REGULARITY OF CODIMENSION-1 MINIMIZING CURRENTS UNDER MINIMAL ASSUMPTIONS ON THE INTEGRAND

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ABSTRACT. In this paper we investigate the regularity theory of codimension-1 integer rectifiable currents that (almost)-minimize parametric elliptic functionals. While in the non-parametric case it follows by De Giorgi-Nash Theorem that $C^{1,1}$ regularity of the integrand is enough to prove $C^{1,\alpha}$ regularity of minimizers, the present regularity theory for parametric functionals assume the integrand to be at least of class C^2. In this paper we fill this gap by proving that $C^{1,1}$ regularity is enough to show that flat almost-minimizing currents are $C^{1,\alpha}$. As a corollary, we also show that the singular set has codimension greater than 2.

Besides the result "per se", of particular interest we believe to be the approach used here: instead of showing that the standard excess function decays geometrically around every point, we construct a new excess with respect to graphs minimizing the non-parametric functional and we prove that if this excess is sufficiently small at some radius R, then it is identically zero at scale $R/2$. This implies that our current coincides with a minimizing graph there, hence it is of class $C^{1,\alpha}$.

1. Introduction

The regularity theory of (almost)-minimizing currents is a classical topic in geometric measure theory that has always received a lot of attention (see for instance [8, 16, 4, 2, 5, 19, 7, 18, 6, 10] and the references therein). The aim of this paper is to focus on the case of codimension-1 integer rectifiable currents that almost minimize parametric elliptic functionals.

The starting point of our investigation is the following observation: let us consider a n-dimensional rectifiable current $T = \overrightarrow{T}\,\|T\|$ in $\mathbb{R}^{n+1}$ that minimizes a variational integral $\mathbb{F}$ of the form

$$\mathbb{F}(T) := \int \mathcal{F}\big(\overrightarrow{T}\big)\, d\|T\|. \tag{1.1}$$

Assuming that $\mathcal{F}$ is elliptic and of class C^2, it is well-known that if T is sufficiently flat inside a ball then T is a $C^{1,\alpha}$ hypersurface inside a smaller ball (see for instance [7]). However, if we assume a priori that T is the graph of Lipschitz some function u, then the functional $\mathbb{F}$ takes the form

$$\int F(\nabla u)\, dx$$

for some (locally) uniformly convex function $F : \mathbb{R}^n \to \mathbb{R}$, and by De Giorgi-Nash Theorem it is enough to assume that F is of class $C^{1,1}$ to obtain that $u \in C^{1,\alpha}$.

The gap between these two results comes from the fact that, when showing regularity of currents, the strategy of the proof goes as follow: exploiting the fact that T is sufficiently flat inside a ball, one first shows that a large part of T can be covered by a Lipschitz graph, and then one proves that this Lipschitz graph almost solves the linearized version of the Euler-Lagrange equation for T.

To explain this better, let us think that T is already the graph of a function u. Then, by minimality, u solves the Euler-Lagrange equation

$$F_{p_ip_j}(\nabla u)\partial_{ij}u = 0,$$

and since u is small in L^∞ one expects that ∇u must also be small in a large fraction of points. Hence $F_{p_ip_j}(\nabla u) \approx F_{p_ip_j}(0)$ in a lot of points, and one would like to say that there is a smooth function v, very close to u, that solves the linear equation

$$F_{p_ip_j}(0)\partial_{ij}v = 0.$$

However, for this last step it is crucial the second derivatives of F to be at least continuous, which explains the original assumption $\mathcal{F} \in C^2$.

It is worth pointing out that this C^2 assumption also appear in several PDE results where one wants to find a "good" limiting equation and use it to get regularity for the original function u (see for instance [1, 3, 17, 11]). To our knowledge, the paper [12] is the only one in this setting where the C^2 assumption is weakened to $C^{1,1}$, but the strategy there crucially uses the fact that one is working with graphs and, in addition, it heavily depends on some specific structural properties of the integrand.

In this paper we finally fill the gap between the parametric and non-parametric setting, proving that $C^{1,1}$ regularity of $\mathcal{F}$ is enough to obtain regularity results for integral currents. Actually, as we shall see, it is enough to assume that $\mathcal{F}$ is $C^{1,1}$ and elliptic only at directions that are close to the "vertical one". This simple observation makes our result flexible enough to be used in situations when ellipticity and regularity of $\mathcal{F}$ are available only in some directions. Also, in this respect, our result can be seen as the natural generalization to currents of Savin's regularity result for flat solutions to fully non-linear equations [17], under minimal assumptions on $\mathcal{F}$.

We shall also see how a minor variant of our argument provides the same regularity result when the currents are only almost-minimizing, and as a corollary we deduce that if $\mathcal{F}$ is $C^{1,1}$ and elliptic in every direction then almost-minimizing currents are regular hypersurfaces outside a singular set of codimension $2+\nu$, for some universal $\nu > 0$.

The paper is structured as follows: in the next section we introduce the notation needed to state our results, and show how our main Theorems 2.1-2.2 implies the estimate on the size of the singular set. Then in Section 3 we collect some preliminary results that are used in Section 4 to prove Theorem 2.1. Finally, in Section 5 we show how our approach can be extended to almost-minimizing currents.

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2. Notation and statement of the results

For the basic definition and properties of integral currents, we refer to the seminal paper [15] and the book [13, Sections 4.1.1-4.1.9, 4.1.24-4.1.28, 4.1.30].

We set the following list of notation:

- points in $\mathbb{R}^{n+1}$ will be denoted by $z=(x,y)$, with $x\in\mathbb{R}^n$ and $y\in\mathbb{R}$.
- $e_1,\dots,e_n,e_{n+1}$ denote the canonical basis of $\mathbb{R}^{n+1}$, and $dx^1,\dots,dx^n,dy$ the dual basis.
- $\Lambda_n(\mathbb{R}^{n+1})$ is the space of n-vectors in $\mathbb{R}^{n+1}$, $\Lambda_n^*(\mathbb{R}^{n+1})$ the space of n-covectors in $\mathbb{R}^{n+1}$, and we set

$$\boldsymbol{e}_0 := e_1\wedge\ldots\wedge e_n \in \Lambda_n(\mathbb{R}^{n+1}),\quad d\boldsymbol{x}^n := dx^1\wedge\ldots\wedge dx^n,$$

$$\hat{\boldsymbol{e}}_i := e_{n+1}\wedge e_1\wedge\ldots\wedge e_{i-1}\wedge e_{i+1}\wedge\ldots\wedge e_n,\quad \forall\, i\in\{1,\dots,n\}.$$

- $\mathcal{C}_R$ denotes the vertical cylinder around the origin of radius R, that is

$$\mathcal{C}_R := B_R\times\mathbb{R},\qquad B_R=B_R(0)\subset\mathbb{R}^n.$$

- $\boldsymbol{p}:\mathbb{R}^{n+1}\to\mathbb{R}^n$ denotes the canonical projection.
- Given a function $h:\mathbb{R}^n\to\mathbb{R}$, we denote by T_h the n-current associated to its graph, that is

$$\langle T_h,\psi\rangle := \int_{\mathbb{R}^n}\langle\psi(H(x)),\wedge_n\nabla H(x)\rangle\,dx,\qquad \forall\,\psi\in C_c^\infty\big(\mathbb{R}^{n+1};\Lambda_n^*(\mathbb{R}^{n+1})\big),$$

where

$$H(x):=(x,h(x))\in\mathbb{R}^{n+1},\qquad \wedge_n\nabla H(x) := \frac{\partial H}{\partial x^1}(x)\wedge\ldots\wedge\frac{\partial H}{\partial x^n}(x)\in\Lambda_n(\mathbb{R}^{n+1}).$$

Notice that

$$\wedge_n\nabla H(x) = \boldsymbol{e}_0 + \frac{\partial h}{\partial x^i}(x)\,\hat{\boldsymbol{e}}_i + O\big(|\nabla h(x)|^2\big). \tag{2.1}$$

- Given a Borel set $A\subset\mathbb{R}^n$, $[\![A]\!]$ denotes the n-current in $\mathbb{R}^n$ obtained by integration over A, that is

$$\langle [\![A]\!],\phi\,d\boldsymbol{x}^n\rangle := \int_A\phi(x)\,dx\qquad \forall\,\phi\in C_c^\infty(\mathbb{R}^n).$$

