C_0(n)\,\sigma(A,B)^{1/n}}\right\}.$$

More recently, in [8, Theorem 1.2 and Remark 3.2], the present authors proved a quantitative stability result when $A=B$: given a measurable set $A \subset \mathbb{R}^n$ with $|A|>0$, set

$$\delta(A) := \frac{\left|\frac{1}{2}(A+A)\right|}{|A|}-1 = \frac{|A+A|}{|2A|}-1. \tag{1.3}$$

Then, a power of $\delta(A)$ dominates the measure of the difference between A and its convex hull $\mathrm{co}(A)$.

Theorem 1.4. *Let $A \subset \mathbb{R}^n$ be a measurable set of positive measure. There exist computable dimensional constants $\delta_n, c_n > 0$ such that if $\delta(A) \leq \delta_n$, then*

$$\delta(A)^{\alpha_n} \geq c_n \frac{|\,\mathrm{co}(A)\setminus A|}{|A|}, \qquad \alpha_n := \frac{1}{8^{n-1}n!\,[(n-1)!]^2}.$$

In addition, there exists a convex set $K \subset \mathbb{R}^n$ such that

$$\delta(A)^{n\alpha_n} \geq c_n \frac{|K\Delta A|}{|A|}.$$

After that, we investigated the general case $A \neq B$. Notice that, after a dilation, one can always assume $|A|=|B|=1$ while replacing the sum $A+B$ by a convex combination $S_t := tA+(1-t)B$. It follows by (1.2) that $|S_t| = 1+\delta$ for some $\delta \geq 0$. The main theorem in [9] is a quantitative version of Christ's result. Since the proof is by induction on the dimension, it is convenient to allow the measures of $|A|$ and $|B|$ not to be exactly equal, but just close in terms of δ. Here is the main result of that paper.

Theorem 1.5. *Let $n \geq 2$, let $A, B \subset \mathbb{R}^n$ be measurable sets, and define $S_t := tA+(1-t)B$ for some $t \in [\tau, 1-\tau]$, $0<\tau\leq 1/2$. There are computable dimensional constants N_n and computable functions $M_n(\tau), \varepsilon_n(\tau) > 0$ such that if*

$$\big||A|-1\big| + \big||B|-1\big| + \big||S_t|-1\big| \leq \delta \tag{1.4}$$

for some $\delta \leq e^{-M_n(\tau)}$, then there exists a convex set $\mathcal{K} \subset \mathbb{R}^n$ such that, up to a translation,

$$A, B \subset \mathcal{K} \qquad \text{and} \qquad |\mathcal{K}\setminus A| + |\mathcal{K}\setminus B| \leq \tau^{-N_n}\delta^{\varepsilon_n(\tau)}.$$

Explicitly, we may take

$$M_n(\tau) = \frac{2^{3^{n+2}}n^{3^n}|\log\tau|^{3^n}}{\tau^{3^n}}, \qquad \varepsilon_n(\tau) = \frac{\tau^{3^n}}{2^{3^{n+1}}n^{3^n}|\log\tau|^{3^n}}.$$

In particular, the measure of the difference between the sets A and B and their convex hull is bounded by a power δ^ϵ, confirming a conjecture of Christ [1].

The result above provides a general quantitative stability for the Brunn-Minkowski inequality in arbitrary dimension. However the exponent degenerates very quickly as the dimension increases (much faster than in Theorem 1.4), and, in addition, the argument in [9] is very long and involved. The aim of this paper is to provide a shorter and more elementary proof when $|A| = |B| > 0$, that we believe to be of independent interest.

After a dilation, one can assume with no loss of generality that $|A| = |B| = 1$. In this case, it follows by (1.2) that $|\frac{1}{2}(A+B)| = 1+\delta$ for some $\delta \geq 0$, and we want to show that a power of δ controls the closeness of A and B to the same convex set $\mathcal{K}$. Again, as in the previous theorem, it will be convenient to allow the measures of $|A|$ and $|B|$ not to be exactly equal, but just close in terms of δ.

Here is the main result of this paper:

Theorem 1.6. *Let $A, B \subset \mathbb{R}^n$ be measurable sets, and define their semi-sum $S := \frac{1}{2}(A+B)$. There exist computable dimensional constants $\delta_n, C_n > 0$ such that if*

$$\big||A|-1\big| + \big||B|-1\big| + \big||S|-1\big| \leq \delta \tag{1.5}$$

for some $\delta \leq \delta_n$, then there exists a convex set $\mathcal{K} \subset \mathbb{R}^n$ such that, up to a translation,

$$A, B \subset \mathcal{K} \qquad \text{and} \qquad |\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \leq C_n \delta^{\beta_n},$$

where

$$\beta_1 := 1, \qquad \beta_n := \frac{1}{2^{6n-5}3^{n-1}n!(n-1)!} \prod_{k=1}^{n} \alpha_k^2 \quad \forall\, n \geq 2,$$

and α_k is given by Theorem 1.4. (Recall that $|S|$ is the outer measure of S if S is not measurable.)

The proof of this theorem is specific to the case $|A|$ near $|B|$. It uses a symmetrization and other techniques introduced by Christ [2, 3], Theorems 1.3 and 1.4, and two propositions of independent interest, Propositions 2.5 and 2.6 below. See Section 3 for further discussion of the strategy of the proof.

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2 Notation and preliminary results

Let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$. Denote by $x = (y,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ a point in $\mathbb{R}^n$, and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\bar{\pi} : \mathbb{R}^n \to \mathbb{R}$ denote the canonical projections, i.e.,

$$\pi(y,t) := y \qquad \text{and} \qquad \bar{\pi}(y,t) := t.$$

Given a compact set $E \subset \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$, and $\lambda > 0$, we use the notation

$$E_y := E \cap \pi^{-1}(y) \subset \{y\} \times \mathbb{R}, \qquad E(t) := E \cap \bar{\pi}^{-1}(t) \subset \mathbb{R}^{n-1} \times \{t\}, \tag{2.1}$$

$$\mathcal{E}(\lambda) := \big\{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(E_y) > \lambda\big\}. \tag{2.2}$$

Following Christ [2], we consider two symmetrizations and combine them. For our purposes (see the proof of Proposition 2.5), it is convenient to use a definition of Schwarz symmetrization that is slightly different from the classical one. (In the usual definition of Schwarz symmetrization $E^*(t) = \emptyset$ whenever $\mathcal{H}^{d-1}\big(E(t)\big) = 0$.)

Definition 2.1. Let $E \subset \mathbb{R}^n$ be a compact set. We define the *Schwarz symmetrization* E^* of E as follows. For each $t \in \mathbb{R}$,

- If $\mathcal{H}^{d-1}\big(E(t)\big) > 0$, then $E^*(t)$ is the closed disk centered at $0 \in \mathbb{R}^{n-1}$ with the same measure.
- If $\mathcal{H}^{d-1}\big(E(t)\big) = 0$ but $E(t)$ is non-empty, then $E^*(t) = \{0\}$.
- If $E(t)$ is empty, then $E^*(t)$ is empty as well.

We define the *Steiner symmetrization* $E^\star$ of E so that for each $y \in \mathbb{R}^{n-1}$, the set $E^\star_y$ is empty if $\mathcal{H}^1(E_y) = 0$; otherwise it is the closed interval of length $\mathcal{H}^1(E_y)$ centered at $0 \in \mathbb{R}$. Finally, we define $E^\natural := (E^\star)^*$.

As for instance in [2, Section 2], both the Schwarz and the Steiner symmetrization preserve the measure of sets, and the $\natural$-symmetrization preserves the measure of the sets $\mathcal{E}(\lambda)$. The following statement collects all these results.

Lemma 2.2. *Let $A, B \subset \mathbb{R}^n$ be compact sets. Then $|A| = |A^*| = |A^\star| = |A^\natural|$,*

$$|A^* + B^*| \le |A + B|, \qquad |A^\star + B^\star| \le |A + B|, \qquad |A^\natural + B^\natural| \le |A + B|,$$

and, for almost every $\lambda > 0$,

$$\big|A \setminus \pi^{-1}\big(\mathcal{A}(\lambda)\big)\big| = \big|A^\natural \setminus \pi^{-1}\big(\mathcal{A}^\natural(\lambda)\big)\big| \qquad \text{and} \qquad \mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big) = \mathcal{H}^{n-1}\big(\mathcal{A}^\natural(\lambda)\big),$$

where $\mathcal{A}(\lambda) := \big\{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(A_y) > \lambda\big\}$ and $\mathcal{A}^\natural(\lambda) := \big\{y \in \mathbb{R}^{n-1} : \mathcal{H}^1(A^\natural_y) > \lambda\big\}$.

Another important fact is that a bound on the measure of $A + B$ in terms of the measures of A and B gives bounds relating the sizes of

$$\sup_y \mathcal{H}^1(A_y), \qquad \sup_y \mathcal{H}^1(B_y), \qquad \mathcal{H}^{n-1}\big(\pi(A)\big), \qquad \mathcal{H}^{n-1}\big(\pi(B)\big).$$

We refer to [9, Lemma 3.2] for a proof.

Lemma 2.3. *Let $A, B \subset \mathbb{R}^n$ be compact sets such that $|A|, |B| \ge 1/2$ and $|\frac{1}{2}(A + B)| \le 2$. There exists a dimensional constant $M > 1$ such that*

$$\frac{\sup_y \mathcal{H}^1(A_y)}{\sup_y \mathcal{H}^1(B_y)} \in (1/M, M), \qquad \frac{\mathcal{H}^{n-1}\big(\pi(A)\big)}{\mathcal{H}^{n-1}\big(\pi(B)\big)} \in (1/M, M),$$

$$\Big(\sup_y \mathcal{H}^1(A_y)\Big)\mathcal{H}^{n-1}\big(\pi(A)\big) \in (1/M, M), \qquad \Big(\sup_y \mathcal{H}^1(B_y)\Big)\mathcal{H}^{n-1}\big(\pi(B)\big) \in (1/M, M).$$

Thus, up a measure preserving affine transformation of the form $(y,t) \mapsto (\tau y, \tau^{1-n}t)$ *with* $\tau > 0$*, all the quantities* $\sup_y \mathcal{H}^1(A_y)$*,* $\sup_y \mathcal{H}^1(B_y)$*,* $\mathcal{H}^{n-1}\big(\pi(A)\big)$*,* $\mathcal{H}^{n-1}\big(\pi(B)\big)$ *are of order one. In particular,*

$$\mathcal{H}^{n-1}\big(\pi(A)\big) + \mathcal{H}^{n-1}\big(\pi(B)\big) + \sup_y \mathcal{H}^1(A_y) + \sup_y \mathcal{H}^1(B_y) \le M. \tag{2.3}$$

In this case, we say that A *and* B *are* M*-normalized.*

The following result of Christ [1, Lemma 4.1] shows that $\sup_t \mathcal{H}^{n-1}\big(A(t)\big)$ and $\sup_t \mathcal{H}^{n-1}\big(B(t)\big)$ are close in terms of δ:

Lemma 2.4. *Let* $A, B \subset \mathbb{R}^n$ *be compact sets, define* $S := \frac{1}{2}(A+B)$*, and assume that* (1.5) *holds for some* $\delta \le 1/2$*. Also, suppose that* A *and* B *are* M*-normalized as defined in Lemma 2.3. Then, there exists a dimensional constant* $C > 0$ *such that*

$$\frac{\sup_t \mathcal{H}^{n-1}\big(A(t)\big)}{\sup_t \mathcal{H}^{n-1}\big(B(t)\big)} \in \big(1 - C\delta^{1/2}, 1 + C\delta^{1/2}\big).$$

Two other key ingredients in our proof of Theorem 1.6 are the following propositions, whose proofs are postponed to Section 4:

Proposition 2.5. *Let* $A, B \subset \mathbb{R}^n$ *be compact sets, define* $S := \frac{1}{2}(A+B)$*, and assume that* (1.5) *holds for some* $\delta \le 1/2$*. Also, suppose that we can find a convex set* $K \subset \mathbb{R}^n$ *such that*

$$|S \Delta K| \le C\delta^{\alpha}$$

for some $\alpha > 0$*, where* $C > 0$ *is a dimensional constant. Then there exists a dimensional constant* $C' > 0$ *such that*

$$|\operatorname{co}(S) \setminus S| \le C'\delta^{\alpha/2n}.$$

Proposition 2.6. *Let* $A, B \subset \mathbb{R}^n$ *be compact sets, define* $S := \frac{1}{2}(A+B)$*, and assume that* (1.5) *holds for some* $\delta \le 1/2$*. Also, suppose that*

$$|\operatorname{co}(S) \setminus S| \le C\delta^{\beta} \tag{2.4}$$

for some $\beta > 0$*, where* $C > 0$ *is a dimensional constant. Then, up to a translation,*

$$|A \Delta B| \le C'\delta^{\beta/2}$$

and there exists a convex set $\mathcal{K}$ *containing both* A *and* B *such that*

$$|\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \le C'\delta^{\beta/2n},$$

for some dimensional constant $C' > 0$.

3 Proof of Theorem 1.6

As explained in [8], by inner approximation[1] it suffices to prove the result when A, B are compact sets. Hence, let A and B be compact, define $S := \frac{1}{2}(A+B)$, and assume that (1.5) holds. We want to prove that there exists a convex set $\mathcal{K}$ such that, up to a translation,

$$A, B \subset \mathcal{K}, \qquad |\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \leq C_n \delta^{\beta_n}.$$

Moreover, since the statement and the conclusions are invariant under measure preserving affine transformations, by Lemma 2.3 we can assume that A and B are M-normalized (see (2.3)).

Ultimately, we wish to show that, up to translation, each of A, B, and S is of nearly full measure in the same convex set. The strategy of the proof is to show first that S is close to a convex set, and then apply Propositions 2.5 and 2.6. To obtain the closeness of S to a convex set, we would like prove that $|\frac{1}{2}(S+S)|$ is close to $|S|$ and then apply Theorem 1.4. It is simpler, however, to construct a subset $\bar{S} \subset S$ such that $|S \setminus \bar{S}|$ is small and $|\frac{1}{2}(\bar{S}+\bar{S})|$ is close to $|\bar{S}|$.

To carry out our argument, one important ingredient will be to use the inductive hypothesis on the level sets $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ defined in (2.2). However, two difficulties arise here: first of all, to apply the inductive hypothesis, we need to know that $\mathcal{H}^{n-1}(\mathcal{A}(\lambda))$ and $\mathcal{H}^{n-1}(\mathcal{B}(\lambda))$ are close. In addition, the Brunn-Minkowski inequality does not have a natural proof by induction unless the measures of all the level sets $\mathcal{H}^{n-1}(\mathcal{A}(\lambda))$ and $\mathcal{H}^{n-1}(\mathcal{B}(\lambda))$ are the nearly same. (See (3.11) below.) Hence, it is important for us to have a preliminary quantitative estimate on the difference between $\mathcal{H}^{n-1}(\mathcal{A}(\lambda))$ and $\mathcal{H}^{n-1}(\mathcal{B}(\lambda))$ for most $\lambda > 0$. For this we follow an approach used first in [2] and readapted in [9], in which we begin by showing our theorem in the special case of symmetrized sets $A = A^\natural$ and $B = B^\natural$ (recall Definition 2.1). Thanks to Lemma 2.2, this will give us the desired closeness between $\mathcal{H}^{n-1}(\mathcal{A}(\lambda))$ and $\mathcal{H}^{n-1}(\mathcal{B}(\lambda))$ for most $\lambda > 0$, which allows us to apply the strategy described above and prove the theorem in the general case.

Throughout the proof, C will denote a generic constant depending only on the dimension, which may change from line to line.