We consider an integrand $\mathcal{F}:\Lambda_n(\mathbb{R}^{n+1})\to\mathbb{R}$, of class C^1 outside the origin, satisfying the following properties: there are positive constants A_0, ϱ_0, such that[1]

$$\mathcal{F}(\lambda\boldsymbol{\xi}) = \lambda\,\mathcal{F}(\boldsymbol{\xi})\qquad \forall\,\lambda\ge 0,\ \boldsymbol{\xi}\in\Lambda_n(\mathbb{R}^{n+1}); \tag{2.2}$$

$$\mathcal{F}(\boldsymbol{\xi})\ge 1\qquad \forall\,\boldsymbol{\xi}\in\Lambda_n(\mathbb{R}^{n+1}),\ |\boldsymbol{\xi}|=1; \tag{2.3}$$

$$\mathcal{F}(\boldsymbol{\eta}) - \langle d\mathcal{F}(\boldsymbol{\xi}),\boldsymbol{\eta}\rangle \ge 0\qquad \forall\,\boldsymbol{\xi},\boldsymbol{\eta}\in\Lambda_n(\mathbb{R}^{n+1}),\ |\boldsymbol{\xi}|=|\boldsymbol{\eta}|=1; \tag{2.4}$$

$$\mathcal{F}(\boldsymbol{\eta}) - \langle d\mathcal{F}(\boldsymbol{\xi}),\boldsymbol{\eta}\rangle \ge \frac{1}{2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2\qquad \forall\,\boldsymbol{\xi},\boldsymbol{\eta}\in\Lambda_n(\mathbb{R}^{n+1}),\ |\boldsymbol{\xi}|=|\boldsymbol{\eta}|=1,\ |\boldsymbol{\xi}-\boldsymbol{e}_0|\le\varrho_0; \tag{2.5}$$

[1] Notice that, by differentiating the homogeneity condition (2.2) with respect to λ and setting $\lambda=1$, we get

$$\mathcal{F}(\boldsymbol{\xi}) = \langle d\mathcal{F}(\boldsymbol{\xi}),\boldsymbol{\xi}\rangle\qquad \forall\,\boldsymbol{\xi}\in\Lambda_n(\mathbb{R}^{n+1}).$$

Hence (2.4) corresponds to the convexity of $\mathcal{F}$, while (2.5) is a uniform convexity assumption for $\mathcal{F}$ at n-vectors $\boldsymbol{\xi}$ sufficiently close to $\boldsymbol{e}_0$. Also, (2.7) asserts that $\mathcal{F}$ is $C^{1,1}$ only at directions sufficiently close to $\boldsymbol{e}_0$.

$$\sup_{|\boldsymbol{\xi}|=1}\Big(\mathcal{F}(\boldsymbol{\xi})+|d\mathcal{F}(\boldsymbol{\xi})|\Big)\le A_0; \tag{2.6}$$

$$\mathcal{F}(\boldsymbol{\eta})-\langle d\mathcal{F}(\boldsymbol{\xi}),\boldsymbol{\eta}\rangle\le A_0|\boldsymbol{\xi}-\boldsymbol{\eta}|^2 \qquad \forall\,\boldsymbol{\xi},\boldsymbol{\eta}\in\Lambda_n(\mathbb{R}^{n+1}),\ |\boldsymbol{\xi}|=|\boldsymbol{\eta}|=1,\ |\boldsymbol{\xi}-\boldsymbol{e}_0|\le\varrho_0. \tag{2.7}$$

Given a function $\mathcal{F}$ as above, we can define a functional on n-dimensional rectifiable currents $\mathbb{F}$ as in (1.1). Notice that when $\mathcal{F}(\boldsymbol{\xi})=|\boldsymbol{\xi}|$ the functional $\mathbb{F}(T)$ coincides with the mass of the current $\mathbb{M}(T)$.

In the whole paper we shall use the notation $a\lesssim b$ whenever $a\le C\,b$ for some constant C depending only on $\mathcal{F}$ and the dimension (such a constant will be called *universal*).

We say that a rectifiable current T is $\mathcal{F}$-minimal inside an open set $\mathcal{O}\subset\mathbb{R}^{n+1}$ if

$$\mathbb{F}(T)\le\mathbb{F}(T+X)$$

for all rectifiable n-currents X with $\partial X=\emptyset$ and $\operatorname{supp}(X)\subset\mathcal{O}$.

To state our main result we shall introduce a notion of excess with respect to graphs. More precisely, given $r>0$ and a C^1 function $u:B_r\to\mathbb{R}$, we define

$$\mathcal{E}(T,r,u):=\frac{1}{r^n}\int_{\mathcal{C}_r}\big|\vec{T}-\vec{U}\big|^2\,d\|T\|,$$

where

$$\vec{U}(z):=\frac{\wedge_n\nabla U(\boldsymbol{p}(z))}{|\wedge_n\nabla U(\boldsymbol{p}(z))|}\qquad\text{for } z\in\mathcal{C}_r,\quad U(x):=\big(x,u(x)\big). \tag{2.8}$$

Theorem 2.1. *Let $\mathcal{F}:\Lambda_n(\mathbb{R}^{n+1})\to\mathbb{R}$ satisfy* (2.2)-(2.7)*, and let T to be a n-dimensional integer rectifiable current in $\mathcal{C}_R$ such that:*

(H1) $0\in\operatorname{supp}(T)$, $\operatorname{supp}(T)\subset\overline{\mathcal{C}}_R$ is compact, $\operatorname{supp}(\partial T)\subset\partial\mathcal{C}_R$;
(H2) T is $\mathcal{F}$-minimal in $\mathcal{C}_R$;
(H3) $\boldsymbol{p}_\#(T\lfloor_{\mathcal{C}_R})=[\![B_R]\!]$.

There exist $\delta>0$ and a function $u:B_{R/2}\to\mathbb{R}$ of class $C^{1,\delta}$ such that if

$$\mathcal{E}(T,R,0)\le\varepsilon_0$$

then

$$\mathcal{E}(T,R/2,u)=0,$$

that is, T coincides inside $\mathcal{C}_{R/2}$ with the n-current associated to the graph of u.

In order to estimate the singular sets of minimizing currents, it will actually be useful to extend the theorem above to almost-minimizing currents. Of course we could have proved Theorem 2.1 above directly in the context of almost-minimizing currents. However, as we already mentioned in the abstract, we believe that the proof of Theorem 2.1 has its own interest, and this is why we have decided to present our results in this way.

Given a non-decreasing function $\omega:\mathbb{R}^+\to\mathbb{R}^+$ with $\omega(0^+)=0$, a rectifiable current T is $(\mathcal{F},\omega)$-minimal inside an open set $\mathcal{O}\subset\mathbb{R}^{n+1}$ if

$$\mathbb{F}(T)\le\mathbb{F}(T+X)+\omega(r)\mathbb{M}(X)$$

for all rectifiable n-currents X with $\partial X = \emptyset$ such that $\mathrm{supp}(X) \subset \mathcal{O}$ and $\mathrm{diam}(\mathrm{supp}(X)) \leq r$.

The following result extends our main Theorem 2.1 to the case of almost-minimizing currents:

Theorem 2.2. *Let $\mathcal{F} : \Lambda_n(\mathbb{R}^{n+1}) \to \mathbb{R}$ satisfy (2.2)-(2.7), and let T to be a n-dimensional integer rectifiable current in $\mathcal{C}_R$ such that:*

- *(H1) $0 \in \mathrm{supp}(T)$, $\mathrm{supp}(T) \subset \overline{\mathcal{C}}_R$ is compact, $\mathrm{supp}(\partial T) \subset \partial \mathcal{C}_R$;*
- *(H2) T is $(\mathcal{F}, \omega)$-minimal in $\mathcal{C}_R$;*
- *(H3) $\boldsymbol{p}_\#(T \lfloor_{\mathcal{C}_R}) = [\![B_R]\!]$.*

Also, set $\Omega(R) := \int_0^R \frac{\sqrt{\omega(r)}}{r}\, dr$ and assume that $\Omega(1) < \infty$.

There exist $r \in (0, 1/8)$, and $\varepsilon_0 = \varepsilon_0(r) > 0$ such that, if

$$R + \mathcal{E}(T, R, 0) \leq \varepsilon_0,$$

then T coincides inside $\mathcal{C}_{R/2}$ with the n-current associated to the graph of a C^1 function $f : B_{R/2} \to \mathbb{R}$. In addition $\|f\|_{C^1(B_{R/2})} \to 0$ as $\varepsilon_0 \to 0$.

Remark 2.3. The modulus of continuity of ∇f in the above theorem depends on ω via (5.1). In particular, if ω decays geometrically (that is, $\omega(R) \lesssim R^\beta$ for some $\beta > 0$) then ∇f is Hölder continuous in $B_{R/2}$.

Remark 2.4. Theorem 2.2 applies also to integrands depending on the variable $z \in \mathbb{R}^{n+1}$, that is $\mathcal{F} = \mathcal{F}(z, \boldsymbol{\xi})$. Indeed, assuming

$$\sup_{|\boldsymbol{\xi}|=1} \|\mathcal{F}(z, \boldsymbol{\xi}) - \mathcal{F}(z', \boldsymbol{\xi})\| \leq \hat{\omega}(|z - z'|) \qquad \forall\, z, z' \in \mathbb{R}^{n+1},$$

any $(\mathcal{F}, \omega)$-minimal current T is $(\mathcal{F}_0, \omega + C\,\hat{\omega})$-minimal for $\mathcal{F}_0(\boldsymbol{\xi}) := \mathcal{F}(0, \boldsymbol{\xi})$ and some universal constant $C > 1$, hence our main theorem applies provided $\int_0^1 \frac{\sqrt{\hat{\omega}(r)}}{r}\, dr < \infty$.