3.1 The case $A = A^\natural$ and $B = B^\natural$

Let $A, B \subset \mathbb{R}^n$ be compact sets satisfying $A = A^\natural$, $B = B^\natural$. Since

$$\pi\big(A(t)\big) \subset \pi\big(A(0)\big) = \pi(A) \quad \text{and} \quad \pi\big(B(t)\big) \subset \pi\big(B(0)\big) = \pi(B) \quad \text{are disks centered at the origin,}$$

applying Lemma 2.4 we deduce that

$$\mathcal{H}^{n-1}\big(\pi(A)\Delta\pi(B)\big) \leq C\delta^{1/2}. \tag{3.1}$$

Hence, if we define

$$\bar{S} := \bigcup_{y \in \pi(A)\cap\pi(B)} \frac{A_y + B_y}{2},$$

[1] The approximation of A (and analogously for B) is by a sequence of compact sets $A_k \subset A$ such that $|A_k| \to |A|$ and $|\operatorname{co}(A_k)| \to |\operatorname{co}(A)|$. One way to construct such sets is to define $A_k := A'_k \cup V_k$, where $A'_k \subset A$ are compact sets satisfying $|A'_k| \to |A|$, and $V_k \subset V_{k+1} \subset A$ are finite sets satisfying $|\operatorname{co}(V_k)| \to |\operatorname{co}(A)|$.

then $\bar{S}_y \subset S_y$ for all $y \in \mathbb{R}^{n-1}$. In addition, using (1.5), (2.3), and (3.1), we have

$$\begin{aligned} 1+\delta \geq |S| &= \int_{\mathbb{R}^{n-1}} \mathcal{H}^1(S_y)\,dy \geq \int_{\pi(A)\cap\pi(B)} \mathcal{H}^1(S_y)\,dy \geq \int_{\pi(A)\cap\pi(B)} \mathcal{H}^1(\bar{S}_y)\,dy \\ &= |\bar{S}| \geq \frac{1}{2}\int_{\pi(A)\cap\pi(B)} \mathcal{H}^1(A_y)\,dy + \frac{1}{2}\int_{\pi(A)\cap\pi(B)} \mathcal{H}^1(B_y)\,dy \\ &\geq \frac{|A|+|B|}{2} - M\mathcal{H}^{n-1}\big(\pi(A)\Delta\pi(B)\big) \geq 1 - C\delta^{1/2}, \end{aligned}$$

which implies (since $\bar{S} \subset S$)

$$|S \setminus \bar{S}| \leq C\delta^{1/2}. \tag{3.2}$$

Furthermore, since each section S_y is an interval centered at $0 \in \mathbb{R}$, for all $y', y'' \in \pi(A)\cap\pi(B)$ such that $\frac{y'+y''}{2} = y$,

$$\bar{S}_{y'} + \bar{S}_{y''} = \frac{A_{y'}+B_{y'}}{2} + \frac{A_{y''}+B_{y''}}{2} = \frac{A_{y'}+B_{y''}}{2} + \frac{A_{y''}+B_{y'}}{2} \subset S_y + S_y = 2S_y,$$

which gives

$$\frac{\bar{S}+\bar{S}}{2} \subset S. \tag{3.3}$$

Recalling (1.3), by (3.2) and (3.3) we obtain that $\delta(\bar{S}) \leq C\delta^{1/2}$. Hence, we can apply Theorem 1.4 to $\bar{S}$ to find a convex set $\bar{K}$ such that

$$|\bar{S}\Delta\bar{K}| \leq C\delta^{n\alpha_n/2}.$$

Hence, by (3.3),

$$|S\Delta\bar{K}| \leq C\delta^{n\alpha_n/2},$$

and using Propositions 2.5 and 2.6 we deduce that, up to a translation, there exists a convex set K such that $A \cup B \subset K$ and

$$|A\Delta B| \leq C\delta^{\alpha_n/8}, \qquad |K \setminus A| + |K \setminus B| \leq C\delta^{\alpha_n/8n}. \tag{3.4}$$

Notice that, because $A = A^\natural$ and $B = B^\natural$, it is easy to check that the above properties still hold with $K^\natural$ in place of K. Hence, in this case, without loss of generality one can assume that $K = K^\natural$.

3.2 The general case

Since, by Theorem 1.2, the result is true when $n = 1$, we may assume that we already proved Theorem 1.6 through $n-1$, and we want to show its validity for n.

Step 1: There exist a dimensional constant $\zeta > 0$ and $\bar{\lambda} \sim \delta^\zeta$ such that we can apply the inductive hypothesis to $\mathcal{A}(\bar{\lambda})$ and $\mathcal{B}(\bar{\lambda})$.

Let $A^\natural$ and $B^\natural$ be as in Definition 2.1 and denote

$$\bar{\alpha} := \frac{\alpha_n}{8}. \tag{3.5}$$

Thanks to Lemma 2.2, $A^\natural$ and $B^\natural$ still satisfy (1.5), so we can apply the result proved in Section 3.1 above to get (see (3.4))

$$\int_{\mathbb{R}^{n-1}} \left|\mathcal{H}^1(A^\natural_y) - \mathcal{H}^1(B^\natural_y)\right| dy \leq \int_{\mathbb{R}^{n-1}} \left|\mathcal{H}^1(A^\natural_y \Delta B^\natural_y)\right| dy = |A^\natural \Delta B^\natural| \leq C\delta^{\bar{\alpha}} \tag{3.6}$$

and

$$K \supset A^\natural \cup B^\natural, \qquad |K \setminus A^\natural| + |K \setminus B^\natural| \leq C\delta^{\bar{\alpha}/n} \tag{3.7}$$

for some convex set $K = K^\natural$.

In addition, because A and B are M-normalized (see (2.3)), so are $A^\natural$ and $B^\natural$, and by (3.7) we deduce that there exists a dimensional constant $R_n > 0$ such that

$$K \subset B_{R_n}. \tag{3.8}$$

Also, by (3.6) and Chebyshev's inequality we obtain that, except for a set of measure $\leq C\delta^{\bar{\alpha}/2}$,

$$\left|\mathcal{H}^1(A^\natural_y) - \mathcal{H}^1(B^\natural_y)\right| \leq \delta^{\bar{\alpha}/2}.$$

Thus, recalling Lemma 2.2, for almost every $\lambda > 0$

$$\mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big) = \mathcal{H}^{n-1}\big(\mathcal{A}^\natural(\lambda)\big) \leq \mathcal{H}^{n-1}\big(\mathcal{B}^\natural(\lambda - \delta^{\bar{\alpha}/2})\big) + C\delta^{\bar{\alpha}/2} = \mathcal{H}^{n-1}\big(\mathcal{B}(\lambda - \delta^{\bar{\alpha}/2})\big) + C\delta^{\bar{\alpha}/2}.$$

Since, by (2.3),

$$\int_0^M \Big(\mathcal{H}^{n-1}\big(\mathcal{B}(\lambda)\big) - \mathcal{H}^{n-1}\big(\mathcal{B}(\lambda + \delta^{\bar{\alpha}/2})\big)\Big)\, d\lambda = \int_0^{\delta^{\bar{\alpha}/2}} \mathcal{H}^{n-1}\big(\mathcal{B}(\lambda)\big)\, d\lambda \leq M\delta^{\bar{\alpha}/2},$$

by Chebyshev's inequality we deduce that

$$\mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big) \leq \mathcal{H}^{n-1}\big(\mathcal{B}(\lambda)\big) + C\delta^{\bar{\alpha}/4}$$

for all λ outside a set of measure $\leq C\delta^{\bar{\alpha}/4}$. Exchanging the roles of A and B we obtain that there exists a set $F \subset [0, M]$ such that

$$\mathcal{H}^1(F) \leq C\delta^{\bar{\alpha}/4}, \qquad \left|\mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big) - \mathcal{H}^{n-1}\big(\mathcal{B}(\lambda)\big)\right| \leq C\delta^{\bar{\alpha}/4} \quad \forall\, \lambda \in [0, \infty] \setminus F. \tag{3.9}$$

Using the elementary inequality

$$\Big(\frac{a+b}{2}\Big)^{n-1} \geq \frac{a^{n-1} + b^{n-1}}{2} - C|a-b|^2 \qquad \forall\, 0 \leq a, b \leq M,$$

and replacing a and b with $a^{1/(n-1)}$ and $b^{1/(n-1)}$, respectively, we get

$$\Big(\frac{a^{1/(n-1)} + b^{1/(n-1)}}{2}\Big)^{n-1} \geq \frac{a+b}{2} - C|a-b|^{2/(n-1)} \qquad \forall\, 0 \leq a, b \leq M \tag{3.10}$$

(notice that $|a^{1/(n-1)} - b^{1/(n-1)}| \leq |a-b|^{1/(n-1)}$). Finally, it is easy to check that

$$\frac{\mathcal{A}(\lambda) + \mathcal{B}(\lambda)}{2} \subset \mathcal{S}(\lambda) \qquad \forall\, \lambda > 0.$$

Hence, by the Brunn-Minkowski inequality (1.2) applied to $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$, using (1.5), (2.3), (3.10), and (3.9), we get

$$
\begin{aligned}
1+\delta \geq |S| &= \int_0^M \mathcal{H}^{n-1}\big(\mathcal{S}(\lambda)\big)\,d\lambda \\
&\geq \frac{1}{2^{n-1}}\int_0^M \Big(\mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big)^{1/(n-1)} + \mathcal{H}^{n-1}\big(\mathcal{B}(\lambda)\big)^{1/(n-1)}\Big)^{n-1}\,d\lambda \\
&\geq \frac{1}{2}\int_0^M \Big(\mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big) + \mathcal{H}^{n-1}\big(\mathcal{B}(\lambda)\big)\Big)\,d\lambda \\
&\quad - C\int_0^M \big|\mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big) - \mathcal{H}^{n-1}\big(\mathcal{B}(\lambda)\big)\big|^{2/(n-1)}\,d\lambda \\
&= \frac{|A|+|B|}{2} - C\delta^{\bar{\alpha}/[2(n-1)]} \\
&\geq 1 - C\delta^{\bar{\alpha}/[2(n-1)]}.
\end{aligned} \tag{3.11}
$$

We also observe that, since $K = K^\natural$, by Lemma 2.2, (3.8), and [2, Lemma 4.3], for almost every $\lambda > 0$ we have

$$
\begin{aligned}
\big|A \setminus \pi^{-1}\big(\mathcal{A}(\lambda)\big)\big| &= \big|A^\natural \setminus \pi^{-1}\big(\mathcal{A}^\natural(\lambda)\big)\big| \\
&\leq \big|K \setminus \pi^{-1}\big(\mathcal{K}(\lambda)\big)\big| + M\,\mathcal{H}^{n-1}\big(\mathcal{A}^\natural(\lambda)\Delta\mathcal{K}(\lambda)\big) \\
&\leq C\lambda^2 + M\,\mathcal{H}^{n-1}(\mathcal{A}^\natural(\lambda)\Delta\mathcal{K}(\lambda)),
\end{aligned} \tag{3.12}
$$

and analogously for B. Also, by (3.7),

$$
\int_0^M \Big(\mathcal{H}^{n-1}\big(\mathcal{A}^\natural(\lambda)\Delta\mathcal{K}(\lambda)\big) + \mathcal{H}^{n-1}\big(\mathcal{B}^\natural(\lambda)\Delta\mathcal{K}(\lambda)\big)\Big)\,d\lambda \leq |K \setminus A^\natural| + |K \setminus B^\natural| \leq C\delta^{\bar{\alpha}/n}. \tag{3.13}
$$

Define

$$
\eta := \min\left\{\frac{\bar{\alpha}}{2(n-1)}, \frac{\bar{\alpha}}{4}\right\}, \tag{3.14}
$$

and note that $\eta \leq \bar{\alpha}/n$. Let $\zeta \in (0,\eta)$ to be fixed later. Then by (3.9), (3.11), (3.12), (3.13), and by Chebyshev's inequality, we can find a level

$$
\bar{\lambda} \in [10\delta^\zeta, 20\delta^\zeta] \tag{3.15}
$$

such that

$$
\big|\mathcal{H}^{n-1}\big(\mathcal{A}(\bar{\lambda})\big) - \mathcal{H}^{n-1}\big(\mathcal{B}(\bar{\lambda})\big)\big| \leq C\delta^\eta. \tag{3.16}
$$

$$
2^{n-1}\mathcal{H}^{n-1}\big(\mathcal{S}(\bar{\lambda})\big) \leq \Big(\mathcal{H}^{n-1}\big(\mathcal{A}(\bar{\lambda})\big)^{1/(n-1)} + \mathcal{H}^{n-1}\big(\mathcal{B}(\bar{\lambda})\big)^{1/(n-1)}\Big)^{n-1} + C\delta^{\eta-\zeta}, \tag{3.17}
$$

$$
\big|A \setminus \pi^{-1}\big(\mathcal{A}(\bar{\lambda})\big)\big| + \big|B \setminus \pi^{-1}\big(\mathcal{B}(\bar{\lambda})\big)\big| \leq C\Big(\delta^{2\zeta} + \delta^{\eta-\zeta}\Big), \tag{3.18}
$$

In addition, from the properties $\mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big) \leq M$ for any $\lambda > 0$ (see (2.3)), $\int_0^M \mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big)\,d\lambda = |A| \geq 1-\delta$, and $s \mapsto \mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big)$ is a decreasing function, we deduce that

$$
\frac{1}{2M} \leq \mathcal{H}^{n-1}\big(\mathcal{A}(\lambda)\big) \leq M \qquad \forall\,\lambda \in \big(0,(2M)^{-1}\big).
$$

The same holds for B and S, hence

$$\mathcal{H}^{n-1}\big(\mathcal{S}(\bar\lambda)\big), \mathcal{H}^{n-1}\big(\mathcal{A}(\bar\lambda)\big), \mathcal{H}^{n-1}\big(\mathcal{B}(\bar\lambda)\big) \in \big[(2M)^{-1}, M\big]$$

provided δ is small enough. Set $\rho := 1/\mathcal{H}^{n-1}\big(\mathcal{A}(\bar\lambda)\big)^{1/(n-1)} \in [1/M^{1/(n-1)}, (2M)^{1/(n-1)}]$, and define

$$A' := \rho\mathcal{A}(\bar\lambda), \qquad B' := \rho\mathcal{B}(\bar\lambda), \qquad S' := \rho\mathcal{S}(\bar\lambda).$$

By (3.17) and (3.16) we get

$$\mathcal{H}^{n-1}(A') = 1, \quad \big|\mathcal{H}^{n-1}(B') - 1\big| \le C\delta^{\eta}, \quad \mathcal{H}^{n-1}(S') \le 1 + C\delta^{\eta-\zeta}.$$

while, by (1.2),

$$\mathcal{H}^{n-1}(S')^{1/(n-1)} \ge \frac{\mathcal{H}^{n-1}(A')^{1/(n-1)} + \mathcal{H}^{n-1}(B')^{1/(n-1)}}{2} \ge 1 - C\delta^{\eta},$$

therefore

$$\big|\mathcal{H}^{n-1}(A') - 1\big| + \big|\mathcal{H}^{n-1}(B') - 1\big| + \big|\mathcal{H}^{n-1}(S') - 1\big| \le C\delta^{\eta-\zeta}.$$

Thus, by the inductive hypothesis of Theorem 1.6, up to a translation there exists a $(n-1)$-dimensional convex set Ω' such that

$$\Omega' \supset A' \cup B', \qquad \mathcal{H}^{n-1}\big(\Omega' \setminus A'\big) + \mathcal{H}^{n-1}\big(\Omega' \setminus B'\big) \le C\delta^{(\eta-\zeta)\beta_{n-1}},$$

and defining $\Omega := \Omega'/\rho$ we obtain (recall that $1/\rho \le M^{1/(n-1)}$)

$$\Omega \supset \mathcal{A}(\bar\lambda) \cup \mathcal{B}(\bar\lambda), \qquad \mathcal{H}^{n-1}\big(\Omega \setminus \mathcal{A}(\bar\lambda)\big) + \mathcal{H}^{n-1}\big(\Omega \setminus \mathcal{B}(\bar\lambda)\big) \le C\delta^{(\eta-\zeta)\beta_{n-1}}. \tag{3.19}$$

Step 2: We apply Theorem 1.2 to the sets A_y and B_y for most $y \in \mathcal{A}(\bar\lambda) \cap \mathcal{B}(\bar\lambda)$.