As mentioned before, Theorem 2.2 allows us to get a bound for the singular sets of minimizing currents provided the $C^{1,1}$ and uniformly convexity assumptions on $\mathcal{F}$ are true at every point. Notice that this corresponds to take $\varrho_0 = 2$ in (2.5) and (2.7), as the inequality $|\boldsymbol{\xi} - \boldsymbol{e}_0| \leq 2$ is always satisfied for $|\boldsymbol{\xi}| = 1$.

To state the corollary we need to introduce the singular set: we first define the unoriented spherical excess

$$\mathscr{E}(T, z, r) := \inf_{|\boldsymbol{\xi}|=1} \frac{1}{r^n} \int_{\mathscr{B}_r(z)} \big|\vec{T} - \boldsymbol{\xi}\big|^2\, d\|T\|,$$

where $\mathscr{B}_r(z) \subset \mathbb{R}^{n+1}$ denotes the ball centered at $z \in \mathbb{R}^{n+1}$ of radius r.

Although our ε-regularity Theorem 2.1 is stated using a cylindrical excess $\mathcal{E}$ instead of $\mathscr{E}$, it is a classical fact that the result is still true with $\mathscr{E}$ in place of $\mathcal{E}$ (maybe, up to reduce the constant ε_0). So we assume that ε_0 is still a regularity scale for $\mathscr{E}$ and we define

$$\mathrm{Sing}(T) := \big\{ z \in \mathrm{supp}(T) \,:\, \liminf_{r \to 0^+} \mathscr{E}(T, z, r) \geq \varepsilon_0 \big\}.$$

The next result gives a bound on the size of the singular set.

Corollary 2.5. *Let $\mathcal{F} : \Lambda_n(\mathbb{R}^{n+1}) \to \mathbb{R}$ satisfy (2.2)-(2.7) with $\varrho_0 = 2$, and let T to be a n-dimensional multiplicity-1 rectifiable current rectifiable current in $\mathcal{C}_R$ such that:*

- $\mathrm{supp}(T) \subset \overline{\mathcal{C}}_R$ *is compact,* $\mathrm{supp}(\partial T) \subset \partial \mathcal{C}_R$*;*

- *T is $(\mathcal{F},\omega)$-minimal in $\mathcal{C}_R$.*

Set $\Omega(R) := \int_0^R \frac{\sqrt{\omega(r)}}{r}\,dr$ and assume that $\Omega(1) < \infty$. Then $\mathrm{Sing}(T)\cap\mathcal{C}_{R/2}$ is relatively closed inside $\mathrm{supp}(T)$, $(\mathrm{supp}(T)\setminus\mathrm{Sing}(T))\cap\mathcal{C}_{R/2}$ is a C^1 hypersurface, and

$$\mathscr{H}^{n-2-\nu}\big(\mathrm{Sing}(T)\cap\mathcal{C}_{R/2}\big) = 0$$

for some universal $\nu > 0$.

Proof. As we shall see, this proof is basically a minor modification of the argument that was first used in [19] and has been recently revised in [9] to estimate the size of the singular set for $C^{2,1}$ integrands. Here we briefly discuss why the proof still works in our setting, referring to the recent paper [9] for more details.

The fact that $\mathrm{Sing}(T)\cap\mathcal{C}_{R/2}$ is relatively closed inside $\mathrm{supp}(T)$ and that $(\mathrm{supp}(T)\setminus\mathrm{Sing}(T))\cap\mathcal{C}_{R/2}$ is a $C^{1,\delta}$ hypersurface follows by Theorem 2.2 and the definition of $\mathrm{Sing}(T)$, so the only point is to prove that $\mathscr{H}^{n-2-\nu}\big(\mathrm{Sing}(T)\cap\mathcal{C}_{R/2}\big) = 0$.

We begin by noticing that it is enough to prove that

$$\mathscr{H}^{n-2}\big(\mathrm{Sing}(T)\cap\mathcal{C}_{R/2}\big) = 0. \tag{2.9}$$

Indeed, a general argument due to Almgren (and written by White in [21, Lemma 5.1]) asserts that every time that one has a family of compact sets $\mathcal{K}$ satisfying suitable scaling and closure properties (in our case, this is the class of singular sets of all $(\mathcal{F},\omega)$-minimal currents as $\mathcal{F}$ varies among all the integrands satisfying (2.2)-(2.7) with $\varrho_0 = 2$), then the set of s such that $\mathscr{H}^s(K) = 0$ for all $K\in\mathcal{K}$ is open. So we are left with proving (2.9).

By a blow-up argument it is enough to prove the result when $\omega\equiv 0$ (equivalently, T minimizes $\mathcal{F}$). The main idea to prove (2.9) is to define the families

$$\mathscr{G} := \{\mathcal{F} : \text{ (2.2)-(2.7) hold with } \varrho_0 = 2\},$$

$$\mathscr{G}_* := \{\mathcal{F}\in\mathscr{G} : \text{(2.9) holds for any } T \text{ as in the statement of the corollary}\},$$

and show that $\mathscr{G}_*$ is both open and closed inside $\mathscr{G}$ for the local uniform convergence of the integrands $\mathcal{F}$. Indeed, noticing that the isotropic integrand $\mathcal{F}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|$ (corresponding to the area functional) belongs to $\mathscr{G}_*$ [14], it follows that $\mathscr{G}_*$ is nonempty and so this openness-closedness property combined with the connectedness of $\mathscr{G}$ (notice that $\mathscr{G}$ is convex) would imply that $\mathscr{G}_* = \mathscr{G}$, concluding the proof. Hence we are left with proving that $\mathscr{G}_*$ is both open and closed inside $\mathscr{G}$.

As shown in the proof of [9, Proposition 1.7], the fact that $\mathscr{G}_*$ is open is an easy consequence of the upper-semicontinuity of singular sets, which in our setting follows from the ε-regularity Theorem 2.1.

Concerning the closedness, the main point is to prove that the validity of (2.9) is equivalent to the fact that the second fundamental form II_T of the hypersurfaces corresponding to $\mathrm{supp}(T)\setminus\mathrm{Sing}(T)$ belongs to L^2 in the interior of $\mathcal{C}_R$. More precisely, one wants to show the following:[2]

Claim: (2.9) is equivalent to

$$\int_{\mathcal{C}_{R/2}\cap(\operatorname{supp}(T)\setminus\operatorname{Sing}(T))} \big|\mathrm{II}_T\big|^2 d\mathscr{H}^n \le C \tag{2.11}$$

where $C > 0$ is a universal constant.

Indeed, assuming the validity of the claim, one argues as follows (see the proof of [9, Proposition 1.8] for more details): if $\mathcal{F}_k$ is a sequence of integrands in $\mathscr{G}_*$ converging to $\mathcal{F}$, given T minimal for $\mathcal{F}$ one can approximate it with a sequence of currents T_k that are almost-minimal for $\mathcal{F}_k$. Since $\mathcal{F}_k \in \mathscr{G}_*$ it follows that $\mathscr{H}^{n-2}\big(\operatorname{Sing}(T_k)\cap\mathcal{C}_{R/2}\big) = 0$ and, thanks to the claim, one gets

$$\int_{\mathcal{C}_{R/2}\cap(\operatorname{supp}(T_k)\setminus\operatorname{Sing}(T_k))} \big|\mathrm{II}_{T_k}\big|^2 d\mathscr{H}^n \le C.$$

Hence, by the lower semicontinuity of the L^2 norm of the second fundamental form[3]

$$\int_{\mathcal{C}_{R/2}\cap(\operatorname{supp}(T)\setminus\operatorname{Sing}(T))} \big|\mathrm{II}_T\big|^2 d\mathscr{H}^n \le C,$$

and applying the converse implication of the claim one deduces that $\mathscr{H}^{n-2}\big(\operatorname{Sing}(T)\cap\mathcal{C}_{R/2}\big) = 0$, proving that $\mathcal{F} \in \mathscr{G}_*$ as desired. Hence we are left with proving the claim.

As shown in [9, Lemma 2.5], the fact that (2.9) implies (2.11) is a simple consequence of [9, Equation (A.13)] combined with a covering argument relying on capacity estimates. In [9] the authors assume that $\mathcal{F} \in C^2$ but actually their proof still works if $\mathcal{F} \in C^{1,1}$. Indeed, although [9, Equation (A.13)] is a consequence of the second variation formula for minimizers of C^2 functionals, the constant appearing there only depend on the L^∞ norm of $D^2\mathcal{F}$. So, if $\mathcal{F}$ is merely $C^{1,1}$ it is enough to approximate T with a family of almost minimizers with C^2 integrands (see for instance the beginning of the proof of [9, Proposition 1.8]) and then pass to the limit into [9, Equation (A.13)]. In this way one proves that (2.9) implies (2.11).