Define $\mathcal{C} := \mathcal{A}(\bar\lambda) \cap \mathcal{B}(\bar\lambda) \subset \mathcal{S}(\bar\lambda)$. By (3.18), (3.19), and (2.3), we have

$$\begin{aligned} |A \setminus \pi^{-1}(\mathcal{C})| + |B \setminus \pi^{-1}(\mathcal{C})| &\le \big|A \setminus \pi^{-1}\big(\mathcal{A}(\bar\lambda)\big)\big| + \big|B \setminus \pi^{-1}\big(\mathcal{B}(\bar\lambda)\big)\big| \\ &\quad + \int_{(\mathcal{A}(\bar\lambda))\setminus(\mathcal{B}(\bar\lambda))} \mathcal{H}^1(A_y)\,dy + \int_{(\mathcal{B}(\bar\lambda))\setminus(\mathcal{A}(\bar\lambda))} \mathcal{H}^1(B_y)\,dy \\ &\le C\Big(\delta^{2\zeta} + \delta^{\eta-\zeta}\Big) + M\Big(\mathcal{H}^{n-1}\big(\Omega \setminus \mathcal{A}(\bar\lambda)\big) + \mathcal{H}^{n-1}\big(\Omega \setminus \mathcal{B}(\bar\lambda)\big)\Big) \\ &\le C\Big(\delta^{2\zeta} + \delta^{\eta-\zeta} + \delta^{(\eta-\zeta)\beta_{n-1}}\Big) \le C\delta^{2\zeta} \end{aligned} \tag{3.20}$$

provided we choose

$$\zeta := \frac{\eta\beta_{n-1}}{3} \tag{3.21}$$

(recall that $\beta_{n-1} \le 1$). Hence, by (1.5) and (3.20),

$$\begin{aligned} \int_{\mathcal{C}} \mathcal{H}^1\Big(S_y \setminus \frac{A_y + B_y}{2}\Big)\,dy &\le \int_{\mathcal{C}} \Big[\mathcal{H}^1(S_y) - \frac{1}{2}\big(\mathcal{H}^1(A_y) + \mathcal{H}^1(B_y)\big)\Big]\,dy \\ &= |S \cap \pi^{-1}(\mathcal{C})| - \frac{|A \cap \pi^{-1}(\mathcal{C})| + |B \cap \pi^{-1}(\mathcal{C})|}{2} \\ &\le |S| - \frac{|A| + |B|}{2} + \frac{|A \setminus \pi^{-1}(\mathcal{C})| + |B \setminus \pi^{-1}(\mathcal{C})|}{2} \\ &\le C\delta^{2\zeta}. \end{aligned} \tag{3.22}$$

Write $\mathcal{C}$ as $\mathcal{C}_1 \cup \mathcal{C}_2$, where

$$\mathcal{C}_1 := \big\{ y \in \mathcal{C} : 2\mathcal{H}^1(S_y) - \mathcal{H}^1(A_y) - \mathcal{H}^1(B_y) \leq \delta^\zeta/2 \big\}, \qquad \mathcal{C}_2 := \mathcal{C} \setminus \mathcal{C}_1.$$

By Chebyshev's inequality and (3.22),

$$\mathcal{H}^{n-1}\big(\mathcal{C}_2\big) \leq C\delta^\zeta, \tag{3.23}$$

while, recalling (3.15),

$$\min\big\{\mathcal{H}^1(A_y), \mathcal{H}^1(B_y)\big\} \geq \bar{\lambda} \geq 10\delta^\zeta > \delta^\zeta/2 \qquad \forall\, y \in \mathcal{C}_1.$$

Hence, by Theorem 1.2 applied to $A_y, B_y \subset \mathbb{R}$ for $y \in \mathcal{C}_1$, we deduce that

$$\mathcal{H}^1\big(\mathrm{co}(A_y) \setminus A_y\big) + \mathcal{H}^1\big(\mathrm{co}(B_y) \setminus B_y\big) \leq \delta^\zeta. \tag{3.24}$$

Let $\hat{\mathcal{C}}_1 \subset \mathcal{C}_1$ denote the set of $y \in \mathcal{C}_1$ such that

$$\mathcal{H}^1\Big(S_y \setminus \frac{A_y + B_y}{2}\Big) \leq \delta^\zeta, \tag{3.25}$$

and notice that, by (3.22) and Chebyshev's inequality, $\mathcal{H}^{n-1}(\mathcal{C}_1 \setminus \hat{\mathcal{C}}_1) \leq C\delta^\zeta$. Then choose a compact set $\mathcal{C}_1' \subset \hat{\mathcal{C}}_1$ such that $\mathcal{H}^{n-1}(\hat{\mathcal{C}}_1 \setminus \mathcal{C}_1') \leq \delta^\zeta$ to obtain

$$\mathcal{H}^{n-1}(\mathcal{C}_1 \setminus \mathcal{C}_1') \leq C\delta^\zeta. \tag{3.26}$$

Step 3: We find $\bar{S} \subset S$ so that $|S \setminus \bar{S}|$ and $\delta(\bar{S})$ are small.

Define the compact set

$$\bar{S} := \bigcup_{y \in \mathcal{C}_1'} \frac{A_y + B_y}{2} \subset \mathbb{R}^n.$$

Observe, thanks to (3.20), (3.23), (3.26), (2.3), and (1.5),

$$\begin{aligned}
2|\bar{S}| &\geq \int_{\mathcal{C}_1'} \mathcal{H}^1(A_y)\, dy + \int_{\mathcal{C}_1'} \mathcal{H}^1(B_y)\, dy \\
&\geq |A| + |B| - |A \setminus \pi^{-1}(\mathcal{C})| - |B \setminus \pi^{-1}(\mathcal{C})| - M\, \mathcal{H}^{n-1}(\mathcal{C} \setminus \mathcal{C}_1') \\
&\geq 2|S| - C\delta^\zeta.
\end{aligned}$$

So, since $\bar{S} \subset S$,

$$|S \Delta \bar{S}| \leq C\delta^\zeta. \tag{3.27}$$

Now, we want to estimate the measure of $\frac{1}{2}(\bar{S} + \bar{S})$. First of all, since

$$S_y = \bigcup_{2y = y' + y''} \frac{A_{y'} + B_{y''}}{2}, \tag{3.28}$$

by (3.25) we get

$$\mathcal{H}^1\bigg(\bigg(\bigcup_{2y=y'+y''}\frac{A_{y'}+B_{y''}}{2}\bigg)\setminus\frac{A_y+B_y}{2}\bigg)\le\delta^\zeta\qquad\forall\,y\in\mathcal{C}_1'. \tag{3.29}$$

Also, if we define the characteristic functions

$$\chi_y^A(\lambda):=\begin{cases}1 & \text{if } (y,\lambda)\in A_y\\ 0 & \text{otherwise,}\end{cases}\qquad \chi_y^{A,*}(\lambda):=\begin{cases}1 & \text{if } (y,\lambda)\in \mathrm{co}(A_y)\\ 0 & \text{otherwise,}\end{cases}$$

and analogously for B_y, by (3.24) we have the following estimate on their convolutions:

$$\begin{aligned}\|\chi_{y'}^{A,*}*\chi_{y''}^{B,*}-\chi_{y'}^{A}*\chi_{y''}^{B}\|_{L^\infty(\mathbb{R})}&\le\|\chi_{y''}^{B,*}-\chi_{y''}^{B}\|_{L^1(\mathbb{R})}+\|\chi_{y'}^{A,*}-\chi_{y'}^{A}\|_{L^1(\mathbb{R})}\\&=\mathcal{H}^1\big(\mathrm{co}(B_{y''})\setminus B_{y''}\big)+\mathcal{H}^1\big(\mathrm{co}(A_{y'})\setminus A_{y'}\big)\\&\le\delta^\zeta<3\delta^\zeta\qquad\forall\,y',y''\in\mathcal{C}_1'.\end{aligned}\tag{3.30}$$