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[2]Notice that, at regular points, T coincides (in some suitable system of coordinates) with the graph of a $C^{1,\delta}$ function u that by minimality solves the Euler-Lagrange equation $\partial_i(F_{p_i}(\nabla u)) = 0$ for some $F : \mathbb{R}^n \to \mathbb{R}$ locally uniformly convex and $C^{1,1}$. Differentiating the Euler-Lagrange equation we obtain that

$$\partial_i\big(a_{ij}\partial_j(\partial_e u)\big) = 0 \qquad \forall\, e \in \mathbb{S}^{n-1},$$

where the coefficients $a_{ij} := F_{p_ip_j}(\nabla u)$ are bounded and uniformly elliptic. So, by Caccioppoli inequality, we deduce that $\partial_e u \in W^{1,2}$ for all $e \in \mathbb{S}^{n-1}$, which yields $u \in C^{1,\delta} \cap W^{2,2}$. Since

$$\big|\mathrm{II}_T\big|^2 = \partial_{ij}u\,\partial_{\ell m}u\,g^{ij}g^{\ell m} \tag{2.10}$$

where $g_{ij} := \delta_{ij} + \partial_i u\,\partial_j u$ and g^{ij} denotes the inverse matrix, it follows that $\mathrm{II}_T \in L^2_{\mathrm{loc}}(\operatorname{supp}(T)\setminus\operatorname{Sing}(T))$. Hence the claim asserts that the validity of (2.9) is equivalent to a global L^2 bound on II_T.

[3]As shown in the proof of [9, Lemma 2.4] it is enough to prove the lower semicontinuity when T_k are represented by the graphs of some functions $u_k \in C^1 \cap W^{2,2}$ that converge in C^1 and weakly in $W^{2,2}$ to some function u representing T (see also footnote 2). Recalling (2.10), the lower semicontinuity follows by the fact that $\big|\mathrm{II}_{T_k}\big|^2$ is a convex function of the Hessian of u_k (so it is lower semicontinuous under weak convergence).

The converse implication is based again on a covering argument together with the fact that if the L^2 norm of $|\mathrm{II}_T|$ is sufficiently small then the point is regular (see [9, Lemmas 2.6 and 2.7]). The validity of this latter fact in our setting is again a consequence of our ε-regularity Theorem 2.1 together with a simple contradiction argument (see the proof of [9, Lemma 2.6] for more details). With respect to the proof in [9, Lemma 2.6] we also need to rule out that a finite union of transversal hyperplanes can minimize an elliptic integrand, a fact proved in [19, Part I, Theorem 1.3] This proves the claim and completes the proof. $\square$

3. Preliminary results

In all this section, (H1)-(H2)-(H3) denote the hypotheses stated in Theorem 2.1.

We begin by stating some basic properties of minimizing currents.

First, as shown for instance in [7, Lemma 4], the following lower density estimate holds:

Lemma 3.1. *Assume (H1)-(H2) hold. Then, if $z_0 \in \operatorname{supp}(T)$, we have*

$$\|T\|(B_r(z_0)) \gtrsim r^n \qquad \forall\, r < R - |z_0|. \tag{3.1}$$

By a classical argument, the above density estimate combined the assumption that $0 \in \operatorname{supp}(T)$ provides a universal L^∞-bound on T.

Lemma 3.2. *Assume (H1) holds and let $u : B_R \to \mathbb{R}$ be a C^1 function with $\|\nabla u\|_{L^\infty(B_R)} \le 1$. There exists a small constant $\varepsilon_1 > 0$ such that if $\mathcal{E} := \mathcal{E}(T, R, u) \le \varepsilon_1$ and T satisfies (3.1), then*

$$\operatorname{supp}(T\lfloor_{\mathcal{C}_{R'}}) \subset \big\{|y - u(x)| \lesssim \mathcal{E}^{1/4n} R\big\}, \tag{3.2}$$

where $R' := \big(1 - \mathcal{E}^{1/2n}\big)\, R$.

Proof. Let $\Phi : \mathcal{C}_R \to \mathcal{C}_R$ be the map

$$\Phi(x, y) := (x, y - u(x)).$$

Since Φ is bi-Lipschitz it is easy to check that the current $\Phi_\# T$ still satisfies the density estimate (3.1). In addition

$$\mathcal{E}(T, R, u) \simeq \mathcal{E}(\Phi_\# T, R, 0),$$

hence the result follows by applying the classical L^∞ bound (see for instance [7, Lemma 5]) to the current $\Phi_\# T$. $\square$

Applying the classical Lipschitz Approximation Lemma (see for instance [2, Lemma 8.12], [7, Lemma 3], [18, Lemma 3]) to the current $\Phi_\# T$ defined in the proof above, we also deduce the following result:

Lemma 3.3 (Lipschitz Approximation Lemma)**.** *Assume that (H1) and (H3) hold, $u : B_R \to \mathbb{R}$ is a C^1 function with $\|\nabla u\|_{L^\infty(B_R)} \le 1$, $\mathcal{E} := \mathcal{E}(T, R, u) \le \varepsilon_1$ (ε_1 as in Lemma 3.2), T satisfies (3.1), and set $R' := \big(1 - \mathcal{E}^{1/2n}\big)\, R$. Then, for any $\gamma \in (0, 1]$ there exists a function $g_\gamma : B_{R'} \to \mathbb{R}$ and a compact set $\mathcal{K}_\gamma \subset B_{R'}$ such that:*

- *(a)* $\mathrm{Lip}(g_\gamma) \le \gamma$, $\|g_\gamma\|_{L^\infty(B_{R'})} \le \mathcal{E}^{1/4n} R$.
- *(b) The currents T and T_{u+g_γ} agree over $\mathcal{K}_\gamma$, that is $T\lfloor_{\boldsymbol{p}^{-1}(\mathcal{K}_\gamma)} = T_{u+g_\gamma}\lfloor_{\boldsymbol{p}^{-1}(\mathcal{K}_\gamma)}$, and the following estimates hold:*

$$\big|B_{R'} \setminus \mathcal{K}_\gamma\big| \lesssim \frac{\mathcal{E}\, R^n}{\gamma^2} \tag{3.3}$$

and

$$\|T\|\big(\mathcal{C}_{R'}\setminus \boldsymbol{p}^{-1}(\mathcal{K}_\gamma)\big)\lesssim \frac{\mathcal{E}\,R^n}{\gamma^2}. \tag{3.4}$$

We now show that the current T is close in L^1 to T_{u+g_γ}, with an estimate which is superlinear in the excess.

Lemma 3.4. *With the same notation as in Lemma 3.3, it holds*

$$\int_{\mathcal{C}_{R'}}|y-u(x)-g_\gamma(x)|\,d\|T\|\lesssim \frac{\mathcal{E}^{1+1/4n}\,R^{n+1}}{\gamma^2}.$$

Proof. We notice that, as a consequence of (3.2) and Lemma 3.3(a),

$$\sup\big\{|y-u(x)-g_\gamma(x)|:\,x\in B_{R'},\,(x,y)\in\operatorname{supp}(T)\big\}\lesssim \mathcal{E}^{1/4n}R.$$

Hence, recalling (3.4), we get

$$\int_{\mathcal{C}_{R'}}|y-u(x)-g_\gamma(x)|\,d\|T\|=\int_{\mathcal{C}_{R'}\setminus\boldsymbol{p}^{-1}(\mathcal{K}_\gamma)}|y-u(x)-g_\gamma(x)|\,d\|T\|$$

$$\lesssim \mathcal{E}^{1/4n}R\,\|T\|\big(\mathcal{C}_{R'}\setminus\boldsymbol{p}^{-1}(\mathcal{K}_\gamma)\big)\lesssim\frac{\mathcal{E}^{1+1/4n}\,R^{n+1}}{\gamma^2}.$$

□

The next result shows that, if we minimize a functional with small $C^{1,\eta}$-boundary data, then global regularity holds under the assumption that the integrand is $C^{1,1}$ and elliptic only in a neighborhood of the origin.

Lemma 3.5. *Let γ,δ,ρ be positive constants, let $F:\mathbb{R}^n\to\mathbb{R}$ be a convex function of class $C^{1,1}$ inside B_ρ and satisfying $D^2F(p)\geq\gamma\,\mathrm{Id}$ for $|p|\leq\rho$. Let $u:B_1\to\mathbb{R}$ minimize*

$$\int_{B_1}F\big(\nabla u(x)\big)\,dx,\qquad u|_{\partial B_1}=g$$

with $g\in C^{1,\eta}(\partial B_1)$ for some $\eta\in[\delta/2,\delta]$. Then there exists a small constant $\kappa>0$, independent of η, such that the following hold provided $\|g\|_{C^{1,\eta}(\partial B_1)}\leq\kappa$:

(1) There exists an exponent $\alpha\in(0,1)$, independent of δ,η,κ, such that

$$\|\nabla u\|_{C^{0,\alpha}(B_{1/2})}\lesssim\|\nabla u\|_{L^2(B_1)}.$$

(2) There exist an exponent $\beta\in(0,1)$, independent of δ,η,κ, such that

$$\|u\|_{C^{1,\zeta}(B_1)}\lesssim\|g\|_{C^{1,\eta}(\partial B_1)},\qquad \zeta:=\min\{\eta,\alpha,\beta\}$$

Proof. Notice that, to prove the above statements, we can assume that $F\in C^{1,1}(\mathbb{R}^n)$ and that $D^2F(p)\geq\frac{\gamma}{2}\,\mathrm{Id}$ for all $p\in\mathbb{R}^n$.

Indeed, consider an integrand $\hat F$ which is uniformly elliptic, of class $C^{1,1}$, and coincides with F for $|p|\leq\rho/2$, and assume that we can prove the above statements for u minimizing $\int_{B_1}\hat F\big(\nabla u(x)\big)\,dx$. Then it follows from (2) that, provided κ is chosen sufficiently small,

$$\|\nabla u\|_{L^\infty(B_1)}\leq\|u\|_{C^{1,\zeta}(B_1)}\leq C_0\,\|g\|_{C^{1,\eta}(\partial B_1)}\leq C_0\kappa\leq\frac{\rho}{2},$$

therefore $F(\nabla u) = \hat{F}(\nabla u)$. This implies that u solves the Euler-Lagrange equation associated to F and so, by the convexity of F, u is the unique minimizer of $\int_{B_1} F\big(\nabla u(x)\big)\,dx$, proving the desired result. Hence we are left with showing the lemma when F is globally $C^{1,1}$ and uniformly convex.

In this case we start by differentiating the Euler-Lagrange equation

$$\partial_i\big(F_{p_i}(\nabla u)\partial_j u\big) = 0 \tag{3.5}$$

to deduce that

$$\partial_i\big(F_{p_i p_j}(\nabla u)\partial_j(\partial_e u)\big) = 0 \qquad \forall\, e \in \mathbb{S}^{n-1},$$

so it follows by De Giorgi-Nash's Theorem that $u \in C^{1,\alpha}_{\rm loc}(B_{1/2})$ for some universal exponent $\alpha > 0$, with

$$\|\nabla u\|_{C^{0,\alpha}(B_{1/2})} \lesssim \|\nabla u\|_{L^2(B_1)}.$$

This proves point (1).

To show point (2), we notice that (3.5) can be rewritten as a linear uniformly elliptic equation with measurable coefficients

$$F_{p_i p_j}(\nabla u)\partial_{ij} u = 0.$$

Hence the global regularity follows in a standard way combining the interior $C^{1,\alpha}$ regularity provided by De Giorgi-Nash's Theorem (see the above discussion) with the $C^{1,\beta}$-boundary estimates for linear equations (see for instance [20, Theorem 1.1], where the result is proved in the more general context of fully nonlinear equations). □

4. Proof of Theorem 2.1

We begin by noticing that, up to perform the dilation $(x, y) \mapsto (x/R, y/R)$, we can assume that $R = 1$ and that

$$T \text{ is } \mathcal{F}\text{-minimal in } \mathcal{C}_1. \tag{4.1}$$

Let α and β be as in Lemma 3.5, and set

$$\delta := \min\left\{\alpha, \beta, \frac{1}{24n}\right\}, \qquad \theta := 1 - \frac{\delta}{4}, \qquad M := \frac{2}{(1-\theta)\delta}. \tag{4.2}$$

We define the series of radii

$$\rho_j := 1/2 + 2^{-j}, \qquad j \geq 1,$$

and we assume, for any $j = 1, \ldots, k$ we are given functions $u_j : B_{\rho_j} \to \mathbb{R}$ such that $\|u_j\|_{C^{1,\delta_j}(B_{\rho_j})} \leq \mu\, 2^{-j}$, where

$$\delta_j := \delta - \frac{1}{M}\sum_{i=1}^{j} \theta^i, \qquad j \geq 1, \tag{4.3}$$

and $\mu > 0$ is a small constants to be fixed later. Notice that, thanks to the definition of M, $\delta_j \in (\delta/2, \delta)$ for all $j \geq 1$.

Then we set $v_k := \sum_{j=1}^k u_j$ and assume that v_k minimizes

$$\int_{B_{\rho_k}} F(\nabla u)\,dx, \qquad F(p) := \mathcal{F}\Big(\boldsymbol{e}_0 + \sum_i p_i \hat{\boldsymbol{e}}_i\Big) \quad \forall\, p = (p_1, \ldots, p_n) \in \mathbb{R}^n, \tag{4.4}$$

and that

$$\mathcal{E}(T,\rho_k,v_k) \le \varepsilon_0^{(1+\delta/2)^k}$$

where ε_0 is a universally small constant (to be fixed later).

Claim: *There exists a function* $u_{k+1} : B_{\rho_{k+1}} \to \mathbb{R}$ *such that* $\|u_{k+1}\|_{C^{1,\delta_{k+1}}(B_{\rho_{k+1}})} \le \mu\, 2^{-(k+1)}$, $v_{k+1} := v_k + u_{k+1}$ *minimizes* (4.4) *in* $B_{\rho_{k+1}}$, *and*

$$\mathcal{E}(T,\rho_{k+1},v_{k+1}) \le \mathcal{E}(T,\rho_k,v_k)^{1+\delta/2}.$$

Notice that the claim implies the theorem. Indeed, thanks to our assumption, we can pick $u_1 \equiv 0$ and apply iteratively the claim above to deduce that

$$\mathcal{E}(T,\rho_k,v_k) \le \varepsilon_0^{(1+\delta/2)^k}, \qquad \|v_k\|_{C^{1,\delta_k}(B_{\rho_k})} \le \mu \qquad \forall\, k \ge 1,$$

so, by letting $k \to \infty$,

$$\mathcal{E}(T,1/2,v_\infty) = 0, \qquad v_\infty := \sum_{i=1}^{\infty} u_i \in C^{1,\delta/2}(B_{1/2}),$$

as desired. Hence our goal will be to prove the claim.

To make the proof easier to follow, we shall split the argument in several steps. Our proof is partly inspired by the arguments in [18].

Step 1: The Lipschitz approximations and some preliminary estimates. Set $\mathcal{E}_k := \mathcal{E}(T,\rho_k,v_k)$, let $\delta > 0$ be as in (4.2), and define $\hat g_\delta : B_{\rho_k'} \to \mathbb{R}$ to be the function g_γ provided by Lemma 3.3 with

$$u := v_k, \qquad R := \rho_k, \qquad \gamma := \mathcal{E}_k^{2\delta}, \qquad \rho_k' := \left(1 - \mathcal{E}_k^{1/4n}\right)\rho_k.$$

Also, let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a convolution kernel with support in B_1 and set

$$\tilde g_\delta := \hat g_\delta * \phi_{\mathcal{E}_k}, \qquad \phi_{\mathcal{E}_k}(x) := \frac{1}{\mathcal{E}_k^n}\,\phi\Big(\frac{x}{\mathcal{E}_k}\Big).$$

Then, thanks to Lemma 3.3(a)-(b) and by standard properties of convolution we have

$$\begin{cases} \|\nabla \tilde g_\delta\|_{L^\infty(B_{\rho_k'-\mathcal{E}_k})} \le \|\nabla \hat g_\delta\|_{L^\infty(B_{\rho_k'})} \le \mathcal{E}_k^{2\delta}, \\ \|\tilde g_\delta - \hat g_\delta\|_{L^\infty(B_{\rho_k'-\mathcal{E}_k})} \le \mathcal{E}_k\,\|\nabla \hat g_\delta\|_{L^\infty(B_{\rho_k'})} \le \mathcal{E}_k^{1+2\delta}, \\ \|\nabla \tilde g_\delta\|_{C^{0,\delta}(B_{\rho_k'-\mathcal{E}_k})} \le \mathcal{E}^{-\delta}\|\nabla \hat g_\delta\|_{L^\infty(B_{\rho_k'})} \le \mathcal{E}_k^{\delta}. \end{cases} \tag{4.5}$$

In addition

$$\|T\|(\mathcal{C}_1) - |B_1| = \frac{1}{2}\int_{\mathcal{C}_1} |\vec{T} - \boldsymbol{e}_0|^2\, d\|T\| \le \mathcal{E}(T,1,0),$$

hence

$$\|T\|(\mathcal{C}_1) \le |B_1| + \mathcal{E} \lesssim 1. \tag{4.6}$$

Notice that, provided ε_0 is sufficiently small, we can ensure that

$$\left(\rho_k' - \mathcal{E}_k\right) - \left(\rho_k - \mathcal{E}_k^{\delta}\right) \gtrsim \mathcal{E}_k^{\delta}. \tag{4.7}$$

Step 2: Finding a good slice to construct a comparison for T. Given $\sigma \in (0,1)$ we define the $(n-1)$-current

$$\langle T,\sigma\rangle := \partial(T\lfloor_{\mathcal{C}_\sigma}).$$

Combining Lemma 3.4 and (4.6) with the Coarea Formula

$$dx\lfloor_{B_1} = \int_0^1 \mathscr{H}^{n-1}\lfloor_{\partial B_\sigma}\,d\sigma, \qquad \|T\lfloor_{\mathcal{C}_1}\| \geq \int_0^1 \|\langle T,\sigma\rangle\|\,d\sigma$$

(see for instance [13, Sections 2.9.19, 3.2.22, 4.2.1]), we deduce the existence of $\sigma \in \big(\rho_k - \mathcal{E}_k^\delta, \rho_k' - \mathcal{E}_k\big)$ such that

$$\int_{\partial\mathcal{C}_\sigma} |y - v_k(x) - \hat{g}_\delta(x)|\,d\|\langle T,\sigma\rangle\| \lesssim \mathcal{E}_k^{1+1/4n-5\delta}, \qquad \mathbb{M}(\langle T,\sigma\rangle) \lesssim \mathcal{E}_k^{-\delta}, \tag{4.8}$$

where $\mathbb{M}(\langle T,\sigma\rangle)$ denotes the mass of the $(n-1)$-current $\langle T,\sigma\rangle$.