Recalling that $\bar\pi:\mathbb{R}^n\to\mathbb{R}$ is the orthogonal projection onto the last component (that is, $\bar\pi(y,t)=t$), denote by $[a,b]$ the interval $\bar\pi\big(\mathrm{co}(A_{y'})+\mathrm{co}(B_{y''})\big)$, and notice that, since by construction

$$\min\{\mathcal{H}^1(A_y),\mathcal{H}^1(B_y)\}\ge\bar\lambda\ge10\delta^\zeta\qquad\forall\,y\in\mathcal{C}_1'$$

(see (3.15)), this interval has length greater than $20\delta^\zeta$. Also, it is easy to check that the function $\chi_{y'}^{A,*}*\chi_{y''}^{B,*}$ is supported on $[a,b]$, has slope equal to 1 (resp. -1) in $[a,a+3\delta^\zeta]$ (resp. $[b-3\delta^\zeta,b]$), and it is greater than $3\delta^\zeta$ in $[a+3\delta^\zeta,b-3\delta^\zeta]$. Hence, since $\bar\pi(A_{y'}+B_{y''})$ contains the set $\{\chi_{y'}^{A}*\chi_{y''}^{B}>0\}$, by (3.30) we deduce that

$$\bar\pi\big(A_{y'}+B_{y''}\big)\supset[a+3\delta^\zeta,b-3\delta^\zeta], \tag{3.31}$$

which implies in particular that

$$\mathcal{H}^1\big(\mathrm{co}(A_{y'})+\mathrm{co}(B_{y''})\big)\le\mathcal{H}^1\big(A_{y'}+B_{y''}\big)+6\delta^\zeta\qquad\forall\,y',y''\in\mathcal{C}_1'. \tag{3.32}$$

Also, by the same argument as in [8, Step 2-a], if we denote by

$$[\alpha_y,\beta_y]:=\bar\pi\big(\mathrm{co}(A_y)+\mathrm{co}(B_y)\big),$$

using (3.25) and (3.31) we have

$$\bar\pi\big(\mathrm{co}(A_{y'})+\mathrm{co}(B_{y''})\big)\subset[\alpha_y-16\delta^\zeta,\beta_y+16\delta^\zeta]\qquad\forall\,y',y'',y=\frac{y'+y''}{2}\in C_1'\,. \tag{3.33}$$

(Compare with [8, Equation (3.25)].)

We now estimate the size of $\big[\frac12(\bar S+\bar S)\big]_y$. Observe that, for all $y\in C_1'$,

$$\begin{aligned}\big[\tfrac12(\bar S+\bar S)\big]_y&=\bigcup_{2y=y'+y'',\,y',y''\in C_1'}\left(\frac{\frac12\big(A_{y'}+B_{y'}\big)+\frac12\big(A_{y''}+B_{y''}\big)}{2}\right)\\&=\bigcup_{2y=y'+y'',\,y',y''\in C_1'}\left(\frac{\frac12\big(A_{y'}+B_{y''}\big)+\frac12\big(A_{y''}+B_{y'}\big)}{2}\right)\\&\subset\frac12\bigg(\bigcup_{2y=y'+y'',\,y',y''\in C_1'}\tfrac12\big(A_{y'}+B_{y''}\big)+\bigcup_{2y=y'+y'',\,y',y''\in C_1'}\tfrac12\big(A_{y'}+B_{y''}\big)\bigg).\end{aligned}$$

Hence, by (3.33) we deduce that each of the latter sets is contained inside the convex set $\{y\} \times [\alpha_y - 16\delta^\zeta, \beta_y + 16\delta^\zeta]$, so also their semi-sum is contained in the same set, and using (3.32) with $y' = y'' = y$ we get

$$\begin{aligned}\mathcal{H}^1\big([(\bar S + \bar S)/2]_y\big) &\le \mathcal{H}^1\Big(\frac{\mathrm{co}(A_y) + \mathrm{co}(B_y)}{2}\Big) + 16\delta^\zeta \\ &\le \mathcal{H}^1\Big(\frac{A_y + B_y}{2}\Big) + 22\delta^\zeta \\ &= \mathcal{H}^1\big(\bar S_y\big) + 22\delta^\zeta \qquad \forall\, y \in C_1'.\end{aligned} \tag{3.34}$$

In order to estimate $\big[\frac12(\bar S + \bar S)\big]_y$ when $y \in \frac{C_1' + C_1'}{2} \setminus C_1'$ we argue as follows: by (3.33) and the fact that $\mathcal{H}^1\big(\mathrm{co}(A_y)\big)$ and $\mathcal{H}^1\big(\mathrm{co}(B_y)\big)$ are universally bounded (see (2.3) and (3.24)), the following holds: if we denote by $c^A(y)$ the barycenter of $\mathrm{co}(A_y)$ (and analogously for B and $\bar S$), we have

$$\big|c^A(y') + c^B(y'') - 2c^{\bar S}(y)\big| \le C \qquad \forall\, y, y', y'' \in \mathcal{C}_1',\ y = \frac{y' + y''}{2}$$

(notice that $\mathrm{co}(\bar S_y) = \mathrm{co}(A_y) + \mathrm{co}(B_y)$). Exchanging the role of A and B and adding up the two inequalities, we deduce that

$$\big|c^{\bar S}(y') + c^{\bar S}(y'') - 2c^{\bar S}(y)\big| \le C \qquad \forall\, y, y', y'' \in \mathcal{C}_1',\ y = \frac{y' + y''}{2}.$$

As shown in [8, Step 3], this estimate combined with the fact that $\mathcal{C}_1'$ is almost of full measure inside the convex set Ω (see (3.19), (3.23), and (3.26)) proves that, up to an affine transformation of the form

$$\mathbb{R}^{n-1} \times \mathbb{R} \ni (y, t) \mapsto (Ty, t - Ly) + (y_0, t_0) \tag{3.35}$$

with $T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, $L : \mathbb{R}^{n-1} \to \mathbb{R}$, $\det(T) = 1$, and $(y_0, t_0) \in \mathbb{R}^n$, the set $\bar S$ is universally bounded, say $\bar S \subset B_R$ for some dimensional constant R. This implies that $\big[\frac12(\bar S + \bar S)\big]_y \subset [-R, R]$, so $\mathcal{H}^1\big(\big[\frac12(\bar S + \bar S)\big]_y\big) \le 2R$.

Hence, since $\frac12(\mathcal{C}_1' + \mathcal{C}_1') \subset \Omega$, by (3.34), (3.19), and (3.21),

$$\begin{aligned}\Big|\frac{\bar S + \bar S}{2} \setminus \bar S\Big| &= \int_{[\frac12(\mathcal{C}_1' + \mathcal{C}_1')] \cap \mathcal{C}_1'} \mathcal{H}^1\big([(\bar S + \bar S)/2]_y\big) - \mathcal{H}^1\big(\bar S_y\big)\, dy \\ &\quad + \int_{[\frac12(\mathcal{C}_1' + \mathcal{C}_1')] \setminus \mathcal{C}_1'} \mathcal{H}^1\big([(\bar S + \bar S)/2]_y\big)\, dy \\ &\le 22\delta^\zeta\, \mathcal{H}^{n-1}(\Omega) + 2R\, \mathcal{H}^{n-1}(\Omega \setminus \mathcal{C}_1') \le C\delta^\zeta,\end{aligned}$$

that is,

$$\delta(\bar S) \le C\delta^\zeta.$$

Step 4: Conclusion.

By the previous step we have that $\delta(\bar S) \le C\delta^\zeta$. Hence, applying Theorem 1.4 to $\bar S$ we find a convex set $\bar{\mathcal{K}}$ such that

$$|\bar S \Delta \bar{\mathcal{K}}| \le C\delta^{n\alpha_n\zeta},$$

so, by (3.27),

$$|S\Delta\bar{\mathcal{K}}| \le C\delta^{n\alpha_n\zeta}.$$

Using this estimate together with Propositions 2.5 and 2.6 we deduce that, up to a translation, there exists a convex set $\mathcal{K}$ convex such that $A \cup B \subset \mathcal{K}$ and

$$|\mathcal{K}\setminus A| + |\mathcal{K}\setminus B| \le C\delta^{\alpha_n\zeta/4n}.$$

Recalling the definition of ζ (see (3.5), (3.14), (3.21)), we see that

$$\beta_n := \frac{\alpha_n\zeta}{4n} = \min\left\{\frac{1}{n-1}, \frac{1}{2}\right\}\frac{\alpha_n^2}{3\cdot 2^6 n}\,\beta_{n-1}.$$

Since $\beta_1 = 1$ (by Theorem 1.2), it is easy to check that

$$\beta_n = \frac{1}{2^{6n-5}3^{n-1}n!(n-1)!}\prod_{k=1}^{n}\alpha_k^2 \qquad \forall\, n \ge 2,$$

concluding the proof.

4 Technical results

As in the previous section, we use C to denote a generic constant depending only on the dimension, which may change from line to line.

4.1 Proof of Proposition 2.5

Assume that

$$|S\Delta K| \le C\delta^{\alpha}$$

for some $\alpha \in (0,1]$. By John's Lemma [16], after a volume preserving affine transformation, we can assume that $B_{r_n} \subset K \subset B_{nr_n}$, with $r_n = r_n(K) > 0$ bounded above and below by positive dimensional constants. Note, however, that with this normalization, we will not be able to assume that A and B are M-normalized, since we have already chosen a different affine normalization.

We want to prove that

$$S \subset (1 + C\delta^{\alpha/2n})K. \tag{4.1}$$

Let $\bar{x}_0 \in S\setminus K$, and set $\rho := \mathrm{dist}(\bar{x}_0, K) = |\bar{x}_0 - \bar{x}_1|$ with $\bar{x}_1 \in K$. With no loss of generality we can assume that $\bar{x}_1 = \tau e_n$, for some $\tau > 0$, $\bar{x}_0 = (\tau+\rho)e_n$, and $K \subset \{x_n \le \tau\}$. We need to prove that $\rho \le C\delta^{\alpha/2n}$.

Let us consider the sets A^*, B^*, S^*, K^* obtained from A, B, S, K performing a Schwarz symmetrization around the e_n-axis (see Definition 2.1). Set $S' := \frac{1}{2}(A^* + B^*)$. Since

$$|S^*\Delta K^*| \le |S\Delta K| \le C\delta^{\alpha},$$

and, by (1.5) (notice that $S' \subset S^*$ and that $|S'| \ge 1 - C\delta$ by (1.2)),

$$|S^*\setminus S'| = |S^*| - |S'| = |S| - |S'| \le C\delta,$$

we get that $|S'\Delta K^*| \le C\delta^\alpha$. In addition, $K^* \subset \{x_n \le \tau\}$, $\bar{x}_1 \in K^*$, and $\bar{x}_0 \in S^*$. Hence, without loss of generality we can assume from the beginning that $A = A^*$, $B = B^*$, $S = \frac{1}{2}(A^* + B^*)$, and $K = K^*$.

For a compact set $E \subset \mathbb{R}^n$, recall the notation $E(t) \subset \mathbb{R}^{n-1} \times \{t\}$ in (2.1), and define $E[s] \subset \mathbb{R}$ by

$$E[s] := \big\{t : \mathcal{H}^{n-1}\big(E(t)\big) \ge s\big\} \tag{4.2}$$

Since $S = \frac{1}{2}(A + B)$ we have

$$\frac{A(t) + B(t)}{2} \subset S(t) \qquad \forall\, t \in \mathbb{R},$$

so, by (1.2) we deduce that

$$S[s] \supset \frac{A[s] + B[s]}{2} \qquad \forall\, s > 0.$$

Hence

$$\mathcal{H}^1\big(A[s]\big) + \mathcal{H}^1\big(B[s]\big) \le 2\mathcal{H}^1\big(S[s]\big) \qquad \forall\, s > 0, \tag{4.3}$$

and integrating with respect to s, by (1.5) we get

$$4\delta \ge 2|S| - |A| - |B| = \int_0^\infty \Big(2\mathcal{H}^1\big(S[s]\big) - \mathcal{H}^1\big(A[s]\big) - \mathcal{H}^1\big(B[s]\big)\Big)\, ds. \tag{4.4}$$

Recall that $K = K^*$, so that the canonical projection $\pi(K)$ onto $\mathbb{R}^{n-1}$ is a ball. We denote it $B_R := \pi(K)$, and note that $R \le n r_n$, with $r_n = r_n(K)$ given by John's lemma at the beginning of this proof. Then, since $|S\Delta K| \le C\delta^\alpha$ we have

$$C\delta^\alpha \ge |S \setminus \pi^{-1}(B_R)| = \int_{\mathcal{H}^{n-1}(B_R)}^\infty \mathcal{H}^1\big(S[s]\big)\, ds,$$

so, by (4.3),

$$|A \setminus \pi^{-1}(B_R)| + |B \setminus \pi^{-1}(B_R)| = \int_{\mathcal{H}^{n-1}(B_R)}^\infty \Big(\mathcal{H}^1\big(A[s]\big) + \mathcal{H}^1\big(B[s]\big)\Big)\, ds \le C\delta^\alpha. \tag{4.5}$$

Hence, recalling that $|A|$ and $|B|$ are $\ge 1 - \delta$, we deduce that

$$\int_0^{\mathcal{H}^{n-1}(B_R)} \mathcal{H}^1\big(A[s]\big)\, ds \ge 1/2, \qquad \int_0^{\mathcal{H}^{n-1}(B_R)} \mathcal{H}^1\big(B[s]\big)\, ds \ge 1/2,$$

and since R is universally bounded (being less than nr_n) and both functions

$$s \mapsto \mathcal{H}^1\big(A[s]\big), \qquad s \mapsto \mathcal{H}^1\big(B[s]\big)$$

are decreasing, there exists a small dimensional constant $c' > 0$ such that

$$\min\big\{\mathcal{H}^1\big(A[s]\big), \mathcal{H}^1\big(B[s]\big)\big\} \ge c' \qquad \forall\, s \in (0, c'). \tag{4.6}$$

Also, by (4.4),

$$\int_0^{c'} \Big(2\mathcal{H}^1\big(S[s]\big) - \mathcal{H}^1\big(A[s]\big) - \mathcal{H}^1\big(B[s]\big)\Big)\, ds \le 4\delta, \tag{4.7}$$

and since $|S\Delta K| \le C\delta^\alpha$ and $K \subset \{x_n \le \tau\}$

$$\int_0^{c'} \mathcal{H}^1\big(S[s] \setminus (-\infty, \tau]\big)\, ds \le |S \setminus \{x_n \le \tau\}| \le C\delta^\alpha. \tag{4.8}$$

Hence, thanks to (4.6), (4.7), (4.8), we use Theorem 1.2 and Chebishev's inequality to find a value

$$\bar{s} \in [\delta^{\alpha/2}, 2\delta^{\alpha/2}] \tag{4.9}$$

such that

$$\mathcal{H}^1\big(\mathrm{co}\big(A[\bar{s}]\big) \setminus A[\bar{s}]\big) + \mathcal{H}^1\big(\mathrm{co}\big(B[\bar{s}]\big) \setminus B[\bar{s}]\big) \le C\delta^{1-\alpha/2} \le C\delta^{\alpha/2}$$

(notice that $\alpha \le 1$) and

$$\mathcal{H}^1\big(S[\bar{s}] \setminus (-\infty, \tau]\big) \le C\delta^{\alpha/2}.$$

Since $\frac{1}{2}(A[\bar{s}] + B[\bar{s}]) \subset S[\bar{s}]$, this implies

$$\frac{\mathrm{co}\big(A[\bar{s}]\big) + \mathrm{co}\big(B[\bar{s}]\big)}{2} \subset (-\infty, \tau + C\delta^{\alpha/2}].$$

Hence, after applying opposite translations along the e_n-axis to A and B, i.e.,

$$A \mapsto A + \ell e_n, \qquad B \mapsto B - \ell e_n,$$

for some $\ell \in \mathbb{R}$, we can assume that

$$\mathrm{co}\big(A[\bar{s}]\big) \subset (-\infty, \tau + C\delta^{\alpha/2}], \qquad \mathrm{co}\big(B[\bar{s}]\big) \subset (-\infty, \tau + C\delta^{\alpha/2}].$$