Now, our goal is to find a "good" comparison for T by gluing the slice $\langle T,\sigma\rangle$ to the graph of a function u minimizing $\mathbb{F}$ inside $\mathcal{C}_\sigma$ with boundary condition $\tilde{g}_\delta$. More precisely, let $u \in C^{1,\delta_k}(\overline{B}_\sigma,\mathbb{R})$ minimize (4.4) with boundary conditions $u = v_k + \tilde{g}_\delta$ on ∂B_σ.[4] Recalling that by assumption $\|v_k\|_{C^{1,\delta_k}} \leq \mu$ and $\delta_k \in (\delta/2,\delta)$, it follows from (4.5), (2.5), and (2.7) that, provided

$$\mu \leq \frac{\kappa}{2}, \qquad \mathcal{E}_k \text{ is sufficiently small,} \tag{4.9}$$

we can apply Lemma 3.5 to deduce that

$$\|u\|_{C^{1,\delta_k}(B_\sigma)} \lesssim \mu + \mathcal{E}_k^\delta. \tag{4.10}$$

Since v_k is also a minimizer, it follows by the maximum principle and Lemma 3.3(a) that

$$\|u - v_k\|_{L^\infty(B_\sigma)} \leq \|\tilde{g}_\delta\|_{L^\infty(\partial B_\sigma)} \leq \mathcal{E}_k^{1/4n},$$

hence by interpolation we get

$$\|u - v_k\|_{C^{1,\delta_{k+1}}(B_\sigma)} \lesssim \mathcal{E}_k^{\frac{1}{4n}\left(1-\frac{1+\delta_{k+1}}{1+\delta_k}\right)} \leq \mathcal{E}_k^{\frac{\delta_k-\delta_{k+1}}{8n}}.$$

Noticing that $\theta(1+\delta/2) > 1$ (see (4.2)) and recalling that $\delta_k - \delta_{k+1} = \frac{1}{M}\theta^{k+1}$ and $\mathcal{E}_k \leq \varepsilon_0^{(1+\delta/2)^k}$, this gives

$$\|u - v_k\|_{C^{1,\delta_{k+1}}(B_\sigma)} \lesssim \varepsilon_0^{\frac{\theta}{8nM}[\theta(1+\delta/2)]^k} \leq \mu\, 2^{-k} \qquad \forall\, k \geq 1 \tag{4.11}$$

provided ε_0 is sufficiently small. We now define the "comparison" current S as

$$S := T_u\lfloor_{C_\sigma}$$

and observe that $S = \vec{S}\,\|S\|$ with

$$\vec{S}(z) = \frac{\wedge_n \nabla W(\boldsymbol{p}(z))}{|\wedge_n \nabla W(\boldsymbol{p}(z))|} \qquad \forall\, z \in \mathcal{C}_\sigma, \quad W(x) := \big(x, u(x) + v_k(x)\big). \tag{4.12}$$

[4]By the homogeneity and ellipticity assumptions (2.2) and (2.4) on $\mathcal{F}$, the function F is convex so existence of a minimizer u is standard.

Step 3: Show that S provides a good approximation of T in terms of the excess. First, by the homogeneity condition (2.2) it follows that $\mathcal{F}(\vec{S}) = \langle d\mathcal{F}(\vec{S}), \vec{S}\rangle$, which gives

$$\begin{aligned}\mathbb{F}(T\lfloor_{\mathcal{C}_\sigma}) - \mathbb{F}(S\lfloor_{\mathcal{C}_\sigma}) &= \int_{\mathcal{C}_\sigma} \mathcal{F}(\vec{T})\, d\|T\| - \int_{\mathcal{C}_\sigma} \mathcal{F}(\vec{S})\, d\|S\| \\ &= \int_{\mathcal{C}_\sigma} \Big(\mathcal{F}(\vec{T}) - \langle d\mathcal{F}(\vec{S}), \vec{T}\rangle\Big)\, d\|T\| \\ &\quad + \int_{\mathcal{C}_\sigma} \langle d\mathcal{F}(\vec{S}), \vec{T}\rangle\, d\|T\| - \int_{\mathcal{C}_\sigma} \langle d\mathcal{F}(\vec{S}), \vec{S}\rangle\, d\|S\|.\end{aligned} \tag{4.13}$$

Noticing that (thanks to (4.12), (2.1), and (4.10))

$$|\vec{S} - \boldsymbol{e}_0| \lesssim |\nabla u| \lesssim \mathcal{E}_k^\delta + \mu, \tag{4.14}$$

assuming $\mathcal{E}_k$ and μ small enough we can use the uniform ellipticity assumption (2.5) to get

$$\int_{\mathcal{C}_\sigma} \Big(\mathcal{F}(\vec{T}) - \langle d\mathcal{F}(\vec{S}), \vec{T}\rangle\Big)\, d\|T\| \geq \frac{1}{2}\int_{\mathcal{C}_\sigma} |\vec{T} - \vec{S}|^2\, d\|T\|. \tag{4.15}$$

To control the last term in (4.13) we consider the map

$$H(t, x, y) := \big(x, (1-t)y + tu(x)\big), \qquad \forall\, t \in [0,1],\, x \in B_\sigma,\, y \in \mathbb{R},$$

and define the $(n+1)$-current

$$X := H_\#\big([0,1] \times T\lfloor_{\mathcal{C}_\sigma}\big).$$

In this way we see that

$$\partial X = (T - S)\lfloor_{\mathcal{C}_\sigma} - \Sigma, \qquad \Sigma := H_\#\big([0,1] \times \langle T, \sigma\rangle\big) \subset \partial\mathcal{C}_\sigma, \tag{4.16}$$

and because of (4.5), (4.8), and the fact that $u = v_k + \tilde{g}_\delta$ on ∂B_σ, it holds

$$\begin{aligned}\mathbb{M}(\Sigma) &\leq \int_{\partial\mathcal{C}_\sigma} |y - v_k(x) - \tilde{g}_\delta(x)|\, d\|\langle T, \sigma'\rangle\| \lesssim \int_{\partial\mathcal{C}_\sigma} |y - v_k(x) - \hat{g}_\delta(x)|\, d\|\langle T, \sigma'\rangle\| \\ &\quad + \int_{\partial\mathcal{C}_\sigma} |\hat{g}_\delta(x) - \tilde{g}_\delta(x)|\, d\|\langle T, \sigma'\rangle\| \lesssim \mathcal{E}_k^{1+1/4n-5\delta} + \mathcal{E}_k^{1+\delta}.\end{aligned} \tag{4.17}$$

We now notice that $d\mathcal{F}$ is 0-homogeneous (since $\mathcal{F}$ is 1-homogeneous), therefore (see (4.12))

$$d\mathcal{F}(\vec{S}) = d\mathcal{F}(\wedge_n dU).$$

Also, because u minimizes F (see (4.4)), the Euler-Lagrange equation for u corresponds to saying that $d\mathcal{F}(\wedge_n dU)$ is a closed n-form, hence

$$\langle \partial X, d\mathcal{F}(\wedge_n dU)\rangle = 0.$$

and by (4.16) and (4.17) we get

$$\big|\langle (T-S)\lfloor_{\mathcal{C}_\sigma}, d\mathcal{F}(\vec{S})\rangle\big| = \big|\langle \Sigma, d\mathcal{F}(\vec{S})\rangle\big| \lesssim \mathcal{E}_k^{1+1/4n-5\delta} + \mathcal{E}_k^{1+\delta}. \tag{4.18}$$

Combining (4.13), (4.15), and (4.18), we conclude that there exists a universal constant C_1 such that

$$\frac{1}{2}\int_{\mathcal{C}_\sigma} |\vec{S} - \vec{T}|^2\, d\|T\| \leq C_1\left(\mathcal{E}_k^{1+1/4n-5\delta} + \mathcal{E}_k^{1+\delta}\right) + \mathbb{F}(T\lfloor_{\mathcal{C}_\sigma}) - \mathbb{F}(S\lfloor_{\mathcal{C}_\sigma}). \tag{4.19}$$

To estimate the last term we simply observe that, since $\partial(T\lfloor_{\mathcal{C}_\sigma}) = \partial(S+\Sigma)$, the minimality condition (4.1) together with (4.17) implies that

$$\mathbb{F}(T\lfloor_{\mathcal{C}_\sigma}) - \mathbb{F}(S\lfloor_{\mathcal{C}_\sigma}) = \mathbb{F}(T\lfloor_{\mathcal{C}_\sigma}) - \mathbb{F}(S\lfloor_{\mathcal{C}_\sigma}+\Sigma) + \mathcal{F}(\Sigma) \lesssim \mathcal{E}_k^{1+1/4n-5\delta} + \mathcal{E}_k^{1+\delta}. \tag{4.20}$$

Combining this bound with (4.19) finally yields

$$\mathcal{E}(T,\sigma,v_k+u) \lesssim \int_{\mathcal{C}_\sigma} |\vec{T} - \vec{S}|^2\, d\|T\| \lesssim \mathcal{E}_k^{1+1/4n-5\delta} + \mathcal{E}_k^{1+\delta} = 2\,\mathcal{E}_k^{1+\delta},$$

where at the last step we used the definition of δ (see (4.2)). In particular, since $\mathcal{E}_k \leq \varepsilon_0^{(1+\delta/2)^k}$ we deduce that $\mathcal{E}_k^{1+\delta} \ll \mathcal{E}_k^{1+\delta/2}$ and

$$\sigma \geq \rho_k - \mathcal{E}_k^\delta \geq \rho_k - 2^{-k-1} = \rho_{k+1},$$

hence

$$\mathcal{E}(T,\rho_{k+1},u) \leq \mathcal{E}_k^{1+\delta/2}.$$

Recalling (4.11), this proves the claim with $u_{k+1} := u - v_k$ and concludes the proof of the theorem.

□

5. Proof of Theorem 2.2

The proof of the ε-regularity theorem for almost-minimizing currents follows the one of Theorem 2.1, with some minor modifications. More precisely, choose $\rho_k := R\,r^k$ for some $r \in (0,1)$ to be fixed later. We claim that there exists a sequence of linear functions $\ell_k : \mathbb{R}^n \to \mathbb{R}$ such that, for all $k \geq 0$,

$$\mathcal{E}_{k+1} \leq r^\alpha\,\mathcal{E}_k + \frac{\omega(\rho_k)}{r^{n+1}} \tag{5.1}$$

and

$$|\nabla \ell_{k+1} - \nabla \ell_k|^2 \leq \frac{\mathcal{E}_k + \omega(\rho_k)}{r^{n+1}}, \tag{5.2}$$

where

$$\mathcal{E}_k := \mathcal{E}(T,\rho_k,\ell_k)$$

and $\alpha > 0$ is given by Lemma 3.5(1). As we shall explain at the end of the section, this claim implies the result.

We set $\ell_0 \equiv 0$ so that $\mathcal{E}_0 = \mathcal{E}(T,R,0) \leq \varepsilon_0$, and we show how to construct ℓ_{k+1} once ℓ_k is given. However, before doing that, we first notice that applying iteratively (5.1) and using that

ω is non-decreasing, one obtains

$$
\begin{aligned}
\mathcal{E}_{k+1} &\le r^{(k+1)\alpha}\mathcal{E}_0 + \frac{1}{r^{n+1}}\sum_{j=0}^{k} r^{j\alpha}\omega(R\,r^{k-j}) \\
&\lesssim r^{(k+1)\alpha}\mathcal{E}_0 + r^{k\alpha}\sum_{j=0}^{k} r^{(j-k)\alpha}\omega(R\,r^{k-j}) \\
&\lesssim r^{(k+1)\alpha}\mathcal{E}_0 + r^{k\alpha}\sum_{j=1}^{k}\int_{r^{k-j}}^{r^{k-j-1}} \omega(R\,t)\,\frac{dt}{t^{1+\alpha}} + r^{k\alpha}\omega(R) \\
&= r^{k\alpha}\Big(\mathcal{E}_0 + \omega(R)\Big) + r^{k\alpha}\int_{r^{k/2}}^{1}\omega(R\,t)\,\frac{dt}{t^{1+\alpha}} + r^{k\alpha}\int_{r^k}^{r^{k/2}}\omega(R\,t)\,\frac{dt}{t^{1+\alpha}} \\
&\lesssim r^{k\alpha}\Big(\mathcal{E}_0 + \omega(R)\Big) + r^{k\alpha}\omega(R)\int_{r^{k/2}}^{1}\frac{dt}{t^{1+\alpha}} + r^{k\alpha}\omega(R\,r^{k/2})\int_{r^k}^{r^{k/2}}\frac{dt}{t^{1+\alpha}} \\
&\lesssim r^{k\alpha}\mathcal{E}_0 + r^{k\alpha/2}\omega(R) + \omega(R\,r^{k/2}).
\end{aligned}
$$

This implies that $\mathcal{E}_k \to 0$ as $k \to \infty$, and in addition $\mathcal{E}_k$ can be made arbitrary small for all $k \ge 1$ provided ε_0 is small enough.

Then, inserting this estimate into (5.2) one deduces that

$$
\begin{aligned}
|\nabla\ell_{k+1}| &\le \sum_{j=0}^{k}|\nabla\ell_{j+1} - \nabla\ell_j| \lesssim \sum_{j=0}^{k}\sqrt{\mathcal{E}_j + \omega(R\,r^j)} \\
&\lesssim \sum_{j=0}^{k}\Big(r^{j\alpha/2}\sqrt{\mathcal{E}_0} + r^{j\alpha/4}\sqrt{\omega(R)} + \sqrt{\omega(R\,r^{j/2})}\Big) \\
&\lesssim \sqrt{\mathcal{E}_0} + \sqrt{\omega(R)} + \int_0^1 \sqrt{\omega(R\,t)}\,\frac{dt}{t},
\end{aligned}
$$

which can also be made arbitrary small provided we choose ε_0 sufficiently small. In particular we can ensure that at every step $|\nabla\ell_k| \le \frac{\kappa}{2}$ with κ as in Lemma 3.5, which allows us to apply Lemma 3.5 (compare with Step 2 in the proof of Theorem 2.1). We now begin the proof of (5.1)-(5.2).

Up to perform the dilation $(x,y) \mapsto (x/\rho_k, y/\rho_k)$ we can assume that $\rho_k = 1$ and that

$$T \text{ is } (\mathcal{F},\omega_k)\text{-minimal in } \mathcal{C}_1 \text{ with } \omega_k(t) := \omega(\rho_k t). \tag{5.3}$$

Then, arguing exactly as in the proof of Theorem 2.1 with ℓ_k in place of v_k we arrive at the validity of (4.19)-(4.20) with the only difference that now T is not $\mathcal{F}$-minimal but $(\mathcal{F},\omega_k)$-minimal, therefore

$$\mathbb{F}(T\lfloor_{\mathcal{C}_\sigma}) - \mathbb{F}(S\lfloor_{\mathcal{C}_\sigma}) \lesssim \mathcal{E}_k^{1+1/4n-5\delta} + \mathcal{E}_k^{1+\delta} + \omega_k(\sigma).$$

Hence, recalling the definition of δ (see (4.2)) we obtain

$$\mathcal{E}(T,\sigma,u) \lesssim \int_{\mathcal{C}_\sigma}|\vec{T} - \vec{S}|^2\,d\|T\| \lesssim 2\,\mathcal{E}_k^{1+\delta} + \omega_k(\sigma). \tag{5.4}$$

We now want to relate $\mathcal{E}(T,\sigma,u)$ with $\mathcal{E}(T,r,\ell_{k+1})$, where $r\in(0,\sigma/4)$ has to be suitably chosen and $\ell_{k+1}(x):=\nabla u(0)\cdot x$. This part of the argument is pretty standard.