Since the sets $s \mapsto A[s],\ B[s]$ are decreasing, we deduce that

$$\mathrm{co}\big(A[s]\big), \mathrm{co}\big(B[s]\big) \subset (-\infty, \tau + C\delta^{\alpha/2}], \qquad \forall\, s \ge \bar{s}. \tag{4.10}$$

We now want to bound $\sup_{s>0} \mathcal{H}^1\big(A[s]\big)$. (Recall that we cannot assume that A and B are M-normalized, since we already made an affine transformation to ensure that $B_{r_n} \subset K \subset B_{nr_n}$.) Since $A = A^*$, we have $\sup_{s>0} \mathcal{H}^1\big(A[s]\big) = \sup_{y \in \mathbb{R}^{n-1}} \mathcal{H}^1(A_y)$, so, by Lemma 2.3,

$$\sup_{s>0} \mathcal{H}^1\big(A[s]\big) \le \frac{M}{\mathcal{H}^{n-1}\big(\pi(B)\big)}, \qquad \frac{\mathcal{H}^{n-1}\big(\pi(A)\big)}{\mathcal{H}^{n-1}\big(\pi(B)\big)} \in (M^{-1}, M). \tag{4.11}$$

In addition, since $\pi(A)$ and $\pi(B)$ are $(n-1)$-dimensional disks centered on the e_n-axis, $|S\Delta K| \le C\delta^\alpha$, and $B_{r_n} \subset K \subset B_{nr_n}$, we easily deduce that

$$\frac{\pi(A) + \pi(B)}{2} = \pi(S) \supset B_{r_n/2}, \tag{4.12}$$

provided δ is small enough. Hence, combining (4.11) and (4.12) we deduce that $\mathcal{H}^{n-1}\big(\pi(B)\big)$ is bounded from away from zero by a dimensional constant, thus

$$\sup_{s>0} \mathcal{H}^1\big(A[s]\big) \le C. \tag{4.13}$$

Hence, by (4.5), (4.10), (4.13), and (4.9),

$$\begin{aligned}|A\setminus\{x_n\le\tau\}| &\le |A\setminus\pi^{-1}(B_R)|+|\pi^{-1}(B_R)\cap\{\tau\le x_n\le\tau+C\delta^{\alpha/2}\}|+\int_0^{\bar s}\mathcal{H}^1\big(A[s]\big)\,ds\\ &\le C\delta^{\alpha}+C\delta^{\alpha/2}+C\bar s\le C\delta^{\alpha/2},\end{aligned}\tag{4.14}$$

and, analogously,

$$|B\setminus\{x_n\le\tau\}|\le C\delta^{\alpha/2}.\tag{4.15}$$

Now, given $r\ge 0$, let us define the sets

$$A'_r:=A\cap\{x_n\le\tau-r\},\qquad B'_r:=B\cap\{x_n\le\tau-r\},\qquad S'_r:=S\cap\{x_n\le\tau-r\}.$$

By (4.14) and (4.15) we know that

$$|A'_0|,|B'_0|\ge 1-C\delta^{\alpha/2},$$

and it is immediate to check that

$$\frac{A'_0+B'_r}{2}\subset S'_{r/2},\qquad \frac{A'_r+B'_0}{2}\subset S'_{r/2}.$$

Also, since K is a convex set satisfying $B_{r_n}\subset K\subset B_{nr_n}$, there exists a dimensional constant $c_n>0$ such that

$$|K\cap\{\tau-r/2\le x_n\le\tau\}|\ge c_n\min\big\{r^n,1\big\}.$$

Hence

$$\begin{aligned}|S'_{r/2}| &\le |S|-|S\cap\{\tau-r/2\le x_n\le\tau\}|\\ &\le |S|+|S\Delta K|-|K\cap\{\tau-r/2\le x_n\le\tau\}|\\ &\le 1+C\delta^{\alpha}-c_n\min\big\{r^n,1\big\},\end{aligned}$$

and by (1.2) applied to A'_r and B'_0 we get

$$\begin{aligned}1-C\delta^{\alpha/2}-C|A\cap\{\tau-r\le x_n\le\tau\}| &\le \frac{|A'_r|^{1/n}+|B'_0|^{1/n}}{2}\le|S'_{r/2}|^{1/n}\\ &\le 1+C\delta^{\alpha}-c_n\min\big\{r^n,1\big\},\end{aligned}$$

which gives

$$C|A\cap\{\tau-r\le x_n\le\tau\}|\ge c_n\min\big\{r^n,1\big\}-C\delta^{\alpha/2}.\tag{4.16}$$

(and analogously for B)

Since the point $\bar x_0=(\tau+\rho)e_n$ belongs to $S=(A+B)/2$, there as to be a point $\bar x\in A\cup B$ such that $\bar x\cdot e_n\ge(\tau+\rho)$. With no loss of generality, assume that $\bar x\in B$. Then, by (4.16) applied with $r=\rho$ we get

$$S\cap\{x_n\ge\tau\}\supset\frac{\bar x+\big(A\cap\{\tau-\rho\le x_n\le\tau\}\big)}{2},$$

so

$$C\delta^{\alpha}\ge|S\cap\{x_n\ge\tau\}|\ge\frac{|A\cap\{\tau-\rho\le x_n\le\tau\}|}{2^n}\ge\frac{c_n}{C}\min\big\{\rho^n,1\big\}-C\delta^{\alpha/2},$$

which implies $\rho\le C\delta^{\alpha/2n}$, proving (4.1).

Hence $\mathrm{co}(S)\subset(1+C\delta^{\alpha/2n})K$, from which the result follows immediately.

4.2 Proof of Proposition 2.6

Since

$$\frac{\mathrm{co}(A)+\mathrm{co}(B)}{2} = \mathrm{co}(S),$$

by (1.2), (2.4), and (1.5) we have

$$\begin{aligned} |\,\mathrm{co}(A)|^{1/n} + |\,\mathrm{co}(B)|^{1/n} &\le |\,\mathrm{co}(A)+\mathrm{co}(B)|^{1/n} \\ &= 2|\,\mathrm{co}(S)|^{1/n} \le 2|S|^{1/n} + C\delta^\beta \\ &\le |A|^{1/n} + |B|^{1/n} + C\delta^\beta \\ &\le |\,\mathrm{co}(A)|^{1/n} + |\,\mathrm{co}(B)|^{1/n} + C\delta^\beta, \end{aligned}$$

from which we deduce that

$$|\,\mathrm{co}(A)\setminus A| + |\,\mathrm{co}(B)\setminus B| \le C\delta^\beta. \tag{4.17}$$

Also, by Theorem 1.3 and the fact that $\big||\,\mathrm{co}(A)| - |\,\mathrm{co}(B)|\big| \le C\delta^{\beta\alpha_n}$ (see (4.17)) we obtain that, up to a translation,

$$|\,\mathrm{co}(A)\Delta\,\mathrm{co}(B)| \le C\left(\delta^{\beta/2} + \delta^\beta\right) \le C\delta^{\beta/2}. \tag{4.18}$$

This estimate combined with (4.17) implies that

$$|A\Delta B| \le C\delta^{\beta/2}.$$

In addition, if we define $\mathcal{K} := \mathrm{co}(A\cup B)$, then we will conclude our argument by showing that

$$|\mathcal{K}\setminus A| + |\mathcal{K}\setminus B| \le C\delta^{\beta/2n}. \tag{4.19}$$

Indeed, by John's Lemma [16], after a volume preserving affine transformation we can assume that $B_r \subset \mathrm{co}(A) \subset B_{nr}$ for some radius r bounded above and below by positive dimensional constants. By (4.18) and a simple geometric argument we easily deduce that

$$\mathrm{co}(B) \subset \left(1 + C\delta^{\beta/2n}\right)\mathrm{co}(A).$$

Thus

$$\mathrm{co}(A)\cup\mathrm{co}(B) \subset \mathcal{K} \subset \left(1 + C\delta^{\beta/2n}\right)\mathrm{co}(A),$$

and (4.19) follows by (4.17) and (4.18).

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