Let $r\in(0,\sigma/4)$ to be fixed later. We have

$$\begin{aligned}\int_{\mathcal{C}_{2r}}\big|\vec{T}-\vec{S}(0)\big|^2\,d\|T\| &\le 2\int_{\mathcal{C}_{2r}}\big|\vec{T}-\vec{S}\big|^2\,d\|T\|+2\int_{\mathcal{C}_{2r}}\big|\vec{S}-\vec{S}(0)\big|^2\,d\|T\|\\ &\le 2\int_{\mathcal{C}_{2r}}\big|\vec{T}-\vec{S}\big|^2\,d\|T\|+2\Big(\sup_{\mathcal{C}_{2r}}\big|\vec{S}-\vec{S}(0)\big|^2\Big)\,\mathbb{M}(T\lfloor_{\mathcal{C}_{2r}}).\end{aligned}\tag{5.5}$$

Observing that

$$\mathbb{M}(T\lfloor_{\mathcal{C}_{2r}})-\sqrt{1+|\nabla\ell_k|^2}\,|B_{2r}|=\frac{1}{2}\int_{\mathcal{C}_{2r}}\big|\vec{T}-\vec{L}_k\big|^2\,d\|T\|\le\mathcal{E}_k$$

where

$$\vec{L}_k:=\frac{\wedge_n\nabla L_k}{|\wedge_n\nabla L_k|},\qquad L_k(x):=\big(x,\ell_k(x)\big),$$

we get

$$\mathbb{M}(T\lfloor_{\mathcal{C}_{2r}})\lesssim r^n+\mathcal{E}_k.$$

Moreover it follows by (2.1) that $\vec{S}=\nabla u+O(|\nabla u|^2)$ and $|\nabla(u-\ell_k)|\lesssim\big|\vec{S}-\vec{L}_k\big|$. Hence, noticing that the $C^{0,\alpha}$ seminorms of ∇u and $\nabla(u-\ell_k)$ are equal, Lemma 3.5(1) and (H3) give

$$\begin{aligned}\sup_{\mathcal{C}_{2r}}\big|\vec{S}-\vec{S}(0)\big|^2 &\lesssim r^{2\alpha}[\nabla u]^2_{C^{0,\alpha}(B_{2r})}=r^{2\alpha}[\nabla(u-\ell_k)]^2_{C^{0,\alpha}(B_{2r})}\\ &\lesssim r^{2\alpha}\|\nabla(u-\ell_k)\|^2_{L^2(B_\sigma)}\lesssim r^{2\alpha}\int_{\mathcal{C}_\sigma}\big|\vec{S}-\vec{L}_k\big|^2\,dx\le r^{2\alpha}\int_{\mathcal{C}_\sigma}\big|\vec{S}-\vec{L}_k\big|^2\,d\|T\|.\end{aligned}$$

Combining these last two estimates with (5.5), we get

$$\begin{aligned}\int_{\mathcal{C}_{2r}}\big|\vec{T}-\vec{S}(0)\big|^2\,d\|T\| &\lesssim\int_{\mathcal{C}_{2r}}\big|\vec{T}-\vec{S}\big|^2\,d\|T\|+r^{2\alpha}\big(r^n+\mathcal{E}_k\big)\int_{\mathcal{C}_\sigma}\big|\vec{S}-\vec{L}_k\big|^2\,d\|T\|\\ &\le\int_{\mathcal{C}_{2r}}\big|\vec{T}-\vec{S}\big|^2\,d\|T\|\\ &\quad+2\,r^{2\alpha}\big(r^n+\mathcal{E}_k\big)\int_{\mathcal{C}_\sigma}\Big[\big|\vec{T}-\vec{S}\big|^2+\big|\vec{T}-\vec{L}_k\big|^2\Big]\,d\|T\|\\ &\lesssim\int_{\mathcal{C}_\sigma}\big|\vec{T}-\vec{S}\big|^2\,d\|T\|+r^{2\alpha}\big(r^n+\mathcal{E}_k\big)\,\mathcal{E}_k,\end{aligned}$$

that together with (5.3) and (5.4) gives

$$\frac{1}{2\,r^n}\int_{\mathcal{C}_{2r}}\big|\vec{T}-\vec{S}(0)\big|^2\,d\|T\|\lesssim\Big(\mathcal{E}_k^\delta+r^{2\alpha}+\frac{\mathcal{E}_k}{r^{n-2\alpha}}\Big)\mathcal{E}_k+\frac{\omega(\rho_k)}{r^n}.\tag{5.6}$$

Since

$$\frac{1}{2\,r^n}\int_{\mathcal{C}_{2r}}\big|\vec{T}-\vec{L}_k\big|^2\,d\|T\|\le\frac{\mathcal{E}_k}{r^n},$$

adding this inequality with (5.6) and using the bound $\mathbb{M}(T\lfloor_{\mathcal{C}_{2r}}) \geq |B_{2r}|$, we obtain

$$|\nabla u(0) - \nabla \ell_k|^2 \lesssim \big|\overrightarrow{S}(0) - \overrightarrow{L}_k\big|^2 \lesssim \frac{\mathcal{E}_k + \omega(\rho_k)}{r^n}.$$

Hence, if we set $\ell_{k+1}(x) := \nabla u(0) \cdot x$ (so that $\overrightarrow{L}_{k+1} = \overrightarrow{S}(0)$) we deduce that

$$\mathcal{E}_{k+1} \leq C_1 \left(\Big[\mathcal{E}_k^\delta + r^{2\alpha} + \frac{\mathcal{E}_k}{r^{n-2\alpha}}\Big]\mathcal{E}_k + \frac{\omega(\rho_k)}{r^n} \right)$$

$$|\nabla \ell_{k+1} - \nabla \ell_k|^2 \leq C_2 \, \frac{\mathcal{E}_k + \omega(\rho_k)}{r^n},$$

where $C_1, C_2 > 0$ are universal constants. We are now ready to select all the constants involved in this proof and complete the argument. First, we choose $r \in (0, 1/8)$ such that

$$C_1 r^{2\alpha} \leq \frac{r^\alpha}{2}, \qquad C_2 r \leq 1.$$

Then, recalling that $\mathcal{E}_k$ can be made arbitrary small for all $k \geq 1$ provided ε_0 is small enough (see the iteration argument described at the beginning of the proof), we take ε_0 sufficiently small so that

$$C_1 \Big[\mathcal{E}_k^\delta + \frac{\mathcal{E}_k}{r^{n-2\alpha}}\Big] \leq \frac{r^\alpha}{2} \qquad \forall\, k \geq 1.$$

Since with these choices $C_2 r \leq 1$ we deduce that (5.1)-(5.2) hold, as desired.

By (5.1)-(5.2), the iteration argument described at the beginning of the proof shows that T is C^1 at the origin. Then, since the same argument can be repeated replacing 0 by any other points $z \in \operatorname{supp}(T) \cap \mathcal{C}_{R/2}$, we deduce that T coincides with a C^1 graph inside $\mathcal{C}_{R/2}$, concluding the proof.

□

References

[1] E. Acerbi; N. Fusco: Local regularity for minimizers of nonconvex integrals. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **16** (1989), 603–636.

[2] W. K. Allard: On the first variation of a varifold. *Ann. of Math. (2)* **95** (1972), 417–491.

[3] G. Anzellotti; M. Giaquinta: Convex functionals and partial regularity. *Arch. Rational Mech. Anal.* **102** (1988), 243–272.

[4] F. J. Almgren Jr.: Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. *Ann. of Math. (2)* **87** (1968), 321–391.

[5] F. J. Almgren Jr.: Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Amer. Math. Soc.* **4** (1976), no. 165, viii+199 pp.

[6] F. J. Almgren Jr. Almgren's big regularity paper. Q-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2. With a preface by Jean E. Taylor and Vladimir Scheffer. *World Scientific Monograph Series in Mathematics, 1.* World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[7] E. Bombieri: Regularity theory for almost minimal currents. *Arch. Rational Mech. Anal.* **78** (1982), no. 2, 99–130.

[8] E. De Giorgi: Frontiere orientate di misura minima. (Italian) *Seminario di Matematica della Scuola Normale Superiore di Pisa*, 1960-61 Editrice Tecnico Scientifica, Pisa 1961, 57 pp.

[9] G. De Philippis; F. Maggi: Dimensional estimates for singular sets in geometric variational problems with free boundaries. *J. Reine Angew. Math.*, to appear.

[10] F. Duzaar; K. Steffen. Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals. *J. Reine Angew. Math.* **546** (2002), 73–138.

[11] M. Colombo; A. Figalli: Regularity results for very degenerate elliptic equations. *J. Math. Pures Appl.* **101** (2014), 94–117.

[12] M. Colombo; A. Figalli: An excess-decay result for a class of degenerate elliptic equations. *Discrete Contin. Dyn. Syst. Ser. S* **7** (2014), no. 4, 631–652.

[13] H. Federer: Geometric measure theory. *Die Grundlehren der mathematischen Wissenschaften, Band 153* Springer-Verlag New York Inc., New York 1969.

[14] H. Federer: The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. *Bull. Amer. Math. Soc.* **76** (1970), 767–771.

[15] H. Federer; W. H. Fleming: Normal and integral currents. *Ann. of Math. (2)* **72** (1960), 458–520.

[16] W. H. Fleming: On the oriented Plateau problem. *Rend. Circ. Mat. Palermo (2)* **11** (1962), 69–90.

[17] O. Savin: Small perturbation solutions for elliptic equations. *Comm. Partial Differential Equations* **32** (2007), 557–578.

[18] R. Schoen; L. Simon: A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals. *Indiana Univ. Math. J.* **31** (1982), no. 3, 415–434.

[19] R. Schoen; L. Simon; F. J. Almgren Jr.: Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals. I, II. *Acta Math.* **139** (1977), no. 3-4, 217–265.

[20] L.Silvestre; B. Sirakov: Boundary regularity for viscosity solutions of fully nonlinear elliptic equations. *Comm. Partial Differential Equations*, **39** (2014), no. 9, 1694-1717. .

[21] B. White: A regularity theorem for minimizing hypersurfaces modulo p. Geometric measure theory and the calculus of variations (Arcata, Calif., 1984), 413-427, *Proc. Sympos. Pure Math.*, **44**, Amer. Math. Soc., Providence, RI, 1986.

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